Spreading of a wave packet in a disordered medium

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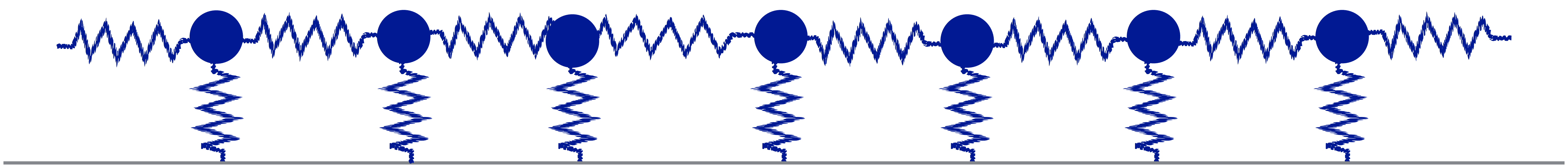
Classical chain of oscillators

$$H(p,q) \;=\; \frac{1}{2}\sum_{x\in\mathbb{Z}} p_x^2 + \omega_x^2 q_x^2 + g(q_x - q_{x+1})^2 + \lambda q_x^4$$



$$\lambda > 0, \qquad \omega_x^2 \text{ i.i.d.}$$

Hamiltonian dynamics:

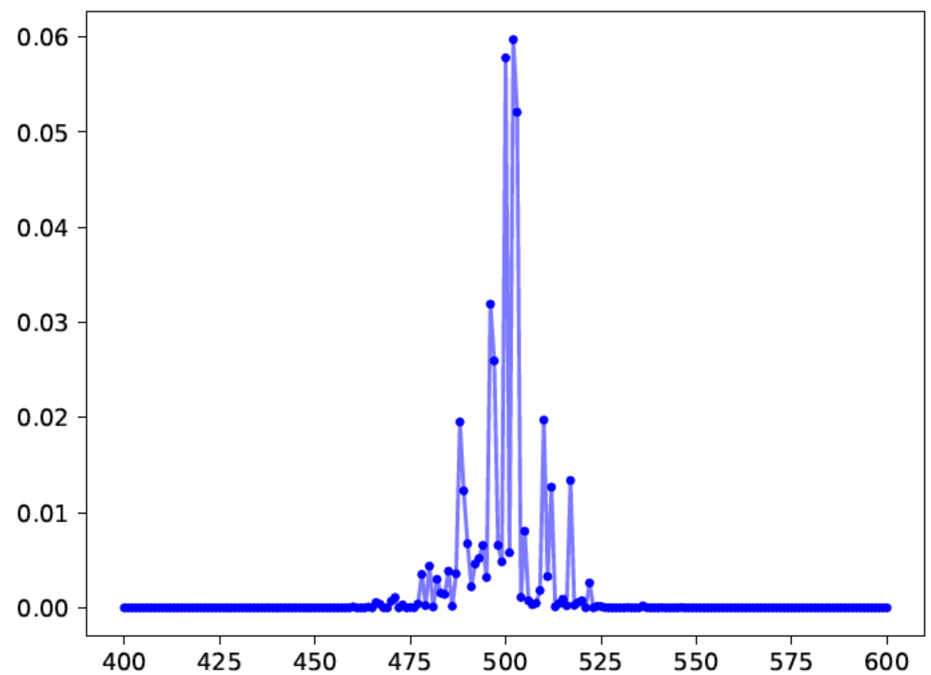
$$\dot{q} \;=\; p, \qquad \dot{p} \;=\; -\nabla_q H(p,q)$$

(nothing special in equilibrium)

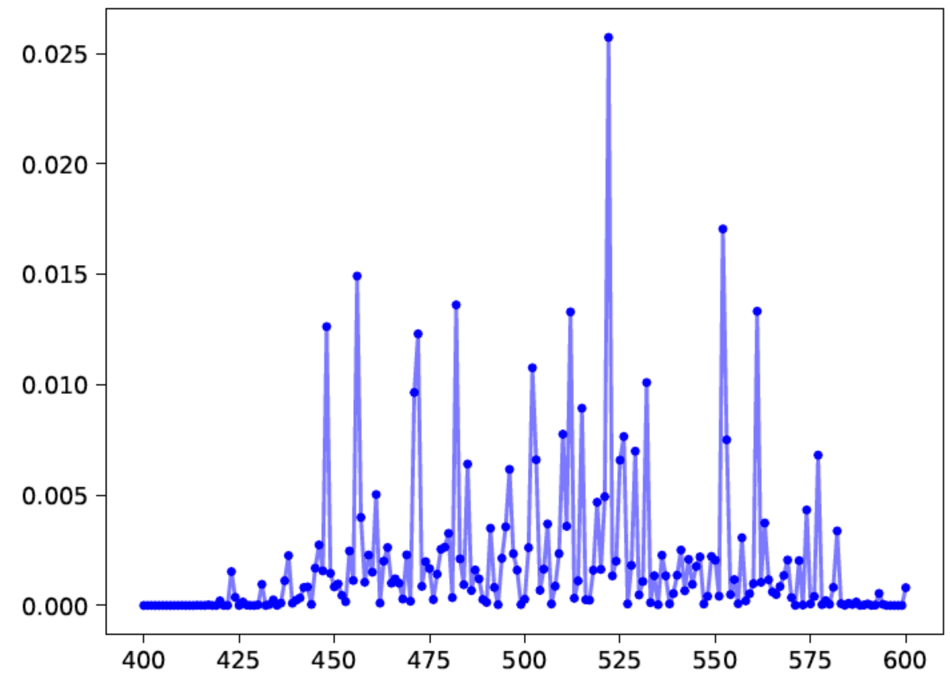
Spreading of a wave packet

The energy is conserved.

Finite quantity of energy in the system (zero temperature)



initial packet



later on

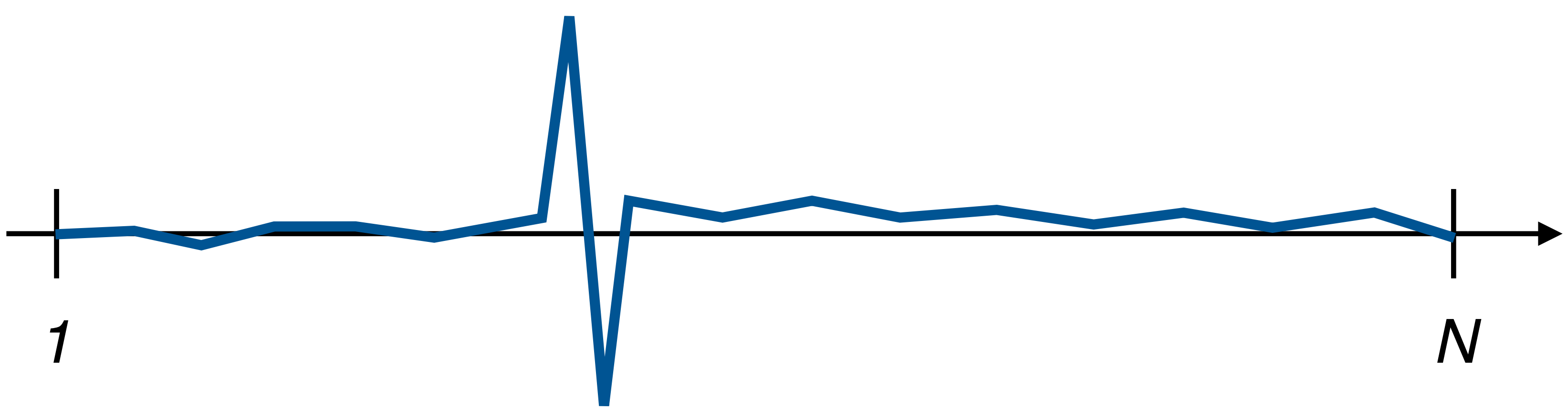
$$H = \sum_x H_x$$

Harmonic case (λ=0)

Linear equations of motion (Anderson localization):

$$\ddot{q} \;=\; -(V - g\Delta)q$$

If $(\omega_x)_{x\in\mathbb{Z}}$ **i.i.d.** , the eigenmodes are **localized :**

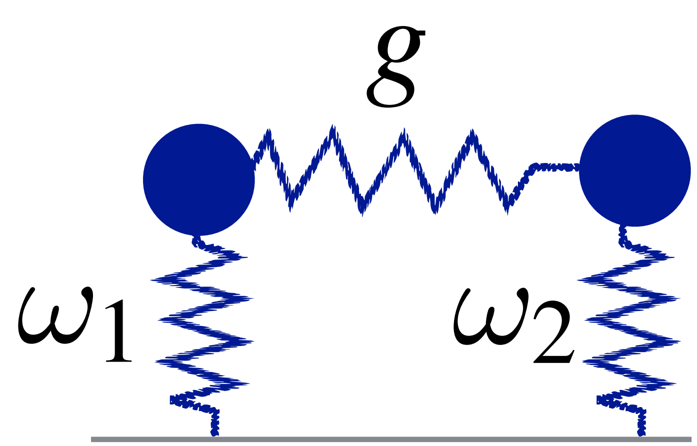


a mode for a chain of size ***N***

$$\mathsf{E}\Big(\sum_{\psi}|\psi(x)\psi(y)|\Big) \;\leq\; \mathrm{C}e^{-c|x-y|}, \qquad \forall x, y$$

Intuition for localization

Oscillators at different frequencies "don't talk to each other", i.e. they are not in resonance

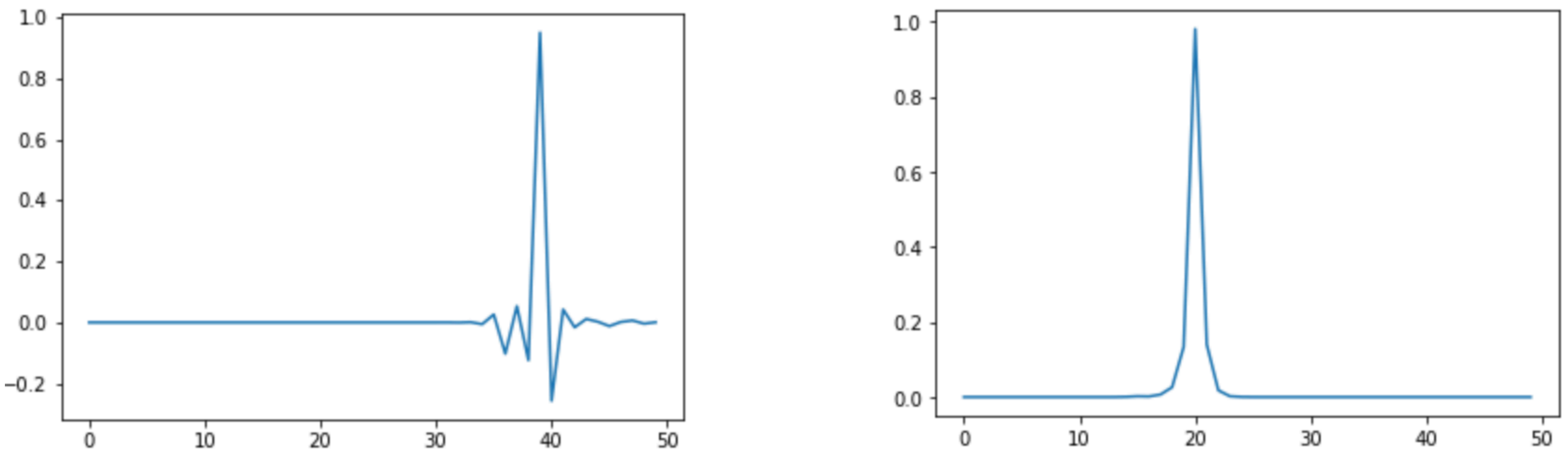


$$-(V - g\Delta) = \begin{pmatrix} -\omega_1 & g \\ g & -\omega_2 \end{pmatrix} \qquad g \ll |\omega_1 - \omega_2|$$

$$\psi_1 = (0.99\ ,\ 0.14), \qquad \psi_2 = (0.14\ ,\ -0.99)$$

Intuition for localization

This still holds true for larger matrices:

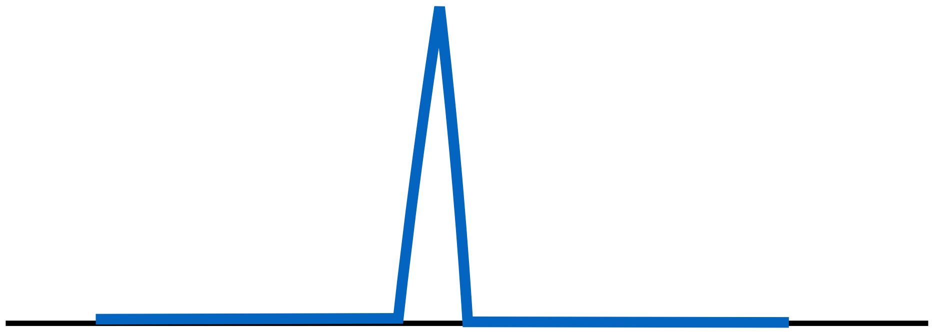


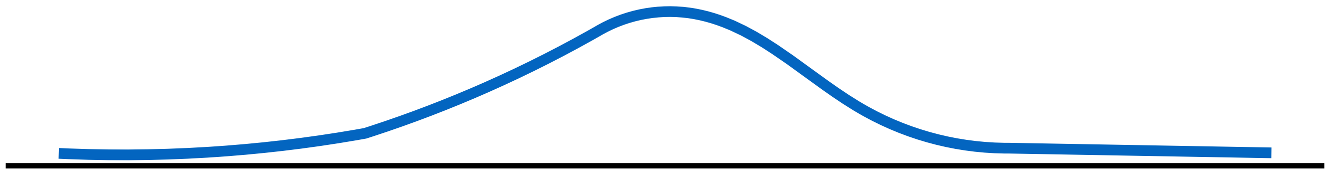
E.g.: two eigenvectors for 50 sites and g=0.1

Can be generalized to the full lattice, and much more…

Harmonic case (λ=0)

The packet **does not** spread (indefinitely):


Initial time


arbitrary long time

Linearity : solution = superposition of localized modes

Anharmonic case ($\lambda \neq 0$)

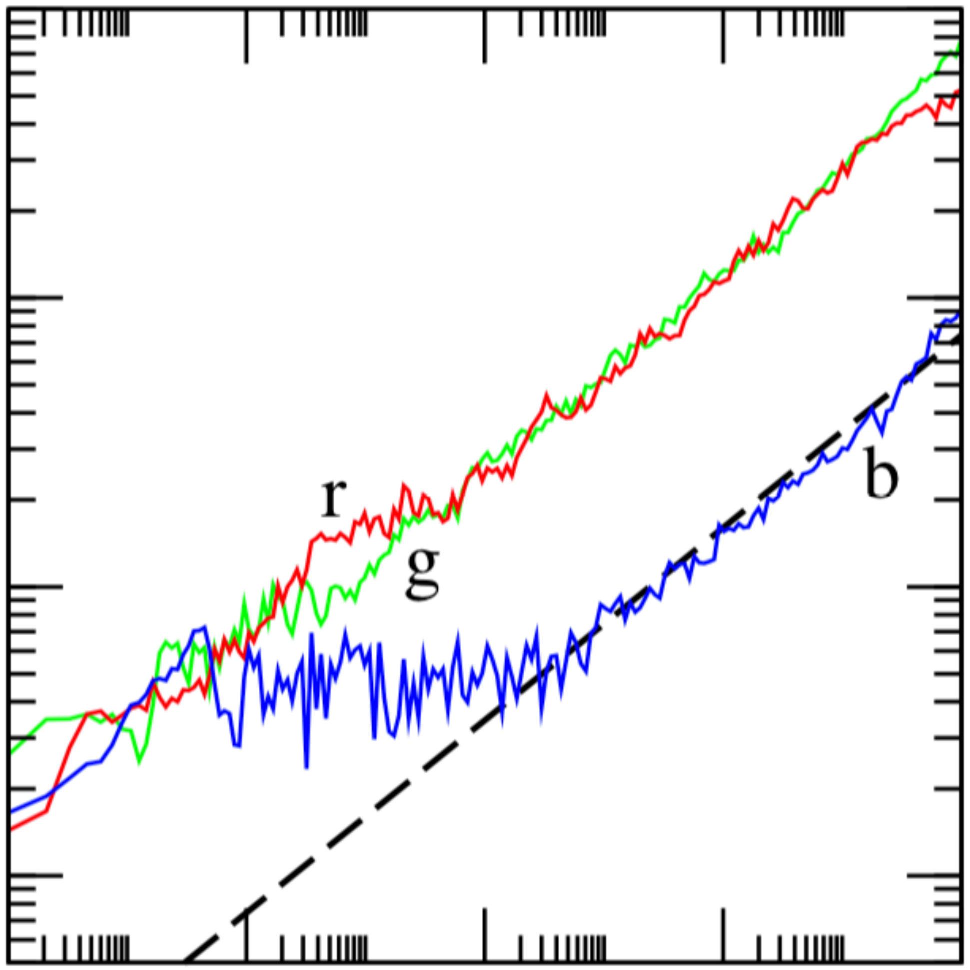
The packet does spread. **At which rate** ?

- Numerical simulations,
- Analytical computations, mathematical results

The main difficulty is, that there is no regime of parameters, where analytical and numerical results agree for a long time.

From S. Fishman, Y. Krivolapov and A. Soffer (2012)

Numerics : power law



m_2

t

-------------- $t^{1/3}$

$$m_2(t) = \sum_x x^2 H_x(t)$$

b, g, r : from low to high energy

(you may perhaps think: from small to large λ)

cf. e.g. Flach et al. (2009, 2014, 2020), and many others

Remark:

Small energy density $\longleftrightarrow$ small effective $\boldsymbol{\lambda}$

Because anharmonic interactions are given by

$$\lambda q^4 = (\lambda q^2) q^2 \simeq (\lambda E) q^2$$

In equilibrium, the effective non-linearity is indeed

$$\lambda T$$

Analytical : slower than a power law!

Theorem (W.-M. Wang and Z. Zhang, 2009)

«The wave packet stays localized for a very long time with very high probability»

$$\forall n \in \mathbb{N},\ \exists \lambda_0 > 0\ :\qquad \lambda < \lambda_0 \quad \Rightarrow \quad \tau \geq \frac{1}{\lambda^n}$$

- τ is the 1st time that 10% of the energy exits some box around the origin
- with probability that goes quickly to **1** as $\lambda \to 0$.

Remarks:

- No proper contradiction with the numerics
- Different (more idealized) model (*atomic limit*):

$$H(p,q) \;=\; \frac{1}{2}\sum_{x\in\mathbb{Z}} p_x^2 + \omega_x^2 q_x^2 + \lambda_1 (q_x - q_{x+1})^2 + \lambda_2 q_x^4$$

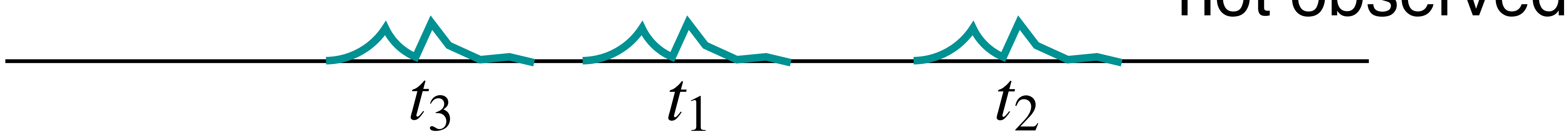
$$\lambda \;=\; \lambda_1 \vee \lambda_2$$

- Several other results of this type. E.g. improved bounds by H. Cong, Y. She and Z. Zhang (2020)

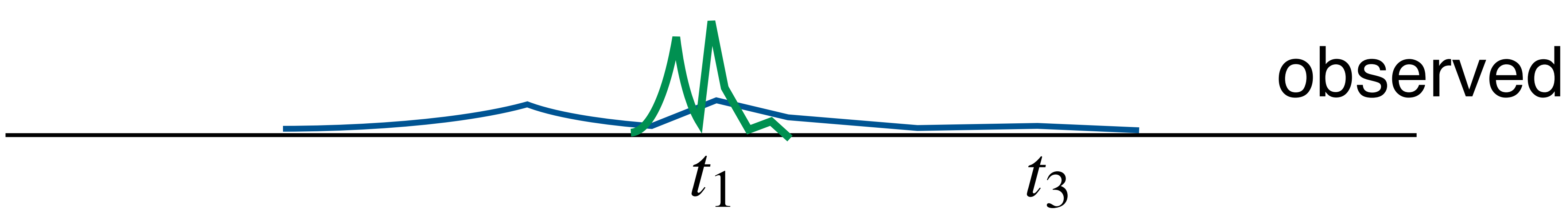
Another analytical result

Let us first contemplate two scenarios for « spreading »:

1. **Wandering** of a hot spot



2. Proper **spreading**



A theorem for yet another model

$$H(p,q) = \frac{1}{2}\sum_{x\in\mathbb{Z}} p_x^2 + \omega_x^2 q_x^2 + \lambda(q_x - q_{x+1})^2 + \lambda_x q_x^4$$

$$\lambda_x = \frac{\lambda}{(1+|x|)^\tau}, \qquad \tau > 0$$

Theorem (J. Bourgain and W.-M. Wang, 2007)

« The packet spreads slower than any power law in time. »

$$\forall n \geq 1, \exists \lambda_0 > 0 : \quad \lambda < \lambda_0 \quad \Rightarrow \quad \sum_{x\in\mathbb{Z}} x^2 H_x(t) \leq t^{1/n}$$

a.s. for all $t \geq 0$ provided that this quantity is finite at $t = 0$

Remark: If the packet properly ***spreads***, the effective λ decays.

Can numerics be misleading?

The observed spreading is actually ***very slow*** :

$$m_2^{1/2}(t) \sim t^{1/6}$$

- When the packet spreads, the effective non-linearity (λq^4) decays.
- If the spreading is slow, you need a lot of time for the effective non-linearity to decay, and so you need a lot of time to change regime and see another power law.

Cheap but not unrealistic to think that numerics were not run for a long enough time

Direct comparison numerics/theory

Recreate a “contradiction” numerics/math
that can be decided

We will define a quantity ***I(t)***

- for the **original model** (technical issues),
- starting from **equilibirum** (simpler),
- that can be controlled by a **theorem**.

Roughly *I(t)* measures the loss of memory in the system

Definition of *I(t)*

Another way to decompose the Hamiltonian at ***λ = 0*** :

$$\sum_x H_x \;=\; \frac{1}{2}\langle p,p\rangle + \langle q,(V-g\Delta)q\rangle$$

$$\;=\; \frac{1}{2}\sum_E |\langle p,E\rangle|^2 + E|\langle q,E\rangle|^2 \;=\; \sum_E H_E$$

with

$$(V-g\Delta)|E\rangle \;=\; E\,|E\rangle$$

The energy of each mode is conserved at ***λ = 0*** :

$$\frac{dH_E}{dt} \;=\; 0 \qquad \forall E$$

Definition of *I(t)*

For the coupled dynamics, $H = H_0 + \lambda H_1$, we define

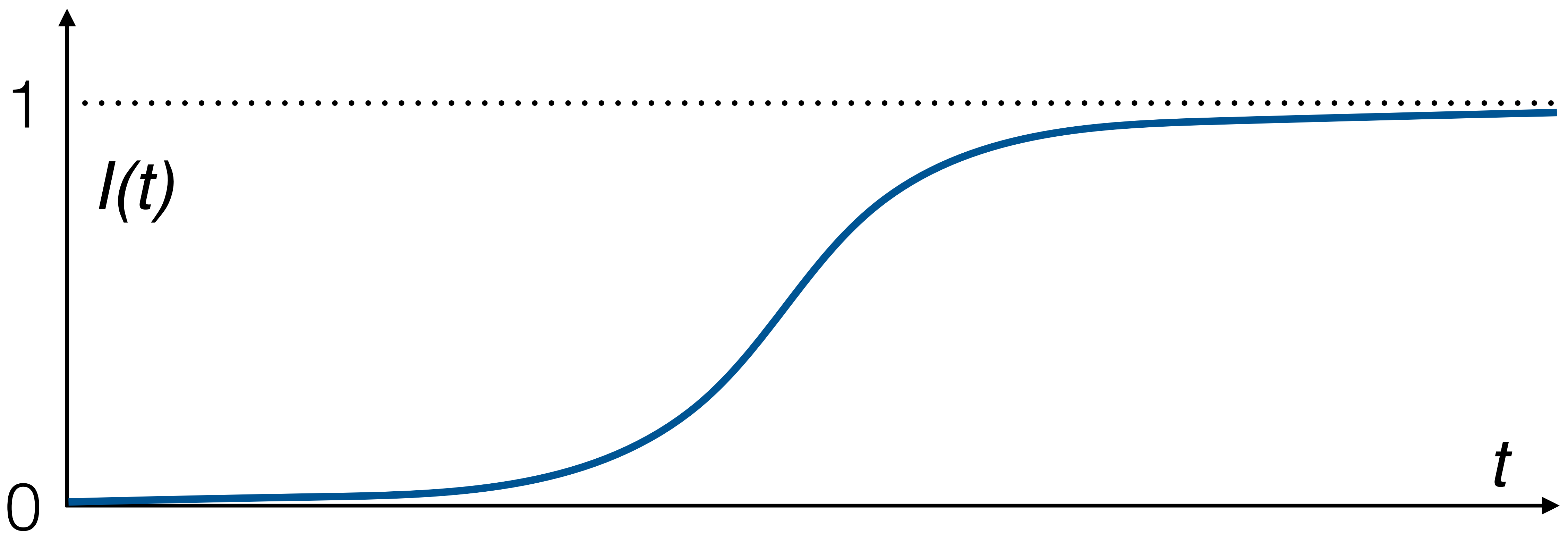
$$I(t) \;=\; \frac{1}{N}\sum_{E}\frac{\langle (H_E(t) - H_E(0))^2\rangle_T}{2\,\mathrm{var}(H_E)}$$

where $\langle f\rangle_T$ is the Gibbs state at temperature ***T***:

$$\langle f\rangle_T \;=\; \frac{1}{Z}\int f(q,p)e^{-H(q,p)/T}dqdp$$

(this is an equilibrium measure for the dynamics)

Expected behavior for *I(t)*



$$I(0) \;=\; 0 \;\; : \text{by definition}$$

$$I(+\infty) \;=\; 1 \;\; : \text{in the large } N \text{ limit, } \langle H_E(t); H_E(0)\rangle_T \to 0 \text{ as } t \to \infty$$

Rigorous bound on *I(t)*

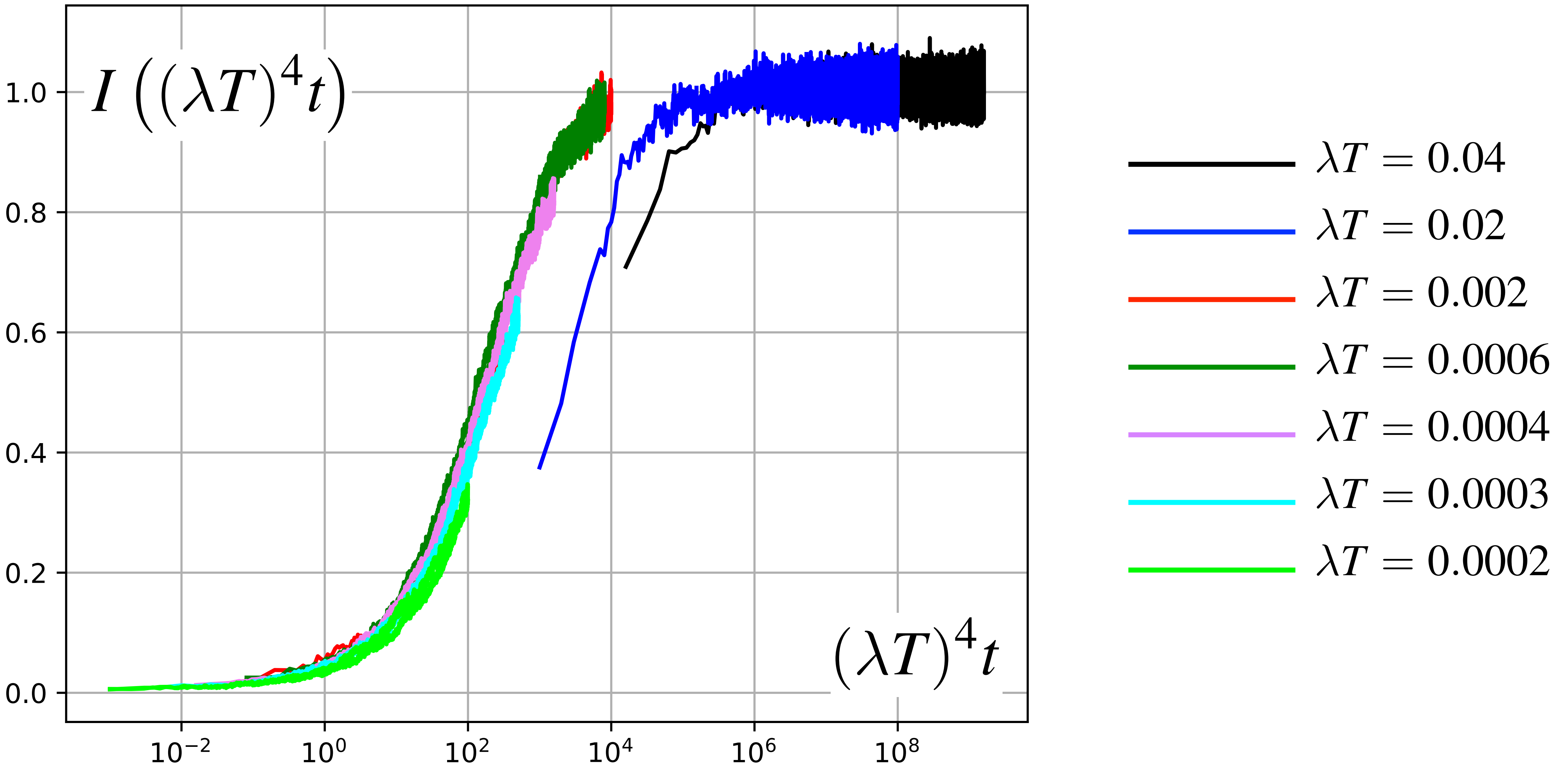
Theorem (W. De Roeck, F. H. and O. Prosniak)**.**
Let $n \in \mathbb{N}$. There exists a deterministic constant $\mathrm{C}_n < +\infty$ such that for all $\lambda \geq 0$ and for all $t \geq 0$,

$$\limsup_{N \to \infty} I_N(t) \;\leq\; \mathrm{C}_n \big(\lambda^{2-a} + (\lambda^n t)^2\big)$$

a.s. with $a < 2$ that can be made explicit.

Remark: We assume that the temperature T is fixed.
Actually $I(\lambda, T) = \bar{I}(\lambda T)$.

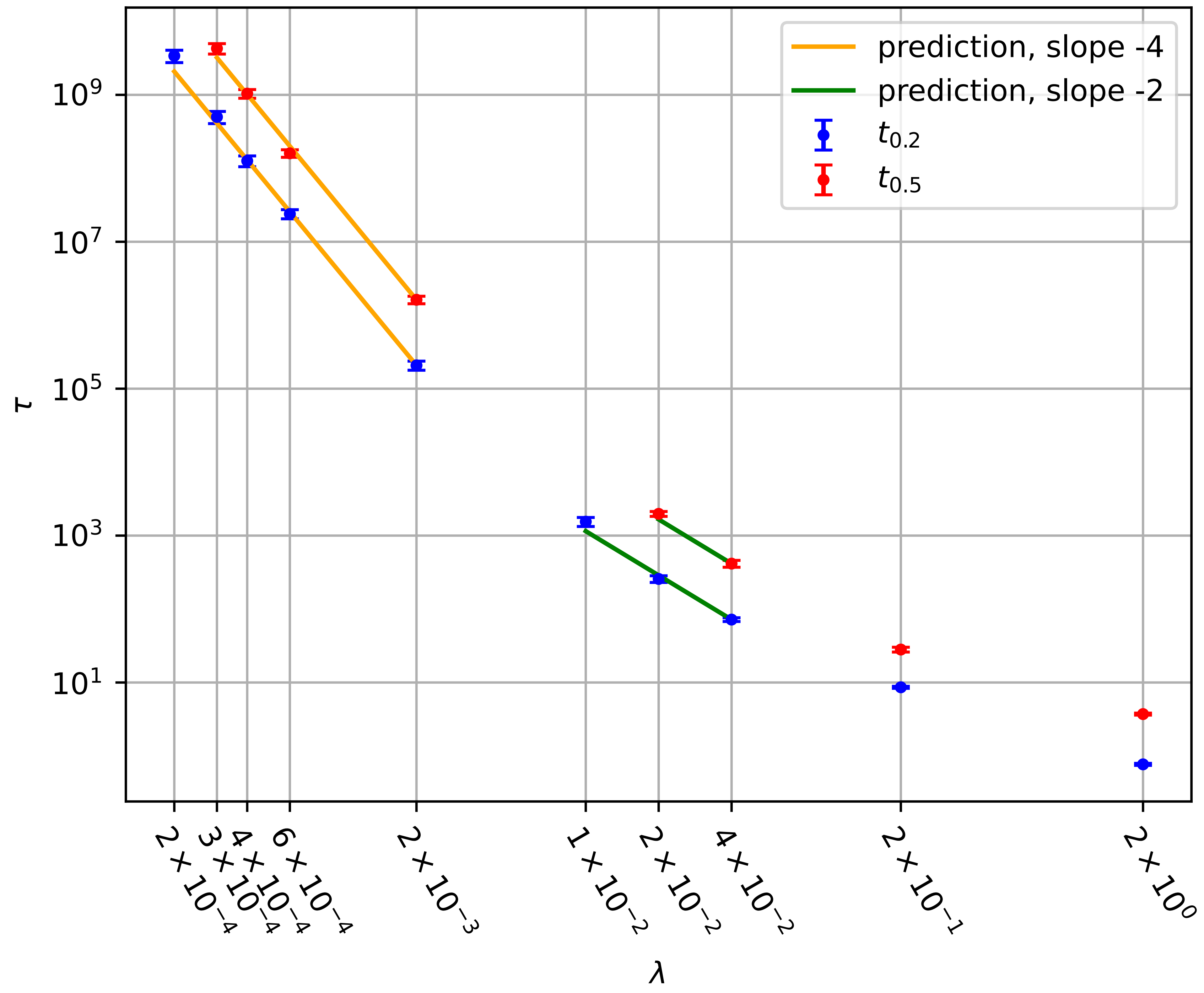
Numerical results for *I(t)*



It would seem that $I(t) = f((\lambda T)^4 t)$, but we know it is not!

(smallest value of λ suggests actually a deviation from this behavior)

Numerical results for *I(t)*



Power law consistent with S. Flach et al.

Back to the spreading of a wave packet. If local equilibrium holds inside the packet:

$$\partial_t E \;=\; \partial_x \big(D(T\lambda) \partial_x E \big)$$

and $T\lambda$ goes to 0 as the packet spreads

For this non-linear diffusion equation, we find

$$m_2(t) \;\sim\; t^{1/3} \qquad \Rightarrow \qquad D(T\lambda) \sim (T\lambda)^4$$

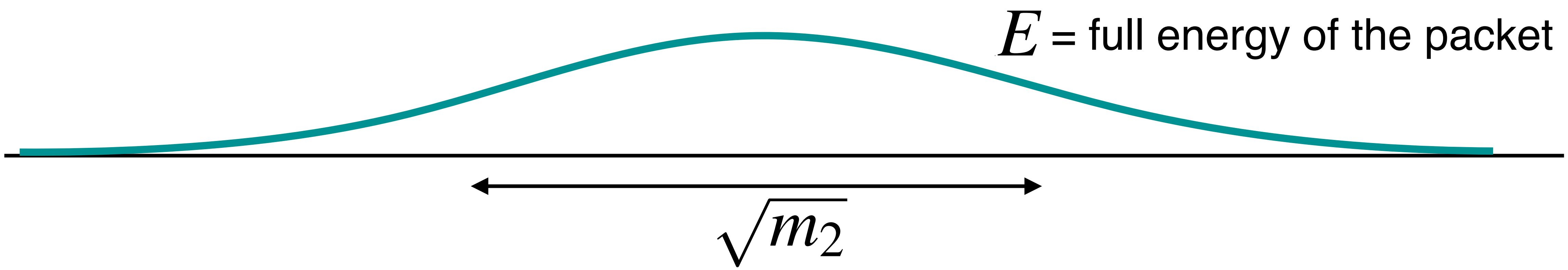
Consistent within linear fluctuating hydrodynamics:

$$I(t) \;=\; f(Dt) \;=\; f\big((T\lambda)^4 t \big)$$

Timescales consistent with S. Flach et al.

Effective temperature of the wave packet:

$$\lambda T \;\leftrightarrow\; \frac{\lambda E}{\sqrt{12 m_2}} \;=\; \text{final energy density}$$



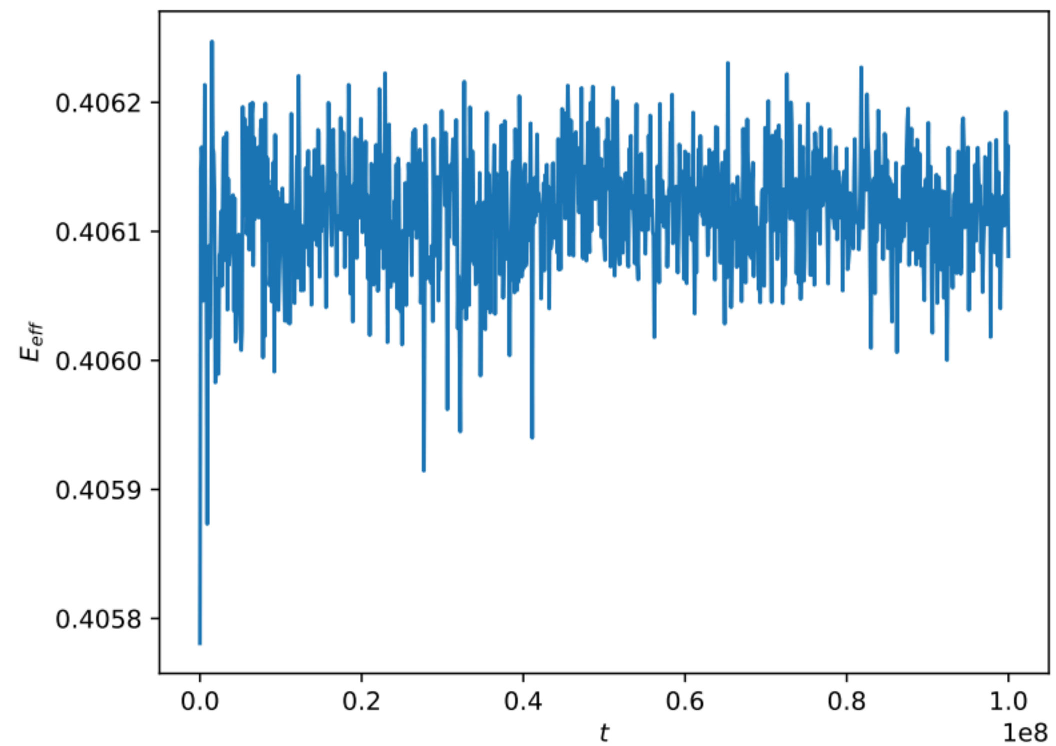
Smallest effective non-linearity that is reached:

λT = 0.0005 ± … (Flach), λT = 0.0002 (us)

Does numerics work at all?

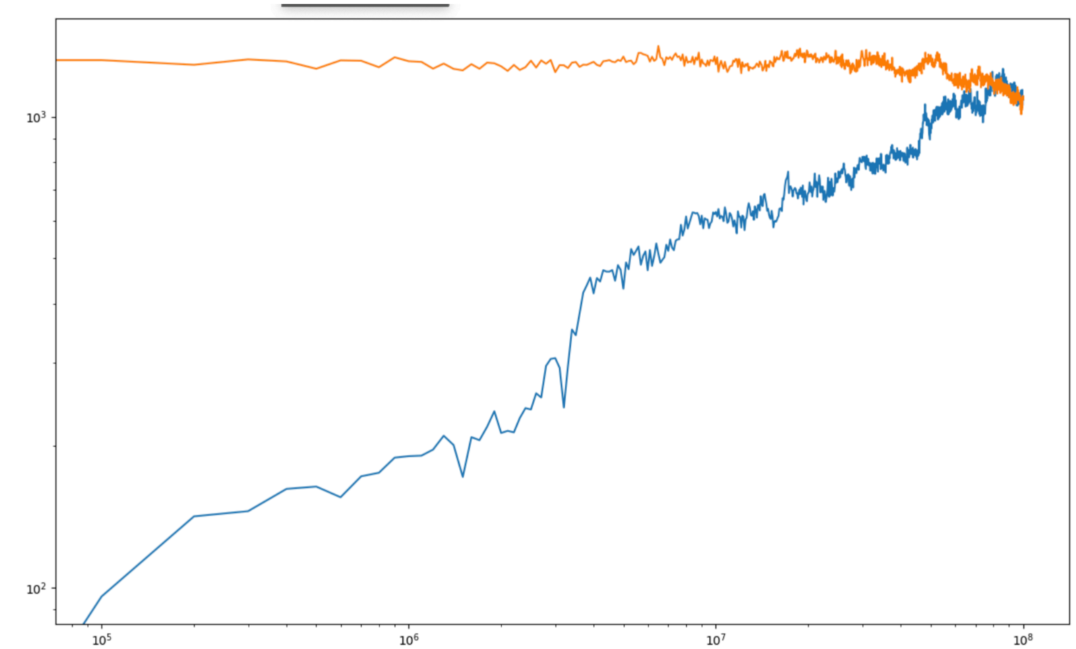
Yes:

- Harmonic chain
- Pseudo-conservation of an effective energy



No:

- Correct time reversal (probably asking too much)



Some catch: Is the packet in local equilibrium?

Applying statistical mechanics is challenging:

- finite amount of energy,
- nearly conserved local quantities

Observation: pre-thermal plateau for the *clean* chain

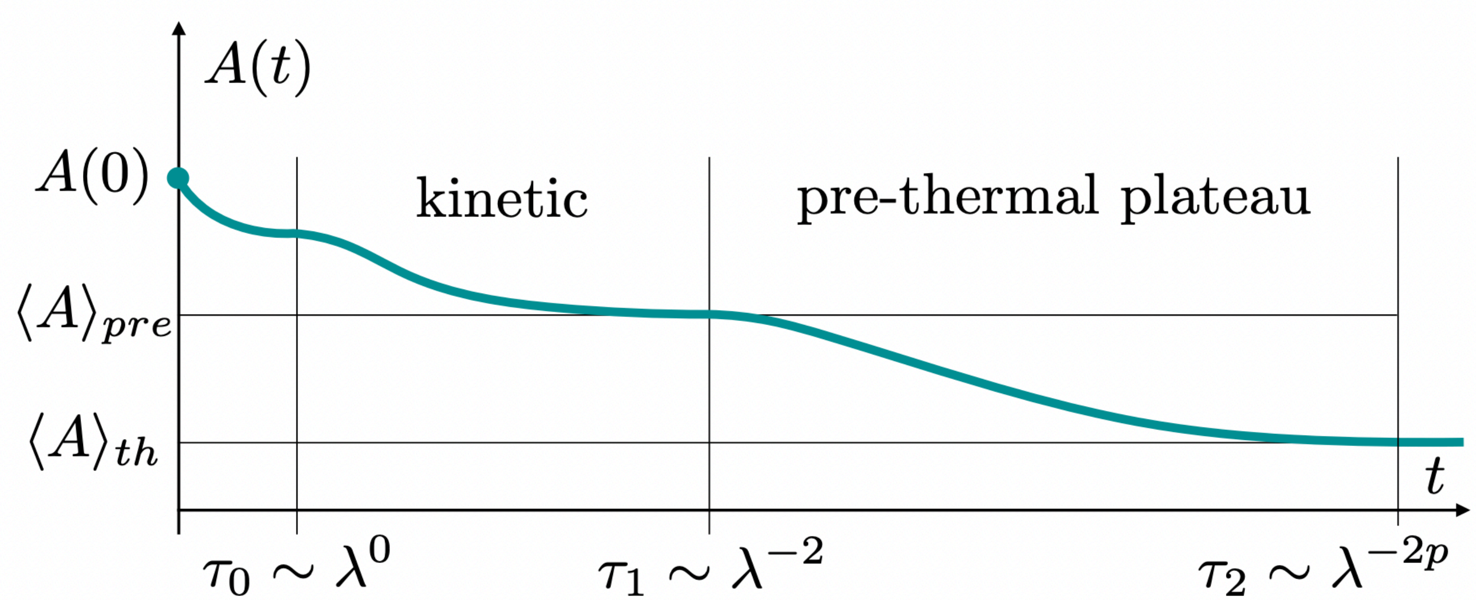
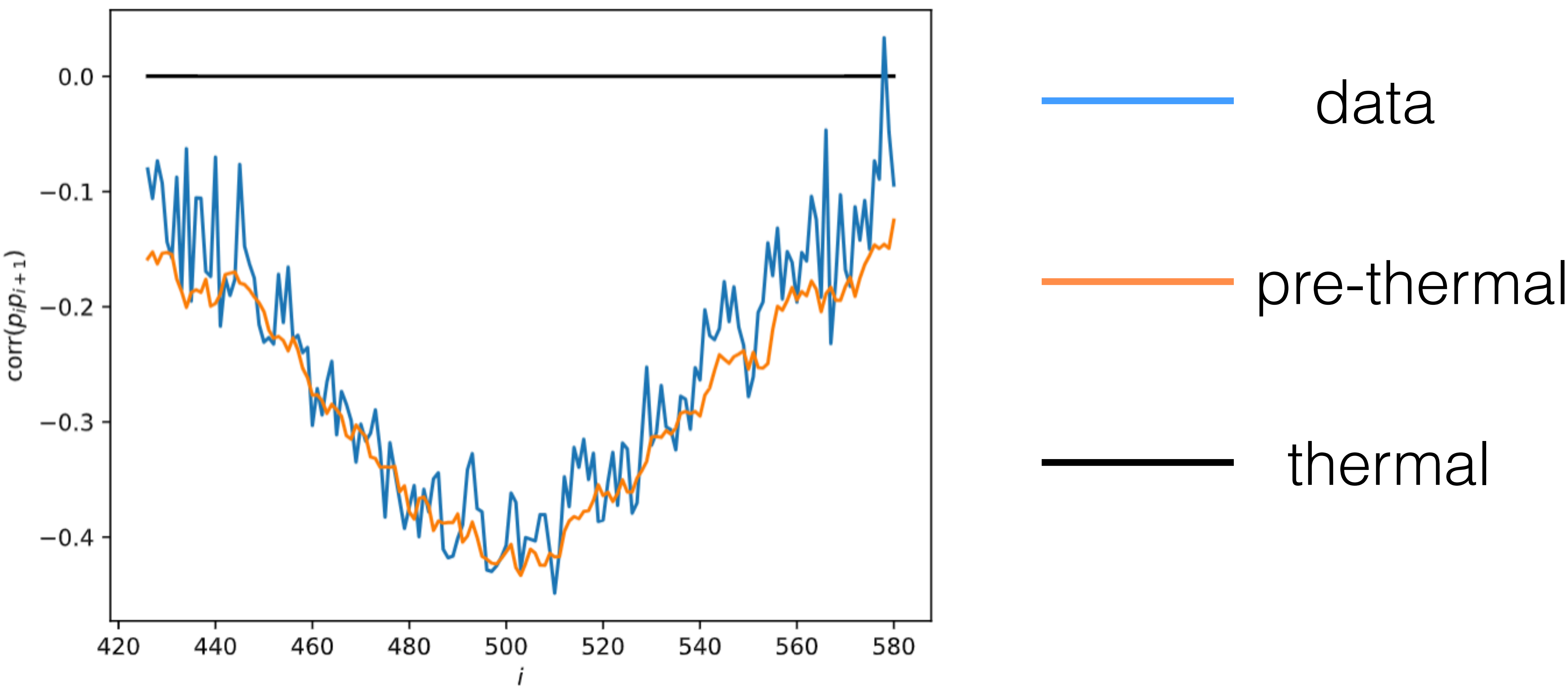


Figure 1. Expected time evolution of a local observable $A(t)$.

H conserved, N pseudo-conserved (number of phonons)

Pre-thermal state in the packet?

Preliminary data suggest the packet is pre-thermal:



It is probably the closest to equilibrium that we can get on these time scales

Challenges with the proof

Perturbative analysis in λ: $\forall n$, find u_n and g_n such that

$$\frac{dH_E}{dt} = \lambda\{H, u_n\} + \lambda^n g_n$$

Hence

$$H_E(t) - H_E(0) = \lambda(u_n(t) - u_n(0)) + \lambda^n \int_0^t ds\, g_n(s)$$

fluctuation ‘dissipation’

Controlling denominators

The perturbative expansion yields *small denominators*:

$$\frac{1}{\sigma_1\nu_1 + \cdots + \sigma_m\nu_m}$$

with

$$\sigma_k \;=\; \pm 1$$

ν_k eigenfrequencies $\;\Leftrightarrow\;$ ν_k^2 eigenvalues of $H = V - g\Delta$

Heuristics: $\nu_1, \ldots, \nu_m$ are nearly i.i.d.

Controlling denominators

- 2 eigenvalues in a system of size L : Minami's estimate

$$\mathsf{P}(\exists \nu_k \neq \nu_{k'} : |\nu_k^2 - \nu_{k'}^2| \leq \gamma) \;\leq\; L^2\gamma$$

- Linear combination of > 2 eigenvalues : **?**

$$\nu_{k_1}^2 + \nu_{k_2}^2 - \nu_{k_3}^2 - \nu_{k_4}^2$$

(would feature if KG would be replaced by DNLS)

- Linear combination of > 2 eigenfrequencies : New bound!

$$\nu_{k_1} + \nu_{k_2} - \nu_{k_3} - \nu_{k_4}$$

Trick to control denominators

Shift the full spectrum:

$$H \;\to\; H + \alpha \mathrm{Id}$$

- Leaves *invariant* : $\nu_{k_1}^2 + \nu_{k_2}^2 - \nu_{k_3}^2 - \nu_{k_4}^2$

- Does *not* leave *invariant* : $\nu_{k_1} + \nu_{k_2} - \nu_{k_3} - \nu_{k_4}$

In our model, the disorder is on the diagonal:

$$H \;=\; V - g\Delta, \qquad V_x = \omega_x^2 \quad \text{i.i.d.}$$

So, we can escape resonances by shifting the whole disorder

Control on denominators

This idea yields a lemma:

Lemma (WDR, FH, OP)**.**
In a system of size L, for any $0 < \varepsilon < 1/L$,

$$\mathsf{P}\left(\min \left| \sum_{k=1}^{m} \tau_k \nu_k \right| \leq \varepsilon \right) \; \leq \; \mathrm{C}_m L^m \varepsilon^{\frac{1}{m+1}}$$

where the minimum runs over m-tuples of all different eigenvalues, and where $\tau_k \in \{-m, \dots, m\}$, $\tau_k \neq 0$ are given.

From a technical point of view, this is the key new result

Conclusion and outlook

- Our mathematical results show that the chain is asymptotically many-body localized: dissipative effects arise as a non-analytic function of λ.

- Comparison with numerical data suggest that state-of-the-art numerics do not capture correctly the asymptotic behavior of the chain.

- Mathematical results are so far limited to the dynamics in equilibrium. We are working to relax this hypothesis.