The Regularity Problem for the Landau Equation

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Landau equation with unknown $f \equiv f(t,v) \geq 0$:

$$\partial_t f(t,v) = \operatorname{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(f(t,v)f(t,w))dw\,, \quad v \in \mathbf{R}^3$$

with the notation:

$$a(z) := \tfrac{1}{8\pi}\nabla^2|z| = \tfrac{1}{8\pi|z|}\Pi(z)\,, \quad \Pi(z) := I - \left(\tfrac{z}{|z|}\right)^{\otimes 2}$$

Nonconservative form

$$\partial_t f(t,v) = \operatorname{trace}\left((a \star_v f(t,v))\nabla_v^2 f(t,v)\right) + f(t,v)^2$$

Open question Global existence of classical solutions or finite-time blow-up for the Cauchy problem with $f\big|_{t=0} = f_{in}$?

A Model Problem? Semilinear Heat Equation

Semilinear heat equation Finite time blow-up for $u \geq 0$ soln of

$$\partial_t u = \Delta_x u + \alpha u^2$$

Kaplan's method (CPAM1963) Ricatti inequality

$$\dot{L}(t) \geq \underbrace{-\lambda_0 L(t)}_{\text{dissipation}} + \underbrace{\alpha L^2(t)}_{\text{NL term}}$$

for the averaged quantity

$$L(t) := \frac{\int_B u(t,x)\phi(x)dx}{\int_B \phi(x)dx} \quad \text{with} \begin{cases} -\Delta\phi = \lambda_0\phi\,, & \phi > 0 \text{ on } B \\ \phi\big|_{\partial B} = 0 \end{cases}$$

Finite time blow-up if $L(0) > \lambda_0/\alpha$ i.e. if NL term>dissipation

"Isotropic" Landau Equation Global existence of nonincreasing and radially symmetric solutions [Gualdani-Guillen 2016] $\alpha = 1$ — see also [Gressman-Krieger-Strain 2012] $\alpha < \frac{74}{75}$

$$\partial_t u = ((-\Delta)^{-1} u)\Delta u + \alpha u^2 , \qquad 0 \leq \alpha \leq 1$$

Idea Any local build-up of u feeds the diffusion coefficient $(-\Delta)^{-1} u$ which offsets the effect of the nonlinear term. Observe that

$$((-\Delta)^{-1} u)\Delta u \quad \text{and} \quad u^2$$

scale identically, both in the size of u and in spatial concentration.

In the isotropic model, diffusion — along with parabolic smoothing? — is expected to be 50% **stronger** than in the true Landau equation.

(1) **Conservation of mass+momentum+energy**

$$\frac{d}{dt}\int_{\mathbf{R}^3} f(t,v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dv = 0$$

(2) **H Theorem** Assuming that $f(t,v) > 0$ a.e., one has

$$\frac{d}{dt}\underbrace{\int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv}_{H\text{-function}}$$

$$= -\underbrace{\int_{\mathbf{R}^6}\underbrace{\frac{f(t,v)f(t,w)}{16\pi|v-w|}\left|\Pi(v-w)\left(\frac{\nabla_v f(t,v)}{f(t,v)}-\frac{\nabla_w f(t,w)}{f(t,w)}\right)\right|^2}_{\frac{1}{4\pi}\left|\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{f(t,v)f(t,w)/|v-w|}\right|^2} dvdw}_{\text{entropy production rate}}$$

THM For each $0 \le f \in L^1_2(\mathbf{R}^3)$ s.t. $f \ln f \in L^1(\mathbf{R}^3)$, one has

$$\underbrace{\int_{\mathbf{R}^3} \frac{\left|\nabla\sqrt{f(v)}\right|^2 dv}{(1+|v|^2)^{3/2}}}_{\text{weighted Fisher info.}} \le C_D + C_D \underbrace{\int_{\mathbf{R}^6} \frac{\left|\Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f(v)f(w)}\right|^2}{|v-w|} dvdw}_{\text{entropy production rate}}$$

with

$$C_D \equiv C_D\left[\int_{\mathbf{R}^3} (1, v, |v|^2, |\ln f(v)|) f(v) dv\right] > 0$$

The Desvillettes inequality replaces the nonlocal entropy production rate with a **local lower bound** thereof with density, viz. the weighted Fisher information rate

$$\frac{\left|\nabla\sqrt{f(v)}\right|^2}{(1+|v|^2)^{3/2}}$$

A Loose Analogy

Dictionary Landau equation/3d Navier-Stokes (Lebesgue exponents)

Navier-Stokes	$u \in L^\infty_t L^2_x\,,$	$\nabla_x u \in L^2_t L^2_x$
Landau	$\sqrt{f} \in L^\infty_t L^2_2(dv)\,,$	$\nabla_v \sqrt{f} \cap L^2_t L^2_{-3}(dv)$

Notation weighted Lebesgue norm $\|g\|^p_{L^p_k} := \int_{\mathbf{R}^3} |g(v)|^p (1+|v|)^k dv$

THM[FG-Gualdani-Imbert-Vasseur AENS2022] The set of singular times for a global weak solution (à la Villani) of the Landau equation has **Hausdorff dimension** $\leq 1/2$.

Similar result for Leray solutions of 3d Navier-Stokes (Hausdorff dim of the set of singular times $< 1/2$) proved by Leray in 1934.

Diffusion Bounds

Pick $\chi \in C^\infty(\mathbf{R})$ such that

$$\mathbf{1}_{[-1,1]} \le \chi \le \mathbf{1}_{[-2,2]}\,, \qquad \text{with } |\chi'| \le 2$$

and define the truncated collision kernel by the formula

$$a_\delta(z) := (1 - \chi(|z|/\delta))a(z)$$

Ellipticity bound Assume that $f \equiv f(v) \ge 0$ satisfies

$$0 < m_0 \le \int_{\mathbf{R}^3} f(v)dv \le M_0\,, \quad \int_{\mathbf{R}^3} \begin{pmatrix} |v|^2 \\ \ln f(v) \end{pmatrix} f(v)dv \le \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}$$

For some $c_0[m_0, M_0, E_0, H_0] > 0$ and $\delta_0[m_0, M_0, E_0, H_0] \in (0,1)$ s.t.

$$\delta < \delta_0 \implies f \star a_\delta(v) \ge \tfrac{c_0}{(1+|v|)^3}\, I\,, \quad v \in \mathbf{R}^3$$

Locally Bounded Landau Solutions are Regular

PARABOLIC CYLINDER Denote $Q_r(t,v) := (t-r^2, t] \times B_r(v)$

If $0 \le f \in L^\infty(Q_R(t_0,v_0))$, then

$$f(t,\cdot)\star a = \underbrace{f(t,\cdot)\star a_\delta}_{L^\infty_t L^1_v \star_v L^\infty_v} + \underbrace{f(t,\cdot)\star(a-a_\delta)}_{=(f\mathbf{1}_{Q_R(t_0,v_0)})\star_v(a-a_\delta)} \in L^\infty(Q_{R-2\delta}(t_0,v_0))$$

With the ellipticity bound, Krylov-Safonov's theorem applied to the nonconservative form of Landau's equation implies that

$$f \in C^{\alpha,\alpha}(Q_{R/2}(t_0,v_0))$$

Then the diffusion matrix $f\star a \in C^{\alpha,\alpha}(Q_{R/4}(t_0,v_0))$ and the parabolic variant of Schauder's estimates implies that

$$f \in C^{1+\alpha/2,2+\alpha}(Q_{R/8}(t_0,v_0))$$

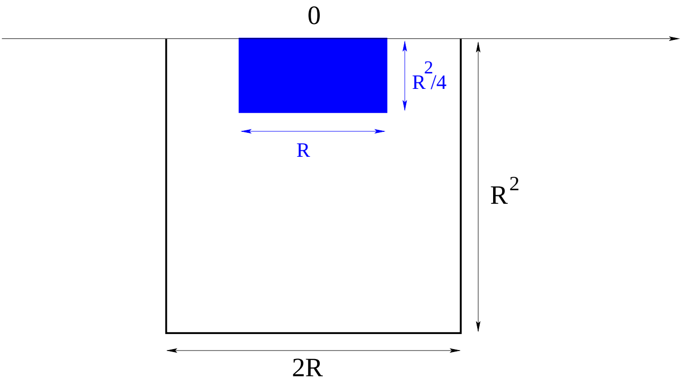

Figure: Nested parabolic cylinders.

Villani's Global Existence Result [ARMA 1998]

With the function χ defined above, set

$$\xi_\delta(v) := \chi(\delta|v|)\,, \qquad \zeta_\delta(v) := \tfrac{1}{X}\tfrac{1}{\delta^3}\chi\left(\tfrac{|v|}{\delta}\right)\,, \quad X := \int_{\mathbf{R}^3}\chi(|v|)dv$$

Villani solutions of the Cauchy problem for the Landau equation with initial data f_{in} are $f \equiv f(t,v) = \lim_{\delta\to 0^+} f_\delta(t,v)$ where

$$(\partial_t - \tfrac{\delta}{2}\Delta_v)f_\delta(t,v) = \operatorname{div}_v \int_{\mathbf{R}^3} a_\delta(v-w)(\nabla_v - \nabla_w)(f_\delta(t,v)f_\delta(t,w))dw$$

$$f_\delta\big|_{t=0} = \zeta_\delta \star (\xi_\delta f_{in}) + \tfrac{\delta}{(2\pi)^{3/2}}e^{-|v|^2/2}$$

THM Any $0 \le f_{in} \in L^1(\mathbf{R}^3;(1+|v|^2)dv)$ s.t. $f_{in}\ln f_{in} \in L^1(\mathbf{R}^3)$ launches a Villani solution of Landau's equation on $[0,+\infty)\times\mathbf{R}^3$

Main Results

Let f be a Villani solution of the Landau equation with initial data f_{in} measurable on $\mathbf{R}^3$ satisfying

$$f_{in} \geq 0 \text{ a.e. and } \quad \int_{\mathbf{R}^3} (1 + |v|^2 + |\ln f_{in}(v)|) f_{in}(v) dv < \infty$$

THM1 If f is radially symmetric, i.e. of the form $f(t, v) = F(t, |v|)$, then $f \in L^\infty_{loc}((0, +\infty) \times (\mathbf{R}^3 \setminus \{0\}))$

THM2 If f is axisymmetric, i.e. if there exists $\omega \in \mathbf{S}^2$ such that

$$f(t, v) = f(t, \mathcal{R}(v - (v \cdot \omega)\omega) + (v \cdot \omega)\omega)$$

for all $\mathcal{R} \in SO((\mathbf{R}\omega)^\perp)$, then $f \in L^\infty_{loc}((0, +\infty) \times (\mathbf{R}^3 \setminus \mathbf{R}\omega))$

Truncated Distribution Function and Entropy

NOTATION set

$$f_+^\kappa := (f-\kappa)_+ = \max(f-\kappa, 0)\,, \quad \ln_+ z := \max(\ln z, 0)$$

DEF For each $g \geq 0$, the truncated entropy generating function is

$$h_+^\kappa(g) = \kappa h_+\left(\frac{g}{\kappa}\right)\,, \qquad h_+(z) := z \ln_+ z - (z-1)_+$$

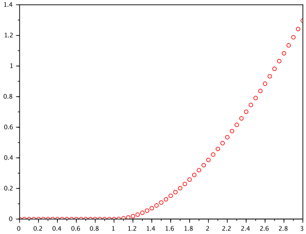

Figure: Graph of the function h_+

Local Truncated Entropy Inequality

Let f be a Villani solution of the Landau-Coulomb equation on $\mathbf{R}^3$.

For all $\Psi \in C_c^\infty((0,T)\times\mathbf{R}^3)$, there exists $\mathcal{N}\subset(0,T)$ negligible s.t.

$$\int_{\mathbf{R}^3} h_+^\kappa(f(t_2,v))\phi(t,v)dv$$

$$\leq \int_{\mathbf{R}^3} h_+^\kappa(f(t_1,v))\phi(t,v)dv + \int_{t_1}^{t_2}\int_{\mathbf{R}^3} h_+^\kappa(f(t,v))\partial_t\phi(t,v)dvdt$$

$$-\int_{t_1}^{t_2}\int_{\mathbf{R}^3}(A\nabla_v f-(\operatorname{div}_v A)f)\Big(\underbrace{\phi\frac{\nabla_v f_+^\kappa}{f}}_{\mathcal{T}_1}+\underbrace{\ln_+(\tfrac{f}{\kappa})\nabla_v\phi}_{\mathcal{T}_2}\Big)(t,v)dvdt$$

for all $\kappa>0$ and all $t_1<t_2\notin\mathcal{N}$, with

$$\phi:=\Psi^2\,,\qquad A(t,v):=\int_{\mathbf{R}^3} f(t,w)a(v-w)dw$$

Scalings

Landau's equation is obviously invariant by translations

$$f(t,v) \text{ solution} \quad \Longrightarrow \quad f(t_0+t, v_0+v) \text{ solution}$$

For all $\lambda > 0$ and $\mu > 0$

$$f(t,v) \text{ solution} \quad \Longrightarrow \quad f_{\lambda,\mu}(t,v) = \lambda f(\lambda t, \mu v) \text{ solution}$$

Action of the scaling on the **l.h.s.** of the key local estimate

$$\int_{B_r(0)} h_+^{\lambda\kappa}(f_{\lambda,\mu}(t,v))dv = \frac{\lambda}{\mu^3}\int_{B_{\mu r}(0)} h_+^{\kappa}(f(\lambda t,\tilde v))d\tilde v$$

$$\int_{-\tau}^0\int_{B_r}\frac{|\nabla_v(f_{\lambda,\mu}(t,v)-\lambda\kappa)_+|^2}{f_{\lambda,\mu}(t,v)}dtdv = \frac{1}{\mu}\int_{-\lambda\tau}^0\int_{B_{\mu r}}\frac{|\nabla_{\tilde v}(f(\tilde t,\tilde v)-\kappa)_+|^2}{f(\tilde t,\tilde v)}d\tilde t d\tilde v$$

Henceforth choose $\mu = \epsilon$ and $\lambda = \mu^2 = \epsilon^2$ so that $\lambda/\mu^3 = 1/\mu$

Let $f \equiv f(t,v)$ be a global Villani solution of Landau's equation. Pick $(t_0, v_0) \in (0,+\infty) \times \mathbf{R}^3$. Henceforth, we study in detail the rescaled solution near (t_0, v_0):

$$f_\epsilon(t,v) := \epsilon^2 f(t_0 + \epsilon^2 t, v_0 + \epsilon v)$$

Critical quantity

$$f_\epsilon(t,\cdot) \star \frac{1}{|\cdot|}(v) = f(t_0 + \epsilon^2 t, \cdot) \star \frac{1}{|\cdot|}(v_0 + \epsilon v)$$

so that

$$\left\| f_\epsilon \star_v \frac{1}{|\cdot|} \right\|_{L^\infty(Q_r(0,0))} = \left\| f \star_v \frac{1}{|\cdot|} \right\|_{L^\infty(Q_{\epsilon r}(t_0,v_0))}$$

Scaled Local Estimate

Let $v_0 \in \mathbf{R}^3$, and assume that

$$0 < m_0 \leq \int_{\mathbf{R}^3} f_{in}(v)dv \leq M_0\,, \quad \int_{\mathbf{R}^3} \begin{pmatrix} |v|^2 \\ \ln f(v) \end{pmatrix} f_{in}(v)dv \leq \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}$$

For each $\epsilon \in (0,1)$, define the scaled quantities

$$\kappa_\epsilon := \epsilon^2 \kappa\,, \qquad \delta_\epsilon := \delta/\epsilon\,, \qquad r_\epsilon := r/\epsilon\,, \qquad f_{\epsilon,+}^{\kappa_\epsilon} := (f_\epsilon - \kappa_\epsilon)_+$$

Pick bump function ϕ with piecewise constant $\nabla_v \phi$, and assume

$$\epsilon \in (0, \min(\tfrac{1}{2}, \sqrt{t_0}))\,, \quad \kappa_\epsilon \in [1,2] \cap \mathbf{Q}\,, \quad r_\epsilon \in (0,2]\,, \quad \delta_\epsilon \in (0,1]$$

There exists $C_0' \equiv C_0'[|v_0|, m_0, M_0, E_0, H_0] \geq 1$ **independent** of ϵ

$$\operatorname*{ess\,sup}_{-r_\epsilon^2 < t \leq 0} \int_{B_{r_\epsilon}} h_+^{\kappa_\epsilon}(f_\epsilon(t,v))dv + \int_{Q_{r_\epsilon}} \frac{|\nabla_v f_{\epsilon+}^{\kappa_\epsilon}(t,v)|^2}{f_\epsilon(t,v)} dtdv$$

$$\leq C_0' \int_{Q_{r_\epsilon + \delta_\epsilon}} \left(\kappa_\epsilon + \frac{1}{\delta_\epsilon^2} + \frac{1}{\delta_\epsilon^2} f_\epsilon \star \frac{1}{|\cdot|} \right) f_\epsilon (\ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon}) + \ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon})^2) dtdv$$

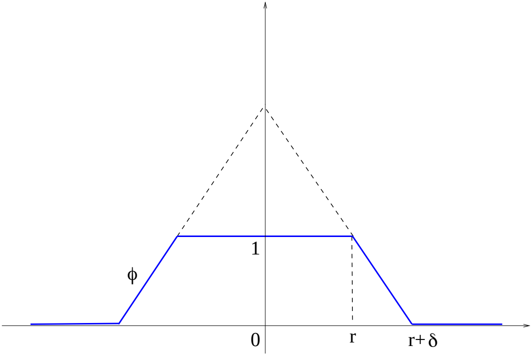

Figure: Graph of the bump function ϕ

De Giorgi's 1st lemma Let $f_\epsilon(t,v)=\epsilon^2 f(t_0+\epsilon^2 t, v_0+\epsilon v)$ be a scaled Villani solution of Landau's equation s.t. $f_{in}:=f(0,\cdot)$ satisfies

$$0<m_0\le\int_{\mathbf{R}^3} f_{in}(v)dv\le M_0\,,\quad \int_{\mathbf{R}^3}\begin{pmatrix}|v|^2\\ \ln f_{in}(v)\end{pmatrix} f_{in}(v)dv\le\begin{pmatrix}E_0\\ H_0\end{pmatrix}$$

Assume

$$\operatorname*{ess\,sup}_{Q_1(0,0)} f_\epsilon(t,\cdot)\star\frac{\mathbf{1}_{B_1(0)^c}}{|\cdot|}\le Z_\epsilon\,,\qquad \text{where } Z_\epsilon\ge 1$$

There exists $\eta_{DG}\equiv\eta_{DG}[m_0,M_0,E_0,H_0,|v_0|]\in(0,1)$ such that

$$\operatorname*{ess\,sup}_{-4<t\le 0}\int_{B_2}(f_\epsilon(t,v)-1)_+dv+\int_{Q_2}|\nabla_v\sqrt{f_\epsilon}|^2\mathbf{1}_{f_\epsilon\ge 1}dtdv\le\frac{\eta_{DG}}{Z_\epsilon^{3/2}}$$

$$\implies f_\epsilon\le 2 \text{ a.e. on } Q_{1/2}$$

Lemma A Let $f \in L^\infty((0,T);L^1(B_R))$ s.t. $\nabla_v f \in L^2((0,T);L^1(B_R))$ of the form

$$f(t,v) = F\Big(t, \underbrace{\sqrt{v_1^2+v_2^2}}_{V_1}, \underbrace{v_3}_{V_2}\Big)$$

Let $t_0 \in (0,T)$, let $V^0 = (V_1^0, V_2^0) \in \mathbf{R}^2$, and let $0 < r < V_1^0 - \rho_0$

$$\int_{t_0-r^2}^{t_0} \int_{|V-V_0|\le r} F(t,V)^2 dV dt$$

$$\le \tfrac{C_S(B_r)^2}{(2\pi\rho_0)^2} \int_{t_0-r^2}^{t_0} \left(\|f(t,\cdot)\|^2_{L^1(B_r(0,V_2^0))} + \|\nabla f(t,\cdot)\|^2_{L^1(B_r(0,V_2^0))} \right) dt$$

where $C_S(B_r)$ is the Sobolev constant for $W^{1,1}(B_r) \subset L^2(B_r)$ in $\mathbf{R}^2$.

Axisymmetric case: Diffusion L^∞ Bound

Lemma B Let $f \geq 0$ be measurable on $(0,T)\times\mathbf{R}^3$ and of the form

$$f(t,v) := F\left(t, \sqrt{v_1^2+v_2^2}, v_3\right)$$

Let $(t_0,v_0)\in(0,T)\times\mathbf{R}^3$ s.t. $\rho_0 := \sqrt{v_1^2+v_2^2} > 0$, and let

$$f_\epsilon(t,v) := \epsilon^2 f(t_0+\epsilon^2 t, v_0+\epsilon v)$$

Then, there exists an absolute constant $C_* > 0$ s.t. for $0<\epsilon<\frac{\rho_0}{6}$

$$\operatorname*{ess\,sup}_{Q_3} f_\epsilon(t,\cdot)\star\frac{1}{|\cdot|} \leq \underbrace{\frac{C_*}{\rho_0}\int_{\mathbf{R}^3} f(1+\ln_+ f)(t,v)dv + C_*\rho_0^2}_{\leq Z_* := Z_*[M_0,E_0,H_0,\rho_0]}$$

Write the scaled key local estimate with $r_\epsilon = 2$ and $\delta_\epsilon = \kappa_\epsilon = \frac{1}{2}$, using the elementary inequality $\ln_+(2y) + (\ln_+(2y))^2 \leq 6y$.

By Lemma B

$$\operatorname*{ess\,sup}_{-4<t\leq 0} \int_{B_2(0)} h_+^{1/2}(f_\epsilon(t,v))dv + \int_{Q_2(0,0)} \frac{|\nabla_v f_{\epsilon+}^{1/2}(t,v)|^2}{f_\epsilon(t,v)} dtdv$$

$$\leq 6C_0'(\tfrac{3}{4} + \tfrac{1}{2}Z_*) \underbrace{\int_{Q_3(0,0)} f_\epsilon(t,v)^2 dtdv}_{2\pi(\rho_0+3)\underbrace{\int_{t_0-9\epsilon^2}^{t_0}\int_{\rho_0-3\epsilon}^{\rho_0+3\epsilon}\int_{v_3^0-3\epsilon}^{v_3^0+3\epsilon}}_{\text{set of vanishing measure}} \underbrace{F^2}_{L^1} dsd\rho dw_3} \to 0 \text{ as } \epsilon \to 0^+$$

by Lemma A+Lebesgue's Thm. Conclude by De Giorgi's 1st lemma. □

Conclusion/Remarks

•We have proved that axisymmetric, or radially symmetric Villani solutions of the Landau-Coulomb equation are regular away from the axis of symmetry, or from the origin in the case of radial solutions

•Proof based on the De Giorgi method applied to the inequality derived from the H Theorem for the Landau equation

•One key ingredient in the proof is the **upper** bound for the diffusion matrix (Lemma B above), which is a critical quantity

Silvestre's Conjecture (2022)

In 2022, Silvestre conjectured that local (in time) classical solutions f of the Landau equation with finite mass, energy and entropy satisfy

$$\|f(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} \le \sqrt{\frac{250}{9\pi}}\|f(0,\cdot)\|_{L^\infty(\mathbf{R}^3)}$$

Idea: as $t \to +\infty$, one expects that

$$f(t,v) \to \frac{\rho}{(2\pi\theta)^{3/2}} e^{-|v-u|^2/2\theta} =: \mathcal{M}[\rho,u,\theta](v)$$

with $\rho, \theta > 0$ and $u \in \mathbf{R}^3$ given in terms of the initial data f_{in} by

$$\rho := \int_{\mathbf{R}^3} f_{in}(v)dv\,, \quad u := \frac{1}{\rho}\int_{\mathbf{R}^3} f_{in}(v)dv\,, \quad \theta := \int_{\mathbf{R}^3} \frac{|v-u|^2}{3\rho} f_{in}(v)dv$$

Then $\|\mathcal{M}[\rho,u,\theta]\|_{L^\infty} = \rho/(2\pi\theta)^{3/2}$, and the variational problem

$$\max_{\substack{0\le f_{in}\le 1\\ \int f_{in}(v)dv=1,\ \int vf_{in}(v)dv=0}} \frac{\rho}{(2\pi\theta)^{3/2}\|f_{in}\|_{L^\infty}}$$ is attained for $f_{in}(v) = \mathbf{1}_{|v|\le R}$