[168r] [In De Morgan's hand] When an equation involves only two variables, it is easy enough to write all differential equations so as to contain nothing but differential coefficients; thus

$y = \log x \qquad \frac{dy}{dx} = \frac{1}{x}$

If we prefer to write $dy = \frac{dx}{x}$, it must be under a new understanding. By $\frac{dy}{dx}$, we mean the limit of $\frac{\Delta y}{\Delta x}$, not any value which $\frac{\Delta y}{\Delta x}$ ever can have, but that which it constantly tends towards, as Δx is diminished without limit. But in $dy = \frac{dx}{x}$, we cannot by dy and dx mean limits, for the limits are zeros; and $0 = \frac{0}{x}$, though very true, is unmeaning

What then do we mean by this

When $y = \log x$, $dy = \frac{dx}{x}$

we mean that $\Delta y = \frac{\Delta x}{x}$, as Δx diminishes without limit, not only diminishes without limit, but diminishes without limit as compared with Δx or Δy. So that, if we call it a, or if

[168v] $\Delta y - \frac{\Delta x}{x} = a$

Then a is useless, and we might as well write 0. For since the processes of the differential calculus always terminate in taking <u>limits of ratios</u> and since

$\frac{\Delta y}{\Delta x} - \frac{1}{x} = \frac{a}{\Delta a}$

(or some transformation of this sort) must come at last, our limiting equation must be

$\frac{dy}{dx} - \frac{1}{x} = \text{Limit of } \frac{a}{\Delta x} = 0$

The truth of every equation differentially written, as $dy = p\,dx$, is always absolutely speaking, only approximate: but the approximation is relatively closer and closer . Understand it as if it were

$dy = (p + \lambda)dx$

where λ diminishes with dx, so that the error made in dy by writing $dy = p\,dx$, namely $\lambda\,dx$, not only diminishes

with dx, but becomes a smaller and
smaller fraction of dx: because
λ diminishes without limit
[169r] All this is conveniently signified in
the language of Leibnitz, namely, that
when dx is infinitely small, $dy - p\,dx$
is as nothing (or infinitely small)
when compared with dx, or dy is
(relatively to its own value) infinitely
near to $p\,dx$.

The differential might easily be
avoided when there are only two variables,
and even when there are more, provided
we only want to use one independent
variable at a time . Thus

$$u = \varphi(x,\, y,\, z)$$

may give the equations

$$\tfrac{du}{dx} = P, \quad \tfrac{du}{dy} = Q, \quad \tfrac{du}{dz} = R$$

But when we want to make
x, y, and z, all vary together,
we have no notion of a differential
coefficient attached to this simulta-
neous variation, unless we suppose some
one new variable on which x, y, and z
all depend, and the variation of which
sets them all varying together.
[169v] If this new variable be t, and
if x, y, and z be severally functions
of t, we have then
[in margin: 'See chapter on Implicit differentiation']

$$\tfrac{d(u)}{dt} = \tfrac{du}{dx} \cdot \tfrac{dx}{dt} + \tfrac{du}{dy}[\cdot]\tfrac{dy}{dt} + \tfrac{du}{dz} \cdot \tfrac{dz}{dt}$$

Thus if $u = xy^2\varepsilon^2$

$$\tfrac{d(u)}{dt} = y^2\varepsilon^2\tfrac{dx}{dt} + 2xy\varepsilon^2\tfrac{dy}{dt} + xy^2\varepsilon^2\tfrac{dz}{dt}$$

But observe that this makes x, y, and z,
(which we want to be independent of one
another) really functions of one another:
thus if $x = t^2$, $y = \log t$, we must
have $y = \log\sqrt{x}$. We might it is
true avoid this by the following suppo-
sition. Let x, y, and z, instead of being
given functions of t, be unassigned and

arbitrary functions, which we can always make whatever functions we please. We can then really hold $\frac{dx}{dt}$ $\frac{dy}{dt}$ $\frac{dz}{dt}$ to be independent of one another, for it is always in our power to assign them any values we like. But this method would be awkward , and would put continual impediments in our way. It is better therefore to avoid that [170r] notation which while it makes the first step by supposing relations to exist between x, y, and z, immediately contradict that supposing by making these relations mean <u>any</u> relations.

If in $\varphi(x,\, y\, z)$ we suppose x, y, and z to be simultaneously altered into $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, then $\varphi(x,\, y\, z)$ takes the value

$$\varphi(x+\Delta x,\, y+\Delta y,\, z+\Delta z)$$

which may be expounded as follows

$$\varphi+\frac{d\varphi}{dx}\Delta x+\frac{d\varphi}{dy}\Delta y+\frac{d\varphi}{dz}\Delta z$$
$$+A\Delta x\Delta y+B\Delta y\Delta z+C\Delta z\Delta x$$
$$+D\overline{\Delta x}^2+E\overline{\Delta y}^2+F\overline{\Delta z}^2$$
$$+\ \&c\ \&c$$

say

$$\varphi+\frac{d\varphi}{dx}\Delta x+\frac{d\varphi}{dy}\Delta y+\frac{d\varphi}{dz}\Delta z+M$$

If it be required that $\varphi=$ constant, or $\varphi=c$ we must have

$$\frac{d\varphi}{dx}\Delta x+\frac{d\varphi}{dy}\Delta y+\frac{d\varphi}{dz}\Delta z+M=0$$

Now if we were to leave out M, and say

$$\frac{d\varphi}{dx}\Delta x+\frac{d\varphi}{dy}\Delta y+\frac{d\varphi}{dz}\Delta z=0$$

we should of course commit an error: but it is one the magnitude of which relatively to Δx, for instance, diminishes [170v] without limit as the increments Δx, Δy, Δz, are diminished without limit. The considerations already given apply here again : because all the terms contain [*sic*] in M, diminish without limit as compared with those which are not [something crossed out] contained in M. This rejection of all terms except those of the first order is always accompanied and

When therefore I say that

$$\varphi=c$$

gives $\frac{d\varphi}{dx}.dx+\frac{d\varphi}{dy}[.]dy+\frac{d\varphi}{dz}.dz=0$

I should, if asked whether this equation is absolutely true, answer no. If then asked why I write it, I should answer that it leads to truth, and for this reason that it is more and more nearly true as dx &c are diminished : not because $\frac{d\varphi}{dx}dx + \&c$ diminishes in that case, though undoubtedly it does so ; but because it diminishes as compared with dx, &c. Hence, when we form ratios and take their limits, it matters nothing, as to the results we obtain, whether we write

$$\frac{d\varphi}{dx}dx + \&c \qquad = -M$$

or

$$\frac{d\varphi}{dx}dx + \&c \qquad = 0$$

marked by writing dx for Δx, dy for Δy, &c.