Abstract

Gravitational instantons are defined as non-compact non-flat complete hyperkähler 4-manifolds with L^2 curvature decay. They have been recently classified and all arise as bubbling limits of K3 surfaces. The Modularization Conjecture posits that any gravitational instanton arises as a moduli space of gauge-theoretic equations. In this talk, I'll focus on a special kind of gravitational instanton: ALG-D_4 gravitational instantons. These can be conjecturally realized as moduli spaces of Hitchin's equations, a system of gauge-theoretic equations on a Riemann surface that is recognized as a central object in mathematics. In joint work 2603.17020 and ongoing work with Rafe Mazzeo, Jan Swoboda, and Hartmut Weiss, we prove the modularization conjecture in this case.

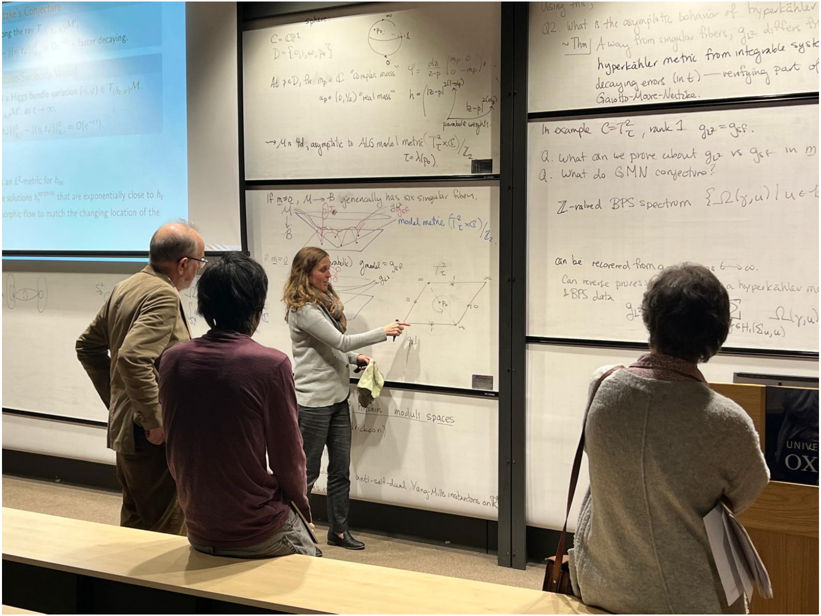
ALG$^{(*)}$ Gravitational Instantons vs. 4d Hitchin moduli spaces

Laura Fredrickson

University of Oregon

Celebration of Nigel Hitchin's 80th Birthday, Oxford, September 2026

Happy Birthday, Nigel!



Oxford, March 2023.

Fundamental Question for Talk:

There are two families of noncompact complete hyperkähler 4-manifolds

- Gravitational instantons of type ALG$^{(*)}$
- 4d Hitchin moduli spaces

How are these fundamental objects related?

Based on:

- 2001.03682, **2603.17020** (108pp) & work-in-progress with R. Mazzeo, J. Swoboda, H. Weiss
- 2407.16798 with B. Collier, R. Wentworth
- 2601.10656 with A. Yae

Gravitational Instantons

Definition

A *gravitational instanton* is a non-compact complete non-flat Riemannian four-manifold with holonomy SU(2) satisfying $\int_M |\mathrm{Riem}|^2 < \infty$.

They automatically have a hyperkähler structure $(M, g, I, J, K, \omega_I, \omega_J, \omega_K)$.

Hyperkähler manifolds are **very rigid!**

Classification of compact 4d hyperkähler manifolds

Compact 4d hyperkähler manifolds are either (1) flat T^4 or (2) $K3$.
For $K3$, $H_2(M, \mathbb{Z}) \simeq \mathbb{Z}^{22}$. The classes $[\omega_I], [\omega_J], [\omega_K]$ encodes the integral of $\omega_\bullet$ over 2-cycles, e.g. $(\subset (\mathbb{R}^3)^{22})$. The famous Torelli Theorem roughly states that M is determined by $[\omega_I], [\omega_J], [\omega_K]$.

Gravitational instantons

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Hyperkähler manifolds are **very rigid!**

Theorem [S. Sun- R. Zhang]

Classification is into 6 types: ALE, ALF, ALG, ALG*, ALH*, ALH.

Integrable Riemannian curvature implies that $|\mathrm{Riem}| = O(r^{-2})$ (Cheeger-Tian). If curvature decay is <u>faster</u> than quadratic, volume growth is r^4, r^3, r^2 or r^1. (G. Chen- X. Chen) Rigidity!

Similarly, Torelli Theorem implies that M is determined by $[\omega_I], [\omega_J], [\omega_K]$. (G. Chen-Viaclovsky-Zhang) Rigidity!

Gravitational instantons

Definition

A *gravitational instanton* is a non-compact complete non-flat Riemannian four-manifold with holonomy SU(2) satisfying $\int_M |\mathrm{Riem}| < \infty$.

For ALE/ALF/ALG/ALH, classification is based on asymptotic volume growth.

ALE: $[\mathrm{Vol} \sim r^4]$ Any X is asymptotic to some standard model $X_\Gamma^\circ = \mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SU(2)$. [Kronheimer]

(using hyperlähler quotient construction in Hitchin–Karlhede–Lindström–Roček *Hyperkähler metrics and supersymmetry* (1987))

Hitchin *Polygons and gravitons* (1979)

ALF: $[\mathrm{Vol} \sim r^3]$ S^1-fibrations over $3d$ spaces, described by Gibbons-Hawking Ansatz.

Hitchin *Monopoles and Geodesics* (1982),

Atiyah–Hitchin *The Geometry and Dynamics of Magnetic Monopoles* (1988),

Cherkis–Hitchin *Gravitational Instantons of Type D_k* (2005)

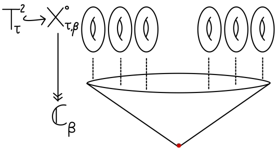
ALG: $[\mathrm{Vol} \sim r^2]$ T^2 fibrations over a cone [Hein]

ALH: $[\mathrm{Vol} \sim r^1]$ T^3 fibrations over $\mathbb{R}$ or $\mathbb{R}^+$.

Noncompact hyperkähler four-manifolds X

G. Chen-X. Chen classified ALG.

ALG: X is asymptotic to some standard model $X^{\circ}_{\tau,\beta}$ fibered over $\mathbb{C}_\beta$ of angle $2\pi\beta$ with fiber T^2_τ. [G. Chen-X. Chen]



What about these other types ALG* and ALH*?

- ALG*: There are examples with $\mathrm{Vol} \sim r^2$ but $\mathrm{Riem} \sim r^{-2}(\log r)^{-1}$ [Hein].
- ALH*: There are examples with $\mathrm{Vol} \sim r^{\frac{4}{3}}$ but $\mathrm{Riem} \sim r^{-2}$ [Hein].

Question

Where do 4d Hitchin moduli spaces fit in to classification of gravitational instantons? (Do they fit?—i.e. do they have integrable Riemannian curvature tensor?)

Modularization Conjecture [Cherkis–Kapustin, Boalch]

All gravitational instantons can be realized as moduli spaces from gauge theory.

Every 4*d* Hitchin moduli space is of type ALG or ALG*. Conversely, every ALG & ALG* hyperkähler metric can be realized as the hyperkähler metric on a Hitchin moduli space.

First data point: Moduli space of two centered SU(2) monopoles on $\mathbb{R}^3$ is ALF-D_0, Atiyah–Hitchin (1988)

Fiber at ∞	Regular	I_0^*	II	II^*	III	III^*	IV	IV^*
Dynkin		D_4	E_8	A_0	E_7	A_1	E_6	A_2
β	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$
C	T_τ^2	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$
D		$\{0,1,\infty,p_0\}$	$\{0,1,\infty\}$	$\{\infty\}$	$\{0,1,\infty\}$	$\{\infty\}$	$\{0,1,\infty\}$	$\{0,\infty\}$
G	$U(1)$	$SU(2)$	$SU(6)$	$SU(2)$	$SU(4)$	$SU(2)$	$SU(3)$	$SU(2)$

The Hitchin moduli spaces associated to the $SU(2)$ theories with $N_f = 0,1,2,3$ are conjecturally of type ALG* with fiber $I^*_{4-N_f}$.

Hitchin moduli spaces

The Hitchin moduli space

Fixed data:

- C, a compact Riemann surface (possibly with punctures D)
- $G = SU(n)$, $G_{\mathbb{C}} = SL(n, \mathbb{C})$
- $E \to C$, a complex vector bundle of rank n with $\mathrm{Aut}(E) = SL(E)$

$\rightsquigarrow$ Hitchin moduli space, $\mathcal{M}$.

Fact #1: $\mathcal{M}$ is a noncompact hyperkähler manifold with metric g_{L^2}
$\Rightarrow$ have a $\mathbb{CP}^1$-family of Kähler manifolds $\mathcal{M}_\zeta = (\mathcal{M}, g_{L^2}, I_\zeta, \omega_\zeta)$.

- $\mathcal{M}_{\zeta=0}$ is the $G_{\mathbb{C}}$-Higgs bundle moduli space
- $\mathcal{M}_{\zeta\in\mathbb{C}^\times}$ is the moduli space of flat $G_{\mathbb{C}}$-connections

Hitchin *The self-duality equations on a Riemann surface* (1987)
Hitchin *Stable bundles and integrable systems*

The Higgs bundle moduli space

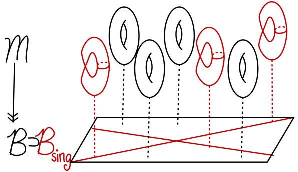
Definition

A **Higgs bundle** is a pair $(\bar{\partial}_E, \varphi)$ consisting of a holomorphic structure $\bar{\partial}_E$ on E and a "Higgs field" $\varphi \in \Omega^{1,0}(C, \mathrm{End}_0 E)$ such that $\bar{\partial}_E \varphi = 0$.

(Locally, $\bar{\partial}_E = \bar{\partial}$ and $\varphi = P\mathrm{d}z$, where P is a tracefree $n \times n$ matrix with holomorphic entries.)

Ex: The $GL(1)$-Higgs bundle moduli space is $\mathcal{M} = \underbrace{\mathrm{Jac}(C)}_{\bar{\partial}_E} \times \underbrace{H^0(K_C)}_{\varphi}$.
For $C = T^2_\tau$, $\mathcal{M} = T^2_\tau \times \mathbb{C}$.

Fact #2: In its avatar as the Higgs bundle moduli space, $\mathcal{M}$ is an algebraic completely integrable system.



Hitchin *Stable bundles and integrable systems* (1987)

Hitchin's equations are equations for a hermitian metric h on E.

Definition

A Higgs bundle $(\bar{\partial}_E, \varphi)$, together with a Hermitian metric h on E, is a **solution of Hitchin's equations** if

$$F_D^\perp + [\varphi, \varphi^{*_h}] = 0.$$

(Here, D is the Chern connection for $(\bar{\partial}_E, h)$.)

There is a correspondence between stable Higgs bundles and solutions of Hitchin's equations. [Hitchin, Simpson, Donaldson]

$$\left\{ \begin{matrix} \text{stable Higgs bundles} \\ (\bar{\partial}_E, \varphi) \end{matrix} \right\} \Big/ \mathcal{SL}(E) \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{soln of Hitchin's eqn} \\ (\bar{\partial}_E, \varphi, h) \end{matrix} \right\} \Big/ \mathcal{SU}(E) =: \mathcal{M}$$

4d Hitchin moduli spaces

Modularization Conjecture

Every $4d$ Hitchin moduli space is of type ALG or ALG*. Conversely, every ALG & ALG* hyperkähler metric can be realized as the hyperkähler metric on a Hitchin moduli space.

Fiber at ∞	Regular	I_0^*	II	II^*	III	III^*	IV	IV^*
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C	T^2_τ	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$
D		$\{0, 1, \infty, p_0\}$	$\{0, 1, \infty\}$	$\{\infty\}$	$\{0, 1, \infty\}$	$\{\infty\}$	$\{0, 1, \infty\}$	$\{0, \infty\}$
G	$U(1)$	$SU(2)$	$SU(6)$	$SU(2)$	$SU(4)$	$SU(2)$	$SU(3)$	$SU(2)$

Here, must have marked points $D \subset C$. Different allowed behaviors:
(1) φ has simple poles "parabolic" [Simpson, Nakajima],
(2) φ has a higher order pole "wild" [Biquard–Boalch].

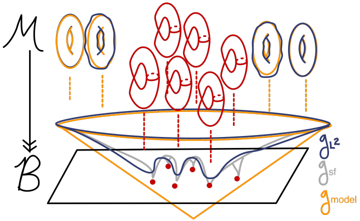
Example: The four-punctured sphere

There is a 12-parameter family of $SU(2)$-Hitchin moduli spaces on $\mathbb{CP}^1 - \{0, 1, \infty, p_0\}$.

Data: At each $p \in \{0, 1, \infty, p_0\}$,

- fix a flag $F_p \subset E_p$.
- fix a "complex mass" $m_p \in \mathbb{C}$ (eigenvalue of $\mathrm{Res}_p\, \varphi$ for F_p)
- fix a "real mass" $\alpha_p \in (0, \frac{1}{2})$ (parabolic weights at p are α_p and $1 - \alpha_p$)

$\rightsquigarrow$ Hitchin moduli space, $\mathcal{M} = \mathcal{M}(\boldsymbol{m}_\bullet, \boldsymbol{\alpha}_\bullet)$.



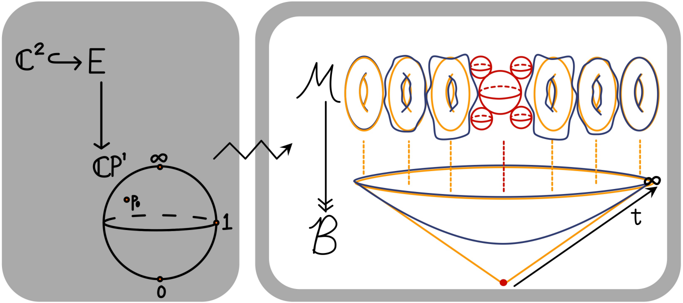
In this case, the model metric has cone angle π and fibers are T^2_τ where $\tau = \lambda(p_0)$. The volume of each fiber is $4\pi^2$. Generically, $\mathcal{M} \to \mathcal{B}$ has 6 singular fibers.

Example: The four-punctured sphere

Now restrict our attention to **strongly parabolic** Hitchin moduli spaces $\mathbf{m} \equiv 0$.

Corollary [F-Mazzeo-Swoboda-Weiss]

The Hitchin moduli spaces $\mathcal{M}(\boldsymbol{\alpha}, \mathbf{0})$ are ALG.



Conjecture/Work-In-Progress [F-Mazzeo-Swoboda-Weiss]

The Hitchin moduli spaces $\mathcal{M}(\boldsymbol{\alpha}, \mathbf{m})$ are ALG.

Proof of ALGness for strongly parabolic Hitchin moduli spaces
2001.03682

Proof of ALGness

A Higgs bundle $(\bar{\partial}_E, t\varphi)$ consists of a holomorphic structure $\bar{\partial}_E$ on E and a Higgs field $t\varphi$. Let $(\dot{\eta}, t\dot{\varphi})$ be a deformation of $(\bar{\partial}_E, \varphi)$.

Hitchin's hyperkähler metric g_{L^2} on $T_{(\bar{\partial}_E, t\varphi)}\mathcal{M}$ is

$$\|(\dot{\eta}, t\dot{\varphi}, \dot{\nu}_t)\|^2_{g_{L^2}} = 2\int_C \left|\dot{\eta} - \bar{\partial}_E \dot{\nu}_t\right|^2_{h_t} + t^2 \left|\dot{\varphi} + [\dot{\nu}_t, \varphi]\right|^2_{h_t},$$

the L^2-norm of harmonic representative. Here, the correction $\dot{\nu}_t$ is the unique solution of

$$\partial_E^{h_t} \bar{\partial}_E \dot{\nu}_t - \partial_E^h \dot{\eta} - t^2 \left[\varphi^{*_{h_t}}, \dot{\varphi} + [\dot{\nu}_t, \varphi]\right] = 0.$$

To understand the hyperkähler metric near the ends, you must understand the **solution of t-rescaled Hitchin's equations** for $t \gg 0$

$$F_D^{\perp} + t^2 \left[\varphi, \varphi^{*_{h_t}}\right] = 0.$$

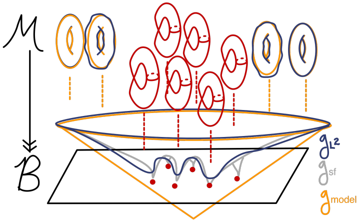
(Here, D is the Chern connection for $(\bar{\partial}_E, h_t)$.)

Idea #1: Semiflat metric is an L^2-metric

The model metric has cone angle π and fibers are T^2_τ where $\tau = \lambda(p_0)$. The volume of each fiber is $4\pi^2$.

It is easy to compare the model metric to a singular hyperkähler metric known as the semiflat metric, introduced by [Freed]. In fact, when $\mathbf{m} = \mathbf{0}$, they agree.

Fundamentally, need to compare g_{L^2} to g_{sf}.



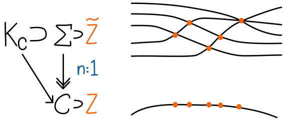
The semiflat metric, from the integrable system structure, on $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$ is an L^2-metric defined using h_∞.

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2 \left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t}$$

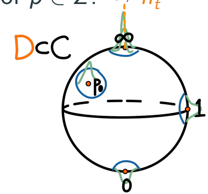
$$\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2 \left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty}$$

Idea #2: Approximate solutions

As $t \to \infty$, $F_{D(\bar{\partial}_E, h_t)}$ concentrates along branch divisor $Z \subset C$. Bubbling!
The limiting metric h_∞ is flat with singularities along Z.



Desingularize h_∞ (singular at Z) by gluing in solutions h_t^{model} of Hitchin's equations on neighborhoods of $p \in Z$. $\rightsquigarrow$ h_t^{approx}.



Perturb h_t^{approx} to an actual solution h_t using a contraction mapping argument.
(Difficulty: Showing the first eigenvalue of linearization is $\geq Ct^{-2}$)

Theorem: $h_t(v, w) = h_t^{\mathrm{approx}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w)$ for $\|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}$.

Bootstrapping to optimal exponential decay

Gaiotto-Moore-Neitzke's Conjecture (Schematically)

Pick a point $u \in \mathcal{B} \simeq \mathbb{C}$ and let $|u| = r$.

$$g_{L^2} - g_{\mathrm{sf}} = \sum_{\gamma \in H_1(\Sigma_u;\mathbb{Z})} \Omega(\gamma, u) \mathrm{e}^{-\ell(\gamma,u)\sqrt{r}}.$$

The first correction is $\Omega(\gamma, u) = 8$ and $\ell(\gamma_0, u) = 2\sqrt{\frac{2}{\mathrm{Im}\tau}}$.

Theorem [F-Mazzeo-Swoboda-Weiss]

Let $\mathcal{M}$ be a strongly-parabolic $SU(2)$ Hitchin moduli space for the four-punctured sphere. The rate of exponential decay for the Hitchin moduli space is as Gaiotto-Moore-Neitzke conjecture:

$$g_{L^2} - g_{\mathrm{sf}} = O(\mathrm{e}^{-2\sqrt{\frac{2}{\mathrm{Im}\tau}}\sqrt{r}}).$$

Corollary [F-Mazzeo-Swoboda-Weiss]

$(\mathcal{M}, g_{L^2})$ is an **ALG metric** asymptotic to the model metric g_{sf}.

Progress on Gaiotto–Moore–Neitzke conjecture

- Mazzeo-Swoboda-Weiss-Witt proved polynomial decay for $SU(2)$-Hitchin moduli space. ['17]
- Dumas-Neitzke proved exponential* decay in $SU(2)$-Hitchin section with its tangent space. ['18]
- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
- **F**-Mazzeo-Swoboda-Weiss proved exponential* decay for $SU(2)$ parabolic Hitchin moduli space. (Higgs field has simple poles along divisor $D \subset C$.) ['20]
- **F**-Mazzeo-Swoboda-Weiss proved sharp rate of exponential decay for $SU(2)$ parabolic Hitchin moduli space on four-punctured sphere. ['20]
- Mochizuki-Szabo proved exponential decay for $SU(n)$-Hitchin moduli space in slightly greater generality. ['23]
- Mochizuki proved sharp rate of exponential decay for $SU(2)$-Hitchin moduli space. ['24]

$*$: Rate of exponential decay is not optimal.

Torelli Parameters

2603.17020

Surjectivity of Torelli Map

$$\left((0,\frac{1}{2})\times\mathbb{C}\right)^4 \text{ \& basis of } H_2(\mathcal{M},\mathbb{Z}) \to \left((\mathbb{R}\times\mathbb{C})^4\right)^{\circ}$$

$$(\boldsymbol{\alpha},\mathbf{m}), \qquad (S_0,S_1,S_2,S_3,S_4) \mapsto \int_{S_i}\omega_I, \int_{S_i}\omega_J + i\omega_K$$

Theorem [G. Chen-Viaclovsky-Zhang]

Periods uniquely characterize ALG$^{(*)}$ gravitational instantons up to diffeomorphism.

Theorem [F-Mazzeo-Swoboda-Weiss]

The periods $\int_{S_i}\omega_I$ are affine linear functions of $\boldsymbol{\alpha}$. The periods $\int_{S_i}\omega_J + i\omega_K$ are linear functions of $\mathbf{m}$. The space of marked Hitchin moduli spaces $(\mathcal{M}(\boldsymbol{\alpha},\mathbf{m}),S_\bullet)$ (nearly) exhaust the entire family of ALG-D_4 gravitational instantons.

Why linear?

Slogan: Periods are linear in moment-map level.

Analogy: Consider the setting of finite-dimensional Kähler quotients for $G \curvearrowright X$, $\mu : X \to \mathfrak{g}^*$. Taking $\mu^{-1}(\alpha)/G$, Duistermaat–Heckman prove that $[\omega_I]$ is an affine function of level α.

$$\bar{\partial}\frac{\mathrm{d}z}{z} = \pi\delta_0 \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Our case: The hyperkähler structure on the Hitchin moduli space is from hyperkähler quotient. Here, can *morally* interpret $\boldsymbol{\alpha}, \mathbf{m}$ as levels of moment map of unitary gauge group:

$$\begin{aligned} M_I(\bar{\partial}_E, \varphi) &= 2i\bar{\partial}_E\varphi \\ &= 2i\sum_{p\in D}\begin{pmatrix} -m_p & n_p \\ 0 & m_p \end{pmatrix}\pi\delta_p d\bar{z}\wedge dz \\ \mu_I(\bar{\partial}_E, \varphi) &= -(F_D^{\perp} + [\varphi, \varphi^*]) \\ &= -\sum_{p\in D}\begin{pmatrix} \alpha_p - \frac{1}{2} & \\ & \frac{1}{2} - \alpha_p \end{pmatrix}\pi\delta_p d\bar{z}\wedge dz \end{aligned}$$

Strategy of Proof: ω_I

- When $\mathbf{m} = \mathbf{0}$, $\mathcal{M}(\boldsymbol{\alpha}, \mathbf{0})$ is fixed by $U(1)$ whose moment map μ solving $\mathrm{d}\mu = \iota_X \omega_I$ is related to integral by

$$\int_{S^2} \omega_I = 2\pi(\mu_{\text{top}} - \mu_{\text{bottom}}).$$

Theorem [Collier-F-Wentworth]

There exists an analytically constructed moduli space of framed Higgs bundles

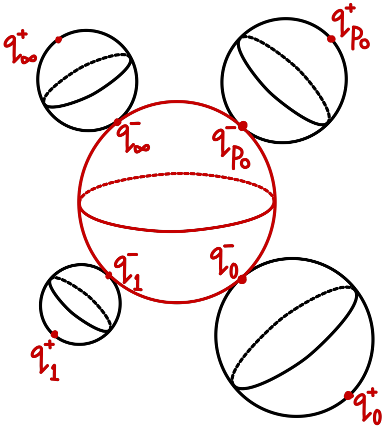
$$\mathcal{P}_*(\boldsymbol{\alpha}) = \sqcup_{\mathbf{m} \in \mathbb{C}^4, \text{framing}} \mathcal{M}(\mathbf{m}, \boldsymbol{\alpha})$$

that (1) admits a $\mathbb{C}^\times$-action and (2) has a holomorphic symplectic structure.

- Then, $\omega_{I,t\mathbf{m}}$ is cohomologous to $\omega_{I,\mathbf{m}}$ for $t \neq 0$. We can extend to $t = 0$ via continuity argument using smooth manifold structure of $\mathcal{P}_*(\boldsymbol{\alpha})$.

Strategy of Proof: $\Omega_I = \omega_J + i\omega_K$

$S_p^2 = [\tau_p^- - \tau_p^+]$ where $\tau_p^\pm$ is determined by Bialynicki-Birula stratification by $\mathbb{C}^\times$-limits.



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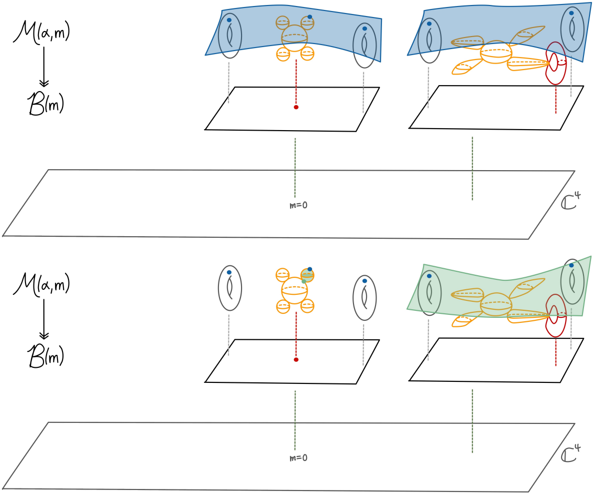
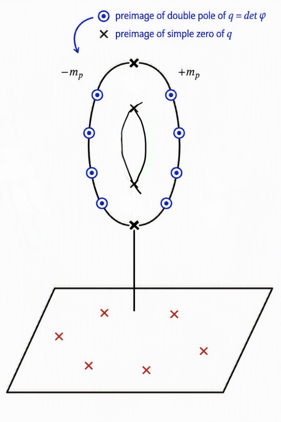


Strategy of Proof: $\Omega_I = \omega_J + i\omega_K$

- $S^2_p = [\tau^+_p - \tau^-_p]$ where $\tau^\pm_p$ is determined by Bialynicki-Birula stratification by $\mathbb{C}^\times$-limits.
- $\Omega_I = \mathrm{d}\lambda_I$, for λ_I tautological 1-form on $\mathrm{Tot}(K_{\mathbb{CP}^1})$, with residue $\pm m_p$ along $\sigma^\pm_p$.



- Compute intersection numbers of S^2_p and $[\sigma^+_{p'} - \sigma^-_{p'}]$.

Summary

Modularization Conjecture

All gravitational instantons can be realized as moduli spaces from gauge theory.

Every $4d$ Hitchin moduli space is of type ALG or ALG*. Conversely, every ALG & ALG* hyperkähler metric can be realized as the hyperkähler metric on a Hitchin moduli space.

[F-Mazzeo-Swoboda-Weiss]

Theorem: The Hitchin moduli spaces $\mathcal{M}(\boldsymbol{\alpha}, \mathbf{0})$ are ALG-D_4.
Conjecture: The Hitchin moduli spaces $\mathcal{M}(\boldsymbol{\alpha}, \mathbf{m})$ are ALG-D_4.
Theorem: The periods $\int_{S_i} \omega_I$ are affine linear functions of $\boldsymbol{\alpha}$. The periods $\int_{S_i} \omega_J + i\omega_K$ are linear functions of $\mathbf{m}$. The space of marked Hitchin moduli spaces $(\mathcal{M}(\boldsymbol{\alpha}, \mathbf{m}), S_\bullet)$ (nearly) exhaust the entire family of ALG-D_4 gravitational instantons.

Clearly, this is not the way to prove the full modularization conjecture!

Application: Instanton Degenerations

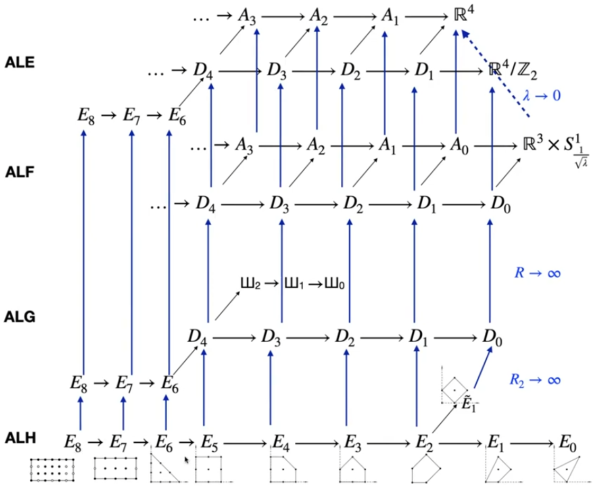
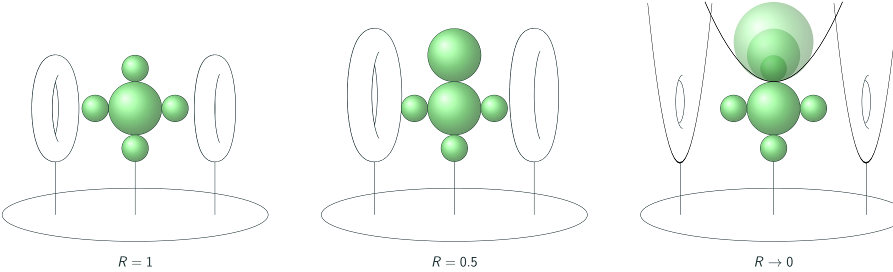


Figure 1: Conjectured Instanton Degenerations (by Sergey Cherkis)

Application: Instanton Degenerations

There is $\mathbb{R}^+_R$-family of Hitchin moduli spaces. The fibers have volume $\frac{4\pi^2}{R}$.
Let $\alpha_p(R) = \frac{1}{2} - R\beta_p$.
Taking $R \to 0$ enlarges the fibers while shrinking the base.



Theorem [F - Yae] (alt. proof Heller- Heller- Meneses)

Let $\mathcal{T}$ be the map from the Nakajima quiver variety "n-hyperpolygon space" to Higgs bundles on n-punctured $\mathbb{CP}^1$.
Then, $\lim_{R\to 0} \mathcal{T}^*(g_R) = g_{\mathrm{HP}}$.

When $n = 4$, this is an ALG-D_4 to ALE-D_4 degeneration.

Happy Birthday, Nigel!

Bootstrapping to optimal exponential decay

LeBrun gave a framework to describe all Ricci-flat Kähler metrics of complex-dimension two with a holomorphic circle action in terms of two functions u, w.

Generalized Gibbons-Hawking Ansatz specialized to our case:

Consider a hyperkähler metric on $T^2_{x,y} \times \mathbb{R}^+_r \times S^1_\theta$ with holomorphic circle action. The hyperkähler metric is

$$g_{L^2} = \mathrm{e}^u u_r(\mathrm{d}x^2 + \mathrm{d}y^2) + u_r \mathrm{d}r^2 + u_r^{-1}\mathrm{d}\theta^2$$

where $u : T^2_{x,y} \times \mathbb{R}^+_r \to \mathbb{R}$ solves

$$\Delta_{T^2} u + \partial_r^2 \mathrm{e}^u = 0.$$

The semiflat metric g_{sf} corresponds to $u_{\mathrm{sf}} = \log r$.

Goal

Show that $u - u_{\mathrm{sf}}$ has conjectured rate of exponential decay.

Bootstrapping to optimal exponential decay

Let $v = u - u_{\mathrm{sf}}$. Then,

$$\underbrace{\Delta_T v + r\partial_r^2 v + 2\partial_r v}_{Lv} = \underbrace{-e^v r(\partial_r v)^2 - (e^v - 1)\left(r\partial_r^2 v + 2\partial_r v\right)}_{Q(v,\partial_r v,\partial_{rr} v)},$$

Observation #1: The first exponentially-decaying function in $\ker L$ decays like $\mathrm{e}^{-2\lambda_T\sqrt{r}}$, where λ_T^2 is the first positive eigenvalue of $-\Delta_{T^2}$. In the torus T_τ^2 with its semiflat metric $\lambda_T^2 = \frac{2}{\mathrm{Im}\ \tau}$.

Observation #2: If $v \sim \mathrm{e}^{-\varepsilon\sqrt{r}}$, then $Q(v, \partial_r v, \partial_{rr} v) \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$.

Solving the non-homogeneous problem $Lv = f$ for $f \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$, we find

$$v \sim \mathrm{e}^{-2\min(\varepsilon,\lambda_T)\sqrt{r}}.$$

Conclusion: $\quad v \sim \mathrm{e}^{-2\lambda_T\sqrt{r}} \quad$ where $\lambda_T = \sqrt{\frac{2}{\mathrm{Im}\ \tau}}$