Intersection Theory on Singular Moduli Spaces of Vector Bundles: A Parabolic Approach

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(Joint work with Olga Trapeznikova)

Cohomology Generators of the Moduli Spaces

Universal Flag Bundles: Let $\mathcal{U}_0$ and $\mathcal{U}_1$ be universal bundles over $P_0(r) \times C$ and $P_1(r) \times C$, respectively.

- On $P_0(r) \times \{p\}$: flag $F_1 \subset F_2 := \mathcal{U}_0|_p$ with $\mathrm{rk}(F_1) = 1$.
- On $P_1(r) \times \{p\}$: flag $G_1 \subset G_2 := \mathcal{U}_1|_p$ with $\mathrm{rk}(G_1) = r - 1$.

Universal Classes:

$$z := c_1 \left(F_1 \otimes \det(\mathcal{U}_0|_p)^{-1/r} \right) \in H^2(P_0(r))$$
$$\xi := c_1 \left(G_2/G_1 \otimes \det(\mathcal{U}_1|_p)^{-1/r} \right) \in H^2(P_1(r))$$

Universal Classes:

- **For $M_1(r)$ (Atiyah-Bott):** Generated by Künneth components $a_k, b_{j,k}, f_k$ of $\mathrm{ch}(\tilde{\mathcal{U}})$.
- **For $P_1(r)$:** Generated by class ξ and pullbacks of $a_k, b_{j,k}, f_k$.
- **For $P_0(r)$:** Generated by class z and pullbacks of $a_k, b_{j,k}, f_k$.

The Jeffrey-Kirwan Formula on $M_1(r)$

The integration on the smooth, coprime space $M_1(r)$ is given by:

Theorem (Jeffrey-Kirwan)

$$\int_{M_1(r)} \exp\left(\sum_{k=2}^{r} \delta_k f_k\right) \prod_{k=2}^{r} a_k^{m_k} \prod_{j_k=1}^{2g} (b_{j_k,k})^{l_{k,j_k}} = \frac{(-1)^{\binom{r}{2}(g-1)}}{r!}$$

$$\sum_{B \in \mathcal{H}_n} iBer_{B,Q} \left[\frac{\prod_{k=2}^{r} \tau_k(x)^{m_k} e^{Q(w_B)}}{\prod_{i<j}(x_i - x_j)^{2g-2} \det(\mathrm{Hess}(Q))} \int_{T^{2g}} \Theta(x, \zeta) \right] (-\bar{c}_B)$$

- Expresses intersection numbers on the smooth coprime moduli space $M_1(r)$ as iterated residues of rational forms on the Cartan subalgebra of $\mathfrak{su}(r)$.
- Key to the pushdown from $P_1(r)$ using the projective bundle formula.

The Iterated Residue Operator $iBer_{B,Q}$

Iterated Residue Definition

For a basis $B = (\beta_{[1]}, \ldots, \beta_{[r-1]}) \in \mathcal{H}_n$ of V^*:

$$iBer_{B,Q}[f(x)](a) := \frac{1}{(2\pi i)^{r-1}} \int_{Z_B} \frac{f(x) e^{Q(\check{a}_B)} dQ(\check{\beta}_{[1]}) \wedge \cdots \wedge dQ(\check{\beta}_{[r-1]})}{(1 - e^{Q(\check{\beta}_{[1]})}) \ldots (1 - e^{Q(\check{\beta}_{[r-1]})})}$$

Coordinate Description (Iterated Residues)

In local coordinates $y_i = x_{\beta_{[i]}}$:

$$iBer_{B,Q}[f(x)](a) = \mathrm{Res}_{y_1=0} \ldots \mathrm{Res}_{y_{r-1}=0} \left[\frac{\hat{f}(y) e^{\hat{Q}(\check{a}_B)} d\hat{Q}(\check{\beta}_{[1]}) \wedge \cdots \wedge d\hat{Q}(\check{\beta}_{[r-1]})}{(1 - e^{\hat{Q}(\check{\beta}_{[1]})}) \ldots (1 - e^{\hat{Q}(\check{\beta}_{[r-1]})})} \right]$$

where the nested cycle is $Z_B = \{x \in V \otimes \mathbb{C} \mid |x_{\beta_{[j]}}| = \varepsilon_j\}$ with $0 < \varepsilon_{r-1} \ll \cdots \ll \varepsilon_1$.
$Q(x) = \sum_{k=2}^r \delta_k \tau_k(x)$ is a polynomial on $V \otimes_{\mathbb{R}} \mathbb{C}$, where τ_k denotes the k-th elementary symmetric polynomial and the δ_k are formal nilpotent parameters with $\delta_2 \neq 0$.

Intersection product on the parabolic space

Theorem (F., Trapeznikova)

$$\int_{\mathcal{P}_0(r)} \exp(\delta_2 f_2 + \delta_3 f_3 + \ldots + \delta_r f_r) x^m \prod_{k=2}^{r} a_k^{m_k} \prod_{j_k=1}^{2g} (b_k^{j_k})^{l_{k,j_k}} = \frac{(-1)^{\binom{r}{2}(g-1)}}{(r-1)!}$$

$$\sum_{\mathbf{B}\in\mathcal{H}_n} \mathrm{iBer}_{\mathbf{B},Q}\left[\frac{\left(\frac{1}{r}\sum_{j=1}^{r}(x_r - x_j)\right)^m \prod_{k=2}^{r}\tau_k(x)^{m_k}}{\prod_{i=1}^{r-1}(x_r - x_i)\prod_{i<j}(x_i - x_j)^{2g-2}\det(\mathrm{Hess}_{\mathbf{B}}(Q))}\int_{T^{2g}}\Theta(x,\zeta)\right](-\bar{c}_B).$$

Roles of z and f_2 in Our Context

- **The class f_2:** Pullback under π of an ample class on $M_0(r)$. It is used to define and calculate the perverse filtration $P_\bullet$ on $H^*(P_0(r))$.
- **The class z:** It is π-relatively ample, and its restriction to each fiber $\pi^{-1}(V) \cong \mathbb{P}^{r-1}$ is the Fubini-Study class $c_1(\mathcal{O}(1))$.
- **The top power $(-z)^{r-1}$:** Acts as a "probe" for integrating along the fibers, reducing the singular intersection pairing on $IH^*(M_0(r))$ to smooth ordinary integrals on $P_0(r)$ via the projection formula:

$$\pi_*(i(\alpha) \cup i(\beta) \cup (-z)^{r-1}) = j^*(\alpha) \cup j^*(\beta) \cdot 1$$

Intersection Product on $IH^*(M_0(r))$

For any $\alpha, \beta \in IH^*(M_0(r))$ of complementary degrees:

The Pairing Formula (F, Trapeznikova '26)

$$\langle \alpha, \beta \rangle = N \cdot \mathrm{Coeff}_{\delta^n} \sum_{B \in \mathcal{H}_n} iBer_{B,Q} \left[\frac{\left(\frac{1}{r}\sum_{j=1}^{r}(x_j - x_r)\right)^{r-1} \prod_{k=2}^{r} \tau_k(x)^{m_k}}{\prod_{i=1}^{r-1}(x_r - x_i) \prod_{i<j}(x_i - x_j)^{2g-2} \det(\mathrm{Hess}(Q))} \int_{T^{2g}} \Theta(x, \zeta) \right] (-\bar{c}_B)$$

- **Normalization factor:**

$$N = \frac{(-1)^{\binom{r}{2}(g-1)} \prod_{k=2}^{r} n_k!}{(r-1)!}$$

- **Exponent $r-1$:** Corresponds to the relatively ample class $(-z)^{r-1}$ along the fiber $\mathbb{P}^{r-1}$.

Specialization to Rank 2 (Kiem's Formula)

For rank $r = 2$, the singular fiber is $\mathbb{P}^1 \cup \mathbb{P}^1$:

Theorem (F.-Trapeznikova '26, Kiem '06)

For classes of the form $\alpha^m f^n \gamma^p$ with $2m + n + 3p = 3g - 3$:

$$\langle \alpha_1, \alpha_2 \rangle = \frac{(-1)^{g+m} n! g! 2^{1+2m+p-g}}{(g-p)!} \operatorname{Res}_{y=0} \left[\frac{y^{2+2m+2p-2g}}{1-e^{-y}} dy \right]$$

where $y = x_1 - x_2$ is the coordinate of the unique residue.

- The iterated residue operator collapses beautifully to a single residue at $y = 0$ in the variable $y = x_1 - x_2$.

Happy 80th birthday Nigel!