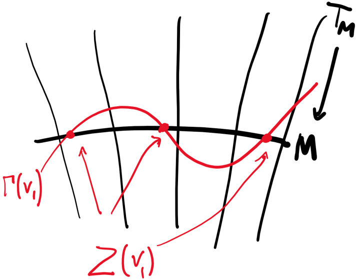
Localisation and commuting vector fields

work in progress with
Francesca Carocci, Yifan Zhao
and Maurício Corrêa

Poincaré-Hopf theorem

The Euler class of a compact oriented C^∞ manifold M is the Poincaré dual of the zero locus of a *generic* vector field v_1,

$$e(M) \cap [M] \;=\; \big[Z(v_1)\big].$$

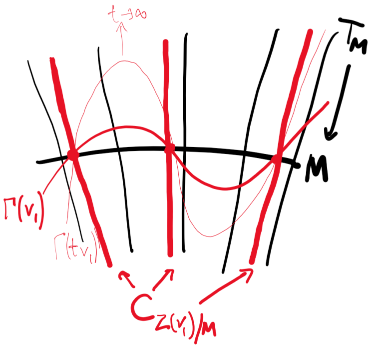


If v_1 is not generic there is still a *localisation* of $e(M)$ to $Z(v_1)$,

$$e(M, v_1) \;\in\; H_0(Z(v_1))\text{ such that }\;\iota_* e(M, v_1) \;=\; e(M) \cap [M].$$

Construction of Fulton-MacPherson refined Euler class

Replace $\Gamma(v_1)$ by $\lim_{t\to\infty}\Gamma(tv_1) =: C_{Z(v_1)/M} \subset T_M|_{Z(v_1)}$



and intersect with the 0-section $0_{T_M|_{Z(v_1)}}$ of $T_M|_{Z(v_1)}$:

$$e(M, v_1) \;:=\; 0^!_{T_M|_{Z(v_1)}}[C_{Z(v_1)/M}] \in A_0(Z(v_1)).$$

- $\iota_* e(M, v_1) = e(M) \cap [M]$,
- if $Z(v_1)$ is smooth then $e(M, v_1) = e(Z_1) \cap [Z_1]$.

More vector fields

If v_2 is a second vector fields *which commutes with* v_1,

$$\mathcal{L}_{v_2} v_1 \;=\; [v_2, v_1] \;=\; 0,$$

then it preserves $Z(v_1) \subset M$ (and so also the cone $C_{Z(v_1)/M}$). So we might hope to localise again to $Z\left(v_2|_{Z(v_1)}\right) = Z(v_1) \cap Z(v_2)$.
(Certainly cannot do this for arbitrary noncommuting v_1, v_2!)

So on a manifold M with commuting vector fields $v_1, \ldots, v_r$ it is natural to ask

if $v_1, \ldots, v_r$ have no common zeros, is $e(M) = 0$?

If M is complex (or almost complex or symplectic, with structure preserved by the flows) and $c(M) \in H^{2\dim_{\mathbb{C}} M}(M)$ is a top degree characteristic class (product of Chern classes) we can ask

can we localise $c(M)$ to $Z(v_1) \cap \cdots \cap Z(v_r)$?

Torus localisation

Yes if $r = 1$, obviously.

Yes if the vector fields induce the action of a torus $T \curvearrowright M$. Then $e(M)$, $c(M)$ lift naturally to $H^*_T(M)$ and we can use the Atiyah-Bott-Berline-Vergne localisation formula.

Lots of good things happen in this case

- $\bigcup_{x \in M \setminus M^T} T^x \subset T$ is a finite union of proper subtori,
- there exists a 1-p.s. $S^1 \subset T$ such that $M^{S^1} = M^T$,
- the $Z(v_i)$ and their intersections are smooth,
- T acts with discrete nonzero weights on their normal bundles,
- we can average over the group (e.g. to show the equivariant Cartan and Weil de Rham complexes are acyclic on $M \setminus M^T$ after localising $H^*(BT) = \mathbb{C}[t_1, \ldots, t_r]$ at $(t_1, \ldots, t_r)$).

Noncompact groups or Lie algebra actions

Let $G \curvearrowright M$ be the action of a connected *abelian* Lie group.

Or let $\mathfrak{g} \to \Gamma(T_M)$ be a Lie algebra homomorphism, where $\mathfrak{g}$ has bracket zero.
(Call the image of a basis $v_i \in \Gamma(T_M)$ so that $[v_i, v_j] = 0$.)

None of the good things are true in the general case

- $\bigcup_{x \in M \setminus M^{\mathfrak{g}}} \mathfrak{g}^x$ can be the whole of $\mathfrak{g}$,
- we can have $M^{\mathbb{R}} \supsetneq M^{\mathfrak{g}}$ for every $\mathbb{R} \subset \mathfrak{g}$,
- the $Z(v_i)$ and their intersections can be singular, nonreduced,
- G can act with zero eigenvalues on their normal bundles,
- we cannot average over noncompact groups G (so although we can form the equivariant Cartan and Weil de Rham complexes, we cannot prove they are acyclic on $M \setminus M^G$ after localising).

In the holomorphic case, Baum-Bott show how to localise to the locus where the rank of the action $\mathfrak{g} \otimes \mathcal{O} \to T_M$ drops rank, but we want to localise to the locus where it drops to zero.

Examples

• $x\partial_y$ on $\mathbb{C}^2_{x,y}$ has fixed locus the y-axis and trivial action on the normal bundle $\langle\partial_x\rangle$.

• $x^2\partial_y$ and $xy\partial_y$ on $\mathbb{C}^2_{x,y}$ have singular fixed loci.

• The tautological sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$ defines an action of $\mathfrak{g} = \mathbb{C}^2$ by translation up the fibres of $M := \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(1)\right) \to \mathbb{P}^1$. Thus its pull back to M is

$$0 \longrightarrow \ker\rho \longrightarrow \mathfrak{g}\otimes\mathcal{O}_M \longrightarrow \operatorname{im}\rho \longrightarrow 0,$$

where $\rho : \mathfrak{g}\otimes\mathcal{O}_M \to T_M$ is the action.
(Can compactify to projective completion $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}\big) = \mathbb{F}_1$ or $\mathbb{P}^2$ and embed in a compact 3-fold on which $\mathfrak{g}$ acts generically freely.)

Thus the stabiliser groups $\ker\rho$ cover $\mathfrak{g}$ and $M^{\mathbb{C}} \supsetneq M^{\mathfrak{g}}$ for every 1-parameter sub Lie algebra $\mathbb{C}\subset\mathfrak{g}$ (similarly for $\mathbb{R}\subset\mathfrak{g}$).

Known results

Given a closed manifold M with r commuting vector fields without common zeros, we know $e(M) = 0$ if

- $r = 1$
- any r and M if induced by a torus action
- $r = 2$, $\dim M = 2$ [Lima, 1964]
- $r = 2$ $\dim M = 3$ (local version) [Bonatti 1992, Bonatti-Santiago 2022]
- $r = 2$, $\dim M = 4$ [Bonatti 1992, Bonatti-Santiago 2025]

Dynamical systems methods, Poincaré-Bendixson theory,...

Bonatti has conjectured the result in all dimensions, but his conjecture seems to have been completely ignored.

This is a shame, as it would give a new type of localisation to calculate certain integrals.

Commuting vector fields on smooth projective varieties

Suppose we have r commuting holomorphic vector fields on a smooth projective variety M of dimension n.

Equivalently a Lie algebra $\mathfrak{g}$ with vanishing Lie bracket, $\dim \mathfrak{g} = r$ and a homomorphism of sheaves of Lie algebras $\rho : \mathfrak{g} \otimes \mathcal{O}_M \to T_M$.

Encode these vector fields in a single twisted vector field $\hat{\rho} : \mathcal{O}_{M \times \mathbb{P}(\mathfrak{g})} \to T_M \boxtimes \mathcal{O}(1)$ on $M \times \mathbb{P}(\mathfrak{g})$, from the composition

$$\pi^*_{\mathbb{P}(\mathfrak{g})} \mathcal{O}(-1) \hookrightarrow \pi^*_{\mathbb{P}(\mathfrak{g})} \mathfrak{g} \longrightarrow \pi^*_M T_M.$$

Transverse case

To begin with let's fantasise that $\hat{\rho}$ is transverse with zero locus $Z \subset M \times \mathbb{P}(\mathfrak{g})$ of codimension n, Poincaré dual to

$$c_n(T_M \boxtimes \mathcal{O}(1)) \;=\; c_n(T_M) + h \cdot c_{n-1}(T_M) + \cdots + h^{r-1} \cdot c_{n-r+1}(T_M)$$

(suppressing some π_M^*s and $\pi_{\mathbb{P}(\mathfrak{g})}^*$s and letting $h := c_1\big(\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)\big)$).

It is $\mathfrak{g}$-invariant and $(r-1)$ dimensional.

- Over any point of $M_0 = M^{\mathfrak{g}}$ the whole fibre $\mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^{r-1} \subseteq Z$ is already of dimension $r-1$, so it is isolated in Z,
- Through any point $x \in M_1$ is a 1-dimensional $\mathfrak{g}$-orbit whose product with $\mathbb{P}(\mathfrak{g}^x) \cong \mathbb{P}^{r-2}$ lies in Z, already of $\dim = r-1$,
- $\ldots$
- Through any point $x \in M_i$ is an i-dimensional $\mathfrak{g}$-orbit whose product with $\mathbb{P}(\mathfrak{g}^x) \cong \mathbb{P}^{r-1-i}$ lies in Z, already of $\dim = r-1$.

So get a finite number of $\dim < r$ orbits from which we read off the terms in $c_n(T_M) + h \cdot c_{n-1}(T_M) + \cdots + h^{r-1} \cdot c_{n-r+1}(T_M)$.

Transverse case too simple?

So no moving stabilisers in this case. $(*)$

$$c_n(T_M \boxtimes \mathcal{O}(1)) \;=\; c_n(T_M) + h \cdot c_{n-1}(T_M) + \cdots + h^{r-1} \cdot c_{n-r+1}(T_M)$$

Cupping with h^{r-1} and pushing down $\pi_M : M \times \mathbb{P}(\mathfrak{g}) \to M$ to get $c_r(T_M)$ — Poincaré dual to intersecting with $M \times \{\mathrm{pt}\}$ — kills all $M_{i>0}$ contributions, leaving only the $M_0 \times \mathbb{P}(\mathfrak{g}) = M^{\mathfrak{g}} \times \mathbb{P}(\mathfrak{g})$ piece.

Gives $e(M) = \mathrm{length}(M^{\mathfrak{g}})$ but for a trivial reason:
Cupping with h^{r-1} is Poincaré dual to restricting to one vector field $\mathbb{C} \cdot v \subset \mathfrak{g}$, where $[v] = \mathrm{pt} \in \mathbb{P}(\mathfrak{g})$ above.
And $M^{\mathfrak{g}} = M^{\mathbb{C} \cdot v}$ by $(*)$ so this is the Poincaré-Hopf theorem again.

General case

When $\hat{\rho}$ is not transverse we can have moving stabilisers: $Z \to \mathbb{P}(\mathfrak{g})$ can be onto even after removing $M^{\mathfrak{g}} \times \mathbb{P}(\mathfrak{g})$.

Need to use a virtual argument (excess intersection theory) before cupping with h^{r-1} or intersecting with $M \times \{\mathrm{pt}\}$.

In the Fulton-Macpherson localisation of $c_n(T_M \boxtimes \mathcal{O}(1))$ to Z,

$$0^!_{T_M \boxtimes \mathcal{O}(1)}[C_{Z/M \times \mathbb{P}(\mathfrak{g})}]$$

individual terms all $\mathfrak{g}$-invariant. Suppose we can take the intersection to be represented by a $\mathfrak{g}$-invariant cycle $[Z]^{\mathrm{vir}}$.

Since $\dim [Z]^{\mathrm{vir}} = r - 1$ we have the same argument as before,

- Any component dominating $\mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^{r-1}$ is finite over it and $\mathfrak{g}$-invariant so must lie over M_0 — i.e. in $M^{\mathfrak{g}} \times \mathbb{P}(\mathfrak{g})$,
- Any component with $(r-2)$-dimensional image in $\mathbb{P}(\mathfrak{g})$ has 1-dimensional fibres so lies over M_1, ...

$\pi_{M*}(h^{r-1} \cap \cdot\,)$ kills all terms except the first, giving a cycle in $M^{\mathfrak{g}}$.

Dichotomy

So if we can do $\mathfrak{g}$-invariant intersection theory we get a localisation.

But if E is an elliptic curve the E-invariant line bundle $\mathcal{O}_E(p-q)$ has no E-invariant first Chern class in $A_0(E)$.

Now $\mathfrak{g}$ generates a closed abelian subgroup $G \subseteq \mathrm{Aut}_0(X)$ with

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \longrightarrow A \longrightarrow 1$$

with either (i) $A = 1$, or
(ii) A an abelian variety acting nontrivally on $\mathrm{Alb}(X)$.

In case (i) G is a product of $\mathbb{C}^*$s and $\mathbb{C}$s and there is a G-invariant intersection theory (Brion, etc).

The action in (ii) has no zeros, so we get a vector field (from a direction in $\mathfrak{g}$) on $X \to \mathrm{Alb}(X)$ with no zeros, so $c_n(T_X) = 0$ and $X^{\mathfrak{g}} = \emptyset$ and our localisation is trivial; there is nothing to prove.

Enumerative algebraic geometry motivation

Two main ways to count curves in Calabi-Yau 3-folds X.

- Stable maps $f : C \to X \rightsquigarrow$ GW theory
- Coherent sheaves on $X \rightsquigarrow$ DT theory
  - (1) ideal sheaves like $I_C \subset \mathcal{O}_X$
  - (2) stable pairs like $\mathcal{O}_X \to \mathcal{O}_C(p_i),\ p_i \in C$
  - (3) sheaves pushed forward from C (1-dimensional sheaves on X)

MNOP conjecture: they all contain the same information (need cohomological DT theory in third case to see GW theory in genus > 0)

We concentrate on (3) ordinary DT theory of dimension 1 sheaves (Katz), equivalent via MNOP to genus 0 GW theory.

Roughly, each C contributes $\overline{\mathrm{Jac}}(C)$ to the moduli space (3), with

$$e\big(\overline{\mathrm{Jac}}(C)\big) \;=\; \begin{cases} 0 & g_g(C) := g(\overline{C}) > 0 \\ 1 & C \cong \mathbb{P}^1 \end{cases}$$

being some kind of count of genus 0 curves in X.

Geometric relationship between moduli spaces

For an irreducible class $\beta \in H_2(X,\mathbb{Z})$, the moduli space $\overline{\mathcal{M}}_0(X,\beta)$ of genus 0 stable maps sits inside the moduli space $M_{(0,0,\beta,1)}(X)$ of 1-dimensional sheaves of class β and $\chi = 1$,

$$\overline{\mathcal{M}}_0(X,\beta) \ \ni \ \big(f:\mathbb{P}^1 \to X\big) \ \longmapsto \ f_*\mathcal{O}_{\mathbb{P}^1} \ \in \ M_{(0,0,\beta,1)}(X).$$

Conversely use the support map $M_{(0,0,\beta,1)}(X) \to \mathrm{Chow}(X,\beta)$ with fibres $\overline{\mathrm{Jac}}(C)$ acted on by $\mathrm{Jac}(C)$.

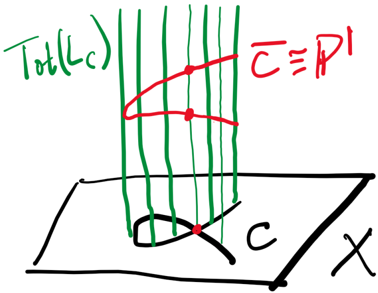
Claim fixed locus is $\overline{\mathcal{M}}_0(X,\beta) \subseteq M_{(0,0,\beta,1)}(X)$.

Firstly, $f_*\mathcal{O}_{\mathbb{P}^1}$ is fixed because f^*L trivial.

Conversely a rank 1 sheaf E on C with $E \cong E \otimes L$ for all $L \in \mathrm{Jac}(C)$ gives a *Higgs pair* $\phi : E \xrightarrow{\sim} E \otimes L$ to which we apply the spectral construction.

Spectral construction

We graph (the eigenvalue of) $\phi : E \xrightarrow{\sim} E \otimes L$ in $\mathrm{Tot}(L)$



More formally, $\phi : E \to E \otimes L$ makes $\mathcal{O}_C$-module E into a $\left(\pi_* \mathcal{O}_{\mathrm{Tot}(L)} = \mathcal{O}_C \oplus L^{-1} \oplus L^{-2} \oplus \dots\right)$-module; i.e. a sheaf on $\mathrm{Tot}(L)$ whose pushdown to C is E.

For generic L is it $\mathcal{O}_{\overline{C}} \cong \mathcal{O}_{\mathbb{P}^1}$ because invariance under $\otimes L$ means $\pi^* L$ trivial for generic L which only happens when $\overline{C} \cong \mathbb{P}^1$.
(Reducible curves ok too. For nonreduced curves gives $\sim$ Nekrasov-Okounkov stable maps of nonreduced curves to the 4-fold $\mathrm{Tot}(L)$).

Infinitesimal spectral construction

So we see $\overline{\mathcal{M}}_0(X,\beta) \subset M_{(0,0,\beta,1)}(X)$ as the fixed locus of the action of the commutative group scheme $\mathrm{Jac}(\mathcal{C}) \to \mathrm{Chow}(X,\beta)$.

To replace groups by vector fields we replace $L \in H^1(\mathcal{O}_C^\times)$ by $v \in H^1(\mathcal{O}_C)$ inducing a vector field $\widetilde{v}$ on $M_{(0,0,\beta,1)}(X)$.

By an **infinitesimal spectral construction** zeros $E \in M_{(0,0,\beta,1)}(X)$ of $\widetilde{v}$ are sheaves $\mathcal{O}_{\mathbb{P}^1}$ on L_v (the *affine bundle* or $\mathbb{C}$-torsor $L_v \to C$ defined by v) which push down to E.

So we now see $\overline{\mathcal{M}}_0(X,\beta) \subset M_{(0,0,\beta,1)}(X)$ as the zero locus of local vector fields $\widetilde{v}$ on $M_{(0,0,\beta,1)}(X)$, or global meromorphic vector fields, and as the moduli space $M_{(0,0,\beta,1)}(L_v)$ of 1-dimensional sheaves on the Calabi-Yau 4-fold L_v (modulo translations by $\mathbb{C}$).

We expect a kind of virtual localisation for the $\mathfrak{jac}(\mathcal{C})$ action on $M_{(0,0,\beta,1)}(X)$ to give something like

$$\begin{aligned} \mathrm{GW}_0(X,\beta) &= \mathrm{DT}^3(0,0,\beta,1)(X) \\ &= \mathrm{DT}^4(0,0,\beta,1)(L_v)/\mathbb{C} \;=\; \mathrm{GW}_0(L_v,\beta)/\mathbb{C}. \end{aligned}$$

Partial explanation: cosection localisation

X is CY^3

$\implies$ $M_{(0,0,\beta,1)}(X)$ has a "symmetric obstruction theory"

$\implies$ obstruction sheaf is $\mathrm{Ob} = \Omega_{M_{(0,0,\beta,1)}(X)}$

$\implies$ $\widetilde{v}$ defines a "cosection"

$$\mathrm{Ob} \xrightarrow{\widetilde{v}} \mathcal{O}_{M_{(0,0,\beta,1)}(X)}$$

$\implies$ [Kiem-Li] localise $[M_{(0,0,\beta,1)}(X)]^{\mathrm{vir}}$ to $Z(\widetilde{v}) = \overline{\mathcal{M}}_0(X,\beta)$.

Thought of as a (-1)-*shifted 1-form*

$$\mathbb{T}_{M_{(0,0,\beta,1)}(X)}[1] \xrightarrow{\widetilde{v}} \mathcal{O}_{M_{(0,0,\beta,1)}(X)}$$

it is *closed*.

By [Kiem-Park] the zero locus of a (-1)-shifted closed 1-form is (-2)-*shifted symplectic* with [BJ, OT] virtual cycle the localised cycle. We expect this (-2)-sss comes from thinking of the zeros as sheaves on the CY fourfold L_v (modulo translation by $\mathbb{C}$).

Invariants

This would give the $\mathrm{DT}^3 = \mathrm{DT}^4$ relation.

Easy to see the (-2)-shifted symplectic DT^4 obstruction theory is a "doubling" of the stable maps obstruction theory in L_v, which should give $\mathrm{DT}^4 = \mathrm{GW}^4$.

Finally the stable maps obstruction theory on L_v (modulo translation), pushed down by $\pi : L_v \to X$, is the stable maps obstruction theory on X, giving $\mathrm{GW}^3 = \mathrm{GW}^4$.

An ongoing project...for now the biggest obstacle is that the cosection localisation can only be done locally over $\mathrm{Chow}(X, \beta)$ or — working globally with meromorphic vector fields — it needs to be iterated or extended from one vector field to many commuting vector fields.

Motivates wanting to localise to zeros of commuting vector fields in both classical and virtual settings.

Happy Birthday Nigel!

With admiration and thanks