Hitchin @80:
Trends in differential and complex geometry

A journey through wobbliness

Ana Peón-Nieto
Universidade de Santiago de Compostela

A brief history

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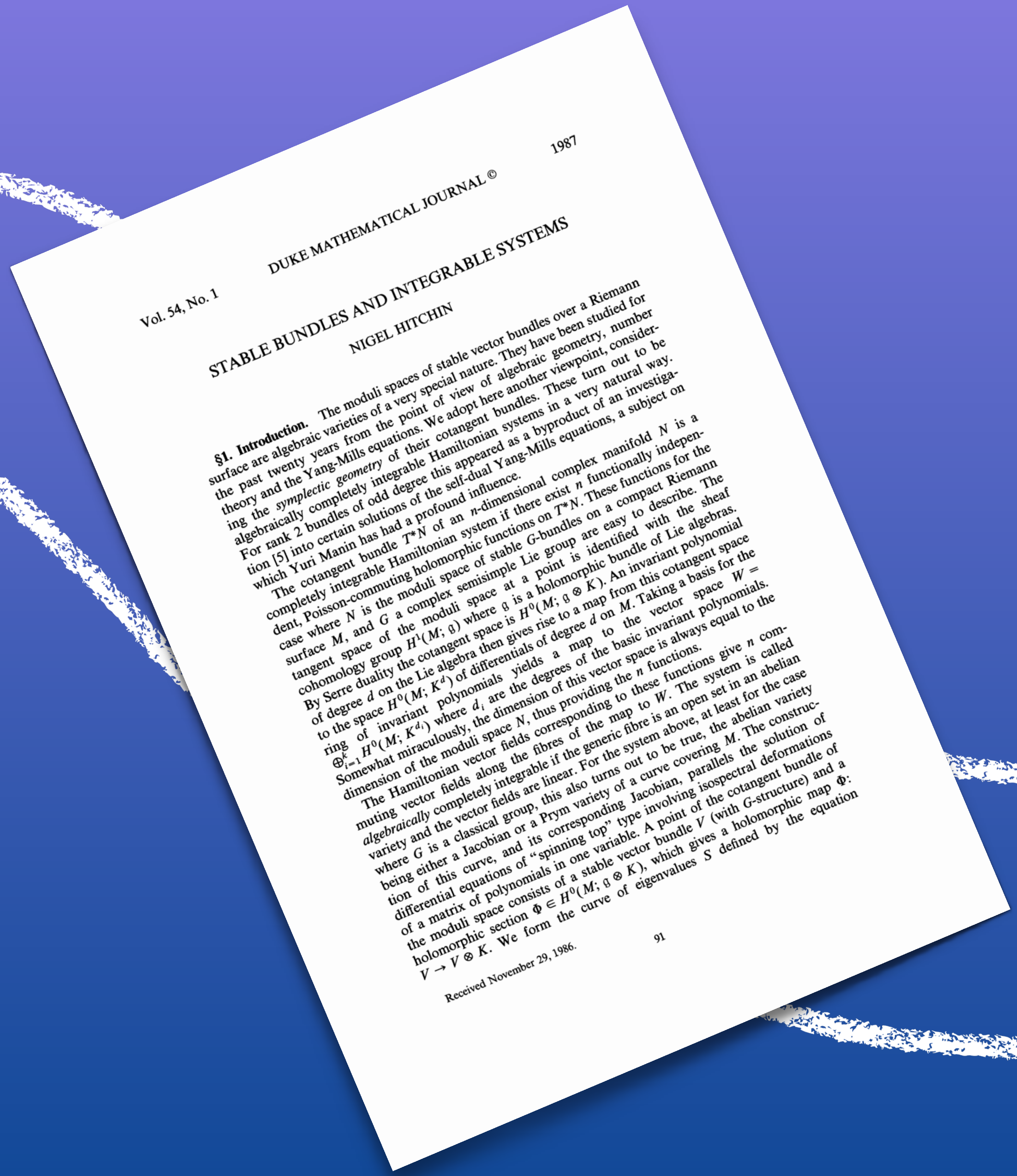
A brief history



Vol. 54, No. 1 DUKE MATHEMATICAL JOURNAL © 1987

STABLE BUNDLES AND INTEGRABLE SYSTEMS

NIGEL HITCHIN

§1. Introduction. The moduli spaces of stable vector bundles over a Riemann surface are algebraic varieties of a very special nature. They have been studied for the past twenty years from the point of view of algebraic geometry, number theory and the Yang-Mills equations. We adopt here another viewpoint, considering the *symplectic geometry* of their cotangent bundles. These turn out to be algebraically completely integrable Hamiltonian systems in a very natural way. For rank 2 bundles of odd degree this appeared as a byproduct of an investigation [5] into certain solutions of the self-dual Yang-Mills equations, a subject on which Yuri Manin has had a profound influence.

The cotangent bundle T^*N of an n-dimensional complex manifold N is a completely integrable Hamiltonian system if there exist n functionally independent, Poisson-commuting holomorphic functions on T^*N. These functions for the case where N is the moduli space of stable G-bundles on a compact Riemann surface M, and G a complex semisimple Lie group are easy to describe. The tangent space of the moduli space at a point is identified with the sheaf cohomology group $H^1(M; \mathfrak{g})$ where $\mathfrak{g}$ is a holomorphic bundle of Lie algebras. By Serre duality the cotangent space is $H^0(M; \mathfrak{g} \otimes K)$. An invariant polynomial of degree d on the Lie algebra then gives rise to a map from this cotangent space to the space $H^0(M; K^d)$ of differentials of degree d on M. Taking a basis for the ring of invariant polynomials yields a map to the vector space $W = \bigoplus_{i=1}^{k} H^0(M; K^{d_i})$ where d_i are the degrees of the basic invariant polynomials. Somewhat miraculously, the dimension of this vector space is always equal to the dimension of the moduli space N, thus providing the n functions.

The Hamiltonian vector fields corresponding to these functions give n commuting vector fields along the fibres of the map to W. The system is called *algebraically* completely integrable if the generic fibre is an open set in an abelian variety and the vector fields are linear. For the system above, at least for the case where G is a classical group, this also turns out to be true, the abelian variety being either a Jacobian or a Prym variety of a curve covering M. The construction of this curve, and its corresponding Jacobian, parallels the solution of differential equations of "spinning top" type involving isospectral deformations of a matrix of polynomials in one variable. A point of the cotangent bundle of the moduli space consists of a stable vector bundle V (with G-structure) and a holomorphic section $\Phi \in H^0(M; \mathfrak{g} \otimes K)$, which gives a holomorphic map Φ: $V \to V \otimes K$. We form the curve of eigenvalues S defined by the equation

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STABLE BUNDLES AND INTEGRA

NIGEL HITCHIN

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LIE GROUPS AND TEICHMÜLLER SPACE

N. J. Hitchin

(Received 9 November 1990; in revised form 5 August 1991)

§1. INTRODUCTION

It is well-known that Teichmüller space is a ball of dimension $(6g-6)$. What is perhaps less well-known is that this result can be interpreted purely in terms of the fundamental group $\pi_1(\Sigma)$ of a compact surface Σ and its representations in $PSL(2, \mathbb{R})$. Every conformal structure on Σ can be uniformized, thereby representing Σ as the quotient of the upper half-plane by a subgroup of $PSL(2, \mathbb{R})$ isomorphic to $\pi_1(\Sigma)$. This describes a homomorphism from $\pi_1(\Sigma)$ to $PSL(2, \mathbb{R})$ well-defined up to conjugation by an element of $PSL(2, \mathbb{R})$. It can be proved (see [4], [6]) that the set of all such homomorphisms constitutes a connected component of the topological space $\mathrm{Hom}(\pi_1(\Sigma); PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$. We conclude, therefore, that there is a component of this space homeomorphic to a Euclidean space $\mathbb{R}^{(6g-6)}$.

In this paper we shall demonstrate the existence of an analogous component—which we shall call the *Teichmüller* component—where we replace the group $PSL(2, \mathbb{R})$ by $PSL(n, \mathbb{R})$ or more generally by the adjoint group of a split real form G^r of any complex simple Lie group G^c (recall that the split real forms of the classical groups are $SL(n, \mathbb{R})$, $SO(n+1, n)$, $Sp(2n, \mathbb{R})$ and $SO(n, n)$). This component is homeomorphic to $\mathbb{R}^{(2g-2)\dim G^r}$ and contains Teichmüller space in the sense that there is a distinguished three-dimensional subgroup $PSL(2, \mathbb{R}) \subset G^r$ which embeds the uniformizing representations of $\pi_1(\Sigma)$ as a subspace of the Teichmüller component. We are therefore considering an extension of ordinary Teichmüller space.

To be more precise, we have:

Theorem A. *Let Σ be a compact oriented surface of genus $g > 1$ and let G^r be the adjoint group of the split real form of a complex simple Lie group G^c. Let $\mathrm{Hom}^+(\pi_1(\Sigma); G^r)$ denote the space of homomorphisms from the fundamental group to G^r which act completely reducibly on the Lie algebra of G^r. Then the quotient of $\mathrm{Hom}^+(\pi_1(\Sigma); G^r)$ by the conjugation action of G^r has a connected component homeomorphic to a Euclidean space of dimension $(2g-2)\dim G^r$.*

Returning to the case of $PSL(2, \mathbb{R})$, there is an easy way to identify this distinguished component. Every homomorphism from $\pi_1(\Sigma)$ to $PSL(2, \mathbb{R})$ defines an associated (flat) principal $PSL(2, \mathbb{R})$- bundle which is topologically classified by its reduction to a maximal compact subgroup—in this case the circle. The first Chern class of this circle bundle then provides an integer invariant and it turns out that Teichmüller space is precisely the subspace for which this integer takes the value $(2g-2)$. (In fact an outer automorphism of $PSL(2, \mathbb{R})$ yields an isomorphic copy of the space corresponding to the value $-(2g-2)$ so that in truth there are *two* components homeomorphic to a ball). In the latter half of this paper we consider whether this feature holds for the Teichmüller component of $PSL(n, \mathbb{R})$ if

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ABELIANIZATION OF HIGGS BUNDLES
FOR QUASI-SPLIT REAL GROUPS

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Abstract. We study the Hitchin map for $G_{\mathbb{R}}$-Higgs bundles on a smooth curve, where $G_{\mathbb{R}}$ is a quasi-split real form of a complex reductive algebraic group G. By looking at the moduli stack of regular $G_{\mathbb{R}}$-Higgs bundles, we prove that it induces a banded gerbe structure on a slightly larger stack, whose band is given by sheaves of tori. This characterization yields a cocyclic description of the fibres of the corresponding Hitchin map by means of cameral data. According to this, fibres of the Hitchin map are categories of principal torus bundles on the cameral cover. The corresponding points inside the stack of G-Higgs bundles are contained in the substack of points fixed by an involution induced by the Cartan involution of $G_{\mathbb{R}}$. We determine this substack of fixed points and prove that stable points are in correspondence with stable $G_{\mathbb{R}}$-Higgs bundles.

1. Introduction

Higgs bundles over a compact Riemann surface for complex reductive Lie groups were introduced by Hitchin in [H1], [H2], and have since been objects of intensive study, due to the rich geometry of their moduli spaces. A particularly interesting aspect is the Hitchin integrable system. This is a fibration by complex Lagrangian tori over the so-called Hitchin base [H2], [BNR], [Si], [Sco], [S], [DG], [N]. This

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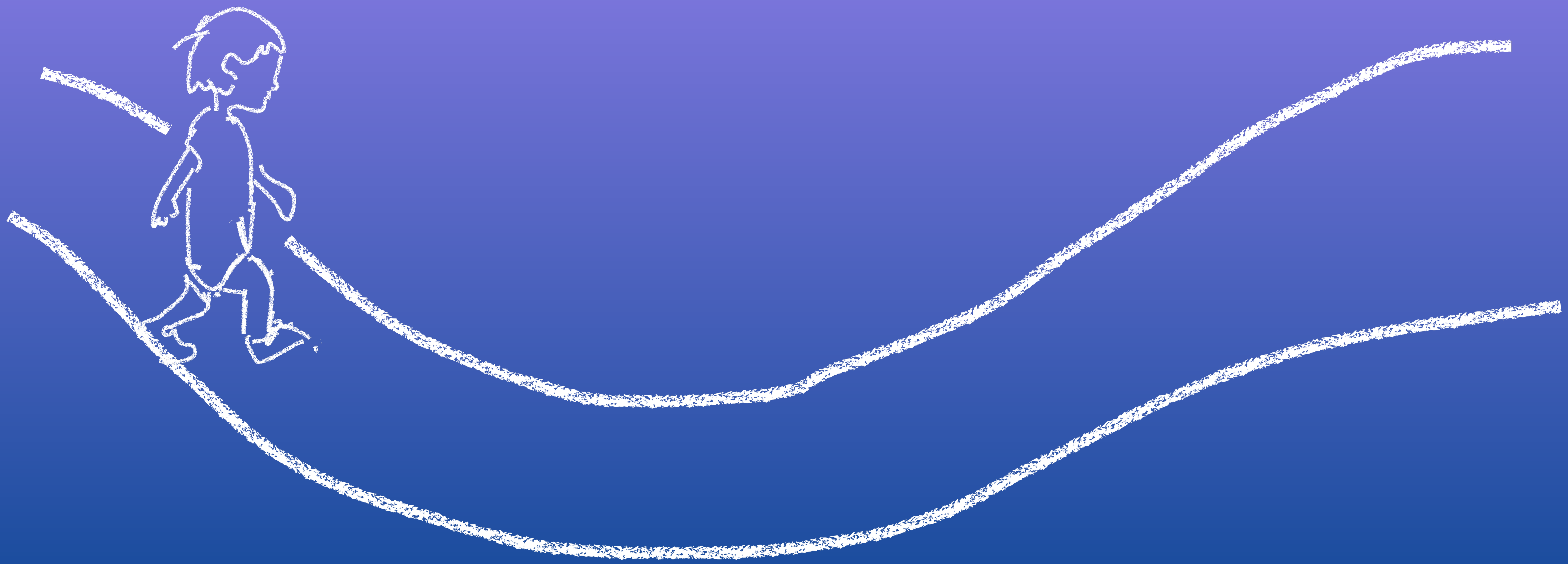

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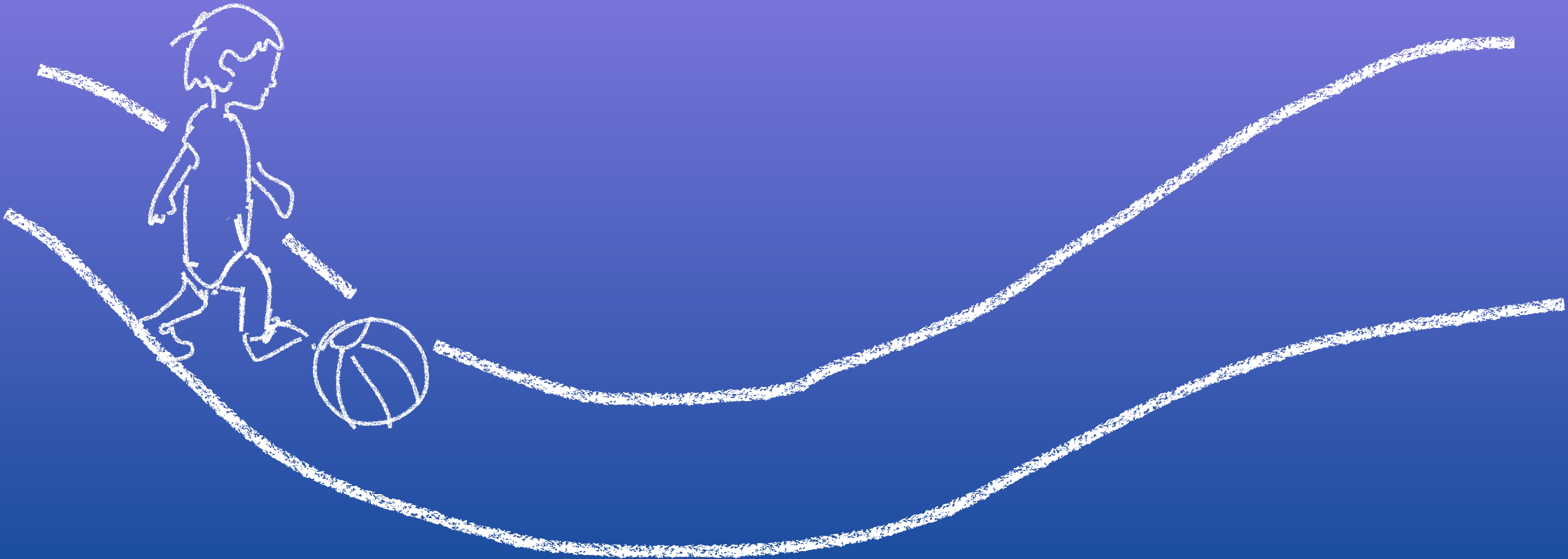

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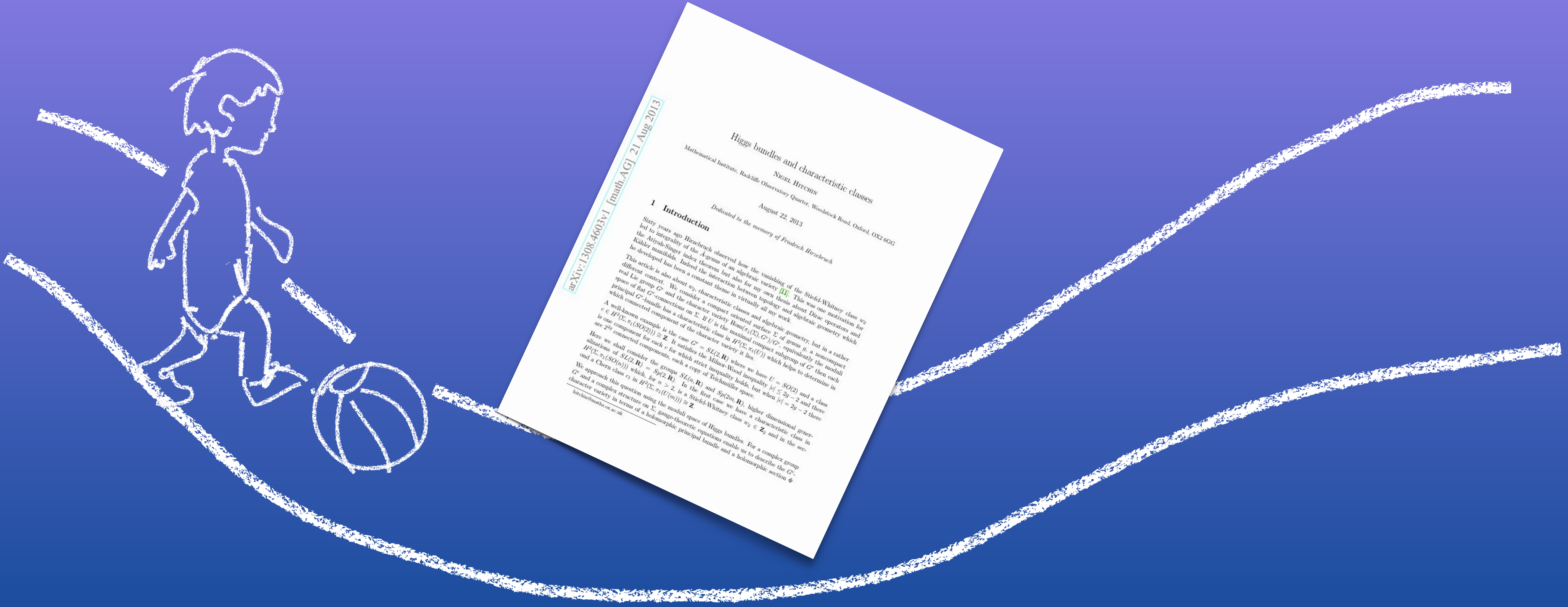

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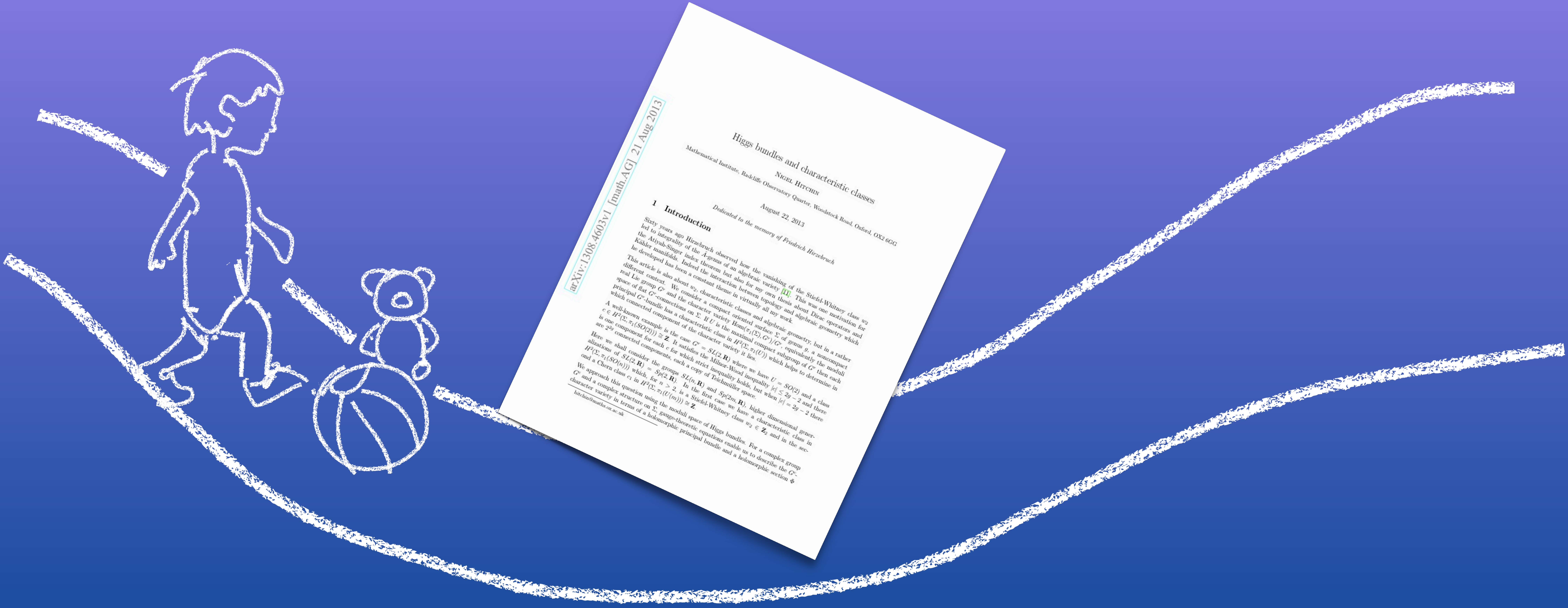

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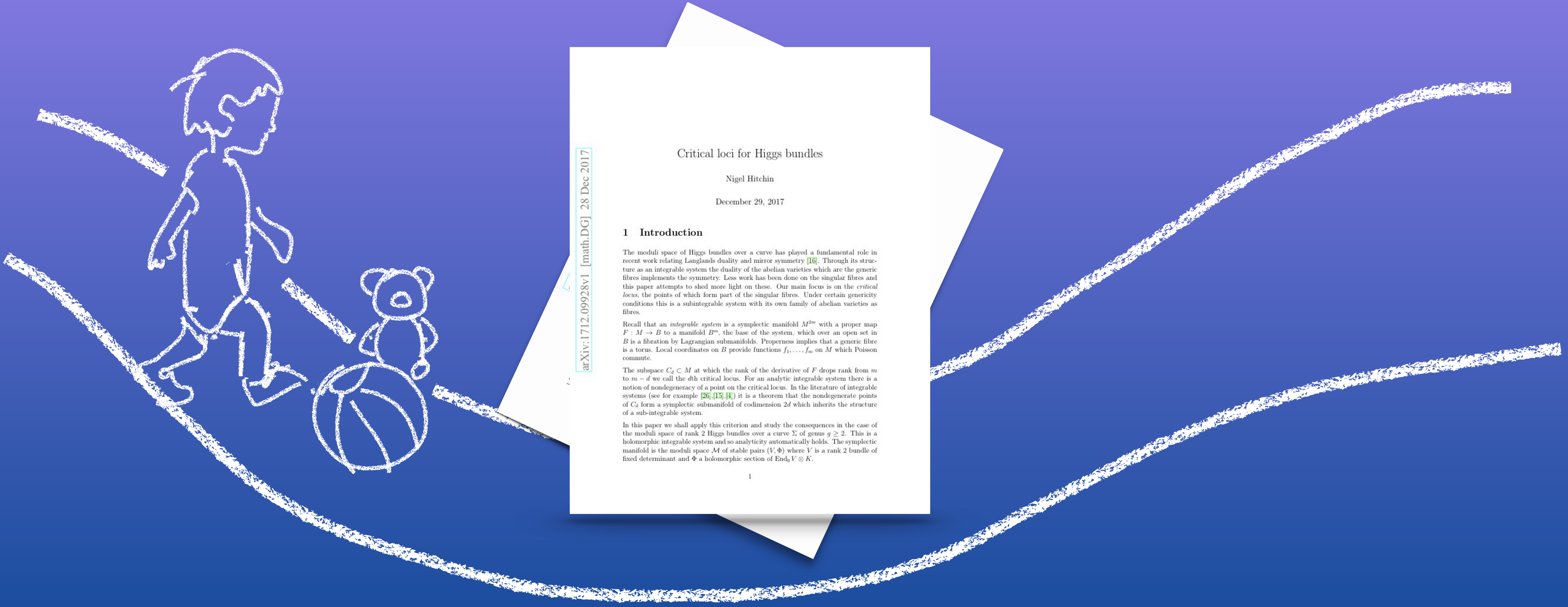

arXiv:1712.09928v1 [math.DG] 28 Dec 2017

Critical loci for Higgs bundles

Nigel Hitchin

December 29, 2017

1 Introduction

The moduli space of Higgs bundles over a curve has played a fundamental role in recent work relating Langlands duality and mirror symmetry [16]. Through its structure as an integrable system the duality of the abelian varieties which are the generic fibres implements the symmetry. Less work has been done on the singular fibres and this paper attempts to shed more light on these. Our main focus is on the *critical locus*, the points of which form part of the singular fibres. Under certain genericity conditions this is a subintegrable system with its own family of abelian varieties as fibres.

Recall that an *integrable system* is a symplectic manifold M^{2m} with a proper map $F : M \to B$ to a manifold B^m, the base of the system, which over an open set in B is a fibration by Lagrangian submanifolds. Properness implies that a generic fibre is a torus. Local coordinates on B provide functions $f_1, \dots, f_m$ on M which Poisson commute.

The subspace $C_d \subset M$ at which the rank of the derivative of F drops rank from m to $m - d$ we call the dth critical locus. For an analytic integrable system there is a notion of nondegeneracy of a point on the critical locus. In the literature of integrable systems (see for example [26],[15],[4]) it is a theorem that the nondegenerate points of C_d form a symplectic submanifold of codimension $2d$ which inherits the structure of a sub-integrable system.

In this paper we shall apply this criterion and study the consequences in the case of the moduli space of rank 2 Higgs bundles over a curve Σ of genus $g \geq 2$. This is a holomorphic integrable system and so analyticity automatically holds. The symplectic manifold is the moduli space $\mathcal{M}$ of stable pairs (V, Φ) where V is a rank 2 bundle of fixed determinant and Φ a holomorphic section of $\mathrm{End}_0 V \otimes K$.

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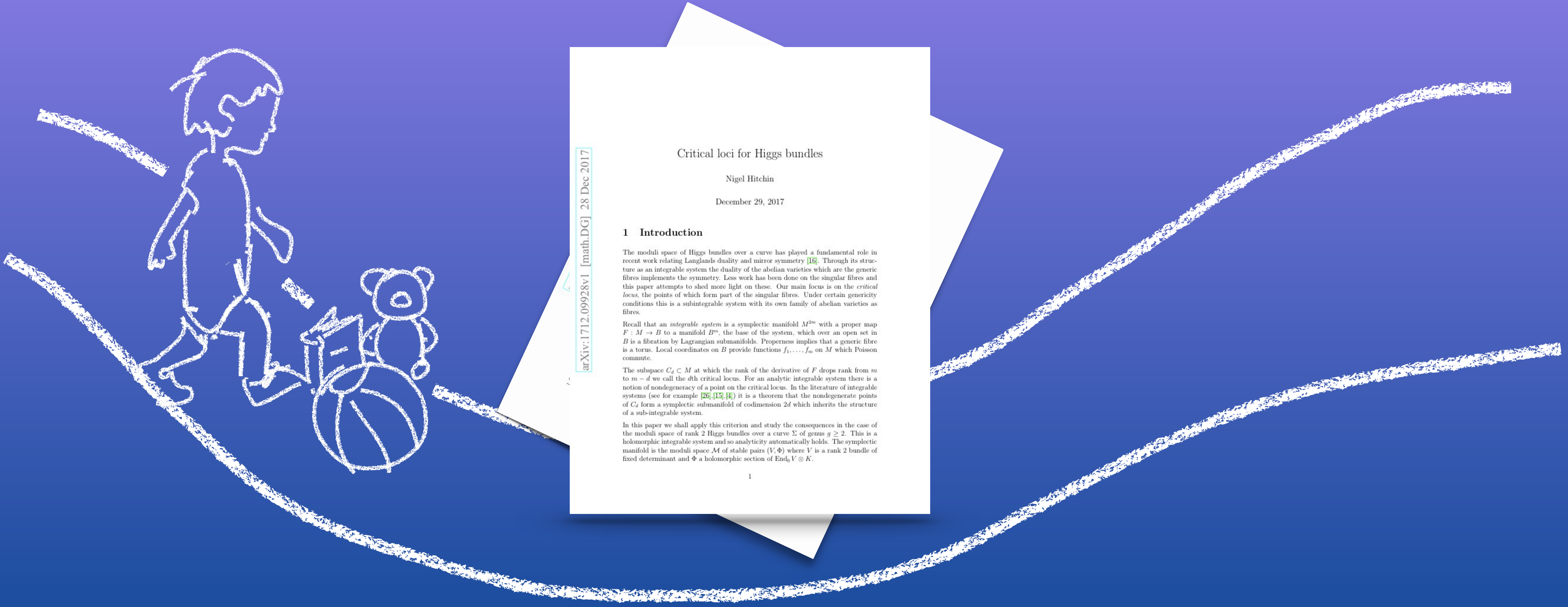

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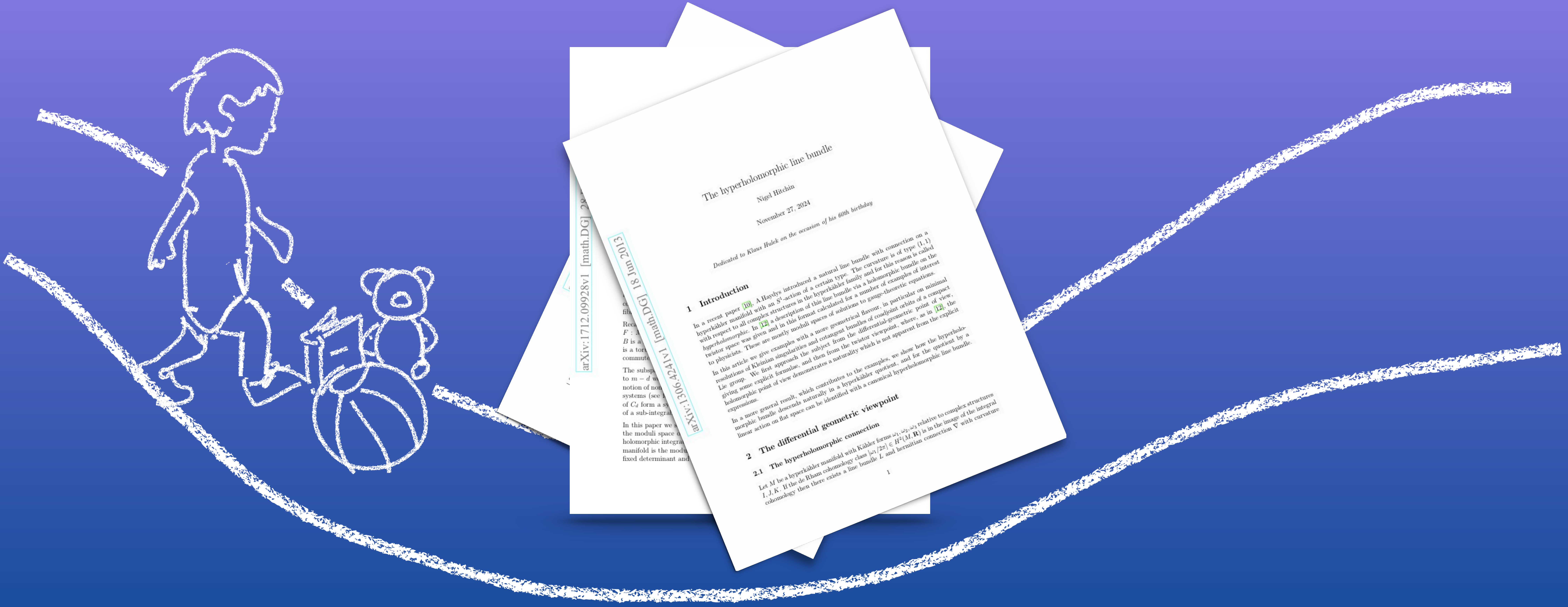

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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Branes on the singular locus of the Hitchin system via Borel and other parabolic subgroups

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Abstract
We study the mirror symmetry on the singular locus of the Hitchin system at two levels. First, by covering it by (supports of) (BBB)-branes, corresponding to Higgs bundles reducing their structure group to the Levi subgroup of some parabolic subgroup P, whose conjectural dual (BAA)-branes we describe. Heuristically speaking, the latter are given by Higgs bundles reducing their structure group to the unipotent radical of P. Second, when P is a Borel subgroup, we are able to construct a family of hyperholomorphic bundles on the (BBB)-brane and study the variation of the dual under this choice. We give evidence of both families of branes being dual under mirror symmetry via an integral functor induced by Fourier–Mukai in the moduli stack of Higgs bundles.

KEYWORDS
Higgs bundles, mirror symmetry

MSC (2020)
14J33, 14D21

1 | INTRODUCTION

1.1 | Brief description

In this paper, we study the action of mirror symmetry on the singular locus of the moduli space M_n of Higgs bundles. We proceed first by describing hyperholomorphic subvarieties covering M_n^{sing}, those become (BBB)-branes after specifying a hyperholomorphic bundle on them. Then, we construct complex Lagrangian subvarieties, supporting (BAA)-branes after being equipped with a flat bundle, and we conjecture that behind these constructions stands a pair of mirror dual branes. Each of the previous pairs of branes is naturally associated with a parabolic subgroup of $GL(n, \mathbb{C})$. When this parabolic is the Borel subgroup, we find ourselves over the locus of totally reducible spectral curves, namely, those with a maximal number of irreducible components, cf. Equation (3.1). A more complete analysis is possible in this case and we are able to construct families of flat (hence hyperholomorphic) bundles giving rise to (BBB)-branes. These (BBB)-branes only intersect Hitchin fibers associated with coarse compactified Jacobians where no Fourier–Mukai transform has been defined. We then consider the Fourier–Mukai transform between the associated stacks and prove that it restricts to a

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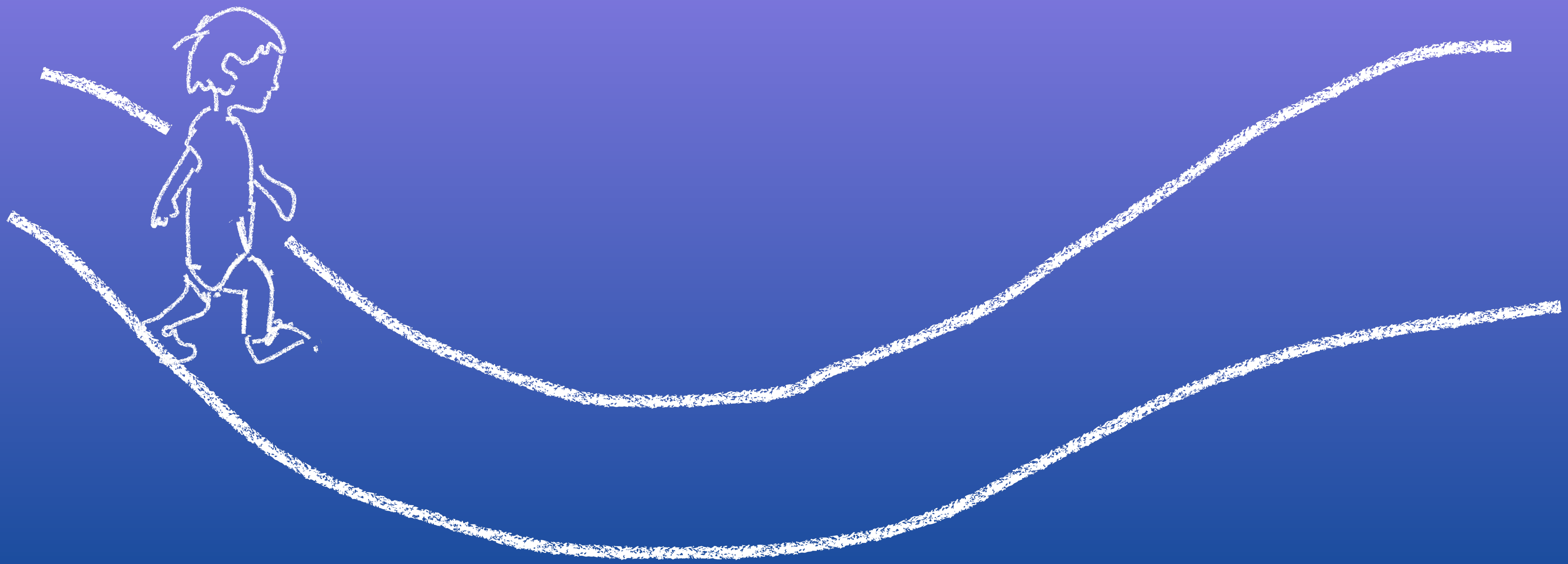

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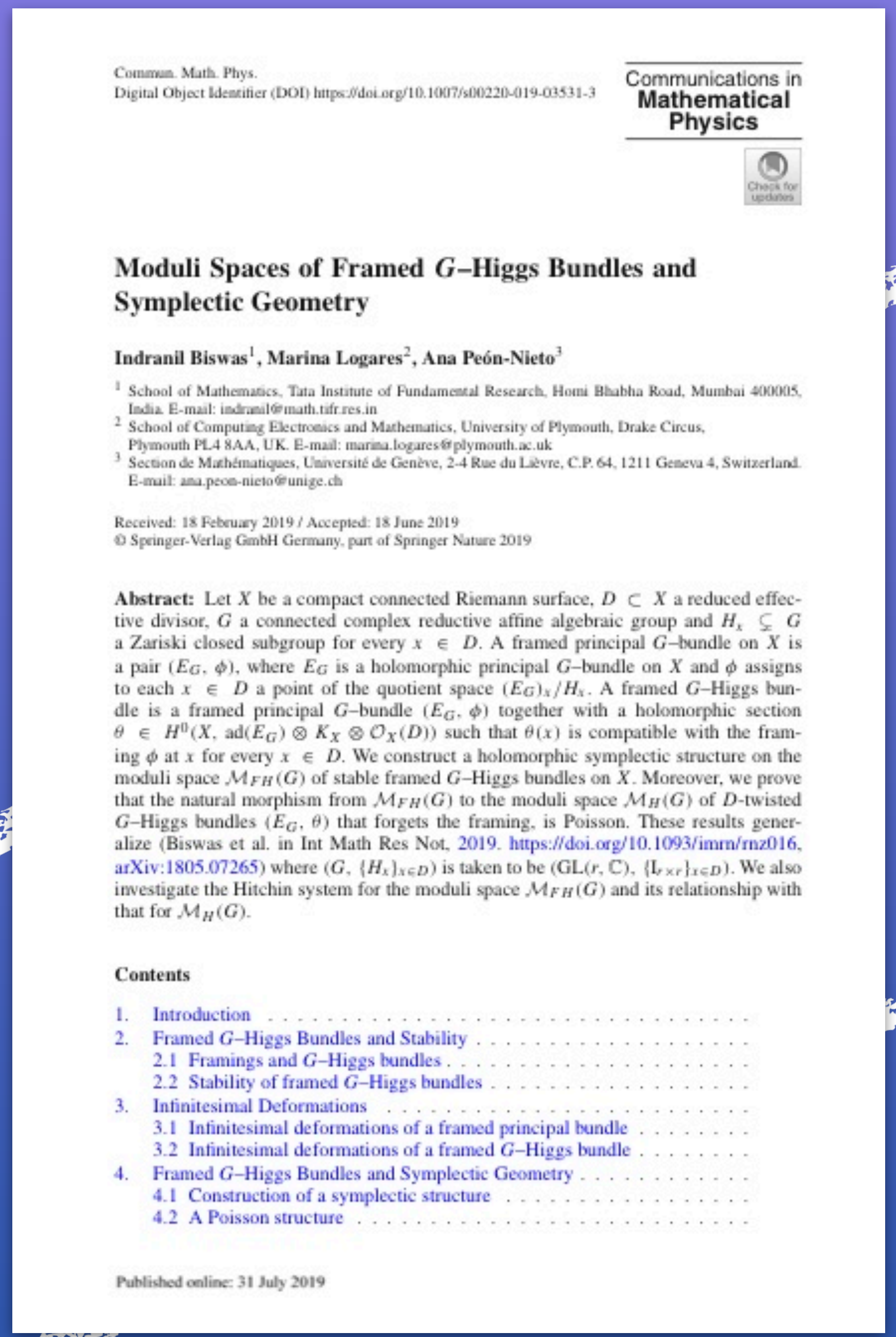
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Communications in Mathematical Physics

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Moduli Spaces of Framed G–Higgs Bundles and Symplectic Geometry

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Abstract: Let X be a compact connected Riemann surface, $D \subset X$ a reduced effective divisor, G a connected complex reductive affine algebraic group and $H_x \subsetneq G$ a Zariski closed subgroup for every $x \in D$. A framed principal G–bundle on X is a pair (E_G, ϕ), where E_G is a holomorphic principal G–bundle on X and ϕ assigns to each $x \in D$ a point of the quotient space $(E_G)_x/H_x$. A framed G–Higgs bundle is a framed principal G–bundle (E_G, ϕ) together with a holomorphic section $\theta \in H^0(X, \mathrm{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D))$ such that $\theta(x)$ is compatible with the framing ϕ at x for every $x \in D$. We construct a holomorphic symplectic structure on the moduli space $\mathcal{M}_{FH}(G)$ of stable framed G–Higgs bundles on X. Moreover, we prove that the natural morphism from $\mathcal{M}_{FH}(G)$ to the moduli space $\mathcal{M}_H(G)$ of D-twisted G–Higgs bundles (E_G, θ) that forgets the framing, is Poisson. These results generalize (Biswas et al. in Int Math Res Not, 2019. https://doi.org/10.1093/imrn/rnz016, arXiv:1805.07265) where $(G, \{H_x\}_{x\in D})$ is taken to be $(\mathrm{GL}(r, \mathbb{C}), \{\mathrm{I}_{r\times r}\}_{x\in D})$. We also investigate the Hitchin system for the moduli space $\mathcal{M}_{FH}(G)$ and its relationship with that for $\mathcal{M}_H(G)$.

Contents

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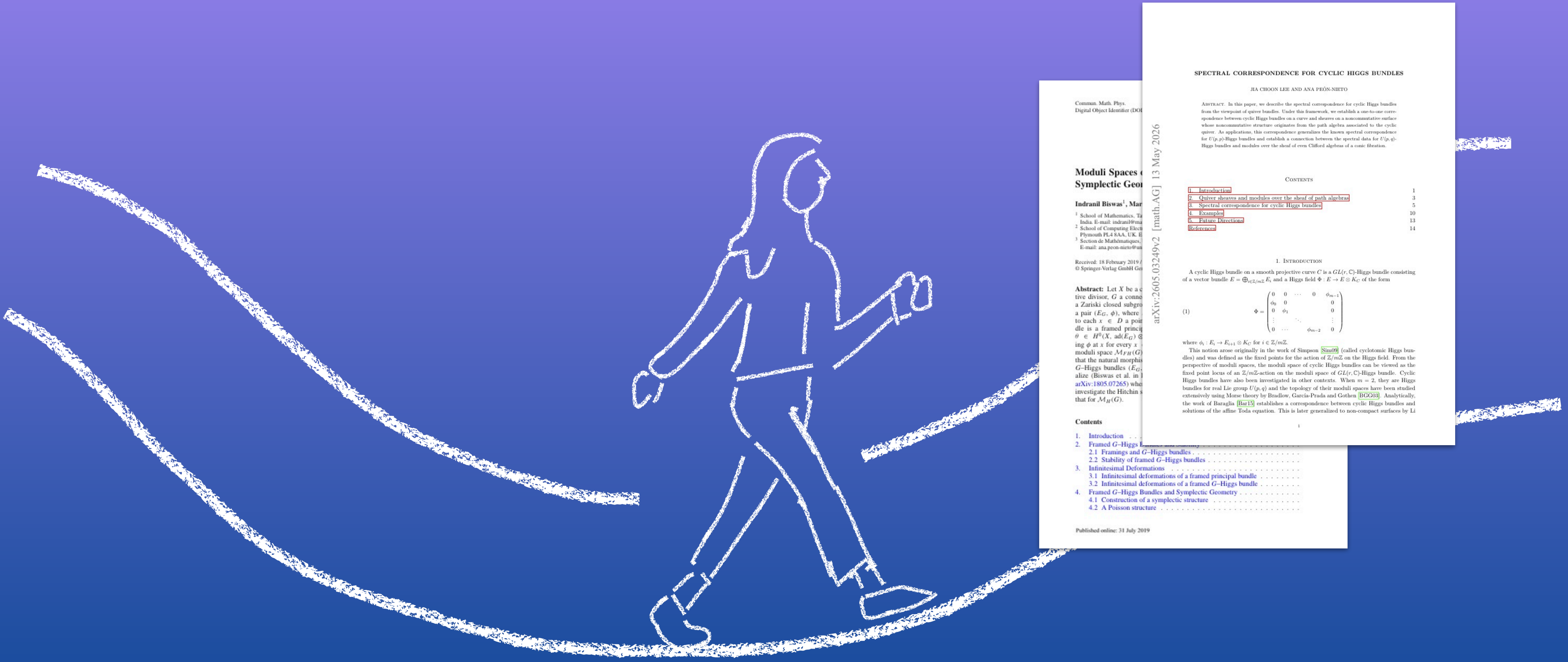

SPECTRAL CORRESPONDENCE FOR CYCLIC HIGGS BUNDLES

JIA CHOON LEE AND ANA PEÓN-NIETO

arXiv:2605.03249v2 [math.AG] 13 May 2026

ABSTRACT. In this paper, we describe the spectral correspondence for cyclic Higgs bundles from the viewpoint of quiver bundles. Under this framework, we establish a one-to-one correspondence between cyclic Higgs bundles on a curve and sheaves on a noncommutative surface whose noncommutative structure originates from the path algebra associated to the cyclic quiver. As applications, this correspondence generalizes the known spectral correspondence for $U(p,p)$-Higgs bundles and establish a connection between the spectral data for $U(p,q)$-Higgs bundles and modules over the sheaf of even Clifford algebras of a conic fibration.

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3. Spectral correspondence for cyclic Higgs bundles 5
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1. INTRODUCTION

A cyclic Higgs bundle on a smooth projective curve C is a $GL(r,\mathbb{C})$-Higgs bundle consisting of a vector bundle $E = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} E_i$ and a Higgs field $\Phi : E \to E \otimes K_C$ of the form

$$\Phi = \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi_{m-1} \\ \phi_0 & 0 & & & 0 \\ 0 & \phi_1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & \phi_{m-2} & 0 \end{pmatrix} \quad (1)$$

where $\phi_i : E_i \to E_{i+1} \otimes K_C$ for $i \in \mathbb{Z}/m\mathbb{Z}$.

This notion arose originally in the work of Simpson [Sim09] (called cyclotomic Higgs bundles) and was defined as the fixed points for the action of $\mathbb{Z}/m\mathbb{Z}$ on the Higgs field. From the perspective of moduli spaces, the moduli space of cyclic Higgs bundles can be viewed as the fixed point locus of an $\mathbb{Z}/m\mathbb{Z}$-action on the moduli space of $GL(r,\mathbb{C})$-Higgs bundle. Cyclic Higgs bundles have also been investigated in other contexts. When $m = 2$, they are Higgs bundles for real Lie group $U(p,q)$ and the topology of their moduli spaces have been studied extensively using Morse theory by Bradlow, García-Prada and Gothen [BGG03]. Analytically, the work of Baraglia [Bar15] establishes a correspondence between cyclic Higgs bundles and solutions of the affine Toda equation. This is later generalized to non-compact surfaces by Li

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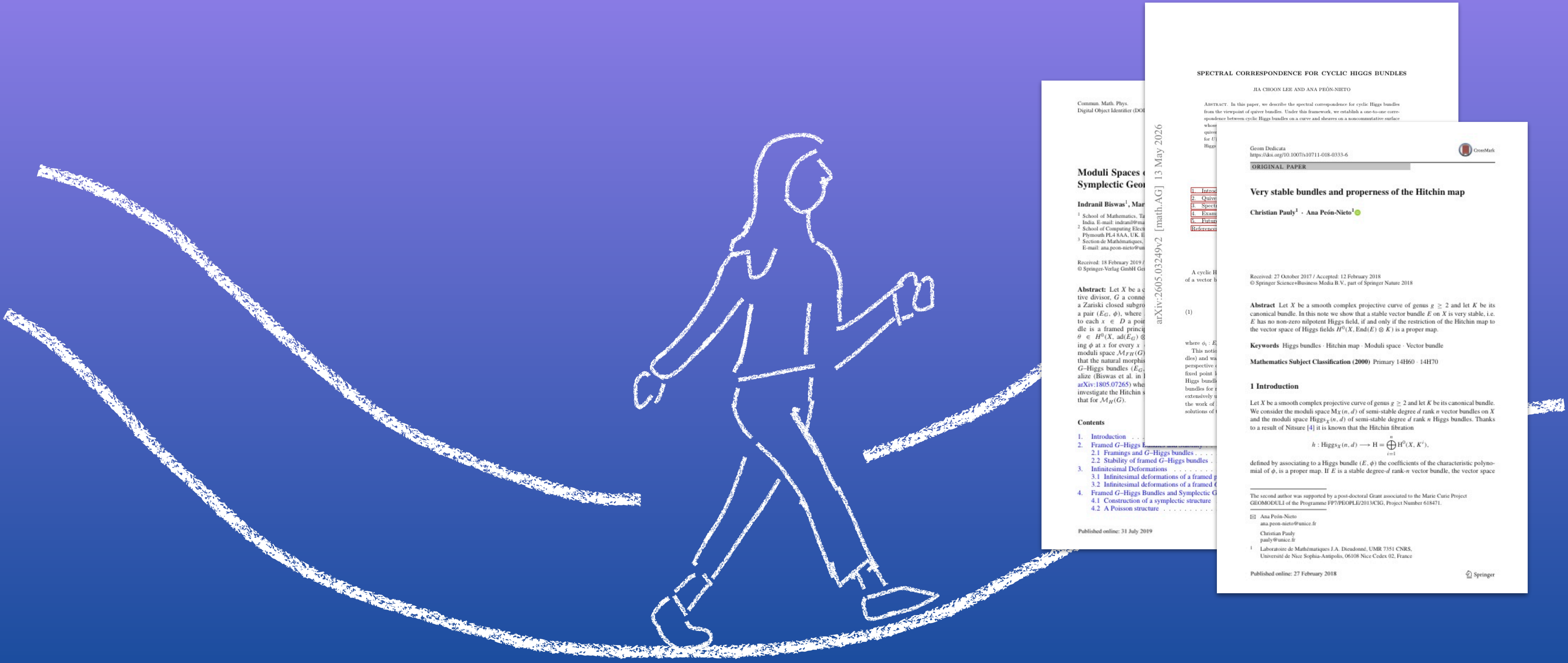

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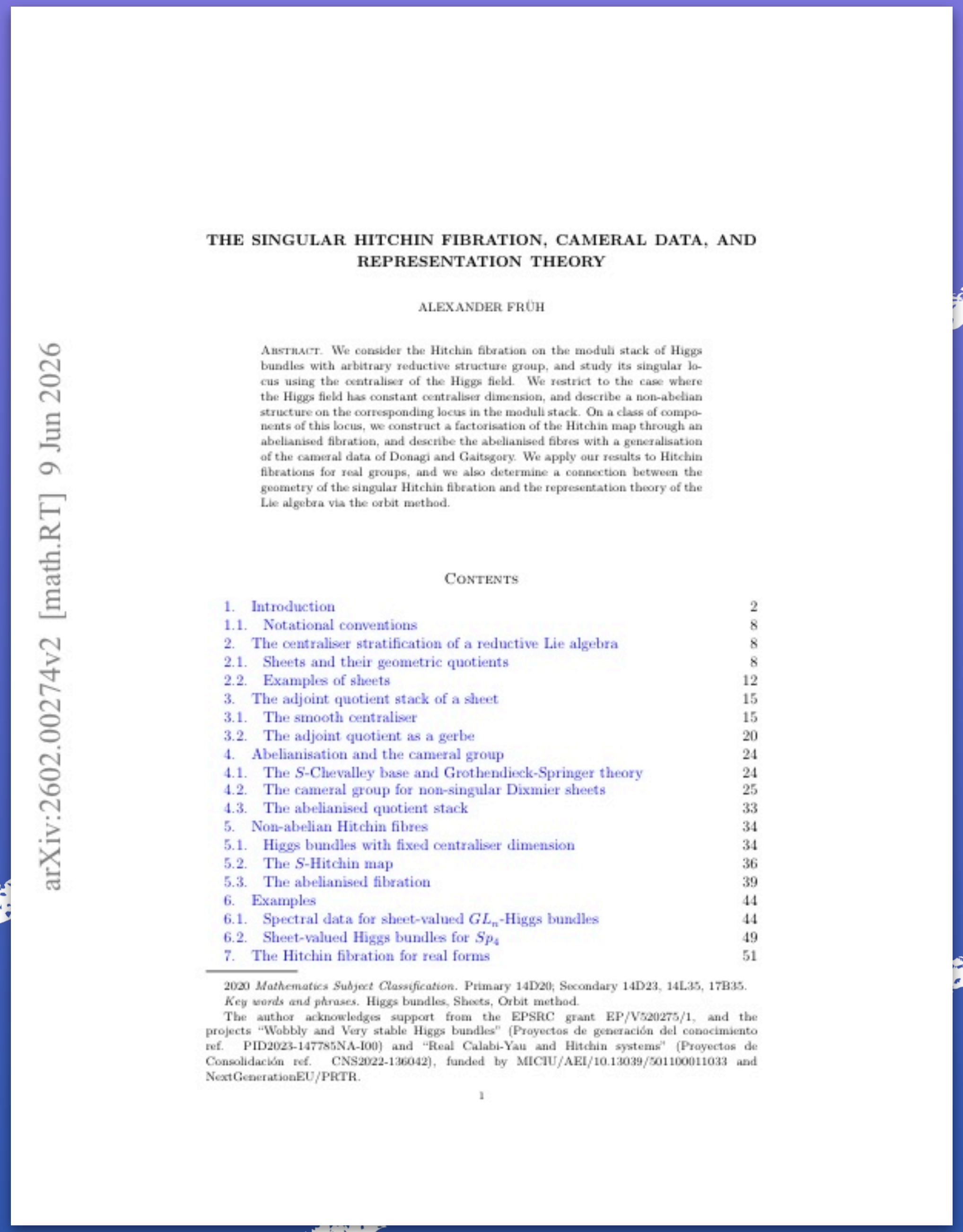

arXiv:2602.00274v2 [math.RT] 9 Jun 2026

THE SINGULAR HITCHIN FIBRATION, CAMERAL DATA, AND REPRESENTATION THEORY

ALEXANDER FRÜH

Abstract. We consider the Hitchin fibration on the moduli stack of Higgs bundles with arbitrary reductive structure group, and study its singular locus using the centraliser of the Higgs field. We restrict to the case where the Higgs field has constant centraliser dimension, and describe a non-abelian structure on the corresponding locus in the moduli stack. On a class of components of this locus, we construct a factorisation of the Hitchin map through an abelianised fibration, and describe the abelianised fibres with a generalisation of the cameral data of Donagi and Gaitsgory. We apply our results to Hitchin fibrations for real groups, and we also determine a connection between the geometry of the singular Hitchin fibration and the representation theory of the Lie algebra via the orbit method.

Contents

2020 *Mathematics Subject Classification.* Primary 14D20; Secondary 14D23, 14L35, 17B35.

Key words and phrases. Higgs bundles, Sheets, Orbit method.

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1

Wobbly Higgs bundles

Wobbly Higgs bundles

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Very stable Higgs bundles, equivariant multiplicity and mirror symmetry

Tamás Hausel[1] · Nigel Hitchin[2]

Received: 4 March 2021 / Accepted: 2 December 2021 / Published online: 21 January 2022

Abstract We define and study the existence of very stable Higgs bundles on Riemann surfaces, how it implies a precise formula for the multiplicity of the very stable components of the global nilpotent cone and its relationship to mirror symmetry. The main ingredients are the Bialynicki-Birula theory of $\mathbb{C}^*$-actions on semiprojective varieties, $\mathbb{C}^*$ characters of indices of $\mathbb{C}^*$-equivariant coherent sheaves, Hecke transformation for Higgs bundles, relative Fourier–Mukai transform along the Hitchin fibration, hyperholomorphic structures on universal bundles and cominuscule Higgs bundles.

Contents

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Wobbly Higgs bundles

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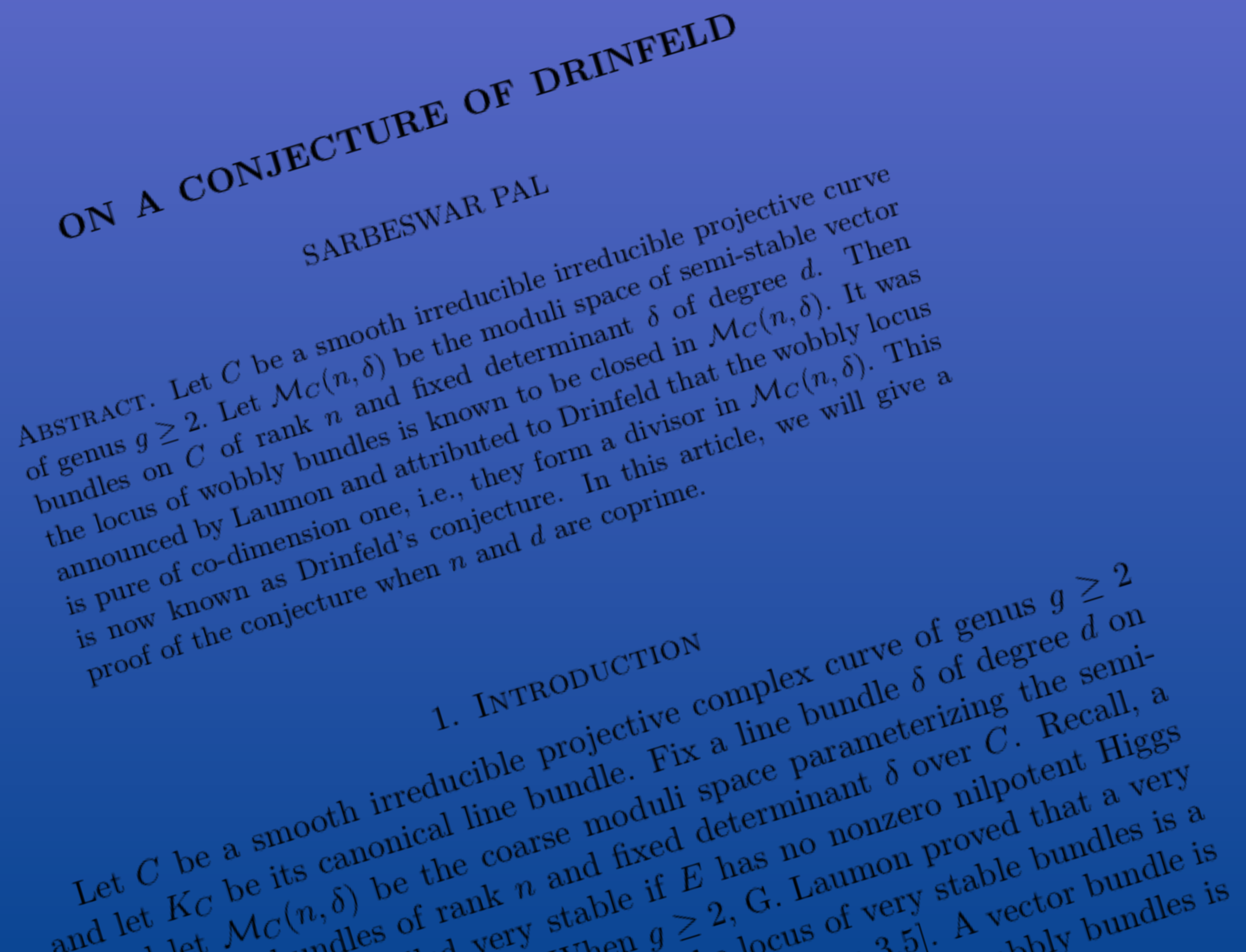
ON A CONJECTURE OF DRINFELD

SARBESWAR PAL

ABSTRACT. Let C be a smooth irreducible irreducible projective curve of genus $g \geq 2$. Let $\mathcal{M}_C(n, \delta)$ be the moduli space of semi-stable vector bundles on C of rank n and fixed determinant δ of degree d. Then the locus of wobbly bundles is known to be closed in $\mathcal{M}_C(n, \delta)$. It was announced by Laumon and attributed to Drinfeld that the wobbly locus is pure of co-dimension one, i.e., they form a divisor in $\mathcal{M}_C(n, \delta)$. This is now known as Drinfeld's conjecture. In this article, we will give a proof of the conjecture when n and d are coprime.

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Let C be a smooth irreducible projective complex curve of genus $g \geq 2$ and let K_C be its canonical line bundle. Fix a line bundle δ of degree d on C and let $\mathcal{M}_C(n, \delta)$ be the coarse moduli space parameterizing the semi-stable bundles of rank n and fixed determinant δ over C. Recall, a vector bundle E is called very stable if E has no nonzero nilpotent Higgs field. When $g \geq 2$, G. Laumon proved that a very stable bundle is a … the locus of very stable bundles is a … A vector bundle is … wobbly bundles is

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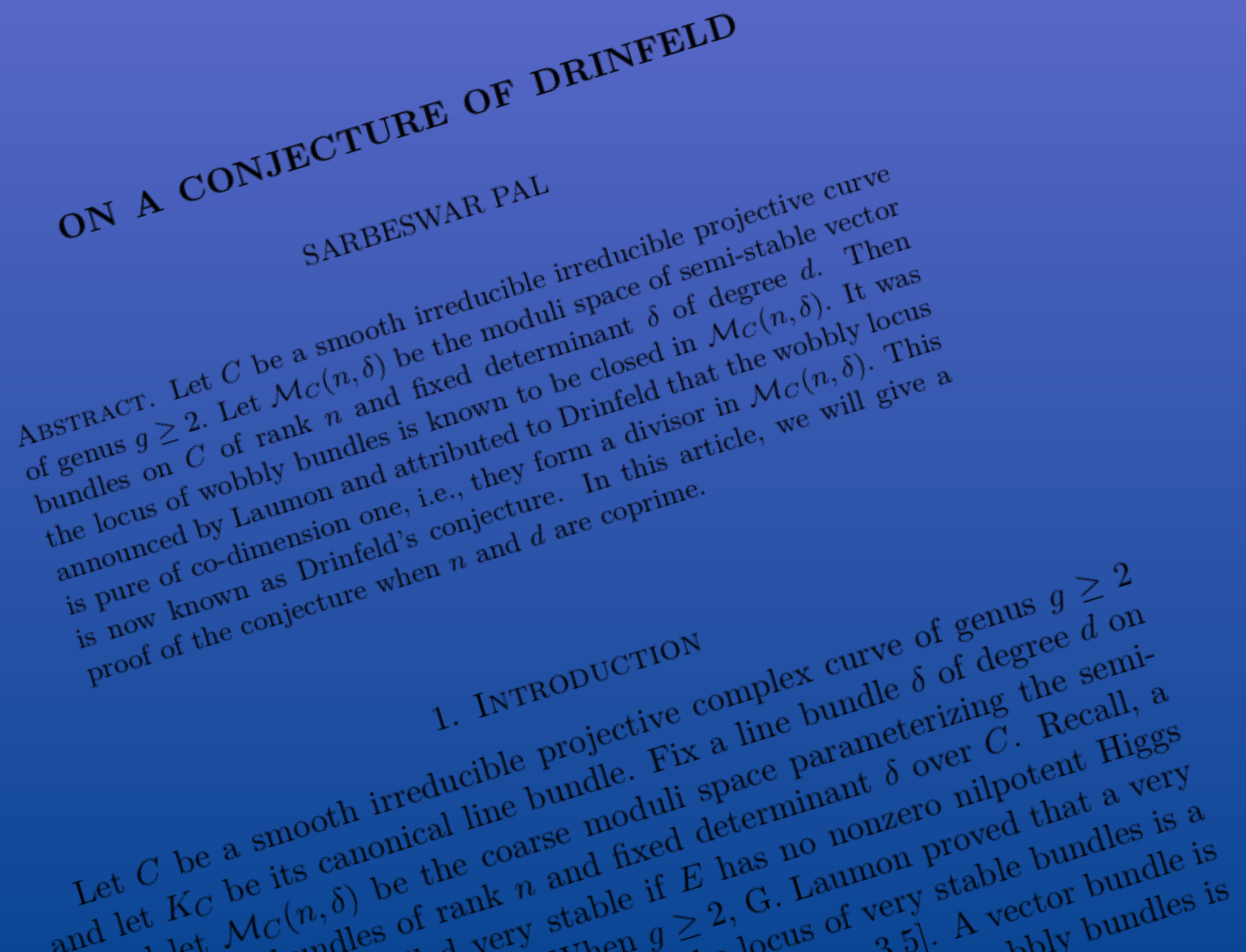
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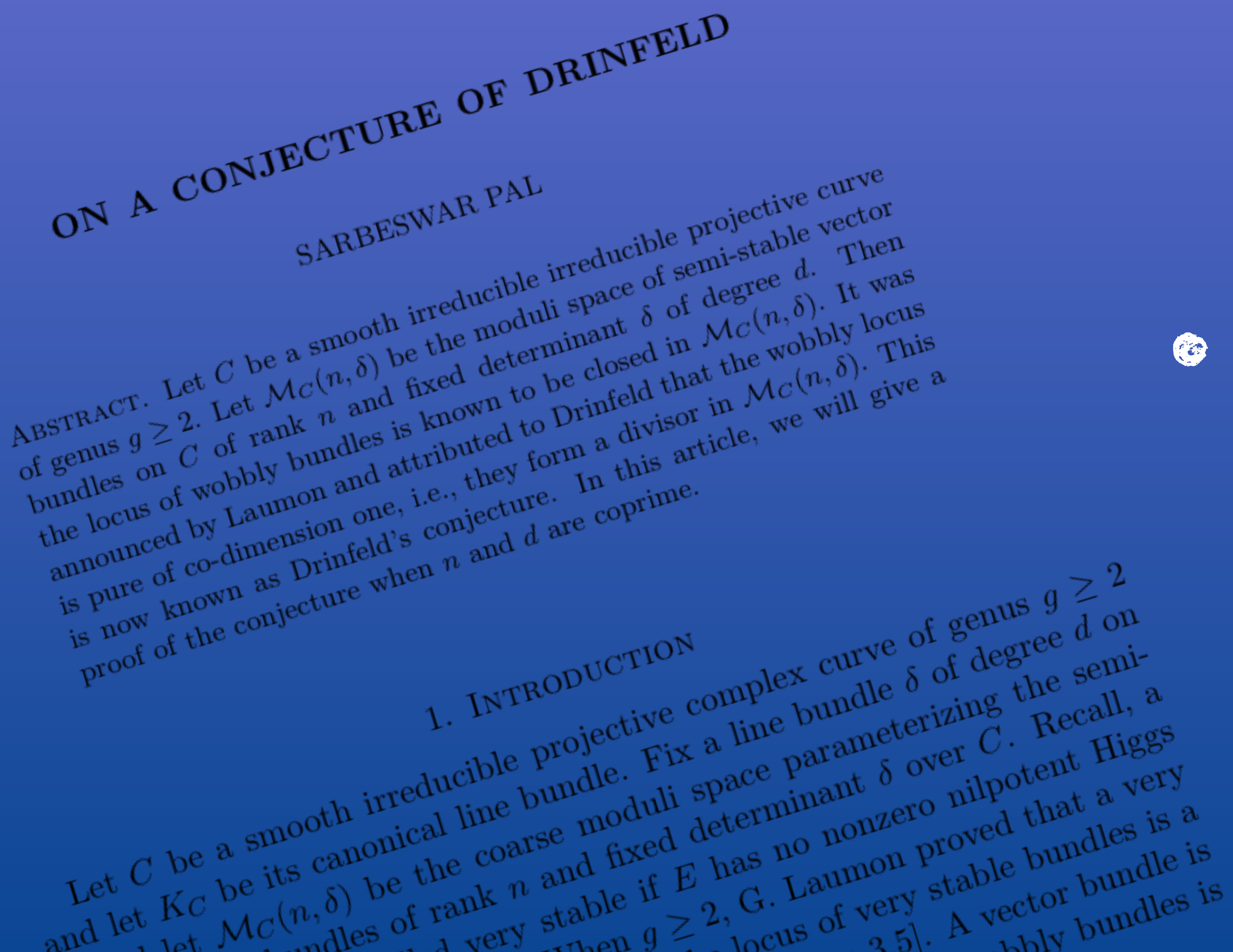
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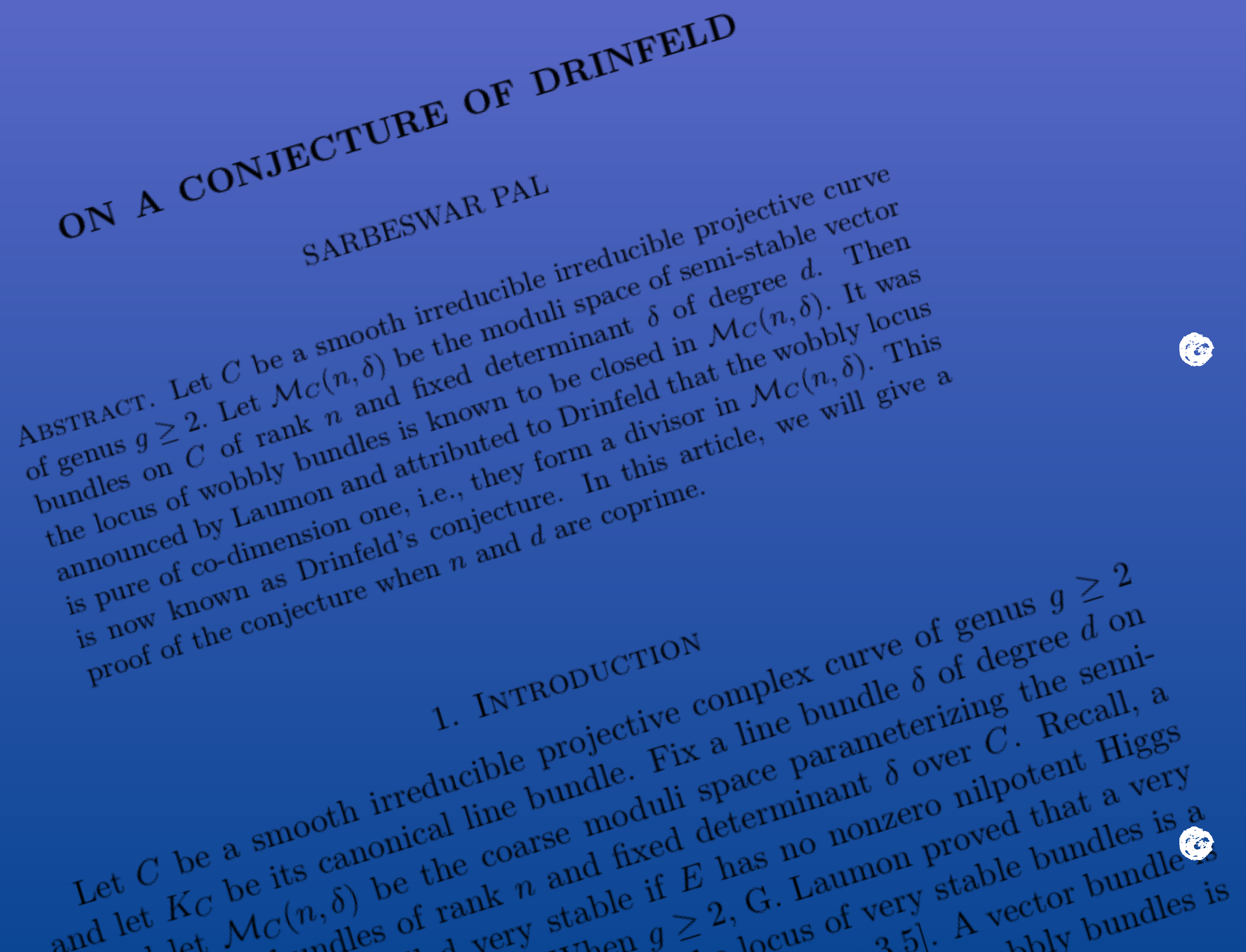
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- ad sends wobbly to wobbly.

Higher Drinfeld's conjecture

Q: Does the conjecture hold for higher fixed point components?

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Type	Invariant	Codim 1 wobbly	Full wobbly
$L_1 \oplus L_2 \oplus L_3$ (φ_2, φ_1)	$0 \leq 2\delta_i + \delta_j \leq 6g-6$	$2\delta_i + \delta_j \neq 6g-6$	—

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$F_1 \oplus F_2$ (φ)	$g-1 < \delta = \deg F_2^* F_1 K < 4g-4$	$\delta \geq 3g-3$	$\delta < 3g-3$

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$\varphi: F_1 \to F_0$

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<u>Example (Früh-Ouaras-PN):</u> $Sp(4,\mathbb{C})$

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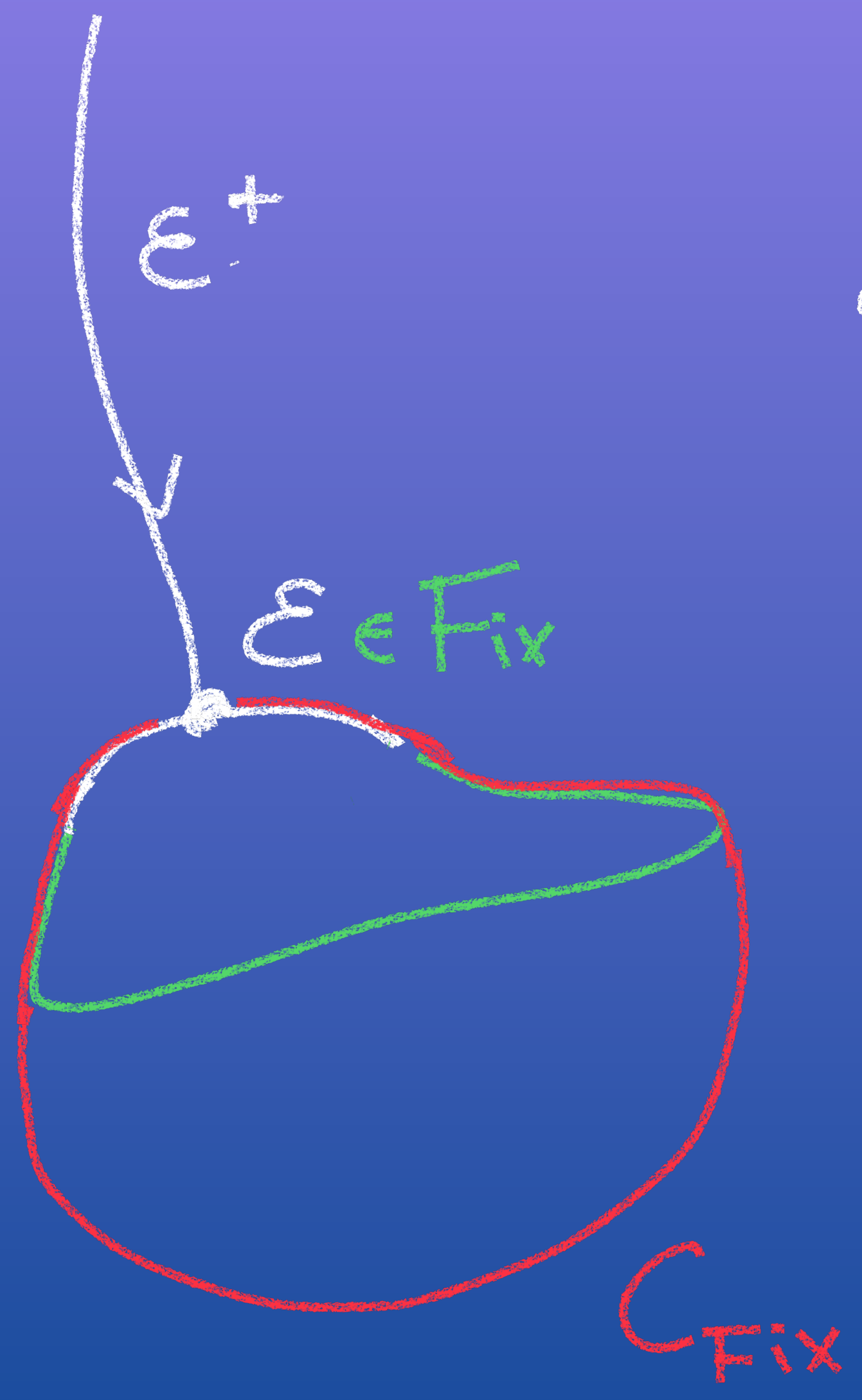
Q: are they real wobbly?

A: Nope! Half the components are, half arent.

Towards a criterion



$$\mathcal{E}^+ = \{ (\mathcal{E}, \Phi) \quad \lim_{t \to 0} (\mathcal{E}, t\Phi) = \mathcal{E} \}$$

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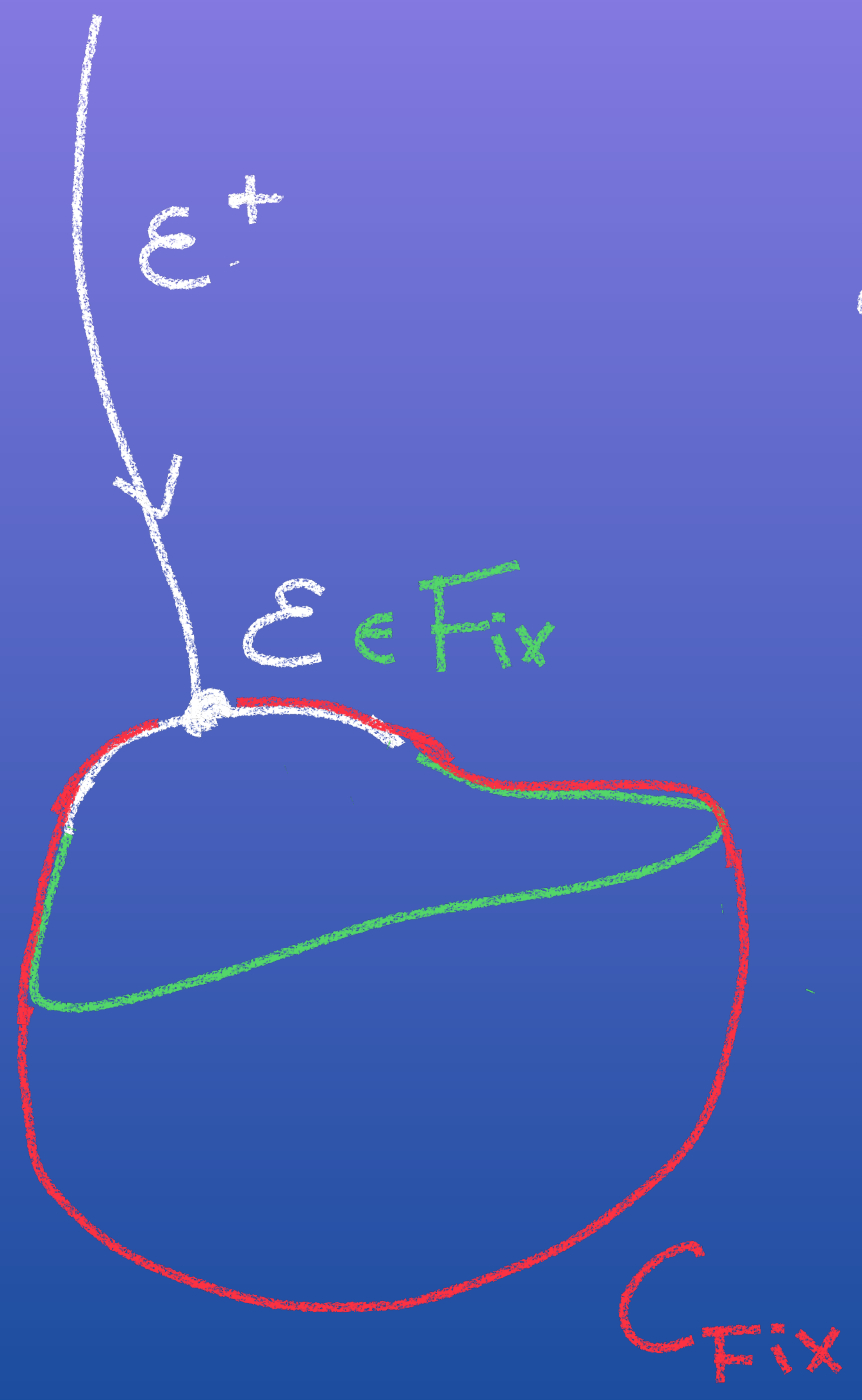

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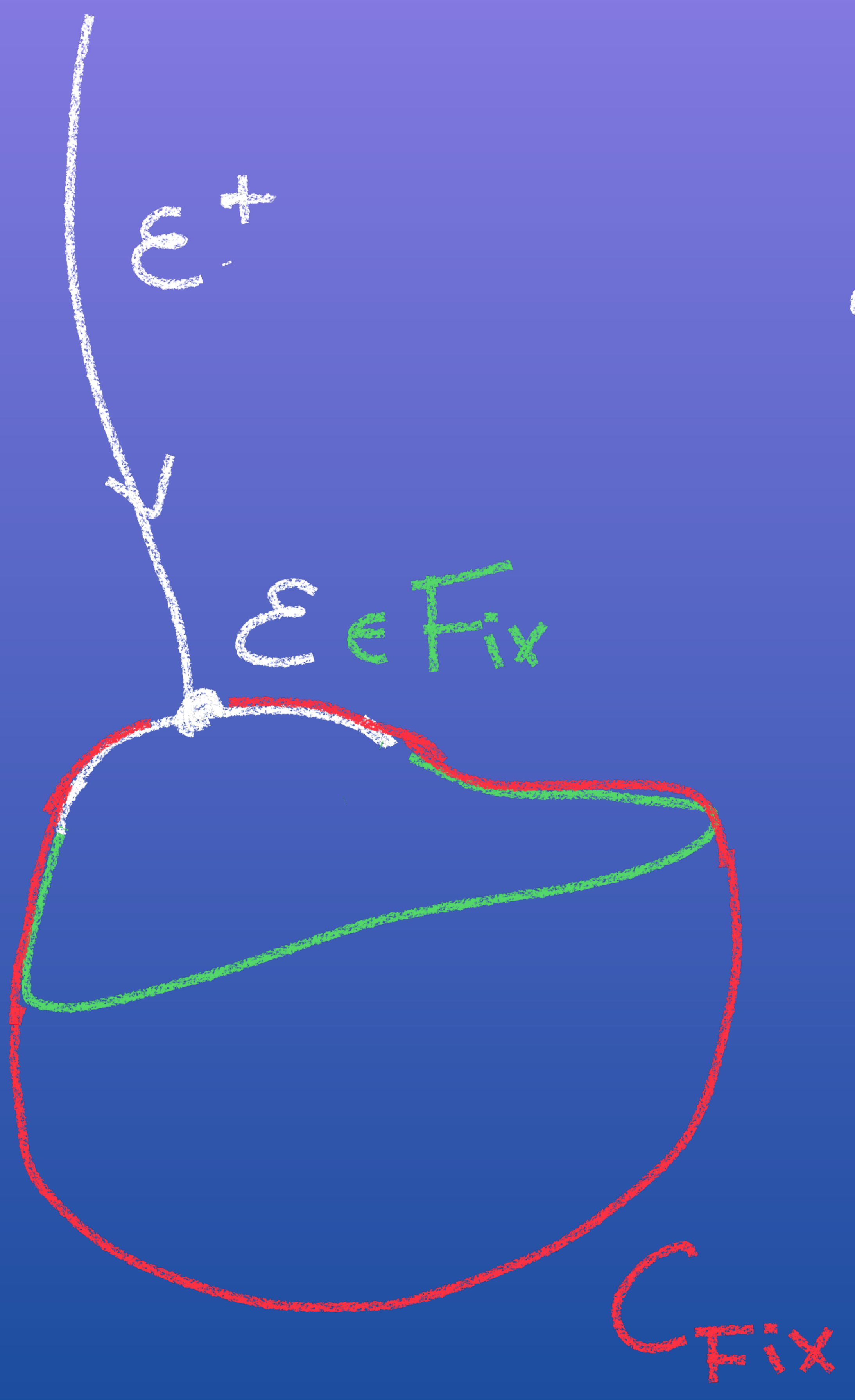
<u>Proposition (Pal–PN):</u> a fixed point $\mathcal{E} \in \mathrm{Fix}$ is wobbly if and only if there exists a non free rational curve $\mathbb{P}^1 \subset C_{\mathrm{Fix}}$ through it or a "non free" Hecke curve.

Towards a criterion: flow weights

$$\mathcal{E}^+ = \{ (\mathcal{E}, \Phi) \ \lim_{t \to \infty} (\mathcal{E}, t\Phi) = \mathcal{E} \}$$



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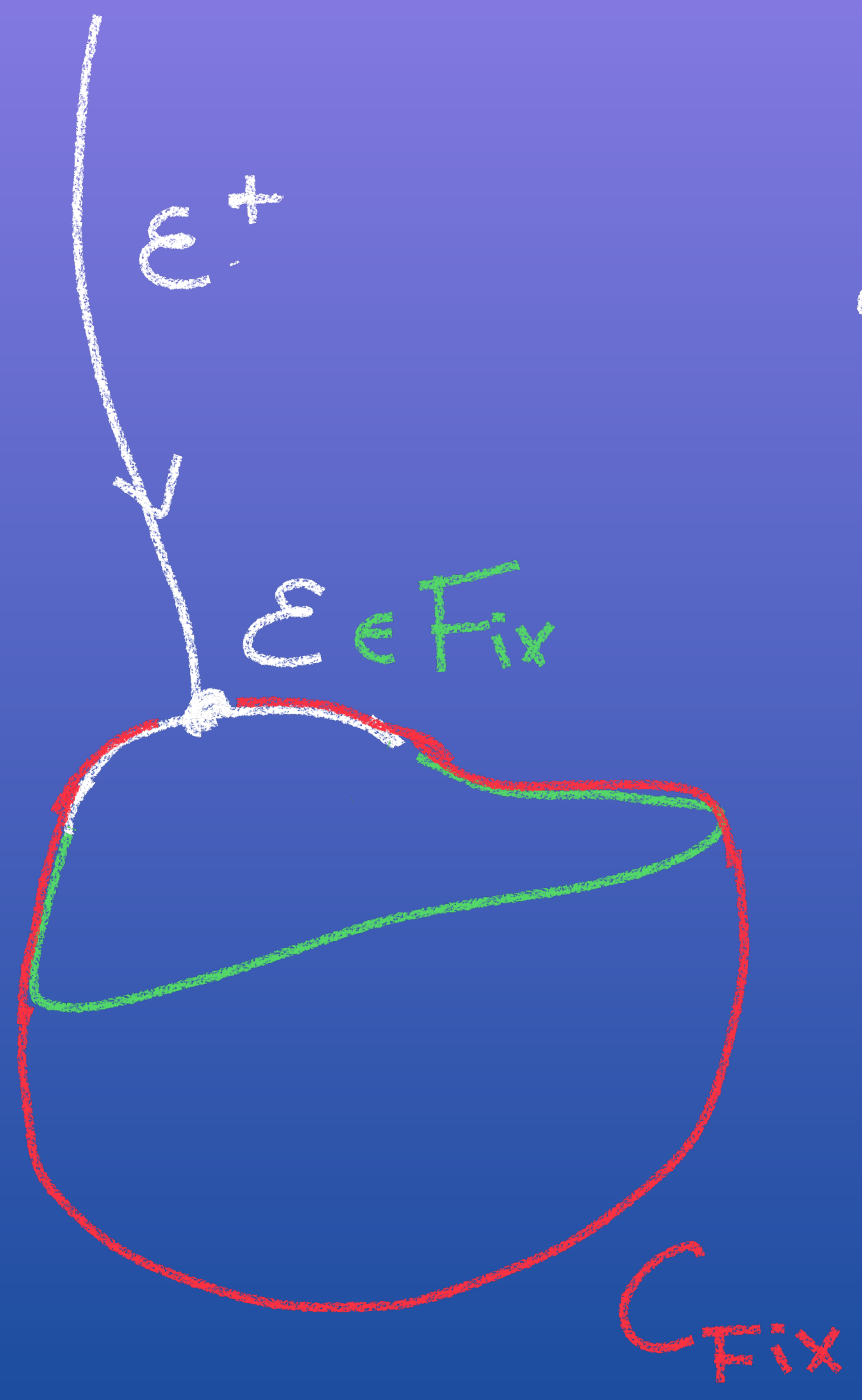
$$\mathcal{E}^+ = \{(\mathcal{E}, \Phi) \ \lim_{t \to 0} (\mathcal{E}, t\Phi) = \mathcal{E}\} = \bigoplus_{\lambda > 0} \mathcal{E}^+_\lambda$$



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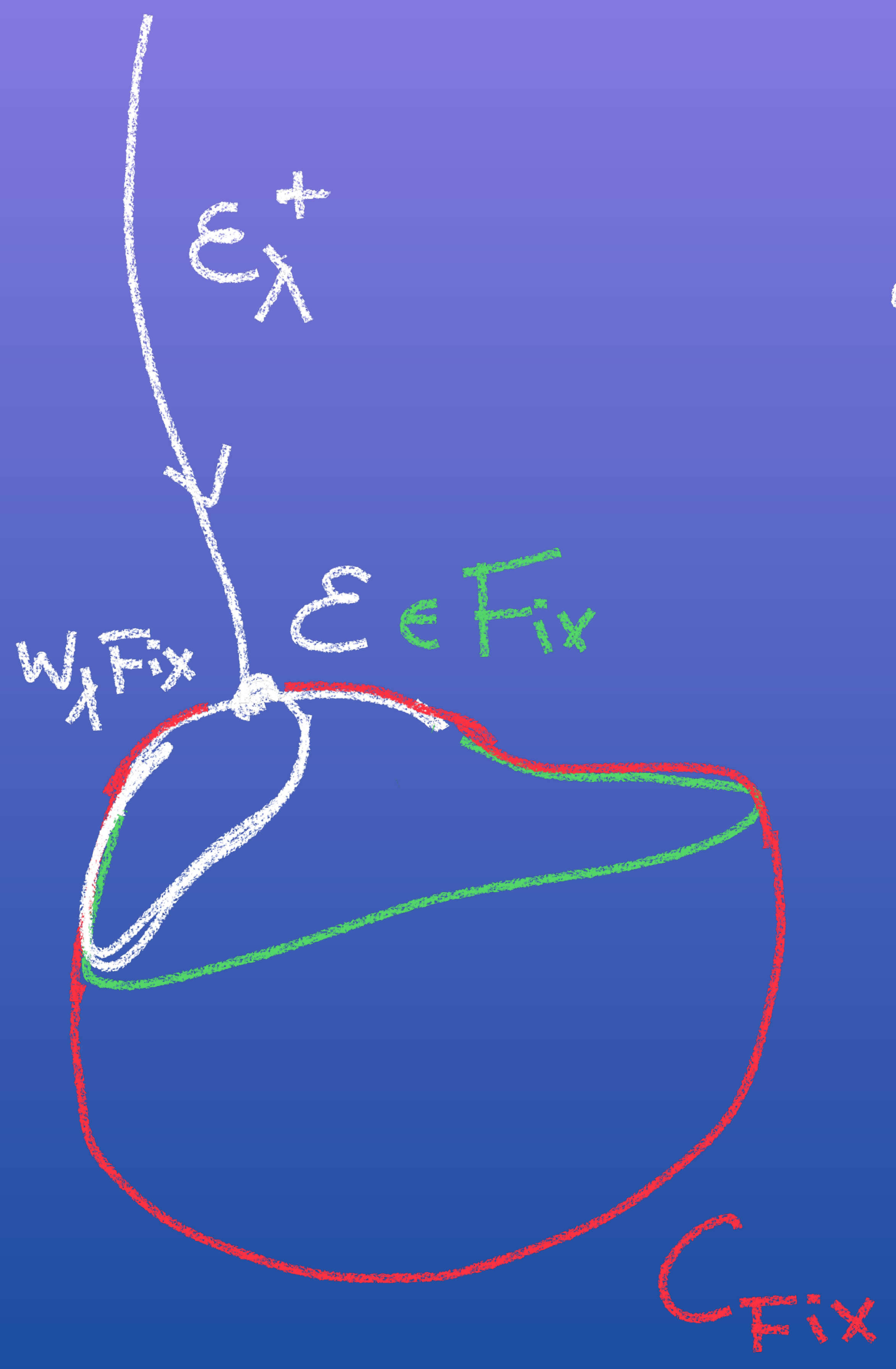

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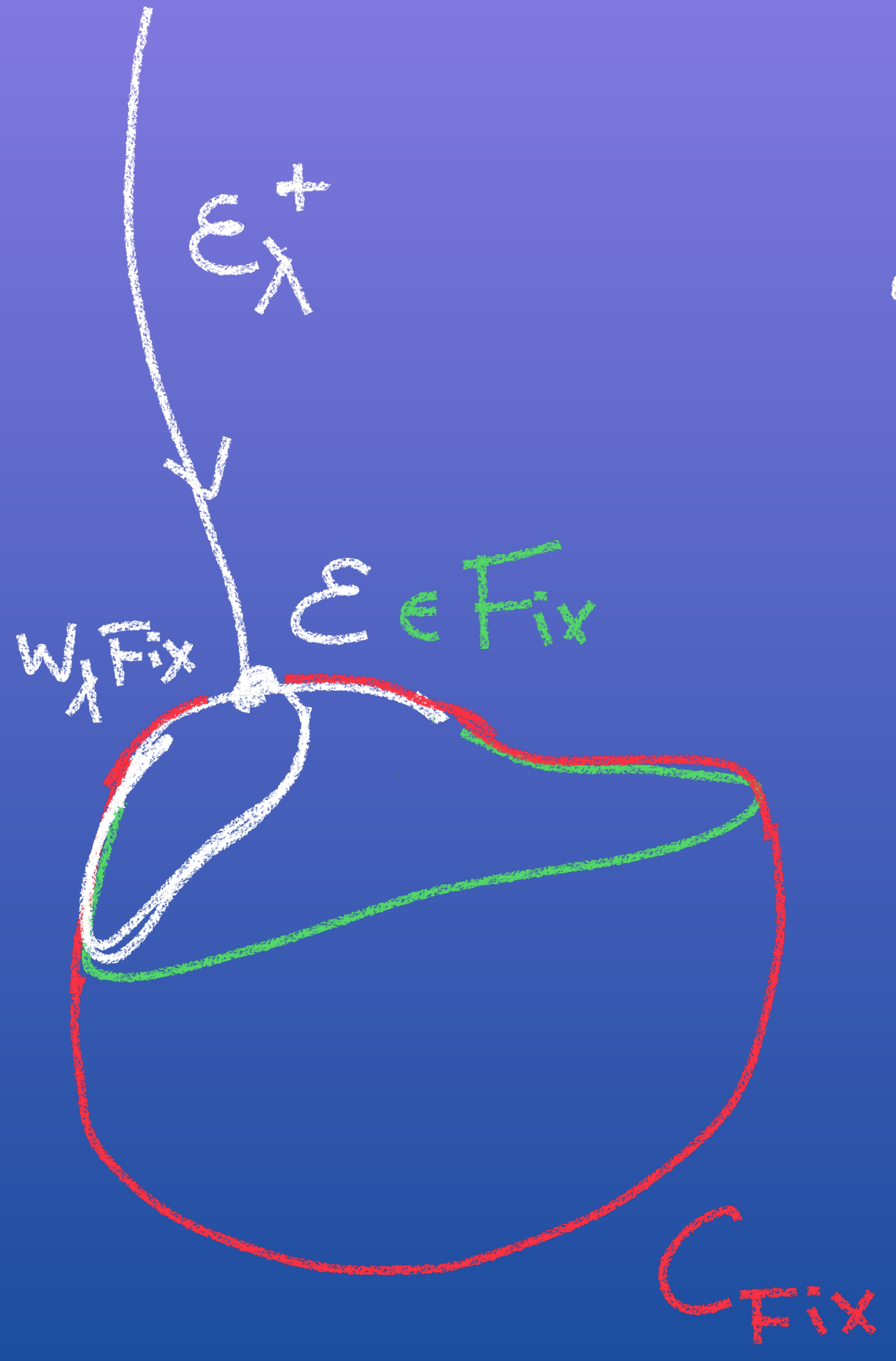

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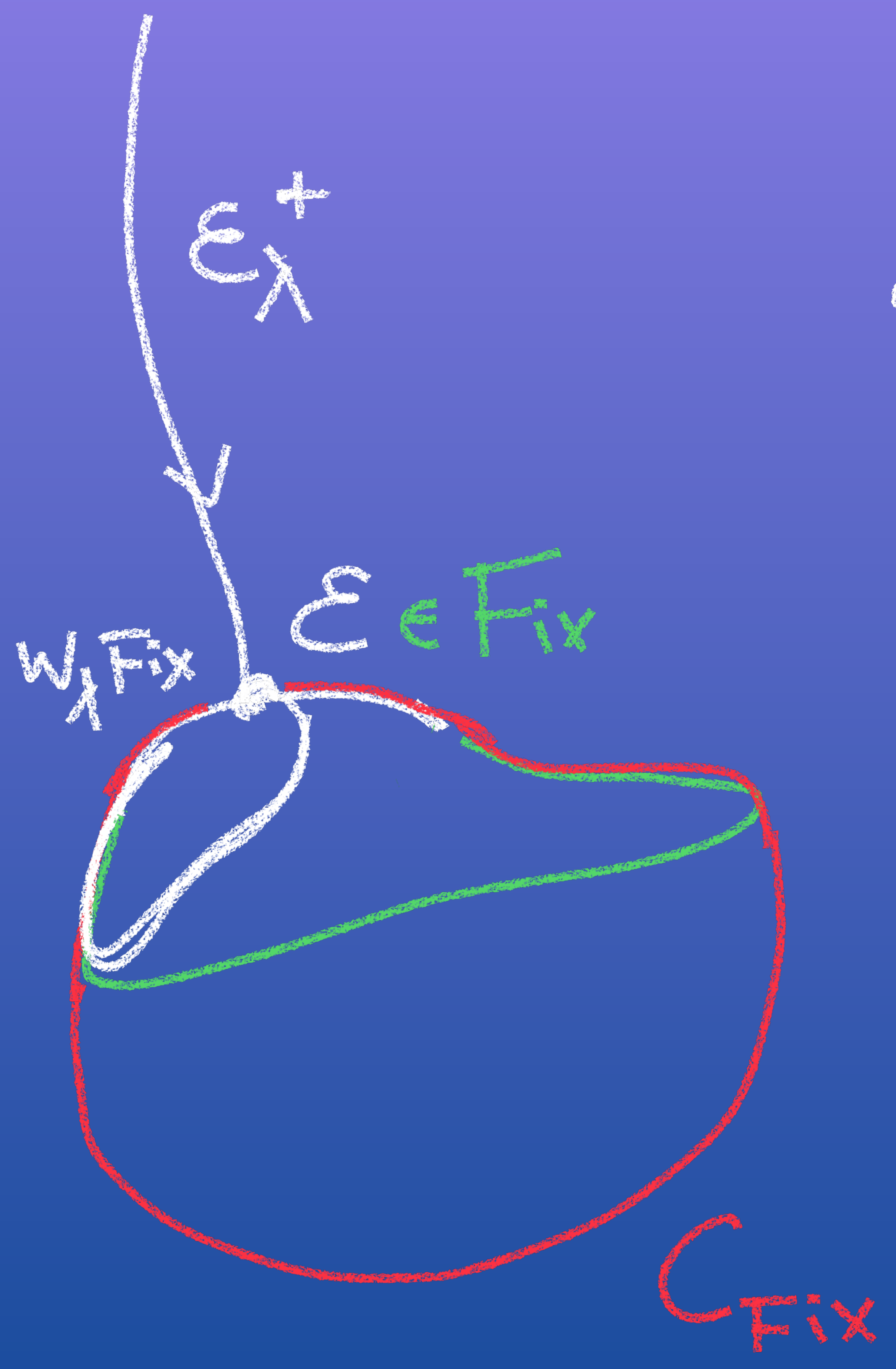

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$\mu < \lambda \qquad W_\mu \subset \overline{W_\lambda}$

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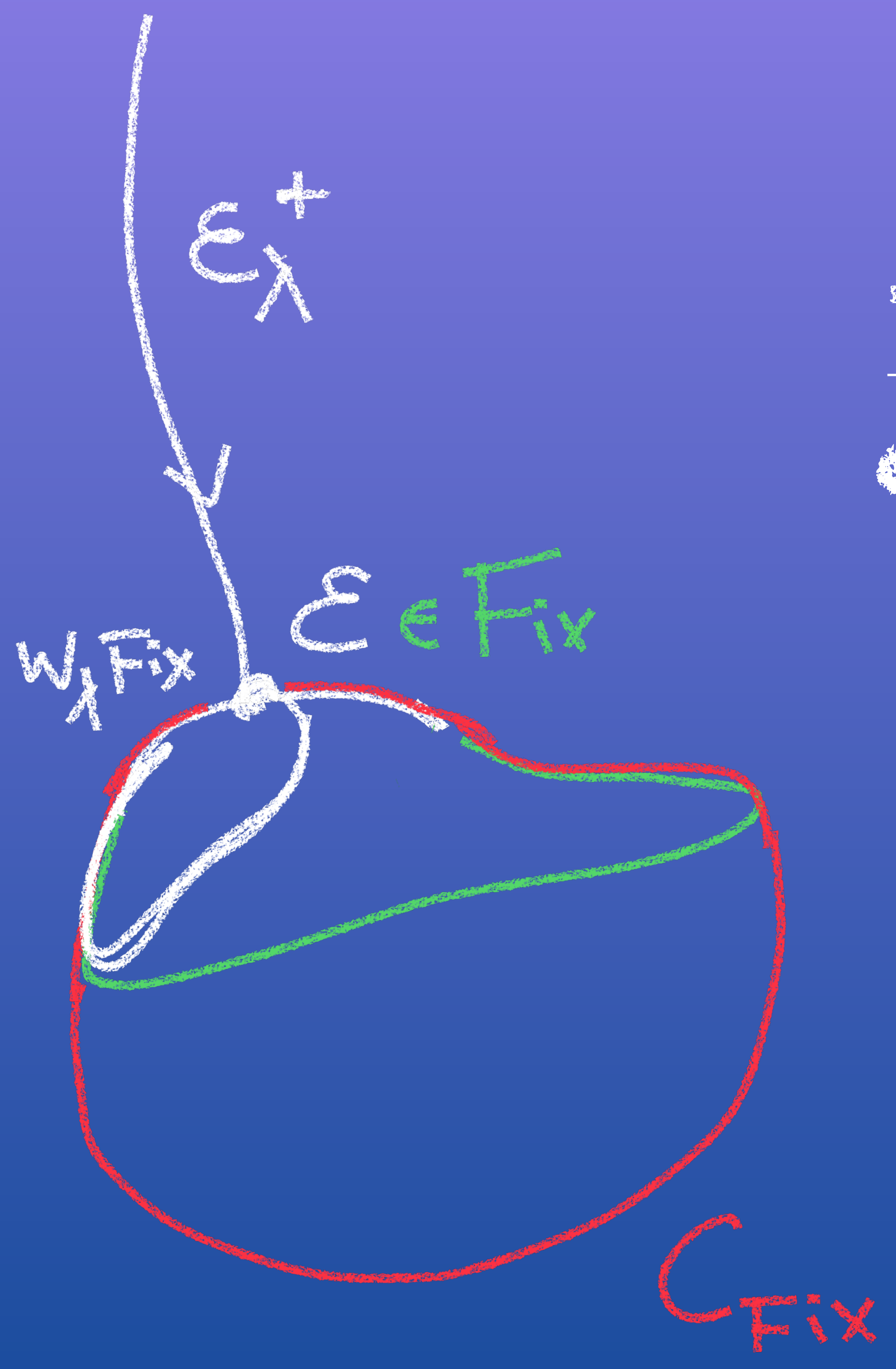

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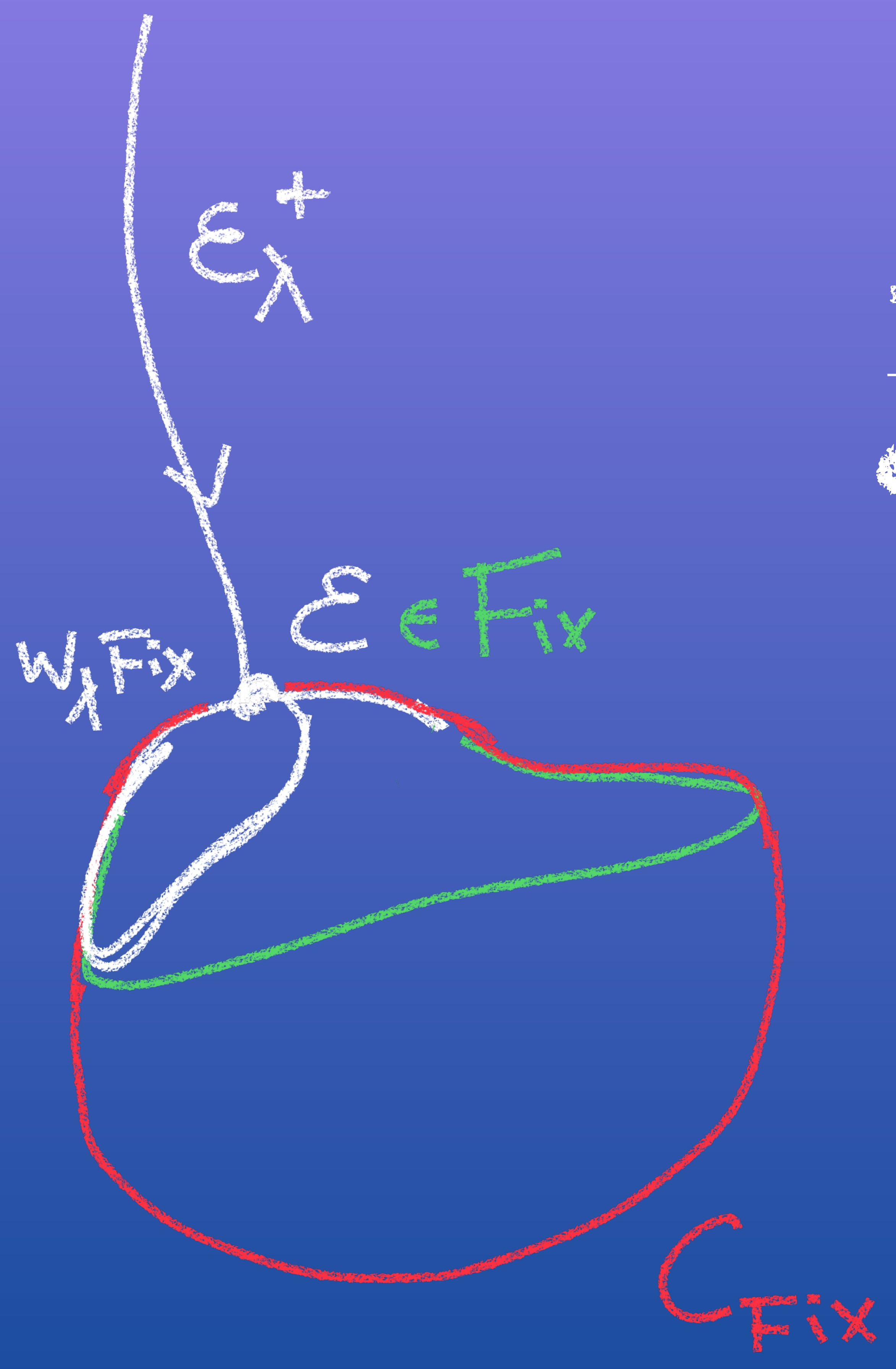

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Moreover

$$\mathrm{codim}_{\mathrm{Fix}} W_1 \mathrm{Fix} > 0$$

Thanks for your attention!

Happiest of b-days, Nigel!