Entropy and classification in dynamical systems

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About Entropy in Large Classical Particle Systems
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Our plan

- Ergodic theory *without* an invariant probability measure
- Dynamical entropy in a topological setting and MMEs
- Classification of surface diffeomorphisms by MMEs
- Newhouse conjecture : MMEs of smooth surface diffeomorphisms
- Behind the curtain : hyperbolicity

Ergodic theory

without an invariant probability measure

Topological dynamical systems

Definition

Topological dynamics (X, f) : homeo f of a compact metric space X with orbits $\mathcal{O}(x) := \{f^n x : n \geq 0\}$

Examples :

▷ $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto A.x$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ (hyperbolic toral autom.)

▷ $\sigma_d : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ on $\Sigma_d := \{1, \ldots, d\}^{\mathbb{Z}}$ (full shift)

▷ $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$, $x \mapsto x + \alpha$ (rotation)

▷ Hénon map on its compact attractor

▷ ...

Asymptotic behavior of orbits

(X, f) topological dynamical system

measure = probability on $\mathcal{B}$ Borel σ-field generated by the open sets

Definition

Empirical measure μ_x^f of $x \in X$: weak limit (if it exists) of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}$

Ergodic basin of measure m : $B(m) := \{x \in X : \mu_x^f \text{ exists and is } m\}$

Examples :

- Any $x \in (\mathbb{Q}/\mathbb{Z})^2$ is T_A-periodic and $\mu_x^{T_A} = \frac{1}{p}(\delta_x + \cdots + \delta_{T_A^{p-1}x})$
- For any $x \in \mathbb{T}^1$, $\mu_x^{R_\alpha} = \mathrm{Leb}_{\mathbb{T}^1}$

Note :

- every $\mu_x^f \in \mathbb{P}(f) := \{m \in \mathbb{P}(X) : f_*(m) := m \circ f^{-1} = m\} \neq \emptyset$
- $\mu_x^{\sigma_d}$ fails to exist for a Baire fat subset of $x \in \Sigma_d$
- not every $m \in \mathbb{P}(f)$ is empirical measure of some point (consider $(\mathbb{T}^1, Id)$)

Asymptotic behavior of orbits

Definition (Irreducibility)

$m \in \mathbb{P}(f)$ is ergodic if $\forall B \in \mathcal{B}\, f^{-1}B = B \implies m(B) = 0$ or 1

Examples of ergodic measures :

$\mathrm{Leb}_{\mathbb{T}^1} \in \mathbb{P}_{\mathrm{erg}}(R_\alpha)$ when $\alpha \notin \mathbb{Q}$; $\delta_0, \mathrm{Leb}_{\mathbb{T}^2} \in \mathbb{P}_{\mathrm{erg}}(T_A)$

Ergodic decomposition :

$\forall \mu \in \mathbb{P}(f)\ \exists P \in \mathbb{P}(\mathbb{P}_{\mathrm{erg}}(f))\ \mu = \int_{\mathbb{P}_{\mathrm{erg}}(f)} \nu\, dP(\nu)$

Theorem (Birkhoff pointwise ergodic theorem)

For all $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$, $\mu(B(\mu)) = 1$

Examples : $B(\mathrm{Leb}_{\mathbb{T}^2})$ contains Lebesgue-a.e. point of $\mathbb{T}^2$

Bernoulli scheme $(p_1, \dots, p_d)^{\mathbb{Z}} \in \mathbb{P}_{\mathrm{erg}}(\sigma_d)$

Corollary

Any topological dynamics (X, f) *has an invariant Borel partition :*

$$X = \bigsqcup_{\mu \in \mathbb{P}_{\mathrm{erg}}(f)} B(\mu) \sqcup X_0 \quad (\forall \mu \in \mathbb{P}(f)\ \mu(X_0) = 0)$$

Dynamical entropy in the topological setting

Topological entropy as counting orbit segments

(X, f) topological dynamics

Counting orbit segments of length $n \geq 1$ at scale $\epsilon > 0$ from $Y \subset X$:
$s_f(\epsilon, n, Y) := \max\{|S| : S \subset Y, x \neq y \in Y \Longrightarrow \exists 0 \leq k < n d(f^k x, f^k y) \geq \epsilon\}$

Definition (Adler-Konheim-MacAndrew 1968, Bowen-Dinaburg)

Topological entropy :

$h_{\text{top}}(f) := \lim_{\epsilon \to 0} h_{\text{top}}(f, \epsilon)$ where
$h_{\text{top}}(f, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log s_f(\epsilon, n, X)$

Lemma (Kushnirenko)

If X is a d-dimensional compact manifold then $h_{\text{top}}(f) \leq d \cdot \log \text{Lip}(f)$

Examples :

- $h_{\text{top}}(R_\alpha) = 0$
- $h_{\text{top}}(\sigma_d) = \log d$
- $h_{\text{top}}(T_A) = \frac{3+\sqrt{5}}{2}$

KS entropy as counting orbit segments

(X, f) topological dynamics

Counting orbit segments of length $n \geq 1$ at scale $\epsilon > 0$ from $Y \subset X$:
$s_f(\epsilon, n, Y) := \max\{|S| : S \subset Y, x \neq y \in Y \Longrightarrow \exists 0 \leq k < n d(f^k x, f^k y) \geq \epsilon\}$

Theorem (Katok)

Kolmogorov-Sinai entropy of $\mu \in \mathbb{P}_{\text{erg}}(f)$: for any $0 < \lambda < 1$,

$h_\mu(f) := \lim_{\epsilon \to 0} h_\mu(f, \epsilon)$ *where*

$h_\mu(f, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log \inf_{\mu(Y) > \lambda} s_f(\epsilon, n, Y)$

Ideology : $h_\mu(f)$ quantifies “how much dynamics” is described by μ

Remark : for $\mu \in \mathbb{P}(f)$, use the ergodic decomposition

Examples :
- $h_{\delta_0}(T_A) = 0$ for $0 \in \mathbb{P}_{\text{erg}}(T_A)$
- $h_{\text{Leb}_{\mathbb{T}^2}}(T_A) = h_{\text{top}}(T_A) = \log \frac{3+\sqrt{5}}{2}$ for $\text{Leb}_{\mathbb{T}^2} \in \mathbb{P}_{\text{erg}}(T_A)$
- $h_{p^{\otimes \mathbb{Z}}}(\sigma_d) = -\sum_{i=1}^{d} p_i \log p_i$ for $(p_1, \ldots, p_d)^{\otimes \mathbb{Z}} \in \mathbb{P}_{\text{erg}}(\sigma_d)$

KS Entropy classifies Bernoulli schemes

Definition

$\mu \in \mathbb{P}(f)$ and $\nu \in \mathbb{P}(g)$ are *measure-preservingly conjugate* if there is a Borel bijection $\psi : X' \to Y'$ with :

- $\mu(X') = \nu(Y') = 1$ and $\psi_*(\mu) = \nu$
- $\psi \circ f = g \circ \psi$

Theorem (Ornstein 1971)

Two Bernoulli schemes $(p_1, \ldots, p_d)^{\mathbb{Z}} \in \mathbb{P}(\sigma_d)$ and $(q_1, \ldots, q_e)^{\mathbb{Z}} \in \mathbb{P}(\sigma_e)$ are measure-preservingly conjugate if and only if they have equal KS entropy :

$$-\sum_{i=1}^{d} p_i \log p_i = -\sum_{j=1}^{e} q_j \log q_j$$

Entropy as limit to embedding

(X, f) topological dynamics

Recall $\sigma_d : \Sigma_d \to \Sigma_d$ left-shift on $\{1, \ldots, d\}^{\mathbb{Z}}$

Definition

A *measure-preserving embedding* $(\mu, f) \mapsto (\Sigma_d, \sigma_d)$ is a Borel injective map $\psi : X' \to \Sigma_d$ with $\mu(X') = 1$ and $\psi \circ f = \sigma \circ \psi$

Theorem (Jewett-Krieger)

Topological dynamics (X, f) and $\mu \in \mathbb{P}(f)$ with μ ergodic

If $h_\mu(f) < \log d$ then $\exists$ measure-preserving embedding $(\mu, f) \mapsto (\Sigma_d, \sigma_d)$

More precisely, a necessary and sufficient condition for the existence of such an embedding is :

$$h_\mu(f) < \log d \text{ or } (\mu, f) \text{ Bernoulli scheme with } h_\mu(f) = \log d$$

$\rightsquigarrow$ Description of the types in $\mathbb{P}_{\text{erg}}(\sigma_d)$

Entropy as limit to embedding

$\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ strongly mixing : $\forall A, B \in \mathcal{B}\ \mu(A \cap f^{-n}B) \to \mu(A)\mu(B)$

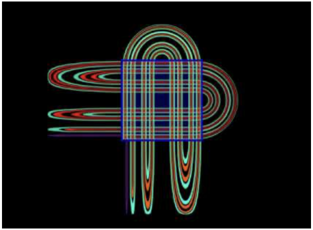
$h_{\mathrm{mix}}(f) := \sup\{h_\mu(f) : \mu \in \mathbb{P}_{\mathrm{erg}}(f)$ is strongly mixing$\}$

Combining Jewett-Krieger theorem with Katok's horseshoe theorem :

Theorem

$g : M \to M$ C^2-diffeomorphism of a compact surface
(X, f) topological dynamics and $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$

If $h_\mu(f) < h_{\mathrm{mix}}(g)$ then $\exists$ probabilistic embedding (μ, f) into (M, g)



Variational principle for the entropy

Theorem (Goodman, Dinaburg)

For any topological dynamics (X, f)

$$h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in \mathbb{P}(f)\} = \sup\{h_\mu(f) : \mu \in \mathbb{P}_{\text{erg}}(f)\}$$

Definition

measure maximizing entropy : $\mu \in \mathbb{P}_{\text{erg}}(f)$ st $h_\mu(f) = \sup_{\nu \in \mathbb{P}(f)} h_\nu(f)$
$\mathsf{MME}(f)$ denotes the set of such ergodic measures

Examples of unique MME :

$\mathrm{Leb}_{\mathbb{T}^2}$ for T_A and $(1/d, \ldots, 1/d)^{\otimes \mathbb{Z}}$ for σ_d

Theorem (Newhouse 1989)

If f is C^∞ smooth, then $\mathsf{MME}(f) \neq \emptyset$

Note : finite smoothness is not enough

Classifying smooth surface dynamics

Borel conjugacy

Definition

Two topological dynamics (X, f) and (Y, g) are *Borel conjugate* if $\exists \psi : X \to Y$ a Borel isomorphism such that $\psi \circ f = g \circ \psi$

Note :
- weakens topological conjugacy (ψ not necessarily continuous)
- $\neq$ measure-preserving conjugacy between conservative dynamics
- implies :
  - $\mu \mapsto \psi_*(\mu)$ is a bijection between $\mathbb{P}_{\mathrm{erg}}(f)$ and $\mathbb{P}_{\mathrm{erg}}(g)$ with paired measures being measure-preservingly conjugate by ψ
  - equal topological entropy
  - equality of the cardinals $\kappa(f, n)$, $\kappa(g, n)$ of the n-periodic points for f and g
  - measure-preserving conjugacy between paired MMEs

Borel conjugacy

Ideology :
- For general topological f, conjugacy types in $\mathbb{P}_{\text{erg}}(f)$ are quite arbitrary
- But smooth surface diffeos with positive entropy are quite rigid

Need :
- $\mu \in \mathbb{P}(f)$ strongly mixing : $\forall A, B \in \mathcal{B}\ \mu(A \cap f^{-n}B) \to \mu(A)\mu(B)$
- free part $X_{\text{free}} := X \setminus \{x \in X : \exists n \geq 1\ f^n x = x\}$

Lemma

Two dynamics are Borel conjugate if and only if :
- their free parts are Borel conjugate
- for all $n \geq 1$, they have the same number $\kappa(f, n) = \kappa(g, n)$ of n-periodic points

Borel conjugacy of surface diffeomorphisms

Using horseshoe's Katok theorem, generator theorems of Hochman, and joint work with Boyle :

Theorem

f, g : C^2 diffeos of compact surfaces with $h_{\mathrm{top}}(f) > 0$, $h_{\mathrm{top}}(g) > 0$
Each has strongly mixing measures with entropy arbitrarily close to their respective topological entropy.
Then the aperiodic parts of f and g are Borel conjugate if and only if both following conditions are satisfied :

1. $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g)$
2. *there is a bijection* $\Phi : \mathsf{MME}(f) \to \mathsf{MME}(g)$ *such that* (μ, f) *and* $(\Phi(\mu), g)$ *are measure-preservingly conjugate*

Condition 1 follows from 2 unless $\mathsf{MME}(f) = \mathsf{MME}(g) = \emptyset$

Question : What are the number and types of the MMEs ?

MMEs of smooth surface diffeomorphisms

The following are standard notions of irreducibility and aperiodicity

Definition

Topologically transitive : $\exists x \in X \; \overline{\{f^n x : n \geq 0\}} = X$

Topologically mixing : $\forall U, V \subset X$ non-empty open $\forall n \gg 1 \; U \cap f^{-n} V \neq \emptyset$

Solving a problem of Newhouse (1990) :

Theorem (B-Crovisier-Sarig)

Let f be a C^∞ diffeomorphism of a compact surface
Assume $h_{\text{top}}(f) > 0$:
- There are finitely many ergodic measures maximizing the entropy
- There is exactly one MME if f is topologically transitive
- The unique MME is Bernoulli if f is topologically mixing

Putting the two theorems together with Ornstein's theorem :

THEOREM (B-CROVISIER-SARIG)

Two topological mixing C^∞ diffeomorphisms f, g of compact surfaces with positive topological entropy are Borel conjugate if and only if the two conditions are satisfied :

1. $h_{\text{top}}(f) = h_{\text{top}}(g)$
2. *for all* $n \geq 1$ $\kappa(f, n) = \kappa(g, n)$

Remarks :

- topological conjugacy does not always hold under above assumptions
- cannot drop "C^∞" or "positive topological entropy"
- dropping "topological mixing" requires other invariants and almost-Borel

Behind the curtain

Ingredients of the proof

Hidden player : Lyapunov exponents for μ-ae $x \in M$

$\lambda(x, v) := \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n.v\| \qquad (x \in M,\ v \in T_xM \setminus 0)$

Ruelle-Margulis inequality :

$$\forall \mu \in \mathbb{P}_{\mathrm{erg}}(f) \quad h_\mu(f) > 0 \implies \mu \text{ has no zero Lyapunov exponent}$$

That is Pesin hyperbolicity

Adding plane topology and smoothness we are able to generalize classical results of **uniformly hyperbolic** dynamics, including the above theorem

More precisely,

- use homoclinic classes of hyperbolic periodic orbits as irreducible pieces of the dynamics
- prove local uniqueness using Sarig's symbolic dynamics
- control pieces with large entropy by using plane topology, dynamical foliations, Sard lemma and Yomdin theory

Conclusion

Conclusion

What we have seen :

- For smooth surface diffeos, positive entropy allows generalizing many result from uniform hyperbolic theory
- This solves Newhouse's problem about MMEs
- Topological entropy classifies in the mixing case

Perspectives :

- quantitative results (exponential mixing, central limit theorem,...)
- classes in higher dimensions

(work in progress with Croviser and Sarig involving a stability property of the Lyapunov exponents)

Thank you!