Tournaments, flags, and spinors

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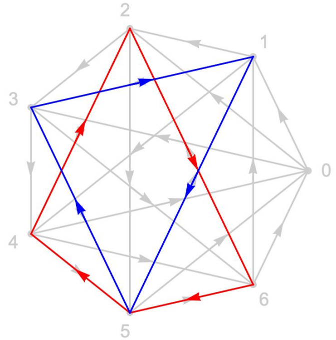

Hitchin80, 03/09/2026

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Key references for this talk

[AS] D. Artacho and U. Semmelmann: Invariant spinors on flag manifolds, Internat. J. Math. 36 (2025), 2550012.

[AHL] I. Agricola, J. Hofmann, M-A. Lawn: Invariant spinors of homogeneous spheres. Differ. Geom. Appl. 89 (2023), 1022014.

[BS] F.E. Burstall and S.M. Salamon: Tournaments, flags, and harmonic maps, Math. Annalen 277 (1987), 249–265.

[H] N.J. Hitchin: Harmonic spinors, Advances Math. 14 (1974). 1–55.

Let be G be a compact simple Lie group, and T a maximal torus in G. Set $\mathbb{F} = G/T = G_c/B$, and let $p = eT$. Then

$$T_p^{1,0}\mathbb{F} \;\cong\; \bigoplus_{i=1}^{k} \mathfrak{g}_i \;\subset\; \mathfrak{g}_c.$$

where $\mathfrak{g}_i$ are the 1-dimensional eigenspaces for positive root vectors $a_1, \dots, a_k$.

One can show that $\mathbb{F}$ has a unique spin structure.

Theorem [AS]. The dimension of the space Ξ of G-invariant spinors on $\mathbb{F}$ equals the number of linear relations $\sum_{i=1}^{k} \varepsilon_i a_i = 0$ with $\varepsilon_i \in \{-1, 1\}$.

A proof relies on expressing the spin representation of T in terms of the $\mathfrak{g}_i$, and we shall do this explicitly for $G = SU(n)$. Artacho-Semmelmann computed

$$\dim \Xi(G_2/T^2) = 4, \quad \dim \Xi(F_4/T^4) = 34\,432, \quad \dim \Xi(E_6/T^6) = 13\,697\,920.$$

Claude (Opus 5) claims that $\dim \Xi(E_8/T^8) \in [2.2, 2.9] \times 10^{29}$ with 85% probability.

The dimension of Ξ is zero if $\rho = \frac{1}{2}\sum a_i$ is *not* in the root lattice. This happens for $\mathrm{SU}(n)$ with n even, for $\mathrm{Sp}(n)$ with $n \equiv 1, 2 \bmod 4$, always for $\mathrm{SO}(2n+1)$, and for $\mathrm{SO}(2n)$ with $n \equiv 2, 3 \bmod 4$. Also for E_7.

Henceforth, assume that n is odd. A point p of

$$\mathbb{F}_n = \frac{\mathrm{SU}(n)}{T^{n-1}} = \frac{\mathrm{U}(n)}{T^n}$$

is (metrically) an orthogonal decomposition $\mathbb{C}^n = L_1 \oplus \cdots \oplus L_n$. We shall not initially distinguish a complex structure. Thus

$$T_p\mathbb{F}_n \;\cong\; \bigoplus_{i<j} \mathfrak{a}_{ij}$$

where $\mathfrak{a}_{ij}$ is the real 2-dimensional subspace underlying the root space

$$L_i^* \otimes L_j = \mathrm{Hom}(L_i, L_j).$$

It corresponds to an edge of the complete graph K_n.

A choice of orientation of each $\mathfrak{a}_{ij}$ determines a G-invariant almost complex structure J on $\mathbb{F}_n$. This corresponds to a tournament, a digraph overlying K_n.

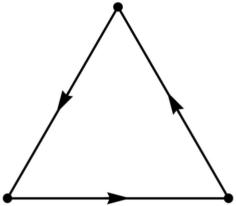
Observation [BS]. J is integrable iff the corresponding tournament has no cycles.

Examples. (i) The labelling defined a complex (indeed, Kähler) structure J for which $L_i^* \otimes L_j$ has type $(1,0)$ whenever $i < j$. This is integrable because

$$[L_i^* \otimes L_j, L_j^* \otimes L_k] \subseteq [L_i^* \otimes L_k]$$

using Lie brackets inside $\mathfrak{g}_c = \mathfrak{sl}(n, \mathbb{C})$.

(ii) If $n = 3$, there are $2^3 = 8$ ACS's on $\mathbb{F}_3$. The two 3-cycles correspond to the non-integrable nearly-Kähler structures $(g_{\mathrm{nk}}, \pm J_2)$, with the pair invariant by the group $\mathfrak{S}_3$ permuting the three maps to $\mathbb{CP}^2$. This also underlies the construction of harmonic maps $S^2 \to \mathbb{CP}^2$.



Definition. A directed subgraph $\mathcal{S}$ of a tournament $\mathcal{T}$ is **balanced** if in-degree = out-degree at each node. If $\mathcal{S}=\mathcal{T}$, we shall also say that $\mathcal{T}$ is **regular** or Eulerian.

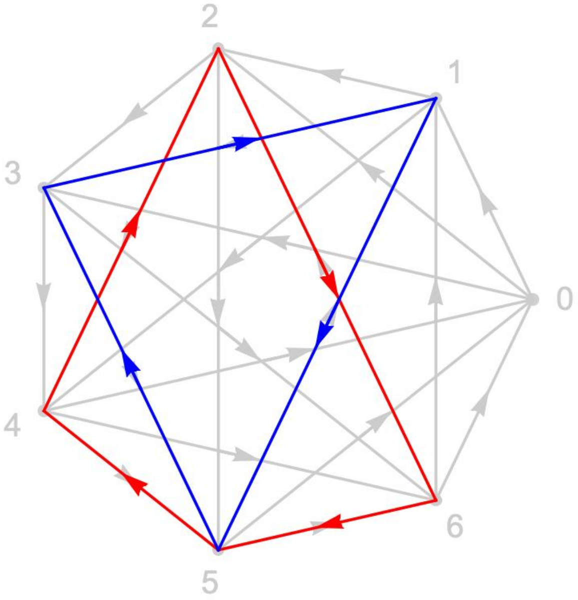
Observations. (i) If $\mathcal{T}$ is a regular n-tournament then $\delta^-=\delta^+=(n-1)/2$.
(ii) If $n=2m+1$ is odd, regular tournaments obviously exist, for example

$$i\to j \;\Leftrightarrow\; j-i\in\{1,2,\ldots,m\} \mod n.$$

(iii) Any regular tournament can be decomposed as a union of directed cycles.
(iv) A tournament is regular iff the corresponding non-integrable ACS has trivial canonical bundle, i.e. $c_1=0$.

Proposition [Akin]. Fix a regular tournament $\mathcal{T}_0$. There is a bijection between the set of balanced subgraphs of $\mathcal{T}$ and the set of *all* regular tournaments!

Idea of proof. Given a balanced subgraph $\mathcal{S}$, reverse the arrows of $\mathcal{T}_0$ along S. That gives a new regular tournament, which we choose to denote by $\mathcal{T}_0\mathbin{\dot{\ominus}}\mathcal{S}$.
In particular, if $\mathcal{T}_0\mathbin{\dot{\ominus}}\varnothing=\mathcal{T}_0$ corresponds to J_0, then $\mathcal{T}\mathbin{\dot{\ominus}}\mathcal{T}$ corresponds to $-J_0$.


2
1
3
0
4
6
5

By hand: the case n=5

Label the nodes $0,1,2,3,4$. Take $\mathcal{T}_0 = (01234)\cdot(02413)$, the circulant tournament. Its balanced subgraphs are listed below; there are none with 1,2,8,9 nodes.

#nodes	cycles or union of cycles	number
0	$\varnothing$	1
3	013, 023, 024, 124, 134	5
4	0123, 0124, 0134, 0234, 1234	5
5	01234, 02413	2
6	$013\cdot024$, $013\cdot124$, $023\cdot124$, $023\cdot134$, $024\cdot134$	5
7	$0123\cdot024 = 0124\cdot023$, $0134\cdot023 = 0234\cdot013$ $0234\cdot124 = 1234\cdot024$, $0123\cdot134 = 1234\cdot013$ $0124\cdot134 = 0134\cdot124$	5
10	$01234\cdot02413$	1
		24

Each entry determines a tournament of order 5, and [we shall see] an invariant pure spinor. It follows that $\dim\Xi_5 = 24$. Whereas $\dim\Xi_7 = 2640\ldots$

A Kähler manifold M has holonomy $\mathrm{U}(m) \subset \mathrm{SO}(2m)$. It is spin iff the canonical bundle $\kappa = \Lambda^{m,0}$ has a square root, in which case the spinor bundle decomposes as

$$\Delta \;\cong\; \kappa^{1/2} \otimes \bigoplus_{q=0}^{m} \Lambda^{0,q}.$$

The Dirac operator can be strung out as a twisted Dolbeault complex. When κ is trivial (meaning M is Calabi-Yau), we can replace $\Lambda^{0,q}$ by $\Lambda^{m-q,0}$, and there are nonzero parallel spinors with $q = 0, m$.

There is an analogous result for holonomy $\mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$, namely

$$\Delta \;\cong\; \bigoplus_{q=0}^{m} \Lambda_0^{m-q} E \otimes S^q H,$$

where E has complex rank $2m$ and H complex rank 2. In 8 real dimensions,

$$\Delta^+ \cong \Lambda_0^2 E \oplus S^2 H, \qquad \Delta^- \cong E \otimes H \cong T.$$

Parallel spinors only exist in the hyperkähler case (in which H is flat).

Let n be odd, and fix a regular n-tournament $\mathcal{T}_0$ and so an ACS J_0 on $\mathbb{F}_n$ with trivial canonical bundle κ. The existence of $\mathcal{T}_0$ confirms that $\mathbb{F}_n$ is spin.

Theorem. The balanced subgraphs of $\mathcal{T}_0$ provide a basis for the space of invariant spinors on $\mathbb{F}_n$.

Idea of proof. Take $\mathcal{T}_0 = (01234)\cdot(02413)$ with $n=5$ as before. Spinors are elements of $\Lambda^{q,0}$, and are therefore generated by *decomposable* $(q,0)$-vectors like

$$\tau = 01\wedge 12\wedge 24\wedge 40\wedge 13\wedge 34\wedge 41 = \tau \in \Lambda^{7,0},$$

associated (up to sign) to $\mathcal{S} = 0124\cdot 134$. Each is a pure spinor because (under Clifford multiplication, i.e. wedging by $(1,0)$ forms and hooking by $(0,1)$ forms) it is annihilated by complex 1-forms spanning a *maximal* isotropic subspace of $T^*_c\mathbb{F}_n$. In the case of τ, by

$$01, 12, 24, 40, 13, 34, 41 \in \Lambda^{1,0} \quad \text{and} \quad \overline{02}, \overline{30}, \overline{23} \in \Lambda^{0,1},$$

which together span the space $\Lambda^{0,1}$ relative to the modified tournament $\mathcal{T}_0 \ominus \mathcal{S}$.

Corollary. The dimension of the space Ξ_n of $SU(n)$-invariant spinors on $\mathbb{F}_n$ equals the number $\mathrm{RT}(n)$ of regular tournaments of order n.

Theorems [B. McKay]. $\mathrm{RT}(n)$ is the constant term in $\prod_{0\leqslant j<k\leqslant n-1} \left(x_j {x_k}^{-1} + x_k {x_j}^{-1}\right)$,

and $\mathrm{RT}(n) \sim \sqrt{n/e}\left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2}$ $(= 8.4 \times 10^{29}$ if $n = 17$, cf. case of $E_8)$.

n	$\mathrm{RT}(n)$
1	1
3	2
5	24
7	2 640
9	3 230 080
11	48 251 508 480
13	9 307 700 611 292 160
15	24 061 983 498 249 428 379 648
17	855 847 205 541 481 495 117 975 879 680

Fix n odd. Let $N(T)$ be the normalizer of $T = T^{n-1}$ in $\mathrm{SU}(n)$. An element of the Weyl group $N(T)/T \cong \mathfrak{S}_n$ acts on $\mathbb{F}_n$ as

$$nT \cdot gT = gTn^{-1} = gn^{-1}T, \quad n \in N(T).$$

More to the point, $\sigma \in \mathfrak{S}_n$ permutes the $\frac{1}{2}n(n-1)$ edges of K_n.

We denote by $|T_\sigma| = \mathrm{fix}(\sigma)$ the set of tournaments fixed by this action. There are none if σ incorporates a cycle τ of length 2ℓ, because τ^ℓ will reflect some edges.

Proposition. Fix n odd. Suppose that $\sigma \in \mathfrak{S}_n$ has only odd cycles of lengths $\ell_1, \ldots, \ell_m$. Set $g_{ij} = \gcd(\ell_i, \ell_j)$ and $L = \frac{1}{2}\sum_i(\ell_i - 1)$. Then $\left|T_\sigma\right|$ equals the coefficient of the monomial $y_1^{(n-\ell_1)/2} \cdots y_m^{(n-\ell_m)/2}$ in

$$2^L \prod_{i<j} (y_i^{\ell_j/g_{ij}} + y_j^{\ell_i/g_{ij}})^{g_{ij}}.$$

Examples. If σ is an n-cycle, the product is vacuous and $|T_\sigma| = 2^{(n-1)/2}$.
If $n = 7$, $|T_\sigma| = 0$ unless σ is the identity, a 7-cycle or a pair of 3-cycles ($|T_\sigma| = 24$).

Using the last example, Burnside's lemma yields

$$|\text{orbits of } \mathfrak{S}_7| = \frac{1}{7!}\sum_{\sigma\in\mathfrak{S}_7} |T_\sigma| = \frac{1}{7!}\left[2640\times 1 + 6!\times 8 + 14\tbinom{6}{3}\times 24\right] = 3.$$

There are in fact three regular 7-tournaments up to isomorphism. Their stabilizers in $\mathfrak{S}_7$ have sizes $7, 21, 3$.

Kelly's conjecture. Every regular tournament is the union of $(n-1)/2$ Hamiltonian cycles. For $n = 7$, we have

$$\mathcal{T}_0 = (0123456)\cdot(0246135)\cdot(0362514) \in \text{fix}(0123456).$$

If $\mathcal{S} = (2456)\cdot(135)$ (see title slide),

$$\begin{aligned} \mathcal{T}_1 = \mathcal{T}_0 \oplus \mathcal{S} &= (014)\cdot(234)\cdot(025)\cdot(315)\cdot(036)\cdot(126)\cdot(465) \\ &= (0126534)\cdot(0231546)\cdot(0361425) \end{aligned}$$

is the 'G_2 tournament', isomorphic to QR_7 (Paley). A third class is represented by

$$\mathcal{T}_2 = \mathcal{T}_1 \oplus (465) = (0123645)\cdot(0261534)\cdot(0314256).$$

$\mathbb{F}_n$ has real dimension $d = n(n-1)$. The Weyl group $\mathfrak{S}_n$ does *not* act directly on the space Ξ_n of invariant spinors, but induces an action on $T_o\mathbb{F}_n$ that has two extensions by $\mathbb{Z}_2$:

$$\begin{tikzcd} \widehat{\mathfrak{S}}_n^{\pm} \arrow[r, hook, shift left] \arrow[r, hook, shift right] \arrow[d, "\pi"'] & \mathrm{Pin}(d) \arrow[d] & \subset \mathrm{Cl}(\mathbb{R}^d) \\ \mathfrak{S}_n \arrow[r, hook] & O(d). \end{tikzcd}$$

Denote by $\widehat{\mathfrak{S}}_n^{+}$ the double cover of $\mathfrak{S}_n$ in $\mathrm{Pin}(d)$ generated by $t_0, \ldots, t_{n-2}$ and $z = -1$, where each t_i is a product of $2n{-}3$ reflections that lifts the 2-cycle $(i, i{+}1)$:

$$z^2 = 1, \quad t_i^2 = 1, \quad (t_i t_{i+1})^3 = 1, \quad t_i t_j = z t_j t_i \ (i{+}1 < j).$$

This group *does* act on Ξ_n.

Lemma. If $\sigma \in \mathfrak{S}_n$ has only odd cycles, it has a unique lift to an element $\widehat{\sigma}$ of the same order in $\widehat{\mathfrak{S}}_n^{+}$, and $\mathrm{tr}(\widehat{\sigma} \mid \Xi_n) = |T_\sigma|$. Otherwise, $\mathrm{tr}(\widehat{\sigma} \mid \Xi_n) = 0$ for either lift.

A character table

For $n = 5$, either extension $\mathfrak{S}_5^{\pm}$ contains the binary icosahedral group $\mathrm{SL}(2, \mathbb{Z}_5)$ of order 120 projecting to A_5. We list *its* irreducible representations for simplicity. Below, $\sigma = (01234)$ is a fixed 5-cycle, $|T_\sigma| = 4$, and z acts as -1 on spin modules:

conj class size →	1	$1z$	20	30	$12(\widehat{\sigma})$	12	20	12	12
χ_1	1	1	1	1	1	1	1	1	1
χ_{3_1}	3	3	0	-1	σ	σ'	0	σ	σ'
χ_{3_2}	3	3	0	-1	σ'	σ	0	σ'	σ
χ_4	4	4	1	0	-1	-1	1	-1	-1
χ_5	5	5	-1	1	0	0	-1	0	0
χ_{2_1}	2	-2	-1	0	$-\sigma'$	$-\sigma$	1	σ'	σ
χ_{2_2}	2	-2	-1	0	$-\sigma$	$-\sigma'$	1	σ	σ'
$\chi_{4'}$	4	-4	1	0	-1	-1	-1	1	1
χ_6	**6**	-6	0	0	**1**	1	0	-1	-1

Note. The first column relates to the extended Dynkin diagram of E_8 via the McKay correspondence. Its dimensions count exceptional curves in the minimal rsolution of $\mathbb{C}^2/G$, and include $2, 3, 4, 5, 6$ because $S^k(\mathbb{C}^2)$ is irreducible for $k \leqslant 5$ and spin for n even.

Irreducible $\widehat{\mathfrak{S}}_n^+$ modules for which z acts faithfully are parametrized by Young diagrams with *strict* partitions $\lambda_1 > \cdots > \lambda_k$ of $n = \sum \lambda_i$, up to twisting with the sign character.

We denote these by $[\lambda_1, \ldots, \lambda_k]^\pm$, though the two are isomorphic if $n - k$ is even. Each has dimension $2^{\lfloor (n-k)/2 \rfloor}$ times

$$\frac{n!}{\lambda_1! \cdots \lambda_k!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j},$$

which counts shifted Young Tableaux of size n. Just one if $k = 1$.

Example. $\dim[4,1]^\pm = 2^1 \times 5 \times \frac{3}{5} = 6$, so $[4,1]^\pm \cong S^5(\mathbb{C}^2)$ restricting to $\mathrm{SL}(2, \mathbb{Z}_5)$:

5	4	3	2
	1		

5	4	3	1
	2		

5	4	2	1
	3		

Corollary. The multiplicity of $[\lambda_1, \dots, \lambda_k]^\pm$ in Ξ_n equals $\frac{1}{n!}\sum_\sigma |T_\sigma|\,\overline{\chi(\widehat{\sigma})}$ where χ is its character, and we only need sum over products of odd cycles in $\mathfrak{S}$.

Example. When $n = 5$, both $[5]$ and $[3,2]$ are absent, whilst $[4,1]^\pm$ has multiplicity $\frac{1}{5!}(24\times 6 + 4!\times 4) = 2$. This confirms that $\Xi_5 \cong 2[4,1]^+ \oplus 2[4,1]^-$.

Conclusion to date. For $7 \leqslant n \leqslant 13$, *all* strict partitions appear as submodules of Ξ_n under the action of $\widehat{\mathfrak{S}}_n^+$. The list for $n = 7$ is

λ	dim	mult	dim(block)
$[7]$	8	8	64
$[6,1]^\pm$	20	8	320
$[5,2]^\pm$	36	20	1440
$[4,3]^\pm$	20	12	480
$[4,2,1]$	28	12	336
			2640

Let ∇ be the Levi-Civita connection, and D the Dirac operator on a Riemannian spin manifold of dimension d. The Licherowicz formula states that

$$D^2 = \nabla^*\nabla + \tfrac{1}{4}s,$$

where s is the scalar curvature. So manifolds with $s > 0$ cannot admit harmonic spinors. Friedrich's theorem states that the first eigenvalue λ_1 of D satisfies

$$\lambda_1^2 \geqslant \frac{d}{4(d-1)} s_{\min}.$$

Equality holds iff the eigenspinor is Killing, in which case

$$\nabla_X \psi = \frac{\lambda_1}{d} X \cdot \psi,$$

an example being $\mathbb{F}_3 = \mathrm{SU}(3)/T^2$ with $d = 6$. If $\psi \in \Xi_n$, then

$$\nabla\psi = (\overline{\nabla} + T)\psi = T\psi$$

where $\overline{\nabla}$ is the canonical cnnection and T is a torsion tensor. In particular, $D\colon \Xi_n \to \Xi_n$ is a linear map.

Write $\mathfrak{su}(n) = \mathfrak{t}^{n-1} \oplus \mathfrak{m}$. The normal homogeneous metric g is defined by

$$\langle X, Y\rangle = -\mathrm{tr}(XY), \quad X, Y \in \mathfrak{m} = T_p\mathbb{F}_n$$

and its Levi-Civita connection by $\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}$. For $X, Y, Z \in \mathfrak{m}$, the Cartan 3-form is

$$B(X, Y, Z) = \langle [X, Y], Z\rangle = -\frac{1}{3}\sum_{i,j,k} \alpha_{ij} \wedge \alpha_{jk} \wedge \alpha_{ki}.$$

Here, α_{ij} is a unit element in $L_i^* \otimes L_j \subset T_c\mathbb{F}_n$. The sum is over all distinct indices with the convention $\alpha_{ji} = -\overline{\alpha_{ij}}$, so that B is real.

Now fix a unit invariant spinor $\psi \in \Xi_n$. Since $\nabla_X \psi = \frac{1}{4}(X \lrcorner B) \cdot \psi$,

$$D\psi = \sum_{r=1}^{d} e_r \cdot \nabla_{e_r}\psi = \frac{3}{4}B \cdot \psi = \frac{1}{4}\sum_{i,j,k} \alpha_{ij} \cdot \alpha_{jl} \cdot \alpha_{li} \cdot \psi.$$

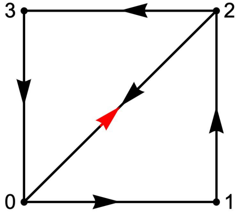
This coincides (up to a constant) with Kostant's cubic Dirac operator $D^{1/3}$. If ψ is pure, there is a tautologous ACS J_ψ with $c_1 = 0$ for which $\psi = 1 \in \Lambda^{0,0} \subset \Xi_n$.

Example. When $n = 3$, $D(1) = \frac{1}{4}\alpha_{01} \wedge \alpha_{12} \wedge \alpha_{20}$.

Theorem. Ξ_n decomposes as the sum of two eigenspaces for $D = D_g$, each of dimension $\frac{1}{2}\mathrm{RT}(n)$, with eigenvalues $\pm\frac{1}{4}\sqrt{3n(n^2-1)}$.

Proof. Since d is even, D interchanges the summands of $\Xi_n = \Xi^+ \oplus \Xi^-$. It suffices to compute $D^2\psi$ for a generic unit pure spinor ψ relative to J_ψ as before. Then

$$D\psi = D(1) = \frac{1}{\sqrt{2}}\sum \alpha_{ij}\wedge\alpha_{jl}\wedge\alpha_{li} \in \Lambda^{3,0},$$



and

$$D^2\psi \in \Lambda^{0,0} \oplus \Lambda^{2,0} \oplus \Lambda^{4,0} \oplus \Lambda^{6,0}.$$

All the degree 4 terms, such as

$$(\alpha_{23}\wedge\alpha_{30}\wedge\alpha_{02}) \lrcorner (\alpha_{20}\wedge\alpha_{01}\wedge\alpha_{12}) = -\alpha_{23}\wedge\alpha_{30}\wedge\alpha_{01}\wedge\alpha_{12}$$

cancel out.

Elements of $\Lambda^{0,0}$ come from fully contracting each directed 3-cycle with its reversal. But *any* regular tournament has a total of $n(n^2-1)/24$ 3-cycles (the dimension of the degree 3 subspace of Ξ_n), each of which contributes $3^2 = 9$ to D^2. $\square$

Two exceptional examples

G_2/T^2 has $\dim \Xi = 4$. Consider the fibration

$$G_2/T^2 \longrightarrow G_2/SU(3) \cong S^6.$$

Its total space has tangent space $(\mathfrak{su}(3)/\mathbb{R}^2) \oplus \mathbb{R}^6$, and an amalgamated ACS with $(1,0)$ space $\mathbb{C}^3 \oplus \mathbb{C}^3$. There are invariant spinors in degrees $0, 3, 3, 6$.

F_4/T^4 has $\dim \Xi = 34,432$. Consider the fibration

$$F_4/T^4 \longrightarrow F_4/Sp(3)Sp(1) = M^{28}$$

whose base has holonomy in $\mathrm{Sp}(7)\mathrm{Sp}(1)$ and total space endowed with a set $\mathscr{A}$ of twistor-type ACS's with $c_1 = 0$. $W(F_4)$ has order 1152 and acts on the set of vertices of the 24-cell (which can be identified with $\mathrm{SL}(2, \mathbb{Z}_3)$), inducing orbits

$$34,432 = 2 \times 128 + 5 \times 384 + 28 \times 1152$$

on $\mathscr{A}$. The first two are distinguished by their spinor degrees as follows:

$$1, 0, 0, 23, 63, 144, 436, 963, 1674, 2735, 3924, 4743, \mathbf{5020}, 4743 \ldots$$
$$1, 0, 0, 23, 72, 135, 379, 1017, 1827, 2627, 3681, 4806, \mathbf{5296}, 4806 \ldots$$

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