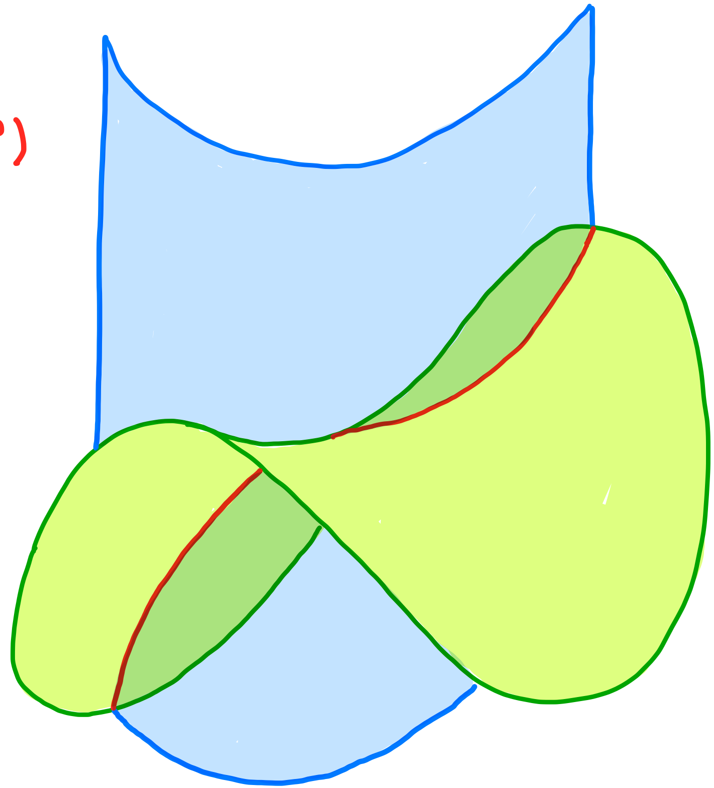
Sklyanin algebras via Lie categories

(t, t^2, t^3)

$x^2 = y$

$xy = z$



Hitchin 80

September 1, 2026

Oxford

Marco Gualtieri

University of Toronto

UPPER CANADA
LOWER CANADA
Abbitibee L.
Patquashagama
Michipicoten Ho.
Michipicoten I.
Cartbuo I.
Montreal I.
LAKE SUPERIOR
White Fish Pt.
Gros Cap
Falls of St. Mary
Forts
L. St. George
St. Joseph
La Cross
Labyrinth Lakes
Montreal R.
Lake Temiscaming
Temiscamings
Lake Nipissing
R. du Francois
Beaver I.
Manatoulin Is.
White Wood I.
Manitou I.
LAKE HURON
Thunder Bay Is.
Thunder Bay
Cabots Head
Gloucester Bay
Troquais Bay
L. Simcoe
Barques Pt.
Roger Pt.
White Rock
MICHIGAN
MICHIGAN TERRITORY
LAKE MICHIGAN
Ft. Clair
L. St. Clair
Detroit
WESTn DISTt.
R. Chenail Ecarte
R. Thames
Suffolk
Essex
Sandwich
Chatham
Maldon
Landguard
Turkey Pt.
Long Pt.
LONDON
Norfolk
Oxford
LONDON DIST.
GORE DIST.
Dundas St.
Burlington Bay
Ft. George
Niagara
Ft. Erie
Buffaloe
Niagara Falls
Mile Pond
Bishing Bay
Trondiquet Bay
Canal
Brighton
Oswego
Oneida L.
HOME DISTt.
York
Gibraltar Pt.
Duffins Creek
Williamsburg
NEWCASTLE
Balsom L.
Shallow Lakes
Sturgeon L.
Rice L.
R. Trent
Hope
Murray
Newcastle
Pt. Edward
S. Peters Bay
Kingston
Ernestown
MIDLAND DISTt.
JOHNSTOWN DIST.
Brockville
Leeds
The Thousand Is.
Sacketts Harbour
LAKE ONTARIO
EASTERN DIST.
Osnabrugh
L. St. Francis
RIVER St. LAWRENCE
Grand or Ottawa River
S.W. Branch
R. du Moine
R. du Nord
L. Chaudiere
Two Mountain L.
S. Eustache
Montreal
Chambly
Dorchester
R. Chateauguay
Philipsburg
Highgate
Plattsburg
Lake Champlain
MONTREAL
William Henry
S. Denis
S. Hyacinthe
R. St. Francis
Nicolet
S. Francis
Megantick L.
DISTRICT OF THREE RIVERS
Three Rivers
S. Peters
L. S. Thomas
S. Anne
R. Chaudiere
R. Becancourt
S. Nicholas
QUEBEC
S. Joseph
Orleans I.
DISTRICT OF QUEBEC
L. St. John
Great Discharge
Saguenay R.
The Great River
Sable R.
Original L.
Montpelier
VERMONT
NEW HAMPSHIRE
Concord
Portsmouth
Augusta
Portland
Casco Bay
Saco Bay
Wells Bay
C. Neddock
Piscataqua Harb.
C. Ann
York
BOSTON
Cape Cod
Northampton
MASSACHUSETTS
Hartford
Providence
Albany
NEW YORK
PENNSYLVANIA
LAKE ERIE
Erie
Pomfret
Newmarket
French T.
Pt. Pelee Is.
Pt. Pelee
Sisters
Sandusky Bay
Ft. Meigs
Maima R.

> "Mathematical truths are in our world. Our invention is to select from all the possible mathematical truths the ones that are really interesting. Now, how and why do we see that something is interesting? Because it is beautiful, it has a nice structure, it produces human emotion. A machine cannot teach us to understand beauty."
>
> — Michael Atiyah, November 2018

“Proofs are to mathematics what spelling (or calligraphy) is to poetry.”

— Vladimir Arnold, 1999

Main Problem

E elliptic curve

$\eta \in \mathrm{Aut}^0(E)$

Find geometric constructions of

1. Algebras $Q_n(E,\eta)$ of Sklyanin 1982, Feigin-Odesskii 1989

   (deformations of $\mathbb{C}P^{n-1}$)

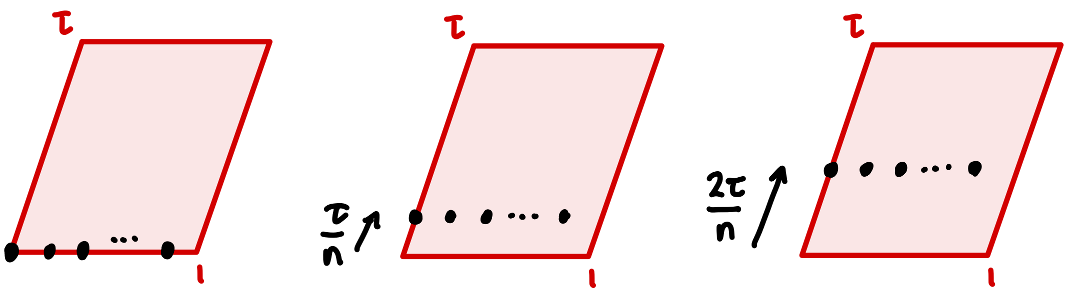
2. the corresponding Elliptic quantum groups of Felder 1994

   (deformations of $PGL_n\mathbb{C}$)

$E = \mathbb{C} / \mathbb{Z} + \tau\mathbb{Z} \qquad \tau \in \mathbb{H} \qquad \eta \in \mathbb{C}$ translation

Definition: $Q_n(E,\eta) = \mathbb{C}\langle x_1, \dots, x_n \rangle / R \qquad \deg x_i = 1$

$$R = \left\langle \sum_{r \in \mathbb{Z}_n} \frac{1}{\theta_{j-i-r}(-\eta)\,\theta_r(\eta)}\, x_{j-r}\, x_{i+r}, \quad i \neq j \in \mathbb{Z}_n \right\rangle$$



$\theta_0 \qquad \theta_1 \qquad \theta_2$

$T: x_i \mapsto x_{i+1}$

$S: x_i \mapsto \varepsilon^i x_i$

$\varepsilon^n = 1$

$$\begin{array}{c} \mathrm{Heis}_n = \mathbb{Z}_n \ltimes \mathbb{Z}_n^2 \\ \cap \\ \mathrm{Aut}\, Q_n(E,\eta) \end{array}$$

Corresponding Poisson structure on, e.g., $\mathbb{CP}^3$:

$$q_1, q_2 \in H^0(\mathbb{P}^3, \mathcal{O}(2))$$

$$\sigma = q_1\, j^1(q_2) - q_2\, j^1(q_1) \quad \text{Wronskian}$$

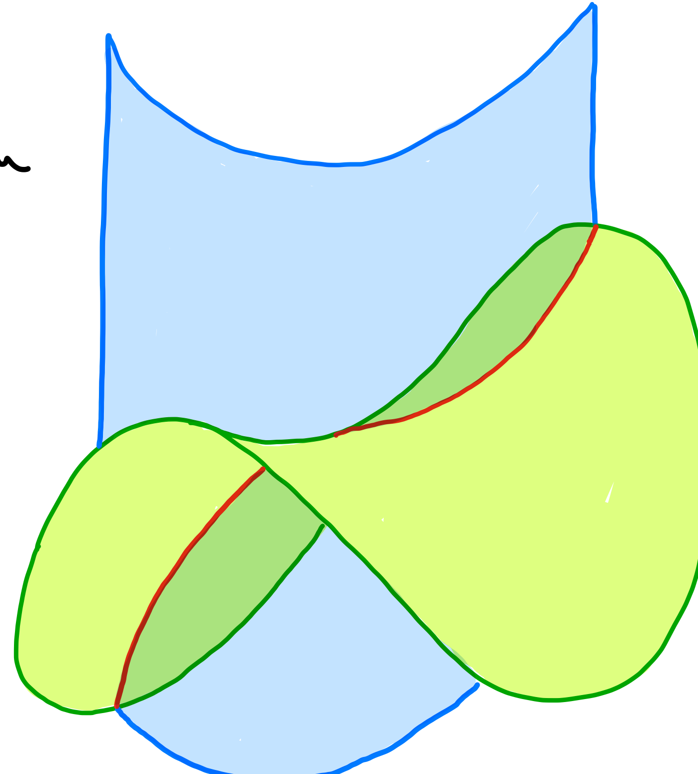
$$\sigma \in H^0(\mathbb{P}^3, T^* \otimes \mathcal{O}(2) \otimes \mathcal{O}(2))$$

$$\begin{aligned} &\parallel \\ T^* &\otimes \Lambda^3 T \\ &\parallel \\ &\Lambda^2 T \end{aligned}$$

Symplectic leaves: quadric pencil

$$\lambda q_1 + \mu q_2 = 0$$

$$[\lambda : \mu] \in \mathbb{P}^1$$



Homogeneous coordinate rings

X projective variety

L ample line bundle $\qquad H^q(X, \mathcal{F} \otimes L^k) = 0$

$q > 0 \quad k \gg 0 \quad \mathcal{F} \in \mathrm{Coh}(X)$

$$A(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k)$$

$f, g \in A^k, A^l \qquad f \cdot g = f \otimes g$

$\mathrm{Coh}(X) \cong A\text{-}\mathrm{Gr}/\mathrm{Tors} \qquad$ (Serre '55)

Twisted Homogeneous coordinate rings

X projective variety, $\eta \in \mathrm{Aut}\, X$

$\mathcal{L}$ η-ample line bundle $\quad H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \eta^*(\mathcal{L}) \otimes \eta^{2*}(\mathcal{L}) \otimes \cdots \otimes \eta^{k-1*}(\mathcal{L})) = 0$

$\quad q > 0 \qquad k \gg 0 \qquad \mathcal{F} \in \mathrm{Coh}(X)$

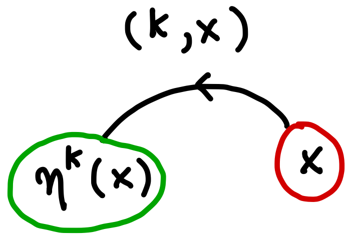
$$A(X, \mathcal{L}, \eta) = H^0(X, \mathcal{O} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \eta^*(\mathcal{L})) \oplus (\mathcal{L} \otimes \eta^*(\mathcal{L}) \otimes \eta^{2*}(\mathcal{L})) \oplus \cdots)$$

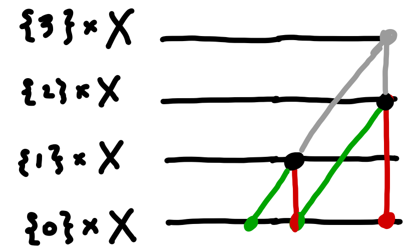
$f, g \in A^k, A^l \qquad f * g := f \otimes \eta^{k*}(g)$

$\mathrm{Coh}(X) \cong A\text{-}\mathrm{Gr}/\mathrm{Tors} \qquad$ (Artin – Van den Bergh '90)

$A(X, L, \eta)$ via Lie category

$$\begin{array}{rcc} \mathcal{C}^1 & = & \mathbb{N} \times X \\ & & {\color{green}t}\downarrow \ \ \downarrow{\color{red}s} \\ \mathcal{C}^0 & = & X \end{array}$$

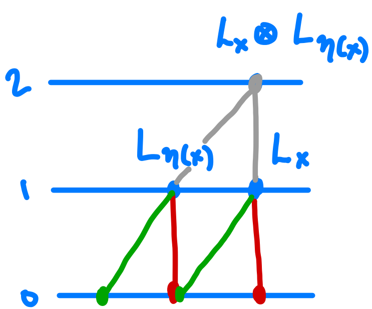




$$m : \mathcal{C}^1 \times_{\mathcal{C}^0} \mathcal{C}^1 \longrightarrow \mathcal{C}^1$$

monoidal sheaf

$$\begin{array}{ccccccccc} \mathcal{L} & = & \mathcal{O} & & L & & L \otimes \eta^*(L) & & L \otimes \eta^*(L) \otimes \cdots \otimes \eta^{k-1\,*}(L) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}^1 & = & \{0\} \times X & & \{1\} \times X & & \{2\} \times X & \ldots & \{k\} \times X \end{array}$$



$\lambda \in \mathrm{Hom}_{\mathcal{C}^1}\left(m_*(\mathcal{L} \boxtimes \mathcal{L}), \mathcal{L} \right)$ associative

Convolution for Lie categories

(joint with Thomas Stanley)

1. Lie category $\mathcal{C}^1 \rightrightarrows \mathcal{C}^0$, $m : \mathcal{C}^1 \times_{\mathcal{C}^0} \mathcal{C}^1 \longrightarrow \mathcal{C}^1$ proper

$(\mathrm{Coh}(\mathcal{C}^1), *)$ monoidal category $\qquad \mathcal{E} \circledast \mathcal{F} = m_*(\mathcal{E} \boxtimes \mathcal{F})$

2. Monoidal sheaf $\mathcal{L} \to \mathcal{C}^1$ $\qquad \lambda : \mathcal{L} \circledast \mathcal{L} \longrightarrow \mathcal{L}$ associative

$A(\mathcal{C}, \mathcal{L}) = H^0(\mathcal{C}^1, \mathcal{L})$ is an associative algebra.

Symmetric monoid of a Riemann surface C

$$\mathcal{C}^1 = \coprod_{k\geq 0} C^{[k]} \qquad C^{[i]} \times C^{[j]} \xrightarrow{m_{ij}} C^{[i+j]} \qquad \text{branched cover}$$

$$\downarrow\downarrow \qquad\qquad (X\,,\,Y) \longmapsto X \amalg Y$$

$$\mathcal{C}^0 = \{*\}$$

R_{ij} = ramification locus

$L \in \mathrm{Pic}^n C \Rightarrow$ monoidal sheaf $L^{[k]} \to C^{[k]}$

$$\lambda = \sum_k \sum_{i+j=k} \lambda_{ij} \in \mathrm{Hom}_{C^{[k]}}\left(m_*\left(L^{[i]} \boxtimes L^{[j]}\right),\ L^{[i+j]} \right)$$

$$\shortparallel$$

$$\mathrm{Hom}_{C^{[i]}\times C^{[j]}}\left(L^{[i]} \boxtimes L^{[j]},\ m^* L^{[i+j]}(R_{ij}) \right)$$

$$\lambda_{ij} := \det \mathcal{D} m_{ij}$$

$$\mathbb{CP}^N$$
$$\cong$$

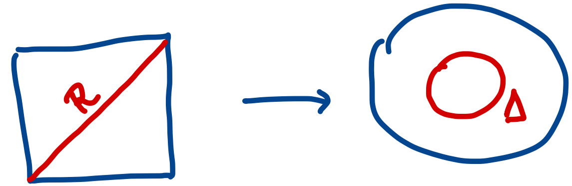
$$A(\mathcal{C}, L) = \bigoplus_{k\geq 0} H^0\left(C^{[k]}, L^{[k]}\right) = \bigoplus_{k\geq 0} \mathrm{Sym}^\bullet H^0(C, L) \Rightarrow \mathbb{P}\left(H^0(C,L)^\vee\right)$$

Symmetric monoid of $\mathbb{CP}^1$

$(C = \mathbb{P}^1, \quad C^{[k]} = \mathbb{P}^k)$

$$\begin{array}{l} \mathcal{C}^1 = \coprod_{k\geq 0} \mathbb{P}^k \\ \downdownarrows \\ \mathcal{C}^0 = \{*\} \end{array}$$

$$m_{i,j} : \mathbb{P}^i \times \mathbb{P}^j \longrightarrow \mathbb{P}^{i+j}$$



$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

Ramification divisor $R_{i,j} \subset \mathbb{P}^i \times \mathbb{P}^j$

$=$ zero set of $\mathrm{Res}_{i,j} := \det(Dm_{i,j})$

Resultant

Symmetric monoid of $\mathbb{CP}^1$

$$\begin{aligned} \mathcal{C}^1 &= \coprod_{k\geq 0} \mathbb{P}^k \\ \downarrow\downarrow & \\ \mathcal{C}^0 &= \{*\} \end{aligned}$$

monoidal sheaf $\mathcal{L} = \coprod_{k\geq 0} \left(\begin{array}{c} \mathcal{O}(n) \\ \downarrow \\ \mathbb{P}^k \end{array} \right)$

$$\lambda \in \mathrm{Hom}_{\mathcal{C}^1}(\mathcal{L} \circledast \mathcal{L}, \mathcal{L})$$

$$\lambda = \sum_{k\geq 0} \sum_{i+j=k} \lambda_{i,j}$$

$$\lambda_{i,j} = \mathrm{Res}_{i,j}$$

$$A(\mathbb{P}^\bullet, \mathcal{O}(n)) = \mathrm{Sym}^\bullet H^0(\mathbb{P}^1, \mathcal{O}(n)) \cong \mathbb{C}[x_0, \dots, x_n]$$

Twisted symmetric monoid: E elliptic curve $\eta \in \mathrm{Aut}^\circ E$

$$E^{[i]} \times E^{[j]} \xrightarrow{m^{\eta}_{ij}} E^{[i+j]}$$

$$(X, Y) \longmapsto \eta^{-j}(X) \amalg \eta^{i}(Y)$$

Monoidal line bundle: $L \in \mathrm{Pic}^{n+1} E$ $\quad L^{[k]} \to E^{[k]}$

$$\lambda = \Sigma \lambda_{ij} \in \mathrm{Hom}_{E^{[k]}}\left(m^{\eta}_{ij*}(L^{[i]} \boxtimes L^{[j]}), L^{[k]} \right)$$

e.g. $\lambda_{11} \in \mathrm{Hom}_{E^{[2]}}\left(m^{\eta}_{11*}(L \boxtimes L), L^{[2]} \right) =$

$$\mathrm{Hom}_{E\times E}\left(L \boxtimes L, m^{\eta *}_{11}(L^{[2]})(R = \Gamma_{\eta^{-2}}) \right) =$$

$$H^0\left(E\times E, \left(L^{-1}\eta^{-1*}L \boxtimes L^{-1}\eta^{*}L\right)(\Gamma_{\eta^{-2}})\right) =$$

$$H^0(E\times E, \Gamma_{\eta^{-(n+2)}})$$

Theorem (M.G., Thomas Stanley): E elliptic curve, $\eta \in \mathrm{Aut}^0(E)$, $L \in \mathrm{Pic}^n E$

1. $L^{[\cdot]} \longrightarrow (E^{[\cdot]}, m_\eta)$ has a canonical monoidal structure.

2. $A(E^{[\cdot]}, m_\eta, L^{[\cdot]}, \lambda) \cong Q_n(E, \eta)$ Coincides with Sklyanin algebra.

3. If L is η-equivariant, then

$$A(C^{[\cdot]}, m_\eta, L^{[\cdot]}, \lambda) = A(\mathbb{P}(H^0(C,L)^\vee), \eta, \mathcal{O}(1))$$

Monoid algebra $\cong$ Twisted coordinate ring.

Many thanks

and

Happy birthday,

Nigel!