# Trends in Differential and Complex Geometry: a celebration of Nigel Hitchin's 80th birthday

## Geometry and Topology of parabolic Higgs bundles

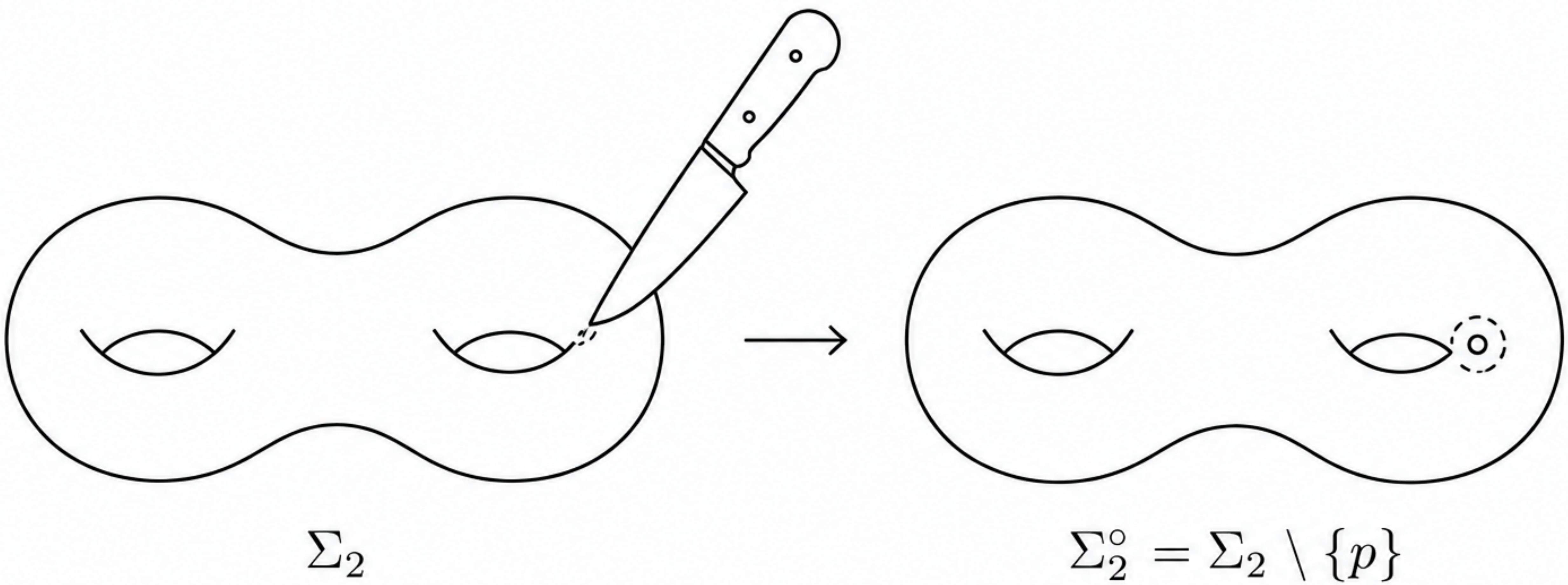
**Marina Logares, Oxford, Thursday 3rd September 2026**

*“In Mathematics everything is symmetric everything fits in, like  a sphere, hence it is not easy to do research. But sometimes, when you find an irregularity, then you can stick the knife in and look what is hidden underneath.”*

Prof. Nigel J. Hitchin

From  "Science Lives" produced by The Simons Foundation

Promt to ChatGPT 5.6 Sol: stick a knife on a Riemann surface




Promt ChatGPT Sol 5.6: draw a genus 2 Riemann surface and a knife and show it makes a punture on it

# Parabolic bundles

$X$ complex projective smooth curve of genus $g$

$E \longrightarrow X$ holomorphic vector bundle

$D \subset X$ reduced effective divisor; $\deg(D) = n$

Parabolic structure on $D$

$$E_x = E_{1,x} \supset E_{2,x} \supset \cdots \supset E_{r(x),x} \supset 0$$
$$0 \leq \alpha_1(x) < \alpha_2(x) < \cdots \alpha_{r(x)}(x) < 1$$

Denote a vector bundle with a parabolic structure by $E_\star$

$$\text{pdeg}(E_\star) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{r(x)} \alpha_i(x) m_i(x)$$

$E_\star$ stable if $\forall\ F_\star \subset E_\star \quad p\mu(F_\star) < p\mu(E_\star)$

For generic weights, semistability coincides with stability. i.e. $\sum_{x \in D} \left( \text{rk}(F) \sum_{i=1}^{r(x)} \alpha_i^E(x) - \text{rk}(E) \sum_{i=1}^{f(x)} \alpha_i^F(x) ) \right) \in \mathbb{Z}$

# Parabolic bundles

Given two parabolic bundles $V_\star$ and $W_\star$ over $(X, D)$ with weights $\alpha = \{\alpha_j(x)\}_{j \in J, x \in D}$ and $\eta = \{\eta_i(x)\}_{i \in I, x \in D}$ a map

$$f : V_\star \longrightarrow W_\star$$

is strongly parabolic if

$$f(V_{x,i}) \subset W_{x,j+1} \text{ whenever } \alpha_i(x) \geq \eta_j(x)$$

parabolic if

$$f(V_{x,i}) \subset W_{x,j+1} \text{ whenever } \alpha_i(x) > \eta_j(x)$$

$V_\star = W_\star$ PEnd$(V_\star)_x = P_x$ parabolic subgroup of $GL(r, \mathbb{C})$

# Parabolic Higgs bundles

Consider $\mathcal{M}(r,d)$ moduli space of stable vector bundles

$T_E\mathcal{M} \cong H^1(X, \mathsf{End}(E)) \implies T_E^*\mathcal{M} \cong H^0(X, \mathsf{End}(E) \otimes K_X)$

Higgs bundle $(E, \phi)$ $\quad \phi \in H^0(X, \mathsf{End}(E) \otimes K_X) \quad \rightsquigarrow \quad \mathcal{M}_H(r,d)$

Consider $\mathcal{M}_P(r,d,\alpha)$ moduli space of stable parabolic vector bundles

$T_E\mathcal{M}_P \cong H^1(X, \mathsf{PEnd}(E))$

$\mathsf{Tr} : \mathsf{End}(E) \otimes \mathsf{End}(E) \to \mathcal{O}_X$ ; $A \otimes B \to \mathsf{Tr}(AB)$ restricts to

$\mathsf{PEnd}(E) \otimes \mathsf{SPEnd}(E) \to \mathcal{O}_X(-D)$ fiberwise non-degenerate $\implies$

$\mathsf{PEnd}(E)^* \cong \mathsf{SPEnd}(E)(D)$ and $\mathsf{SPEnd}(E)^* \cong \mathsf{PEnd}(E)(D)$

$T_E^*\mathcal{M}_P \cong H^0(X, \mathsf{SPEnd}(E) \otimes K_X(D))$

Parabolic Higgs bundle $(E_\star, \phi)$ $\quad \phi \in H^0(X, \mathsf{SPEnd}(E) \otimes K_X(D)) \quad \rightsquigarrow \quad \mathcal{M}_{PH}(r,d,\alpha)$

# Geometry

# Higgs bundles

$T^*_E(r,d)\mathscr{M} \subset \mathscr{M}_H(r,d)$ Liouville symplectic holomorphic structure

$h : \mathscr{M}_H(r,d) \longrightarrow B = \bigoplus_{i=1}^{r} H^0(X, K_X^i)$ abelian fibres & $\dim B = \frac{1}{2}\dim \mathscr{M}_H \Longrightarrow$ ACIS

$T^*_{E_\star}\mathscr{M}_P(r,d,\alpha) \subset \mathscr{M}_{PH}(r,d,\alpha)$ Liouville symplectic holomorphic structure

$h : \mathscr{M}_{PH} \longrightarrow B = \bigoplus_{i=1}^{r} H^0(X, K_X^i(D^{i-1})$ abelian fibres & $\dim B = \frac{1}{2}\dim \mathscr{M}_{PH} \Longrightarrow$ ACIS

Moreover,

**E. Markmann**, “Spectral curves and integrable systems”, Compositio Math. 93 (1994), no. 3, 255–290.

**F. Bottacin,** “Symplectic geometry on moduli spaces of stable pairs”, Ann. Sci. École Norm. Sup. (4) 28 (1995), no. 4, 391–433. He does also $\mathscr{M}_{PH}$

Bottacin and Markman treat the more general meromorphic/twisted situation:
$(E,\phi) \quad \phi \in H^0(X, \mathrm{End}(E) \otimes L) \rightsquigarrow$ Poisson structures

# MR1300764 (95k:14013)

Reviewed

Markman, Eyal (1-MI)

**Spectral curves and integrable systems.**

Compositio Math. **93** (1994), no. 3, 255–290.

## Review

Hitchin's abelianization program, namely the introduction of the spectral-curve technique to translate line bundle into vector bundle theory [cf. N. J. Hitchin, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126; MR0887284; Comm. Math. Phys. **83** (1982), no. 4, 579–602; MR0649818] for vector bundles over $\mathbf{P}^1$, gave rise to tremendous progress as regards moduli problems of vector bundles for the first time after their inception. Thus, the Hitchin system [Duke Math. J. **54** (1987), no. 1, 91–114; MR0885778] continues to receive enormous attention and generalizations (typically to other types of bundles or varieties). This paper's set-up is the same as was developed by F. Bottacin [Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 4, 391–433; ``Symplectic geometry on moduli spaces of stable pairs'', Thèse d'État, Univ. Paris XI (Paris-Sud), Orsay, 1993; per revr.]. The main result is the following: for any line bundle $L$ over a Riemann surface $\Sigma$ with $L\otimes K^{-1}$ effective ($K$ is the canonical bundle over $\Sigma$) the moduli space of stable Higgs pairs $(E,\phi)$, where $E$ is a (fixed rank, degree) vector bundle over $\Sigma$ and $\phi\in H^0(\Sigma,\mathscr{E}nd\,E\otimes L)$, is an algebraically completely integrable system. The choice of Poisson structure on the space depends on a nonzero section $s\in H^0(\Sigma,L\otimes K^{-1})$ and the fibres of the system are Jacobians of spectral curves. As with Bottacin's work, the study of the Poisson structure involves the hypercohomology of a complex that keeps track of deformations of Higgs pairs [cf. G. E. Welters, Compositio Math. **49** (1983), no. 2, 173–194; MR0704390]. For generalizations to principal bundles see a paper by G. Faltings [J. Algebraic Geom. **2** (1993), no. 3, 507–568; MR1211997].

**Reviewer:** Previato, Emma

# Weakly Parabolic/ Framed… Higgs bundles

$(E_\star, \phi), \quad \phi \in H^0(X, \mathsf{PEnd}(E_\star)) \rightsquigarrow \mathscr{M}^w_{PH}(r,d,\alpha)$

Recall $\mathsf{PEnd}(E)^* = \mathsf{SPEnd}(E)(D)! \rightsquigarrow \dim \mathscr{M}^w_{PH}(r,d) = (2g-2+n)r^2+1$

**Theorem**[Martens, L. 2010]

$\mathscr{M}^w_{PH}$ holomorphic Poisson structure

$h : \mathscr{M}^w_{PH} \longrightarrow \bigoplus_{i=1}^{r} H^0(X, K^i_X(D^i))$ Poisson ACIS

where $\dim B = k + l$ and $\dim \mathscr{M}^w_{PH} = 2k+l$ note $\mathscr{M}_{PH}(r,d,\alpha)$ is a symplectic leaf

**Theorem**[Biswas, L. Peón-Nieto,2020]

$(E, \phi, \varphi) \quad \varphi : \mathcal{O}_D^{\oplus r} \xrightarrow{\cong} E|_D$ framing $\neq$ level structures.

$h^{-1}(b)$ torsor over $\mathsf{Jac}(X_b)$

$$\begin{tikzcd} \mathscr{M}_{FH}(r,d) \arrow[d] \arrow[drr, "h_F"] & & \\ \mathscr{M}_H(r,d) \arrow[rr, "h"] & & B = \bigoplus_{i=1}^r H^0(X, K^i(D^i)) \end{tikzcd}$$

Playful map

$$
\begin{tikzcd}
 & & \mathcal{M}_{PH} \arrow[r, hook] \arrow[dll] \arrow[ddl] & \mathcal{M}_{PH}^{\omega} \arrow[dddl] \\
\mathcal{M}_{H} \arrow[dd] & & \mathcal{M}_{K(D)} \arrow[ll] \arrow[dd] & \\
 & \bigoplus H^0(K^i D^{i-1}) \arrow[dr, bend left] & & \\
\bigoplus H^0(K^i) \arrow[ur] \arrow[rr] & & \bigoplus H^0(K^i D^i) & \\
\mathcal{M}_{FH} \arrow[uuu] \arrow[urr] & & & \mathcal{M}_{FPH} \arrow[lll] \arrow[ul]
\end{tikzcd}
$$

and applying $\Phi$ to obtain

$$\Phi\omega^{2m-1} \in H^0(M; V_1 \otimes V_1).$$

Hence, a generic fibre of the map $p$ is an open set in the Prym variety of $S$ and the system is integrable.

*Remark.* Note the similarity between the cases of $G = \mathrm{Sp}(m, \mathbb{C})$ and $\mathrm{SO}(2m+1, \mathbb{C})$. In both cases the projection $p$ maps to the same vector space of differentials on $M$ and the generic fibre is the same Prym variety: locally the two systems are equivalent. Globally, the difference in identifying the fibres as Prym varieties (compare (5.13) and (5.21)) will affect the structure of the moduli spaces. This may be a convenient context in which to bring in the duality theory of Lie groups by means of which $\mathrm{Sp}(m, \mathbb{C})$ and $\mathrm{SO}(2m+1, \mathbb{C})$ are viewed as dual groups.

$G = SO(2m+1, \mathbb{C})$ $\quad$ ${}^LG = Sp(m, \mathbb{C})$ !!!!

Nigel, did you already know GLC?

# Topology

# Parabolic Higgs bundles

**H.U. Boden and K. Yokogawa** "Moduli spaces of parabolic bundles and parabolic K(D) pairs over smooth curves I" Internat. J. Math. 7 (1996), no. 5, 573–598. Betti numbers for $SL(2,\mathbb{C})$ via Morse theory - do not depend on weights.

**O. García-Prada, P. Gothen, V. Muñoz** "Betti numbers of the moduli space of rank 3 parabolic Higgs bundles" Mem. Amer. Math. Soc. 187 (2007), no. 879

Poincaré polynomial for $g = 0, n = 3$:

$P_t(\mathscr{M}_{PH}^{GL(3,\mathbb{C})}) = 7t^2 + 1 = P_t(\mathscr{M}_{PH}^{SL(3,\mathbb{C})})$ (because Jac$(\mathbb{P}^1) = pt$

Poincaré polynomial for $g = 1$ $n = 1$

$P_t\left(\mathscr{M}_{PH}^{GL(3,\mathbb{C}}\right) = 1 + 2t + 4t^2 + 8t^3 + 13t^4 + 20t^5 + 24t^6 + 18t^7 + 6t^8$ . (dim $\mathscr{M}_{PH}^{GL(3,\mathbb{C}} = 8$)

$P_t\left(\mathscr{M}_{PH}^{SL(3,\mathbb{C}}\right) = 1 + 3t^2 + 2t^3 + 6t^4 + 6t^5 + 22t^6$ . (dim $\mathscr{M}_{PH}^{SL(3,\mathbb{C}} = 6$)

# Parabolic Higgs bundles

Non-parabolic **O. García-Prada, J. Heinloth, A. Schmitt**, “On the motives of moduli of chains and Higgs bundles” J. Eur. Math. Soc. (JEMS) 16 (2014), no. 12, 2617–2668. explicit formula for the motive of the space of semi-stable Higgs bundles of rank 4 and odd degree.

Parabolic **A. Mellit**, “Poincaré polynomials of character varieties, Macdonald polynomials and affine Springer fibers”, Ann. of Math. **192** (2020). Formulas for the number of stable (parabolic) Higgs bundles and the Poincaré polynomials of the moduli spaces. Arbitrary rank

**Viet Cuong Do**, “On the motives of moduli of parabolic chains and parabolic Higgs bundles”, Bull. Sci. Math. **193** (2024), art. 103449. Explicitly modelled on GHS.

**What about parabolic G-Higgs bundles?** (G real Lie group)

**O. García-Prada, M. L., V. Muñoz**, “Moduli spaces of parabolic $U(p,q)$ Higgs bundles” Q. J. Math. 60 (2009), no. 2, 183–233.

**M. L.** “Betti numbers of parabolic $U(2,1)$ Higgs bundles” Geom. Dedicata 123 (2006), 187–200.


Happy Birthday from Daniel Hernández Ruipérez

moduli $\mathcal{M}$ studied in Ref. [5]. It is known that for fixed rank, the moduli spaces $\mathcal{M}$ of stable parabolic Higgs bundles, corresponding to different choices of degrees and generic weights, have the same Betti numbers [5]. Our computations for $p + q = 3$ produce counterexamples to this type of phenomena for the submanifolds $\mathcal{U}$, that is, they provide an example of the dependence of these moduli spaces on the generic weights of the parabolic structure.

M. L. "Betti numbers of parabolic $U(2,1)$ Higgs bundles" Geom. Dedicata 123 (2006), 187–200.

# Parabolic G-Higgs bundles

**O.Biquard, O. García-Prada, I. Mundet i Riera**, "Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group" Adv. Math. 372 (2020), 107305, 70 pp. Establish the correspondence with representations of the fundamental group of the punctured surface in G with arbitrary holonomy around the punctures. Future: higher order poles in the Higgs field.

**P. Gothen, A. Oliveira**, "Topological mirror symmetry for parabolic Higgs bundles" J. Geom. Phys. 137 (2019), 7–34. Proof of topological mirror symmetry for strongly parabolic $SL(n,\mathbb{C})$ / $GL(n,\mathbb{C})$ $n = 2,3$

**G. Kydonakis, H. Sun, L. Zhao**, "Topological invariants of parabolic G-Higgs bundles" Math. Z. 297 (2021), no. 1-2, 585–632. Show that this moduli space admits a nontrivial connected component in the case where G is the split real form of a complex simple Lie group. In deph $\mathrm{Sp}(2n,\mathbb{R})$

**G. Kydonakis, H. Sun, L. Zhao**, "Monodromy of rank 2 parabolic Hitchin systems", J. Geom. Phys. 171 (2022), Paper No. 104411, 18 pp. Number of connected components of the moduli space of polystable G-Higgs bundles such that the weights in the parabolic structure can be written as a fraction with denominator 2 $G = GL(2,\mathbb{C}), SL(2,\mathbb{C}), PSL(2,\mathbb{C})$

Inspired in **L. Schaposnik** "Monodromy of the $SL_2$ Hitchin fibration" Internat. J. Math. 24 (2013), no. 2, 1350013, 21 pp. as well as **D. Baraglia L. Schaposnik**, "Monodromy of rank 2 twisted Hitchin systems and real character varieties" Trans. Amer. Math. Soc. 370 (2018), no. 8, 5491–5534. Connected components of $GL(2,\mathbb{R})$, $SL(2,\mathbb{R})$, $PGL(2,\mathbb{R})$, $PSL(2,\mathbb{R})$; $Sp(4,\mathbb{R})$ and $SO(2,3)$-character varieties with maximal Toledo invariant, and $SO(2,2)$

# What about SU(2,1)?

# Parabolic U(p,q) Higgs bundles

Consider$C_U^\bullet$ : $\mathrm{PEnd}(V_\star) \oplus \mathrm{PEnd}(W_\star) \xrightarrow{\mathrm{ad}(\phi)} (\mathrm{SPHom}(V_\star, V_\star) \oplus \mathrm{SPHom}(W_\star, V_\star)) \otimes K(D)$

then

$T_{(E,\phi)}\mathcal{U} = \mathbb{H}^1(C_U^\bullet)$

Hence $\dim \mathcal{U}(p,q,a,b,\alpha,\eta) = \dim \mathbb{H}^1(C_U^\bullet) = 1 - \chi(c_U^\bullet)$

$\mathbb{H}^0(C_U^\bullet) = \mathbb{C}$, and $\mathbb{H}^2(C_U^\bullet) = 0$

$$\dim \mathcal{U}(p,q) = 1 + (p+q)^2(g-1) + \sum_{x \in D} f_x(V \oplus W)$$

where $f_x(V \oplus W) = \dfrac{1}{2}\left((p+q)^2 - \sum_\nu \left(m_\nu^V(x) + m_\nu^W(x)\right)^2\right) = \dim GL_{\mathbb{C}}(n)/P_x$

full flags $\dim \mathcal{U}(p,q,a,b,\alpha,\eta) = 1 + (p+q)^2(g-1) + \dfrac{(p+q)^2 - (p+q)}{2}n$

# Parabolic U(p,q) Higgs bundles

Possible dimensions of the moduli space for low genus:

| $(g,n)$ | possible dimensions for $\mathcal{U}(2,1)$ |
|---|---|
| $(0,3)$ | $0,1$ |
| $(0,4)$ | $0,1,2,3,4$ |
| $(1,1)$ | $1,3,4$ |

where we used that

$$f_x = \begin{cases} 3, & V:(1,1) \text{ and } \eta \neq \alpha_1, \alpha_2, \\ 2, & V:(1,1) \text{ and } \eta = \alpha_1 \text{ or } \alpha_2, \\ 2, & V:(2) \text{ and } \eta \neq \alpha, \\ 0, & V:(2) \text{ and } \eta = \alpha. \end{cases}$$

# Parabolic SU(p,q) Higgs bundles

Fix $\Lambda \in \mathrm{Pic}(X)$

consider $d\det : \mathscr{U}(p,q,a,b,\alpha,\eta) \longrightarrow T^*\mathrm{Pic}(X)$,

$$V_\star \oplus W_\star, \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} \mapsto (\det(V_\star \oplus W_\star), \mathrm{tr}(\phi))$$

define $\mathscr{U}^\Lambda = \det^{-1}(\Lambda)$ in particular, we define

$\mathscr{S}\mathscr{U}(p,q,a,b,\alpha,\eta) = \det^{-1}(\mathscr{O}_X)$

indeed $\mathscr{S}\mathscr{U}(p,q) = \mathscr{U}(p,q) \cap \mathscr{N}_{SL_{\mathbb{C}}(p+q)}$

closed subvariety of $\mathscr{U}(p,q)$

# Parabolic SU(p,q) Higgs bundles

Notice that

pardeg(det($V_\star \oplus V_\star$) = pardeg(det($V_\star$) $\otimes$ $\det(W_\star)$) = pardeg($V_\star$) + pardeg($W_\star$)

hence

$$a + b + \sum_{x \in D} \left( \sum_{i=1}^{p(x)} \alpha_i(x) m_i(x) + \sum_{j=1}^{q(x)} \eta_j(x) m_j(x) \right) = 0$$

thus

$$\sum_{x \in D} \left( \sum_{i=1}^{p(x)} \alpha_i(x) m_i(x) + \sum_{j=1}^{q(x)} \eta_j(x) m_j(x) \right) = 0 = -(a+b) \text{ so } (a+b) \leq 0$$

# Parabolic SU(p,q) Higgs bundles

$C^\bullet_{SU} : (\mathrm{PEnd}(V_\star) \oplus \mathrm{PEnd}(W_\star))_0 \xrightarrow{\mathrm{ad}(\phi)} (\mathrm{SPHom}(V_\star, V_\star) \oplus \mathrm{SPHom}(W_\star, V_\star)) \otimes K(D)$

where $(\mathrm{PEnd}(V_\star) \oplus \mathrm{PEnd}(W_\star))_0 = \{(a, b) \in \mathrm{PEnd}(V_\star) \oplus \mathrm{PEnd}(W_\star) \,|\, \mathrm{tr}(a) + \mathrm{tr}(b) = 0\}$

$T_{(E,\phi)}\mathcal{SU} = \mathbb{H}^1(C^\bullet_{SU})$.

Hence $\dim \mathcal{SU}(p, q, a, b, \alpha, \eta) = \dim \mathbb{H}^1(C^\bullet_{SU}) = -\chi(C^\bullet_{SU})$

since $\mathbb{H}^0(C^\bullet_{SU}) = 0$, and $\mathbb{H}^2(C^\bullet_{SU}) = 0$

$0 \longrightarrow C^0_{SU} \longrightarrow C^0_U \longrightarrow \mathcal{O}_X \longrightarrow 0 \implies \chi(C^0_{SU}) = \chi(C^0_U) + g - 1$

$\dim \mathcal{SU}(p, q) = \mathbb{H}^1(C^\bullet_{SU}) = \chi(C^1_{SU}) - \chi(C^0_{SU}) = \chi(C^1_U) - (\chi(C^0_U) + g - 1) =$
$\mathbb{H}^1(C^\bullet_U) - g = \dim \mathcal{U}(p, q) - g$

# Parabolic SU(p,q) Higgs bundles

$$\dim \mathcal{SU}(p,q) = 1 - g + (p+q)^2(g-1) + \sum_{x \in D} f_x(V \oplus W)$$

where

$$f_x(V \oplus W) = \frac{1}{2}\left( (p+q)^2 - \sum_{\nu} \left(m_\nu^V(x) + m_\nu^W(x)\right)^2 \right) = \dim GL_{\mathbb{C}}(p+q)/P_x = \dim SL_{\mathbb{C}}(p+q)/Q_x$$

where $Q_x = P_x \cap SL_{\mathbb{C}}(p+q)$

full flags

$$\dim \mathcal{SU}(p,q,a,b,\alpha,\eta) = 1 - g + (p+q)^2(g-1) + \frac{(p+q)^2 - (p+q)}{2} n$$

# Morse function

$S^1 \times M \longrightarrow M$ Hamiltonian action.

$$\mu(E,\phi) = -\frac{1}{2}\|\phi\|^2 = -i\int_X \mathrm{tr}(\phi\phi^\star)$$

$$f : M \longrightarrow \mathbb{R}, \quad f(E,\phi) = \frac{1}{2}\|\phi\|^2$$

$f$ is proper (Uhlenbeck, Hitchin; Biquard, Boden-Yokogawa—parabolic)

**Frankel** $f : M \longrightarrow \mathbb{R}$ proper moment map for a Hamiltonian circle action $\Longrightarrow$ it is a perfect Bott-Morse function.

Recall if Z is a Hausdorff space and $f : Z \to \mathbb{R}$ is proper and bounded below then $f$ attains a minimum on each connected component of Z. Therefore, if the subspace of local minima of f is connected then so is Z.

# Connected components U(p,q) and SU(p,q)

The **Toledo invariant** for $U(p,q)$ parabolic Higgs bundles is

$$\tau = 2\frac{q\,\text{pdeg}(V_\star) - p\,\text{pdeg}(W_\star)}{p+q}.$$ (Does not depend on the flag)

In the $SU(p,q)$ case reduces to

$$\tau = 2\text{pdeg}(V_\star) = -2\text{pdeg}(W_\star)$$

**Milnor-Wood inequalities**:

$$|\tau| \leq \min\{p,q\}(2g-2+n)$$

in the $SU(p,q)$ case reduces to $|\text{pdeg}(V_\star)| \leq \frac{1}{2}\min\{p,q\}(2g-2+n)$ and

$$|\text{pdeg}(W_\star)| \leq \frac{1}{2}\min\{p,q\}(2g-2+n)$$

The maximal value for the Toledo invariant is investigated through the understanding of something known as triples

# Connected components

$\tau < 0 \iff \gamma = 0 \;\; \tau > 0 \iff \beta = 0 \quad \rightsquigarrow E_1 \xrightarrow{\varphi} E_0$

**Theorem**[García Prada, L., Muñoz, 2009] $g > 0$ (full flag on $E_\star = V_\star \oplus W_\star$)

$\tau_L = \min p, q(2g-2+n) - \dfrac{|p-q|}{p+q}\epsilon$ with $\epsilon > 0$ explicitly computable, s.t. the subvariety of local minima for the Morse function $\mathcal{N}(p,q,a,b,\alpha,\eta)$ ((2g-2)-stable triples)

is non-empty and connected $\iff |\tau| < \tau_L$

The moduli space $\mathcal{U}(p,q,a,b,\alpha,\eta)$ is non-empty whenever $|\tau| < \tau_L$

The moduli space $\mathcal{U}(p,q,a,b,\alpha,\eta)$ is empty whenever $|\tau| > \tau_L$

For the moduli space $\mathcal{SU}(p,q,a,b\,.\,\alpha,\eta)$ we need to understand triples with fixed determinant (which was done in GGM! in rank 3)

**Theorem**[L. 2026] For $\mathcal{SU}(2{,}1{,}a,b\,.\,\alpha,\eta)$ it goes as above, general $p,q$ in progress.

# Morse theory

The critical points of $f$ are exactly the fixed points of the circle action on a moduli space $\mathscr{M}$

At a critical point $p \in \mathscr{M}$ is

$$T_p\mathscr{M} = \bigoplus_{\ell \in \mathbb{Z}} T_\ell, \qquad d\rho_{e^{i\theta}}(v) = e^{i\ell\theta}v, \quad v \in T_\ell\,.$$

Moreover, the eigenvalue $\ell$ subspace for the Hessian of $f$ is the same as the weight $-\ell$ subspace for the infinitesimal circle action on the tangent space.

Hence the negative eigenspace of the Hessian is

$$T_p^-\mathscr{M} = \bigoplus_{\ell > 0} T_\ell\,.$$

Therefore the Morse index is $\lambda(p) = \dim_{\mathbb{R}} T_p^-\mathscr{M} = 2\sum_{\ell>0} \dim_{\mathbb{C}} T_\ell\,.$

As $f$ is a perfect Bott-Morse function we get

$P_t(\mathscr{M}) = \sum_F t^{\lambda(F)} P_t(F)$ where $F$ are the critical subvarieties

# Poincaré polynomials U(2,1)

Fix $\alpha_1 < \alpha_2 < \eta$. S3

For g=1 n=1 a=b=0 S3(a) $\eta - \alpha_2 > \alpha_2 - \alpha_1$

$$P_t\big(\mathcal{U}(2,1)\big) = (1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$$

For g=1 n=1 a=b=0 S3(b) $\eta - \alpha_2 < \alpha_2 - \alpha_1$

$$P_t\big(\mathcal{U}(2,1)\big) = (1+t^4) + t^2(1+t)^4 = 1 + 4t + 7t^2 + 8t^3 + 7t^4 + 4t^5 + t^6 .$$

$$P_t\big(\mathcal{U}(2,1)_{S3(b)}\big) - P_t\big(\mathcal{U}(2,1)_{S3(a)}\big) = t^2(1+t)^4 .$$

The difference crossing a wall is precisely the (1,1,1) critical component that disapears when crossing the wall, which is $X \times X$ with Morse index 2

# Poincaré polynomials SU(2,1)

Type (1,1,1) critical subvarieties

$\lambda(SU(2,1) : (1,1,1) = 2g - 2 + 4d_0 + 2b + 2(n - v)$

where $v = \sharp\{x \in D; \alpha_0(x) \leq \alpha_2(x)\}$

It corresponds to compute $\dim \mathbb{H}^1((C^\bullet_{SU})_1)$ where $(C^\bullet_{SU})_1$ is

$\mathrm{PHom}(E_0, E_2) \longrightarrow 0$

# Poincaré polynomials SU(2,1)

$$P_t(SU(2,1):(1,1,1)) =$$

$$= \sum_{\epsilon \in (S_2)^n} \sum_{d_0} t^{\lambda(d_0,\epsilon)} \left[ P_t(S^{m_1}X) P_t(S^{m_2}X) + (3^{2g}-1) \binom{2g-2}{m_1} \binom{2g-2}{m_2} t^{m_1+m_2} \right].$$

$$m_1 = b - d_0 + 2g - 2 + v_1 \quad m_2 = -d_0 - 2b + 2g - 2 + v_2$$

$$v_1 = \sharp\{x \in D; \alpha_{\omega(1)}(x) < \eta(x)\} \;\; v_2 = \sharp\{x \in D; \eta(x) < \alpha_{\omega(2)}(x)\}$$

In particular for $g = 1$ and $n = 1$

| | $U(2,1)$ | $SU(2,1)$ |
|---|---|---|
| $S3(a)$, minimum | $(1+t)^4$ | $(1+t)^2$ |
| $S3(a)$, $(1,1,1)$ | $0$ | $0$ |
| Total | $(1+t)^4$ | $(1+t)^2$ |

| | $U(2,1)$ | $SU(2,1)$ |
|---|---|---|
| $S3(b)$, minimum | $(1+t)^4$ | $(1+t)^2$ |
| $S3(b)$, $(1,1,1)$ | $t^2(1+t)^4$ | $t^2(1+t)^2$ |
| Total | $(1+t)^4(1+t^2)$ | $(1+t)^2(1+t^2)$ |

Hence $P_t\big(U(2,1)_{S3(a/b)}\big) = P_t(\mathrm{Jac}X)\, P_t\big(SU(2,1)_{S3(a/b)}\big)$

# Poincaré polynomials SU(2,1)

g=1 n=1 S3(a) $\alpha_1 < \alpha_2 < \eta \quad \eta - \alpha_2 > \alpha_2 - \alpha_1$

$$P_t\big(\mathcal{SU}(2,1)\big) = (1+t)^2$$

g=1 n=1 S3(b) $\alpha_1 < \alpha_2 < \eta \quad \eta - \alpha_2 < \alpha_2 - \alpha_1$

$$P_t\big(\mathcal{SU}(2,1)\big) = (1+t)^2(1+t^2)$$

Both have same (2,1) contribution, they differ on the (1,1,1) contribution which for S3(a) is empty and for S3(b) is $X$ with Morse index 2, that is

$$P_t(\mathcal{SU}(2,1)_{S3(b)}) - P_t(\mathcal{SU}(2,1)_{S3(a)}) = t^2(1+t)^2 = t^2 P_t(X)\,.$$

A candidate for $\mathcal{SU}(2,1)_{S3(a)}$ is $X \times \mathbb{C}^2$

A candidate for $\mathcal{SU}(2,1)_{S3(b)}$ is $X \times T^*\mathbb{P}^1$ .

# Happy birthday Nigel!