Completeness of left-invariant geodesic flows

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Trends in Differential and Complex Geometry
A celebration of Nigel Hitchin's 80th birthday
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Some years ago...

N.Hitchin, Compact four-dimensional Einstein manifolds, J. Differential Geom. 9 (1974) 435–441.

Theorem (Hitchin-Thorpe inequality)

Let (M, g) be a compact, oriented, four-dimensional Riemannian manifold, which is Einstein. Then

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

where $\chi(M)$ is the Euler characteristic of M and $\tau(M)$ is the signature of M. If

$$\chi(M) = \frac{3}{2}|\tau(M)|$$

then (M, g) is Ricci-flat and it is either flat or its universal cover is a K3 surface.

Some years later...

A.C. Ferreira, Riemannian geometry with skew torsion, DPhil Thesis, Oxford University (2010).

Theorem

Let (M, g) be a compact, oriented, four-dimensional Riemannian manifold, which is Einstein with skew torsion. Then

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

where $\chi(M)$ is the Euler characteristic of M and $\tau(M)$ is the signature of M. If

$$\chi(M) = \frac{3}{2}|\tau(M)|$$

then (M, g) is Ricci-flat and it is either flat or its universal cover is a K3 surface.

More recently...

[EFR] A. Elshafei, A.C. Ferreira, H. Reis,
Geodesic completeness of pseudo and holomorphic-Riemannian metrics on Lie groups
Nonlinear Anal. Theory Methods Appl. (2023)

[EFSZ] A. Elshafei, A.C. Ferreira, M. Sánchez, A. Zeghib,
Lie groups with all left-invariant semi-Riemannian complete
Trans. Amer. Math. Soc. (2024)

[CFZ] S. Chaib, A.C. Ferreira, A. Zeghib,
The completeness problem on the pseudo-homothetic Lie group
Differ. Geom. Appl. (2025)

[CF] S. Chaib, A.C. Ferreira
The completeness problem on 3-dimensional non-unimodular Lie groups
Comm. Anal. Geom. (to appear)

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Geodesic equations

(M,g) pseudo-Riemannian manifold, ∇ the Levi-Civita connection
$\mathcal{X}_\gamma(M)$: vector fields along a curve γ in M

Covariant derivative: $\nabla_{\dot\gamma} : \mathcal{X}_\gamma(M) \longrightarrow \mathcal{X}_\gamma(M)$

γ is called a **geodesic** if it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$.

In local coordinates

$$\ddot\gamma^k(t) + \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0$$

where Γ^k_{ij} are the Christoffel symbols.

A maximal geodesic $\gamma : I \longrightarrow M$ is called

$\rightarrow$ complete if $I = \mathbb{R}$.

$\longrightarrow$ (M,g) is said to be complete if all of its maximal geodesics are complete.

Riemannian vs. pseudo-Riemannian geometry

Hopf-Rinow theorem

For a Riemannian manifold (M, g) the following condition are equivalent:

(MC) As a metric space with Riem. distance d, (M, d) is Cauchy-complete.

(GC) (M, g) is geodesically complete.

(HB) Every closed and bounded subset of M is compact.

Companion results:
For a complete Riemannian manifold (M, g)

$\rightarrow$ Any two points can be joined by a minimizing geodesic segment.

$\rightarrow$ The length of any diverging curve is infinite.

Theorem

A Riemannian homogeneous manifold (M, g) is geodesically complete.

However:

PR manifolds can fail to be complete even in the compact case!

Example: Clifton-Pohl torus

$(M = \mathbb{R}^2 \backslash \{(0,0)\}, g)$, $g = 2\dfrac{dxdy}{x^2+y^2}$

Geodesic equations: $\ddot{x} = \frac{2x}{x^2+y^2}(\dot{x})^2$, $\ddot{y} = \frac{2y}{x^2+y^2}(\dot{y})^2$

The curve $\alpha(t) = \left(\frac{1}{1-t}, 0\right)$ is an incomplete (null) geodesic.

$\mu(x,y) = 2(x,y)$ is an isometry, $\Gamma = \{\mu^n\}$ is properly discontinuous.

$T = M/\Gamma$ is a torus which is incomplete with the induced metric.

Nevertheless...

The following classes of manifolds are geodesically complete:

- $\to$ PR symmetric spaces
- $\to$ Compact PR homogenous spaces (Mardsen 1973)
- $\to$ Compact PR globally conformal to homogeneous (Sánchez-Romero 1994)
- $\to$ Compact PR of index v with v timelike conformal vector fields which are pointwise independent. (Sánchez-Romero 1995)
- $\to$ 2-step nilpotent Lie groups (Guediri 1994)
- $\to$ Compact Lorentzian of constant curvature (Carriére 1989, Klinger 1996)
- $\to$ Compact (Lorentzian) pp-waves (Leistner-Schliebner 2016)
- $\to$ Compact (Lorentzian) Brinkmann waves (Mehidi-Zeghib 2026)

Euler-Arnold formalism on Lie groups

Euler and later Arnold:

$\rightarrow$ described the motions of a rigid body as geodesics of a Lie group in the context of perfect fluids.

$\rightarrow$ Key idea:

$$\begin{array}{c} \text{Geodesic curves} \\ \text{on Lie group} \end{array} \quad \overset{1:1}{\longleftrightarrow} \quad \begin{array}{c} \text{Integral curves of vector field} \\ \text{on Lie algebra} \end{array}$$

$\rightarrow$ Key advantage:

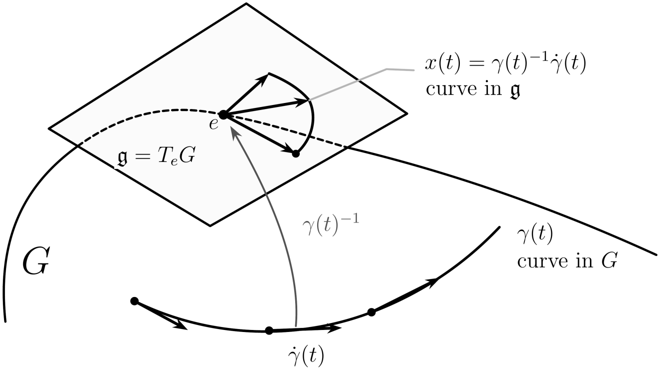
geodesics seen as curves in $\mathbb{R}^n$ with standard topology;

easier to control behaviour at infinity.

The associated curve

(G, g) be a Lie group with Lie algebra $\mathfrak{g}$.

$\gamma(t)$ curve in G and $x(t)$ the associated curve in $\mathfrak{g}$ given by $x(t) = (L_{\gamma^{-1}(t)})_* \dot{\gamma}(t) := \gamma^{-1}(t)\dot{\gamma}(t)$



The Euler-Arnold theorem

Theorem (Arnold, 1966)

Let (G, g) be a Lie group equipped with a left-invariant PR metric and $\mathfrak{g}$ be the Lie algebra of G.

Then $\gamma(t)$ is a geodesic of (G, g) if and only if $x(t) = \gamma^{-1}(t)\gamma(t)$ is a curve in $\mathfrak{g}$ which is a solution of the following ODE

$$\dot{x}(t) = \mathrm{ad}^{\dagger}_{x(t)} x(t)$$

where $\dagger$ is the formal adjoint w.r.t to g_1.

The ODE above is called the Euler-Arnold equation.

V. I. Arnold, "Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits," Annales de l'Institut Fourier (Grenoble) 16 (1966), no. 1, 319–361.

Quadratic Lie groups

A Lie group is called **quadratic** if it admits a bi-invariant metric.

Let G be quadratic with bi-invariant metric B then

$$\begin{array}{ccc} \text{LI PR metrics} & \xleftrightarrow{1:1} & B\text{-self-adjoint linear isomorphisms } \Phi \\ \text{on Lie group } G & & \text{on Lie algebra } \mathfrak{g} \end{array}$$

via the equation

$$q(x, y) = B(\Phi x, y).$$

Corollary (Lax equation)
If G is quadratic and Φ the isomorphism associated to q then

$\gamma(t)$ is a geodesic of (G, q) iff $y(t) = \Phi^{-1}x(t)$ is an integral curve of

$$\dot{y}(t) = [y(t), \Phi^{-1}y(t)].$$

First integrals

The Lax equation come with two (generically independent) first integrals given by

$$\begin{cases} F(y) = B(y, y) \\ G(y) = B(y, \Phi^{-1} y) \end{cases}$$

Proposition (Elshafei-F-Reis 2023)
Let G be a quadratic Lie group.
For every possible signature, there exist infinitely many LI PR complete metrics.

However...

$\mathrm{SL}(2, \mathbb{R})$ has many incomplete Lorentzian metrics (Bromberg-Medina 2008)

Corollary (Elshafei-F-Reis 2023)
If G is a semisimple non-compact Lie group.
Then G admits infinitely many incomplete LI PR metrics.

Clairaut first integrals

Well-known facts:

- $\rightarrow$ If γ is geodesic, X is Killing field, then $g(\dot{\gamma}(t), X_{\gamma(t)}) = \text{constant}$.
- $\rightarrow$ If X is a right-invariant vector field, then X is Killing for any LI metric.

A construction inspired by J. Marsden:

Let (e_i) be a basis of $\mathfrak{g}$. Denote by Y_i the extension of e_i as a right-invariant vector field.

Then $\omega^i = g(Y_i, \cdot)$ is a first integral of the geodesic equations. The (ω^i) span T^*G.

Definition (Elshafei, F., Sánchez, Zeghib, 2024)

The *Clairaut metric* associated to g from (e_i) is the Riemannian metric defined by

$$h = \sum_i \omega^i \otimes \omega^i$$

Transformation law:

For $u, v \in \mathfrak{g}$

$$\omega^i(p.u) = g_p(Y_i(p), p.u) = g_p(ei.p, p.u) = g_1(\mathrm{Ad}_{p^{-1}}(e_i), u) = \omega_1^i(((\mathrm{Ad}_p)^\dagger)^{-1}(u))$$

and a concrete expression for h is

$$h(p.u, p.v) = \sum_i \; g_1((\mathrm{Ad}_{p^{-1}})(e_i), u) \; g_1((\mathrm{Ad}_{p^{-1}})(e_i), v).$$

$\rightarrow$ h is not left-invariant (nor right-invariant unless g is bi-invariant)

Definition

Two Riemannian metrics R and $\hat{R}$ are said to be bi-Lipschitz bounded if there exists a constant $c > 0$ s.t. $\quad c\,R \leq \hat{R} \leq R/c$.

Proposition (EFSZ 2024)

Let h and $\hat{h}$ be two Clairaut metrics associated to g from the bases (e_i) and $(\hat{e}_i)$, resp., then h and $\hat{h}$ are bi-Lipschitz bounded.

Wick rotation

(G, g) PR Lie group.

→ Choose an ON basis (e_i) for g_1

→ Construct a LI Riem. metric $\tilde{g}$ by imposing that (e_i) is ON for $\tilde{g}_1$.

⟶ $\tilde{g}$ is obtained from g by *Wick rotation*.

More precisely, consider

→ the linear map ψ such that $\psi(e_i) = \epsilon_i e_i$

→ $\tilde{g}$ be the LI Riem. metric such that $\tilde{g}_1(e_i, e_j) = \delta_{ij} = g_1(e_i, \psi(e_j))$.

Proposition (EFSZ 2024)

Let $g, \tilde{g}$ be Wick rotated metrics and h the Clairaut metric associated to g from (e_i). Then, h is unique (independent of the chosen (e_i)) and

$$h_p(p.u, p.v) = \tilde{g}_1(\mathrm{Ad}^*_{p^{-1}}(\psi(u)), \mathrm{Ad}^*_{p^{-1}}(\psi(v)))$$

Remark: If g is Riemannian then $\tilde{g} = g$.

Completeness

→ $\omega^i(\dot\gamma(t))$ is constant for any inextensible geodesic γ of g

→ thus $h(\dot\gamma(t), \dot\gamma(t))$ is constant

Theorem (EFSZ 2024)

The left-invariant pseudo-Riemannian metric g is complete if its associated Clairaut metric h is complete.

Proof: The curve γ restricted to any bounded interval $I \subset \mathbb{R}$ has finite h-length. Thus, by the completeness of h, γ is continuously extensible to the closure of I and, then, it is extensible as a geodesic of g. $\square$

Corollary

Any bi-invariant PR metric g on G is complete.

Corollary (Marsden 1973)

Any left-invariant PR metric g on a compact Lie group G is complete.

The action of $\mathrm{Aut}(G)$

Lie's second theorem : $\mathrm{Aut}(\mathfrak{g})$ is in 1:1 correspondence with $\mathrm{Aut}(\tilde{G})$.

$\longrightarrow$ $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathrm{Sym}(\mathfrak{g})$ by

$$(\varphi.m)(u,v) = m(\varphi^{-1}u, \varphi^{-1}v).$$

with $\varphi \in \mathrm{Aut}(\mathfrak{g}), m \in \mathrm{Sym}(\mathfrak{g})$ and $u, v \in \mathfrak{g}$.

Proposition (EFSZ 2024)

G connected Lie group, g LI PR metric, $\varphi \in \mathrm{Aut}(\mathfrak{g})$, g^{φ} the LI metric such that $(g^{\varphi})_1 = \varphi.g_1$.
Then:

(1) g^{φ} is complete if and only if so is g.

(2) LI PR metrics in each orbit of $\mathrm{Sym}(\mathfrak{g})$ are all complete or incomplete.

(3) Clairaut metrics associated to LI metrics on the same orbit are all bi-Lipschitz bounded.

A question

$\longrightarrow$ What classes of Lie groups have all of their indefinite LI metrics incomplete?

Example

$\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{Id}} \mathbb{R}^n$ almost abelian Lie algebra

$\widetilde{G} = \mathbb{R} \ltimes \mathbb{R}^n$ with multiplication

$$(t, x).(s, y) = (t + s, x + \mathrm{e}^t y).$$

* All Riemannian LI metrics are equivalent up to automorphism and homothety to the hyperbolic metric (constant sectional curvature -1)
* All other LI PR are incomplete.

$\longrightarrow$ Are there other examples?

Another question

$\longrightarrow$ What classes of Lie groups have all of their PR LI metrics complete?

Positive answers

- $\rightarrow$ Abelian
- $\rightarrow$ Compact (Marsden 1973)
- $\rightarrow$ 2-step nilpotent (Guediri 1994)
- $\rightarrow$ $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ (Bromberg-Medina 2008)

* What about k-step nilpotent Lie algebras, $k \geq 3$?

* What is special about $\mathrm{SO}(2) \ltimes \mathbb{R}^2$?

Example

(Guediri 1994)

There exists a unique (up to isomorphism) 3-step nilpotent Lie algebra in dim 4: $\mathfrak{nil}_4$

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_4$$

Consider the left-invariant Lorentzian metric determined by

$$\langle e_1, e_1 \rangle = -1, \qquad \langle e_i, e_i \rangle = 1 \quad (i = 2, 3, 4), \qquad \langle e_i, e_j \rangle = 0 \quad (i \neq j).$$

Writing

$$x(t) = x_1(t)e_1 + x_2(t)e_2 + x_3(t)e_3 + x_4(t)e_4,$$

the Euler-Arnold equation becomes

$$\begin{cases} \dot{x}_1 = x_3(x_2 + x_4), \\ \dot{x}_2 = x_1 x_3, \\ \dot{x}_3 = x_1 x_4, \\ \dot{x}_4 = 0. \end{cases}$$

For the initial condition $x(0) = 2e_1 + e_2 + 2e_3 + e_4$, the solution is lightlike and

$$x_1(t) = \frac{2}{(1-t)^2}, \quad x_2(t) = \frac{2}{(1-t)^2} - 1 \quad x_3(t) = \frac{2}{1-t}, \quad x_4(t) = 1.$$

Groups of linear growth

Let R be a LI Riemannian metric on a Lie group G.

Consider the map

$$r : G \longrightarrow \mathbb{R} \qquad \text{where} \qquad r(p) = d_R(1,p)$$

and d_R is the distance induced by the Riemannian metric R.

Definition (EFSZ 2024)

A Lie group G has (at most) linear growth if there exist

→ a LI Riemannian metric R on G

→ a Euclidean scalar product with norm $\|\cdot\|$ on $\mathfrak{g}$

$$\text{s.t.} \quad \frac{\|u\|}{a + b\,r(p)} \leq \|\mathrm{Ad}_p(u)\| \leq (a + b\,r(p))\|u\|$$

for some constants $a, b \geq 0$, for every $p \in G$ and for every $u \in \mathfrak{g}$.

Remark: since R is LI, $r(p) = r(p^{-1})$, the two inequalities are equivalent,
linear growth is independent of choice of LI R and $\|\cdot\|$.

Preservation of completeness

Proposition

(M, R) connected complete Riemannian manifold.
Choose $x_0 \in M$ and let $M \ni x \mapsto d_R(x)$ be the distance function from x_0.
If h is a Riemannian metric on M with pointwise norm $\|\cdot\|_h$ satisfying:

$$\|v_x\|_h \geq \frac{\|v_x\|_R}{a + b\, d_R(x)}, \qquad \forall x \in M, v_x \in T_x M$$

for some constants $a, b \geq 0$, then h is complete.

Remarks

→ $\varphi(r) = a + b\, r$ can be replaced with any Lipschitz function φ s.t. $\int_0^\infty \frac{1}{\varphi(r)}\, dr = \infty$.

→ similiar types of bounds have been known since the 70s
[Abraham-Marsden, Foundations of mechanics, 1987]

Linear growth of Clairaut metrics

Theorem (EFSZ 2024)

All the left-invariant pseudo-Riemannian metrics of a Lie group with linear growth are geodesically complete.

Proof (sketch)

Take LI PR g, choose ON basis, construct Wick rotated $\tilde{g}$ and Clairaut h.
From the definition of h, can be computed that

$$h_p(p.u, p.u) \geq \frac{\tilde{g}_1(u,u)}{\|\mathrm{Ad}_p\|^2}$$

Since $\|\mathrm{Ad}_p\| \leq a + b\, r(p)$ then

$$\|v_p\|_h \geq \frac{\|v_p\|_{\tilde{g}}}{a + b\, r(p)}$$

and the result follows. $\square$

Some groups of linear growth

Theorem (EFSZ 2024)

The following classes of Lie groups have linear growth

- → abelian, compact [in fact, bounded growth]
- → 2-step nilpotent
- → the semidirect product $K \ltimes_\rho \mathbb{R}^n$ where K is *pseudo-compact* and $\rho(K)$ is pre-compact in $\mathrm{GL}(n, \mathbb{R})$
- → a subgroup or the direct product of the groups above

Corollary

All of the groups above have all their LI PR metrics geodesically complete.

Proposition (EFSZ 2024)

A k-step nilpotent Lie group, with $k \geq 3$, does not have linear growth.

Example: the 4D oscillator Lie algebra

Let

$$\mathfrak{osc} = \mathbb{R}e_0 \ltimes \mathfrak{h}_3, \qquad \mathfrak{h}_3 = \operatorname{span}\{e_1, e_2, e_3\},$$

where

$$[e_0, e_1] = e_2, \qquad [e_0, e_2] = -e_1, \qquad [e_1, e_2] = e_3.$$

A special Lorentzian metric

Define g by

$$g(e_0, e_3) = 1, \qquad g(e_1, e_1) = g(e_2, e_2) = 1,$$

with all other scalar products zero.
g has signature $(3, 1)$ and is ad-invariant. Hence

$$\dot{x} = \operatorname{ad}_x^\dagger x = 0,$$

so g is complete.

At the group level, the geodesics through the identity are one-parameter subgroups.

An incomplete Lorentzian metric

Define a left-invariant metric q by

$$q(e_0, e_1) = -1, \qquad q(e_2, e_2) = q(e_3, e_3) = 1,$$

with all other scalar products zero.

Write

$$x(t) = x_0(t)e_0 + x_1(t)e_1 + x_2(t)e_2 + x_3(t)e_3.$$

On the invariant set $x_0 = x_3 = 1$, the Euler-Arnold equation reduces to

$$\dot{x}_1 = x_2(1 + x_1), \qquad \dot{x}_2 = 1 + x_1.$$

It admits the spacelike solution

$$x(t) = e_0 + \left(\frac{2}{(1-t)^2} - 1\right) e_1 + \frac{2}{1-t} e_2 + e_3.$$

Thus q is geodesically incomplete.

Idempotents

Definition

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a quadratic homogeneous vector field.
X is called an idempotent if X is a non-trivial solution of $F(X) = X$.

Proposition

(Bromberg-Medina 2008)

If X_0 is an idempotent of $\dot{X} = F(X)$ then $t \longmapsto X_0\alpha(t)$ with $\dot{\alpha} = \alpha^2$, $\alpha(0) = 1$, is the solution with initial condition X_0.
If Q is a quadratic first integral of $\dot{X} = F(X)$ then $Q(X_0) = 0$.

Corollary

If $\dot{X} = F(X)$ is the EA equation of a PR metric and X_0 is an idempotent of $F(X)$, then X_0 yields a lightlike incomplete geodesic.

An obstruction to completeness

Proposition (EFSZ 2024)

Let q be a LI PR metric on G and suppose that $\dot{x} = \mathrm{ad}_x^\dagger x$ has an idempotent $x_0 \in \mathfrak{g}$. Then ad_{x_0} has a non-zero real eigenvalue.

Proof (sketch)

Since x_0 is an idempotent then, for any $y \in \mathfrak{g}$, $q(x_0, y - \mathrm{ad}_{x_0} y) = 0$. Thus the map $y \longmapsto y - \mathrm{ad}_{x_0} y$ has non-trivial kernel and there exists a y_0 such that $\mathrm{ad}_{x_0} y_0 = y_0$. $\square$

Conversely... (CF 2026)

If there exist an $x_0 \in \mathfrak{g}$ such that ad_{x_0} has a non-zero real eigenvalue then there exist PR metrics q such that x_0 is an idempotent of $\dot{x} = \mathrm{ad}_x^\dagger x$.

Corollaries

- $\rightarrow$ Nilpotent Lie groups have no idempotents.
- $\rightarrow$ If $\mathfrak{aff}(\mathbb{R})$ is a subalgebra of $\mathfrak{g}$ then G has incomplete metrics
- $\rightarrow$ In particular, completely solvable non-nilpotent Lie algebras admit incomplete metrics

3D Lie algebras

→ simple: $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$

→ solvable: semi-direct product $\mathbb{R} \ltimes_\varphi \mathbb{R}^2$
classification in terms of normal form of φ up to scaling

- A Lie algebra $\mathfrak{g}$ is said to be unimodular if $\mathrm{tr}(\mathrm{ad}_x) = 0,\ \forall x \in \mathfrak{g}$

→ In dimension 3: $\mathbb{R}^3$, $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{heis}(3)$, $\mathfrak{e}(2)$, $\mathfrak{e}(1,1)$

- 3-dim unimodular LA have been classified in terms of completeness (Bromberg-Medina 2008)

→ $(\mathfrak{g}, g)$ is incomplete iff has idempotents

→ on $\mathfrak{g}$ the set of complete metric is closed

- However, (CFZ 2025, CF 2026)
  - → There exist incomplete metrics with no idempotents on non-unimodular Lie algebras
  - → The set of complete metrics in not necessarily closed (nor open)
  - → There exists a complete metric on $\mathfrak{e}(1,1)$ with no positive definite quadratic first integral

Theorem (CF 2026)
All non-unimodular 3D Lie algebras admit incomplete metrics.

- Questions:
  - → Do all non-unimodular Lie algebras admit incomplete metrics?
  - → Do all k-step nilpotent Lie algebras, $k \geq 3$, admit incomplete metrics?
  - → Is our list of Lie groups of linear growth the list of all Lie groups of linear growth?
  - → Is linear growth equivalent to completeness?

Feliz aniversário, Nigel !