Large deviations for hard spheres in the low density limit

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Microscopic description of fluids (Newton)

Gas: $N \gg 1$ particles evolving and interacting in a d-dimensional domain.

- The particles are all identical spheres of mass 1 and diameter $\varepsilon > 0$.
- The particles evolve in a periodic box of size 1 denoted $\mathbb{T}^d := [0,1]^d$.
- The particles interact elastically at each binary collision and there is no other type of interaction nor forcing.

Microscopic description of fluids (Newton)

For a gas made of N particles, they are undistinguishable and labeled by integers $i \in \{1, \ldots, N\}$.

Denote by $(x_i, v_i) \in \mathbb{T}^d \times \mathbb{R}^d$ the position and velocity of particle i for $1 \leq i \leq N$.

Denote by $X_N := (x_1, \ldots, x_N) \in \mathbb{T}^{dN}$ the set of positions and by $V_N := (v_1, \ldots, v_N) \in \mathbb{R}^{dN}$ the set of velocities of the particles.

Denote by $Z_N := (z_1, \ldots, z_N) \in \mathbb{T}^{dN} \times \mathbb{R}^{dN}$ the set of configurations of the particles, with for each particle $z_i := (x_i, v_i)$.

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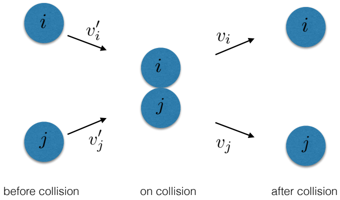
The positions and velocities of the system of N particles obey Newton's laws

$$\boxed{\forall i \in [1, \ldots, N], \qquad \frac{dx_i(t)}{dt} = v_i(t), \qquad \frac{dv_i(t)}{dt} = 0,}$$

provided that the *exclusion condition* $|x_i(t) - x_j(t)| > \varepsilon$ is satisfied for all $j \neq i$.

Microscopic description of fluids (Newton)



If $|x_i - x_j| = \varepsilon$ then

$$v_i' = v_i - \omega^{i,j} \cdot (v_i - v_j)\, \omega^{i,j}$$
$$v_j' = v_j + \omega^{i,j} \cdot (v_i - v_j)\, \omega^{i,j} \,,$$

where

$$\omega^{i,j} := \frac{x_i - x_j}{|x_i - x_j|} \,.$$

The phase space is

$$\mathcal{D}_N^\varepsilon := \Big\{ Z_N \in \mathbb{T}^{dN} \times \mathbb{R}^{dN} \,/\, \forall i \neq j,\, |x_i - x_j| > \varepsilon \Big\} \,.$$

Mesoscopic description of fluids (Boltzmann)

The distribution function $f = f(t, x, v)$ of a particle satisfies

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

with

$$Q(f,f)(v) := \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \big(f(v')f(v_1') - f(v)f(v_1)\big)\big((v - v_1)\cdot\omega\big)_+ \, d\omega dv_1$$

and

$$v' := v - \omega \cdot (v - v_1)\,\omega$$
$$v_1' := v_1 + \omega \cdot (v - v_1)\,\omega \,.$$

Questions

How to derive this equation from the system of particles ?

Can one describe fluctuations and **large deviations** from that limit ?

The grand canonical setting

We denote by $W_N^\varepsilon(t, Z_N)$ the probability density of finding $N \geq 0$ hard spheres of diameter ε at configuration Z_N at time t. It solves

$$\partial_t W_N^\varepsilon + V_N \cdot \nabla_{X_N} W_N^\varepsilon = 0 \quad \text{on} \quad \mathcal{D}_N^\varepsilon \,,$$

with specular reflection on the boundary.

The grand canonical setting

The initial probability density is defined on the configurations (N, Z_N) as

$$\boxed{\frac{1}{N!} W_N^{\varepsilon 0}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon} \frac{\mu_\varepsilon^N}{N!} \prod_{i=1}^N f^0(z_i) \, \mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)}$$

with $\mu_\varepsilon > 0$, and where the normalization constant $\mathcal{Z}^\varepsilon$ is given by

$$\mathcal{Z}^\varepsilon := 1 + \sum_{N \geq 1} \frac{\mu_\varepsilon^N}{N!} \int_{\mathcal{D}_N^\varepsilon} dZ_N \prod_{i=1}^N f^0(z_i) \, .$$

We denote by $\mathbb{P}_\varepsilon$ the probability and $\mathbb{E}_\varepsilon$ the expectation with respect to this initial measure.

The grand canonical setting

Let $\mathcal{N}$ be the total number of particles, we want that

$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon (\mathcal{N}) \, \varepsilon^{d-1} = 1 \, ,$$

which ensures that the *low density limit* holds, i.e. that the inverse mean free path is of order 1. Thus from now on we set

$$\boxed{\mu_\varepsilon := \varepsilon^{-(d-1)}.}$$

The grand canonical setting

The (rescaled) n-particle correlation function is

$$F_n^\varepsilon(t, Z_n) := \mu_\varepsilon^{-n} \sum_{p=0}^{\infty} \frac{1}{p!} \int dz_{n+1} \dots dz_{n+p} \, W_{n+p}^\varepsilon(t, Z_{n+p})$$

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For any test function h_n, the following holds :

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n \\ i_j \neq i_k, j \neq k}} h_n\big(z_{i_1}^\varepsilon(t), \dots, z_{i_n}^\varepsilon(t)\big)\Big) = \mathbb{E}_\varepsilon\Big(\delta_{\mathcal{N} \geq n} \frac{\mathcal{N}!}{(\mathcal{N}-n)!} h_n\big(z_1^\varepsilon(t), \dots, z_n^\varepsilon(t)\big)\Big)$$

$$= \sum_{p=n}^{\infty} \int dZ_p \, \frac{W_p^\varepsilon(t, Z_p)}{p!} \, \frac{p!}{(p-n)!} \, h_n\big(Z_n\big)$$

$$= \mu_\varepsilon^n \int dZ_n \, F_n^\varepsilon(t, Z_n) \, h_n(Z_n) \, .$$

The grand canonical setting

For any φ, ψ defined on $\mathbb{T}^d \times \mathbb{R}^d$ set

$$\langle \varphi, \psi \rangle := \int dz \, \varphi(z)\psi(z) \, .$$

Then in particular

$$\frac{1}{\mu_\varepsilon} \mathbb{E}_\varepsilon \Big(\sum_{i=1}^{\mathcal{N}} h\big(z_i^\varepsilon(t)\big) \Big) = \langle F_1^\varepsilon(t), h \rangle \ .$$

We note in the following the empirical distribution at time t

$$\pi_t^\varepsilon(h) := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} h\big(z_i^\varepsilon(t)\big) \, .$$

Convergence result: Lanford's theorem

Theorem [Lanford, 1974]

Recall

$$\frac{1}{N!} W_N^{\varepsilon 0}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon} \frac{\mu_\varepsilon^N}{N!} \prod_{i=1}^N f^0(z_i) \, \mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)$$

and assume that f_0 is a continuous probability such that

$$\left\| f_0 \exp(\mu_0 + \frac{\beta_0}{2} |v|^2) \right\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq 1 \,,$$

for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$.

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for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$. In the limit $\mu_\varepsilon \to \infty$, **the one-particle density $F_1^\varepsilon(t)$ converges uniformly to the solution $f(t)$ of the Boltzmann equation** with initial data f_0, on a time interval $[0, T_0]$ where T_0 depends only on the parameters β_0, μ_0.

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Furthermore for each n, **the n-particle correlation function $F_n^\varepsilon(t)$ converges almost everywhere to** $f^{\otimes n}(t)$ on the same time interval.

Lanford's theorem is a law of large numbers

Proposition

For all test functions h,

$$\forall \delta > 0\,, \quad \mathbb{P}_\varepsilon \left(\left| \pi_t^\varepsilon(h) - \langle f(t), h \rangle \right| > \delta \right) \xrightarrow[\mu_\varepsilon \to \infty]{} 0\,.$$

Computing the variance for any test function h, we get that

$$\mathbb{E}_\varepsilon \Big(\big(\pi_t^\varepsilon(h) - \langle F_1^\varepsilon(t), h \rangle \big)^2 \Big)$$

$$= \mathbb{E}_\varepsilon \Big(\frac{1}{\mu_\varepsilon^2} \sum_{i=1}^{\mathcal{N}} h^2\big(z_i^\varepsilon(t)\big) + \frac{1}{\mu_\varepsilon^2} \sum_{i \neq j} h\big(z_i^\varepsilon(t)\big) h\big(z_j^\varepsilon(t)\big) \Big) - \langle F_1^\varepsilon(t,z), h \rangle^2$$

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$$= \frac{1}{\mu_\varepsilon}\langle F_1^\varepsilon(t), h^2\rangle + \int F_2^\varepsilon(t, Z_2)\, h(z_1)h(z_2)\, dZ_2 - \langle F_1^\varepsilon(t,z), h\rangle^2\,,$$

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and the convergence to 0 follows from the fact that F_2^ε converges to $f^{\otimes 2}$ and F_1^ε to f almost everywhere.

Large deviations

Goal: quantify the probability of an untypical event. A large deviations principle holds if

$$\boxed{\mathbb{P}_\varepsilon\left(\pi_t^\varepsilon \approx \varphi_t,\ \ t \in [0, T]\right) \asymp \exp\left(-\mu_\varepsilon \mathcal{F}(T, \varphi)\right).}$$

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Define the cumulant generating function

$$\Lambda_t^\varepsilon(e^h) := \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon\Big(\exp\left(\mu_\varepsilon\, \pi_t^\varepsilon(h)\right)\Big).$$

One expects $\mathcal{F}$ to be obtained by taking the Legendre transform of

$$\lim_{\varepsilon\to 0} \Lambda_t^\varepsilon\Big(\exp\Big(g(t) - \int_0^t (\partial_s g + v\cdot\nabla_x g)\, ds\Big)\Big).$$

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In the following we shall

- study the limit of Λ_t^ε ;
- prove the LDP;
- identify $\mathcal{F}$.

Large deviations

One can prove that

$$\mathbb{E}_\varepsilon\Big(\exp\big(\pi_t^\varepsilon(h)\big)\Big) = 1 + \sum_{n\geq 1} \frac{\mu_\varepsilon^n}{n!} \int dZ_n \, F_n^\varepsilon(t, Z_n) \left(e^{h/\mu_\varepsilon} - 1\right)^{\otimes n} (Z_n) \, .$$

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It turns out that

$$\begin{aligned}\Lambda_t^\varepsilon(e^h) &:= \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon\Big(\exp\big(\mu_\varepsilon\, \pi_t^\varepsilon(h)\big)\Big) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int dZ_n\, f_n^\varepsilon(t, Z_n) \left(e^h - 1\right)^{\otimes n} (Z_n)\,,\end{aligned}$$

where the f_n^ε are **(rescaled) cumulants**. To identify the limit of Λ_t^ε we are going to describe the cumulants f_n^ε and their limits.

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For instance $f_1^\varepsilon = F_1^\varepsilon$ and $f_2^\varepsilon = \mu_\varepsilon\big(F_2^\varepsilon - F_1^\varepsilon \otimes F_1^\varepsilon\big)$.

Cumulants

Definition

Let $(G_n)_{n\geq 1}$ be a family of distributions of n variables invariant by permutation of the labels of the variables. The (rescaled) cumulants associated with $(G_n)_{n\geq 1}$ form the family $(g_n)_{n\geq 1}$ defined, for all $n \geq 1$, by

$$g_n = \mu_\varepsilon^{n-1} \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} (-1)^{s-1}(s-1)!\, G_\sigma \,.$$

The map from $(G_n)_{n\geq 1}$ to $(g_n)_{n\geq 1}$ is bijective and

$$\forall n \geq 1\,, \qquad G_n = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} \mu_\varepsilon^{-(n-s)} g_\sigma \,.$$

Cumulants measure **departure from factorization**.

Cumulants

Cumulants are supported on "clusters", or "connected graphs".
For instance consider the exclusion condition

$$\Phi_n(x_1,\ldots,x_n) := \prod_{1\le i\neq j\le n} \mathbf{1}_{|x_i-x_j|>\varepsilon} \,.$$

For $n=1$, we set $\Phi_1(x_1)\equiv 1$.

Proposition [Penrose '67]

The cumulants of (Φ_n) are equal to

$$\varphi_n(x_1,\ldots,x_n) = \mu_\varepsilon^{n-1} \sum_{G\in\mathcal{C}_n} \prod_{\{i,j\}\in E(G)} \left(-\mathbf{1}_{|x_i-x_j|\le\varepsilon}\right),$$

where $\mathcal{C}_n$ is the set of connected graphs with n vertices. Moreover

$$|\varphi_n(x_1,\ldots,x_n)| \le \sum_{T\in\mathcal{T}_n} \prod_{\{i,j\}\in E(T)} \mathbf{1}_{|x_i-x_j|\le\varepsilon} \,,$$

where $\mathcal{T}_n$ is the set of *minimally connected* graphs with n vertices.

Dynamical cumulants

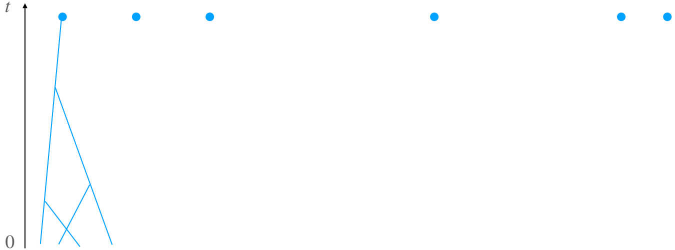
Recall that
$$\Lambda_t^\varepsilon(e^h) := \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon\Big(\exp\big(\mu_\varepsilon\, \pi_t^\varepsilon(h)\big)\Big).$$
Then
$$\Lambda_t^\varepsilon(e^h) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dZ_n\, f_n^\varepsilon(t, Z_n)\, \big(e^h - 1\big)^{\otimes n} (Z_n)\,,$$
if the series is absolutely convergent, where $(f_n^\varepsilon)_{n\geq 1}$ is the family of rescaled **dynamical** cumulants associated with $(F_n^\varepsilon)_{n\geq 1}$.

Graphical construction of dynamical cumulants

n particles at time t

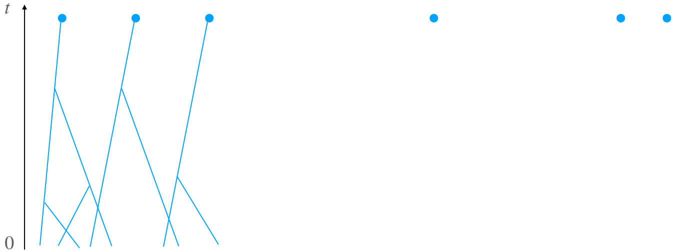
Graphical construction of dynamical cumulants

Backward collision tree of particle 1



Graphical construction of dynamical cumulants

External recollision (between two collision trees) $\rightarrow$ forest



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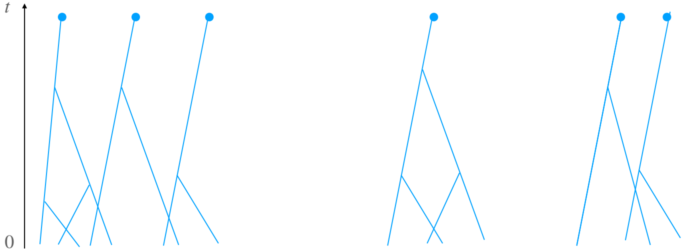
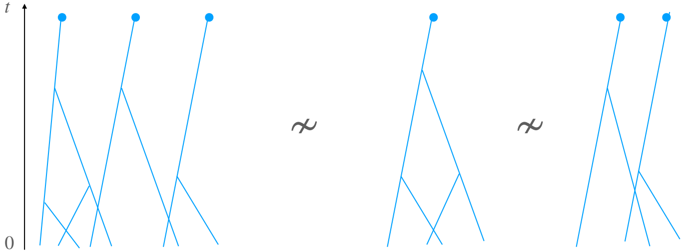


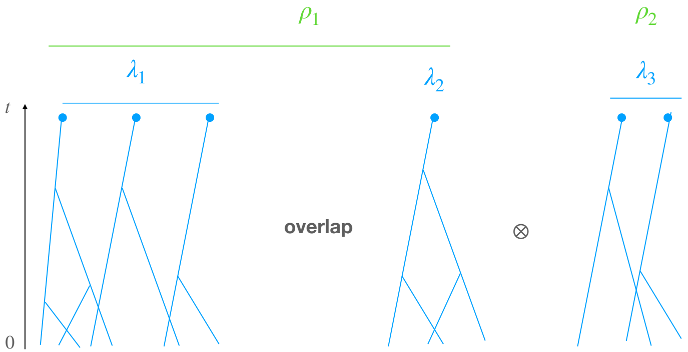
Graphical construction of dynamical cumulants

Non-intersecting forests are correlated



Graphical construction of cumulants

Overlapping forests → jungle



Dynamical cumulants

$$f^{\varepsilon}_{n,[0,t]}(h^{\otimes n}) = \int dZ_n \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda \in \mathcal{P}_n^\ell} \sum_{r=1}^{\ell} \sum_{\rho \in \mathcal{P}_\ell^r} \int \Big(\prod_{i=1}^{\ell} d\mu(\Psi^\varepsilon_{\lambda_i}) \times \mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big) \Delta\!\!\!\Delta_{\lambda_i} \Big) \, \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r}) \, .$$

We have written

$$d\mu(\Psi^\varepsilon_n) := \sum_{m \geq n} \sum_{a \in \mathcal{A}^{\pm}_{n,m}} dT_m d\Omega_m dV_m \prod_{k=1}^{m} \Big(s_k \left((v_k - v_{a_k}(t_k)) \cdot \omega_k \right)_+ \Big) \, .$$

Dynamical correlations are encoded in the **collision trees**, and in the **external recollisions and overlaps** (= *clusterings*) between trees.

Dynamical cumulants and the cumulant generating function

Recall that

$$\Lambda_t^\varepsilon(e^h) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dZ_n \, f_n^\varepsilon(t, Z_n) \left(e^h - 1\right)^{\otimes n} (Z_n) \, .$$

One can show thanks to the tree inequality that cumulants are bounded for short times:

$$|f^\varepsilon_{n,[0,t]}(h^{\otimes n})| \leq C^n t^{n-1} n! \, .$$

Limit dynamical cumulants and the cumulant generating function

Limit cumulants are supported on *minimally connected graphs*.

Summing over n, one can treat all connections in a symmetric way.

$$\boxed{\Lambda_{[0,t]}(e^h) + 1 = \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{T \in \mathcal{T}_K^{\pm}} \int d\mu_{\rm sing}^{T}(\Psi_{K,0})(e^h)^{\otimes K}(\Psi_{K,0}) f^{0\otimes K}(\Psi_{K,0}^0)\,,}$$

where

$$d\mu_{\rm sing}^{T} := dx_K^* dV_K \prod_{e=\{q,q'\}\in E(T)} s_e\Big(\big(v_q(\tau_e) - v_{q'}(\tau_e)\big)\cdot\omega_e\Big)_+ d\tau_e d\omega_e\,.$$

The Hamilton-Jacobi equation

We write

$$\mathcal{I}(t,g) := \Lambda_{[0,t]}\Big(\exp\big(g(t) - \int_0^t (\overbrace{\partial_s g + v\cdot\nabla_x g}^{Dg(s)})(s,z(s))ds\big)\Big)\,.$$

One can prove that the series defining $\mathcal{I}(t,g)$ is well defined for a short time and for g satisfying appropriate bounds.

Note that

$$\langle \frac{\partial \mathcal{I}(t,g)}{\partial g(t)}, \Gamma\rangle = \sum_K \frac{1}{K!} \sum_{T\in\mathcal{T}_K^{\pm}} \sum_{i=1}^K \int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0})\Gamma(z_i(t))$$

$$\times \left(e^{g(t)-\int_0^t D_s g ds}\right)^{\otimes K}(\Psi_{K,0})\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})\,.$$

The Hamilton-Jacobi equation

It solves formally

$$\partial_t \mathcal{I}(t,g) = \mathcal{H}\Big(\frac{\partial \mathcal{I}(t,g)}{\partial g(t)}, g(t)\Big)$$
$$\text{with} \quad \mathcal{H}(\varphi, p) = \frac{1}{2}\int \varphi(z_1)\varphi(z_2)\Big(e^{\Delta p} - 1\Big)\, d\mu(z_1, z_2, \omega)\,,$$

with

$$\Delta p(z_1, z_2, \omega) := p(z_1') + p(z_2') - p(z_1) - p(z_2)\,,$$
$$d\mu(z_1, z_2, \omega) := \delta_{x_1 - x_2}\, \big((v_1 - v_2)\cdot\omega\big)_+ d\omega\, dv_1\, dv_2 dx_1\,,$$

and with initial condition

$$\mathcal{I}(0,g) = \langle f^0, e^{g(0)} - 1\rangle\,.$$

The Hamilton-Jacobi equation

It solves formally

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$$\text{with} \quad \mathcal{H}(\varphi,p) = \frac{1}{2}\int \underbrace{\varphi(z_1)\varphi(z_2)}_{\text{independent until clustering time}} \underbrace{\Big(e^{\Delta p}-1\Big)}_{\text{jumps of } g \text{ at clustering time}} d\mu(z_1,z_2,\omega)\,,$$

with

$$\Delta p(z_1,z_2,\omega) := p(z_1') + p(z_2') - p(z_1) - p(z_2)\,,$$
$$d\mu(z_1,z_2,\omega) := \delta_{x_1-x_2}\,\Big((v_1-v_2)\cdot\omega\Big)_+ d\omega\, dv_1\, dv_2 dx_1\,,$$

and with initial condition

$$\mathcal{I}(0,g) = \langle f^0, e^{g(0)} - 1\rangle\,.$$

The Hamilton-Jacobi equation

Notice that

$$f_1(t) := \lim_{\mu_\varepsilon \to \infty} F_1^\varepsilon(t) = \frac{\partial \mathcal{I}(t,0)}{\partial g(t)}$$

and thanks to the HJ equation one finds that f_1 satisfies the **Boltzmann equation**.

One can actually also compute the **equation for the limit covariance** by differentiating $\mathcal{I}$ twice (thanks to the bounds on the cumulants we find the limit fluctuation field is Gaussian).

The large deviation functional

Recall $\langle \varphi(t), \psi(t) \rangle := \int \varphi(t,z)\psi(t,z)dz$ and define

$$\langle\!\langle \varphi, \psi \rangle\!\rangle := \int_0^t \langle \varphi(s), \psi(s) \rangle ds\,.$$

Set

$$\mathcal{F}(t,\varphi) := \sup_g \Big\{ -\langle\!\langle \varphi, Dg \rangle\!\rangle + \langle \varphi(t), g(t) \rangle - \mathcal{I}(t,g) \Big\}\,,$$

where the sup is taken on functions satisfying appropriate bounds.

A large deviation theorem: upper bound

Define $D([0,T],\mathcal{M})$ the space of trajectories with values in the space of measures.

Theorem: Upper bound [BGSRS 2023]

In the limit $\mu_\varepsilon \to \infty$, the empirical measure satisfies the following large deviation upper bound: for any closed set $\mathbf{F} \subset D([0,T],\mathcal{M})$,

$$\limsup_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon \left(\pi^\varepsilon \in \mathbf{F}\right) \leq - \inf_{\varphi \in \mathbf{F}} \mathcal{F}(T, \varphi) .$$

Proof of the upper bound

It follows rather classical methods. Thanks to a tightness argument, it suffices to prove the result for **compact sets in the weak topology** defined by the open sets

$$\mathbf{O}_{\delta,g}(\nu) := \Big\{\nu' \in D([0,T],\mathcal{M}) \ : \\ \Big|\Big(\langle\!\langle \nu', Dg \rangle\!\rangle - \langle \nu'_T, g_T \rangle\Big) - \Big(\langle\!\langle \nu, Dg \rangle\!\rangle - \langle \nu_T, g_T \rangle\Big)\Big| < \delta/2\Big\}.$$

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Set $\delta > 0$. For any g there holds

$$\mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\right) \le \exp\Big(\mu_\varepsilon \frac{\delta}{2} + \mu_\varepsilon \langle\!\langle \varphi, Dg\rangle\!\rangle - \mu_\varepsilon \langle \varphi(T), g(T)\rangle\Big) \\ \times \mathbb{E}_\varepsilon\Big(\exp\Big(-\mu_\varepsilon \langle\!\langle \pi^\varepsilon, Dg\rangle\!\rangle + \mu_\varepsilon \langle \pi^\varepsilon_T, g(T)\rangle\Big)\Big)$$

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$$\leq \exp\Big(\mu_\varepsilon \frac{\delta}{2} + \mu_\varepsilon \langle\langle \varphi, Dg \rangle\rangle - \mu_\varepsilon \langle \varphi(T), g(T) \rangle + \mu_\varepsilon\, \mathcal{I}^\varepsilon(T,g) \Big),$$

with

$$\mathcal{I}^\varepsilon(T,g) := \Lambda^\varepsilon_{[0,T]}\big(e^{g - \int_0^T Dg} \big) := \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon\Big(\exp\big(\mu_\varepsilon\, \pi^\varepsilon_{[0,T]}(g - \int_0^T Dg) \big) \Big).$$

Proof of the upper bound

Passing to the limit produces

$$\limsup_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\Big(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\Big) \le \delta/2 + \langle\!\langle \varphi, Dg \rangle\!\rangle - \langle \varphi(T), g(T) \rangle + \mathcal{I}(T,g)\,.$$

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But

$$\boxed{\mathcal{F}(t,\varphi) := \sup_g \Big\{ -\langle\!\langle \varphi, Dg \rangle\!\rangle + \langle \varphi(t), g(t)\rangle - \mathcal{I}(t,g) \Big\}\,,}$$

so if $\varphi \in \mathbf{F}$ then there exists g such that

$$\mathcal{F}(T,\varphi) \le -\langle\!\langle \varphi, Dg \rangle\!\rangle + \langle \varphi(T), g(T)\rangle - \mathcal{I}(T,g) + \frac{\delta}{2}\,.$$

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$$\mathcal{F}(T,\varphi) \leq -\langle\!\langle \varphi, Dg \rangle\!\rangle + \langle \varphi(T), g(T)\rangle - \mathcal{I}(T,g) + \frac{\delta}{2}\,.$$

This completes the proof: we recover

$$\limsup_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\Big(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\Big) \leq -\mathcal{F}(T,\varphi) + \delta\,,$$

and it suffices to apply this to a finite covering of $\mathbf{F} \subset \cup_{i\leq K} \mathbf{O}_{\delta,g_i}(\varphi_i)$ and to let δ go to 0.

A large deviation theorem: lower bound

One needs to **restrict the class of observables** to the set $\mathcal{R}$ defined by functions φ such that for some p,

$$D_t\varphi(z) = \int \big(\varphi(z')\varphi(z_2')\exp(-\Delta p) - \varphi(z)\varphi(z_2)\exp(\Delta p)\big)\, d\mu_z(z_2,\omega)$$

with $\quad \varphi(0) = f^0 e^{p(0)}$.

The restriction to $\mathcal{R}$ implies that the supremum defining $\mathcal{F}$ is reached for some g.

Theorem: Lower bound [BGSRS 2023]

In the limit $\mu_\varepsilon \to \infty$, the empirical measure satisfies the following large deviation lower bound: for any open set $\mathbf{O} \subset D([0,T],\mathcal{M})$,

$$\liminf_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}\right) \geq - \inf_{\varphi\in\mathbf{O}\cap\mathcal{R}} \mathcal{F}(T,\varphi)\,.$$

Identification of the Large Deviation Functional

For a 1D stochastic system, F. Rezakhanlou proved in 1998 (and F. Bouchet conjectured in 2020 for Boltzmann) that the Large Deviation Functional is

$$\boxed{\widehat{\mathcal{F}}(t,\varphi) := \widehat{\mathcal{F}}(0,\varphi_0) + \sup_p \left\{ \langle\!\langle p, D\varphi \rangle\!\rangle - \int_0^t \mathcal{H}\big(\varphi(s), p(s)\big)\, ds \right\},}$$

with

$$\widehat{\mathcal{F}}(0,\varphi_0) := \int dz \ \left(\varphi_0 \log\left(\frac{\varphi_0}{f^0}\right) - \varphi_0 + f^0 \right)$$

and where the Hamiltonian is given by

$$\mathcal{H}(\varphi, p) := \frac{1}{2} \int d\mu(z_1, z_2, \omega)\varphi(z_1)\varphi(z_2)\big(\exp\left(\Delta p\right) - 1\big)\,.$$

Identification of the Large Deviation Functional

It turns out that $\widehat{\mathcal{F}} = \mathcal{F}$ in $\mathcal{R}$. The proof follows from the fact that the action

$$\widehat{\mathcal{I}}(t,g) := \langle f^0, (e^{p_t(0)} - 1)\rangle + \langle\!\langle D_s(p_t - g), \varphi_t \rangle\!\rangle + \int_0^t \mathcal{H}(\varphi_t(s), p_t(s))ds$$

associated with the Hamiltonian system

$$D_s\varphi_t = \frac{\partial \mathcal{H}}{\partial p}(\varphi_t, p_t)\,, \quad \text{with} \quad \varphi_t(0) = f^0 e^{p_t(0)}\,,$$
$$D_s(p_t - g) = -\frac{\partial \mathcal{H}}{\partial \varphi}(\varphi_t, p_t)\,, \quad \text{with} \quad p_t(t) = g(t)\,.$$

satisfies the same Hamilton-Jacobi equation as $\mathcal{I}$. This allows to prove the result on $\mathcal{R}$.

Some open questions

- Improve the existence time of those results (Lanford and fluctuations OK at equilibrium).
- Improve the understanding of the Hamilton-Jacobi equation. Better functional setting ? Equation at fixed ε ?
- What information does the Hamilton-Jacobi equation (not) retain – in terms of the original cumulants for instance. Is there conservation of entropy at the level of cumulants ?
- Clarify the restriction on the lower bound (cf G. Basile, D. Heydecker)