Bundles on elliptic curves, del Pezzo surfaces and q-difference equations

Tom Bridgeland

University of Sheffield

Joint work with Pierrick Bousseau and Luca Giovenzana

Motivation

The work of Gaiotto-Moore-Neitzke (2008) uncovered unexpected links

$$\{\text{DT invariants of a } \mathrm{CY}_3 \text{ category } \mathcal{D}\} \longleftrightarrow \{\text{Geometry of a HK manifold } \mathcal{M}\}.$$

Example 1. $\mathcal{M}$ is an $\mathrm{SL}_2(\mathbb{C})$ Hitchin moduli space, and $\mathcal{D} = D^b \operatorname{Lag}(X)$ where

$$X = \{uv = y^2 - q(x)\} \subset \mathbb{C}^2 \times T^*C.$$

Globally $q(x)$ is a meromorphic quadratic differential on a Riemann surface C.

Example 2. $\mathcal{M}$ is a space of doubly periodic monopoles, and $\mathcal{D} = D^b \operatorname{Lag}(X)$ where

$$X = \{uv = f(x, y)\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2.$$

These categories include derived categories $D^b \operatorname{Coh}(\omega_Z)$ of local del Pezzo surfaces.

Non-abelian Hodge theory

The Hitchin manifold has different incarnations:

- Dolbeault space $\mathcal{M}_{Dol}$: moduli of Higgs bundles (E, Φ) on C.
- de Rham space $\mathcal{M}_{dR}$: moduli of flat connections (E, ∇) on C.
- Betti space $\mathcal{M}_{Betti}$: moduli of local systems $\rho\colon \pi_1(C) \to \mathrm{SL}_2(\mathbb{C})$.

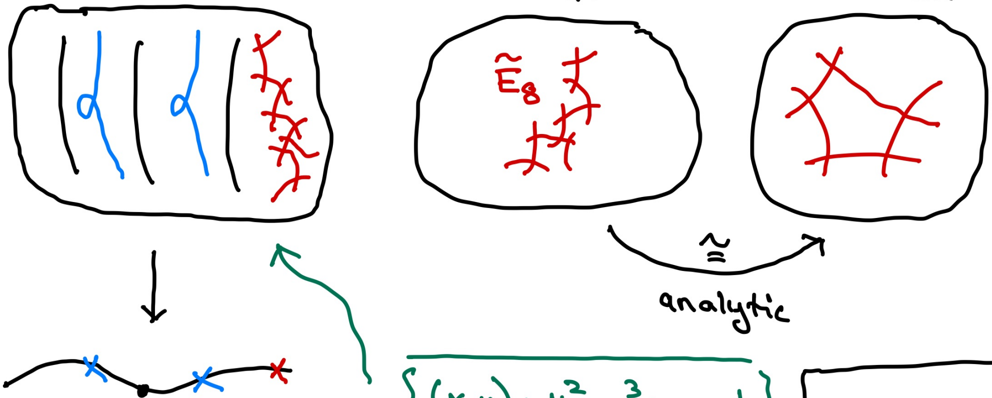
The moduli of λ-connections gives $\mathcal{M}_{Dol}$ (for $\lambda = 0$) and $\mathcal{M}_{dR}$ (for $\lambda = 1$).

The Hitchin map $\mathcal{M}_{Dol} \to H^0(C, \omega_C^{\otimes 2})$ is an abelian fibration.

The Riemann-Hilbert map gives an analytic isomorphism $\mathcal{M}_{dR} \cong \mathcal{M}_{Betti}$

Remark. Once we allow higher-order poles, the definition of $\mathcal{M}_{Betti}$ is less obvious.

$\mathcal{M}_{Dol}$ $\mathcal{M}_{dR}$ $\mathcal{M}_{Betti}$



$$\mathcal{M}_{dR} \xrightarrow[\text{analytic}]{\cong} \mathcal{M}_{Betti}$$

$b \in B = \mathbb{C}$

$$\{(x,y) : y^2 = x^3 + ax + b\}$$

$a \in \mathbb{C}$ <u>fixed</u>.

$$D_{\Gamma_3}(\bullet \to \bullet)$$

Definition of q-difference connections

Passing from Example 1 to 2 we replace flat connections by q-difference connections.

Take $q \in \mathbb{C}^*$ with $|q| \neq 1$ and let $\mathbb{Z}$ act on $\mathbb{C}^*$ via $m_q \colon z \mapsto qz$.

A q-**difference connection** is a $\mathbb{Z}$-equivariant algebraic vector bundle on $\mathbb{C}^* = \mathbb{G}_m$.

Equivalently, it is an algebraic bundle E together with an isomorphism $\theta \colon E \to m_q^*(E)$.

Taking a trivialisation $E \cong \mathcal{O}_{\mathbb{C}^*}^{\oplus n}$ we can write θ as a matrix $A \in \mathrm{GL}_n(\mathbb{C}[z^{\pm 1}])$.

The equation for a $\mathbb{Z}$-invariant section is then $y(qz) = A(z)y(z)$.

Gauge transformations $P(z) \in \mathrm{GL}_2(\mathbb{C}[z^{\pm 1}])$ act by $A(z) \mapsto P(qz)A(z)P(z)^{-1}$.

Taking $q = 1$ gives a multiplicative Higgs bundle: a bundle with an automorphism.

We can also allow finite singularities by replacing $\mathbb{C}[z^{\pm 1}]$ by $\mathbb{C}(z)$.

Non-abelian Hodge theory in the q-difference setting

We can now consider:

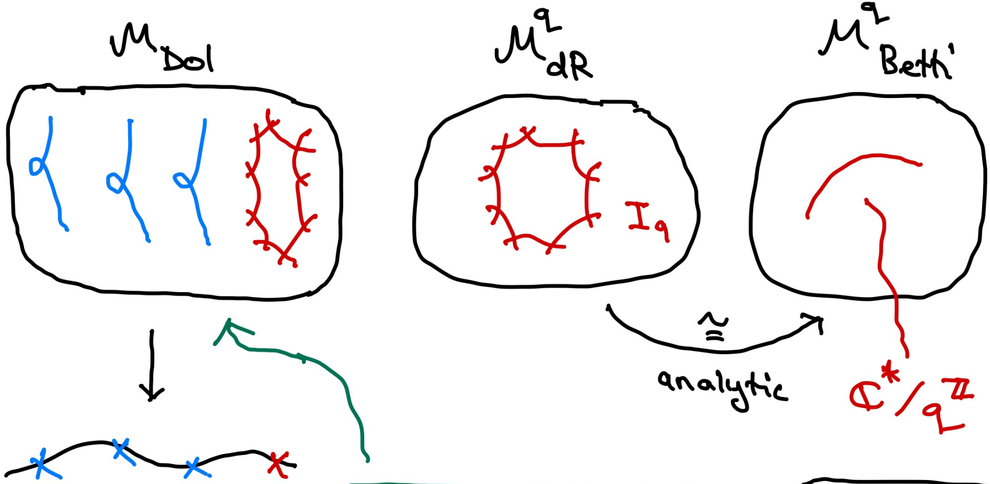
- Dolbeault space $\mathcal{M}_{Dol}$: moduli of multiplicative Higgs bundles.
- de Rham space $\mathcal{M}_{dR}$: moduli of q-difference connections.
- Betti space $\mathcal{M}_{Betti}$: moduli of "monodromies" of q-difference connections.

Our aim is to define $\mathcal{M}_{Betti}$ and compute it in an interesting class of examples.

A $\mathbb{Z}$-equivariant *holomorphic* bundle on $\mathbb{C}^*$ is equivalent to a bundle on the elliptic curve

$$C_q := \mathbb{C}^*/q^{\mathbb{Z}} = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau), \qquad q = e^{2\pi i \tau}.$$

The algebraic structure gives extra data: filtrations by rates of decay of sections at $0, \infty$.



$$\left\{(x,y) : x + y + \frac{1}{xy} = b\right\}$$

$$D^b(\omega_{\mathbb{P}^2})$$

Double ascending filtrations

Let C be an elliptic curve. Given $E \in \mathrm{Coh}(C)$ we define the slope

$$\mu(E) := d(E)/r(E) \in \mathbb{Q} \cup \{\infty\}.$$

A **double ascending filtration** of $E \in \mathrm{Coh}(C)$ is a pair of filtrations

$$0 = P_0 \subset P_1 \subset \cdots \subset P_p = E, \qquad 0 = Q_0 \subset Q_1 \subset \cdots \subset Q_q = E,$$

such that $U_i = P_i/P_{i-1}$ and $V_i = Q_i/Q_{i-1}$ are semistable, with

$$\mu(U_1) < \cdots < \mu(U_p), \qquad \mu(V_1) < \cdots < \mu(V_q).$$

Our main theorem will describe moduli spaces of such objects.

Riemann-Hilbert for q-connections

Coherent sheaves equipped with double ascending filtrations form a category $\mathrm{Coh}(C)^{++}$.

It is defined as a full subcategory of the category of $(\mathbb{Q} \cup \{\infty\})^2$- filtered coherent sheaves.

Haiden, Katzarkov and Simpson proved that $\mathrm{Coh}(C)^{++}$ is an abelian category.

Fix $q \in \mathbb{C}^*$ with $|q| \neq 1$ and set $C_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Conjecture (Kontsevich-Soibelman). There is an equivalence of categories

$$\text{q-diff}_{\mathbb{C}[z^{\pm 1}]} \cong \mathrm{Vect}(C_q)^{++}_{<\infty}.$$

There should also be a version relating holonomic q-difference modules to $\mathrm{Coh}(C_q)^{++}$.

Derived category perspective: boomerangs

A **boomerang** in $D^b\,\mathrm{Coh}(C)$ is a sequence of maps

$$0 = R_0 \xrightarrow{f_1} R_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-1}} R_{h-1} \xrightarrow{f_h} R_h = 0,$$

such that $F_i = \mathrm{Cone}(f_i)$ is semistable for each $1 \leq i \leq n$ and

$$\phi(F_1) < \phi(F_2) < \cdots < \phi(F_h) < \phi(F_1) + 2.$$

One can go from double ascending filtrations to boomerangs:

$$0 = P_0 \subset P_1 \subset \cdots \subset P_p = E, \qquad 0 = Q_0 \subset Q_1 \subset \cdots \subset Q_q = E,$$

$$0 = P_0 \to P_1 \to \cdots \to P_{p-1} \to P_p \to E/Q_1 \to \cdots \to E/Q_{q-1} \to E/Q_q = 0.$$

Remark. Boomerangs make sense in any Δ-category equipped with a stability condition.

Double filtrations and boomerangs

There is a rotation operation on boomerangs acting on semistable factors as

$$\left(F_1, \cdots, F_h\right) \mapsto \left(F_2, \cdots, F_h, F_1[2]\right).$$

Using the above construction we can identify

- Double ascending filtrations in $\mathrm{Coh}(C)$,
- Boomerangs in $D^b\,\mathrm{Coh}(C)$ with phases $\phi(F_i)$ in the interval $(0, 2]$,
- Boomerangs in $D^b\,\mathrm{Coh}(C)$ up to rotation.

Thus $\mathcal{M}_{Betti}$ also parameterises boomerangs up to rotation on $C_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Note that auto-equivalences of $D^b\,\mathrm{Coh}(C_q)$ then give an action of $\mathrm{SL}_2(\mathbb{Z})$.

Numerical invariants: balanced seeds

A double ascending filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_p = E, \qquad 0 = Q_0 \subset Q_1 \subset \cdots \subset Q_q = E,$$

is called **polystable** if the factors $U_j = P_j/P_{j-1}$ and $V_k = Q_k/Q_{k-1}$ are all polystable.

$$\bigoplus_{j=1}^{p} U_j = \bigoplus_{j\in J} S_j^{\oplus m_j}, \qquad \bigoplus_{k=1}^{q} V_k = \bigoplus_{k\in K} T_k^{\oplus n_k}, \qquad S_{j_1} \ncong S_{j_2}, \; T_{k_1} \ncong T_{k_2}.$$

The associated **balanced seed** $\mathbf{w} = \{w_i : i \in I\} \subset \mathbb{Z}^2$ is the collection of vectors

$$\mathbf{w} = \left\{ \mathrm{ch}(S_j^{\oplus m_j}) : j \in J \right\} \cup \left\{ -\mathrm{ch}(T_k^{\oplus n_k}) : k \in K \right\}, \qquad \mathrm{ch}(E) := (r(E), d(E)).$$

Note that $\sum_{i\in I} w_i = 0$. In the non-polystable case the definition of $\mathbf{w}$ is more complicated.

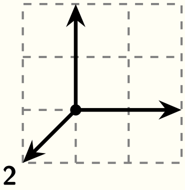
Pictorial representation: decorated lattice polygons

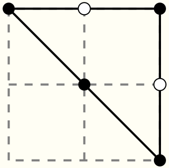
A balanced seed is a collection of nonzero vectors $\{w_i : i \in I\} \subset \mathbb{Z}^2$ satisfying $\sum_{i \in I} w_i = 0$.

There is an associated convex lattice polygon with edges $\sqrt{-1} \cdot w_i$.

It has a natural decoration: the boundary lattice points are labelled black or white.

Example. Take $\mathbf{w} = \{(2,0),(0,2),(-1,-1)^2\} \subset \mathbb{Z}^2$.





Finer invariants: K-lifts

Consider the Chern character map

$$ch = (r, d) \colon K_0(C) = \mathbb{Z}^2 \oplus Pic^0(C) \longrightarrow \mathbb{Z}^2.$$

Given a balanced seed $\mathbf{w} = \{w_i : i \in I\} \subset \mathbb{Z}^2$ we write

$$w_i = \ell_i \cdot v_i, \qquad \ell_i \in \mathbb{N}, \qquad v_i \in \mathbb{Z}^2 \text{ primitive.}$$

A K-**lift** of $\mathbf{w}$ is a collection $\mathbf{s} = \{s_i : i \in I\} \subset K_0(C)$ such that

$$\mathrm{ch}(s_i) = v_i, \qquad \sum_{i \in I} \ell_i \cdot s_i = 0.$$

When $\mathbf{w}$ comes from a double ascending filtration it has a canonical K-lift

$$\mathbf{s} = \{[S_j] : j \in J\} \cup \{-[T_k] : k \in K\} \subset K_0(C).$$

Moduli spaces of double ascending filtrations

Theorem.

Let C be an elliptic curve and $\mathbf{w} = \{w_i : i \in I\} \subset \mathbb{Z}^2$ a balanced seed. Then there exists

- a smooth projective variety $\mathrm{Base}_{\mathbf{w}}(C)$ parameterising all K-lifts of $\mathbf{w}$,
- a finite-type algebraic stack $\mathrm{Boom}_{\mathbf{w}}(C)$ of double ascending filtrations of type $\mathbf{w}$,
- an open substack $\mathrm{Boom}^{\circ}_{\mathbf{w}}(C) \subset \mathrm{Boom}_{\mathbf{w}}(C)$ of polystable points,
- a canonical map

$$\pi \colon \mathrm{Boom}_{\mathbf{w}}(C) \to \mathrm{Base}_{\mathbf{w}}(C).$$

Balanced seeds of del Pezzo type

There is a (partially-defined) operation of mutation on balanced seeds.

There are 10 mutataion equivalence classes which we call of del Pezzo type.

These are in bijection with the 10 deformation types of del Pezzo surface.

The associated polygons include the 14 reflexive lattice polygons.

Main Theorem - short version. (Pierrick Bousseau, Luca Giovenzana, -)

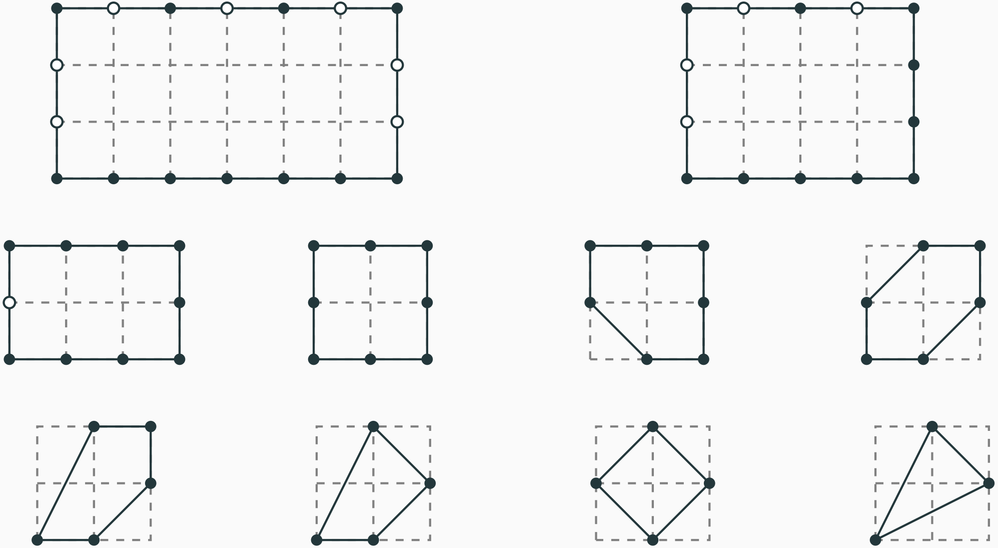
If **w** is of del Pezzo type then the general fibre of

$$\pi\colon \mathsf{Boom}_{\mathbf{w}}(C) \to \mathsf{Base}_{\mathbf{w}}(C)$$

is isomorphic to $(Z \setminus C) \times B\mathbb{C}^*$, where

- Z is a del Pezzo surface of the relevant type,
- $C \hookrightarrow Z$ is an embedding whose image is an anti-canonical divisor.

Decorated polygons from the 10 mutation-classes of del Pezzo type



Example: part I

Take the balanced seed $\mathbf{w} = \{(1,-2),(1,1),(-2,1)\}$.

It is of del Pezzo type and is in the mutation equivalence-class corresponding to $\mathbb{P}^2$.

Double ascending filtrations of this numerical type take the form

$$0 = P_0 \subset P_1 \subset P_2 = E, \qquad 0 = Q_0 \subset Q_1 = E,$$

with semistable factors $U_1 = P_1$, $U_2 = P_2/P_1$ and $V_1 = E$, and

$$\mathrm{ch}(U_1) = (1,-2), \qquad \mathrm{ch}(U_2) = (1,1), \qquad \mathrm{ch}(V_1) = (2,-1).$$

Since all the vectors are primitive the double filtration is automatically polystable.

Example: part II

The stable objects U_1, U_2, V_1 are determined by their classes in $K_0(C)$. Thus

$$\mathrm{Base}_{\mathbf{w}}(C) = \mathrm{Pic}^{-2}(C) \times \mathrm{Pic}^1(C) \cong C \times C.$$

The fibre of $\pi\colon \mathrm{Boom}_{\mathbf{w}}(C) \to \mathrm{Base}_{\mathbf{w}}(C)$ over a point (U_1, U_2) parameterises sequences

$$0 \longrightarrow U_1 \longrightarrow V_1 \longrightarrow U_2 \longrightarrow 0.$$

There is a $\mathbb{P}^2$ family of nonzero maps $f\colon U_1 \to V_1$ up to scale.

All these maps are injective, and if the quotient is torsion-free then it is isomorphic to U_2.

But for each point $x \in C$ there is a line bundle $L \in \mathrm{Pic}^0(C)$ and a sequence

$$0 \longrightarrow U_1 \longrightarrow V_1 \longrightarrow \mathcal{O}_x \oplus L \longrightarrow 0.$$

This gives an embedding $C \hookrightarrow \mathbb{P}^2$ and the moduli space $\mathbb{P}^2 \setminus C$ is the complement of these.

Torelli theorem for del Pezzo pairs

Take a balanced seed **w** of del Pezzo type and a del Pezzo $Z_{\mathbf{w}}$ of the corresponding type.

Introduce a space $\mathrm{Pairs}_{\mathbf{w}}(C)$ of **framed anti-canonical pairs** parameterising

(i) a del Pezzo Z, and an embedding $C \hookrightarrow Z$ whose image is an anti-canonical divisor,

(ii) an isometry $\theta\colon H^2(Z_{\mathbf{w}}, \mathbb{Z}) \to H^2(Z, \mathbb{Z})$ with $\theta(K_{Z_{\mathbf{w}}}) = K_Z$.

Theorem. $\mathrm{Pairs}_{\mathbf{w}}(C)$ is a smooth quasi-projective variety, with a dense open embedding

$$\mathrm{Pairs}_{\mathbf{w}}(C) \longrightarrow \mathrm{Base}_{\mathbf{w}}(C)/\operatorname{Pic}^0(C)$$

Universal family of anti-canonical complements

Main Theorem - full version. (Pierrick Bousseau, Luca Giovenzana, -)

There is a Cartesian diagram

$$\begin{tikzcd} B\mathbb{C}^* \times (\mathcal{Z} \setminus \mathcal{C}) \arrow[r, hook] \arrow[d] & \mathrm{Boom}^\circ_{\mathbf{w}}(C)/\operatorname{Pic}^0(C) \arrow[d] \\ \mathrm{Pairs}_{\mathbf{w}}(C) \arrow[r, hook] & \mathrm{Base}_{\mathbf{w}}(C)/\operatorname{Pic}^0(C) \end{tikzcd}$$

where $\mathcal{Z} \setminus \mathcal{C} \to \mathrm{Pairs}_{\mathbf{w}}(C)$ is the universal family of complements $Z \setminus C$.

Key idea for the proof. Take a geometric exceptional collection $(E_1, \cdots, E_n)$ on Z.

Given a point $x \in Z$ we get a filtration of $\mathcal{O}_x$ by shifts of sums of the dual objects F_i.

If $x \in Z \setminus C$ then restricting to C we get a filtration of 0, which is in fact a boomerang.

Happy Birthday, Nigel!