Dynamic Large Deviations: In search of Matching Bounds

Kac's Process and the Zero-Range Process

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Introduction

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- Relation to aspects of the original PDE?

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Restricted Large Deviations: Kac's Process

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- Total cost $\mathcal{I}(\mu_\bullet, w) = H(\mu_0|\gamma) + \mathcal{J}(\mu_\bullet, w)$.

Theorem 1: Positive Result

The rate function $\mathcal{I}$ written above captures at least some of the correct large deviations behaviour:

Theorem (H, 2021; see also Basile-Benedetto-Bertini-Orierri, 2021)

- *The variables* $(\mu^N_\bullet, w^N)$ *are exponentially tight in* $\mathbb{D} \times \mathcal{M}(E)$.
- *For all* $\mathcal{A} \subset \mathbb{D} \times \mathcal{M}(E)$ *closed,*

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where $\mathcal{X} = \{(\mu_\bullet, w) : \langle |v|^2 + |v_\star|^2, w\rangle < \infty\}$.

Positive Result: Main ideas

- **Upper bound:** variational formulation

$$\mathcal{I}(\mu_\bullet, w) = \sup\left\{\Xi(\mu_\bullet, w, \varphi, f, g)_T : \varphi \in C_{b,v}, f \in C^{1,0}_{b,t,v}, g \in C_c(E)\right\}$$

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- **Argument exploits uniqueness for (BE_K)**.

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- Argument: suppression of collisions in various 'bad' regions of collision space, then convolution $\mu_t \mapsto g_\delta \star \mu_t$.
    - Truncation argument critically uses $\langle 1 + |v|^2 + |v_\star|^2, w \rangle < \infty$.

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Theorem (H, 2021; see also Basile-Bernadetto-Bertini-Caglioti 2021)
Let $\Theta : [0, T] \to [0, \infty)$ be a bounded energy profile satisfying certain technical assumptions. Then the set

$$\mathcal{A}_\Theta = \{(\mu_\bullet, w) : \mu_0 = \gamma, \mu_\bullet \text{ solves (BE)}, w = \overline{m}_\mu, \langle |v|^2, \mu_t \rangle = \Theta(t)\}$$

is compact, nonempty, and $\mathcal{I} = 0$ on $\mathcal{A}_\Theta$. For some $\mathcal{V} \supset \mathcal{A}_\Theta$,

$$\liminf \frac{1}{N} \log \mathbb{P}\left((\mu^N_\bullet, w^N) \in \mathcal{V}\right) < 0 = -\inf\{\mathcal{I}(\mu_\bullet, w) : (\mu_\bullet, w) \in \mathcal{A}_\Theta\}$$

while

$$\inf_{\mathcal{U} \supset \mathcal{A}_\Theta} \liminf \frac{1}{N} \log \mathbb{P}\left((\mu^N_\bullet, w^N) \in \mathcal{U}\right) > -\infty.$$

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$$\liminf \frac{1}{N} \log \mathbb{P}\left((\mu^N_\bullet, w^N) \in \mathcal{V}\right) < 0 = -\inf\{\mathcal{I}(\mu_\bullet, w) : (\mu_\bullet, w) \in \mathcal{A}_\Theta\}$$

while

$$\inf_{\mathcal{U} \supset \mathcal{A}_\Theta} \liminf \frac{1}{N} \log \mathbb{P}\left((\mu^N_\bullet, w^N) \in \mathcal{U}\right) > -\infty.$$

....so such behaviour cannot be excluded, but the rate function predicts the exponential occurrence wrongly.

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- Using martingale arguments, Pozvner inequality..., all distributional limits $(\mu_\bullet, w)$ are supported on $\mathcal{A}_\Theta$, so for all open $\mathcal{U} \supset \mathcal{A}$,

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- LDP rate function **still** not correct for other Boltzmann kernels (e.g. cutoff Maxwell Molecules).

Zero-Range Process: An Example with Matching Bounds

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for some Φ determined by the jump rate g.

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Key ideas: Renormalised kinetic solutions (generalising DiPerna-Lions, Ambrosio, LeBris-Lions).

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- In particular, $\mathcal{I} = \infty$ outside of $\mathcal{R}$.

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Let η_0^N be drawn from a local equilibrium $\rho \in C(\mathbb{T}^d, (0,\infty))$. Then we have ***matching*** *large deviations upper and lower bounds with speed N^d/χ_N, and the rate function given by taking $\mathcal{I}_0(u_0) := \alpha H(u_0|\rho)$ and nonlinearity $\Phi(u) = u^\alpha$.*

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- **Innovations: remove paths outside $\mathcal{R}$ by trajectorial estimates, recovery sequences for paths inside $\mathcal{R}$.**

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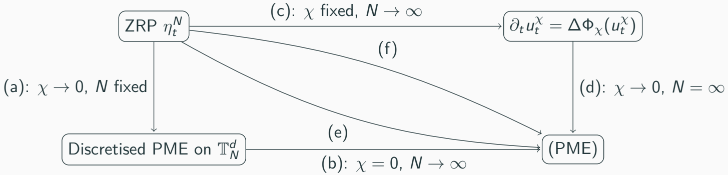
- Possible limits:



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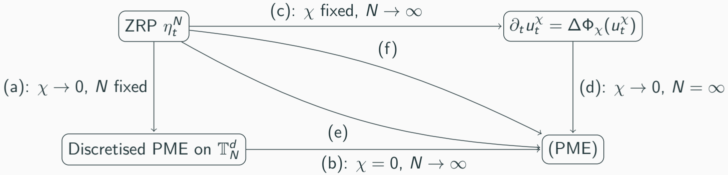

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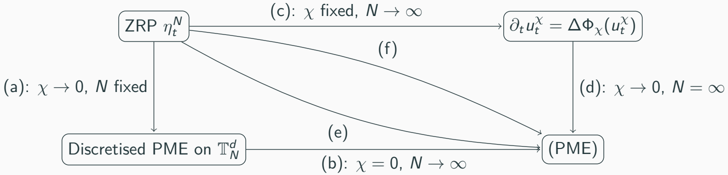

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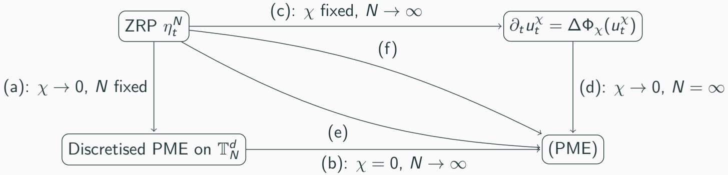

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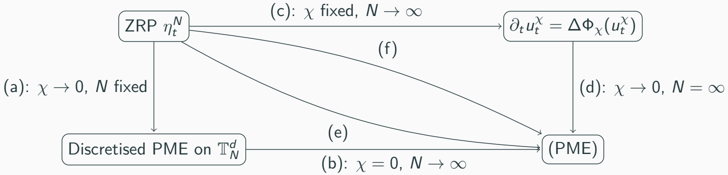

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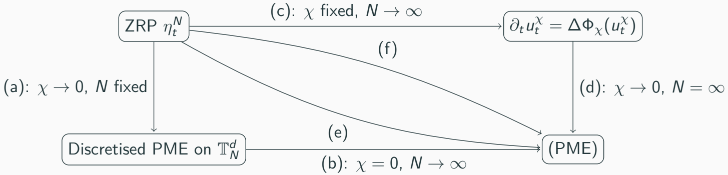

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- Still formally, $|\alpha D\mathcal{H}(u)|^2_u = \alpha \mathcal{D}_\alpha(u)$.

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- "Improbability of starting at nonequilibrium u_0 and evolving forwards by (PME) = Improbability of evolving via backwards (PME) into u_0".

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Theorem (H.-Gess 2023)

Let $u_\bullet \in \mathbb{D}$ with $\mathcal{H}(u_0) < \infty$. Then we have the identity

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allowing both sides to be infinite, where

$$\mathcal{A}(u_\bullet) = \frac{1}{2}\inf\left\{\|\theta\|^2_{L^2_{t,x}} : \partial_t u_t + \nabla\cdot(\frac{1}{2}u_t^{\alpha/2}\theta_t) = 0\right\}.$$

In particular, the functional on the right-hand side is nonnegative, and vanishes if and only if $u_\bullet$ is a solution to (PME).

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- The unique optimisers for g, θ are characterised by

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- After some manipulations,

$$\mathcal{J}(u_\bullet) = \frac{1}{2} \left(\alpha \mathcal{H}(u_T) - \alpha \mathcal{H}(u_0) + \left\| \Pi[u_\bullet] \nabla u^{\alpha/2} \right\|^2_{L^2_{t,x}} + \mathcal{A}(u_\bullet) \right).$$

Gradient Flow: Sketch Proof, 2

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  - ($\star$): Could be done purely by PDE tools from Fehrman-Gess - but not obvious starting from PME!

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