[42r] My dear Lady Lovelace

Mr. Frend's death (which took place on Sunday Morning) has made me answer your letter later than I should otherwise have done. The family are all well, and have looked forward to this termination for some time. My wife will answer your letter on a part of this.
Number and Mag$^{\text{n}}$. pp. 75, 76. The use of this theorem is shown in what follows. It proves that any quantity which lies between two others is either one of a set of mean proportionals between those two, or as near to one as we please. It is not self evident that the base of Napiers system, as given by himself is ε or $1+1+\frac{1}{2}+\cdots$ as we learn from the modern mode of presenting the theory. The last sentence in the book (making V a linear unit) would show that Napier's notion was to take k in such a manner that x shall expound [?] $1+x$ ['without' crossed out] or rather that the smaller x is the more nearly shall
x expound $1+x$. If this were accurately done, we should have

$$k^x = 1+x \quad \text{or} \quad \frac{k^x-1}{x} = 1$$

and this is to be nearer to the truth the smaller x is. Now when the common theory is known, it is known that $k=\varepsilon$ gives

$$\frac{\varepsilon^x-1}{x} = 1+\frac{x}{2}+\frac{x^2}{2\cdot 3}+\cdots \qquad \text{and limit of } \frac{\varepsilon^x-1}{x} = 1$$

while $\frac{k^x-1}{x} = \log k + \frac{\overline{\log k}|^2 .x}{2} + \cdots \qquad$ and limit of $\frac{k^x-1}{x} = \log k$
where the log. has this very base ε. Having proved these things, it is then obvious that, $\log k$ being never 1 except when k is the base or ε, the last paragraph cannot consist with any other value of k except ε. In this book (Num & Mag.) I must refer you to the algebra, which I do not in the Diff. Calc., many matters of series, until the whole doctrine is reestablished.

Now ['as' crossed out] as to the Diff. Calc. You do not see that θ is a function of a and h. Let us take the simplest case of the original theorem which is

[42v] $\qquad \varphi(a+h) = \varphi a + h\ \varphi'(a+\theta h) \qquad (1)$

Now 1. Why should θ be independent of a and h, we have never proved it to be so : all we have proved is that <u>one</u> of the numerical values of θ is < 1, or that this equation (1) can be satisfied by a value of $\theta < 1$. As to what θ is, let ψ be the inverse function of φ' so that $\psi\varphi' x = x$. Then

$$\frac{\varphi(a+h)-\varphi a}{h} = \varphi'(a+\theta h)$$

$$\psi\left(\frac{\varphi(a+h)-\varphi(a)}{h}\right) = \psi\varphi'(a+\theta h) = a+\theta h$$

$$\theta = \frac{\psi\left(\frac{\varphi(a+h)-\varphi a}{h}\right)-a}{h}. \left\{ \begin{array}{l} \text{Say that this is not a function} \\ \text{of } a \text{ and } h\text{, if you dare} \end{array} \right.$$

For example $\quad \varphi x = c^x$

$\qquad \varphi' x = c^x . \log c$

$c^{a+h} = c^a + h \log c\ c^{a+\theta h}$
$c^{a+\theta h} = \frac{c^{a+h}-c^a}{h \log c}$
$(a+\theta h) \log c = \log \frac{c^{a+h}-c^a}{h \log c}$
$\theta = \frac{\log \frac{c^{a+h}-c^a}{h \log c} - a \log c}{h \log c}$
$\quad = \frac{\log(c^h-1)-\log(h \log c)}{h \log c}$

In this particular example θ happens to be a function of h only, not of a : but you must remember that in every case where we speak of a quantity as being generally a function of a, we do not mean thereby to deny that it may be in particular case, not a function of a at all : just as [43r] when we say that there is a number (x) which satisfies certain conditions, we do not thereby exclude the extreme case in which $x = 0$.

Look at the question of differences in this manner. Any thing which has been proved to be true of u_n relatively to u_{n-1}[,] u_{n-2} &c has also been proved to be true of Δu_n relatively to Δu_{n-1}[,] Δu_{n-2} &c. For in the set

$$\begin{array}{llll} u_0 & & & \\ & \Delta u_0 & & \\ u_1 & & \Delta^2 u_0 & \\ & \Delta u_1 & & \text{\&c} \\ u_2 & & \Delta^2 u_1 & \\ & \Delta u_2 & & \\ u_3 & & & \end{array}$$

the first column may be rubbed out and the second column becomes the first &c. It is obvious that the $m+1$, $m+2$, &c columns are formed from the mth precisely as the 2nd, 3rd &c are formed from the first. If then I show that up to $n = 7$, for instance

[$u \ldots$ crossed out] $u_n = u_0 + n\Delta u_0 + \cdots\cdots$

I also show (writing Δu_n for u_n) that $\Delta u_n = \Delta u_0 + n\Delta(\Delta u_0) + \cdots$
Perhaps you had better let the question of discontinuity rest for the present, and take the result as proved for continuous functions. You will presently see in a more natural manner the entrance of discontinuity

The paper which I return is correct

Yours very truly
ADeMorgan

69 Gower St
Mond^y^ Ev^g^

[43v] [Note to AAL from SDM]