Three papers of Nigel Hitchin on Higgs bundles

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Trends in Differential and Complex Geometry:
a celebration of Nigel Hitchin's 80th birthday
University of Oxford
31 August – 4 September 2026

Ayuda CEX2023-001347-S financiada por:
MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
AGENCIA ESTATAL DE INVESTIGACIÓN
CSIC Consejo Superior de Investigaciones Científicas
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EXCELENCIA SEVERO OCHOA

Nigel and author, Oxford, August 1991

Nigel, Sir Michael, author and his students, Madrid, 2006

Nigel, author and his students, Madrid, 2022

Nigel and his students, Restaurante Jai Alai, Madrid, 2016

Nigel and Nedda, Restaurante Jai Alai, Madrid, 2016

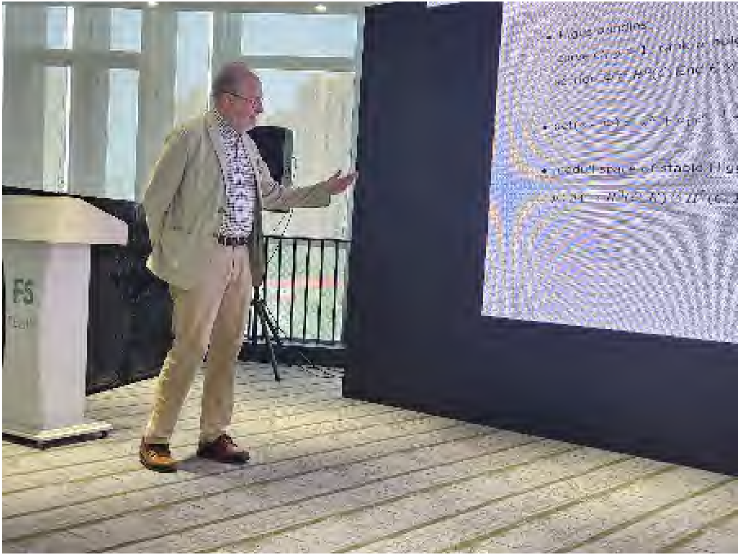
Conference for Nigel's 70th birthday, ICMAT, Madrid, 2016

ICMAT Labs associated to Nigel

- First ICMAT Hitchin Lab (2012–2015).
- ICMAT Donaldson–Hitchin Lab (2016–2019).
- ICMAT Hitchin–Ngô Labs (2020–2023 & 2024–2028)

Nigel, Châu and author, ICMAT, Madrid 2023

Hitchin–Ngô Lab at Feishu, Shanghai (2025–)



Nigel at Feishu, Shanghai, 2026

Hitchin–Ngô Lab at Feishu, Shanghai (2025–)

Nigel at USTC, Shanghai, 2026

Workshop of the first Hitchin Lab, Miraflores, March 2013

Workshop of the Donaldson–Hitchin Lab, ICMAT, 2018

Workshop of the Hitchin-Ngô Lab, ICMAT, 2023

The three papers are:

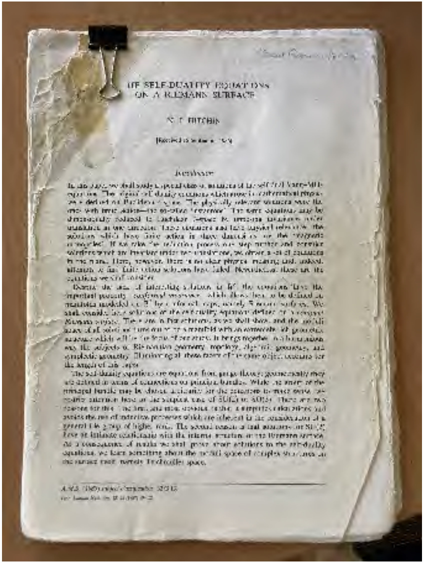
[1] N.J. Hitchin, The self duality equations on a Riemann surface, *Proc. London Math. Soc*, **55** (3) (1987), 59–126.

[2] N.J. Hitchin, Stable bundles and integrable systems, *Duke Math. J.*, **54** (1987), 91–114.

[3] N.J. Hitchin, Lie groups and Teichmüller space, *Topology* **31** (1992), 449–473.

I will certainly mention **other papers** by Nigel on Higgs bundles.

Nigel's PLMS paper on Higgs bundles, 1987



[1] N.J. Hitchin, The self-duality equations on a Riemann surface, Proc. LMS 55 (1987), 59-126.

Higgs bundles and Hitchin's self-duality equations

- X: **compact Riemann surface** of genus $g \geq 2$ with canonical line bundle K
- G: **reductive complex Lie group** with Lie algebra $\mathfrak{g}$
- A G-**Higgs bundle** on X is a pair (E, Φ) consisting of a holomorphic principal G-bundle $E \to X$ and $\Phi \in H^0(X, \mathrm{ad}(E) \otimes K)$ — the **Higgs field**.

Hitchin's self-duality equations

Let (E, Φ) be a G-Higgs bundle. **Hitchin's self-duality equations** for a reduction h of the structure group of E from G to a maximal compact subgroup $H < G$ are given by

$$F_A + [\Phi, \Phi^*] = 0.$$

Here F_A is the curvature of the unique H-connection A compatible with the holomorphic structure of E.

- A $\mathrm{GL}(n, \mathbb{C})$-**Higgs bundle** is equivalent to a pair (V, Φ), where V is rank n **holomorphic vector bundle** and

$$\Phi : V \to V \otimes K.$$

- For $G = \mathrm{SL}(n, \mathbb{C})$ we add the conditions $\det V = \mathcal{O}$ and $\mathrm{Tr}\, \Phi = 0$.
- In [1], Nigel focuses on the **rank 2 case**. He defines **stability** of (V, Φ) and proves the following.

Theorem (Hitchin [1])

An indecomposable rank 2 Higgs bundle (V, Φ) admits a solution to Hitchin's equations if and only if it is stable.

- He then computes the **Betti numbers** of the moduli space of stable Higgs bundles of **rank** 2 and **degree** 1.

Hitchin's functional

- Let $\mathcal{M}$ be the moduli space of stable **rank = 2** and **degree = 1** Higgs bundles.
- In [1] Nigel defines the functional $f : \mathcal{M} \to \mathbb{R}$ given by

$$f(V, \Phi) = \|\Phi\|^2_{L^2}$$

  where the L^2-norm of Φ is computed using the metric on V solving the Hitchin's equations.
- The functional f is a proper **Bott–Morse function** and Nigel computes the **Betti numbers** of $\mathcal{M}$ localizing at the **critical subvarieties** of f.
- Nigel shows that the critical subvarities of f are the **fixed points** of the action of $\mathbb{C}^*$ on $\mathcal{M}$ given by

$$\lambda \cdot (V, \Phi) := (V, \lambda\Phi) \text{ for } (V, \Phi) \in \mathcal{M} \text{ and } \lambda \in \mathbb{C}^*.$$

The fixed points are:

- **Type 1**: $(V, \Phi) \in \mathcal{M}$ such that $\Phi = 0$. This subvariety is isomorphic to the **moduli space of stable vector bundles** of rank 2 and degree 1, whose Poicaré polynomial had been computed by **Newstead** (Topology, 1968).
- **Type 2**: $(V, \Phi) \in \mathcal{M}$ such that $V = L_0 \oplus L_1$, where L_0 and L_1 are **holomorphic line bundles**, and

$$\Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \tag{1}$$

where $\varphi : L_0 \to L_1 \otimes K$.
These subvarieties are basically spaces of effective divisors of certain degrees, that is, **symmetric products** of X, whose Poincaré polynomials had been computed by **Macdonald** (Topology, 1962).

CONTEMPORARY MATHEMATICS

The Lefschetz Centennial Conference

Proceedings on Algebraic Geometry

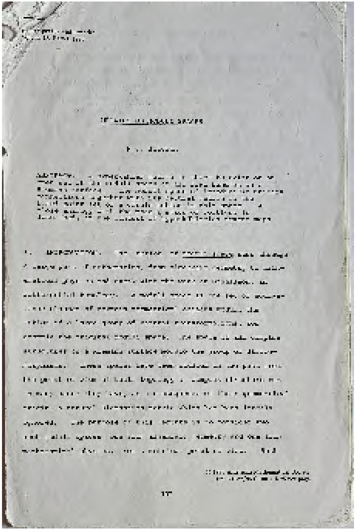
Proceedings of The Lefschetz Centennial Conference held December 10–14, 1984

AMERICAN MATHEMATICAL SOCIETY

VOLUME 58

Part I

Nigel's paper in Contemporary Mathematics, 1986



N.J. Hitchin, Metrics on moduli spaces, Contemporary Mathematics, Vol. 58, Part I (1986) 157-178 .

Nigel's paper in Contemporary Mathematics, 1986

If the flat unitary connection at which we are linearizing is irreducible, then standard deformation arguments [2, 8] show that (5.3) can be solved for nearby flat connections. We then obtain a hyperkähler metric on the $n^2(4g - 4)$ dimensional space of equivalence classes of flat connections, in a neighbourhood of the flat unitary ones.

Note that we have described only one of the complex structures of the hyperkähler metric, arising from the product complex structure on $M = \mathcal{A} \times \bar{\mathcal{A}}$. If we denote by $\partial_1 + \bar{\partial}_1$ the unitary connection corresponding to $\bar{\partial}_1 \in \mathcal{A}$ and $\partial_2 + \bar{\partial}_2$ the unitary connection compatible with $\partial_2 \in \bar{\mathcal{A}}$, then the isomorphism

$$(\bar{\partial}_1, \partial_2) \to (\tfrac{1}{2}(\bar{\partial}_1 + \bar{\partial}_2), \tfrac{1}{2}(\partial_2 - \partial_1))$$

from $\mathcal{A} \times \bar{\mathcal{A}}$ to $\mathcal{A} \times \Omega^{1,0}(\mathfrak{g})$ defines another constant complex structure on $M = \mathcal{A} \times \bar{\mathcal{A}}$. This time it is a product of a complex affine space with a complex vector space, so the group of complex gauge transformations act differently.

Let us put $\bar{\partial} = \frac{1}{2}(\bar{\partial}_1 + \bar{\partial}_2)$, $\phi = \frac{1}{2}(\partial_2 - \partial_1)$ then with respect to this complex structure, the moment maps are:

$$\left.\begin{aligned} \mu_1(\bar{\partial}, \phi) &= F + \tfrac{1}{2}[\phi, \phi^*] \\ \mu_2 + i\mu_3(\bar{\partial}, \phi) &= \bar{\partial}\phi \end{aligned}\right\} . \tag{5.4}$$

In this form the equations clearly admit a $U(1)$ action:

$$\phi \to e^{i\theta}\phi .$$

It is an action which only fixes this complex structure (and its conjugate) and so is not an obvious action on the space

Nigel's paper in Contemporary Mathematics, 1986

The equations (5.4) in this setting appear to have no non-trivial solutions. (We may point out that they are in fact the self-duality equations in $\mathbb{R}^4$ which are invariant under an $\mathbb{R}^2$ translation group). Thus in the moment map formalism $\mu^{-1}(0)$ may be empty.

Consider however $F_0 = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{pmatrix} \in \Omega^2(\mathfrak{g})$.

The subgroup of $SU(2)$ gauge transformations which preserve F_0 consists of $U(1)$ gauge transformations, $\mathcal{G}_0$. From the quotient construction we expect then to obtain a hyperkähler metric on

$$\mu^{-1}(F_0,0,0)/\mathcal{G}_0 .$$

This is the moduli space of solutions to the equations

$$\left.\begin{aligned} \bar\partial\phi &= 0 \\ F + \tfrac{1}{2}[\phi,\phi^*] &= F_0 \end{aligned}\right\} \qquad (6.1)$$

If we again look at the circle action $\phi \to e^{i\theta}\phi$, then at a fixed point in the moduli space the $SU(2)$ connection defined by $\bar\partial$ will reduce to a $U(1)$ connection A with curvature

$$F = \begin{pmatrix} F_A & 0 \\ 0 & -F_A \end{pmatrix}$$

and, relative to this decomposition, we must have

$$\phi = \begin{pmatrix} 0 & \phi dz \\ 0 & 0 \end{pmatrix}$$

according to the first equation of (5.5).

Nigel's paper in Contemporary Mathematics, 1986

The equations (6.1) then become

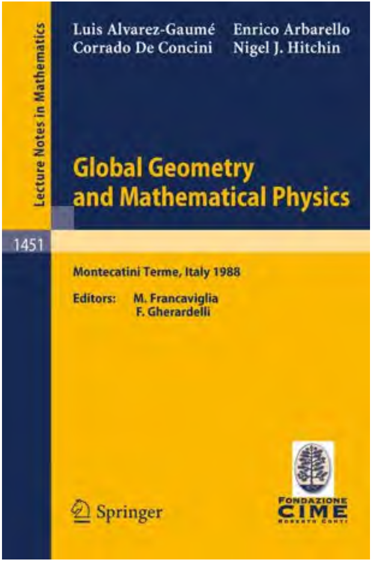
$$\left.\begin{aligned} \bar{\partial}_A \phi &= 0 \\ F_A &= \tfrac{1}{2}(|\phi|^2 - 1) \end{aligned}\right\}$$

which are the vortex equations (3.1). Here, from the work of Jaffe and Taubes, we have non-trivial solutions but as yet the appropriate elliptic estimates which will give a good deformation theory to produce solutions of the general equations (6.1) for non-abelian vortices have not been worked out.

Nevertheless, formally speaking, we expect that the moduli space of vortices should be the fixed point set of an isometric circle action and hence a <u>totally geodesic submanifold of a hyperkähler manifold</u>.

7. TWISTOR THEORY. Why, one might ask, are hyperkähler metrics so interesting? In the examples we have given the hyperkähler metric exists on a manifold twice the dimension of the original manifold we were considering and there may well be a theorem that any analytic Kähler manifold can be embedded as the fixed point set of a circle in a hyperkähler manifold. This would not say anything special about the Kähler metric itself. The answer is not so much that the <u>local</u> metric structure of moduli spaces is special, but that the hyperkähler metrics in which they sit arise in a natural way and moreover, thanks to twistor theory, have a <u>holomorphic</u> description. This raises the hope that it may be possible to go directly from holomorphic data - stable bundles on a Riemann surface - to the hyperkähler metric.

Nigel’s paper in Lecture Notes in Mathematics, 1990



Nigel's paper in Lecture Notes in Mathematics, 1990

THE GEOMETRY AND TOPOLOGY OF MODULI SPACES

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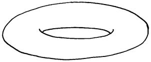
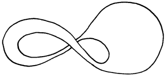
1. What is a moduli space?

To obtain an idea of what a moduli space is, we can go back to the original example of the space of moduli of elliptic curves (or tori). Suppose we ask ourselves the question: "What is the set of all conformal structures on a 2-dimensional torus?" Immediately our mental image is of a torus of revolution:



However, a moment's thought tells us that applying any diffeomorphism to the torus gives us another conformal structure:



This observation forces us to modify the original question to "What is the set of equivalence classes of conformal structures under the notion of equivalence by diffeomorphism?", or alternatively "What are the orbits of the group of diffeomorphisms acting on the space of conformal structures?"

N.J. Hitchin, The geometry and topology of moduli spaces, in ''Global geometry and mathematical physics (Montecatini Terme, 1988)'', LNM, 1451, Springer, Berlin, (1990), 1-48.

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to give

$$|d_A\phi|^2 = *(d'_A d''_A\phi,\phi) + *2(F_A\phi,\phi) + |d'_A\phi|^2 + *d(d'_A\phi,\phi)$$

$$= 2|d''_A\phi|^2 + 2*F_A|\phi|^2 + *d\theta \qquad (12.1)$$

where $\int_{\mathbb{R}^2} *d\phi$ can be evaluated in terms of the asymptotic data by Stokes' theorem.

Using this rearrangement of terms we write

$$|F_A|^2 + (1-|\phi|^2)^2 = |F_A - *(1-|\phi|^2)|^2 + 2(F_A,*(1-|\phi|^2))$$

$$= |F_A - *(1-|\phi|^2)|^2 + 2*F_A - 2*F_A|\phi|^2 .$$

Evaluating the integrals of both $*d\theta$ and $*F_A = *dA$ in terms of the behaviour at infinity, one finds

$$\int_{\mathbb{R}^2} |F_A|^2 + (1-|\phi|^2)^2 + |d_A\phi|^2$$

$$= \int |F_A - *(1-|\phi|^2)|^2 + 2|d''_A\phi|^2 + 2\pi k$$

and so the action is bounded below by $2\pi k$ with equality if and only if the Abelian vortex equation is satisfied:

$$\left.\begin{aligned} d''_A\phi &= 0 \\ F_A &= *(1-|\phi|^2) \end{aligned}\right\} . \qquad (12.2)$$

This is beguilingly similar to the special solution of the self-duality equations (11.3), together with the holomorphicity of the quadratic differential q, but there is an essential difference. Equation (12.2) involves $*1$, the volume 2-form of the flat metric on $\mathbb{R}^2$ and the curvature of the line bundle L. Equation (11.3) involves only one connection and metric.

The vortex equations are not particular solutions to the self-dual Yang-Mills equations in $\mathbb{R}^4$, and so we cannot expect the methods we have at our disposal there to apply. It is true that they may be interpreted as $SO(3)$-invariant solutions to the self-dual Yang-Mills equations on $S^2 \times \mathbb{R}^2$, but there again we have no special solution techniques to apply. Even the case of the rotationally symmetric 1-vortex solution is not known in finite terms.

What is known about vortices is their moduli space, which was essentially calculated by Jaffe & Taubes. What they proved was that

N.J. Hitchin, The geometry and topology of moduli spaces, in ''Global geometry and mathematical physics (Montecatini Terme, 1988)'', LNM, 1451, Springer, Berlin, (1990), 1-48.

Abelian vortices on a compact Riemann surface

- X compact Riemann surface
- ω_X Kähler form on X (normalised $\mathrm{vol}(X) = 2\pi$)
- $L \to X$ holomorphic line bundle over X
- $\varphi \in H^0(X, L)$ holomorphic section of L

Vortex equations

for a Hermitian metric h on L:

$$i\Lambda F_h + |\varphi|_h^2 - \tau = 0$$

- $F_h \in \Omega^2(X)$ curvature of Chern connection of h on L
- $\Lambda F_h \in C^\infty(X)$ contraction of F_h with ω_X
- $|\cdot|_h \in C^\infty(X)$ positive norm on L associated to h
- $\tau \in \mathbb{R}$ real parameter

- Integrating, we obtain

$$2\pi \deg L + \|\varphi\|^2_{L^2} = \tau,$$

which implies

$$\deg L \leq \tau.$$

- In my **D.Phil Thesis (Oxford 1991)** I gave two proofs (Comm. Math. Phys., 1993 and Bull. LMS, 1994) of the following.

Theorem

Existence of solutions to vortex equations $\iff$ $\deg L \leq \tau$

- **Other proofs** using different methods had been given by Noguchi (J. Math. Phys., 1987) and Bradlow (Comm. Math. Phys., 1990) .

- Pairs (L, φ) over the compact Riemann surface X are in **one-to-one** correspondence with **rank** 2 **holomorphic vector bundles** V over $X \times \mathbb{P}^1$ of the form

$$0 \to L \to V \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

- The rank 2 bundle V is SU(2)-**equivariant** for the action of SU(2) trivially on X and in the standard way on $\mathbb{P}^1$
- Consider the SU(2)-**invariant Kähler metric** on $X \times \mathbb{P}^1$ given by

$$\omega_\tau = \tau\omega_X \oplus \omega_{\mathbb{P}^1}^{FS},$$

where $\tau > 0$, and $\omega_{\mathbb{P}^1}^{FS}$ is the Fubini–Study metric.

Theorem (OGP, Comm. Math. Phys. 1993)

$$\begin{array}{ccc} \text{solution to vortex eqn. on } X & \longleftrightarrow & \deg L < \tau \\ \Updownarrow & & \Updownarrow \\ \mathrm{SU}(2)\text{-inv. } \omega_\tau\text{-HYM metric} & \xleftrightarrow{\text{Donaldson–Uhlenbeck–Yau}} & V \ \omega_\tau\text{-stable} \end{array}$$

- Hence, vortices on X are SU(2)-**dimensional reductions** of **ASD instantons** on $X \times \mathbb{P}^1$.
- This generalises results of **Witten** (Theory. Phys. Rev. Lett., 1977) for vortices on the hyperbolic real plane and **Taubes** (Comm. Math. Phys., 1980) for vortices on $\mathbb{R}^2$.
- More generally, there is a one-to-one correspondence between **triples** (V_0, V_1, φ), where V_0 and V_1 are holomorphic vector bundles over X and $\varphi : V_0 \to V_1$, and holomorphic vector bundles on $X \times \mathbb{P}^1$ given by

$$0 \to V_1 \to V \to V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

- There are **vortex equations** for hermitian metrics on V_0 and V_1, which, similarly to the abelian case, appear as **dimensional reduction of the HYM equation** (OGP, D.Phil. Thesis, 1991 and Int. J. Math., 1994).
- In the paper above and a subsequent paper joint with **Steve Bradlow** (Math. Ann., 1996), we prove **existence therems** for the vortex equations for triples in terms of a certain **stability condition depending on a real parameter** for the triple.
- SU(2)**-equivariant holomorphic vector bundles** on $X \times \mathbb{P}^1$ are in **bijective correspondence** with **chains** over X given by

$$V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} V_m$$

- There are **vortex equations** and **stability conditions** involving m real **parameters** solving the **chain vortex equations** (Álvarez-Cónsul–OGP, Int. J. Math. 2001).

Chains and Higgs bundles

Let $\mathcal{M} := \mathcal{M}(n,d)$ be **moduli space of stable Higgs bundles** of rank n and degree d, with n and d coprime. The **fixed points** of $\mathcal{M}(n,d)$ under the $\mathbb{C}^*$**-action** are given by:

- $(V,\Phi) \in \mathcal{M}$ with $\Phi = 0$: **stable vector bundles** of rank n and degree d.
- $(V,\Phi) \in \mathcal{M}$ with

$$V = V_0 \oplus \cdots \oplus V_m \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \varphi_0 & 0 & 0 & \dots & 0 \\ 0 & \varphi_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varphi_{m-1} & 0 \end{pmatrix}.$$

- These are in bijective correspondence with K**-twisted chains**

$$V_0 \xrightarrow{\varphi_0} V_1 \cdots \xrightarrow{\varphi_{m-1}} V_m$$

that is, $\varphi_i : V_i \to V_{i+1} \otimes K$ for $i = 0, \cdots, m-1$.

Moduli spaces of chains and localization

- This was exploited by **Gothen** (Int. J. Math., 1994), as part of his 1995 PhD Thesis (**supervised by Nigel**), to compute the Betti numbers of $\mathcal{M}$ for $n = 3$ applying Bott-Morse localization, and using a result of **Thaddeus** on rank 2 stable pairs (Inv. Math., 1994).
- Further study of moduli spaces of chains was carried out by **Bradow–GP–Gothen** (Math. Ann., 2004) and **Álvarez-Cónsul–GP–Schmitt** (Math. Res. Pap., 2006), culminating in the computation of the **Grothendieck motive** of arbitrary moduli spaces of chains by **G-Heinloth-Schmitt** (JEMS, 2014) and **G-Heilonth** (Duke J. Math., 2013).
- This was used in the above papers to give a **recursive formula** for the **Grothendieck motive** (and hence Betti numbers) of the moduli space $\mathcal{M}(n, d)$ of Higgs bundles of rank n and degree d with n and d coprime, by means of **Bialynicki-Birula localization**.

Non-abelian Hodge correspondence

- A very important feature of Higgs bundles on initiated by Nigel in [1] is their relation with **representations of the fundamental group** of the Riemann surface.
- Let G be a **semisimple complex Lie group**, and X be a compact Riemann surface. The G-**character variety** of the fundamental group of X is defined as the **GIT quotient**

$$\mathcal{R}(G) = \mathrm{Hom}(\pi_1(X), G) /\!\!/ G.$$

- Let $\mathcal{M}(G)$ be the moduli space of polystable G-Higgs bundles.

The non-abelian Hodge correspondence (NAHC)

establishes a **homeomorphism**

$$\mathcal{M}(G) \cong \mathcal{R}(G).$$

Non-abelian Hodge correspondence (NAHC)

- This is proved for $G = \mathrm{SL}(2, \mathbb{C})$ by Nigel in [1], combining his existence theorem for the Hitchin's equations with a result of **Donaldson** (Proc. London Math. Soc., 1987) on the existence of a **harmonic metrics** on **reductive** flat $\mathrm{SL}(2, \mathbb{C})$-bundles on X.
- For arbitrary semisimple complex Lie group G the NAHC follows combining theorems of **Simpson** (J. Amer. Math. Soc., 1988) and **Corlette** (J. Differential Geom., 1988).
- In [1] Nigel initiates also the use of Higgs bundle theory to study **character varieties** of the fundamental group of X for **real groups**.
- He fully analyses the case of $\mathrm{SL}(2, \mathbb{R})$.

$\mathrm{SL}(2,\mathbb{R})$-Higgs bundles

- In [1], Nigel shows that Higgs bundles $(V, \Phi) \in \mathcal{M}$ corresponding to representations of $\pi_1(X)$ in $\mathrm{SL}(2,\mathbb{R})$ have the form $V = L \oplus L^{-1}$, where L is a **holomorphic line bundle** and
$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{2}$$
where $\beta \in H^0(X, L^2K)$ and $\gamma \in H^0(X, L^{-2}K)$.
- The **degree** $d = \deg L$ coincides with the **Euler number** of the corresponding representation ρ, and applying **semistability** of (V, Φ), Nigel shows that
$$|d| \leq g - 1 \quad \text{where } g \text{ is the genus of } X.$$
- This was proved by **Milnor** for the Euler number of ρ (Comment. Math. Helv., 1958).

$\mathrm{SL}(2,\mathbb{R})$-Higgs bundles

- In [1], Nigel gives a explicit description of the moduli spaces of $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles for fixed d in terms of **symmetric products** of X.
- Showing in particular that this moduli space is **connected** for $|d| < g-1$, and has 2^{2g} **connected components** if $|d| = g-1$, each one isomorphic to the space $H^0(X,K^2)$ of **quadratic differentials** of X.
- He also identifies each of these components with $|d| = g-1$ with the **Teichmüller space** of the underlying smooth compact surface.
- The counting of components of the $\mathrm{SL}(2,\mathbb{R})$-character variety and the identification with Teichmüller space of each component when $|d| = g-1$ had been carried out also by **Goldman** (Inv. Math., 1988).

Higgs bundles for real forms

- Let G be a **semisimple complex Lie group**. A real subgroup $G_{\mathbb{R}} \subset G$ is called a **real form** of G if $G_{\mathbb{R}}$ is the **fixed point subgroup** G^{σ} of an **antiholomorphic involution** $\sigma : G \to G$.
- Given σ, there exists a **compact antiholomorphic involution** τ of G commuting with σ, and

$$\theta := \sigma\tau$$

  defines a **holomorphic involution** of G.
- Let $H := G^{\theta}$ the fixed point subgroup (reductive) and consider the **Cartan decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

- A $G_{\mathbb{R}}$**-Higgs bundle** on X is pair (E, φ), where $E \to X$ is a holomorphic principal H-bundle on X and $\varphi \in H^0(X, E \times_H \mathfrak{m} \otimes K)$.

NAHC for real forms

- We define the $G_{\mathbb{R}}$**-character variety** of $\pi_1(X)$ as

$$\mathcal{R}(G_{\mathbb{R}}) := \mathrm{Hom}^{\mathrm{reductive}}(\pi_1(X), G_{\mathbb{R}})/G_{\mathbb{R}}.$$

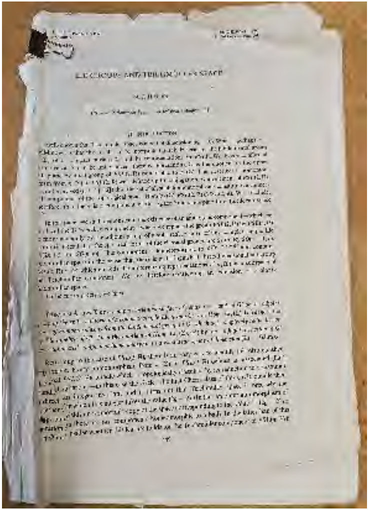
- And consider the **moduli space** $\mathcal{M}(G_{\mathbb{R}})$ **of polystable** $G_{\mathbb{R}}$**-Higgs bundles**.
- The Non-abelian Hodge correspondence can be extended to this situation by applying **Corlette** (1988) and a indirect application of **Simpson** (1988) or a direct approach by **GP–Gothen–Mundet i Riera** (2009), to obtain

Non-abelian Hodge correspondence

There is a homeomorphism

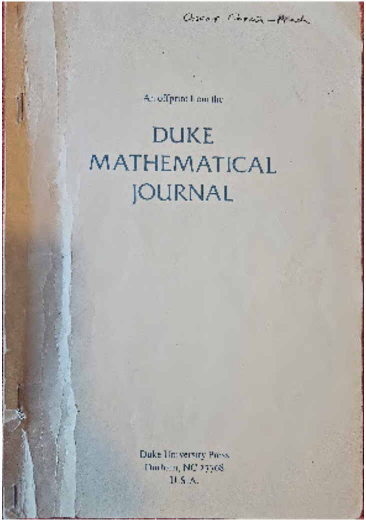
$$\mathcal{M}(G_{\mathbb{R}}) \cong \mathcal{R}(G_{\mathbb{R}}).$$

Nigel's Topology paper on Higgs bundles, 1992



[3] N.J. Hitchin, Lie groups and Teichmüller space,Topology 31 no. 3 (1992), 449-473.

Nigel's Duke paper on Higgs bundles, 1987



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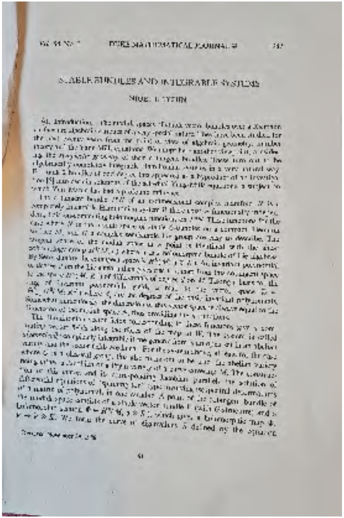
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STABLE BUNDLES AND INTEGRABLE SYSTEMS

[2] N.J. Hitchin, Stable bundles and integrable systems, Duke Math. J. 54 no. 1 (1987), 91–114.

Hitchin map and split real forms

- Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$. And let $\mathcal{M}(G)$ be the **moduli space of G-Higgs bundles**.
- In the **Duke paper [2]**, Nigel introduces a map

$$p : \mathcal{M}(G) \to \bigoplus_{i=1}^{r} H^0(X, K^{d_i}),$$

  defined for (E, Φ) by evaluating a basis of the algebra of G-**invariant polynomials** of $\mathfrak{g}$ on $\Phi \in H^0(X, \mathrm{Ad}(E) \otimes K)$.
- Here $r = \operatorname{rank} G$ and $\{d_1, \cdots, d_r\}$ are the **exponents** of G.
- In the **Topology paper [3]**, Nigel constructs a **section** of this map, and shows that the image is a **connected component** of the moduli space of Higgs bundles for a **split real form** of G (**Hitchin component**).

Hitchin components

- Nigel's construction uses a **principal embedding** of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{g}$ and the so-called **Kostant section**.
- For $G = \mathrm{SL}(2,\mathbb{C})$ the Hitchin map was already defined in [1]. A Hitchin section is determined by choosing a square root of K, and the Hitchin components are the moduli spaces with $|d| = g-1$ identified with **Teichmüller space**.
- Every complex semisimple Lie group has a split real form **Examples**: $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n)$, $\mathrm{SO}(n,n+1)$
- **Labourie** (Inv. Math., 2006) introduces the concept of **Anosov representation**, and shows that the representations in the Hitchin component are **discrete and faithful**. This is exactly the property that characterizes the representations of $\pi_1(X)$ in $\mathrm{SL}(2,\mathbb{R})$ in the Teichmüller components: **Fuchsian representations**.
- This is the beginning of **higher rank Teichmüller theory**!!!

Higgs bundles for Hermitian real forms

- Let $G_{\mathbb{R}} \subset G$ be **Hermitian real form**. This means that, if $H_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a maximal compact subgroup, $G_{\mathbb{R}}/H_{\mathbb{R}}$ is a **Hermitian symmetric space**.
- **Classical** simple groups of Hermitian type: $\mathrm{SU}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}(2,n)$. Two **exceptional** real forms E_6^{-14} and E_7^{-25}.
- Like for $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$, there is an integral invariant τ similar to the Euler number, called the **Toledo invariant**, and the **Milnor–Wood inequality**:

Theorem Milnor–Wood inequality

Let (E,φ) be a **semistable** $G_{\mathbb{R}}$-Higgs bundle with Toledo invariant τ. Then

$$|\tau| \leq \operatorname{rank}(G_{\mathbb{R}}/H_{\mathbb{R}})(2g-2).$$

- This is proved for the classical groups by **Bradlow–GP–Gothen** (J. Differential Geom., 2003), and **GP–Gothen–Mundet i Riera** (J. Topology, 2013); and in general by **Biquard–GP–Rubio** (J. Topology, 2017)). Special attention is given in these papers to **maximal Toledo components**, that is, the subvariety $\mathcal{M}_{\max}(G_{\mathbb{R}}) \subset \mathcal{M}(G_{\mathbb{R}})$ with $\tau = \operatorname{rank}(G_{\mathbb{R}}/H_{\mathbb{R}})(2g-2)$.

Theorem (Cayley Correspondence)

Let $G_{\mathbb{R}}$ be such that $G_{\mathbb{R}}/H_{\mathbb{R}}$ is a Hermitian symmetric space of **tube type**. Then

$$\mathcal{M}_{\max}(G_{\mathbb{R}}) \cong \mathcal{M}_{K^2}(G'_{\mathbb{R}}),$$

where $G'_{\mathbb{R}}$ is a real form $H_{\mathbb{R}}^{\mathbb{C}}$ related to the **Shilov boundary** of $G_{\mathbb{R}}/H_{\mathbb{R}}$ and $\mathcal{M}_{K^2}(G'_{\mathbb{R}})$ is the moduli space of K^2-twisted $G'_{\mathbb{R}}$-Higgs bundles.

Search for other higher Teichmüller spaces

- Work of **Burger–Iozzi–Wienhard** (2003, 2010), and **Burger–Iozzi–Labourie–Wienhard** (2006) shows that the components of $\mathcal{R}(G_{\mathbb{R}})$ corresponding to the maximal Toledo components are higher Teichmüller components.
- **Other higher Teichmüller spaces** emerged from a generalization of the **Cayley correspondence** for the the moduli spaces of $\mathrm{SO}(p,q)$-Higgs bundles in

  M. Aparicio-Arroyo, S. Bradow, B. Collier, O. García-Prada, P.P. Gothen and A. Oliveira, $\mathrm{SO}(p,q)$-Higgs bundles and higher Teichmüller components, Inv. Math. 218 (2019) 197-299.
- And a construction of a general Cayley correspondence in

  S. Bradlow, B. Collier, O. García-Prada, P.B. Gothen and A. Oliveira, A general Cayley correspondence and higher rank Teichmüller spaces, Annals of Math. 200 (2024) 803-892

  giving a Higgs bundle parametrization of simmingly all higher Teichmüller spaces (thanks to work on **positive representations** by **Beyrer, Guichard, Labourie, Pozzetti & Wienhard**, 2021, 2024).

- In [BBGGO] we show that there are some special $\mathfrak{sl}_2$-**triples** of $\mathfrak{g}$ (which we call **magical**) that canonically define real forms $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$. For these there are **integers** $m_1, \cdots, m_c, \cdots, m_N$, and a real form $\mathfrak{g}'_{\mathbb{R}}$ of a certain Lie subalgebra of $\mathfrak{g}$, for which we have the following.

- In [BBGGO] we show that there are some special $\mathfrak{sl}_2$-**triples** of $\mathfrak{g}$ (which we call **magical**) that canonically

Theorem (BCGGO, Annals of Math. 2024)

Consider be a magical $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, with corresponding canonical real form $\mathfrak{g}_{\mathbb{R}}$, and let $G'_{\mathbb{R}}$ be a real Lie group with Lie algebra $\mathfrak{g}'_{\mathbb{R}}$. If $G_{\mathbb{R}}$ is a Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, then there is a map, called **Cayley map**

$$\Psi : \mathcal{M}_{K^{m_c}}(G'_{\mathbb{R}}) \times \bigoplus_{i=1,\, i\neq c}^{N} H^0(K^{m_i}) \hookrightarrow \mathcal{M}(G_{\mathbb{R}}),$$

so that the image $\mathcal{H}_e(G_{\mathbb{R}}) := \operatorname{Im}\Psi$ is **open** and **closed** in $\mathcal{M}(G_{\mathbb{R}})$, and hence consists of a union of connected components of $\mathcal{M}(G_{\mathbb{R}})$, called **Cayley components**.

The classification of magical $\mathfrak{sl}_2$-triples coincides with the classification of positivity structures given by **Guichard–Wienhard** (2018).

Real forms associated to magical $\mathfrak{sl}_2$-triples

Theorem (BCGGO, Annals of Math. 2024)

Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$ be a real form. Then $\mathfrak{g}_{\mathbb{R}}$ is the **canonical real form** associated to a magical $\mathfrak{sl}_2$-triple if and only if it is one of the following:

1. $\mathfrak{g}$ is any type and $\mathfrak{g}_{\mathbb{R}}$ is its **split real form**;
2. $\mathfrak{g}$ has type A_{2n-1}, B_n, C_n, D_n, D_{2n}, or E_7 and $\mathfrak{g}_{\mathbb{R}}$ is **Hermitian of tube type**, i.e. $\mathfrak{g}_{\mathbb{R}}$ is one of the following:
   - $\mathfrak{su}_{n,n}$,
   - $\mathfrak{so}_{2,p}$ (with $2 + p = 2n + 1$),
   - $\mathfrak{sp}_{2n}\mathbb{R}$,
   - $\mathfrak{so}_{2,p}$ (with $2 + p = n$),
   - $\mathfrak{so}^*_{4n}$, or
   - the real form of E_7 of Hermitian type;
3. $\mathfrak{g}$ has type B_n or D_n and $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{so}_{p,q}$ with $1 < p < q$ and $p + q = 2n + 1$ or $p + q = 2n$ respectively;
4. $\mathfrak{g}$ has type E_6, E_7, E_8 or F_4 and $\mathfrak{g}_{\mathbb{R}}$ is its **quaternionic real form**.

Back to $\mathbb{C}^*$-fixed points: G-Hodge bundles

- Recall that is an **action** of $\mathbb{C}^*$ on the moduli space of G-Higgs bundles given for $\lambda \in \mathbb{C}^*$ by

$$\lambda \cdot (E, \Phi) = (E, \lambda\Phi).$$

  A G-Higgs bundle is G-**Hodge bundle** if it is a **fixed point** under this action.

- If (E, Φ) is a G-Hodge bundle, there is $\mathbb{Z}$-**grading**

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \quad \text{where} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

  so that E reduces to a G_0-bundle E_{G_0} and $\varphi \in H^0(X, E_{G_0} \times_{G_0} \mathfrak{g}_1) \otimes K)$ (**Simpson**).

- Via de non-abelian Hodge correspondence, G-Hodge bundles are in bijection with **variations of Hodge structure**.

G-Hodge bundles

In O. Biquard, B. Collier, O. García-Prada, and D. Toledo, Arakelov–Milnor inequalities and maximal variations of Hodge structure, Compositio Mathematica 159 (2023) 1005–1041.

- We initiate a **systematic study** of the moduli spaces of G-Hodge bundles exploting, in particular, the fact that $\mathfrak{g}_1$ is **prehomogeneous vector space** for the action of G_0.
- We define a **Toledo invariant** for a G-Hodge bundle of any type, and prove an analogue of the Milnor–Wood inequality (**Arakelov–Milnor inequality**).
- Prove a **Cayley correspondence** and **rigidity results** when the Toledo invariant is maximal.

Jointly with **Brian Collier** and **Jochen Heinloth** we are studying the topology of moduli spaces of G-Hodge bundles, in particular, to extend the **localization programme initiated by Nigel** in [1] for $G = \mathrm{SL}(2, \mathbb{C})$ to an arbitrary semisimple complex Lie group G.

Cyclic G-Higgs bundles

- Let θ be an **automorphism** of G of order m and let ζ be an m-th **root of unity**. Consider the action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathcal{M}(G)$ given by

$$(E, \Phi) \mapsto (\theta(E), \zeta\theta(\Phi)).$$

- A G-Higgs bundle $(E, \Phi) \in \mathcal{M}(G)$ is called **cyclic** (more precisely θ-**cyclic**) if it is a **fixed point** under this action.
- Cyclic G-Higgs bundles have been studied by many authors with various different motivations.
- I have studied cyclic G-Higgs bundles with **S. Ramanan** in several papers since the early 2000s (**Hitchin for real!**). In particular, in Proc. London Math. Soc., 2019., we show that if (E, Φ) is a θ-cyclic G-Higgs bundle there is a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$ so that E reduces to a G^θ-bundle E_0 and $\Phi \in H^0(X, E_0 \times_{G^\theta} \mathfrak{g}_1)$.

Cyclic G-Higgs bundles and Vinberg θ-pairs

- The pairs $(G^\theta, \mathfrak{g}_1)$ are called **Vinberg θ-pairs**, and hence we have to study moduli spaces of $(G^\theta, \mathfrak{g}_1)$-Higgs pairs $\mathcal{M}(G^\theta, \mathfrak{g}_1)$ (recent paper with **Miguel González**).
- A **main result** of Vinberg's theory (1976) is a **version of the Chevalley restriction theorem** for Vinberg θ-pairs. There is a **Cartan subspace** $\mathfrak{a} \subset \mathfrak{g}_1$ and a **little Weyl group** $W(\mathfrak{a})$ so that $\mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$ is a **polynomial ring**, and for which we have an **isomorphism of invariant polynomial rings** $\mathbb{C}[\mathfrak{g}_1]^{G^\theta} \to \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$,
- One has then **Hitchin-type** maps
$$h : \mathcal{M}(G^\theta, \mathfrak{g}_1) \to \bigoplus_{i=1}^r H^0(X, K^{d_i}).$$
- The case $m = 2$ corresponds to Higgs bundles for real forms: Peón-Nieto (2013), Schaposnik (2013), Hitchin–Schaposnik (2014), GP–Peón-Nieto–Ramanan (2017), GP–Peón-Nieto (2021), Hameister–Morrissey (2024).
- Basically **nothing is known** for $m > 2$...

Nigel, Peter Higgs, Peter Gothen and author, Edinburgh, 2009

HAPPY BIRTHDAY, NIGEL!