The Birch and Swinnerton-Dyer Conjecture and Related Problems

Abstracts

L3

Ashay Burungale (University of Texas)

Title: A proof of Sylvester's conjecture

Abstract: Sylvester's conjecture, originating in his 1879 study of ternary cubic equations, asserts that every prime $p \equiv 4, 7, 8 \pmod 9$ is the sum of two rational cubes. We discuss its recent resolution by Yin for primes $p \equiv 4, 7 \pmod 9$, and by Burngale and Tian for the remaining case $p \equiv 8 \pmod 9$.The proofs are rooted in Heegner's work and lie at the intersection of explicit complex multiplication and Shimura reciprocity, the Yuan–Zhang–Zhang generalization of the Gross–Zagier formula, auxiliary Rankin–Selberg constructions for Heegner points, and classical cubic descent along with its derived variant. They also rely on the resolution of the unbounded denominators conjecture, as well as recent rank-zero results and converse theorems towards the Birch–Swinnerton–Dyer conjecture for auxiliary CM elliptic curves.

Ana Caraiana (Imperial College London)

Title: Recent advances on modularity

Abstract: The modularity of rational elliptic curves is a celebrated result in its own right, via the link to Fermat's last theorem. It is also an essential input for the rank part of the Birch and Swinnerton-Dyer conjecture in analytic ranks 0 and 1. In this talk, I will survey recent advances on modularity over more general number fields, due to various researchers. I will particularly focus on elliptic curves over imaginary quadratic fields and on rational abelian surfaces.

John Cremona (University of Warwick)

Title: Hyperelliptic densities

Abstract: I will report on ongoing work with Manjul Bhargava and Tom Fisher concerning local and global densities for hyperelliptic curves. The talk will give a brief survey of the methods used and will mention some earlier joint results of a similar nature before describing our current results for hyperelliptic curves.

Noam Elkies (Harvard University)

Title: The quest for elliptic curves of high rank

Abstract: In 1922 Mordell proved that $E(\mathbf{Q})$ is a finitely-generated abelian group for every elliptic curve E over $\mathbf{Q}$. Mazur, 55 years later, determined the list of all torsion groups T that can appear. We still do not know which ranks r are possible, or even whether we should expect r to be bounded.

One way to try to surmise the answer is trying hard to find examples of large rank. We describe the history, diverse ingredients, and current state of the ongoing quest to find curves *E* that have large *r,* or whose rank is large given a condition such as the structure of *T* or the size (conductor, height, etc.) of *E*.

While this quest does not bear directly on BSD (the conjecture of Birch and Swinnerton-Dyer that is the main theme of this conference), it is informed by BSD at several junctures.

Kaivalya Kulkarni (Harvard University)

Title: Prime Quadratic Twists of Elliptic Curves

Abstract: We show that for an elliptic curve over Q with surjective mod-2 residual representation and 4-Selmer rank 0 or 2, there exist infinitely many prime quadratic twists of E with 4-Selmer rank (and hence Mordell-Weil rank) 0. This proves new cases of a conjecture which predicts that for any elliptic curve E/Q, there exist infinitely many primes p such that either Ep or E−p has Mordell-Weil rank 0. Similarly, we show that for an elliptic curve E/Q with surjective mod 2 residual representation and 4-Selmer rank 1 or 3, there exist infinitely many prime quadratic twists with 4-Selmer rank 1. The methods behind the proofs of the above results are inspired by work of A. Smith, and a key result in this work involves a classical invarianttheoretic formula for the difference of two Cassels-Tate pairings stemming from a classical invariant theoretic formulation of the Cassels-Tate pairing on the 2-Selmer group of an elliptic curve due to T. Fisher.

Marco Sangiovanni Vincentelli (Columbia University)

Title: Theta classes on modular curves and Euler systems

Abstract: I will present joint work with A. Burungale constructing new Euler systems for elliptic modular forms over an imaginary quadratic field. In a similar fashion to Kato’s celebrated construction, our method imposes no local condition on the modular form at p. i will present the key new ingredient i.e., a change in geometric perspective from previous work of Lei—Loeffler—Zerbes: we construct p-adic étale cohomology classes on modular curves arising from theta series. The key feature of our new Euler system is that it verifies a “rank 0 p-adic Gross—Zagier formula", which is surprisingly analogous to Kato’s celebrated explicit reciprocity law.

Ari Shnidman (Temple University)

Title: Sums of two rational cubes

Abstract: I'll discuss work with Alpoge and Bhargava, showing that a positive proportion of integers n can be written as a sum of two cubes while a positive proportion cannot. The proof involves combining the circle method with geometry of numbers, to compute the average size of the 2-Selmer group in a cubic twist family of elliptic curves. Together with parity results and converse theorems of Burungale--Skinner, we deduce the result. I'll also discuss more recent results on the distribution of 2-Selmer groups in the family of all elliptic curves over Q.

Alex Smith (Northwestern University)

Title: BSD implies Goldfeld's conjecture

Abstract: Given an elliptic curve E/Q, we show that 50% of the quadratic twists of E have 2^∞ Selmer corank 0, and 50% have 2^∞ Selmer corank 1. As one consequence, we show that BSD implies Goldfeld's conjecture on the distribution of analytic ranks in quadratic twist families.

This theorem was proved in three batches over a period of eight years. We go over the work that made each of these improvements possible, and we explain the difficulties in generalizing this result to e.g. higher dimensional abelian varieties.

Naomi Sweeting (MIT)

Title: New bounds on adjoint Selmer groups for Hilbert modular forms

Abstract: Let f be a cuspidal Hilbert modular eigenform of parallel weight 2, without CM.
The Selmer groups for the adjoint p-adic Galois representations attached to f have a rich history, including (to name just a few) works of Flach, Wiles, Taylor-Wiles, Kisin, and more recently Newton and Thorne. In particular, these Selmer groups are now all known to be finite. However, it is still not known in general that their lengths are given by appropriate L-values or congruence numbers, as predicted by the Tamagawa number conjecture. In this talk, I will report on joint work in progress with Chris Skinner, in which we bound the adjoint Selmer group for f in terms of the congruence number in new cases. Most notably, we are able to get a near-sharp bound even when the level of f is arbitrarily bad at p. The proof uses a system of auxiliary Galois cohomology classes, constructed by combining an idea of Skinner and Venkatesh with level-raising techniques inspired by bipartite Euler systems.

Ye Tian (Chinese Academy of Sciences)

Title: Refined Tunnell–Saito Theory and Non-vanishing Applications

Abstract: The classical Tunnell–Saito theorem determines which member of a quaternionic pair supports a given toric period. I will discuss a refinement that further identifies the relevant eigenspace of the normalized self-twist. I will also explain its applications to non-vanishing in anticyclotomic families, with particular attention to the exceptional CM case. This talk is based on joint work with A. Burungale, W. He, and X. Ye.

Ziquan Yang (Chinese University of Hong Kong)

Title: Producing cycles across characteristics

Abstract: I will discuss two complementary approaches to the Tate conjecture and to BSD over function fields: a constructive approach, in which one tries to produce algebraic cycles from cohomological data, and a passive approach, in which one shows that the relevant obstructions vanish. I will explain how these two perspectives appear in several of my works from recent years (with Hamacher, Zhao, Guo, and Urbanik), and conclude with very recent joint work with Xie on a new proof of Faltings' isogeny theorem based on equidistribution.

Shou-Wu Zhang (Princeton University)

Title: From Euclid to Gross-Zagier: A Mathematical Journey Across Two Millennia

Abstract: This lecture traces two mathematical developments that evolved largely independently over nearly two thousand years. One begins with Euclid's parametrization of right triangles and culminates in the arithmetic of elliptic curves and Heegner points. The other starts with Euler's and Dirichlet's work on prime numbers and leads to the theory of L-functions and their special values. The Gross–Zagier formula reveals a remarkable connection between these two worlds by expressing the arithmetic height of a Heegner point as the central derivative of an L-function. I will also describe some of its arithmetic applications and its far-reaching generalizations to Shimura curves and Shimura varieties. The story illustrates a recurring theme in mathematics: ideas developed in distant areas, often centuries apart, can ultimately converge to illuminate the same arithmetic phenomena.

David Zywina (Cornell University)

Title: Elliptic curves of low rank over number fields

Abstract: For a fixed number field K, we are interested in constructing many elliptic curves over K of low rank. Our main result says that if r is a nonnegative integer that is at most 4, then there are infinitely many elliptic curves over K of rank r. Our curves will be found by specializing well-chosen explicit families. We will use a result of Kai, which generalizes work of Green, Tao and Ziegler to number fields, to carefully choose our curves in the families.