Entropy Methods in Particle Systems

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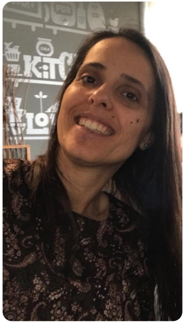
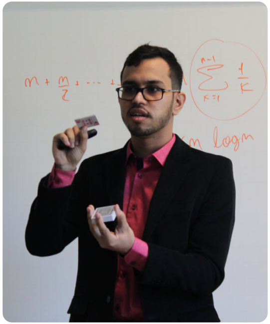


Patrícia Gonçalves



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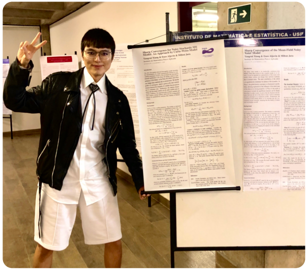
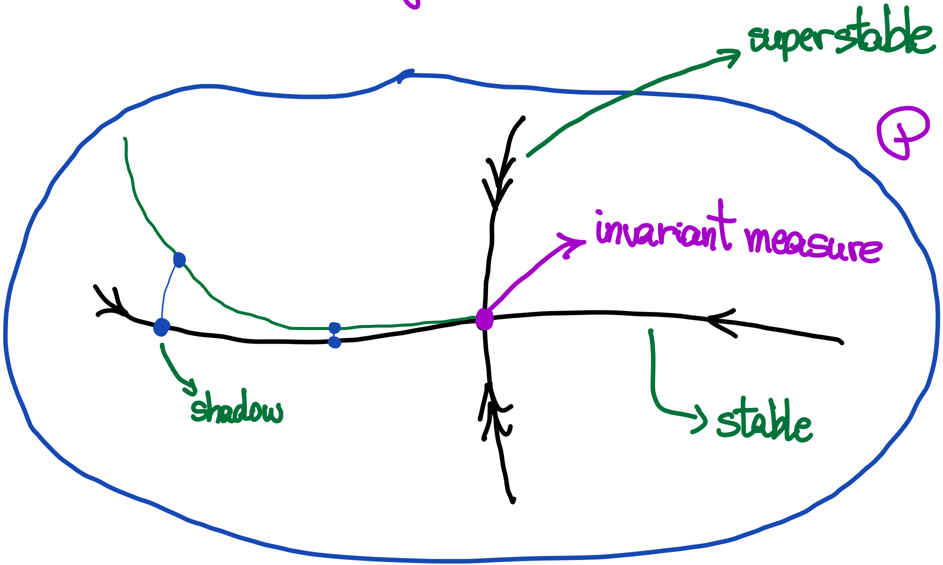


⑪ Dimension reduction of Markov chains

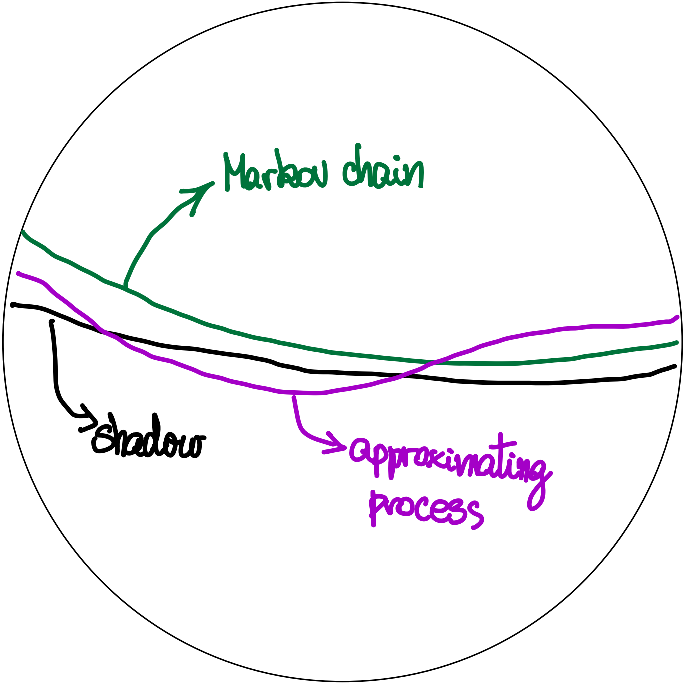


Scaling limits of stochastic systems

Convergence to the shadow

Description of the shadow

Convergence of the shadow



Markov chain
shadow
approximating process

- **General strategy**

  - Understand "the shadow"
  - Approximate the shadow by a tractable object
    ↳ martingale, PDE, integrable system, ...
  - Forget about the shadow and compose the chain with the auxiliary process "directly"

Relative entropy method (Yau):

- Relative entropy between the law of the chain and its shadow decreases in time

- Substitute shadow by approximate process

Setup

Ω: finite state space

$(\omega_t;\, t\geq 0)$: continuous-time, irreducible Markov chain in Ω

not fundamental

L: generator of $(\omega_t;\, t\geq 0)$

$$Lf(\omega) := \sum_{\sigma\in\Omega} r(\omega,\sigma)\big(f(\sigma)-f(\omega)\big)$$

jump rate

- Carré du champ:

Γ : carré du champ associated to L

$$\Gamma f := Lf^2 - 2fLf$$

<u>Obs.</u>: $\Gamma f(\omega) = \sum_{\sigma \in \Omega_0} r(\omega,\sigma)\big(f(\sigma)-f(\omega)\big)^2$

explains the denomination

- ℓ-variation:

$$\Gamma^\ell f(\omega) = \sum_{\sigma \in \Omega_0} r(\omega,\sigma)\big(f(\sigma)-f(\omega)\big)^\ell$$

- Relative entropy:

$$H(\mu|\nu) := \int f \log f \, d\nu \, ; \quad f := \frac{d\mu}{d\nu}$$

Pinsker's inequality: $\|\mu-\nu\|_{TV} \leq \sqrt{2H(\mu|\nu)}$

Donsker-Varadhan: $\int g \, d\mu \leq \frac{1}{\gamma}\left(H(\mu|\nu) + \log \int e^{\gamma g} d\mu\right)$

full support

Reference measures: $(\mu_t \, ; \, t \in [0,T])$; $\psi_t := \frac{d\mu_t}{d\bar{\mu}}$

Density: $f_t :=$ density of ω_t wrt μ_t

Yau's inequality:

$$H(t) := \int f_t \log f_t \, d\mu_t = H(\mathcal{L}aw(\omega_t) \,|\, \mu_t);$$

$$H'(t) \leq -\int \Gamma \sqrt{f_t} \, d\mu_t + \mathbb{E}[\mathbb{F}_t(\omega_t)];$$

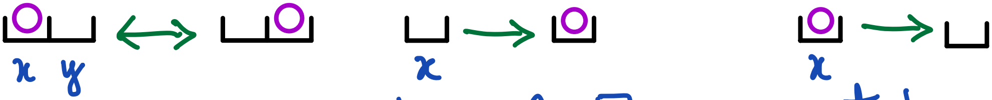
$$\mathbb{F}_t := \frac{1}{\psi_t} (L^* - \partial_t) \psi_t$$

entropy production

adjoint wrt $L^2(\bar{\mu})$

dissipation (energy)

Example: Reaction-diffusion model



$x \sim y$; rate n^2

rate $a + \frac{\lambda}{2d} \sum_{y \sim x} \eta_y$

rate b

Diffusion (exclusion)

Reaction (contact)

$\Omega_n := \{0,1\}^{\mathbb{T}_n^d}$

discrete torus

- ~~Hydrodynamic~~ Scaling Limit Equation equation:

$$\partial_t u = \Delta u + F(u); \qquad F(u) := (a+\lambda u)(1-u) - bu$$

only one zero $u^* \neq 0,1$ in $[0,1]$

[De Masi, Ferrari, Lebowitz 86]

[Kipnis, Olla, Varadhan '90]

- Reference measure:

$$\mu_t := \bigotimes_{x \in \mathbb{T}^d_n} \mathrm{Bern}\left(u\left(t, \frac{x}{n}\right)\right)$$

better a discrete version of HDE

Thm.: (J.-Menezes) (Quantitative hydrodynamics)

$$H'(t) \leq C(u(t))\left(H(t) + g_d(n)\, n^{d-2}\right),$$

where

$$g_d(n) := \begin{cases} n, & d=1 \\ \log n, & d=2 \\ 1, & d\geq 3. \end{cases}$$

In particular, for every $f \in \mathcal{C}(\mathbb{T}^d)$, if $H(0) \leq C g_d(n)\, n^{d-2}$, then

$$\mathbb{E}\left[\left(\frac{1}{n^d} \sum_{x\in\mathbb{T}_n^d} f\left(\frac{x}{n}\right)\left(\eta_x(t) - u\left(t,\frac{x}{n}\right)\right)\right)^2\right] \leq \frac{C(t,u_0)\, g_d(n)}{n^2}.$$

Rmk: $C(t,u_0)$ grows exp. fast

Application: CLT for NESS

ν^n: invariant measure of $(\eta(t)\,;\, t \geq 0)$ in Ω_n

Thm Wrt ν^n, for every $f \in \mathcal{C}(\mathbb{T}^d)$, in $d \leq 3$,

$$\lim_{n\to\infty} \frac{1}{n^{d/2}} \sum_{x \in \mathbb{T}^d_n} f\left(\frac{x}{n}\right)\left(\eta_x(t) - u\left(t, \frac{x}{n}\right)\right) = X(f),$$

where X is a Gaussian process with non-trivial covariance structure.

Stein method

- Quantitative version of CLT
  ↳ Notes by Ley-Reinert-Swan, ...
- Our aim: Stein method for Markov chains

Dynkin's formula

$f : \Omega \times [0,T] \to \mathbb{R}$: test function

$$\boxed{M_t^f := f(t,\omega_t) - f(0,\omega_0) - \int_0^t (\partial_s + L) f(s,\omega_s)\,ds \quad \text{is a martingale}}$$

Question: Is M^f_T close to Gaussian?

First intuition: If $\frac{d}{dt}\langle M^f\rangle_t$ is close to constant, then M^f_T is close to Gaussian

Wrong! $\rightsquigarrow$ Poisson process

$\hookrightarrow$ but Poisson IS close to Gaussian

Second intuition: If jumps of M^f_T are small (compared to $\langle M^f\rangle_t$), then M^f_T is close to Gaussian

Thrm.: Let $\phi : [0,T] \to [0,\infty)$ be given. Define

$$\sigma_{t,T}^2 := \int_t^T \phi(s)\,ds$$

$$\gamma_2 := \int_0^T \frac{\mathbb{E}[|\Gamma f(\omega_t) - \phi(t)|]}{\sigma_{t,T}}\,dt$$

$$\gamma_3^1(t_3) := \int_0^{T-t_3} \frac{\mathbb{E}[|\Gamma^3 f(\omega_t)|]}{\sigma_{t,T}^2}\,dt$$

$$\gamma_3^2(t_3) := \int_{T-t_3}^T \mathbb{E}[|\Gamma^3 f(\omega_t)|]\,dt$$

$$\gamma_4^1(t_4) := \int_0^{T-t_4} \frac{\mathbb{E}[\Gamma^4 f_t(\omega_t)]}{\sigma_{t,T}^{3/2}}\, dt$$

$$\gamma_4^2(t_4) := \int_{T-t_4}^{T} \mathbb{E}[\Gamma^4 f_t(\omega_t)]\, dt.$$

Then,

$$\boxed{\mathrm{Wass}_1(\mu_T^f, \mathcal{N}(0,\sigma_{0,T}^2)) \leq C \max\{\gamma_2+\gamma_3^1+\gamma_4^1, (\gamma_3^2)^{1/3}, (\gamma_4^2)^{1/4}\}.}$$