Kleinian groups beyond the Riemann sphere

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Joint work with Ángel Cano, and in parts also with
W. Barrera and J.-P.Navarrete

The goal is:

Understand the geometry and dynamics

of discrete groups in $\mathrm{PSL}(n+1, \mathbb{C})$

Recall the classical limit set Λ: the set of accumulation points of orbits.

Closed non-empty set that splits the Riemann sphere in two invariant sets:

The limit set Λ is where the dynamics concentrates. This idea lies at the very heart of holomorphic dynamics.

The discontinuity region Ω lies at the heart of uniformization theory and its connections with complex geometry.

The classical miracle

$$\mathsf{PSL}(2,\mathbb{C}) \cong \mathsf{Aut}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathsf{M\ddot{o}b}(2,\mathbb{C}) \cong \mathsf{Conf}^+(S^2) \cong \mathsf{Isom}^+(\mathbb{H}^3_{\mathbb{R}}).$$

$$\begin{array}{ccc} \text{complex analysis} & \longleftrightarrow & \text{hyperbolic geometry} \\ \updownarrow & & \updownarrow \\ \text{dynamics} & \longleftrightarrow & \text{topology} \end{array}$$

Great richness!

Natural to think of higher dimensions: a dichotomy

- The conformal-hyperbolic viewpoint: *Conf* (S^n)
- The holomorphic viewpoint $\mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$

In the late 1990s, Verjovsky and Seade introduced the following notion.

Definition

A *complex Kleinian group* is a discrete subgroup

$$\Gamma < \mathrm{PSL}(n+1,\mathbb{C})$$

acting on $\mathbb{P}^n_{\mathbb{C}}$.

This framework contains several important families:

$$PU(n,1), \qquad PU(p+1,n-p), \qquad \mathrm{Aff}(n,\mathbb{C}),$$

Also:

- A wide variety of representations of real hyperbolic groups.
- Groups obtained via twistor theory.
- Complex Schottky groups, etc.

A common projective framework for several geometries.

Since the early 2000s we have a team working to lay down a systematic study of the subject.

The classical theory in $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$ is a guiding principle.

And also genuinely new phenomena appear.

We will discuss:

i) First, the dynamics.

ii) Then, the geometry.

iii) Finally, a global view.

From dimension one to higher dimensions

Recall that for $\mathrm{PSL}(2,\mathbb{C})$ a fundamental result is

the convergence property.

In $\mathbb{P}^n_{\mathbb{C}}$, one does not have this classical property.

The asymptotic geometry becomes richer and more diverse.

The natural asymptotic objects are *projective flags*.

One has

$$\boxed{\text{points} \quad + \quad \text{projective lines} \quad + \quad \cdots \quad + \quad \text{incidence.}}$$

Different aspects of the dynamics give rise to different limit sets:

$$\boxed{\text{Equicontinuity: its complement is } \Lambda_{\mathrm{Myr}}(G)}$$

$$\boxed{\text{Proper discontinuity: Gives rise to } \Lambda_{\mathrm{Kul}}(G) = L_0 \cup L_1 \cup L_2}$$

$L_0 =$ closure of points with infinite isotropy,

$L_1 =$ closure of accumulation points of orbits in $\mathbb{P}^n_{\mathbb{C}} \setminus L_0$

$L_2 =$ closure of accumulation set of orbits of compact sets in $\mathbb{P}^n_{\mathbb{C}} \setminus (L_0 \cup L_1)$

$$\boxed{\begin{aligned}\text{minimality} \quad \longrightarrow \quad \Lambda_{\mathrm{CoG}}(G) &= \text{closure of fixed points} \\ &\quad\ \text{of proximal elements}\end{aligned}}$$

At first glance, these are quite different constructions:

Yet, we will see that
these are actually all manifestations
of the same unifying principle:
the pseudo-projective boundary of the group

We now focus on $\mathbb{P}^2_{\mathbb{C}}$.

The first step is to compactify

$$\mathrm{PSL}(3, \mathbb{C})$$

in a way that remembers how projective transformations degenerate.

The pseudo-projective compactification

Introduced by Cano–Seade (J. Geom. Anal. 2010)

$$SP(3,\mathbb{C}) = \big(M_{3\times 3}(\mathbb{C}) \setminus \{0\}\big)/\mathbb{C}^*.$$

Thus

$$SP(3,\mathbb{C}) \cong \mathbb{P}^8_{\mathbb{C}},$$

and

$$\mathrm{PSL}(3,\mathbb{C}) \subset SP(3,\mathbb{C})$$

is an open dense subset.

Every

$$S \in SP(3,\mathbb{C})$$

induces a partially defined projective map

$$S : \mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S) \longrightarrow \mathbb{P}^2_{\mathbb{C}}.$$

The boundary consists of singular projective maps.

Pseudo-projective limits

After passing to a subsequence, every sequence of pairwise distinct elements

$$g_n \in \mathrm{PSL}(3, \mathbb{C})$$

has a pseudo-projective limit

$$g_n \longrightarrow S \in SP(3, \mathbb{C}).$$

Moreover, $g_n \longrightarrow S$ uniformly on compact subsets of

$$\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S).$$

Thus the first asymptotic data are

$$\boxed{\mathrm{Ker}(S) \qquad \text{and} \qquad \mathrm{Im}(S).}$$

But the pseudo-projective limit leaves open one question:

What happens on the exceptional set $\mathrm{Ker}(S)$?

The classical convergence property

For a divergent sequence

$$(g_m) \subset \mathrm{PSL}(2, \mathbb{C}),$$

after passing to a subsequence, there exist points

$$p^+, p^- \in \mathbb{P}^1_{\mathbb{C}}$$

such that

$$g_m \longrightarrow p^-$$

uniformly on compact subsets of

$$\mathbb{P}^1_{\mathbb{C}} \setminus \{p^+\},$$

while

$$g_m^{-1} \longrightarrow p^+$$

uniformly on compact subsets of

$$\mathbb{P}^1_{\mathbb{C}} \setminus \{p^-\}.$$

One repelling point, one attracting point: there is only one asymptotic scale.

In higher dimensions: stratified convergence

In higher rank, the first pseudo-projective limit need not contain all the asymptotic information.

The exceptional set

$$\mathrm{Ker}(S)$$

may carry dynamics at a slower scale.

After successive renormalizations, one obtains

A nested family of projective subspaces
together with limiting maps on the successive strata.

The classical convergence property becomes:

Stratified convergence

Write the Cartan decomposition in terms of the singular values:

$$g_m = k_m \operatorname{diag}\left(e^{\lambda_{1,m}}, e^{\lambda_{2,m}}, e^{\lambda_{3,m}}\right)\ell_m, \qquad \lambda_{1,m} \geq \lambda_{2,m} \geq \lambda_{3,m}.$$

There are two Cartan gaps:

$$\delta_1(m) = \lambda_{1,m} - \lambda_{2,m}, \qquad \delta_2(m) = \lambda_{2,m} - \lambda_{3,m}.$$

For a divergent sequence, at least one of them tends to infinity.

Hence, after passing to a subsequence, there are exactly three asymptotic regimes:

$$\boxed{(\infty, \infty), \qquad (\text{bounded}, \infty), \qquad (\infty, \text{bounded}).}$$

These three regimes give exactly the three possible stratified convergence types in $\mathbb{P}^2_{\mathbb{C}}$.

The asymptotic trichotomy theorem:

Trichotomy for sequences in $\mathrm{PSL}(3,\mathbb{C})$

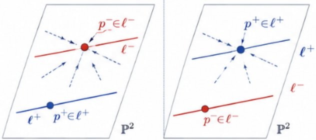
Let $(g_n) \subset \mathrm{PSL}(3,\mathbb{C})$ with no convergent subsequence. **Exactly one of the following holds** (up to subsequence):

1. **Full flags.** There exist two full flags $(p^{\pm}, \ell^{\pm})$ in $\mathbb{P}^2$, with $p^{\pm} \in \ell^{\pm}$, such that g_n converges on $\mathbb{P}^2 \setminus \ell^+$ to the constant map p^-, and g_n^{-1} converges locally uniformly on $\mathbb{P}^2 \setminus \ell^-$ to p^+.
2. **Mixed type (point–line or line–point).** Either there exist a point $p^+ \in \mathbb{P}^2$ and a line $\ell^- \subset \mathbb{P}^2$ such that, outside p^+, (g_n) converges to a pseudo–projective map with image ℓ^-, while (g_n^{-1}) collapses $\mathbb{P}^2 \setminus \ell^-$ to p^+; or, dually, there exist a line $\ell^+ \subset \mathbb{P}^2$ and a point $p^- \in \mathbb{P}^2$ such that, outside ℓ^+, (g_n) collapses to p^-, while (g_n^{-1}) converges to a pseudo–projective map with image ℓ^+.

1. Full flags

$g_n : \mathbb{P}^2 \setminus \ell^+ \to p^-$ (collapse to the point p^-)

$g_n^{-1} : \mathbb{P}^2 \setminus \ell^- \to p^+$ (collapse to the point p^+)



Limits: $g_n \to p^-$ on $\mathbb{P}^2 \setminus \ell^+$ and $g_n^{-1} \to p^+$ on $\mathbb{P}^2 \setminus \ell^-$.

2. Mixed type: (point–line) or (line–point)

(point–line)

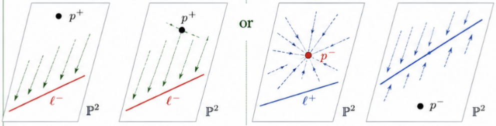
$g_n : \mathbb{P}^2 \setminus \{p^+\} \to \ell^-$ (pseudo–projective, image ℓ^-)

$g_n^{-1} : \mathbb{P}^2 \setminus \ell^- \to p^+$ (collapse to p^+)

(line–point)

$g_n : \mathbb{P}^2 \setminus \ell^+ \to p^-$ (collapse to p^-)

$g_n^{-1} : \mathbb{P}^2 \setminus \{p^-\} \to \ell^+$ (pseudo–projective, image ℓ^+)



In both variants one map collapses off a line to a point, and the other converges off that point to a map with image the line.

Definition

Let $G < \mathrm{PSL}(3,\mathbb{C})$ be discrete. Its pseudo–projective boundary $\partial_{SP}(G)$ is the set of all limits in $SP(3,\mathbb{C})$ of divergent sequences in G.

Theorem

Let G be a discrete group with at least three lines in general position in its limit set $\Lambda_{Kul}(G)$. Then:

$$\boxed{\Lambda_{\mathrm{Kul}}(G) = \Lambda_{\mathrm{Myr}}(G) = \bigcup_{[\ell]\in\Lambda_{\mathrm{CoG}}(\check{G})} \ell.}$$

where the union on the right runs over the Conze-Guivarc'h limit set of the dual action.

Several notions of limit set arise from one common asymptotic geometry.

The geometric viewpoint

I) Quasi-cocompact groups

Definition

$G < \mathrm{PSL}(3,\mathbb{C})$ discrete is quasi-cocompact if there is an invariant domain $U \subset \mathbb{P}^2_{\mathbb{C}}$ on which G acts properly discontinuously and such that U/G is compact.

(Cano–Seade, *Geometriae Dedicata* 168, 2014)

Theorem (Reminiscent of Kobayashi-Ochiai's theorem)

If $G < \mathrm{PSL}(3,\mathbb{C})$ is quasi-cocompact, then
G is either complex hyperbolic or virtually affine, and moreover: the possible Kulkarni regions of discontinuity, the corresponding groups, and their quotient orbifolds are classified.

The classification

Cano–Seade, *Geometriae Dedicata* 168 (2014)

Kulkarni region	**Compact quotient (up to finite cover)**
$\mathbb{C}^2$	complex torus or primary Kodaira surface
$\mathbb{C}^2 \setminus \{0\}$	complex torus or primary Hopf surface
$\mathbb{C}^* \times \mathbb{C}$	complex torus
$(\mathbb{C}^*)^2$	complex torus
$\mathbb{C}^* \times (H^+ \cup H^-)$	Inoue surface(s)
$D \times \mathbb{C}^*$	elliptic surface(s)
$\mathbb{H}^2_{\mathbb{C}}$	compact complex hyperbolic surface

Thus, complex Kleinian groups provide projective uniformizations of several fundamental classes of compact complex surfaces.

Beyond the quasi-cocompact picture

What can we say about the geometry of the discontinuity region for arbitrary

$$G < \mathrm{PSL}(3, \mathbb{C})\,?$$

First a preparatory result:

Theorem (Barrera, Cano, Navarrete)

- *$\Lambda_{Kul}(G)$ always contains at least 1 projective line, and if it has more than 1 line, then it is a union of lines.*
- *If $\Lambda_{Kul}(G)$ contains more than 3 lines, then it contains infinitely many lines.*
- *If $\Lambda_{Kul}(G)$ contains more than 4 lines in general position, then it contains infinitely many lines in general position.*

The geometry of the ordinary set

Theorem

Let $G < \mathrm{PSL}(3,\mathbb{C})$ be discrete.
If $\Lambda_{\mathrm{Kul}}(G)$ contains at least four projective lines in general position, then every connected component of

$$\Omega_{\mathrm{Kul}}(G)$$

is complete Kobayashi hyperbolic and hyperbolically embedded in $\mathbb{P}^2_{\mathbb{C}}$. Furthermore, every such component is Stein, pseudoconvex, and taut.

There are two remarkably different mechanisms: Let $\nu(G)$ be the greatest number of lines in general position in $\Lambda_{\mathrm{Kul}}(G)$.
Then:

$$\boxed{\begin{array}{lll} \nu(G) = 4 & \Longrightarrow & \Omega_0 \simeq \mathbb{H} \times \mathbb{H}, \\ \nu(G) > 4 & \Longrightarrow & \text{arbitrarily many lines in general position.} \end{array}}$$

Many lines force hyperbolicity

When $\nu(G) > 4$, the dynamics produce arbitrarily many projective lines in general position.

Choose five of them:

$$\ell_1, \ldots, \ell_5 \subset \Lambda_{\mathrm{Kul}}(G).$$

Then

$$\Omega_{\mathrm{Kul}}(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\ell_1 \cup \cdots \cup \ell_5).$$

By Green's theorem,

$$\mathbb{P}^2_{\mathbb{C}} \setminus (\ell_1 \cup \cdots \cup \ell_5)$$

is Kobayashi hyperbolic.

The asymptotic line configuration controls the intrinsic geometry of the ordinary set.

The preceding results raise a natural question:

$\boxed{\text{How general or exceptional is this general-position regime?}}$

The answer is given by a deformation argument based on Schottky-type dynamics introduced by Conze-Guivarc'h.

Zariski density

Let

$$F_r = \langle x_1, \ldots, x_r \rangle, \qquad \mathrm{Hom}(F_r, \mathrm{PSL}(3,\mathbb{C})) \simeq \mathrm{PSL}(3,\mathbb{C})^r.$$

Let $\mathcal{K}_r$ denote the locus of representations whose image is a non-virtually-cyclic complex Kleinian group, and set

$$\mathcal{H}_r = \{\rho \in \mathcal{K}_r : \Omega_{\mathrm{Kul}}(\rho(F_r)) \text{ is Kobayashi hyperbolic}\}.$$

Theorem

$$\boxed{\mathcal{H}_r \text{ is Zariski dense in } \mathcal{K}_r.}$$

A global view in dimension 2

The following are different manifestations
of a single asymptotic projective geometry.

Dynamics	Complex geometry	Zariski genericity
Limit sets correspond to flags	Ω_{Kul} complete Kobayashi geometry	Kobayashi hyperbolicity is Zariski dense

Beyond the Riemann sphere.

In dimension 1 we have Sullivan's dictionary:

Kleinian groups $\longleftrightarrow$ holomorphic dynamics.

In complex projective dimension two, the above presentation gives some items for a Sullivan type dictionary:

Holomorphic dynamics	**Complex Kleinian groups**
Julia set	Λ_{Kul}
first Julia set	Λ_{CoG}
Fatou set	Ω_{Kul}
equicontinuity region	$\mathrm{Eq}(G)$
repelling periodic points	attracting/repelling data of loxodromic elements
generic Kobayashi geometry of the Fatou set	generic Kobayashi geometry of Ω_{Kul}

The emerging picture

In $\mathbb{P}^n$, one actually has a stratified convergence property with important dynamical implications. One should expect an entire hierarchy

$$\boxed{\text{points} \subset \text{lines} \subset \cdots \subset \text{hyperplanes},}$$

organized by partial flags and corresponding to the successive asymptotic scales of the dynamics.

In higher rank, the asymptotic boundary is no longer described by a single limit set: it naturally leads to a hierarchy of limit sets organized by flags.

Happy 80th birthday, Nigel!