[149r]

Ashley-Combe
Nov^{r} 27th

Dear M^{r} De Morgan. I have I believe made
some little progress towards the comprehension of
the Chapter on Notation of Functions, & I enclose
you my Demonstration of one of the Exercises at
the end of it : "Show that the equation $\varphi(x+y) =$
"$= \varphi x + \varphi y$ can be satisfied by no other solution
"than $\varphi x = ax$." At the same time I am
by no means satisfied that I do understand
these Functional Equations perfectly well, because
I am completely baffled by the other Exercise :
"Shew that the equation $\varphi(x+y) + \varphi(x-y) =$
"$= 2\varphi x \times \varphi y$ is satisfied by $\varphi x = \frac{1}{2}(a^x + a^{-x})$
"for every value of a". _
I do not know when I have been so tantalized
by anything, & should be ashamed to say <u>how</u>
much time I have spent upon it, in vain.
[149v] These Functional Equations are complete Will-o'-
-the-Wisps to me. The moment I fancy I have
really at last got hold of something tangible
& substantial, it all recedes further & further
& vanishes again into thin air. _
But now for this perplexing $\varphi x = \frac{1}{2}(a^x + a^{-x})$.
I be<u>lieve</u> I have left no method untried ; but
I cannot get further than as below, with any
certainty :_

$$\varphi(x+y) + \varphi(x-y) = 2\varphi x \times \varphi y$$
$$\because 2\varphi x = \frac{\varphi(x+y)+\varphi(x-y)}{\varphi y}$$

Since x and y may have a<u>ny</u> values whatever,
(at least such I conclude is of course intended), let
$y = 0$. We have then

$$2\varphi x = \frac{\varphi(x)+\varphi(x)}{\varphi(0)}$$
$$\because 2\varphi x \times \varphi(0) = \varphi(x) + \varphi(x)$$
$$\text{or } 2\varphi x \times \varphi(0) = 2\varphi x$$
$$\varphi(0) \text{ must } = 1, \text{ or } = a^0, \text{ since } a^0 \text{ is}$$

the only function of 0 which can = 1
I think so far is correct in itself, but whether
[150r] it be the ['right' inserted] road to the rest is another question.
At any rate, <u>I</u> have not succeeded in proving

it such. To assume that since $\varphi(0) = a^0$,
$\varphi(x+y) = a^{x+y}$, $\varphi(x-y) = a^{x-y}$, $\varphi y = a^y$
appears to me scarcely warrantable ; and
besides in that case it must be equally
assumed that $\varphi x = a^x$, (there being the same
ground for the one assumption as for the others),
and we should then have,

$$a^x = \tfrac{a^{x+y}+a^{x-y}}{a^y}$$
$$\because a^x = \tfrac{a^{x+y}}{a^y} + \tfrac{a^{x-y}}{a^y}$$
$$\because a^x = a^x + a^{x-2y}$$

most clearly absurd, independent of it's [*sic*] being
discordant with the book. __
Once I thought I had hit on something very
clever indeed, and wrote as follows :

$$\varphi(x+y) = \varphi(\overline{x+y}.1)$$
$$= \{\varphi(1)\}^{x+y} \text{ by equation } (\varphi a)^n = \varphi(na)$$

page 205, entirely forgetting that the φ of that
equation had nothing whatever to do with the
$\underline{\varphi}$ of any other equations ; (a disagreeable truth
[150v] which did not occur to me until 24 hours
later). I then had $\varphi(x-y) = \varphi(\overline{x-y}.1) = \{\varphi(1)\}^{x-y}$
$\varphi(y) = \varphi(y.1) = \{\varphi(1)\}^y$, and

$$2\varphi x = \tfrac{\{\varphi(1)\}^{x+y}+\{\varphi(1)\}^{x-y}}{\{\varphi(1)\}^y}$$
$$= \tfrac{\{\varphi(1)\}^{x+y}}{\{\varphi(1)\}^y} + \tfrac{\{\varphi(1)\}^{x-y}}{\{\varphi(1)\}^y}$$
$$= \{\varphi(1)\}^x + \{\varphi(1)\}^{x-2y}, \text{ and supposing}$$

$x = y$, then $2\varphi x = \{\varphi(1)\}^x + \frac{1}{2}\{\varphi(1)\}^{-x}$
or calling $\varphi(1) = a$, $\varphi x = \frac{1}{2}(a^x + a^{-x})$
But besides my unwarrantable assumption of
$\varphi(\overline{x+y}.1) = \{\varphi(1)\}^{x+y}$, there was t<u>his</u> in the
result which was unsatisfactory, that it was
necessary to assume $x = y$, and the result seemed
to hold good in t<u>hat</u> case al<u>one</u>. Also, when to
verify, I tried $x = 1$, $\because \varphi(1) = \frac{1}{2}(a^1 + a^{-1})$,
which ought to have come out $a = a$, I could
make neither head or tail of it. _ Well, I
abandoned this, & tried all sorts of other resources.
[151r] I understand to work out something by means
similar to those in page 205 and in the
Problem I send ; but equally unsuccessfully.
I also in equation (A) page 204, changed
$\varphi(x+y)$ into $\varphi(x-y)$ and investigated this,

thinking I might derive a hint possibly from it.
[something crossed out] In short, many & various are the experiments
I have made, but I will not detail any
more. Indeed I think you may be possibly
heartily sick of what I h<u>ave</u> detailed. But
I wished to show you that I have not failed
from want of t<u>rying</u>, at least ; & also to give
you the chance of smiling at my expence [*sic*]. ____

I shall have to trouble you with another
letter shortly, on other knotty points. Really I
do not give you a Sinecure. Your letters are
however well bestowed, in as far as the use
they are of to me, can make them so, and
the great encouragement that such assistance
is to me to continue my Studies with zeal
& spirit. We are to return to Surrey very
[151v] soon. I expect to have occasion to trouble
you again before we go, & after that I shall
hope to see you & M^rs^ De Morgan in Town,
where I intend to be for two or three days
in about a fortnight. __

Yours very truly
A. A. L