Some aspects of specific relative entropy

(in the study of log-gases)

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"Entropy in large classical systems"
CMI, Oxford Sept. 23

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specific

=

per unit volume

Specific?

P, Q "infinite-vol." objects

P_Λ, Q_Λ "finite-vol." restrictions

usual rel. ent.

$$\mathcal{E}(P|Q) := \lim_{|\Lambda|\to+\infty} \frac{H(P_\Lambda|Q_\Lambda)}{|\Lambda|}$$

specific rel. ent.

$|\Lambda|$ ← volume

[cf. "Shannon's entropy" for stationary processes (J.R. Chazottes's talk)]

Log-gases? $(d = 1$ or $2)$

$$\Sigma_N = \left[-\frac{N}{2}, \frac{N}{2}\right] \text{ or } D\left(0, \sqrt{\frac{N}{\pi}}\right) \quad \left(|\Sigma_N| = N\right)$$

X_N = N-tuple $\underbrace{(x_1, \dots, x_N)}_{\text{state}}$ of points in Σ_N

$$f_N = \sum_{i=1}^{N} \delta_{x_i} - \mathbb{1}_{\Sigma_N} dx$$

purely atomic ≥ 0 measure

negative uniform background

$$\underbrace{F_N(X_N)}_{\text{energy}} := \frac{1}{2} \iint_{x \neq y} -\log|x-y| \, df_N(x) \, df_N(y)$$

Log-gas Gibbs measure

state

X_N, $F_N(X_N)$ energy $+$ $\beta > 0$ inverse temp.

$$d\mathbb{P}_N^\beta(X_N) = \frac{1}{Z_N^\beta} e^{-\beta F_N(X_N)} dX_N$$

Gibbs measure of the log-gas

partit° funct°

Boltzmann's factor

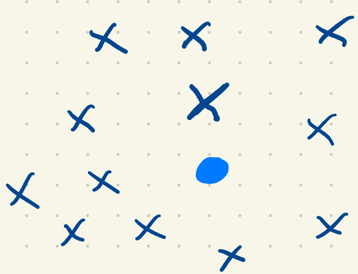
Prob. Measure on $\Sigma_N \times \dots \times \Sigma_N \subset (\mathbb{R}^d)^N$.

Why?

(electrostatics ...)

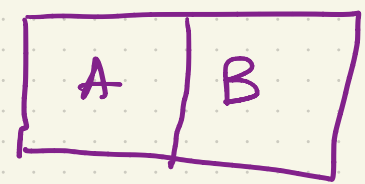
- $-\log$ is Coulomb kernel in $d=2$
- $\mathbb{P}_N^\beta \longleftrightarrow$ random eigenvalues in $d=1$
- Long-range + singular interaction

many classical results fail to apply



Energy of a point in an infinite config?

Energy of $A+B$

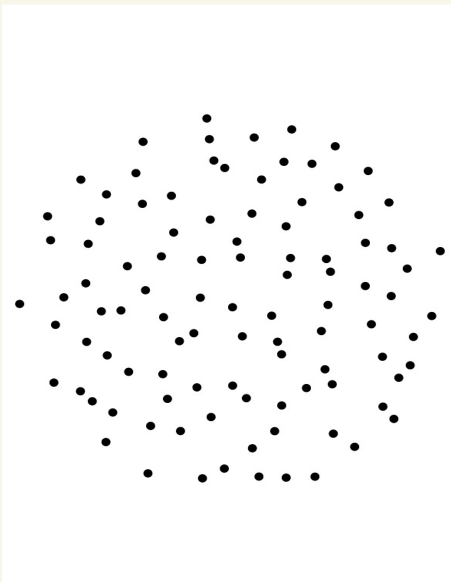


vs.

Energy A + Energy B?

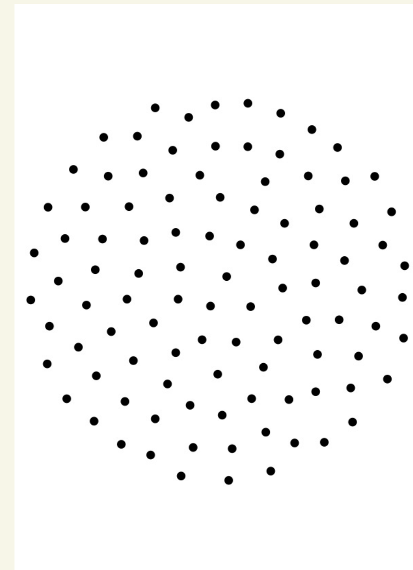
Questions

Dependency // β of the microscopic behavior?



$\beta = 5$

? ?



$\beta = 400$

Large N limit?

$$\mathbb{P}_N^\beta \in \mathcal{P}(\Sigma_N^N) \qquad \boxed{e^{-\beta F_N}\, dx^{\otimes N}} \text{ density}$$

$$= \underset{\mu \in \cdots}{\operatorname{argmin}} \Big(\underbrace{\beta\, \mathbb{E}_\mu[F_N] + H(\mu \mid dx^{\otimes N})}_{\text{free energy}} \Big)$$

(unique)

As $N \to +\infty$

a) $X_N \in \Sigma_N^N \longrightarrow$? X

b) $dx^{\otimes N} \longrightarrow$ Ref. measure for X ?

c) $F_N \longrightarrow F$?

d) $\mathbb{P}_N^\beta \longrightarrow \mathbb{P}$?

e) unique min of $\beta\, \mathbb{E}_P[F] + H(\cdot \mid \ )$?

Large N limit (I) : Point config.

$X_N \in \Sigma_N^N \to$ finite point config Conf_N

$X_N \longleftrightarrow \sum_{i=1}^{N} \delta_{x_i}$ purely atomic (loc.) finite measure

$\mathrm{Conf}_N \subset \mathrm{Conf}$ ("point configurations")

(+) Local topology on Conf, allowing

$$X_N \xrightarrow[N\to\infty]{} X \text{ in } \mathrm{Conf}.$$

Large N limit (II) : Poisson

$dx^{\otimes N}$ Lebesgue measure on Σ_N (box of size N)

$=$ law of N iid uniform points on Σ_N

$\xrightarrow[N\to\infty]{}$ Poisson point process

(Think of $\mathrm{Ber}(n, \frac{1}{n}) \xrightarrow[n\to\infty]{} \mathrm{Poisson}(1)$)

Law of # points in Λ = Poisson$(|\Lambda|)$

$\oplus$ if $\Lambda \cap \Lambda' = \emptyset$
the config in Λ, Λ' are $\perp\!\!\!\perp$

Large N limit (III) : Energy

$$X_N \longrightarrow F_N(X_N) = \frac{1}{2} \iint_{x \neq y} -\log|x-y| \, d\mu_N(x) \, d\mu_N(y)$$

$\downarrow$ X infinite point config

$\downarrow$?

$$\frac{F_N(X_N)}{N}$$ maybe ?

('10's)

(Sandier–Serfaty)

There exists an energy functional $F : \mathrm{Conf} \to \mathbb{R}$ such that $\frac{F_N}{N}$ Γ-converges to F.

① "Γ-$\underline{\lim}$"

$$X_N \to X \implies \boxed{\varliminf_{N\to\infty} \frac{F_N(X_N)}{N} \geq F(X)}$$

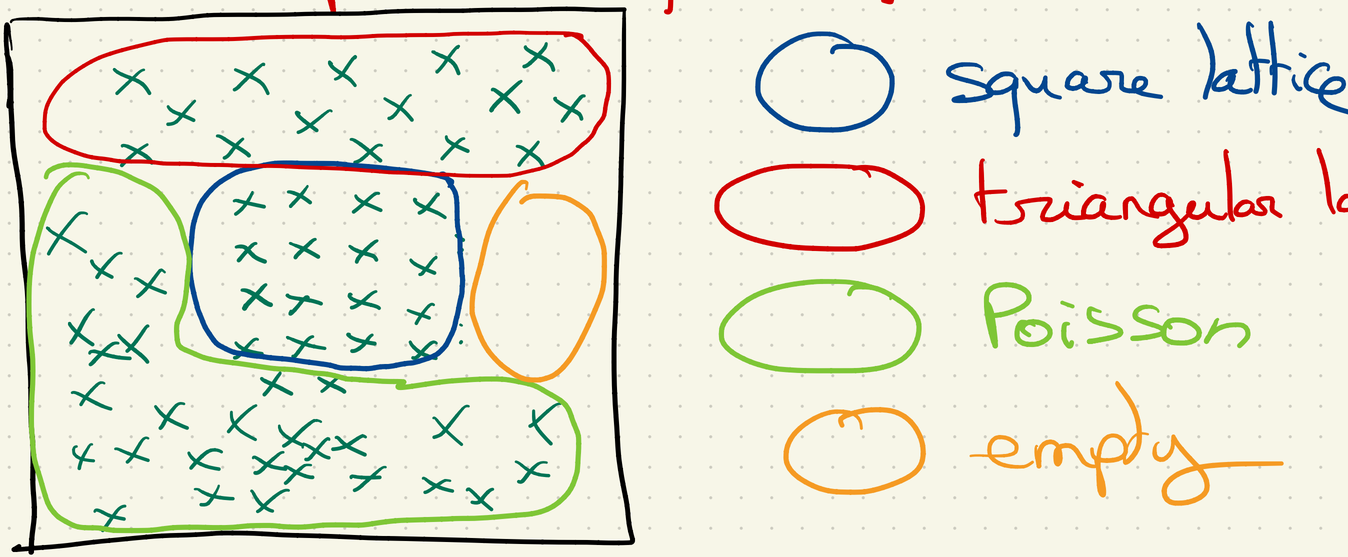
② "Γ-$\overline{\lim}$" X given

$$\boxed{\exists (X_N \to X) \quad \varlimsup_{N\to\infty} F_N/N(X_N) \leq F(X)}$$

Large N limit (IV): infinite vol. limit

$$\mathbb{P}_N^\beta \in \mathcal{P}(\mathrm{Conf}_N) \xrightarrow[N\to\infty]{} \mathbb{P} \in \mathcal{P}(\mathrm{Conf})?$$

Local top. on Conf ⇔ information near 0 only



Empirical field

Corresponds to "average" microscopic behavior

$X_N \in (x_1, \dots, x_N) \in \Sigma_N^N$ $\quad$ X_N "seen from" "centered at" t

$\downarrow$

$\mathcal{C}_N = \sum_{i=1}^N \delta_{x_i} \in \mathrm{Conf}_N$

$t \in \mathbb{R}^d$, let $\mathcal{C}_{N,t} = \sum_{i=1}^N \delta_{x_i - t} \in \mathrm{Conf}_N$

$$\boxed{\mathrm{Emp}_N := \frac{1}{|\Lambda_N|} \int_{\Lambda_N} \delta_{\mathcal{C}_{N,t}} \, dt} \in \mathcal{P}(\mathrm{Conf}_N) \subset \mathcal{P}(\mathrm{Conf})$$

Large deviations for emp. fields ?

$X_N \longmapsto \mathrm{Emp}_N \in \mathcal{P}(\mathrm{Conf})$ random

$\mathbb{P}_N^\beta(\mathrm{Emp}_N \approx P)$ for $P \in \mathcal{P}(\mathrm{Conf})$ fixed ?

$$= \frac{1}{Z_N^\beta} \int_{\Sigma_N \times \dots \times \Sigma_N} \mathbb{1}_{\mathrm{Emp}_N \approx P} \, e^{-\beta F_N(X_N)} \, dx^{\otimes N}$$

① Guess F_N from "$\mathrm{Emp}_N \approx P$" ?

② Large dev for Emp_N without interactions ?

Large dev without interactions

$$\int_{\Sigma_N \times \cdots \times \Sigma_N} \mathbb{1}_{\mathrm{Emp}_N \simeq P} \, dx^{\otimes N} \ ?$$

N iid points X_N $\longrightarrow$ Emp_N ? Should look like Poisson

Large deviations ($P \neq$ Poisson) ?

Reminder: Sanov's theorem

μ prob. meas on $\mathcal{X}$

$Y_1, \dots, Y_n$ iid r.v. of law μ

Empirical measure $\boxed{\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}} \in \mathcal{P}(\mathcal{X})$

$\mu_n \approx \mu$ (LLN)

Sanov $\mathbb{P}[\mu_n \approx \nu] \approx e^{-n H(\nu|\mu)}$

$H(\nu|\mu)$ rel. entropy ≥ 0 as "rate function"

(cf. C. Mouhot's talk) cf. Boltzmann's formula

Sanov for empirical fields

Π Poisson process (prob. measure on Conf)

Empirical field $\frac{1}{|\Lambda_N|} \int_{t \in \Lambda_N} \delta_{X-t} \, dt$

"Sanov" [Föllmer, ?]

$$\mathbb{P}\,[\text{Emp. field} \simeq P] \simeq e^{-|\Lambda_N| \, \mathcal{E}(P|\Pi)}$$

$$\mathcal{E}(P|\Pi) = \lim_{N \to \infty} \frac{1}{|\Lambda|} H(P_{|\Lambda} \,|\, \Pi_{|\Lambda})$$

= specific relative entropy

Existence? Yes (Fekete).

(Extension : replace Π in Λ_N by $|\Lambda_N|$ iid points)

$$\int_{\Sigma_N \times \cdots \times \Sigma_N} \mathbb{1}_{\mathrm{Emp}_N \approx P} \, dx^{\otimes N} \; ?$$

Poisson

$$\text{“}=\text{”} \quad e^{-\underbrace{|\Lambda_N|}_{=N} \mathcal{E}(P \mid \Pi)}$$

specific relative entropy

Comment: ① $\mathcal{E}(P|\Pi) = 0 \Rightarrow P = \Pi$

② P really needs to be stationary

③ P not abs. cont // Π !!!

$\mathbb{P}_N^\beta(\mathrm{Emp}_N \approx P)$ for $P \in \mathcal{P}(\mathrm{Conf})$ fixed?

$$= \frac{1}{Z_N^\beta} \int_{\Sigma_N \times \cdots \times \Sigma_N} \mathbb{1}_{\mathrm{Emp}_N \approx P}\, e^{-\beta F_N(X_N)}\, dx^{\otimes N}$$

Guess F_N from "$\mathrm{Emp}_N \approx P$"? If yes:

$$= \frac{1}{Z_N^\beta}\, e^{-\beta \text{ Energy when } \mathrm{Emp}_N \approx P} \int_{\Sigma_N \times \cdots \times \Sigma_N} \mathbb{1}_{\mathrm{Emp}_N \approx P}\, dx^{\otimes N}$$

?

LDP without interact°.

Γ-$\underline{\lim}$ for $\frac{F_N}{N} \longrightarrow F : \mathrm{Conf} \to \mathbb{R}$

$\Rightarrow$ if $\mathrm{Emp}_N \approx P$ then

$$\underline{\lim}_{N \to \infty} \frac{F_N}{N}(X_N) \geq \mathbb{E}_{\underbrace{P}_{\in \mathcal{P}(\mathrm{Conf})}}[F]$$

Converse is false : not enough information in "$\mathrm{Emp}_N \approx P$".

?

① local vs. global info $\Leftrightarrow$ Screening (Sandier - Serfaty)

② singularity $\Leftrightarrow$ Quasicontinuity

Replace config-wise statement

"If X_N is s.t. $\mathrm{Emp}_N \simeq P$ then $\frac{F_N(X_N)}{N} \leq F(X) + \varepsilon$"

by "quasi continuity" (Bodineau-Guionnet)

"Among all X_N's s.t. $\mathrm{Emp}_N \simeq P$, a significant fraction is s.t. $\frac{F_N(X_N)}{N} \leq F(X) + \varepsilon$"

The log. volume must be comparable to the specific entropy

$- \log \{\text{good microstates}\} = - \log \{\text{all microstates}\} + o(N)$

(Rem. this is analogous to the Γ-Tim construction
of a "recovery sequence"
but you need a volume of good microstates,
that matches the total volume
at log. scale)

$\mathbb{P}_N^\beta$ finite Gibbs measure of a log-gas

$\downarrow$

Empirical field = average micro. behavior

$\downarrow$ (LDP, L. - Serfaty '15)

Concentrates on minimizers of rate function

(Free energy of P) $\beta \, \mathbb{E}_P[F] + \mathcal{E}(P|\Pi)$

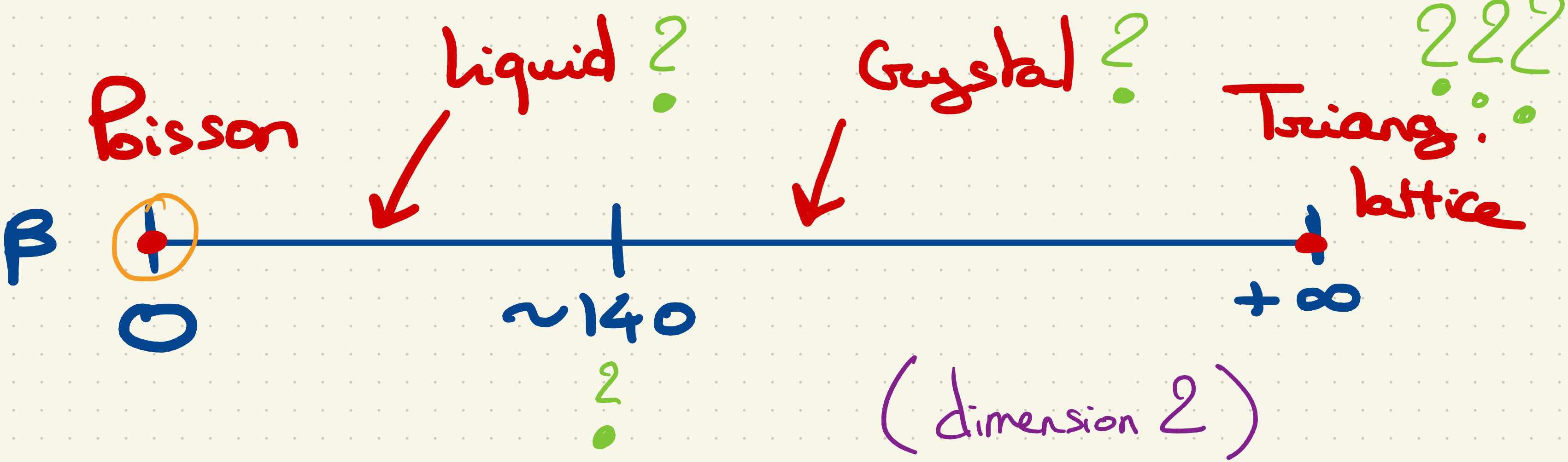
$\in \mathcal{P}(\mathrm{Conf})$

$F : \mathrm{Conf} \to \mathbb{R}$

Specific rel. entropy

Poisson

In principle, one could understand (some of) the phase portrait by minimizing the free energy functional.



- For $\beta \in (0, +\infty)$ the entropy is finite
  energy

$\longrightarrow$ No "perfect" order.

$\longrightarrow$ Not Poisson

$$\beta \mathbb{E}_P[F] + \mathcal{E}(P|\Pi)$$

However

- As $\beta \to 0$, minimizers satisfy

$$\mathcal{E}(P_\beta|\Pi) \to 0 \quad (L.\ '16)$$

$$\mathcal{E}(P|\Pi) = \sup_\Lambda \frac{H(P_\Lambda|\Pi_\Lambda)}{|\Lambda|} \quad \text{(subadditivity)}$$

$\Longrightarrow$ $P_\beta|_\Lambda \xrightarrow[\beta \to 0]{TV} \Pi_\Lambda$ by Pinsker's inequality, $\forall \Lambda$.

It suffices to construct candidates $(Q_n)_n$ s.t.

$$\begin{cases} \mathcal{E}(Q_n \mid \Pi) \xrightarrow[n\to\infty]{} 0 \\ \mathbb{E}_{Q_n}[F] \text{ is finite } \forall n. \end{cases}$$

not trivial

Then $\beta\, \mathbb{E}_{P_\beta}[F] + \mathcal{E}(P_\beta \mid \Pi)$

$$\leq \beta\, \mathbb{E}_{Q_n}[F] + \mathcal{E}(Q_n \mid \Pi)$$

($\overset{+}{F}$ bounded below) adjust n, β as desired...

$\Longrightarrow \mathcal{E}(P_\beta \mid \Pi) \longrightarrow 0$ as $\beta \to 0$

"Fun fact" for 1d case

the limit of $\mathbb{P}_N^\beta$ is known (without averaging)

"Sine_β process" [Valko-Virag]

(Coupled SDE's, counting explosions ...)

$\mathrm{Sine}_\beta \xrightarrow[\beta\to 0]{} 0$ [Allez-Dumaz '14]

$\Uparrow$

argmin free energy $\xrightarrow[\beta\to 0]{} 0$

$\beta = +\infty$?

$X =$

- In $d=1$, F is minimized at $\mathbb{Z}$ (lattice)
  and argmin $\mathbb{E}_P[F]$ among stationary P
  is $\underline{P} = P_{\mathbb{Z}}$ = randomly shifted lattice.

- As $\beta \to +\infty$, minimizers of free energy
  tend to $P_{\mathbb{Z}}$.

"Of course" $\mathcal{E}(P_{\mathbb{Z}} | \Pi) = +\infty$

OK to estimate

Need to construct $(P_n)_n$ s.t. $\mathcal{E}(P_n | \Pi) < +\infty$
$\mathbb{E}_{P_n}[F] \longrightarrow$ minimal energy.

$\beta = +\infty$ ($d=2$)

- In $d=2$, the minimization of F is a major open problem.

Conjecture: triangular lattice is minimizing.

(true among lattices...)

- We know that minimizers have some good properties
  - × Minimal separation between points
  - × "Small fluctuations"

<u>Question</u>: $\mathbb{P}$ stationary minimizing $\mathbb{E}_{\mathbb{P}}[F] \Rightarrow \mathbb{P}$ has infinite entropy ??

One difficulty is that $\mathcal{E}(\mathbb{P}|\Pi)$ grows "slowly" with the weirdness of $\mathbb{P}$.

- $\mathbb{P}$ = Lattice $\Longrightarrow \mathcal{E}(\mathbb{P}|\Pi) = +\infty$
- $\mathbb{P} = \delta$-perturbation of a lattice || Impossible for Poisson

  $\Longrightarrow \mathcal{E}(\mathbb{P}|\Pi) \simeq -\log \delta$
- "All the points separated by c" || impossible for Poisson!

  ($\mathbb{P}$-a.s.)

  $\Longrightarrow \mathcal{E}(\mathbb{P}|\Pi) \geq c' > 0 \ldots$

$\beta \in (0, +\infty)$

Basic question: (non)-uniqueness of free energy minimizers $\longleftrightarrow$ "phase" $\overset{?}{\longleftrightarrow}$ "Gibbs state"

In finite vol. $\mu \longmapsto \beta \mathbb{E}_\mu[F_N] + H(\mu \mid dx^{\otimes N})$

linear strict-convex

In inf. vol. $P \longmapsto \beta \mathbb{E}_P[F] + \boxed{\mathcal{E}(P \mid \Pi)}$

affine ...

$\Rightarrow$ Possibly many minimizers. Need to find convexity elsewhere (Displacement convex?)

Role of specific rel. entropy (in general)

- Finite Gibbs measure $\mathbb{P}_N^\beta$ energy / ref. measure (finite volume)

$\sup_N \mathcal{E}(\mathbb{P}_N^\beta | \Pi) < +\infty \implies \exists$ of limit points

- $P_\beta = \lim \mathbb{P}_N^\beta$ (up to subsequence) $= e^{-\beta \text{Energy}} d\text{Vol}$?

NO

(sol. of) ⓐ

- DLR equations ("Gibbs states")

ⓑ

P Gibbs state $\iff$ P minimizes free energy

$\iff \mathcal{E}(P | P_\beta) = 0$

"Var. pple" for Gibbs states.

For log gases ⓐ ⓑ make sense... separately ...

Some rigidity properties (I)

- "Number variance"

$$\frac{\mathrm{Var}\left[\,|X \cap \Lambda|\,\right]}{|\Lambda|} \xrightarrow[|\Lambda| \to +\infty]{} \,?$$

* P is "hyperuniform" when $\boxed{\text{Variance in } B(0,R) = o(R^d)}$
(S. Torquato)
or
"Superhomogeneous"
(J. Lebowitz)

(Poisson case $= |B(0,R)|$)

Some rigidity properties (II)

$\varphi : \mathbb{R}^d \to \mathbb{R}$ test function

$\hookrightarrow$ "linear statistic" $\sum_{x \in X} \varphi(x)$

$|X \cap \Lambda| =$ lin. stat. for $\varphi = \mathbb{1}_\Lambda$.

What about $\mathrm{Var}\left[\sum_{x \in X} \varphi(x)\right]$ for other / more general functions φ?

In particular if $\varphi = \varphi_R = \overline{\varphi}(\cdot / R)$?

Some rigidity properties (III)

"À la Ghosh–Peres" [S. Ghosh's PhD Thesis – '12]

- $\mathbb{P}$ point process, $\Lambda \subset \mathbb{R}^d$ bounded
  - Take an observable $\varphi : \mathcal{X} \longrightarrow \mathbb{R}$, Λ-local
  - $\mathbb{P}$ is φ-rigid if $\exists g$, Λ^c-mesurable

$$\varphi(X) = \varphi(X_\Lambda) \overset{?}{\underset{?}{=}} g(X_{\Lambda^c}) \quad (\mathbb{P}\text{ a.s.})$$

Ex. Number rigid, center-of-mass-rigid ...

Entropy vs. rigidity

Predicted in physics literature

	Poisson	Lattice	Log-gas
Number variance in $B(0,R)$	R^2	R	$o(R^2)$* R??
Variance of $\sum_{x \in X} \varphi_R(x)$, $\varphi_R = \bar{\varphi}(\cdot/R)$, $\bar{\varphi}$ smooth	R^2	$O(1)$	$O(1)$** $\Downarrow$
Rigidity à la G-P	$\emptyset$	Fully	Number-rigid
Entropy // Poisson	0	$+\infty$	$\in (0, +\infty)$

• • •

* L. 2023 ** L.-Serfaty, Bauerschmidt-Nikula-Yau, Bourgade, Serfaty ('16-'22)

Comments / questions

- Those rigidity properties hold for the Gibbs measure (minimizer of free energy), not simply under the condition of finite energy.
- Strong forms of rigidity are not incompatible with finite specific relative entropy.
- What is "too rigid for Poisson" ??
- Does "$\mathbb{P}$ is center-of-mass rigid" $\Rightarrow$ $\mathbb{P}$ has infinite entropy ?

Thank you for your attention!