9/29/23

Clay, Oxford

Integrable many-particle systems: generalized Entropy and Scattering

Herbert Spohn

TUMünchen

1. Entropy of integrable systems

many particles fine-tuned

Stat Mech

Gibbs entropy $\overset{\text{Legendre}}{\Longleftrightarrow}$ free energy

- conservation laws N, H

$F(\mu\beta,\beta) = -\log Z(\mu\beta,\beta)$ $\qquad Z(\mu\beta,\beta) = \int dq\,dp\; e^{-\beta(H-\mu N)}$

- integrable system extensive $Q^{[0]}$, $Q^{[1]}$, $Q^{[2]}$, ...

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad N \qquad P \qquad H$

$$\Rightarrow \left\| Z(\vec{\mu}) = \int dq\,dp\; e^{-\sum_{n\geq 0} \mu_n Q^{[n]}} \right\|$$

GGE generalized Gibbs

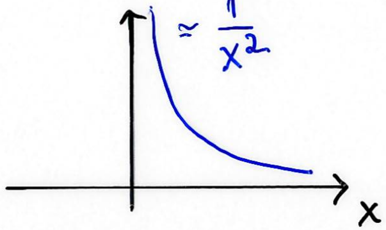
thermal $\mu_n = 0$ for $n \geq 3$

Entropy is functional!

shows in linearized GHD

2. Calogero fluid

$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \frac{1}{2} \sum_{i \neq j = 1}^{N} V_{ca}(q_i - q_j) \qquad V_{ca}(x) = \frac{1}{\sinh^2 x}$$



- Lax matrix

$$L_{ij} = \delta_{ij} p_j + i (1 - \delta_{ij}) \frac{1}{\sinh(q_i - q_j)}$$

$N \times N$

$L \psi_j = \lambda_j \psi_j$ eigenvalues $\Rightarrow$ locally conserved fields $\Leftarrow$

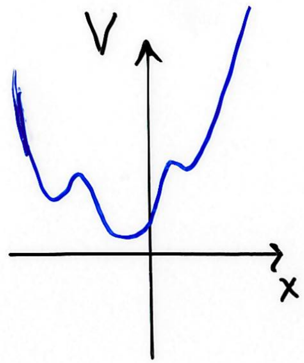
$$Q^{[n]} = \operatorname{tr} L^n = \sum_{j=1}^{N} (L^n)_{jj} = \sum_{j=1}^{N} (\lambda_j)^n$$

- density $Q^{[n]}(x) = \sum_{j=1}^{N} (L^n)_{jj} \, \delta(q_j - x)$ $\rightsquigarrow$ $\partial_t Q^{[n]}(x,t) + \partial_x \mathfrak{J}^{[n]}(x,t) = 0$

$N = \operatorname{tr} L^0$, $P = \operatorname{tr} L$, $H = \operatorname{tr} L^2$

$\rightsquigarrow$ GGE $\quad e^{-\operatorname{tr}(V(L))}$

formally $V(x) = \sum_{n \geq 0} \mu_n x^n$



"confining potential"

GGE $\quad dq\,dp\ e^{-\operatorname{tr} V(L)}\ e^{-\sum_{j=1}^{N} e^{-\ell} \cosh q_j}$

forces size ℓ

$\ell \to \infty \quad \frac{\ell}{N} = \nu$ fixed

L is random matrix under GGE

- density of states $\quad \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda_j - w) \xrightarrow[\ell \to \infty]{} \rho_p(w)\,dw$ non random

RESULT $\quad \rho \geq 0, \ \int \rho\, dw = \nu,$ HS (2023)

$$\mathcal{F}(\rho) = \int dw\, \rho(w) \Big(\underbrace{V(w) - 1 + \log \rho(w)}_{\text{ideal gas}} - \underbrace{\log\Big(1 + \int dw' \rho(w')\, \phi_{ca}(w - w')\Big)}_{\text{interaction}} \Big)$$

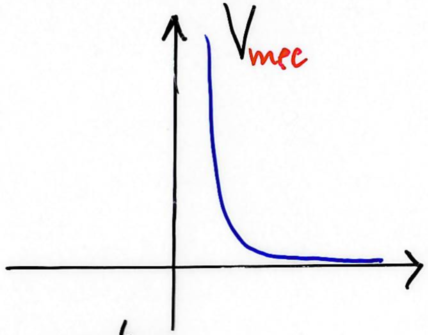
velocities $e^{-V(w)}$

$\phi_{ca}(w) = -\log\left(1 + \frac{1}{w^2}\right)$ minimizer ρ_p

$$\lim_{\ell \to \infty} \frac{1}{\ell} F_\ell(\nu, V) = \mathcal{F}(\rho_p)$$

scattering theory

Hubacher 1989

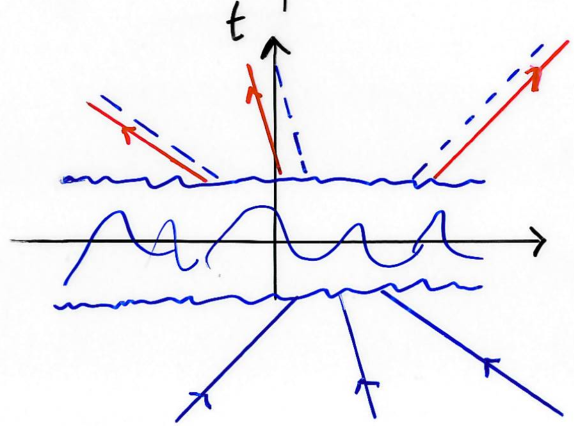


$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \frac{1}{2} \sum_{i \neq j = 1}^{N} V_{mec}(q_i - q_j)$$

$t \to \pm \infty$

$$\frac{1}{t} q_j(t) \to p_j^{\pm}$$

$$\lim_{t \to \pm\infty} q_j(t) - p_j t = \phi_j^{\pm}$$



relative shift $\phi^+_{N-j+1} - \phi^-_j = \alpha_j$

quasi-particles ONLY integrable

- integrable Calogero

$$p^+_{N-j+1} = p^-_j = \lambda_j$$

$$\alpha_j = -\sum_{i=1}^{j-1} \phi_{ca}(w_i - w_j) + \sum_{i=j+1}^{N} \phi_{ca}(w_i - w_j)$$

2- particle scattering shift

WHY?

holds also for

- Toda chain
- Lieb - Liniger δ-Bose

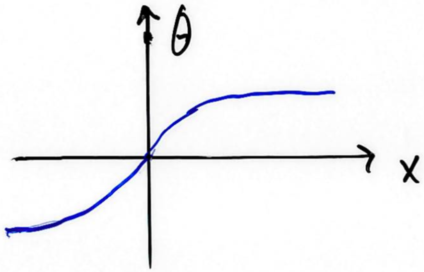
3. No surprise

integrable quantum models

Lieb-Liniger $\quad H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + c \, \frac{1}{2} \sum_{i \neq j = 1}^{N} \delta(x_i - x_j)$ $\quad c > 0$ repulsive $\quad p_j = -i \frac{\partial}{\partial x_j}$

eigenvalues $\quad$ Bethe ansatz

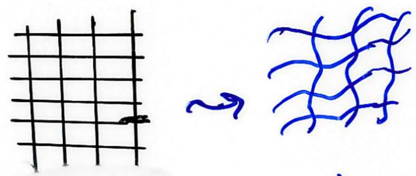
Bethe equations $\quad 2\pi I_j = \ell k_j + \sum_{i=1}^{N} \theta(k_j - k_i)$ $\quad$ *



integer input $\{I_1 < \dots < I_N\}$ $\qquad$ output $\{k_1 < \dots < k_N\}$

Bethe roots

GGE $\quad e^{-\sum_{j=1}^{N} V(k_j)} \times$ counting



free $\qquad$ interacting

thermal $\quad V(x) = x^2 \beta$

Yang, Yang 1969

Yang, Yang free energy functional

$$\theta'(w) = \frac{2c}{w^2 + c^2}$$

• Lewis et al 1989

2-particle scattering shift

* mean field (empirical measure)

4. Big surprise Ablowitz-Ladik (AL) lattice

$\alpha_j \in \mathbb{D} = \{ z \mid |z| \leq 1 \}$ $\qquad \frac{d}{dt} \alpha_j = i (1 - |\alpha_j|^2)(\alpha_{j-1} + \alpha_{j+1})$

integrable discretized defocusing NLS

Mazzuca, Grava 2022

HS 2022

Mazzuca, Memin 2022

Poisson brackets

$$\{ f, g \} = i \sum_j \rho_j^2 \left(\partial_{\bar{\alpha}_j} f \, \partial_{\alpha_j} g - \partial_{\alpha_j} f \, \partial_{\bar{\alpha}_j} g \right)$$

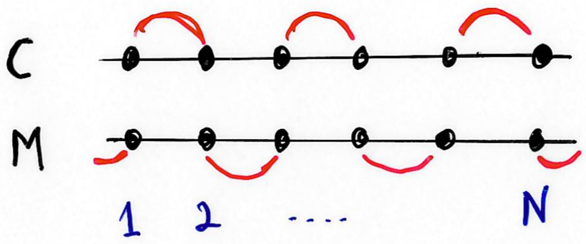
$\rho_j^2 = (1 - |\alpha_j|^2)$

hamiltonian $\quad H = - \sum_j (\alpha_j \bar{\alpha}_{j+1} + \bar{\alpha}_j \alpha_{j+1})$ $\qquad \frac{d}{dt} \alpha_j = \{ \alpha_j, H \}$

GGE ring $j = 1, \ldots, N$, N even

Lax matrix

I. Nenciu 2005



2x2 block $\quad \Xi = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}$ $\qquad L = CM$

unitary

- L is CMV matrix Cantero, Moral, Velazquez 2005

eigenvalues $z_1, \dots, z_N$, $|z_j| = 1$

→ locally conserved fields $\mathrm{tr}\, L^n = Q^{[n]}$

⇝ GGE $\prod_{j=1}^{N} d^2\alpha_j \, (\rho_j^2)^{-1+P} \, e^{-\mathrm{tr}\, \hat{V}(L)}$ | $\hat{V}$ real-valued

pressure $P > 0$ parameters P, V

$z_j = e^{i\vartheta_j}$ DOS $\frac{1}{N} \sum_{j=1}^{N} \delta(\vartheta_j - w)$

- free energy functional

$$\mathcal{F}(\rho) = \int_0^{2\pi} dw\, \rho(w) \Big(V(w) + \log \rho(w) - \int_0^{2\pi} dw' \log|e^{iw} - e^{iw'}| \rho(w') \Big)$$

$\rho \geq 0$, $\int_0^{2\pi} dw\, \rho(w) = P$ free energy $F = \mathcal{F}(\rho_P)$

scattering shift $\quad \log | e^{iw} - e^{iw'} |^2$

Brollo, HS 2023

|| scattering of WHAT ? ||

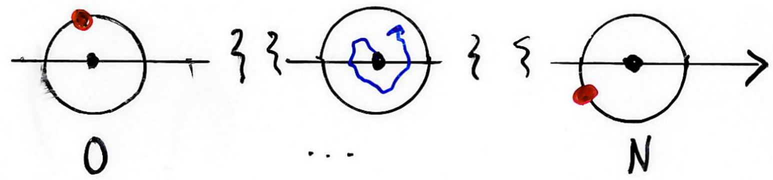
based on Moser 1975

Killip, Nenciu 2007

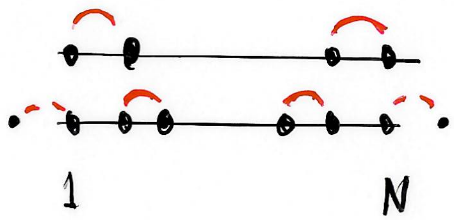
- closed AL chain , ring ⇝ quasi-periodic motion

⇒ open AL chain

$$e^{i\vartheta_-}, \alpha_1, \dots, \alpha_{N-1}, e^{i\vartheta_+}$$



Lax



$$\lim_{t\to\infty} \alpha_j(t) = (-1)^{j+1}\, \bar{z}_1 \cdots \bar{z}_j$$

$$\lim_{t\to-\infty} \alpha_j(t) = (-1)^{j+1}\, \bar{z}_{N-j+1} \cdots \bar{z}_N$$

|| exponentially fast ||

$$\mathrm{tr}\, L = \sum_{j=1}^{N} e^{i\vartheta_j} = \sum_{j=1}^{N} \cos\vartheta_j + i \sum_{j=1}^{N} \sin\vartheta_j \qquad \sin\vartheta_j = \lambda_j$$

energy momentum

$$\lambda_N < \dots < \lambda_1 \quad !!$$

almost surely

$$\lim_{t\to\infty} \frac{1}{t} \log \rho_j^2(t) = \lambda_{N-j+1} - \lambda_{N-j}$$

$$\left[\!\left[\; -\log \rho_j^2 \Longleftrightarrow q_{j+1} - q_j = r_j \; \right]\!\right]$$

$$\lim_{t\to-\infty} \frac{1}{t} \log \rho_j^2(t) = \lambda_{j+1} - \lambda_j$$

$$\lim_{t\to\infty} \rho_{N-j}^2(t)\, \rho_j^2(-t)\, e^{2(\lambda_{j+1}-\lambda_j)t}$$

$$= |z_{j+1} - z_j|^4 \prod_{m=1}^{j-1} \left| \frac{z_{j+1} - z_m}{z_j - z_m} \right|^2 \prod_{m=j+2}^{N} \left| \frac{z_j - z_m}{z_{j+1} - z_m} \right|^2$$

implies $\phi_{AL} = \log |e^{iw} - e^{iw'}|^2$

modified AL (Schur flow) energy $\rightsquigarrow$ momentum

$$\tilde{H} = -i \sum_j (\alpha_{j-1}\bar{\alpha}_j - \bar{\alpha}_{j-1}\alpha_j) = -i \,\mathrm{tr}\,(C - C^*)$$

$\Rightarrow$ real data propagate $\Leftarrow$ $\alpha_j \in [-1, 1]$

$$\frac{d}{dt}\alpha_j = \rho_j^2 (\alpha_{j+1} - \alpha_{j-1})$$

b.c. $\alpha_0 = \pm 1$, $\alpha_N = \pm 1$

2-fold degenerate eigenvalues $z, \bar{z}$

order

$\cos \lambda_j$

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 \dots \lambda_j > \lambda_{j+1}$$

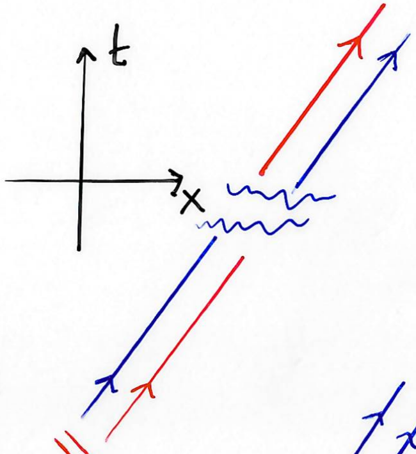
$$\lim_{t\to\infty} \alpha_j(t) = \frac{1}{2}(z_j + \bar{z}_j) \quad \text{inside } [-1, 1], \quad j \text{ odd}$$

$$= 1, \quad j \text{ even}$$

- higher order corrections $\Leftarrow$

$$r_j = -\log \rho_j^2$$

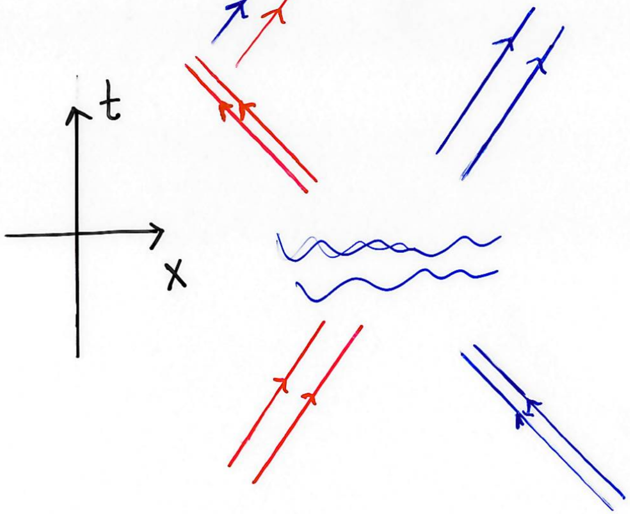
- one - body



shift $z, \bar{z}$ $\quad z = e^{i\vartheta}$

$\log \sin^2 \vartheta$

- two - body



shift $z_1 = e^{i\vartheta_1}, \; z_2 = e^{i\vartheta_2}$

$$\phi_{1,2} = \log|2\cos\vartheta_1 - 2\cos\vartheta_2|$$

- agrees with GGE

$$\mathcal{F}(\rho) = \int_0^{\pi} dw\, \rho(w) \Big(V(w) + \log\rho(w) + \log\sin w - \mathcal{P}\int_0^{\pi} dw\, \rho(w') \log\left(2|\cos w' - \cos w|\right)\Big)$$

1- body $\qquad$ 2-body

5. Summary / future

generalized free energy $\Longleftrightarrow$ scattering shift

$\rightarrow$ Toda chain, Calogero fluid, Ablowitz-Ladik chain

low density

further models

- continuum NLS
- Korteweg-de Vries
- box-ball

scattering shift ✓

|| two solitons ||

particle based
soliton based } GHD