Analytic versions of the (quantum) geometric Langlands correspondence

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Based on joint work with

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and an incredible amount of inspiration from **Nigel's** wonderful papers!

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Geometric Langlands-correspondence – schematic view

Schematically, the geometric Langlands-correspondence is often presented as relation

$$\boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G(C)} \quad \leftrightarrow \quad \boxed{{}^{\mathrm{L}}\mathfrak{g}\text{-local systems on } C}$$

where C: Riemann surface, compact, or with punctures,

- G: complex semi-simple Lie group with Lie algebra $\mathfrak{g}$.
- $\mathcal{D}$-modules on $\mathrm{Bun}_G(C)$ (often represented by PDE)
- ${}^{\mathrm{L}}\mathfrak{g}$: Langlands dual of Lie algebra $\mathfrak{g}$.
- Local systems: pairs $(\mathcal{E}, \nabla)$, where
  - $\mathcal{E}$ holomorphic ${}^{\mathrm{L}}G$-bundle,
  - ∇ connection on $\mathcal{E}$.

First instances formulated and proven in work of Drinfeld and Beilinson-Drinfeld, full proof presented recently (Gaitsgory, Raskin et.al.).

Analytic versions:

Consider <u>single-valued</u> and <u>square-integrable</u> $\mathcal{D}$-modules on $\mathrm{Bun}_G(C)$.

Does there exist a nice description of the ${}^{\mathrm{L}}\mathfrak{g}$-local systems they correspond to?

Hitchin's integrable system

Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}(C)$ (C: Riemann surface): Moduli space of pairs $(\mathcal{E}, \varphi)$,

- $\mathcal{E}$: holomorphic $G = (P)SL(2)$-bundle on C,
- $\varphi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$.

$\mathcal{M}_{\mathrm{Hit}}(C)$ has a canonical Poisson structure.

Integrability: Let $q := \frac{1}{2}\mathrm{tr}(\varphi^2) \in H^0(C, K^2)$. We have

$$q(z) = \sum_{r=1}^{3g-3+n} H_r Q_r(z), \qquad \{H_r, H_s\} = 0.$$

Quantisation (Beilinson-Drinfeld):

There exist differential operators H_r, $r = 1, \ldots, 3g-3+n$ on $K_{\mathrm{Bun}}^{1/2}$, satisfying

$$[\,\mathsf{H}_r, \mathsf{H}_s\,] = 0, \qquad r, s = 1, \ldots, d, \qquad D = 3g-3+n,$$

and having Hitchin's Hamiltonians H_r as their symbols.

Geometric Langlands-correspondence – case of opers

An important special case of the geometric Langlands-correspondence:[1]

$$\boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G(C)} \quad \leftrightarrow \quad \boxed{{}^{\mathrm{L}}\mathfrak{g} \text{ opers } (\mathcal{E}_{\mathrm{op}}, \nabla_E) \text{ on } C}$$

with G: complex group with Lie algebra $\mathfrak{g}$, ${}^{\mathrm{L}}\mathfrak{g}$: Langlands dual Lie algebra of $\mathfrak{g}$,

$\mathcal{D}$-**modules** represented by the **Hitchin eigenvalue equations**

$$\mathsf{H}_r \Psi_E = E_r \Psi_E,$$

and **opers** $\chi_E = (\mathcal{E}_{\mathrm{op}}, \nabla_E)$, for $\mathfrak{g} = \mathfrak{sl}_2$,

$$\nabla_E = dz\Big(\partial_z + \begin{pmatrix} 0 & t_E \\ 1 & 0 \end{pmatrix}\Big), \qquad t_E = t_0 + q_E, \qquad q_E(z) = \sum_{r=1}^{D} E_r Q_r(z),$$

$$0 \to K^{\frac{1}{2}} \to \mathcal{E}_{\mathrm{op}} \to K^{-\frac{1}{2}} \to 0.$$

[1] Drinfeld; Beilinson-Drinfeld

Analytic Langlands correspondence

For given oper χ_E we may consider the pair of complex conjugate eigenvalue equations

$$\mathsf{H}_r\,\Psi_E = E_r\Psi_E, \qquad \bar{\mathsf{H}}_r\,\Psi_E = \bar{E}_r\Psi_E, \qquad r = 1,\dots,d.$$

Look for solutions in $K_{\mathrm{Bun}}^{1/2} \otimes \bar{K}_{\mathrm{Bun}}^{1/2}$, smooth away from "wobbly" bundles[2], locally

$$\Psi_E(\boldsymbol{x},\bar{\boldsymbol{x}}) = \sum_{k,l} C_{kl}\,\psi_k(\boldsymbol{x})\bar{\psi}_l(\bar{\boldsymbol{x}}),$$

$(\boldsymbol{x} = (x_1,\dots,x_D)$: local coordinates on $\mathrm{Bun}_G^{\mathrm{vs}}(C))$, which are

single-valued, and furthermore **square-integrable**.

Conjecture:[3,4] – Geometric answer to a hard harmonic analysis problem –

$$\boxed{\begin{Bmatrix}\text{single-valued } \& \\ L^2\text{-normalisable}\end{Bmatrix} \text{ Hitchin eigen-}\mathcal{D}\text{-modules}} \quad \leftrightarrow \quad \boxed{\text{Real opers}}$$

[2]Wobbly: Admitting nilpotent Higgs fields.

[3]Single-valuedness: J.T., 2011, 2017.

[4]Square-integrability: Etingof-Frenkel-Kazhdan, 2019.

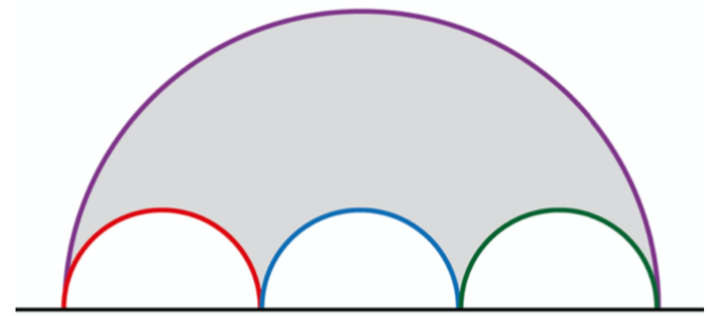
Real opers $\simeq$ real projective structures

A **real** oper is an oper with real **monodromy**. Real opers $\chi_E = (\mathcal{E}_{\rm op}, \nabla_E)$ are in one-to-one correspondence to a certain class of hyperbolic metrics on C.

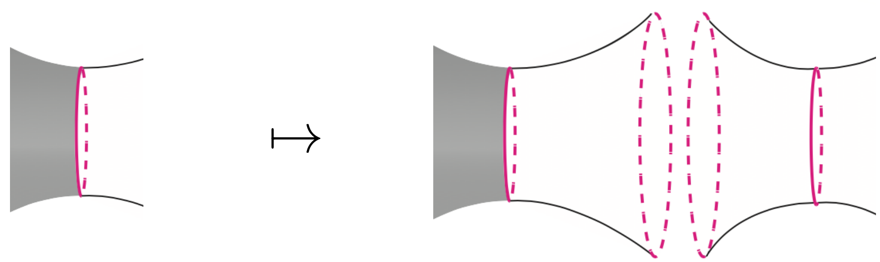
$$(\partial_u^2 - t(u))\phi(u,\bar{u}) = 0$$

$$\rightsquigarrow ds^2 = \frac{du d\bar{u}}{(\phi(u,\bar{u}))^2} \ \text{ has } \ R = -1,$$

$\rightsquigarrow$ uniformisation, e.g. for $C = C_{1,1}$:



Other real opers obtained by **grafting**, inserting pairs of "trumpets":[5]



Classification[6]: All real opers can be obtained in this way, classified by counting singularity lines $\simeq$ even[7] half-integer **measured laminations** $\lambda \in \mathcal{ML}_C^{\rm ev}(\frac{1}{2}\mathbb{Z})$.

[5] Pictures from arXiv:2102.03936 on a different, but somewhat related topic.

[6] W. Goldman; Review of relevant results: Dumas, arXiv:0902.1951.

[7] The natural projection to $H_1(C, \mathbb{Z}/2)$ vanishes.

Separation of variables – computing Analytic Langlands explicitly

Theorem[8] for $g = 0$, **conjecture** for $g > 0$:

There exist **unitary** integral transformations of the form

$$\Psi_E(\boldsymbol{x}) = \int d\mu_{\mathrm{Skl}}(\boldsymbol{u})\; K(\boldsymbol{x}, \boldsymbol{u})\, \Phi_E(\boldsymbol{u}),$$

$\mathbf{u} = (u_1, \ldots, u_D; \bar{u}_1, \ldots, \bar{u}_D)$, such that

$$\begin{aligned} \mathsf{H}_r\, \Psi_E &= E_r \Psi_E, \\ \bar{\mathsf{H}}_r\, \Psi_E &= \bar{E}_r \Psi_E, \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} (\partial_{u_r}^2 - t_E(u_r))\Phi_E(\boldsymbol{u}) &= 0, \\ (\bar{\partial}_{\bar{u}_r}^2 - \bar{t}_E(\bar{u}_r))\Phi_E(\boldsymbol{u}) &= 0. \end{aligned}$$

$\Phi_E(\boldsymbol{u})$ and therefore $\Psi_E(\boldsymbol{x})$ are **single-valued iff** monodromy of $\partial_u^2 - t_E(u)$ is **real**.

It follows that

$$\Phi_E(\boldsymbol{u}) = \prod_{r=1}^{D} \phi_E(u_r, \bar{u}_r), \qquad \begin{aligned} (\partial_u^2 - t_E(u))\phi_E(u, \bar{u}) &= 0, \\ (\bar{\partial}_{\bar{u}}^2 - \bar{t}_E(\bar{u}))\phi_E(u, \bar{u}) &= 0. \end{aligned}$$

[8] Ambrosino-J.T.

Geometry of SOV

The proof is based on the geometry used in Drinfeld's first proof of the geometric Langlands correspondence:

$$\begin{tikzcd}
 & \mathcal{N}_{\Lambda,d} \arrow[dl, dashed, "i"'] \arrow[dr, "j"] & & \mathcal{N}^{\vee}_{\Lambda,d} \arrow[dl, "j^{\vee}"] \arrow[r, "\mathsf{SoV}"] & C^{[m]} \\
\mathcal{N}_{\Lambda} & & \mathrm{Pic}^d & &
\end{tikzcd}$$

- $\mathcal{N}_{\Lambda}$: moduli space of stable rank-2 bundles, determinant Λ,
- $\mathcal{N}_{\Lambda,d}$: pairs $(L, \mathbf{x})$, with L degree d line bundle, $\mathbf{x} \in \mathbb{P}H^1(L^2\Lambda^{-1})$,
- $\mathcal{N}^{\vee}_{\Lambda,d} \simeq \mathbb{P}H^0(KL^{-2}\Lambda)$,
- SoV: Maps elements $c \in H^0(KL^{-2}\Lambda)$ to divisor of its zeros.

Analytic implementation of SOV

Roadmap for SoV:

$$
\begin{tikzcd}
 & \mathcal{N}_{\Lambda,d} \arrow[dl, dashed, "i"'] \arrow[dr, "j"] & & \mathcal{N}^{\vee}_{\Lambda,d} \arrow[dl, "j^{\vee}"] \arrow[r, "\mathsf{SoV}"] & C^{[m]} \\
\mathcal{N}_{\Lambda} & & \mathrm{Pic}^d & &
\end{tikzcd}
$$

- Take solution to $(\partial_u^2 - t_E(u))\phi_E(u,\bar{u}) = 0$ on C,
- $\rightsquigarrow$ "functions" $\Phi_E(\boldsymbol{u}) = \prod_{r=1}^{D} \phi_E(u_r, \bar{u}_r)$ on $C^{[m]}$,
- change variables according to SoV $\rightsquigarrow$ "functions" $\Xi_E(\boldsymbol{k})$ on $H^0(KL^{-2}\Lambda)$,
- apply Fourier-transform $\rightsquigarrow$ "functions" $\Psi_E(\boldsymbol{x})$ on $\mathcal{N}_{\Lambda,d}$,
- descending to $\mathcal{N}_{\Lambda}$.

Analytic Langlands $\simeq$ Harmonic analysis on Bun_G

Theorem for $g = 0$ (Ambrosino, J.T.), **conjecture** for $g > 0$:

i) Composition above defines unitary maps

$$\mathsf{SoV} : L^2(C^{[m]}, d\mu_{\mathrm{Skl}}) \to L^2(\mathrm{Bun}_{\mathrm{PSL}(2)}(C), |K_{\mathrm{Bun}}|).$$

(ii) The Hilbert space $L^2(\mathrm{Bun}_{\mathrm{PSL}(2)}(C), |K_{\mathrm{Bun}}|)$ admits the spectral decomposition

$$L^2(\mathrm{Bun}_{\mathrm{PSL}(2)}(C), |K_{\mathrm{Bun}}|) \simeq \bigoplus_{\lambda \in \mathcal{ML}_C^{\mathrm{ev}}(\frac{1}{2}\mathbb{Z})} \mathcal{H}_\lambda,$$

into eigenspaces $\mathcal{H}_\lambda$ of H_r with $\dim \mathcal{H}_\lambda = 1$, labelled by real opers χ_{E_λ}.

Proof uses Hecke-operators[9], and combines foundational results of Etingof-Frenkel-Kazhdan[10] with SoV.

[9] Defined by averaging functions on Bun_G over orbits of Hecke modifications

[10] Compactness of Hecke operators, relation Hecke operators to Hitchin's Hamiltonians

Quantum Analytic Langlands correspondence

Complex Chern-Simons theory I

$$S = \frac{\mathrm{i}s}{2} S_{\mathrm{CS}}(\mathcal{A}) - \frac{\mathrm{i}s}{2} S_{\mathrm{CS}}(\bar{\mathcal{A}}), \qquad s \in \mathbb{R},$$

with $\mathcal{A}$: complex $\mathrm{SL}(2,\mathbb{C})$-connection on on $M^3 = I \times C$, C: Riemann surface.

Hyperkähler structures on $\mathcal{M}_{\mathrm{flat}}(\mathrm{SL}(2,\mathbb{C}), C)$ (Nigel) $\rightsquigarrow$ Quantisation schemes:

- **Geometric:** Based on polarisation induced from $\mathcal{A} = \mathcal{A}^{1,0} + \mathcal{A}^{0,1}$.

  $\rightsquigarrow$ Hilbert spaces $\mathcal{H}_C = L^2(\mathrm{Bun}_G(C))$, using $\mathcal{A}^{0,1} \sim E \in \mathrm{Bun}_G$.

  $\rightsquigarrow$ Complex structure dependence of $\mathcal{H}_C$!

- **Topological:** Based on relation $\mathcal{M}_{\mathrm{flat}}(\mathrm{SL}(2,\mathbb{C}), C) \sim \mathcal{M}_{\mathrm{char}}(G, C)$.

  Quantisation can be based on structure of $\mathcal{M}_{\mathrm{char}}(G, C)$ as cluster variety[11].

  $\rightsquigarrow$ Hilbert space $\mathcal{K}$ **independent** of complex structure!

Relation to Complex Chern-Simons theory suggests isomorphisms $\mathcal{H}_C \simeq \mathcal{K}$ of Hilbert spaces with relevant module structures (skein-algebra and mapping class group).

[11] Fock-Goncharov

Complex Chern-Simons theory II

Geometric Quantisation

(Hitchin 1990, Axelrod-DellaPietra-Witten 1991, Witten 1991) $\rightsquigarrow$

- Bundle of Hilbert spaces $L^2(\mathrm{Bun}_G, |K|^{1+\mathrm{i}s})$ over $\mathcal{M}_{g,n}$,
- unitary connection

$$(k+2)\frac{\partial}{\partial z_r}\Psi_k(\boldsymbol{x},\boldsymbol{z}) = \mathsf{H}_r\Psi_k(\boldsymbol{x},\boldsymbol{z}), \qquad k+2 = \mathrm{i}s,$$

  with $\boldsymbol{z} = (z_1,\dots,z_d)$: local coordinates for Teichmüller space $\mathcal{T}(C)$.

Topological (Schur-)Quantization (Gaiotto-J.T.; Ambrosino-Gaiotto) $\rightsquigarrow$

- Algebra $\mathcal{D}_s = \mathcal{A}_s \times \mathcal{A}_s^{\mathrm{op}}$ generated by line operators, $\mathcal{A}_s \simeq \mathrm{Skein}_q(C)$,
- canonical unitary representation on $\mathcal{H}_{\mathrm{Schur}}$, generated from spherical vector,
- canonical basis $\{\mathcal{L}_\lambda, \lambda \in \mathcal{ML}_C^{\mathrm{ev}}(\frac{1}{2}\mathbb{Z})\}$ for $\mathcal{A}_s$

  $\rightsquigarrow$ canonical basis $\{e_\lambda;, \lambda \in \mathcal{ML}_C^{\mathrm{ev}}(\frac{1}{2}\mathbb{Z})\}$ for $\mathcal{H}_{\mathrm{Schur}}$.

Quantum deformation of SOV $\sim$ quantum analytic Langlands?

There exists a one-parameter family of integral transformations[12]

$$\Psi_k(\boldsymbol{x},\boldsymbol{z}) = \int d\mu_{\mathrm{Skl}}(\boldsymbol{u})\; \mathcal{K}_b(\boldsymbol{x},\boldsymbol{u})\Phi_b(\boldsymbol{u},\boldsymbol{z}), \qquad k+2 = -\frac{1}{b^2},$$

with coordinates $\boldsymbol{z} = (z_1,\dots,z_d)$ for $\mathcal{T}(C)$, intertwining solutions to

KZB-equations

$$(k+2)\frac{\partial}{\partial z_r}\Psi_k(\boldsymbol{x},\boldsymbol{z}) \overset{(*)}{=} \mathsf{H}_r\Psi_k(\boldsymbol{x},\boldsymbol{z}),$$

representation theory of $\widehat{\mathfrak{sl}}_{2,k}$

BPZ-equations

$$\left(b^2\frac{\partial^2}{\partial u_r^2} - \mathcal{T}_r(u_r)\right)\Phi_b(\boldsymbol{u},\boldsymbol{z}) = 0.$$

representation theory of Virasoro algebra

$(*)$ is a natural deformation of Hitchin eigenvalue equations, in the sense that

$$\left[\begin{array}{c} \Psi_k(\boldsymbol{x},\boldsymbol{z}) = e^{\frac{1}{k+2}S(\boldsymbol{z})}\psi(\boldsymbol{x},\boldsymbol{z}) + \dots \\[2ex] \mathsf{H}_r\Psi_k(\boldsymbol{x},\boldsymbol{z}) = (k+2)\dfrac{\partial}{\partial z_r}\Psi_k(\boldsymbol{x},\boldsymbol{z}) \end{array}\right] \underset{k\to-2}{\Rightarrow} \left[\begin{array}{c} \dfrac{\partial}{\partial z_r}S(\boldsymbol{z}) = E_r(\boldsymbol{z}), \\[2ex] \mathsf{H}_r\psi(\boldsymbol{x},\boldsymbol{z}) = E_r(\boldsymbol{z})\psi(\boldsymbol{x},\boldsymbol{z}). \end{array}\right]$$

[12] Building on previous work of Feigin-Frenkel-Stoyanovsky; Ribault-J.T.; Hikida-Schomerus.

Quantum deformation of SOV $\rightsquigarrow$ quantum analytic Langlands?

Main analytic problem: How to construct the BPZ-solutions Φ_b appearing in

$$\boxed{\Psi_k(\boldsymbol{x},\boldsymbol{z}) = \int d\mu_{\rm Skl}(\boldsymbol{u})\ \mathcal{K}_b(\boldsymbol{x},\boldsymbol{u})\Phi_b(\boldsymbol{u},\boldsymbol{z})} \qquad ?$$

$\Psi_k(\boldsymbol{x},\boldsymbol{z})$ and $\Phi_b(\boldsymbol{u},\boldsymbol{z})$ have **rigorous** definitions as path integrals:

$\Phi_b(\boldsymbol{u},\boldsymbol{z})$: Single valued BPZ-solutions = Liouville correlation functions[13],

$$\Phi_b(\boldsymbol{u},\boldsymbol{z}) = \Big\langle \prod_{l=1}^{d} V_{-1/2b}(u_l) \Big\rangle_{\rm Liou}, \quad S_{\rm L} = \frac{1}{4\pi}\int_C d^2x\left(|\nabla\phi(x)|^2 + 4\pi\mu e^{2b\phi(x)}\right).$$

$$\Big\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r) \Big\rangle_{\rm Liou} := \int_{\phi:C\to\mathbb{R}} \mathcal{D}[\phi]\ e^{-S_{\rm L}[\phi]} \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r)}.$$

$\Psi_k(\boldsymbol{x},\boldsymbol{z})$: Partition function of H_3^+-WZNW model[14].

[13] Rigorously constructed by David, Kupiainen, Rhodes, Vargas in 2014.

[14] Gawedzki-Kupiainen 1989; Construction & relation to Liouville theory: Guillarmou-Kupiainen-Rhodes 2025

Quantum Analytic Langlands Correspondence

Line operators $\mathcal{L}_\lambda$ have realisation in Liouville CFT[15]

Idea of definition: Quantum deformation of grafting, described as[16]

- Bubbling (pair-creation of chiral degenerate fields),
- analytic continuation (of position of one degenerate field along closed path),
- de-bubbling (fusion of degenerate fields).

$$\boxed{\text{Limit } b^2 \to \infty \text{ of } \mathcal{L}_\lambda \;\simeq\; \text{grafting along } \lambda.}$$

Operators $\mathcal{L}_\lambda$ generate an infinite family of solutions $\mathcal{L}_\lambda \Psi_k(\boldsymbol{x}, \boldsymbol{z})$ to KZB-equations,

labelled by measured laminations $\lambda \in \mathcal{ML}_C^{\rm ev}(\frac{1}{2}\mathbb{Z})$ = labels of **real opers**!

$$\boxed{\text{KZB-solutions } \; \mathcal{L}_\lambda \Psi_k(\boldsymbol{x}, \boldsymbol{z})} \quad \leftrightarrow \quad \boxed{\text{Canonical basis elements } \mathcal{L}_\lambda}$$

– First version of Quantum Analytic Langlands Correspondence –

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[15] E. Verlinde; Alday-Gaiotto-Gukov-Tachikawa-Verlinde; Drukker-Gomis-Okuda-J.T., ...

[16] Surgery of branched projective structures: Calsamiglia-Deroin-Francaviglia; T. Figiel, unpublished.

Quantum Analytic Langlands II

Relation to complex Chern-Simons suggests that expected isomorphisms $\mathcal{H}_C \simeq \mathcal{K}$,

$$\Psi(\boldsymbol{x}) = \sum_{\lambda \in \mathcal{ML}_C^{\mathrm{ev}}(\frac{1}{2}\mathbb{Z})} c_\lambda \, \langle \boldsymbol{x}, e_\lambda \rangle, \qquad \Psi \in \mathcal{H}_C, \quad e_\lambda \in \mathcal{K},$$

can be described by identifying

$$\langle \boldsymbol{x}, e_\lambda \rangle = \mathcal{L}_\lambda \Psi_k(\boldsymbol{x}, \boldsymbol{z}).$$

$\rightsquigarrow$ basis for harmonic analysis of $\mathcal{H}_C \simeq L^2(\mathrm{Bun}_G, |K|^{1+\mathrm{i}s})$.

– Quantum Analytic Langlands $\simeq$ Harmonic analysis of $\mathcal{H}_C$ –

Happy birthday, Nigel,

and all the best for the years to come!