Mean-field limit of (non)-exchangeable multi-agent systems

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Goal:

Mean-field limit for multi-agent systems

$\Longrightarrow$ Singular (Coulomb) interaction agents

Major open problem

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Connetivities of dense graphs

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Type of cooperations (symmetry)

Connetivities of dense graphs

Results:

Novel hierarchy of observables

New L^p estimates of the marginals of the system

Allowing very singular interaction kernels

Non-exchangeable agents on sparse-graphs

$\rightarrow$ New concept of limits: extended graphons

A general class of N-agents interacting system:

$$dX_i = \sum_{j=1}^{N} w_{ij}^N K(X_i - X_j)\, dt + \sqrt{2\sigma}\, dW_i$$

$$X_i(0) = X_i^0 \in \mathbb{R}^d$$

where

X_i	positions/activities
w_{ij}	weights/connectivities
W_i	N independent Wiener processes
K	interaction kernel

Classical mean-field theory:

$$dX_i = \frac{1}{N}\sum_{j=1}^{N} K(X_i - X_j)\,dt + \sqrt{2\sigma}\,dW_i$$

$$X_i(0) = X_i^0 \in \mathbb{R}^d$$

$w_{ij} \sim \frac{1}{N}$ exchangeable particles

K interaction kernel

$$\downarrow$$

$$\frac{\partial f}{\partial t} + \mathrm{div}\left(f(K * f) - \sigma\nabla f\right) = 0$$

Classical mean-field theory: **Second order systems**

$$\frac{d}{dt}X_i(t) = V_i(t)$$
$$dV_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,dt + \sqrt{2\sigma}\,dW_i$$

Rigorous limit $\downarrow$ $N \to \infty$

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + K * \rho\cdot\nabla_v f = \sigma\Delta_v f$$

Vlasov-Fokker-Planck equation

distribution: $f(t,x,v)$

density: $\rho(t,x) = \int f(t,x,v)\,dv$

Classical mean-field theory: Bibliographic background

$K \in W^{1,\infty}$ + f is smooth enough

McKean 1967, Braun-Hepp 1977, Dobrushin 1979, ...

Neunzert, Sznitmann, ..., Golse

Estimates on the trajectories distance one by one

$$\left(\frac{1}{N} \mathbb{E} \left(\sum_{i=1}^{N} \left| (X_i, V_i) - (\bar{X}_i, \bar{V}_i) \right|^p \right) \right)^{\frac{1}{p}}$$

where $(\bar{X}_i, \bar{V}_i)$ are N identical copies of

$$\frac{d}{dt} \bar{X}(t) = \bar{V}(t)$$

$$d\bar{V}(t) = \left(K \star \int f(t, \bar{X}, v)\, dv \right) dt + \sqrt{2\sigma}\, dW$$

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$$\|\mu_N - f\|_{W^{-1,1}} \leq c\, e^{t\,\|\nabla K\|_{L^\infty}} \|\mu_N^0 - f^0\|_{W^{-1,1}}$$

$$\mu_N(t,x,v) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}(x)\delta_{V_i(t)}(v)$$

(empirical measure)

$\Longrightarrow$ This trajectorial approach requieres $K \in W^{1,\infty}$

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(empirical measure)

$|K| \leq \frac{c}{|x|^\theta},\ \theta < 1$

Hauray-Jabin 2015

$\rightarrow$ only in the deterministic case

$\rightarrow$ in the stochastic case

$d = 1$

Hauray-Salem 2019

$\rightarrow$ in the stochastic case

$d = 1$

Hauray-Salem 2019

$d > 1$: Truncated kernels

Huang-Liu-Pickl 2020, Pickl *et al.* $\theta < \frac{1}{d},\ d = 3$

Typically $\phi_\varepsilon = \dfrac{C}{(\varepsilon_N + |x|)^{d-2}}$, $K_\varepsilon = -\nabla\phi_\varepsilon$, $d \geq 3$

where $\varepsilon_N = N^{-\theta}$

with the critical scale $\theta = \frac{1}{d}$

Interaction kernel:

$K = -\nabla\phi$, for a repulsive and nonnegative ϕ

Main example:

Coulombian interactions

$$\rightarrow \ \phi = -c \ln|x|, \quad \text{if } d = 2$$

$$\rightarrow \ \phi = \frac{c}{|x|^{d-2}}, \quad \text{if } d \geq 3$$

For simplicity:

$$X_i \in \mathbb{T}^d$$

$$V_i \in \mathbb{R}^d$$

Classical mean-field theory: Statistical approach

- New object: We put individual trajectories aside

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- New object: We put individual trajectories aside
- Full joint law at time t: $f_N(t, x_1, v_1, \dots, x_N, v_N)$

  satisfies the Liouville equation:

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N + \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \sigma \sum_i \Delta_{v_i} f_N,$$

We need velocity decay to control ρ_N

Statistical approach

- Introduce a joint law of the "trajectories" as a kind of projection: k-point marginals

- Marginal laws of $X_1, V_1, \ldots, X_k, V_k$ at time t:

$$f_{k,N}(t, x_1, v_1, \ldots, x_k, v_k) =$$
$$\int_{\mathbb{T}^{d(N-k)} \times \mathbb{R}^{d(N-k)}} f_N(t, x_1, v_1 \ldots, x_N, v_N)\, dx_{k+1}\, dv_{k+1} \ldots dx_N\, dv_N.$$

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- Question: $f_{k,N} \rightharpoonup f^{\otimes k}$?

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i \le k} \frac{1}{N} \sum_{j \le k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$
$$+ \frac{N-k}{N} \sum_{i \le k} \nabla_{v_i} \cdot \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{k+1,N} K(x_i - x_{k+1}) dx_{k+1} dv_{k+1}$$
$$= \sigma \sum_{i \le k} \Delta_{v_i} f_{k,N}.$$

Critical new idea

the new term has the same scaling as the convolution at the limit

Moment propagation:

$$e^{\sum_{i \le k} (1+|v_i|^2)} \qquad .$$

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\leq k} \frac{1}{N} \sum_{j\leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$
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$$e^{e_k(x_1,v_1,\ldots,x_k,v_k)} = e^{\sum_{i\leq k}(1+|v_i|^2)+\frac{1}{N}\sum_{i,j\leq k}\phi(x_i-x_j)}.$$

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\le k} \frac{1}{N} \sum_{j\le k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$
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Critical new idea

the new term has the same scaling as the convolution at the limit:

$\Longrightarrow$ behaves well with (weighted) L^p norms

$$\|f_{k,N}\|^q_{L^q_{\lambda(t)e_k}} = \int_{\mathbb{T}^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q\, e^{\lambda(t)\, e_k}$$

$$e_k(x_1, v_1, \ldots, x_k, v_k) = \sum_{i\le k}(1 + |v_i|^2) + \frac{1}{N} \sum_{i,j\le k} \phi(x_i - x_j).$$

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\leq k} \frac{1}{N} \sum_{j\leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$
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Avoid loosing a derivative in v_i

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Avoid loosing a derivative in v_i

In parallel:

- Lacker (2021) studies the propagation of marginals on 1st (2nd) order systems with non-degenerate diffusion using the relative entropy, but require interaction kernels K in some exponential Orlicz space
- Jabin-Poyato-S, 2021, for non-exchangeable systems

New results: $d = 2, 3$.

Assume $K \in L^p(\mathbb{T}^d)$, $p > 1$, and define

$$\lambda(t) = \frac{1}{\Lambda\,(1+t)}, \qquad L = \frac{C}{\lambda(1)^\theta}\|K\|_{L^p}^q, \quad q > p'.$$

Consider a renormalized solution f_N to the Liouville eq satisfying the Gaussian decay with $f_N^0 \in L^\infty(\mathbb{T}^{dN} \times \mathbb{R}^{dN})$, such that

$$\int_{\mathbb{T}^{kd}\times\mathbb{R}^{kd}} |f_{k,N}^0|^q\, e^{\lambda(0)\, e_k} \leq F_0^k, \qquad \sup_{t\leq 1} \int_{\mathbb{T}^{Nd}\times\mathbb{R}^{Nd}} |f_N|^q\, e^{\lambda(t)\, e_N} \leq F^N$$

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Then, one has that

$$\sup_{t \leq T} \int_{\mathbb{T}^{kd} \times \mathbb{R}^{kd}} |f_{k,N}|^q \, e^{\lambda(t)\, e_k} \leq 2^k \, F_0^k + F^k \, 2^{2k-N-1}$$

New results (Bresch, Jabin & JS)

Assume that (d=2)

- $K = -\nabla\phi \in L^p(\mathbb{T}^d)$, for some $p > 1$,
- $\int_{\mathbb{T}^d} e^{\theta\,\phi(x)}\,dx < +\infty, \quad \theta > 0,$
- Let f be the unique smooth solution to the Vlasov equation with initial data $f^0 \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ & $\int_{\mathbb{T}^d\times\mathbb{R}^d} f^0\, e^{\beta\,|v|^2} < \infty$
- $f^0_{k,N} \rightharpoonup (f^0)^{\otimes k}$ in L^1, $\quad f^{\otimes k} = \prod_{i=1}^{k} f(t, x_i, v_i)$
- $\|f^0_{k,N}\|_{L^\infty(\mathbb{T}^{dN}\times\mathbb{R}^{dN})} \le M^k$, for some $M > 0,\ \forall k < N$

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Then, there exists T^* such that

$$f_{k,N} \rightharpoonup f^{\otimes k}, \quad \text{in } L^q_{loc}\left([0,\ T^\star] \times \mathbb{T}^{kd} \times \mathbb{R}^{kd}\right)$$

Strong propagation of Chaotic/Tensorized law

Remarks

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  Quantitative estimate for the case $K \in L^2$.

- Extension to the stochastic case of mildly singular kernels

- Does not apply directly in $d \geq 3$ to i.i.d. X_i^0 as $\int e^{\lambda\phi(x)} dx = +\infty,\ \forall\lambda > 0$, if $\phi = \frac{c}{|x|^{d-2}}$.

New results: **First order systems**

$$dX_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sqrt{2\sigma}\, dW_i$$

$$\downarrow$$

$$\frac{\partial f}{\partial t} + \operatorname{div}\left(f(K * f) - \sigma \nabla f\right) = 0$$

Applications:

- Keller-Segel, Euler / Navier-Stokes

  Fetecau, Huang-Liu, Pickl, Bresch-Jabin-Wang,...

  Chorin, Goodman, Beale-Majda, Cottet, Fournier, S.,

  Hauray, Jabin-Wang, Wynter,

  Rosenzweig, Duerinckx, Serfaty ($u \in W^{1,\infty}$),...

New results: First order models (Bresch, Jabin & JS).

Assume that

- $K \in L^p(\mathbb{T}^d)$, for some $p > 1$, $\boxed{K \sim \frac{1}{|x|^s}}$ $s < d$
- $(\mathrm{div} K)_- \in L^\infty(\mathbb{T}^d)$,
- Let f be the unique smooth solution to the transport equation with initial data $f^0 \in C^\infty(\mathbb{T}^d)$,
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Then, there exists T^* such that

$$\boxed{f_{k,N} \rightharpoonup f^{\otimes k}, \ \text{in } L^q_{loc}\left([0, \ T^\star] \times \mathbb{T}^{kd}\right)}$$

$\Longrightarrow$ Strong propagation of chaos

Non-exchangeable systems:

A prototype to study non-exchangeable systems is

$$dX_i = \sum_{j=1}^{N} w_{ij}^N \, K(X_i - X_j) \, dt + \cancel{\sqrt{2\sigma} \, dW_i},$$

where

$\frac{1}{N} \to w_{ij}^N$ non-necessarily symmetric interaction weights

Assumptions on the interaction weights:

Objective: stay within the mean-field limit

$$\max_{1\le i\le N} \sum_{j=1}^{N} |w_{ij}^N| = O(1)$$

Total interaction with any object must be finite

$$\max_{1\le i,j\le N} |w_{ij}^N| \stackrel{N\to\infty}{\longrightarrow} 0$$

Individual coefficients should be small

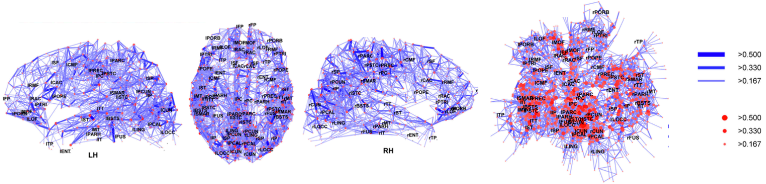
Some examples:

Brain neural networks

Many neurons: Human brains contain $\sim 86 \cdot 10^9$ neurons.

Sparseness & modularity: Each neuron has synaptic connections with only $7 \cdot 10^3$ neurons. Human Connectome is organized into structural cores and modules.

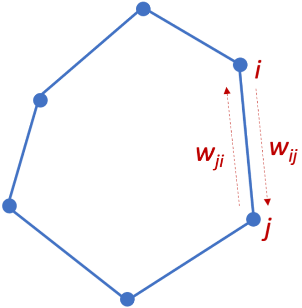
Synchronization: emerges as a consequence of the interplay of the local dynamics in regions with large intra connectivity and the topology of non-symmetric connectivities.



Some examples:

Agents on a graph

Brain structure, Epidemiology, Machine Learning, ...



Sparseness: We don't expect to have many interactions

$w_{ij} = 0$ for most i, j.

There is another scale within the graph: $\forall i$, the number $\#j$ of order N^θ such that $w_{ij} \sim \frac{1}{N^\theta}$

$\Longrightarrow$ w_{ij} satisfy the assumptions and the law of large numbers still applies

How different is the study of the mean-field limit of the classical case with respect to this context?

Strategies regarding the structure of w_{ij}: **simple structure**

For $w_{ij} = m_j$, define

$$\nu_N = \sum_{j=1}^{N} m_j \, \delta(x - X_j)$$

which verifies (symmetrization)

$$\frac{\partial \nu_N}{\partial t} + \mathrm{div}\left((K \star \nu_N)\nu_N\right) = \sigma \Delta \nu_N$$

$\Longrightarrow$ In Fluid Mechanics gives the total vorticity,

but ν_N can lose the probability measure character

and modulated energy techniques could give problems.

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For m_j random and i.i.d. Then $m_j \sim \frac{1}{N}$

Defining $\tilde{X}_j = m_j X_j \Longrightarrow$ case of exchangeable systems

extend the entropy approaches

Strategies regarding the structure of w_{ij}^N:

When w_{ij}^N do not have any simple structure

- It no longer seems possible to define an empirical measure

$\rightsquigarrow$ Try to incorporate w_{ij}^N as part of the state

Can we write

$$w_{ij}^N = w(w_{ij}^N) = w(\xi_i, \xi_j), \quad \xi_j \in [0, 1]$$

for some distribution of ξ_j and some kernel w?

$\Longrightarrow$ we go back to a classical mean-field context for which the regularity of w is crucial

Strategies regarding the structure of w_{ij}^N:

We would like to solve the mean-field limit equation associated with this new kernel

$$\partial_t \bar{f}(t,x,\xi) + \mathrm{div}_x \left(\bar{f}(t,x,\xi) \int_0^1 w(\xi,\zeta) \int_{\mathbb{R}^d} K(x-y) \bar{f}(t,dy,d\zeta) \right) = \sigma \Delta \bar{f}(t,x,\xi)$$

Strategies regarding the structure of w_{ij}^N:

We would like to solve the mean-field limit equation associated with this new kernel

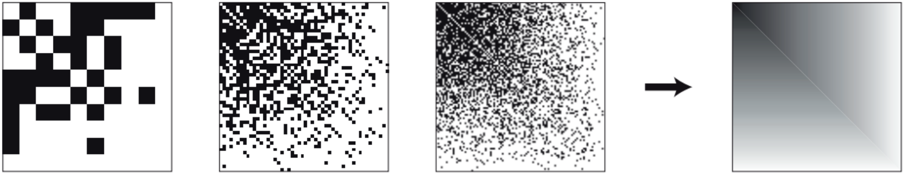
$$
\begin{aligned}
&\partial_t \bar{f}(t,x,\xi) \\
&+ \mathrm{div}_x \left(\bar{f}(t,x,\xi) \int_0^1 w(\xi,\zeta) \int_{\mathbb{R}^d} K(x-y)\bar{f}(t,dy,d\zeta) \right) \\
&= \sigma \Delta \bar{f}(t,x,\xi)
\end{aligned}
$$

- There is no initial data
- We are not interested in $\bar{f}$, but in $\int_0^1 \bar{f}(t,x,d\xi)$ which should be the limit of the “empirical measure”
- Need for an analytical context: Graph Theory

Graphons as dense graph limits (Lovasz, Szegedy, '06)

Large-scale limits of symmetric dense graphs

Let $G = (V, E, W_N)$ a finite graph with vertex $V = [\![1; N]\!]$, edges E, and adjacency matrix $W_N = (w_{ij}^N)_{ij}$



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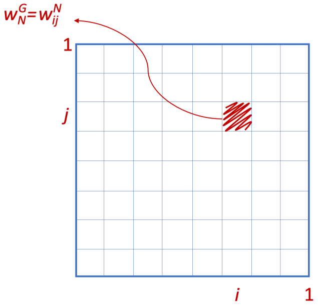
From w_{ij}^N we construct a piecewise constant function w_N^G



$$w_N^G(\xi, \zeta) = \sum_{i,j=1}^{N} w_{ij}^N \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta)$$

After permutation of indexes
$w_N^G \to w \in L^\infty([0,1]^2)$, w symmetric,
for dense graphs $|E(G_N)| \approx |V(G_N)|^2$.

Large-scale limits of symmetric dense graphs

Let $G = (V, E, W_N)$ a finite graph with vertex $V = [\![1; N]\!]$, edges E, and adjacency matrix $W_N = (w_{ij}^N)_{ij}$

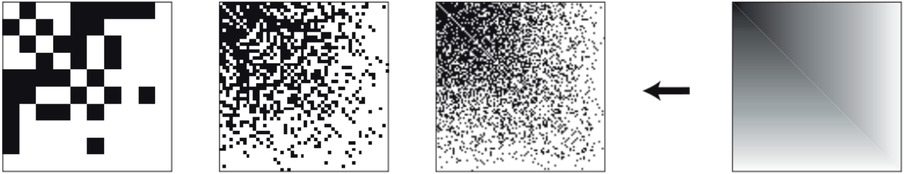
- $$w_N^G(\xi, \zeta) = \sum_{i,j=1}^{N} w_{ij} \mathbb{1}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta)$$
- $w_N^G \to w \in \left\{ L^\infty([0,1]^2),\ w \text{ symmetric } \right\} \equiv$ Graphons, for dense graphs $|E(G_N)| \approx |V(G_N)|^2$.
- The cut metric: $\delta_\square(w, w_N^G)$ there exists a measure preserving map ϕ_N on $[0, 1]$ st
$$\left\| w - w^N_{\phi_N(i),\phi_N(j)} \right\|_{L^\infty \to L^1} \overset{N\to\infty}{\longrightarrow} 0$$

What about mean-field limits on symmetric dense graphs?

Medvedev '18-'19

- For any graphon w, we can find finite graphs G_N approximating $0 \leq w \leq 1$, with weights

$$w_{ij}^N = N \int_{[\frac{i-1}{N},\frac{i}{N})\times[\frac{j-1}{N},\frac{j}{N})} w(\xi,\zeta)d\xi\, d\zeta,$$



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$$w_N^G \overset{N \to \infty}{\longrightarrow} w \text{ in } L^1([0,\ 1]^2) \quad \Longrightarrow \quad \delta_\square(w, w_N^G) \to 0$$

- $K \in W^{1,\infty}$

- Stability estimate:

$$\frac{d}{dt} \int_0^1 \int_{\mathbb{R}^d} |f_N - f| \, dx \, d\xi \leq C_1 \int_0^1 \int_{\mathbb{R}^d} |f_N - f| \, dx \, d\xi + C_2 \, \|w_N^G - w\|_{L^\infty \to L^1}$$

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$\Longrightarrow$ $\mu_N \rightharpoonup \int_0^1 f(t,x,\xi) d\xi$ and f satisfies

$$\partial_t f(t,x,\xi) + \mathrm{div}_x \left(f(t,x,\xi) \int_0^1 \int_{\mathbb{R}^d} w(\xi,\zeta)\, K(x-y) f(t,y,\zeta)\, dy\, d\zeta \right) = 0$$

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$$w_{ij}^N = N \int_{[\frac{i-1}{N}, \frac{i}{N}) \times [\frac{j-1}{N}, \frac{j}{N})} w(\xi, \zeta) d\xi \, d\zeta, \quad \boxed{\sup_{i,j} N |w_{ij}^N| = O(1)}$$

$$w_N^G \xrightarrow{N \to \infty} w \text{ in } L^1([0,\ 1]^2) \quad \Longrightarrow \quad \delta_\square(w, w_N^G) \to 0$$

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What about mean-field limits on sparse graphs, $|E(G_N)| \approx |V(G_N)|$?

- $\sup_{N\in\mathbb{N}} \max_i \sum_j |w_{ij}^N| + \max_j \sum_i |w_{ij}^N| \le C, \qquad \lim_{N\to\infty} \max_{i,j} |w_{ij}^N| = 0.$
- $K \in W^{1,1} \cap W^{1,\infty}$
- X_i^0 independent (but not *i.i.d.*) *st* their laws f_i^0 verify

$$\sup_{N\in\mathbb{N}} \max_i \left\{ \int_{\mathbb{R}^d} |x|^2 f_i^0(x)\, dx,\ \|f_i^0\|_{W^{1,1}\cap W^{1,\infty}} \right\} < \infty$$

What about mean-field limits on sparse graphs, $|E(G_N)| \approx |V(G_N)|$?

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$$\sup_{N\in\mathbb{N}} \max_i \left\{ \int_{\mathbb{R}^d} |x|^2 f_i^0(x)\,dx,\ \|f_i^0\|_{W^{1,1}\cap W^{1,\infty}} \right\} < \infty$$

$\Longrightarrow$ the mean-field limit is described by *(extended graphon)*

$$w \in L_\xi^\infty([0,\ 1], \mathcal{M}_\zeta([0,\ 1])) \cap L_\zeta^\infty([0,\ 1], \mathcal{M}_\xi([0,\ 1]))$$
$$f \in L_{t,\xi}^\infty([0,\ T] \times [0,\ 1],\ W_x^{1,1} \cap W_x^{1,\infty}(\mathbb{R}^d))$$

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \mathbb{E}\, W_1 \left(\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{X_i(t)}, \int_0^1 f(t,\cdot,\xi)\,d\xi \right) = 0.$$

Jabin, Poyato, JS

Idea of the proof

Propagation of independence

$$\frac{d\bar{X}_i}{dt} = \sum_{j=1}^{N} w_{ij} \int_{\mathbb{R}^d} K(\bar{X}_i - y)\bar{f}_j(t,y)\,dy,$$

$$\bar{X}_i(0) = X_i^0 \quad \text{(independent)}$$

$\bar{f}_i(t,\cdot) = \mathrm{Law}(\bar{X}_i(t))$ verify

$$\partial_t \bar{f}_i + \mathrm{div}_x \left(\bar{f}_i(t,x) \sum_{j=1}^{N} w_{ij} \int_{\mathbb{R}^d} K(x-y)\,\bar{f}_j(t,dy) \right) = 0,$$

$$\bar{f}_i(0,x) = f_i(0,x).$$

$\Longrightarrow$

$$\mathbb{E}\, W_1 \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)},\ \frac{1}{N} \sum_{i=1}^{N} \bar{f}_i(t,\cdot) \right) \leq C_1(t) \max_{i,j} |w_{ij}|^{1/2} + \frac{C_2(t)}{N^{\theta_d}}.$$

Consequences of the propagation of independence:

- For general connectivities w_{ij}^N,

$\not\Longrightarrow$ the limit of the corresponding 1-particle distribution

$$\frac{1}{N}\sum_{i=1}^{N}\bar{f}_i(t,x).$$

- The equation for 1-particle distribution evolves

  a kind of 2-particle distribution

$$\frac{1}{N^2}\sum_{i,j=1}^{N} w_{ij}^N \bar{f}_i(t,x)\bar{f}_j(t,x)$$

$\Longrightarrow$ There is a hierarchy of equations indexed by trees

Graphon-like reformulation

- We can recast the equation for $\bar{f}_i$ using graphons:

$$w_N(\xi,\zeta) := \sum_{i,j} N w_{ij}^N \mathbb{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi)\mathbb{1}_{[\frac{j-1}{N},\frac{j}{N})}(\zeta),$$

$$f_N(x,\xi) := \sum_{i} \bar{f}_i(t,x)\mathbb{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi).$$

- (w_N, f_N) is a solution of the generalized Vlasov equation

$$\boxed{\partial_t f_N(t,x,\xi) + \mathrm{div}_x\left(f_N(t,x,\xi)\int_0^1 w_N(\xi,\zeta)\int_{\mathbb{R}^d} K(x-y) f_N(t,dy,d\zeta)\right) = 0}$$

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Q1) Compactness of (w_N, f_N)

Q2) Identification of the limit (w, f) in an appropriate topology.

Q3) The limit (w, f) satisfies the generalized Vlasov equation

$\rightsquigarrow$ **New compactness result in the spirit of Lovasz & Szegedy.**

New hierarchy of observables

- For finite tree T (and not any arbitrary graph), we define

$$\tau(T, w_N, f_N)(t, x_1, \ldots, x_{|T|}) = \int_{[0,\ 1]^{|T|}} \prod_{(k,l)\in\mathcal{E}(T)} w_N(\xi_k, \xi_l) \prod_{m=1}^{|T|} f_N(t, x_m, \xi_m)\, d\xi_1 \ldots d\xi_{|T|}$$

These observables include the 1-particle distribution for the tree $T = T_1$ with only one vertex:

$$\tau(T_1, w_N, f_N)(t, x) = \frac{1}{N} \sum_{i=1}^{N} \bar{f}_i(t, x).$$

- Our observables completely entangle the kernel with the initial conditions

New hierarchy of observables

- A critical point is that these observables solve an independent hierarchy of equations

$$\partial_t \tau(T, w_N, f_N) + \sum_{i=1}^{|T|} \mathrm{div}_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z)\, \tau(T+i, w_N, f_N)(t, x_1, \dots, x_{|T|}, z) dz \right) = 0$$

where $T + i$ denotes the tree obtained from T by adding a leaf on the i-th vertex

- They naturally extend the notion of marginals, and hierarchy of marginals to non-exchangeable systems
If the particles were exchageable, observables would depend only on the number of nodes $\to$ classical hierarchy

New hierarchy of observables

- A critical point is that these observables solve an independent hierarchy of equations

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- The kernel w does not appear explicitly in this equation, it only appears in the definition of observable

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- We don't prove the convergence of f_N to f in a direct sense. This must be inferred from the convergence of $\tau(T, w_N, f_N)$ that gives the correct topology for the convergence

$\Longrightarrow$ we only need that $\tau(T, w_N, f_N)(t = 0, x_1, \ldots, x_{|T|})$ converges

New hierarchy of observables

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$$\partial_t \tau(T, w_N, f_N) + \sum_{i=1}^{|T|} \mathrm{div}_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z)\, \tau(T+i, w_N, f_N)(t, x_1, \ldots, x_{|T|}, z) dz \right) = 0$$

  where $T+i$ denotes the tree obtained from T by adding a leaf on the i-th vertex

- In our strategy there are two types of limits:

  1) Propagation of independence, limit of zero correlation: the coupled system: the PDE for $\bar{f}_i$ and the DS for $\bar{X}_i$

  2) Push the graph to infinity: the limit of $\lim_{N\to\infty} \tau(T, w_N, f_N)$

What about the metric to estimate compactness? Stability

- For $\lambda > 0$ and any $w, \tilde{w} \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ $f^0, \tilde{f}^0 \in L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x)$, we define

$$\mathcal{D}_\lambda((w, f^0), (\tilde{w}, \tilde{f}^0)) = \sup_{\text{trees } T} \lambda^{|T|/2} \left\| \tau(T, w, f^0) - \tau(T, \tilde{w}, \tilde{f}^0) \right\|_{L^2(\mathbb{R}^{d|T|})}$$

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- Stability:

  $f, \tilde{f} \in L^\infty_t([0,\ t_*]; L^\infty_\xi(W^{1,1}_x \cap W^{1,\infty}_x))$ unique solutions of

$$\partial_t f(t,x,\xi) + \mathrm{div}_x \left(f(t,x,\xi) \int_0^1 \int_{\mathbb{R}^d} w(\xi,\zeta)\, K(x-y) f(t,y,\zeta)\, dy\, d\zeta \right) = 0$$

$$\Longrightarrow \left\| \int_0^1 f\, d\xi - \int_0^1 \tilde{f}\, d\xi \right\|_{L^\infty_t([0,\ t_*], L^2_x)} \lesssim \frac{1}{(\ln|\ln \mathcal{D}_\lambda((w, f^0), (\tilde{w}, \tilde{f}^0))|)^{\frac{1}{2}}_+}$$

Compactness result towards extended graphons

- Consider $w_N \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f_N \in L^\infty_\xi(W^{1,1}_x \cap W^{1,\infty}_x)$

$$\sup_{N\in\mathbb{N}} \|w_N\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} < \infty, \quad \sup_{N\in\mathbb{N}} \|f_N\|_{L^\infty_\xi(W^{1,1}_x \cap W^{1,\infty}_x)} < \infty$$

- Then, there is a subsequence $\{N_k\}_{k\in\mathbb{N}}$ and there are $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f \in L^\infty_\xi(W^{1,1}_x \cap W^{1,\infty}_x)$ st

$$\boxed{\tau(T, w_{N_k}, f_{N_k}) \to \tau(T, w, f) \quad \text{in} \quad L^p_{\text{loc}}\left(\mathbb{R}^{d|T|}\right),\ 1 \le p < \infty}$$

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$$\boxed{\tau(T, w_{N_k}, f_{N_k}) \to \tau(T, w, f) \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^{d|T|}),\ 1 \le p < \infty}$$

- Reformulation $\tau(T, w, f) = \int_0^1 F(w, f)\, d\xi$ in terms of a countable algebra of transforms $F \in \mathcal{F}$ consistent with the adding-leafs process.
- New compactness-by-rearrangement lemma reminiscent of Szemerédi lemma proving that $F(w_N, f_N)$ must converge in L^p_{loc} modulo measure-preserving rearrangements w.r.t. ξ.
- Invariance under rearrangements of $\tau(T, w, f)$.

Compactness result towards extended graphons

- Consider $w_N \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f^0_N \in L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x)$

$$\sup_{N\in\mathbb{N}} \|w_N\|_{L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi} < \infty, \quad \sup_{N\in\mathbb{N}} \|f^0_N\|_{L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x)} < \infty$$

- Then, there is a subsequence $\{N_k\}_{k\in\mathbb{N}}$ and there are $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f^0 \in L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x)$ st

$$\mathcal{D}_\lambda((w_N, f^0_{N_k}), (w, f^0)) \to 0, \quad \text{as } N \to \infty$$

for a sufficiently small $\lambda > 0$.

The identification of the limit

1) Once the limit (w, f^0) has been calculated

$w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$ and $f^0 \in L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x)$

2) We solve

$$\partial_t f(t,x,\xi) + \mathrm{div}_x \left(f(t,x,\xi) \int_0^1 \int_{\mathbb{R}^d} w(\xi,\zeta)\, K(x-y) f(t,y,\zeta)\, dy\, d\zeta \right) = 0$$

to get $f \in L^\infty([0,\ t_*], L^\infty_\xi (W^{1,1}_x \cap W^{1,\infty}_x))$

3) The stability result gives

$$\left\| \int_0^1 f\, d\xi - \frac{1}{N_k} \sum_i \bar{f}_i \right\|_{L^\infty_t([0,\ t_*], L^2_x)} \lesssim \frac{1}{(\ln|\ln \mathcal{D}_\lambda((w, f^0), (w_{N_k}, f^0_{N_k}))|)_+^{\frac{1}{2}}} \to 0$$

$\Longrightarrow$ W_1-convergence by the propagation of moments.

The identification of the limit

4) Therefore, we can conclude that

$$\mathbb{E} W_1\left(\frac{1}{N_k}\sum_i \delta_{X_i}, \int_0^1 f(t,\cdot,\xi)\,d\xi\right)$$

$$\leq \mathbb{E} W_1\left(\frac{1}{N_k}\sum_i \delta_{X_i}, \frac{1}{N_k}\sum_i \bar{f}_i(t,\cdot)\right)$$

$$+\, W_1\left(\frac{1}{N_k}\sum_i \bar{f}_i(t,\cdot), \int_0^1 f(t,\cdot,\xi)\,d\xi\right) \to 0$$

Conclusions and perspectives

Advantages

- Joint approach mean-field and large-graph limit.
- Large class of sparse non-symmetric graphs.
- Weights not coming as discretization
  of fixed continuous objects.
- Better results in the noisy case:

$$\|\tau(T, w, f) - \tau(T, w_N, f_N)\|_{L^2(\mathbb{R}^{d|T|})} \to 0$$

Open questions

- What if $K \notin W^{1,\infty}$?
  only used in the propagation of independence

Thanks for your attention!

Further previous results in the non-exchangeable case

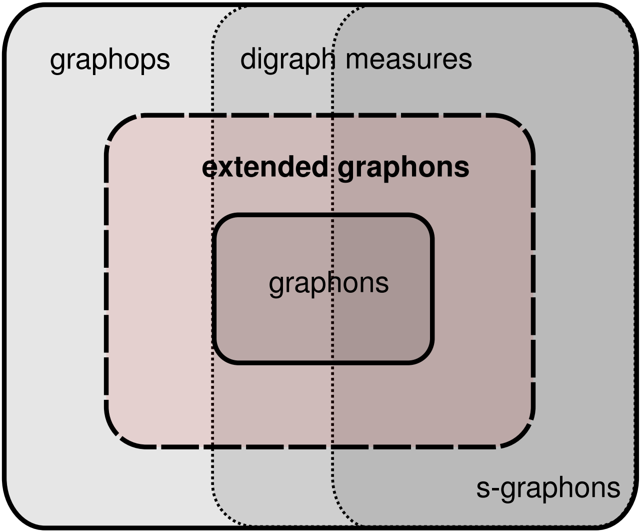
- ▷ **Dense graph limits** $|E(G_N)| \approx |V(G_N)|^2$.
  (LGL) Lovasz, Szegedy '06: Graphons $w \in L^\infty([0, 1]^2)$.
  (MFL) Chiba, Medvedev '19, Kaliuzhnyi-Verbovetski, Medvedev '18.

- ▷ **Sparse graph limits** $|E(G_n)| \approx |V(G_n)|$.
  (LGL) Benjamini, Schram '01: Graphings.

- ▷ **Intermediate density** $|V(G_n)| \ll |E(G_n)| \lesssim |V(G_n)|^2$.
  (LGL) Borgs, Chayes, Cohn, Zhao '18-'19: L^p graphons.

- ▷ **Intermediate density** $|V(G_n)| \lesssim |E(G_n)| \lesssim |V(G_n)|^2$.
  (LGL) Backhausz, Szegedy '20: Graphops
  (LGL) Kunszenti-Kovasz, Lovasz, Szegedy '19: s-graphons
  (LGL) Kuehn, Xu '21: Digraph-measures
  (MFL) Gkogkas, Kuehn '20; Kuehn, Xu '21.

Comparison of the various graph limit theories



Entropy solution

- A function $f_N \in L^\infty([0,\ 1] \times \mathbb{T}^{dN} \times \mathbb{R}^{dN})$ satisfying the Gaussian decay

$$\sup_{t\leq 1} \int_{\mathbb{T}^{dN}\times\mathbb{R}^{dN}} e^{\beta \sum_{i\leq N} |v_i|^2} f_N \, dx_1 \, dv_1 \dots dx_N \, dv_N \leq V^N,$$

Entropy solution

- A function $f_N \in L^\infty([0,\ 1] \times \mathbb{T}^{dN} \times \mathbb{R}^{dN})$ satisfying the Gaussian decay is an entropy solution iff all marginals $f_{k,N}$ for $1 \leq k \leq N$ verify

$$\int_0^T \int_{\mathbb{T}^{dk} \times \mathbb{R}^{dk}} e^{\lambda\, e_k}\, |f_{k,N}|^{q-1} \, \text{sign}\,\left(f_{k,N}\right)\, L_k\left[f_{k,N}\right]\, dx_1\, dv_1 \dots dx_k\, dv_k\, dt \geq 0,$$

Reduced energy

$$e_k(x_1, v_1, \dots, x_k, v_k) = \sum_{i \leq k} (1 + |v_i|^2) + \frac{1}{N} \sum_{i,j \leq k} \phi(x_i - x_j).$$

e_k is invariant under the advection component

$$L_k = \sum_{i \leq k} v_i \cdot \nabla_{x_i} + \frac{1}{N} \sum_{i,j \leq k} K(x_i - x_j) \cdot \nabla_{v_i}$$

Entropy solution

- A function $f_N \in L^\infty([0,\ 1] \times \mathbb{T}^{dN} \times \mathbb{R}^{dN})$ satisfying the Gaussian decay is an entropy solution iff all marginals $f_{k,N}$ for $1 \leq k \leq N$ verify

$$\int_0^T \int_{\mathbb{T}^{dk} \times \mathbb{R}^{dk}} e^{\lambda\, e_k}\, |f_{k,N}|^{q-1} \operatorname{sign}\,\left(f_{k,N}\right)\, L_k\left[f_{k,N}\right]\, dx_1\, dv_1 \ldots dx_k\, dv_k\, dt \geq 0,$$

- Renormalized or mild solutions can offer a natural way to prove it, while it is automatically satisfied if we have classical solutions.