LEVERHULME TRUST

ICREA

Categorical approach to decorated Betti Moduli Space

In collaboration with Benedetta Facciotti & Nikita Nikolaev

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BARCELONATECH

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Betti moduli space : Σ surface with boundary.

$$\underbrace{\mathcal{L}oc_G(\Sigma) \simeq \mathcal{R}ep_G(\Pi_1(\Sigma)) \simeq \mathcal{R}ep_G(\pi_1(\Sigma, p)) \simeq \mathrm{Hom}(\pi_1(\Sigma, p), G)/G}_{\mathcal{M}_B}$$

Betti moduli space : Σ surface with boundary.

$$\underbrace{\mathrm{Loc}_G(\Sigma) \simeq \mathrm{Rep}_G(\Pi_1(\Sigma)) \simeq \mathrm{Rep}_G(\pi_1(\Sigma,p)) \simeq \mathrm{Hom}(\pi_1(\Sigma,p),G)/G}_{\mathcal{M}_B}$$

Capturing higher order poles : (Σ, P) surface with marked boundary

- Deligne, Simpson, Boalch, Fock-Goncharov, Goncharov-Shen
- Gualtieri-Li-Pym
- Boalch
- Chekhov, M.M., Rubtsov

Today : categorical framework to decorated Betti moduli space that specializes to each of the above

Introductory example: $\dfrac{dy}{dz} = \dfrac{1}{z}\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} y$

basis of solutions: $y_i = z^{\lambda_i} e_i$ analytic in $U \subset \mathbb{C}^\times$

$\Rightarrow$ **Linear local system** $\mathcal{E}^\circ$ on $\mathbb{C}^\times$

$$\mathcal{E}^\circ(U) = \begin{cases} \langle y_1, y_2, y_3 \rangle_{\mathbb{C}} & \text{if } U \subset \mathbb{C}^\times \text{ is simply connected} \\ 0 & \text{otherwise} \end{cases}$$

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$\Rightarrow$ Linear local system $\mathcal{E}^\circ$ on $\mathbb{C}^\times$

$$\mathcal{E}^\circ(\mathring{U}) = \begin{cases} \langle y_1, y_2, y_3 \rangle_{\mathbb{C}} & \text{if } \mathring{U} \subset \mathbb{C}^\times \text{ is simply connected} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Linear local system $\mathcal{E}$ on $\Sigma = \mathbb{C}^\times \sqcup C_0 \sqcup C_1$

$\mathrm{Loc}_G(\Sigma)$

$\mathcal{L}oc_G(\Sigma)$

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basis of solutions: $y_i = z^{\lambda_i} e_i$ analytic in $U^\circ \subset \mathbb{C}^\times$

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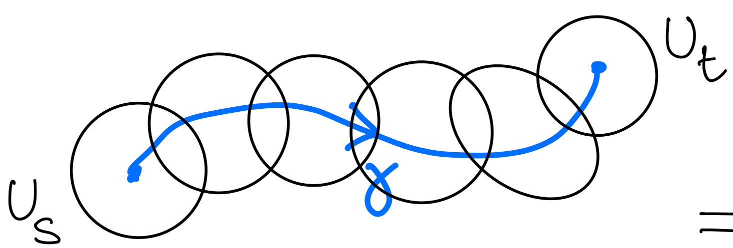
$\Rightarrow$ Linear local system $\mathcal{E}$ on $\Sigma = \mathbb{C}^\times \sqcup C_0 \sqcup C_1$ $\quad \mathrm{Loc}_G(\Sigma)$, $\mathcal{L}oc_G(\Sigma)$

$\forall\, \gamma$ path in Σ, we can transport $y(z)$ along it $\Rightarrow$



$\mathrm{hol}_\gamma : \mathcal{E}(U_s) \simeq \mathcal{E}(U_t)$

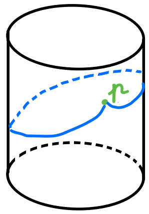
$\Rightarrow (\mathcal{E}, \mathrm{hol}) \longrightarrow (E := \mathcal{E} \otimes \mathcal{O}_X, \phi)$

sections $e \in \Gamma(U, E)$ $\quad e(u) = c_1(u) y_1 + c_2(u) y_2 + c_3(u) y_3$

$c_i : U \to \mathbb{C}$ continuous.

$\Rightarrow$ Representation of $\pi_1(\Sigma)$ $\Rightarrow (E, \phi)$, $\phi : \pi_1(\Sigma) \to GL(E)$

$\mathrm{Rep}_G(\pi_1(\Sigma))$, $\mathcal{R}ep_G(\pi_1(\Sigma))$

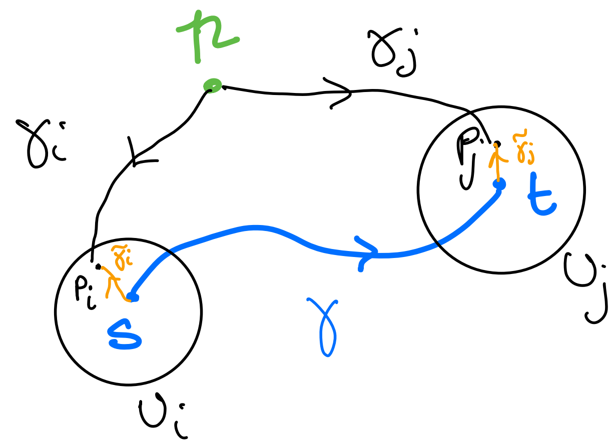


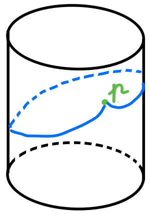
$\Rightarrow$ discretely generated model

$$\mathrm{Rep}_G(\Pi_1(\Sigma)) \xrightarrow{\sim} \mathrm{Rep}_G(\pi_1(\Sigma, p)) \quad \text{Morita equivalence}$$

$$\begin{array}{ccc} \cup & & \cup \\ (E, \phi) & & (E_p, \Phi_p) \end{array}$$

- Pick a cover $\{U_i\}$, U_i simply connected
- Pick $p_i \in U_i$
- Pick γ_i : $\gamma_i(0) = p$, $\gamma_i(1) = p_i$





$\Longrightarrow$ discretely generated model

$$\mathrm{Rep}_G(\Pi_1(\Sigma)) \xrightarrow{\sim} \mathrm{Rep}_G(\pi_1(\Sigma, p)) \qquad \text{Morita equivalence}$$

$$\begin{array}{ccc} \mathrm{Rep}_G(\Pi_1(\Sigma)) & & \mathrm{Rep}_G(\pi_1(\Sigma,p)) \\ \cup & & \cup \\ (E, \Phi) & & (E_p, \Phi_p) \end{array}$$

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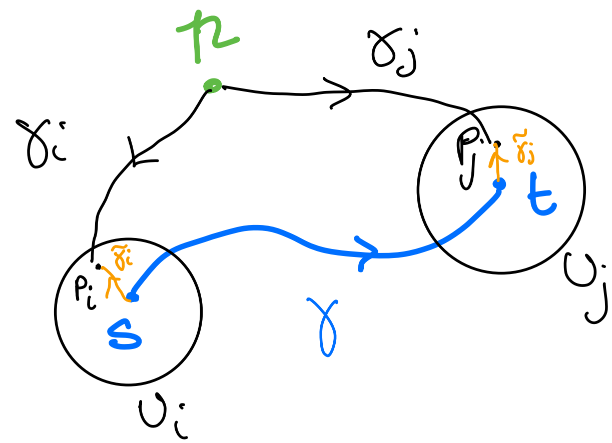

Choose a trivialization $E_p \simeq \mathbb{C}^n$ $\Longrightarrow$

$$G \ltimes \mathrm{Hom}(\pi_1(\Sigma, p), G)$$

$$\mathcal{X} := \mathrm{Hom}(\pi_1(\Sigma, p), G)/G \qquad \text{character variety}$$

Betti moduli space

$$\mathcal{L}oc_G(\Sigma) \xrightarrow{\sim} \mathcal{R}ep_G(\Pi_1(\Sigma)) \xrightarrow{\sim} \mathcal{R}ep_G(\pi_1(\Sigma, p)) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(\Sigma, p), G)/G$$

Existing generalizations

$\Sigma \longmapsto (\Sigma, P)$

Betti moduli space

$$\mathcal{L}oc_G(\Sigma) \xrightarrow{\sim} \mathcal{R}ep_G(\Pi_1(\Sigma)) \xrightarrow{\sim} \mathcal{R}ep_G(\pi_1(\Sigma, p)) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(\Sigma, p), G)/G$$

$\downarrow$ $\mathcal{L}oc_G^{Fi}(\Sigma, P)$ — Deligne, Simpson, Fock, Goncharov, Shen

$\downarrow$ $\mathcal{R}ep_G(\hat{\Pi}_1(\Sigma, P))$ — Gualtieri, Li, Pym

$\downarrow$ $\mathcal{S}to\mathcal{R}ep_G(\pi_1(\hat{\Sigma}, P))$ — Boalch

$\downarrow$ $\mathrm{Hom}(\pi_1(\Sigma, P), G)/U_P$ — Chekhov, M.M., Rubtsov

we would like to establish a relationship between all these approaches

Categorical approach: every moduli space is the underlying set of a groupoid. Equivalences of groupoids imply bijections of moduli spaces.

$\boxed{\text{Main theorem [Facciotti–M.M.–Nikolaev]}}$ for $\square \in \{Fi, Fr, PFr\}$

$$\mathrm{Loc}_G^{\square}(\Sigma, P) \xrightarrow{\sim} \mathrm{Rep}_G^{\square}(\Pi_1(\Sigma), P) \xrightarrow{\sim} \mathrm{Rep}_G^{\square}(\pi_1(\Sigma, P)) \xrightarrow{\sim} K_P^{\square} \ltimes R_G(\Sigma, P)$$

Main theorem [Facciotti–M.M.–Nikolaev]

For $\square \in \{\mathcal{F}i, \mathcal{F}r, P\mathcal{F}r\}$

$$\mathrm{Loc}_G^{\square}(\Sigma, P) \xrightarrow{\simeq} \mathrm{Rep}_G^{\square}(\Pi_1(\Sigma), P) \xrightarrow{\simeq} \mathrm{Rep}_G^{\square}(\pi_1(\Sigma, P)) \xrightarrow{\simeq} K_P^{\square} \ltimes R_G(\Sigma, P)$$

Decorated Betti moduli space

$$\boxed{\mathcal{L}oc_G^{\square}(\Sigma, P) \simeq \mathcal{R}ep_G^{\square}(\Pi_1(\Sigma), P) \simeq \mathcal{R}ep_G^{\square}(\pi_1(\Sigma, P)) \simeq \mathcal{X}_G^{\square}}$$

$\square = \mathcal{F}i$: Fock, Goncharov

$\square = \mathcal{F}r$: Goncharov, Shen

$\square = \mathcal{F}i$: Chekhov, M.M., Rubtsov

Main theorem [Facciotti-M.M.-Nikolaev] For $\square \in \{\mathcal{F}i, \mathcal{F}r, \mathcal{PF}r\}$

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$\square = \mathcal{F}i$ Fock Goncharov

$\square = \mathcal{F}r$ Goncharov Shen

Gualtieri Li-Pym

[Facciotti, work in progress]

Boalch

$\square = \mathcal{F}i$ Chekhov, M.M. Rubtsov

Consequences

- Fock Goncharov coordinates can be used to describe objects in each of these moduli spaces
- Cluster variety structure is intrinsic.

Example $E \to \mathbb{P}^1$. rank n vector bundle

∇ connection on E with pole divisor $D = 1\times 0 + 2\times\infty$

$$\nabla = d - \left(U + \frac{V}{z}\right)dz$$

$V \in \mathfrak{g}$, $U \in \mathfrak{h}$

Horizontal sections: $\displaystyle \frac{dY}{dz} = \left(U + \frac{V}{z}\right)Y$

Example

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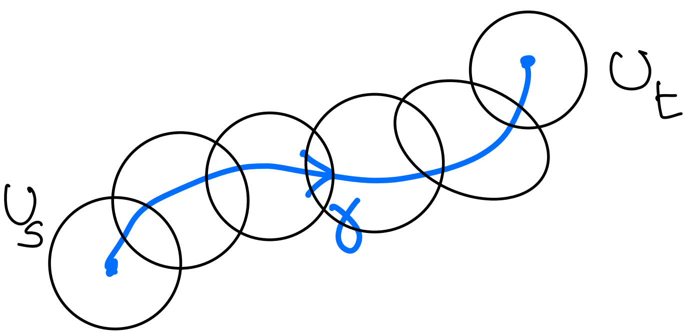
$V \in \mathfrak{g}$, $U \in \mathfrak{h}$ $\quad \leftarrow \mathcal{M}_{dR}$

Horizontal sections: $\dfrac{dY}{dz} = \left(U + \dfrac{V}{z}\right)Y$

Local behavior near 0: $Y_0(z) = F_0(z)\, z^{H}$

$\Rightarrow$ Local system $\mathcal{E}$ on $\Sigma = \mathbb{C}^\times \sqcup C_0 \sqcup C_1$ $\quad \leftarrow \mathrm{Loc}_G(\Sigma)$



$\mathcal{E}(U_s) \simeq \mathcal{E}(U_t)$

$\Rightarrow$ $\forall$ path we define an isomorphism of vector spaces.

$\mathrm{Rep}_G(\Pi_1(\Sigma))$

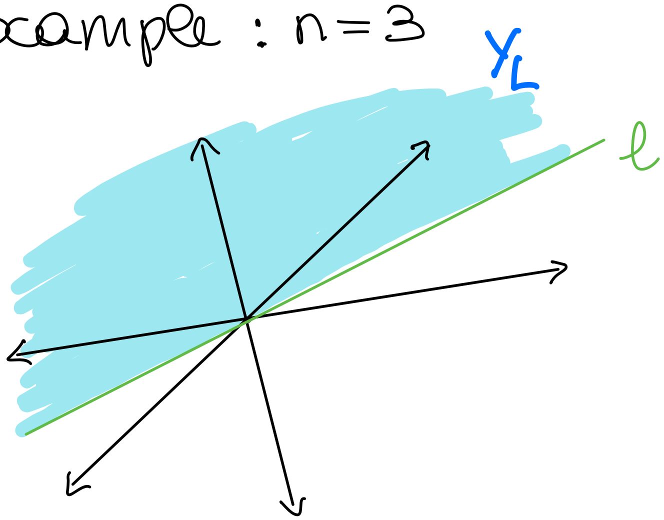
$$\mathrm{Loc}_G(\Sigma) \xrightarrow{\sim} \mathrm{Rep}_G(\Pi_1(\Sigma))$$

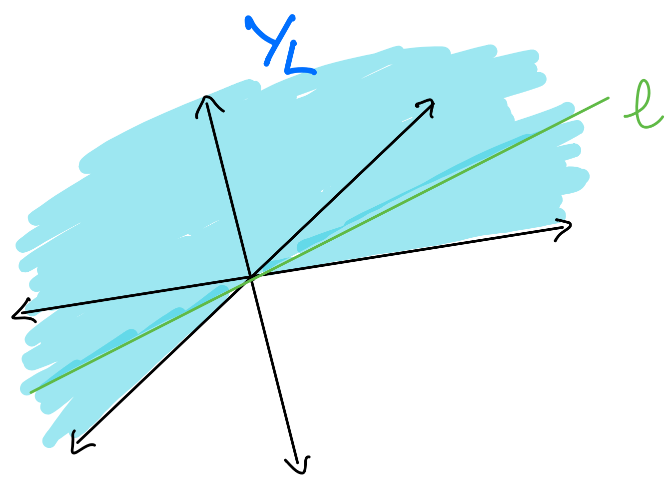
Local behaviour at ∞

$$Y_\infty^f(z) = \underbrace{F_\infty(1/z)}_{\uparrow\ \text{divergent series!}}\, e^{Uz}\, z^{V_D}$$

- Existence of holomorphic solutions is only guaranteed in sectors of opening π
- Stokes rays : $\mathrm{Re}(zu_i - zu_j) = 0$. We can extend uniquely to larger sectors as long as we don't cross them.

Example : $n = 3$





$Y_L \sim Y_\infty^f$

Local behaviour at ∞

$$Y^f_\infty(z) = \underbrace{F_\infty(1/z)}_{\uparrow\ \text{divergent series!}}\, e^{Uz}\, z^{V_D}$$

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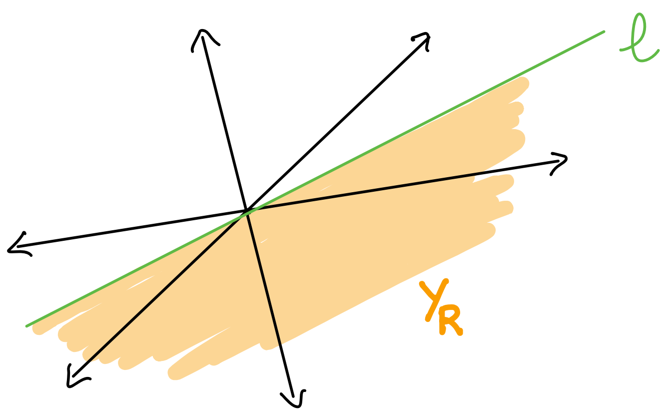
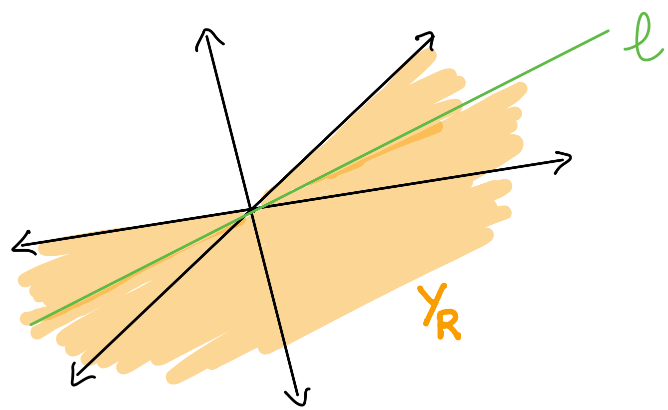


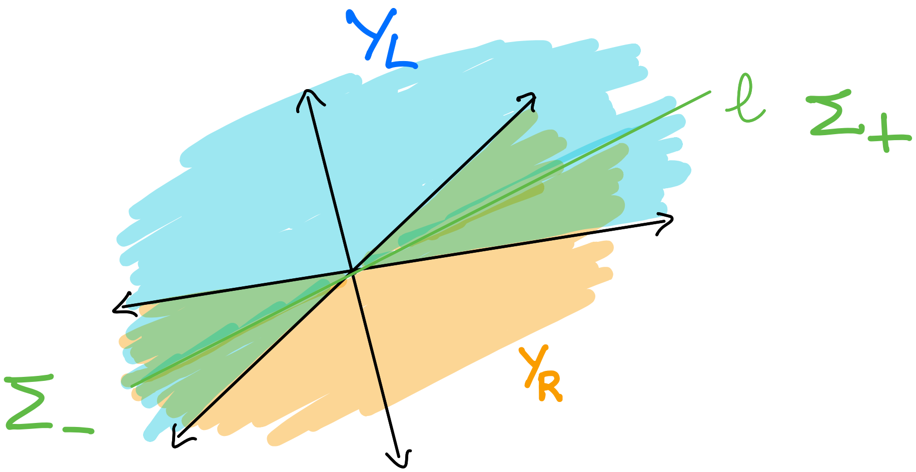


$Y_R \sim Y^f_\infty$

Local behaviour at ∞

$$Y_\infty^f(z) = \underbrace{F_\infty(1/z)}\, e^{Uz}\, z^{V_\Delta}$$

divergent series!



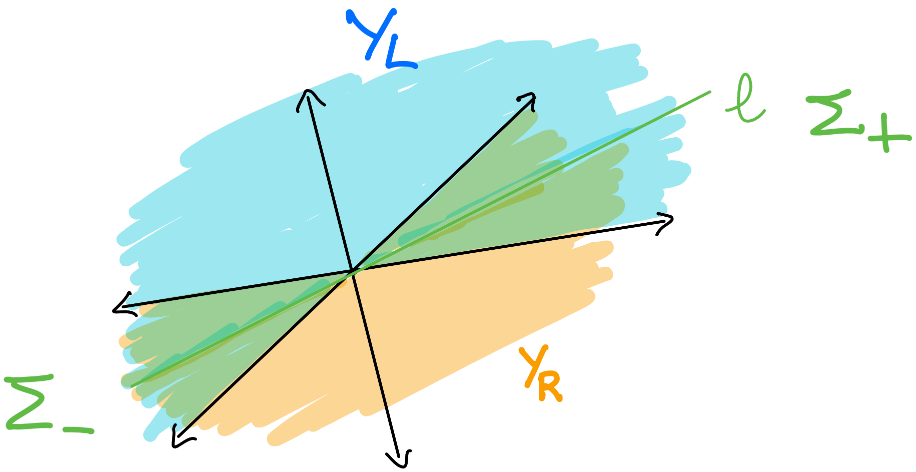
Stokes matrices

$$Y_L(z) = Y_R(z)\, S_- \qquad z \in \Sigma_-$$

$$Y_R(z) = Y_L(z)\, S_+ \qquad z \in \Sigma_+$$

Local behaviour at ∞

$$Y_\infty^f(z) = \underbrace{F_\infty(1/z)}_{\uparrow\ \text{divergent series!}}\, e^{Uz}\, z^{V_\Delta}$$



Stokes matrices

$Y_L(z) = Y_R(z)\, S_- \qquad z \in \Sigma_-$

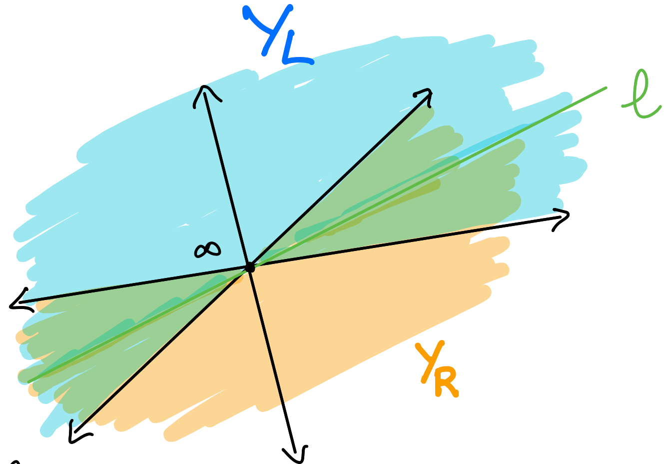
$Y_R(z) = Y_L(z)\, S_+ \qquad z \in \Sigma_+$

Ideally:

$\Sigma_+, \Sigma_-, S_+, S_-$

$$\mathcal{L}oc_G(\Sigma) \xrightarrow{\sim} \mathcal{R}ep_G(\Pi_1(\Sigma))$$

Step 1

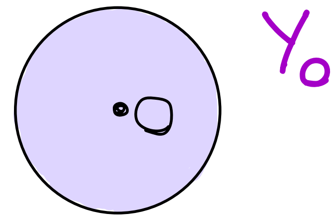


Σ_+

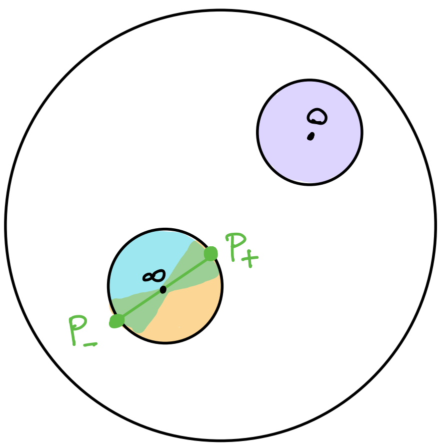
$e^{zu_1} \gg e^{zu_2} \gg \dots \gg e^{zu_n}$

Σ_-

$e^{zu_1} \ll e^{zu_2} \ll \dots \ll e^{zu_n}$

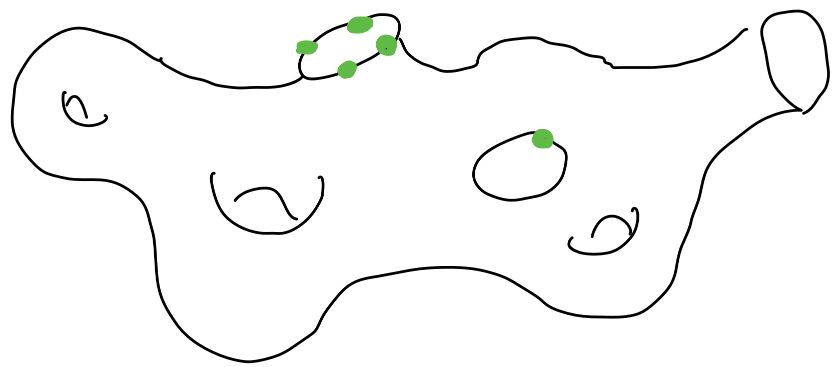


Real oriented blow up:
with marked points



In general:

(Σ, P)

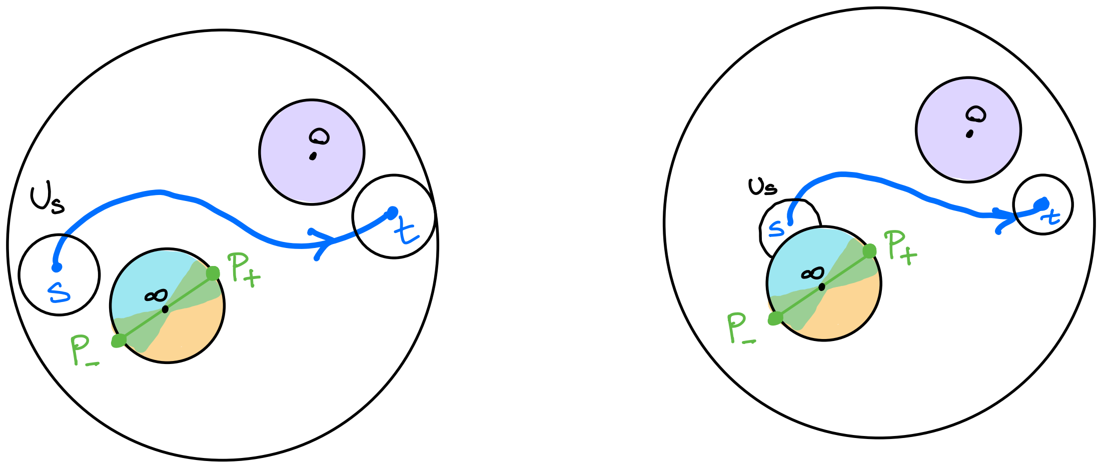


Local system $\mathcal{E}$: $Y_0, Y_L, Y_S \rightsquigarrow$ complete transversal flags on Σ_+.

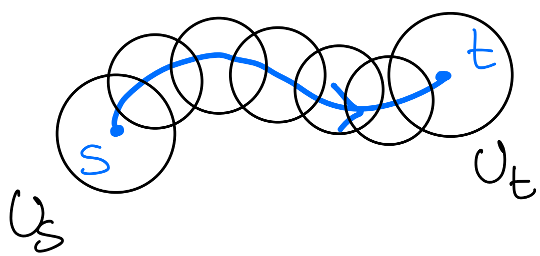
$\boxed{\mathrm{Loc}_G^{Fi}(\Sigma, P)} \ni (\mathcal{E}, F_P)$

[Deligne, Simpson]

Step 2



$$\begin{array}{c} E \\ \downarrow \\ (\Sigma, P) \end{array} \qquad E = \mathcal{E} \otimes \mathcal{O}_{\Sigma}$$



$\forall\, \gamma$, we have an isomorphism

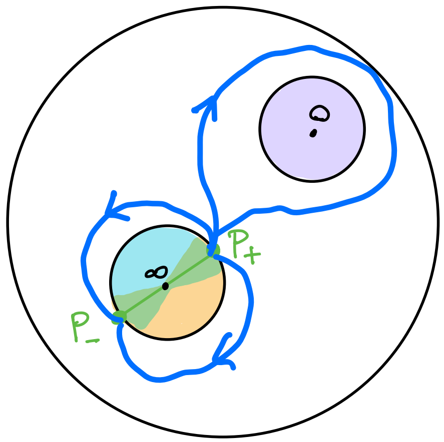
$$\phi_\gamma : E_s \simeq E_t$$

$$\boxed{\mathrm{Rep}^{Fi}(\Pi_1(\Sigma), P)} \ni (E, F_p, \Phi)$$

theorem: $\mathrm{Loc}_G^{Fi}(\Sigma, P) \xrightarrow{\sim} \mathrm{Rep}^{Fi}(\Pi_1(\Sigma), P)$

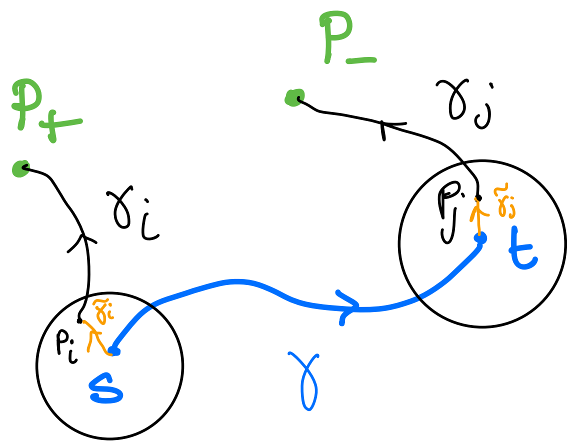
Step 3 (discretely generated model)

$$\mathrm{Rep}_G^{Fi}(\Pi_1(\Sigma)) \longrightarrow \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))$$



- Cover $\Sigma \setminus \partial\Sigma$ by open balls B_i
- Choose $p_i \in B_i$
- Fix γ_i : $\gamma_i(0) = p_i$, $\gamma_i(1) \in P$

$\Rightarrow$



$[\tilde\gamma_i]$, $[\tilde\gamma_j]$ are unique $\Rightarrow \gamma_j \tilde\gamma_j \gamma \tilde\gamma_i^{-1} \gamma_i^{-1} \in \Pi_1(\Sigma)$

$$\boxed{\mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))} \ni (E_P, F_P, \Phi_P)$$

theorem: $\mathrm{Rep}_G^{Fi}(\Pi_1(\Sigma), P) \xrightarrow{\sim} \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))$

Step 4 (Matrices)

Given $(E_P, F_P, \Phi_P) \in \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))$

$\forall p \in P$ choose $\mathcal{S}_p : E_p \longrightarrow \mathbb{C}^n \Longrightarrow \forall \gamma \in \pi_1(\Sigma, P)$.

$$
\begin{tikzcd}
E_{\gamma(0)}, F_{\gamma(0)} \arrow[r, "\Phi_\gamma"] \arrow[d, "\mathcal{S}_{\gamma(0)}"'] \arrow[dr, phantom, "\circlearrowright"] & E_{\gamma(1)}, F_{\gamma(1)} \arrow[d, "\mathcal{S}_{\gamma(1)}"] \\
\mathbb{C}^n, F_{st} \arrow[r, "\rho"] & \mathbb{C}^n, F_{st}
\end{tikzcd}
$$

$\rho \in \mathrm{Hom}(\pi_1(\Sigma, P), GL_n(\mathbb{C}))$

$F_{st} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$
$\cdots \subset \langle e_1, \dots, e_n \rangle$

Step 4 (Matrices) Given $(E_P, F_P, \Phi_P) \in \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))$

$\forall\, p \in P$ choose $\delta_p : E_p \longrightarrow \mathbb{C}^n \Rightarrow \forall\, \gamma \in \pi_1(\Sigma, P)$.

$$
\begin{tikzcd}
E'_{\gamma(0)}, F'_{\gamma(0)} \arrow[r, "\Phi'_\gamma"] & E'_{\gamma(1)}, F'_{\gamma(1)} \\
E_{\gamma(0)}, F_{\gamma(0)} \arrow[u, "\varphi_{\gamma(0)}"] \arrow[r, "\Phi_\gamma"] \arrow[d, "\delta_{\gamma(0)}"'] \arrow[ur, phantom, "\circlearrowright"] & E_{\gamma(1)}, F_{\gamma(1)} \arrow[u, "\varphi_{\gamma(1)}"'] \arrow[d, "\delta_{\gamma(1)}"] \\
\mathbb{C}^n, F_{st} \arrow[r, "\rho"] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{C}^n, F_{st}
\end{tikzcd}
$$

$\rho \in \mathrm{Hom}(\pi_1(\Sigma, P), GL_n(\mathbb{C}))$

$F_{st} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$
$\cdots \subset \langle e_1, \ldots, e_n \rangle$

Step 4 (Matrices)

Given $(E_P, F_P, \Phi_P) \in \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P))$

$\forall\, p \in P$ choose $\rho_p : E_p \to \mathbb{C}^n \implies \forall\, \gamma \in \pi_1(\Sigma, P)$.

$$\begin{tikzcd}
\mathbb{C}^n, F_{st} \arrow[r, "\rho'"] & \mathbb{C}^n, F_{st} \\
E'_{\gamma(0)}, F'_{\gamma(0)} \arrow[u, "\rho'_{\gamma(0)}"] \arrow[r, "\Phi'_\gamma"] & E'_{\gamma(1)}, F'_{\gamma(1)} \arrow[u, "\rho'_{\gamma(1)}"'] \\
E_{\gamma(0)}, F_{\gamma(0)} \arrow[u, "\varphi_{\gamma(0)}"] \arrow[r, "\Phi_\gamma"] \arrow[d, "\rho_{\gamma(0)}"'] & E_{\gamma(1)}, F_{\gamma(1)} \arrow[u, "\varphi_{\gamma(1)}"'] \arrow[d, "\rho_{\gamma(1)}"] \\
\mathbb{C}^n, F_{st} \arrow[r, "\rho"] \arrow[uuu, bend left=60, "g_{\gamma(0)}"] & \mathbb{C}^n, F_{st} \arrow[uuu, bend right=60, "g_{\gamma(1)}"']
\end{tikzcd}$$

$\rho \in \mathrm{Hom}(\pi_1(\Sigma, P), GL_n(\mathbb{C}))$

$F_{st} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle$

$\rho'_\gamma = g_{\gamma(1)}\, \rho_\gamma\, g^{-1}_{\gamma(0)}$

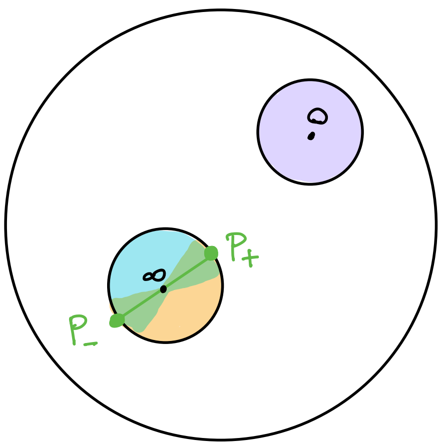
Mixed conjugation groupoid: $B_P \ltimes \mathrm{Hom}(\pi_1(\Sigma, P), GL_n(\mathbb{C}^n))$

Step 4 (Matrices)

$\forall p \in P$ choose $\rho_p : E_p \longrightarrow \mathbb{C}^n \Rightarrow \forall \gamma \in \pi_1(\Sigma, P)$.

$$\begin{tikzcd}
\mathbb{C}^n, {\color{red}F_{st}} \arrow[r, "\rho'"] & \mathbb{C}^n, {\color{red}F_{st}} \\
\mathbb{C}^n, {\color{red}F_{st}} \arrow[r, "\rho"] \arrow[u, "g_{\gamma(0)}"] & \mathbb{C}^n, {\color{red}F_{st}} \arrow[u, "g_{\gamma(1)}"']
\end{tikzcd}$$

$$\rho'_\gamma = g_{\gamma(1)} \rho_\gamma g^{-1}_{\gamma(0)}$$



$$(B_{p+}, B_{p-}) \times (GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))$$

Theorem: $\mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P)) \xrightarrow{\sim} B_P \times \mathrm{Hom}(\pi_1(\Sigma, P), GL_n(\mathbb{C}))$

Summary:

$$\mathrm{Loc}_G^{Fi}(\Sigma, P) \xrightarrow{\simeq} \mathrm{Rep}_G^{Fi}(\Pi_1(\Sigma), P) \xrightarrow{\simeq} \mathrm{Rep}_G^{Fi}(\pi_1(\Sigma, P)) \xrightarrow{\simeq} B_P \ltimes \mathrm{Hom}(\pi_1(\Sigma, P), G)$$

Summary:

$$\mathrm{Loc}_G^{Fi}(\Sigma, P) \xrightarrow{\simeq} \mathrm{Rep}_G^{Fi}\big(\Pi_1(\Sigma), P\big) \xrightarrow{\simeq} \mathrm{Rep}_G^{Fi}\big(\pi_1(\Sigma, P)\big) \xrightarrow{\simeq} B_P \ltimes \mathrm{Hom}(\pi_1(\Sigma, P), G)$$

$\Rightarrow$ bijection on the underlying moduli spaces.

Decorated Betti moduli space

$$\boxed{\mathcal{L}oc_G^{\square}(\Sigma, P) \simeq \mathcal{R}ep_G^{\square}(\Pi_1(\Sigma), P) \simeq \mathcal{R}ep_G^{\square}(\pi_1(\Sigma, P)) \simeq \mathcal{X}_G^{\square}}$$

Summary:

$$\mathrm{Loc}^{Fi}_G(\Sigma,P) \xrightarrow{\simeq} \mathrm{Rep}^{Fi}_G\big(\Pi_1(\Sigma),P\big) \xrightarrow{\simeq} \mathrm{Rep}^{Fi}_G\big(\pi_1(\Sigma,P)\big) \xrightarrow{\simeq} B_P \ltimes \mathrm{Hom}\big(\pi_1(\Sigma,P),G\big)$$

$\Rightarrow$ bijection on the underlying moduli spaces.

Decorated Betti moduli space

$$\boxed{\mathcal{L}oc^{\square}_G(\Sigma,P) \simeq \mathcal{R}ep^{\square}_G\big(\Pi_1(\Sigma),P\big) \simeq \mathcal{R}ep^{\square}_G\big(\pi_1(\Sigma,P)\big) \simeq \mathcal{X}^{\square}_G}$$

Framed and projectively framed case

$$\mathrm{Loc}^{\square}_G(\Sigma,P) \xrightarrow{\simeq} \mathrm{Rep}^{\square}_G\big(\Pi_1(\Sigma),P\big) \xrightarrow{\simeq} \mathrm{Rep}^{\square}_G\big(\pi_1(\Sigma,P)\big) \xrightarrow{\simeq} K^{\square}_P \ltimes \mathrm{Hom}\big(\pi_1(\Sigma,P),G\big)$$

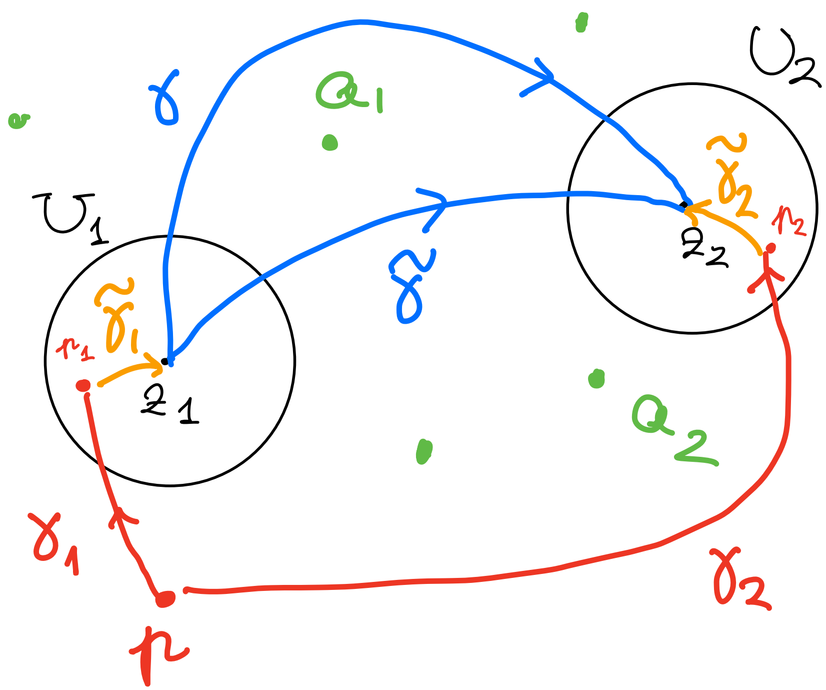
$$K_P^{Fr} = \sqcup\, U_P \qquad K_P^{PFr} = U_P^{\times} := (\mathbb{C}^{\times})^{|P|} \times U_P$$

Happy birthday !!

Miura equivalence: $\mathrm{Rep}_G(\Pi_1(\Sigma)) \xrightarrow{\sim} \mathrm{Rep}_G(\pi_1(\Sigma, p))$

$$\begin{array}{ccc} \cup & & \cup \\ (E, \phi) & & (E_p, \Phi_p) \end{array}$$

$\uparrow$ monodromy

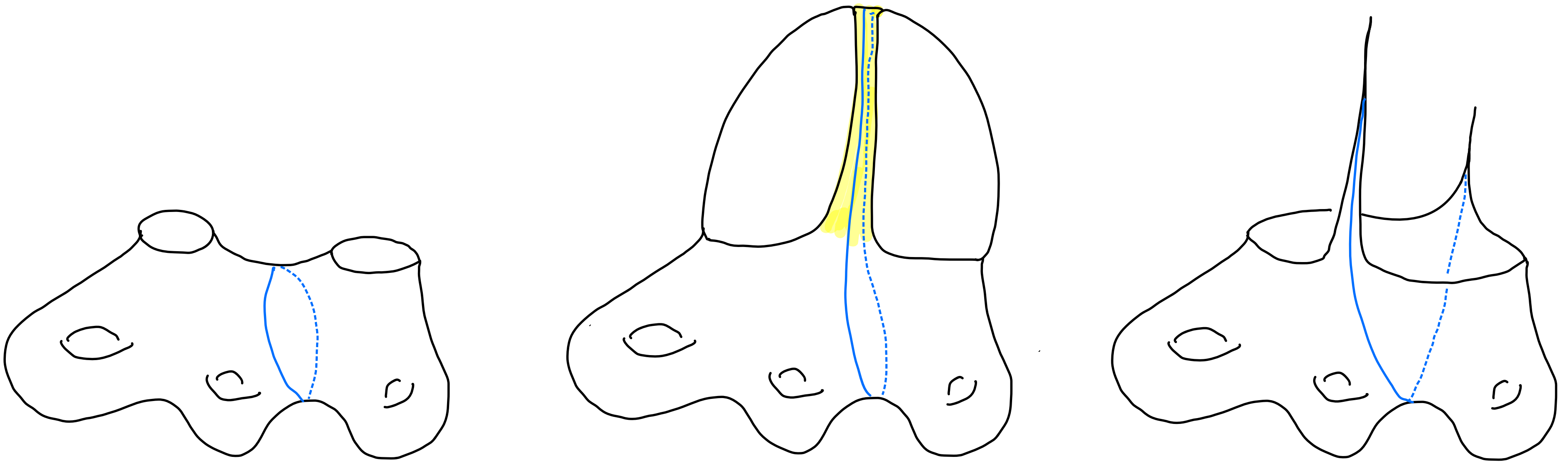


$$\gamma \longleftrightarrow \gamma_2^{-1} \tilde{\gamma}_2^{-1} \gamma \tilde{\gamma}_1 \gamma_1$$

confluence

[Chekhov, M.M., Rubtsov 2016]

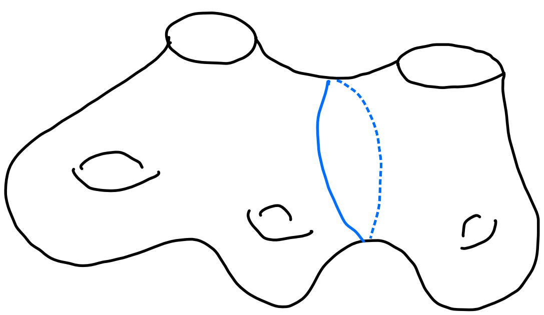
$\mathrm{Hom}(\pi_1(\Sigma_{g,s}), G)/G$ for $G = \mathrm{PSL}_2(\mathbb{R})$ Teichmüller space

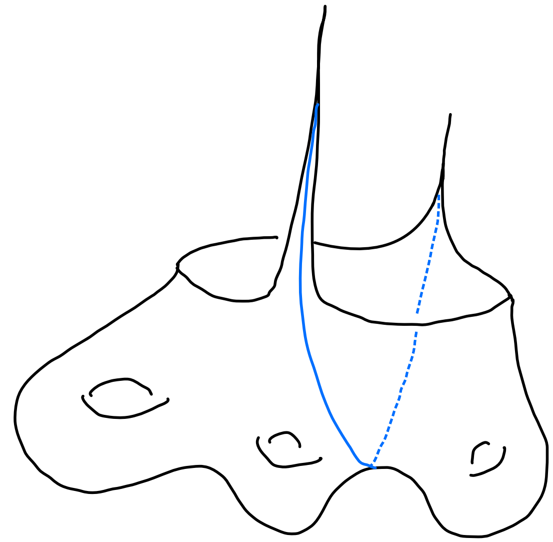


Conference

[Chekhov, M.M., Rubtsov 2016]

$\mathrm{Hom}(\pi_1(\Sigma_{g,s}), G)/G$ for $G = PSL_2(\mathbb{R})$ Teichmüller space





$\Sigma_{g,s}$

$\pi_1(\Sigma_{g,s}, x_0)$

$\mathrm{Hom}(\pi_1(\Sigma_{g,s}, x_0), G)$

$\mathrm{Hom}(\pi_1(\Sigma_{g,s}, x_0), G)/G$

$\Sigma_{g,s,m}$

$\pi_1(\Sigma_{g,s,m}, P)$

$\mathrm{Hom}(\pi_1(\Sigma_{g,s,m}, P), G)$

$\mathrm{Hom}(\pi_1(\Sigma_{g,s,m}, P), G)/U_P$