Transcripts of Folios 1-179

Box 170

The Lovelace Byron Papers
Bodleian Library, Oxford

by

Christopher Hollings

[1r] My dear Lady Lovelace

I have of course but little to say
on your report of progress up to the 21st.

With regard to your music, if you have any
wish to begin the study of acoustics, you may find
an elementary compendium of the most material
points directly connected with music, in the following
articles of the Penny Cyclopaedia

*Acoustics*, *Cord*, *Harmonics*, *Pipe*

The last will appear at the end of this month
and contains a correction to be made in each
of the first and second. I should recommend to you, out of the same
Cyclop$^{\text{a}}$ the articles

*Infinite*; *Nothing*; *Limit*; *Negative and Impossible Quantities*;

*Fractions, Vanishing*;

[1v] With regard to the only point to which you
have alluded your finding that ‘I do not insist
in zero being *something*’ the matter is as follows.
Zero is *something* though not some *quantity*, which
is what you here mean by *thing*.
Some writers on the differential calculus use
not 0, but $\frac{0}{0}$, in an absolute sense, as standing
for a quantity. The difference between them and
others consists mostly in phraseology. One person
will say that $\frac{a^2-a^2}{a-a}$ *is* $2a$; a second will say that
the appearance of $\frac{a^2-a^2}{a-a}$ *denotes a misapprehension to have
taken place during the process, which being avoided in a
repetition of the process,* $2a$ *comes out rationally as the
answer*. If the first be pressed to prove his assertion,
he will tell you that the two words underlined are
the abbreviation of all that is afterwards underlined.

You will see more of this abbreviation in the article
*Infinite* to which I have referred. Absolute modes
of speaking, which are false, are conventionally
used in abbreviation of circumlocutions.

I forwarded to Lord Lovelace the other day a prospectus
of a Society now in process of formation for
[2r] printing scientific manuscripts which have not
been hitherto printed. The works of this society
are not to be published, but only distributed

among the members. As you will certainly take
an interest in the results of its labors, if you
continue your studies, I should recommend
<u>your</u> being a member, with Lord Lovelace as
your proxy.
There is another Society also forming, of the same
kind, and on the same terms, the <u>Percy</u> Society
for the publication of ancient inedited
songs and ballads.

I remain
Faithfully yours
<u>ADeMorgan</u>

3 Grotes' Place
Blackheath
Thursd$^{y}$ m$^{g}$

My wife desires kind remembrances.

[3r] My dear Lady Lovelace

The solution of the $t^3 + t^2 + t$ problem is correct
and I have no doubt from it that you fully understand the problem
of the stone.
The difficulty you meet with in the variable coefficient would
be fatal to the process if the coefficient increased without limit
as the value of $k$ diminishes without limit.
But in this case $t$ which you call a variable, is really
a given constant, for it is not $t$ which varies, but the time
which is first $t$, then $t + k$, then $t + 2k$, &c.
There is a little incorrectness in the phraseology of variable
quantities. A quantity varies, it is first $x$, and then $x + a$;
here it is usual to say that $x$ varies, whereas it is not
$x$, but the magnitude which $x$ represents, which changes
and is no longer represented by $x$, but by $x + a$.
Thus if we pass in thought from 10 seconds to 11 eleven [*sic*]
seconds, we say let 10 vary, and become 11. Now 10 is a
fixed symbol, and so is 11; it is we ourselves who vary our
supposition, and pass from one to the other.
But even if the coefficients of $k$ were variable, it
would not vitiate the result, as may be thus shown
Let $a + vk$ be an expression in which $a$ is a constant
$k$ diminishes without limit, and $v$ at the same time
varies, say increases. Let $v$ always remain finite, that
is, let it not increase without limit while $k$ diminishes
[3v] Suppose for instance, that it never exceeds a
certain number (no matter how great) say a
million. Then $vk$ never exceeds $1000,000.k$. Now if
$k$ may be made as small as we please, so may $1000,000k$,
and still more $vk$, which is less, or at least not greater.
That is $vk$ diminishes without limit, and
$a + vk$ has the limit $a$.
But if $v$ increased without limit as $k$
diminished, the case might be altered (not necessarily
would. For example

1. Let $v =$ [$\frac{1}{k}$ crossed out] $\frac{1+k}{k}$
   $a + vk = a + 1 + k$ and the limit is $a + 1$

2. Let $v = \frac{1+k}{k^2}$
   $a + vk = a + \frac{1+k}{k}$, and increases without limit

3. Let $v = \frac{1+k}{\sqrt{k}}$

$a + vk = a + (1+k)\sqrt{k}$, and the limit is $a$, as at first.

And $\frac{1+k}{k}$, $\frac{1+k}{k^2}$, $\frac{1+k}{\sqrt{k}}$ all increase without limit as $k$ diminishes without limit.

[4r] Your correction of the press is right. On the cover of no 12 you will see a list of errata.

[diagram here in original] When two lines at right angles are made standards of position, and when the position of a point is determined by its perpendicular distance from the lines, as $PA$, $PB$, then $PA$ and $PB$ are called <u>co-ordinates</u> of $P$. One of them, as $BP$ is usually found by its equal $OA$, and called the <u>abscissa</u> of $P$, the other, $AP$, is called the ordinate.

I have written something on the paper which I return.

My wife desires kind remembrances, and with our united remembrance to Lord Lovelace

I remain

Yours very truly

<u>ADeMorgan</u>

3 Grotes' Place Blackheath

August 1, 1840

Should you decide on supporting our old printing Society, I shall be very happy to have Lord Lovelace's name inserted. We are beginning with an Old Saxon treatise on astronomy with a translation.

[5r] My dear Lady Lovelace

I should be as able as willing to see you in town on Friday, but have first heard that M[r] Frend is not so well as he has been, and am going to Highgate to day to see how he is. In consequence, having various matters to complete definitively by the 16th instant, I shall find it impossible to go to town again this week.

With regard to the second chapter, pray remember that you are not supposed to know, or to want to know, what differentiation is, but only that there is a process of that name, which is to be learnt by rule for the present, [5v] as an exercise in algebraical work.

With regard to the logarithms, in the first place, Bourdon is too long. If you will look at the chapter in my algebra, you will find it shorter.

In the equation

$$a^b = c$$

$b$ <u>is called</u> the logarithm of $c$ to the base $a$. This is the <u>meaning</u> of the term. But for convenience the series $1 + 1 + \frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{2\times 3\times 4} + \&c$ ad inf or $2.7182818\cdots$ (called $\varepsilon$) is the base always used <u>in theory</u>; while when assistance in calculation is the object, 10 is always the base; thus if

$\varepsilon^x = y$ $\quad x$ is the logarithm of $y$

[6r] Thus $\quad a = \log b$ is by definition synonymous with $\quad b = \varepsilon^a \quad \varepsilon$ being $2.7182818\cdots$

I remain

Yours very truly

<u>ADeMorgan</u>

3 Grotes' Place, Blackheath

Wednesday M[g]

[7r] My dear Lady Lovelace

The Theorem in page 16 can be easily proved when the following is proved

[$\frac{a+b}{}$ crossed out] $\frac{a+a'}{b+b'}$ lies between $\frac{a}{b}$ and $\frac{a'}{b'}$

$\frac{a+a'}{b+b'} = \frac{a(1+\frac{a'}{a})}{b(1+\frac{b'}{b})} = \frac{a}{b} \times \frac{1+\frac{a'}{a}}{1+\frac{b'}{b}}$

Now if $\frac{a}{b}$ be greater than $\frac{a'}{b'}$

$ab' \cdots\cdots\cdots\cdots\cdots\ a'b$

$\frac{b'}{b} \cdots\cdots\cdots\cdots\cdots\ \frac{a'}{a}$ whence $\frac{1+\frac{a'}{a}}{1+\frac{b'}{b}}$ is less than 1

or $\frac{a}{b} \times \frac{1+\frac{a'}{a}}{1+\frac{b'}{b}}$ is less than $\frac{a}{b}$

or $\frac{a+a'}{b+b'}$ is less than $\frac{a}{b}$

Similarly, it may be shown that if $\frac{a}{b}$ be less than $\frac{a'}{b'}$ $\frac{a+a'}{b+b'}$ is greater than $\frac{a}{b}$. You will now I think, not have much difficulty in proving the whole. Page 48 [or 28?] contains the general view of this theorem

Page 29. Our conclusions are really the same. To say that [diagram in original] is a r$^{\text{t}}$ angled triangle, is to say that $OP$ is straight and not curved. The following however will explain

[7v] [diagram in original] By the tangent of $\angle POM$ is meant the fraction $\frac{PM}{OM}$, which is, by similar triangles, the same thing for every point of $OP$.

If then $PM = \frac{2}{3}OM$, always, we have $\frac{PM}{OM} = \frac{2}{3}$ always, or the direction $OP$ is always such as to make the angle $POM$ the same, namely that angle which has $\frac{2}{3}$ for its tangent. To see all this fully something of Trigonometry and the application of algebra to geometry is required.

The Differential and Integral Calculus deal in the same elements, but the former separates one element from the mass and examines it, the latter puts together the different elements to make the whole mass.

The examination of $PQ\,MN$ (p. 29) with a view to the relation between $OM$ and $MP$ is a case of the first: the summation of the rectangles in page 30, of the second.

Page 32. The reference is unnecessary.

The first series $1 + 4 +$ &c is finite, the second infinite. It is not easy to see à priori why one problem should be attainable with given means and another not

so. It is stated here with a view to the following
common misapprehension.
[8r] It is thought that Newton and Leibnitz had some
remarkable new conception of principles, which is not
true. Archimedes and others ['and others' inserted] had a differential and integral
calculus, but not an algebraical system of sufficient
power to express very general truths.
Many persons before Newton knew, for instance that if
$\frac{(x+h)^n-x^n}{h}$ could be developed for any value of $n$,
the tangents of a great many curves could be drawn
and they knew this upon principles precisely the same
as Newton and Leibnitz knew it. But Newton
<u>did</u> develope $\frac{(x+h)^n-x^n}{h}$ and <u>did</u> that which they
could not <u>do</u>.

It was the additions made to the powers of algebra
in the seventeenth century, and not any new
conceptions of quantity, which made it worth while
to attempt that organization which has been called
the Differential Calculus

I should recommend your decidedly continuing the
Differential Calculus, warning you that you will
have long digressions to make in Algebra and
Trigonometry. I should recommend you to get my
Trigonometry, but not to attempt anything till I
send you a sketch of what to read in it. The
Algebra you must go through at some time or
[8v] other, adding to it the article

"Negative and impossible quantities
in the Penny Cyclopaedia.
I have no doubt of being able to talk this
matter over with you in town when you
arrive
In the mean while, as mechanical expertness
in differentiation is of the utmost consequence,
and as it is the most valuable exercise in
algebraical manipulation which you can
possibly have, I should recommend your
thoroughly acquiring and keeping up the
Chapter you are now upon.

Yours very truly
<u>ADeMorgan</u>

3 Grotes' Place

Monday Aug$^{st}$ 17/40

M$^{r}$ Frend is rather better. I will add Lord Lovelace's name to my list of members.

[9r] My dear Lady Lovelace

I am in the middle of arranging my books and can only just get room to write a short note.

I received yours relative to the inquiry about the study of Math$^{cs}$ but did not answer as you would have left town and I did not know your country address.

I don't think you will want the 'Study of Math$^{cs}$. The corresponding subject in the [9v] Algebra or Trigonometry will be what you want.

The continuity question you must consider well (because you will not allow yourself to skip) and perhaps your <u>ultimate</u> difficulties will not be altogether the same as those you commence with. The curve is laid down right and I have appended a remark to what I return.

Yours very truly
<u>ADeMorgan</u>

69 G.S.

[10r] I have now returned to town. My wife is gone to Highgate for a fortnight

[11] [diagram down left-hand side of original]

[in Lovelace's hand] The successive values of $x$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, 4 are selected & the corresponding function $x^2$ represented on perpendicular lines drawn from the extremities of the respective line representing $\underline{x}$.
By writing the successive perpendicular extremities $\frac{1}{16}$, $\frac{1}{4}$, $\frac{9}{16}$, 1, $\frac{9}{4}$, 4 &c &c, a curve appears to be produced.
In lines 13 & 14, this <u>curve</u> (according to <u>my</u> interpretation) is alluded to as "the representation of a function, or functions".

This does not appear to me to hold good, as <u>I</u> should have said that the <u>perpendicular straight lines</u> are the representation of the function, & I do not see any <u>precise relation</u> that the existing curve holds to them.

[in De Morgan's hand] The precise relation is that this one curve, and no other, belongs to $y = x^2$. Of course there could be no <u>visible</u> relation unless to a person whose eye was so good a [Judge crossed out] judge of length that he could <u>see</u> the ordinate increasing with the square of the abscissa.

[12r] My dear Lady Lovelace

You have got through the matter about which you write better than I should have expected.

I have finished what you sent as you will see

With regard to the curve, I drew it as containing every possible sort of singular point. Its equation would be enormously complex

There must be an infinite number of different equations which belong to a curve of a similar form, but the question 'given the more general [12v] form of a curve, required the equations which may belong to such form' is a very difficult one.

I will merely give you a glimpse

Required an equation to a curve such that it passes through the following points $P\ Q\ R$

[diagram in original] at $P$ let $x = a,\ y = A$

$$\begin{array}{lll} Q & x = b & y = B \\ R & x = c & y = C \end{array}$$

[the next formula and the following line of text stretch across 12v and 13r]

$$y = A\tfrac{(x-b)(x-c)}{(a-b)(a-c)} + B\tfrac{(x-c)(x-a)}{(b-c)(b-a)} + C\tfrac{(x-a)(x-b)}{(c-a)(c-b)} + \left\{\begin{array}{c}\text{any function of } x \text{ which}\\ \text{does not become infinite}\\ \text{when } x = a, \text{ or } b, \text{ or } c\end{array}\right\} \times (x-a)(x-b)(x-c)$$

Here is an infinite number of equations which you will find to satisfy the conditions

I have to thank you for very good partridges received from Ockham

With kind remembrances to Lord Lovelace

I am Yours very truly

ADeMorgan

I have heard of Lady Byron by $M^r$ Phitton [?] who left her safe at Fountainebleu

[14r] My dear Lady Lovelace

Your inquiries were received just after I had dispatched the receipt for Lord L's subscription to the Hist. Soc.

While I think of it (the Hist. Soc. reminds me) Nicolas Occam, or Ockham, or of Ockham, who flourished about 1350, took his name I rather think, from the same place as your little boy. He was a mathematician, and one of the most remarkable English metaphysicians before Locke. It is very likely that the late Ld King may on both accounts, Ockham and metaphysics, [', Ockham and metaphysics,' inserted] have collected something about him, or that Lord Lovelace may be in possession of something relating to him. If so, it can certainly be made useful. His logic was printed very early but is so scarce that I have never been able to get sight of a copy.

Now to your queries. Festina lente, and above all never estimate progress by the number of pages. You can hardly be a judge of the progress you make, and I should say that it is more likely you progress rapidly upon a point that makes you think for an hour, than upon an hour's quick reading, even when you feel satisfied. That which you say about the comparison of what you do with what you see can be done was equally said by Newton when he compared himself to a boy who had picked up a few pebbles from the shore; and the last words of Laplace were 'Ce que nous connaissons est peu de chose; ce que nous ignorons est immense' So that you have respectable authority for supposing that you will never get rid of that feeling; and it is no use trying to catch the horizon.

[14v] Peacocks examples will be of more use than any book.

As to the functional equation. You must distinguish in algebra questions of quantity from questions of form. For example "given $x + 8 = 10$, required $x$," is a question of quantity but "given $x$, an arbitrary variable, required a function of $x$ in which if the function itself be substituted for $x$, $x$ shall be the result" is a question of form, independent of value, for it is to be true for all values of $x$. One solution is

$$\frac{1-x}{1+x}$$

for $x$ substitute the function itself, this gives $\frac{1+\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}$

or $\frac{1+x-(1-x)}{1+x+(1-x)}$ or $\frac{2x}{2}$ or $x$.

Another solution is $1 - x$, since $1 - (1 - x)$ is $x$; a third is $-x$, since $-(-x)$ is $x$.

Now suppose $\varphi(x+y) = \varphi x + \varphi y$

$x^2$ does <u>not</u> satisfy this; $\overline{x+y}|^2$ is not $x^2+y^2$

$ax$ does $\qquad a(x+y)$ is $ax+ay$

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Let $\varphi x = x^a \qquad \varphi(xy) = (xy)^a = x^a y^a = \varphi x + \varphi y$

$\varphi x = a^x \qquad \varphi x \times \varphi y = a^x \times a^y = a^{x+y} = \varphi(x+y)$

[15r] $\varphi x = ax+b \qquad \frac{\varphi x - \varphi y}{\varphi x - \varphi z} = \frac{ax+b-(ay+b)}{ax+b-(az+b)} = \frac{ax-ay}{ax-az} = \frac{x-y}{x-z}$

A functional equation is one which has for its [something crossed out] <u>unknown</u> the form proper to satisfy a certain condition Example. What function of $x$ is that which is not altered by changing $x$ into $1-x$, let $x$ be what it may. Or, required $\varphi x$ so that

$$\varphi x = \varphi(1-x)$$

One solution is $\varphi x = 1 - 2x + 2x^2$

for $\varphi(1-x) = 1 - 2(1-x) + 2(1-x)^2$

$= 1 - 2 + 2x + 2 - 4x + 2x^2$

$= 1 - 2x + 2x^2$ as before.

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The equation of a curve means that equation which must necessarily be true of the coordinates of every point in it, and obviously depends upon 1. The point chosen from which to measure coordinates 2. The direction chosen for the coordinates. 3. The nature and position of the curve. For example let the curve be a circle, the point chosen its center, and the axes of coordinates two lines at right angles. Let the [diagram in original] radius be $a$; then at every point $x$ and $y$ must be the two sides of a right angled triangle whose hypotenuse is $a$; or

$$x^2 + y^2 = a^2$$

which being an equation true at every point [15v] of the circle, is called the equation of the circle

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My wife returns to day from Highgate.
M^r Frend continues very comfortable, and neither mends nor grows worse. I hope Ld Lovelace and the little people are well. The old Ockham will be a poor example for the young one, though

he was a monk, as I suppose. I would have been nothing else had I lived in his day

Yours very truly

ADeMorgan

69 Gower St.
Monday Sep$^{t}$ 15/40

[16r] My dear Lady Lovelace

You have taken a proper time to begin with Incommensurables and if the subject interests you, I should recommend you to continue. You understand of course that your Diff[l] Calculus must be delayed from time to time while you make up those points of Algebra and Trigonometry which you have left behind.

D. C. p. 53. As in page 22 refers to the method of proving that if $P = 2Q$, lim. of $P = 2$. lim. of $Q$

In similar way it may be shown that if

$\frac{\Delta u}{\Delta x} \cdot \frac{\Delta x}{\Delta u} = 1$ lim. of $\frac{\Delta u}{\Delta x} \times$ lim. of $\frac{\Delta x}{\Delta u} = 1$

With reference to your remark remember that

$\frac{\Delta u}{\Delta x} \cdot \frac{\Delta x}{\Delta u} = 1$ and $\frac{a}{b} \times \frac{b}{a} = 1$ <u>are</u> the same proposition

But $\frac{du}{dx} \times \frac{dx}{du} = 1$ and $\frac{a}{b} \times \frac{b}{a} = 1$ are <u>not</u> the same

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$\frac{\Delta u}{\Delta x} \times \frac{\Delta x}{\Delta u} = 1$ by common algebra $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$

But we cannot say $\frac{du}{dx} \times \frac{dx}{du} = \frac{du}{dx}\frac{dx}{du} = 1$

because $\frac{du}{dx}$ is a mere symbol to denote limit of $\frac{\Delta u}{\Delta x}$ and $du$ and $dx$ have no separate meaning

[16v] N. & M. p. 17

The erratum exists [‘but the misprint is’ crossed out] and must be set right as you propose

[‘for $\frac{q_1}{p_1} - \frac{q_2}{p_2}$’ crossed out]

The lengthiness of the proof arises from the necessity of adapting a very common algebraical theory to Euclid’s method.

You should try some of the examples of differentiation in Peacock’s book. Remember that there are some misprints in it. You will not have to go <u>through</u> it to try a little of everything.

When the article <u>Proportion</u> appears in the Penny Cycl. which it will in a few weeks, I recommend your attention to it

With remembrances to Lord Lovelace I am

Yours truly

<u>ADeMorgan</u>

69 Gower St.

Sunday M[g] Sept[r] 27/40

[18r] My dear Lady Lovelace

First as to your <u>non</u> mathematical question: I do not think an English jury would have found Mad. Laffarge guilty; but the presumptions of guilt and innocence can only be perfectly made by those who have the same national opinions and feelings as the accused. I think it very possible that a Frenchman may know a Frenchman to be guilty upon grounds which an Englishman would not understand; for instance, a particular act may be such as a Frenchman may know a Frenchman would not do, unless he had committed a murder before; and such acts may have been proved for aught I (who have not read much of the trial) can tell. It is the same thing in our courts of justice: judges and counsel can by experience make things which would appear to you or me almost indifferent, carry very positive conclusions to their own minds.

Now as to the part of your difficulty contained between X X in the remarks. It matters nothing (p. 210) whether $\frac{n-p}{p+1}x$ is negative or positive. If you perfectly understand why I neglect the sign in the fractional case, the same reason applies to the negative one. When $n$ is negative then $n-p$ ($p$ being essentially $+$) is necessarily negative. <u>Consequently</u>, ($x$ being positive) the terms alternate in sign from the very beginning, whereas when $n$ is positive and fractional, they do not begin to alternate until $p$ passes $n$. But our matter is to determine convergency, and if a series of positive terms be convergent, so will be the series of similar terms alternating. It is on the <u>absolute</u> <u>magnitude</u> of $\frac{n-p}{p+1}x$, independent of sign, that it depends whether the terms shall ultimately diminish so as to create convergence, or not.
Now the limit of this is $x$, whence follows as in the book
[18v] It is important to remember in results which <u>depend</u> <u>entirely</u> on limits that they have nothing to do with any vagaries which the quantity tending to a limit chooses to play, provided that, when it has sown its wild oats, it settles down into a steady approach to its limit. The sins of its youth are not to be remembered against it.
Now $\frac{n-p}{p+1}x$ when $n$ is positive, remains positive until $p$ passes $n$ and then settles into incurable negativeness. But when $x$ is ['positive' crossed out] negative, it is negative from the beginning
Now this matters nothing as to a result which depends only on the limit to which $\frac{n-p}{p+1}x$ approaches as $p$ is increased without limit.

As to the point marked $B$, remember that placing this doubtful assumption, namely the expansibility of functions of $x$ in whole powers of $x$, out of doubt by instances,

has been a prevailing vice of algebraical writers, and
one which is to be carefully avoided. It was once thought, by instance, that $x^2+x+41$ must be a prime number, whenever $x$ is a whole number, for

$$\begin{array}{ll}
x=0 & x^2+x+41 = 41 \text{ a prime n}^{\text{o}} \\
\ \ =1 & \qquad\qquad\quad = 43 \cdots\cdots\cdots \\
\ \ =2 & \qquad\qquad\quad = 47 \cdots\cdots\cdots \\
\ \ =3 & \qquad\qquad\quad = 53 \cdots\cdots\cdots \\
\ \ =4 & \qquad\qquad\quad = 61 \cdots\cdots\cdots \\
\ \ =5 & \qquad\qquad\quad = 71 \cdots\cdots\cdots \\
\ \ =6 & \qquad\qquad\quad = 83 \cdots\cdots\cdots \\
\ \ =7 & \qquad\qquad\quad = 97 \cdots\cdots\cdots
\end{array}$$

Now $x^2+x+41$, though it gives nothing but prime numbers up to $x=39$ inclusive, yet gives a composite number when $x=40$; for it then is

$$40\times 40+40+41 \quad \text{or} \quad 41\times 40+41 \text{ or } 41\times 41$$

and when $x=41$ it is

$$41\times 42+41 \quad \text{or} \quad 43\times 41$$

and for higher values it gives sometimes prime numbers sometimes not, like other functions. So much for instances.

C. The supposition as to the meaning of the non-arithmetical roots is right (Read from p. 109 "We shall now proceed" to p. 113 inclusive). When we use $\sqrt{-1}$, which we must do at present, if at all, without full explanation, it is to be remembered that we say two expressions are equal when they are <u>algebraically</u> the same, that is, when each side has all the algebraical properties of the other. ['M' crossed out] Numerical accordance must not be looked for when one or both sides are numerically unintelligible, and algebraical accordance merely means that everything which is true of one side is true of the other. It is then unnecessary to consider any restrictions which may be necessary when numerical accordance is that which is denoted by =.
But this is touching on even a higher algebra than the one before you
Suppose

$$\tfrac{1}{1-x} = 1+x+x^2+x^3+\cdots\cdot \text{ ad inf.}$$

which is certainly true in the arithmetical sense when [19v] $x<1$. But if $x>1$, say $x=2$, we have

$$\tfrac{1}{1-2} \text{ or } -1 = 1+2+4+8+16+\&c$$

which, arithmetically considered is absurd. But nevertheless
$-1$ and $1+2+4+8+$ &c have the same properties

This point is treated in the chapter on the meaning of the sign =.

My wife desires to be kindly remembered

I remain

Yours very truly

ADeMorgan

69 Gower St.

Thursday Ev$^{g}$ Oct$^{r}$ 15/40

It is fair to tell you that the use of divergent series is condemned altogether by some modern names of very great note. For myself I am fully satisfied that they have an algebraical truth wholly independent of arithmetical considerations; but I am also satisfied that this is the most difficult question in mathematics.

[20r] My dear Lady Lovelace

With regard to the error in Peacock you will see that you have omitted a sign. It is very common to suppose that if $\varphi x$ differentiated gives $\psi x$, then $\varphi(-x)$ gives $\psi(-x)$, but this should be $\psi(-x) \times$ diff.co.$(-x)$ or $\psi(-x) \times -1$. Thus

$$y = \varepsilon^x \qquad \frac{dy}{dx} = \varepsilon^x$$
$$y = \varepsilon^{-x} \qquad \frac{dy}{dx} = \varepsilon^{-x} \times (-1) = -\varepsilon^{-x}$$

As to the note, my copy of Peacock wants a few pages at the beginning by reason of certain thumbing of my own and others in 1825. I remember however that there is a note which I did not attend to, nor need you. But if curiosity prompts, pray sent it to me in writing.

As to $du = \varphi(x).dx$, you should not have written it $du = \varphi(x)$ as you proposed but

$$\frac{du}{dx} = \varphi x$$

The differential coeff$^{\text{t}}$ is the limit of $\frac{\Delta u}{\Delta x}$, and is a Total symbol. Those whose [*sic*] write

$$y = x^2 \quad \therefore \quad dy = 2xdx \text{ make an error}$$

but if $dy = 2xdx + \alpha$ be the truth, $\alpha$ diminishes without limit as compared with $dx$, when $dx$ diminishes. Consequently $\alpha$ is of no use in finding [20v] any limit, and those who use differentials, as they are called, do not differ at the end of their process from those who make limiting ratios as they go along. You can however for the present transform Peacock's formula $du = A\,dx$ into $\frac{du}{dx} = A$.

There is the erratum you mention in Alg. p. 225

As to p. 226

$$\frac{1+b}{1-b} = \frac{1+x}{x}$$
$$(1+b)x = (1-b)(1+x)$$
$$x + bx = 1 + x - b - bx$$
$$bx = 1 - b - bx$$
$$2bx + b = 1$$
$$(2x+1)b = 1 \qquad b = \frac{1}{2x+1}$$

Verification $$\frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x+1}} = \frac{2x+1+1}{2x+1-1} = \frac{2x+2}{2x}$$
$$= \frac{2(x+1)}{2x} = \frac{x+1}{x}$$

[21r] p. 212. To shew that for instance

$$n\,\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{n-3}{4}\,\frac{n-4}{5} + m\,n\,\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{n-3}{4} + m\,\frac{m-1}{2}\,n\,\frac{n-1}{2}\,\frac{n-2}{3}$$
$$+m\,\frac{m-1}{2}\,\frac{m-2}{3}\,n\,\frac{n-1}{2} + m\,\frac{m-1}{2}\,\frac{m-2}{3}\,\frac{m-3}{4}\,n + m\,\frac{m-1}{2}\,\frac{m-2}{3}\,\frac{m-3}{4}\,\frac{m-4}{5}$$

$= \overline{m+n}\ \frac{m+n-1}{2}\ \overline{\frac{m+n-2}{3}}\ \frac{m+n-3}{4}\ \frac{m+n-4}{5}$ without actual multiplication
$m$ and $n$ being whole numbers

1. $m\,\frac{m-1}{2}\,\frac{m-2}{3}\cdots\cdot\frac{m-(r-1)}{r}$ is the number of ways in which $r$ can be taken out of $m$ (see chapter on combinations in the Arithmetic)

If then we denote by $(a, b)$ the number of ways in which $a$ can be taken out of $b$, we have to prove that

$$(5, n) + (1, m) \times (4, n) + (2, m) \times (3, n) + (3, m) \times (2, n)$$
$$(4, m) \times (1, n) + (5, m) = (5, m + n)$$

Suppose ['the' crossed out?] $m + n$ counters to be divided into two parcels, one containing $m$ and the other $n$ counters

He who would take 5 out of them must either take

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 out of the | $m$ | and | 5 out of the | $n$ | |
| or | 1 | $\cdots$ | $m \cdots$ | 4 | $\cdots\cdots$ | $n$ |
| or | 2 | $\cdots$ | $m \cdots$ | 3 | $\cdots\cdots$ | $n$ |
| | 3 | $\cdots$ | $m \cdots$ | 2 | $\cdots\cdots$ | $n$ |
| | 4 | $\cdots$ | $m \cdots$ | 1 | $\cdots\cdots$ | $n$ |
| | 5 | $\cdots$ | $m \cdots$ | 0 | $\cdots\cdots$ | $n$ |

Now if to take say the third of these cases, we can take two of $m$ in $a$ ways and 3 out of $n$ in $b$ ways [21v] we can do both together in $a \times b$ ways. For if for instance there are 12 things in one lot and 7 in another, we can take one out of each lot in $12 \times 7$ ways, since any one of the twelve may come out with any one of the seven

Hence the number of distinct ['distinct' inserted] ways of bringing 2 out of $m$ and 3 out of $n$ together is

$$(2, m) \times (3, n)$$

I think you will now be able to make out that the preceding theorem is true when $m$ and $n$ are whole, whence, by the reasoning in the book it must be true when they are fractional.

This reasoning you do not see. It is an appeal to the nature of the method by which algebraical operations are performed. There is no difference of operation in the fundamental rules (addition subt$^{\text{n}}$ mult$^{\text{n}}$ & div$^{\text{n}}$) whether the symbols be whole nos or fractions. Hence if a theorem be true when the letters are <u>any</u> wh. nos, it remains true when they are fractions

For example, suppose it proved that for all whole nos

$$(a + b) \times (a + b) = a \times a + 2a \times b + b \times b$$

we should then, if we performed the operation $(a+b)\times(a+b)$ remembering that $a$ and $b$ are whole numbers find $a\times a+$ &c

$$\begin{array}{l} a+b \\ \underline{a+b} \\ aa+ab \\ \qquad +ab+bb \\ \hline aa+2ab+bb \end{array}$$

[22r] Now in no part of this operation are you required to stop and do ['or omit' inserted] anything because the letters are whole numbers which you would not do or not omit if they were fractions. Consequently, the reservation that the letters are whole numbers cannot affect the result which if true with it is true without. This principle requires some algebraical practice to see the necessity of its truth.

The notation of functions is very abstract. Can you put your finger upon the part of Chapt. X at which there is difficulty

The equation

$$\varphi(x)\times\varphi(y)=\varphi(x+y)$$

is supposed to be <u>universally</u> true for all values of $x$ and $y$. You have hitherto had to deal with equations in which <u>value</u> was the thing sought: now it is not <u>value</u>, but <u>form</u>. Perhaps you are thinking of the latter when it ought to be of the former.

With our remembrances to L.' Lovelace I am

Yours very truly

<u>ADeMorgan</u>

69 Gower St. No$^{\text{r}}$ 14/40

[24r] My dear Lady Lovelace

I send back your worked question.
The second is right the first wrong in two places

I should recommend you to get out of the habit of writing $d$ thus ['$d$' with flourish at top of stem] or thus ['$d$' with flourish and double stem]. If you have much to do with the Diff. Calculus, it will make a good deal of difference in time. The best way is to form all the letters as like those in the book as you can.

Pages 13–15 of Elementary Illustrations bear closely on the distinction of $\frac{du}{dx} = a$ and $du = adx$

It is not because $\overline{1+x}^m \times \overline{1+x}^n = \overline{1+x}^{m+n}$ when $m$ & $n$ are whole numbers that the same is true for fractions but because a certain other property which is therefore true makes it necessary in the case of fractions.

For instance, the logic is as follows

$A$ is true when $m$ is a whole number
Whenever $A$ is true, $B$ is true
$B$ is of that nature, that if true when $m$ is a whole number, it is also true when $m$ is a fraction.
When $B$ is true $C$ is true
$\therefore C$ is true if $A$ be true when $m$ is a whole number

Thus, when $m$ and $n$ are whole numbers,

$$\overline{1+x}^m = 1 + mx + m\,\tfrac{m-1}{2}\,x^2 + \text{\&c} \qquad \overline{\;1+x\;}|^n = 1 + nx + \text{\&c}$$

[24v] But $\overline{1+x}^m \times \overline{1+x}^n = \overline{1+x}^{m+n}$ always
Therefore when $m$ and $n$ are whole numbers
$(1 + mx + \text{\&c}) \times (1 + nx + \text{\&c}) = 1 + \overline{m+n}x + \text{\&c}$
but this last property (without any reference to its mode of derivation) is true of $m$ and $n$ fractional or negative if true of whole numbers.

Hence, if $1 + mx + \text{\&c}$ be called $\varphi m$

$$\varphi m \times \varphi n = \varphi(m+n)$$

But this (again by an independent process) is shown to be never universally true unless

$$\varphi m = c^m,\ c \text{ being independent of } m$$

whence $c^m = 1 + mx + m\,\frac{m-1}{2}\,x^2 + \cdots$.
But since $c$ is independent of $m$, what it is when $m = 1$, it is always. Therefore

$$c^1 = 1 + 1.x + 1.\tfrac{1-1}{2}\,x^2 + \cdots \qquad \text{or } c = 1 + x$$

---

Let $\varphi(xy) = x \times \varphi y$ be always true
required $\varphi x$

Make $y = 1$, then $\varphi(x) = x \times \varphi(1)$

$\varphi(1)$ is not yet determined; let it be $c$. The equation is either true for all values of $c$, or for some (not all)

$\varphi x = cx$

If any particular value were needed for $c$, it would be found by making $c = 1$. Do this and we have

$\varphi(1) = c$, or $c = c$, which is true for all values of $c$.

---

Suppose

$$\varphi(x) = \varphi(1).x + 2\varphi(1) - \overline{\varphi(1)}|^2$$

to be true for all values of $x$. It is then true when $x = 1$, giving

$$\varphi(1) = \varphi(1) + 2\varphi(1) - \overline{\varphi(1)}|^2$$

or $\overline{\varphi(1)}|^2 - 2\varphi(1) = 0$

or $\varphi(1) = 2$ only

$$\varphi(x) = 2x + 2.2 - 2^2 = 2x$$
$$\varphi(1) \qquad\qquad = 2$$

which agree with each other.

---

You must remember than [*sic*] when a form is universally true, it is true in all particular cases. Then in $2x = x + x$, I have a perfect right to say this is true when $x = 1$ and $\therefore 2 = 1 + 1$, and true when $x = 20$, or $40 = 20 + 20$: though I do ['do' crossed out] not thereby say that $x$ can be 10 and 1 both at once.

---

$$\frac{1}{m}\log z = \left(z^{\frac{1}{m}} - 1\right) - \frac{1}{2}\left(z^{\frac{1}{m}} - 1\right)^2 + \cdots.$$

when $m$ increases $z^{\frac{1}{m}} - 1$ diminishes
It is not $\log z$ which $z^{\frac{1}{m}} - 1$ approaches to, but $\frac{1}{m}\log z$, which also diminishes as $m$ increases
I see that in page 219 it is thus

$$\log z = \frac{1}{m}\left\{z^m - 1 - \frac{1}{2}\overline{z^m - 1}^n + \cdots\right\}$$

and $m$ diminishes without limit. Now as $m$ diminishes [25v] $\frac{1}{m}$ increases, and the assertion is that as $m$ diminishes, and therefore $z^m - 1$, the

product $\frac{1}{m}(z^m - 1)$
has one factor continually increasing & the other diminishing, so that their product approaches without limit to $\log z$.

In page 187, it is shown that if $x$ be made sufficiently small, any term of $a + bx + cx^2 + \cdots$ may be made to contain all the rest as often as we please, that is may be made as great as we please compared with the sum of all the rest. Consequently $m$ being small
$z^m - 1$ may be made as great as we please compared with the sum of all the terms of the series which follow, even if all were positive, still more when they counterbalance each other by difference of sign.

---

p. 205

If $\quad \varphi(x+y) = \varphi x + \varphi y \quad$ be <u>always</u> true (hypothesis)

It is true when $x = 0$

It is also true when $y = -x$

This equation being always true, is the representation of a collection of an infinite number of truths

I do not say that these truths coexist

Put it thus. Let $\varphi$ be such a function that, if $a$, $b$, $c$, $d$, &c be any quantities whatever

$\varphi(a+b) = \varphi a \times \varphi b$
$\varphi(b+c) = \varphi b \times \varphi c$
$\varphi(d+e) = \varphi d \times \varphi e \qquad$ &c &c

That is let $\varphi(x+y) = \varphi(x) \times \varphi(y)$
for all values of $x$ and $y$

1. Let $a = 0$, then $\varphi b = \varphi(0) \times \varphi(b)$
or $\varphi(0) = 1$

Let $b = -c$, then $\varphi(0) = \varphi(b) \times \varphi(c)$
or $1 = \varphi(b) \times \varphi(-b)$

and so on.

There is a want of distinction between an equation made true by choice of values and one which is true of itself, independently of all values

$x = 3 - x \qquad x = \frac{3}{2}$, and then only
$x = 3x + a - (2x + a)$ is true for <u>all</u> values of $x$, though it cannot have more than one at a time.

There is the erratum in the Trigonometry, as you say

Yours very truly
ADeMorgan

69 Gower St.
Friday Ev[g]

[27r] My dear Lady Lovelace

I can soon put you out of your misery about p. 206.

You have shown correctly that $\varphi(x+y) = \varphi(x) + \varphi(y)$ can have no other solution than $\varphi x = ax$, but the preceding question is not of the same kind; it is not show that there can be no other solution except $\frac{1}{2}\left(a^x + a^{-x}\right)$ but show that $\frac{1}{2}\left(a^x + a^{-x}\right)$ is a solution: that is, try this solution

$$\varphi(x+y) = \tfrac{1}{2}\left(a^{x+y} + a^{-x-y}\right)$$
$$\varphi(x-y) = \tfrac{1}{2}\left(a^{x-y} + a^{-x+y}\right)$$
$$\varphi(x+y) + \varphi(x-y) = \tfrac{1}{2}\left(a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y}\right)$$
$$\begin{aligned} 2\varphi x.\varphi y &= 2.\tfrac{1}{2}\left(a^x + a^{-x}\right).\tfrac{1}{2}\left(a^y + a^{-y}\right) \\ &= \tfrac{1}{2}\left(a^x + a^{-x}\right)\left(a^y + a^{-y}\right) \\ &= \tfrac{1}{2}\left(a^{x+y} + a^{-x+y} + a^{x-y} + a^{-x-y}\right) \end{aligned}$$

the same as before.

[27v] To prove that this can be the only solution would be above you

I think you have got all you were meant to get from the chapter on functions. The functional equations which can be fully solved are few in number

Yours very truly

ADeMorgan

69 G.S.
Mon$^{y}$ M$^{g}$

[29r] My dear Lady Lovelace
I have made some additional notes on your papers.
[diagram in original] The meaning of $\frac{\theta}{\sin\theta}$ is as follows
$\theta : 1$ and $1 : \sin\theta$ compounded
give it in arithmetic
In fact $\frac{a}{b}$ in arithmetic is another way of writing $a : b$.
In geometry $AB : AO$ is $\theta$[:]1
and $AO$ or $OB : BM$ is $\sin\theta$
The compounded ratio is that of $AB : BM$ which approaches without limit to the ratio of 1 to 1 as $AB$ is diminished
Your notion of the ratio approximating to unity is correct. The term 'ratio approximating to $a$' is a mixture of the geometrical and [29v] arithmetical mode of speaking, it should be 'ratio approximating to
$a : 1$.
I think you have got over the difficulty of that part of the subject
I was sorry to have been out when Lord Lovelace called, and could not get down to S[t] James' Square till you had gone. With best remembrances I am
Yours very truly
ADeMorgan

[31r] My dear Lady Lovelace

You are right about the writing down of the terms:

$$\frac{z}{(2n-2)(2n-3)}$$

is the $n$th term divided by the $(n-1)$th and the $\overline{n+1}$th divided by the $n$th is $\frac{z}{2n(2n-1)}$ as you make it.

If I understand you correctly [31v] you are now satisfied about all the rest

Suppose you try at what term convergency begins in the following series

$$1+\frac{x}{2.4}+\frac{x^2}{2.4.6.8}+\frac{x^3}{2.4.6.8.10.12}$$
$$+\cdots\cdots$$

when $x=100,000$

With remembrances to Lord Lovelace

I remain

Yours truly

<u>ADeMorgan</u>

69 G.S.

Thursday

[32r] You will see the alteration I have made in your paper If you do not see it clearly, write again for the sort of point contained in it is one of importance.

[33r] My dear Lady Lovelace

I return the papers about series which
are all right, the old one is as you suppose

With reference to your remarks on the diff[l] calculus

1. You observe that

$$\frac{\varphi(x+n\theta+\theta)-\varphi(x+n\theta)}{\theta}$$

differs from $\frac{\varphi(x+\theta)-\varphi(x)}{\theta}$
in that while $\theta$ diminishes, $x+n\theta$ varies. So it [second 'it' crossed out] is,
and if $n$ be finite and fixed, it might be shown that the
limits of the two are the same. But if $n$ increase while
$\theta$ diminishes, in such manner that $n\theta$ is either equal to
or approaches the limit $a$, then the first fraction has
the same limit as $\frac{\varphi(x+a+\theta)-\varphi(x+a)}{\theta}$

To illustrate this, let $\varphi x$ be the ordinate of a
curve, the abscissa being $x$. If $x$ remains fixed, the triangle
[diagram in original] (blotted) diminishes without
limit with $\theta$; but if while
$\theta$ diminishes, the point $A$ moves
in toward $B$, so as continually to
approach $B$, and to come as near as
[33v] we please to it, and yet never absolutely to reach
$B$ as long as $\theta$ has any value, it is obvious that
the small triangle would ride along the curve,
perpetually diminishing its dimensions, and
continually approaching in figure nearer and
nearer to the figure of as small a triangle at $B$.
All this necessarily follows from the notion
of continuity
[diagram in original]

2. You want to extend what I have said
about continuous functions to all possible cases,
not being able to imagine a function which changes
its values suddenly. But for this you must
wait till you come to the mathematics of disconti-
nuous quantity. It is perfectly possible though the
calculation would be laborious, to find an algebraical
function which from $x = 1$ to $x = 2$ increases like the
ordinate of a straight line, from $x = 2$ to $x = 3$ draws the
[diagram in original] likeness of a human profile in a
different place, from $x = 3$ to $x = 4$
draws a part of a circle, from
$x = 4$ to $x = 5$ is nothing, and from

[34r] $x = 5$ to $x = 6$ makes any odd combination of lines or curves, perfectly irregular. None of the notions incidental to continuity must be applied to such a function

3. Your proof of the diff.co. of $x^n$ is correct, <u>but it assumes the</u> <u>binomial</u> <u>theorem</u>. Now I endeavor to establish the diff.calc. without any assumption of an infinite series, in order that the theory of series may be established upon the differential calculus

Besides, if you take the common proof of the binomial theorem, you are reasoning in a circle, for that proof requires that it should be shown that $\frac{v^n - w^n}{v-w}$ has the limit $nv^{n-1}$ as $w$ approaches $v$. This is precisely the proposition which you have deduced <u>from</u> the binomial theorem.

Pray send your point about the exponential theorem.

And thank Lord Lovelace for pheasants and hare duly received this morning

Yours very truly

<u>ADeMorgan</u>

69 G.S. Wed$^{\text{y}}$

[35r] My dear Lady Lovelace

We shall be happy to
see you on Monday Evening, and
Lord Lovelace too if he be not
afraid of the algebra
Your points in your letter are
are [should be 'I'] think, clear enough in
your own head. A little addition
however may be made as follows.

You are not [something crossed out] to think that
because $x$ must be diminished
without limit to prove a conclusion
that conclusion is only true for
small values of $x$, or for $x = 0$.

For example suppose I know that

$$(a+x)(a-x) = P + Qx + Rx^2$$

but of $P$ $Q$ and $R$ <u>I only know that
they are independent of $x$</u>. What
[35v] therefore they are for any
one value of $x$, they are for
any other. I find them thus
Since the preceding is by hypothesis
true for <u>all</u> values of $x$, and
since altering $x$ does not alter
$P$ $Q$ or $R$, I take $x = 0$
to begin with

$$a^2 = P \text{ when } x = 0$$

but $a$ and $P$ are independent
of $x$, therefore what relation
exists when $x = 0$ exists always
or $\quad a^2 = P$
Let $x = a$

$$0 = P + Qa + Ra^2$$

Let $x = -a$

$$0 = P - Qa + Ra^2$$

[something crossed out] subtract $\qquad 2Qa = 0$ or $Q = 0$
Here are two values of $x$ made use
[36r] of.

Add

$$2P + 2Ra^2 = 0$$

$$R = -\frac{P}{a^2} = -1$$

whence

$$\begin{aligned}(a+x)(a-x) &= a^2 + 0.x - x^2 \\ &= a^2 - x^2\end{aligned}$$

<u>if</u> it must be of the form

$$P + Qx + Rx^2$$

We are much obliged by your
invitation to Ockham, but
I am closely tied up by
lectures & other things. Even
at such times as Xtmas I am
generally very busy
With kind remembrances to
Lord Lovelace I am

Yours very truly
<u>ADeMorgan</u>

[37r] My dear Lady Lovelace
If you look back to page 48, you will there see that $\frac{a+a'+a''+\cdots}{b+b'+b''+\cdots}$ always lies between the greatest & least of $\frac{a}{b}$[,] $\frac{a'}{b'}$ &c whatever the signs of $a$[,] $a'$ &c may be, provided that $b$, $b'$ &c are all of one sign. That is the reason why $\varphi x$ need not continually increase or decrease in the next chapter

The paper you have sent me is correct. In page 70, the reasons are given for [37v] avoiding the common proof of Taylor's Theorem, and 71 &c contains the amended proof.

Of $\frac{\varphi(a+h)}{\psi(a+h)} = \frac{\varphi'(a+\theta h)}{\psi'(a+\theta h}$ [bracket missing in last denominator] it cannot only be said that it turns out useful. A beginner can hardly see why a diff$^{l}$ coeff$^{t}$ itself should be of any use

Yours truly

ADeMorgan

Feb$^{y}$ 6/41

[38r] My dear Lady Lovelace

I have added a note or two to your papers.

As to the subject of continuity, it must be as much as possible your object now to remember while proving the things which are true of continuity to remember that they are not false of ['conti' crossed out?] dis continuous [*sic*] functions, because true of continuous ones. Thus, you will afterwards see that

$$\varphi(a+h) = \varphi a + \varphi'(a+\theta h).h$$

is only an algebraical translation of the following geometrical theorem

"Every continuous and ordinary arc of a curve has somewhere a tangent parallel to its chord"

[38v] [diagram in original]

But this is not always false of discontinuous curves

[diagram in original]

Neither is the algebraical theorem false of them.

The best way at present, is to mark that discontinuous functions are now excluded only because we have no language to express them in. This will come intime [*sic*] : you will have enough of them when you come to apply math^cs to the theory of heat.

My wife has duly received [39r] your letter & is much obliged to you & Miss King.

Yours truly

ADeMorgan

69 G.S.

Feb^y 11/41 [?]

[40r] My dear Lady Lovelace

There is a misprint
in page 83, I find, line 6 from
the bottom. For $x + w$ read
$x + 1$. Your result is correct
for an in crement [*sic*] [something crossed out] $w$.

My last letter was
all ready when yours arrived
and I had no time to make
any addition. My wife
begs me to give her kind
regards and to say that she
knows her mother will
wish her to be with her
[40v] on Sunday Evening (the day
after Mr. Frend's funeral) so
that she will be glad if you
can come any other evening
in the week

With kind remembrances
to Lord Lovelace

I remain

Yours very truly

ADeMorgan

69 Gower St.
Wed$^{y}$ Ev$^{g}$

[41r] [blank]

[42v] [Small note — either where letter was addressed after folding or simply a librarian's label]

A DeMorgan
to [?] Lady L

[42r] My dear Lady Lovelace

Mr. Frend's death (which took place on Sunday Morning) has made me answer your letter later than I should otherwise have done. The family are all well, and have looked forward to this termination for some time. My wife will answer your letter on a part of this.

Number and Mag$^{\text{n}}$. pp. 75, 76. The use of this theorem is shown in what follows. It proves that any quantity which lies between two others is either one of a set of mean proportionals between those two, or as near to one as we please. It is not self evident that the base of Napiers system, as given by himself is $\varepsilon$ or $1+1+\frac{1}{2}+\cdots$ as we learn from the modern mode of presenting the theory. The last sentence in the book (making V a linear unit) would show that Napier's notion was to take $k$ in such a manner that $x$ shall expound [?] $1+x$ ['without' crossed out] or rather that the smaller $x$ is the more nearly shall $x$ expound $1+x$. If this were accurately done, we should have

$$k^x = 1 + x \quad \text{or} \quad \frac{k^x-1}{x} = 1$$

and this is to be nearer to the truth the smaller $x$ is. Now when the common theory is known, it is known that $k=\varepsilon$ gives

$$\frac{\varepsilon^x-1}{x} = 1 + \frac{x}{2} + \frac{x^2}{2\cdot 3} + \cdots \cdot \qquad \text{and limit of } \frac{\varepsilon^x-1}{x} = 1$$

while $\frac{k^x-1}{x} = \log k + \frac{\overline{\log k}|^2 . x}{2} + \cdots$ and limit of $\frac{k^x-1}{x} = \log k$

where the log. has this very base $\varepsilon$. Having proved these things, it is then obvious that, $\log k$ being never 1 except when $k$ is the base or $\varepsilon$, the last paragraph cannot consist with any other value of $k$ except $\varepsilon$. In this book (Num & Mag.) I must refer you to the algebra, which I do not in the Diff. Calc., many matters of series, until the whole doctrine is reestablished.

Now ['as' crossed out] as to the Diff. Calc. You do not see that $\theta$ is a function of $a$ and $h$. Let us take the simplest case of the original theorem which is

[42v] $\qquad \varphi(a+h) = \varphi a + h\ \varphi'(a+\theta h) \qquad (1)$

Now 1. Why should $\theta$ be independent of $a$ and $h$, we have never proved it to be so : all we have proved is that <u>one</u> of the numerical values of $\theta$ is $< 1$, or that this equation (1) can be satisfied by a value of $\theta < 1$. As to what $\theta$ is, let $\psi$ be the inverse function of $\varphi'$ so that $\psi\varphi' x = x$. Then

$$\frac{\varphi(a+h)-\varphi a}{h} = \varphi'(a+\theta h)$$

$$\psi\left(\frac{\varphi(a+h)-\varphi(a)}{h}\right) = \psi\varphi'(a+\theta h) = a + \theta h$$

$$\theta = \frac{\psi\left(\frac{\varphi(a+h)-\varphi a}{h}\right)-a}{h}. \left\{ \begin{array}{l}\text{Say that this is not a function}\\ \text{of } a \text{ and } h\text{, if you dare}\end{array}\right.$$

For example $\quad \varphi x = c^x$

$\qquad \varphi' x = c^x . \log c$

$c^{a+h} = c^a + h \log c\ c^{a+\theta h}$
$c^{a+\theta h} = \frac{c^{a+h}-c^a}{h \log c}$
$(a+\theta h)\log c = \log \frac{c^{a+h}-c^a}{h \log c}$
$\theta = \frac{\log \frac{c^{a+h}-c^a}{h \log c} - a \log c}{h \log c}$
$\quad = \frac{\log(c^h-1) - \log(h \log c)}{h \log c}$

In this particular example $\theta$ happens to be a function of $h$ only, not of $a$ : but you must remember that in every case where we speak of a quantity as being generally a function of $a$, we do not mean thereby to deny that it may be in particular case, not a function of $a$ at all : just as [43r] when we say that there is a number ($x$) which satisfies certain conditions, we do not thereby exclude the extreme case in which $x = 0$.

Look at the question of differences in this manner. Any thing which has been proved to be true of $u_n$ relatively to $u_{n-1}$[,] $u_{n-2}$ &c has also been proved to be true of $\Delta u_n$ relatively to $\Delta u_{n-1}$[,] $\Delta u_{n-2}$ &c. For in the set

$$\begin{array}{llll} u_0 & & & \\ & \Delta u_0 & & \\ u_1 & & \Delta^2 u_0 & \\ & \Delta u_1 & & \&c \\ u_2 & & \Delta^2 u_1 & \\ & \Delta u_2 & & \\ u_3 & & & \end{array}$$

the first column may be rubbed out and the second column becomes the first &c. It is obvious that the $m+1$, $m+2$, &c columns are formed from the $m$th precisely as the 2nd, 3rd &c are formed from the first. If then I show that up to $n = 7$, for instance

[$u$ ... crossed out] $u_n = u_0 + n\Delta u_0 + \cdots\cdots$

I also show (writing $\Delta u_n$ for $u_n$) that $\Delta u_n = \Delta u_0 + n\Delta(\Delta u_0) + \cdots$

Perhaps you had better let the question of discontinuity rest for the present, and take the result as proved for continuous functions. You will presently see in a more natural manner the entrance of discontinuity

The paper which I return is correct

Yours very truly
ADeMorgan

69 Gower St
Mond$^{y}$ Ev$^{g}$

[43v] [Note to AAL from SDM]

[45] [in ADM's hand]

$\int_a^{a'} f\,ds$

$\int_a^{a'} \varphi s.ds$ $\quad$ $\varphi s$ meaning $f$

$\int f\,ds$

$\int \frac{dv}{dt}.ds$

$\int \frac{dv.ds}{dt}$

Negative & Impossible Qu. $\int dv.\frac{ds}{dt}$

Operation $\int dv.v$

Relation $\int v\,dv$ [diagram in original]

$v^2 = 2\int_a^{a'} f\,ds + C$ $\qquad$ $dy/dx$

$V^2 = 0 + C$

$v^2 - V^2 = 2\int_a^{a'} f\,ds$

$v^2 = 2\int_a^{a'} f\,ds + V^2$

$v^2 = 2\int f\,ds + C$

[46r]
[something written vertically here — belongs at end of letter so transcribed there]

Ockham
Monday. 4th Sepr

Dear Mr De Morgan
Will you send on the enclosed to Mrs De Morgan. It explains the arrangements I have made in case she comes here, & also that Lord L__ & myself have delayed our own departure until Satdy next. Our household & children however are already gone.

Now to mathematical business : I think you have hit the right nail on the head, altho' my confused notion of Differentials was [46v] not the only piece of puzzle & mistiness which constituted the impediments towards my comprehension of $X\frac{du}{dx} + Y\frac{du}{dy} = U$. The rectifi==cation of this however, has I believe given me the key to the remaining difficulties. If you will be kind enough to read the ['following' crossed out] enclosed observations, you will be able to judge how far I now take a just view of the matter.

I find that when (a long time ago) I studied Chapter V, I never gave due importance to the conclusion

$$\Delta.u = \frac{du}{dx_1}\Delta x_1 + \frac{du}{dx_2}\Delta x_2 + \frac{du}{dx_3}\Delta x_3 + \&c + \{(\Delta x)^2, (\Delta x_1 \Delta x_2), \&c, \&c\}$$

deduced at the bottom of page 87, as the result of pages 86, 87. My [47r] whole attention was given to the subsequent theorem of page 90, "If $u$ be a function of $t$ in different

"ways &c, &c", which I con=
=ceived to be the only object in
view, & that the equation of
page 87 was of no consequence
in itself, but merely means to
an end. This seems to have been
an egregious blunder, since the
whole theory of Differentials
rests on the very part which
I ['comparatively' inserted] neglected, from ['fancying it' inserted] a merely subsidiary
theory. I believe I am not wrong
in this present view of the
matter.

I cannot help here remarking
a circumstance which I ['believe' crossed out] think is
almost invariably true respecting
all my difficulties & confusions
in studying. They are without any
[47v] exception that I can recal [*sic*], from
misapprehension of the meaning
of some symbol, or ['of' inserted] some phrase
or definition ; & on no occasion
from either any error in my
reasoning, or ['from' inserted] any difficulty in
carrying on ['any' crossed out] chains of deductions
correctly, however complicated or
profound or lengthy ['these may be' inserted]. I therefore
have lately begun to ask myself,
whenever I am stopped, whether
I clearly understand what the
subjects of the reasoning are ; &
to go carefully over every verbal
[something crossed out] & symbolic representative of a
thing or an idea, with the
question respecting each, "now what
"does it mean, & how was it got?
"Am I sure of this, in each instance
"involved in the subject?" This
may save me much future trouble.

I will send you my remarks tomorrow,
[vertical text on 46r] as I want to look
over them once more

first ; & today I
have had enough of
these subjects.

Yours very truly

A.A.L

[48r]

Ashley
Sun$^{dy}$ 13$^{th}$ Sep$^{r}$
[1840] [added in pencil by later reader]

Dear M$^{r}$ De Morgan

I am very much obliged by your remarks & additions. I believe I now understand as much of the points in question as I am intended to understand <u>at present</u>. I am much inclined to agree with the paragraph in page 48 ; for though the conclusions must be [48v] admitted to be most perfectly correct & indisputable, logically speaking, yet there is a something <u>intangible</u> & a little unsatisfactory too, about the proposition. ____

I expect to gain a good deal of new light, & to get a good <u>lift</u>, in studying from page 52 to 58 ; ____ though probably I shall be a long time about this. I could wish I went on quicker. That is __ I wish a human head, or <u>my</u> head at all events, could take in [49r] a great deal <u>more</u> & a great deal more <u>rapidly</u> than is the case ; __ and if <u>I</u> had made my own head, I would have proportioned it's [*sic*] wishes & ambition a little more to it's [*sic*] capacity. ____ When I sit down to study, I generally feel as if I could <u>never</u> be tired ; _ as if I could

go on for ever. _ I say
to myself constantly, “Now today
I will get through so & so”;
and it is very disappointing
to find oneself after an
hour or two quite wearied,
& having accomplished perhaps
[49v] about one twentieth part of
one’s intentions, _ perhaps not
that. When I compare
the very little I do, with the
very much _ the infinite I
may say _ that there is to
be done ;_ I can only
hope that hereafter in some
future state, we shall be
cleverer than we are now. _
I am
afraid I do not understand
what you were kind enough
to write about the Curve;
and I think for this reason,
that I do not know what
[164r] the term equation to a curve
means. __ Probably with some
study, I should deduce that
meaning myself ; but having
plenty else to attend to of
more immediate consequence,
I do not like to give my
time to a mere digression
of this sort. _ I should
much like at some future
period, (when I have got
rid of the common Algebra
& Trigonometry which at
present detain me), to
attend particularly to this
subject. __ At present, you
[164v] will observe I have four
distinct things to [something crossed out]
carry on at the same

time ; _ the Algebra ; ____
Trigonometry ; __ Chapter 2nd of the
Differential Calculus ; _ & the
mere practice in Differentiation.

This last reminds me
that my bookseller has at
last & with much difficulty
got me Peacock's Book ; &
I hope it will be of
great use, for it's [*sic*] cost is
£2..12..6 ! __ It was
originally 30s. _ It is
[163r] coming here next week.

By the bye I have a
question to ask upon pages
203 & 204 of the Algebra.
In consequence of a reference
to page 203, in the 9th line
of the 25th page of the
Trigonometry, I was induced
to look & see what it
related to. Reading on
afterwards to the bottom of
the page, I found
"A f u n c t i o n a l e q u a t i o n is an
"equation which is necessarily
"true of a function or functions
"for every value of the letter
"which it contains. Thus if,
[163v] "$\varphi x = ax$, we have $\varphi(bx) =$
"$abx = b \times \varphi x$, or
"$\varphi(bx) = b\varphi x$"
"is always true when $\varphi x$
"means $ax$." _
So far I think is clear ;
but then what follows, _
"Thus &c
"If $\varphi x = x^\alpha$ $\qquad \varphi\alpha \times \varphi y = \varphi(\alpha y)$
" $\varphi x = a^x$ $\quad \ldots \quad \varphi x \times \varphi y = \varphi(x+y)$
" $\varphi x = ax + b$ $\ \ldots \ \frac{\varphi x - \varphi y}{\varphi x - \varphi z} = \frac{x-y}{x-z}$
" $\varphi x = ax$ $\qquad \varphi x + \varphi y = \varphi(x+y)$

I cannot trace the connection. I suppose there is something I have not understood, in the explanation of the Functional Equation. _ I hope before very long to have something further to send you upon Chapter 2$^{nd}$ of the Calculus, either of success or of enquiry. __

Has M$^{rs}$ De Morgan returned yet, & how is M$^{r}$ Frend? _

With many thanks,

Yours very truly

A. A. Lovelace

[50r]

S$^{t}$ James' Sq$^{re}$
Saturday Morning

Dear M$^{r}$ De Morgan. I
hope you have not by
this time come to the
conclusion that I have
drowned the Differential
Calculus at least (if not
myself with it also) in
the Seine or the Channel.
I am setting about work
again now in good
[50v] earnest. To say the truth
I find myself a little
bit rusty & awkward.
I have been some days
getting my head in again,
& it is not yet even
at all what it was.
However this will soon
mend, I doubt not. I
very easily lose a
habit, but then I very
easily re-form it. But
I am a little vexed at
this interruption. I was
going on so nicely. I
[51r] now write because I am
in need of a little help
from you. Some points
from page 100 to page
103 are not perfectly
clear to me ; & as my
head is not at this
moment in it's [*sic*] best
working habits, I think
instead of plaguing myself
more about the misty
parts, I had better
apply to you ; for I

am inclined to get a little worried about it, [51v] which is always to be avoided.

I should prefer seeing you to writing, on this occasion. I have at present no evening ; for tomorrow (Sun$^{dy}$) I happen to have some engagements at home.

But I would suggest, (if it is not trespassing too much on your time & kindness) that I will send my carriage to bring you here on Tuesday Morning at $\frac{1}{2}$ past [52r] eleven o'clock. I propose sending for you thus, in order to facilitate your coming. Or would you prefer instead that I should go to Gower S$^{t}$ at that hour on Tues$^{dy}$?

Will you tell M$^{rs}$ De Morgan, with my kindest remembrances, that (if agreeable to her) I rather hope to be able to spend Sunday Evening (I mean tomorrow week) with her & you But my arrangements ['for that time' inserted] are not yet quite fixed ; so I am not sure yet what I shall be able to do.

Pray believe me

Yours very truly

A. A. Lovelace

[54r]

S^t James' Square
Friday Morning

Dear M^r De Morgan. I
send you a large packet
of papers :
1 : Some Remarks & Queries on
the subjects of a portion of
pages 75 & 76 (Differential Calculus)
2 : an Abstract of the demonstration
of the Method of finding the
$n^{\text{th}}$ Differential Co-efficient by
means of the Formula
Limit of $\frac{\Delta^n u}{(\Delta x)^n} = u^{(n)}$
[54v] 3 : Some objections & enquiries
on the subjects of pages 83,
84, 85. _
4 : Two enquiries on two
Formulae in page['s' crossed out] 35 of the
"Elementary Illustrations". __
In addition to all
this, I have a word to
say on two points in your
last letter.
Firstly : that $\theta$ is a function
of $a$ & $h$, (or in all cases
a function of one at any
rate of these quantities), is
very clearly shown by you
in reply to my question.
But I still do not see exactly
[55r] the use & aim of this fact
being so particularly pointed
out in the parenthesis at the
top of page 80. It does
not appear to me that
the subsequent argument is
at all affected by it. __
Secondly : I still am not
satisfied about the Logarithms,
I mean about the peculiarity

which constitutes a Naperian
Logarithm in what I call
the Geometrical Method, _
the method in your
Number & Magnitude. I
am [‘now’ inserted] satisfied of the following :
that there is nothing in
the Geometrical Method to
[55v] lead to the precise determination
of $\varepsilon$ ; _ that $\varepsilon$ is arrived
at by other means, Algebraical
means ; & then identified
with the $k$ on $HL$ of the
Geometrical Method. What
constitutes a Naperian Logarithm
in the Geometrical view, is
“taking $k$ so that $x$ shall
“expound $1 + x$, or rather that
“the smaller $x$ is, the more
“nearly shall $x$ expound $1 + x$.
But in this definition there
are two points that are still
misty to me : I do not
see in what, (beyond the
mere fact itself), these
Logarithms differ from those
[56r] in which $x$ does not
expound $1 + x$. I cannot
perceive how this one
peculiarity in them, involves
any others, or imparts to
them any particular use,
or simplicity, not belonging
to other logarithms. _
Also, I do not comprehend
the doubt implied as to
the absolute theoretical
strictly-mathematical existence
of a construction in which
$x$ shall expound $1 + x$.
It appears to me that,
whether practically with a

pair of good compasses, or
theoretically with a pair of
[56v] mental compasses, I can as
easily as may be take any
[diagram in original]
line I please $MQ$ greater than
$OK$ or $V$, measure their difference
$PQ$ which call $x$, then on
$OH$ $(= OK)$ lay down a portion
$OM$ equal to this difference $x$
(not that I pretend this is
correctly done in my figure,
which is only roughly inked
down at the moment), &
['finally' inserted] stick up $MQ$ on the point
$M$. Then $x$ expounds $MQ$ or
[57r] $V + x$, or $1 + x$. I can see
no difficulty in accomplishing
this, or any reason why these
can be only an approximation
to it. Neither do I very
clearly perceive that the
Base $k$ would be
necessarily influenced by this
proceeding. __
In short I take the real
truth to be that this view
of Exponents being wholly new
to me, there is some little
link which has escaped me,
or to which at any rate I
have not given it's [*sic*] due
importance. But I think
I have now fully explained
[57v] what it is that I do not
understand.
Believe me
Yours very truly
A. A. Lovelace

[I think this is in fact a letter to *Mrs.* De Morgan. The reference in the letter to *Mr.* De Morgan is pretty clear evidence for this. As for the salutation, the 'r' of what appears to be 'Mr' curls back on itself very slightly, in a way that is suggestive of AAL's way of writing 'Mrs'. Compare what we find here with, for example, her 'Mr' in Box 170, fol. 54r, and the 'Mrs' in Box 171, fol. 5r. Other points to mention, although these are by no means conclusive, are that the letter is much chattier than her usual letters to ADM, and that the letter ends 'Yours most sincerely', which, based on a glance through the letters from Box 171, was AAL's usual way of ending letters to Mrs. De Morgan — her standard sign-off to Mr. De Morgan seems to have been 'Yours most truly'.]

[58r]

S$^{t}$ James' Square
6 o'clock
Sunday Evening

My Dear M$^{rs}$ De Morgan

I am very sorry
indeed at the extremely
dispiriting account you
give of yourselves ; & also
I am much disappointed
at losing the pleasure
of spending this evening
with you. __ I was
out when your note arrived,
[58v] but I hear from the
servants that it came
just after half past four
o'clock. __
Can I, (as far as you
at present foresee), spend
<u>next</u> Sunday Evening
with you? I hope so,
for I believe it will
be my last ['Sunday' inserted] previous to
going to Paris ; & I
have no week-day evening
disengaged to offer. ____

[59r] Another of the little
mathematical <u>epochs</u>, that
are so interesting to me,
has arrived ; I mean

another Chapter of the
Differential Calculus is
just satisfactorily completed ;
& I have a whole bundle
of papers to submit to
M$^{r}$ De Morgan upon it,
& one or two questions
to ask on some trifling
points in it that are
not perfectly clear. But
[59v] as it is not of a nature
that immediately presses,
all this may I think
wait till I can come
to you. _ Meantime I
shall begin Chapter VI,
on Integration &c. ____

I have since I
saw you had such a
wonderful quantity of
occupations of various
sorts to get thro', that
I feel a little surprised
how I have managed to
[60r] ['got thro" crossed out] accomplish, even this much
of the mathematics. __
However I do not think
anything will ever manage
to oust the latter. Indeed
the last fortnight is
rather a convincing
proof that nothing can.

I have been out
either to the Opera,
German Opera, or
somewhere or other, every
night. I have had
music-lessons every morning,
[60v] & practised my Harp too
for an hour or two ; &
I have been on horseback
nearly every day also. I

might add many sundries
& et-ceteras to this list. _
I must however maintain
that the Differential
Calculus is king of the
company ; __ & may it
ever be so ! __

Believe me

Yours most sincerely

A. Ada Lovelace

[62r]
Ashley-Combe
$10^{\text{th}}$ Nov r

Dear Mr De Morgan. The
last fortnight has been spent
in total idleness, mathematically
at least ; for we have had
company & been as they say
gadding about. __ I must
set too [*sic*] now & work up
arrears. _ But I have a
batch of questions & remarks
to send. __
First — on Peacock's Examples,
which I have only now begun
[62v] upon:
What does he mean by adding
$d\,x$ to every solution? _ It
appears to me a work of
supererogation. _ I take the
very first example in the
book as an instance, and
the same applies to all : _
Let $u = ax^3 + bx^2 + cx + e$:
it's [*sic*] differential, or $du =$
$= 3ax^2dx + 2bx\,dx + cdx$
or $(3ax^2 + 2bx + c)\,dx$.
I should have written, & in
fact did write : it's [*sic*] differential
or $du = 3ax^2 + 2bx + c$.
I suppose that this form
[63r] is used under the supposition
that $x$ itself may be a
function. ____
My result & the book's do
not agree in one particular
in the $9^{\text{th}}$ example, page 2,
& I am inclined to think
it is a misprint in the
latter : the Books says :
Let $u = x^2(a + x)^3(b - x)^4$
$du = \{2ab - (6a - 5b)x - 9x^2\}x(a + x)^2(b - x)^3dx$

and I say :
$du = \{2ab - (6a - 5b)x - x^2\}x(a + x)^2(b - x)^3dx$
In case it may save you
trouble, I enclose my working
out of the whole. __
I do not the least understand
[63v] the note in page 2. Not
one of the three theorems it
contains is intelligible to me.
I conclude you to have
the Book by you ; but if
not I can copy out the
note & send it to you. __
Secondly — to go to your
Algebra : I think there is
an evident erratum page 225,
line 8 from the bottom, where
$1 + x + \frac{x-\frac{1}{n}}{2} + \frac{x-\frac{1}{n}}{2} \cdot \frac{x-\frac{2}{n}}{3} + \&c$
should certainly be
$1 + x + x\,\frac{x-\frac{1}{n}}{2} + x\,\frac{x-\frac{1}{n}}{2} \cdot \frac{x-\frac{2}{n}}{3} + \&c.$
I have a little difficulty
in page 226, the last line,
[64r] "let $\frac{1+b}{1-b} = \frac{1+x}{x}$ which gives $b =$
"$= \frac{1}{2x+1}$". —
In the first place I do not
feel satisfied that the form
$\frac{1+b}{1-b}$ is capable of being
changed into the form
$\frac{1+x}{x}$. There are three
suppositions we may make
upon it , (supposing that
it is capable of this second
form) . $x$ may be less
than $b$, in which case
the denominator must also
be less than $1 - b$, and less
in a certain given proportion,
in order that the Fractional
[64v] Expression may remain the
same . _ $x$ may $= b$, in
which case the second form
can only be true on the

supposition that $1 - b = x =$
$= b$ , or $b = \frac{1}{2}$. _
$x$ may be <u>greater</u> than $b$,
in which case the denominator
of the second form must also
be <u>greater</u> than $1 - b$, in
a certain given proportion,
in order that the Fractional
expression may remain the
same. __
But secondly supposing $\frac{1+b}{1-b}$
to be under all circumstances
[65r] susceptible of the form $\frac{1+x}{x}$ ,
I cannot deduce from this
equation $b = \frac{1}{2x+1}$ . ____
Your last letter, on
the Binomial Theorem, was
quite satisfactory to me, but
I have some remarks to make
on the s<u>econd</u> proof of it,
pages 211 to 213. I think
you <u>well</u> observe in the
note page 213, that the two
proofs supply each other's
deficiencies ; for I like neither
of them t<u>ake</u>n <u>singly</u>. __
The latter one is what I
should call rather cumbrous,
especially the verification of
$\varphi n \times \varphi m = \varphi(n + m)$ by
[65v] actual multiplication in page
212, which is an exceedingly
awkward & inconvenient
process in my opinion. __
Then I am not at all
sure that I like the
assumption in the last
paragraph of page 212. __
It seems to me somewhat
a large one , & much
more wanting of proof than
many things which in

Mathematics are rigorously &
scrupulously demonstrated.
But these inconsistencies
have always struck me
occasionally, and are perhaps
only in reality the inconsistencies
[66r] in a beginner's mind , &
which long experience &
practice are requisite to do
away with. __
The end of Euler's proof,
page 213, is not agreeable
to me, and for this reason,
that I cannot feel properly
satisfied as yet with the
little Chapter on Notation
of Functions, and upon the
full comprehension of this
depends the force of the
latter part of this proof. __
I do not know why it
is exactly , but I feel I
only half understand that
[66v] little Chapter X , and it has
already cost me more trouble
with less effect than most
things have . I must study
it a little more I suppose.
I hope soon I
may be able to return to
your Differential Calculus. _
At the same time , I never
more felt the importance of
not being in a hurry. _
I fancy great proficiency
in Mathematical Studies is
best attained by time ; __
constantly & continually doing
a little . _ If this is so,
surely then the University
[67r] cramming system must be
very prejudicial to a real

progress in the long run,
particularly when one considers
how very very little School-boys
are [‘generally’ inserted] prepared on first going
to the Universities, with
anything like distinct
mathematical or even
arithmetical notions of the
most elementary kind. __

I am now
puzzling over the Composition
of Ratios , but I hope in
a day or two more I shall
get successfully over that. __
It plagues me a good deal.
[67v] I believe I thought some
years ago, that I understood
it ; but I am inclined to
think I certainly never did.

You see just at
this moment I am full of
unsatisfactory obstacles ; but
I doubt not they will soon
yield . ____
With kindest remembrances to
M^rs^ De Morgan , I am

Yours very truly

A. A. L

I think there is an erratum
in your Trigonometry, page 34, line 7
from the top :
“let $NOM = \theta\odot$, $MOP = \varphi\odot$ &c”
should be . . . $\underline{\underline{N}}OP = \varphi\odot$ &c

[68r]

Ockham Park
Ripley
Surrey
Friday . 17$^{\text{th}}$ Dec $^{\text{r}}$ ['1840' added in pencil by later reader]

Dear M$^{\text{r}}$ De Morgan. This
is very mathematical weather.
When one cannot exercise one's
muscles out of doors , one is
peculiarly inclined to exercise
one's brains in-doors.
Accordingly I have been
setting vigorously to work
again , with much satisfaction.

But I am sorry to
say that I am sadly obstinate
[68v] about the Term at which
Convergence begins for the
Series (A) page 231 of Algebra,
and which we thought had
been made clear by you
on Monday Evening . I must
persist in thinking Convergence
begins at $\frac{x^{33}}{2.3.4....32.33}$,
and I have enclosed my
Demonstration of my view of
the case, & which seems to
me as clear as possible. __

At any rate this will
show you at once where my
blunder is ; if I have
blundered, for I can hardly
think I have, so very palpable
[69r] does the proof appear to me.

The other night I had
unluckily been 10 days or
more without thinking about
anything of the kind , and
consequently the ['whole' inserted] thread of the
argument was not present to
me ; or I might have
spared you & myself this

trouble now. __
I have thought with much
pleasure of my agreeable
evening on Monday ,__
mathematical & educational.
Yours most truly
A. A. Lovelace

[69v] [a child's scribbling?]

[70r]
Ockham
Tuesday. 22nd Dec r

Dear Mr De Morgan I now see exactly my mistake. I had overlooked that the Series in question is not one in successive Powers of $x$ [‘like that in page 185’ inserted], but only in successive even powers of $x$.

I used once to regret these sort of errors, & to speak of time lost over them. But I have materially altered my mind on this subject. I often gain more from the discovery of a mistake of this sort, than from 10 acquisitions made at [70v] once & without any kind of difficulty. __
There is still one little thing in your Demonstration not perfectly clear to me. __ At the end you remark that “our result “gave the 503d Term instead of “the 502nd, which arose from “taking the whole number next “above $\sqrt{x+\frac{1}{4}}$ instead of an “intermediate fraction.” __
In examining the equation
$n =$ next whole number above
$$\frac{5}{4}+\frac{1}{2}\sqrt{1000,000+\frac{1}{4}}$$
I see clearly that $\frac{5}{4}+\frac{1}{2}(1001)$ is greater than $\frac{5}{4}+\frac{1}{2}\sqrt{1000,000+\frac{1}{4}}$; that the true answer would be $\frac{5}{4}+\frac{1}{2}\left(1000+\frac{1}{a}\right)$, $\frac{1}{a}$ being some [71r] fraction. We should then have had,
$n =$ nearest whole number above

$\frac{5}{4} + 500 + \frac{1}{2a}$, instead of =
$= \frac{5}{4} + 500 + \frac{1}{2}$
But, since $\frac{5}{4}$ is greater than 1,
the result must exceed 501 even
if we neglected the $\frac{1}{4}$ altogether;
and therefore at any rate $n$
(the next whole number above
$\frac{5}{4} + \frac{1}{2}\sqrt{Z + \frac{1}{4}}$), must be 502, &
$n + 1$ consequently $= 503$ . __
I do not therefore see that the
fact of taking 1001 instead of
the real square part of $Z + \frac{1}{4}$
does account for the discrepancy
in question . ____
[71v] I have now some question to
put respecting certain operations
with Incommensurables. Thanks to
your Treatise I think I understand
['this subject' inserted] pretty tolerably now. But there
are still one or two points of
Practical Application which I
am [something crossed out] busy in working up
previous to leaving the subject
altogether as a direct study, &
which I find not quite plain
sailing. __
I have been writing out in
the Mathematical Scrap-Book,
a full explanation of the
operations with Incommensurables
analogous to those of Multiplication,
Division, Raising of Powers &c,
and a day or two ago I was
[72r] about completing it with that
analogous to the extraction of
Roots, when I found I did
not fully understand the
process, _ that is beyond the
consideration of one Mean
Proportional. I have written
out & enclose my explanation
for one Mean Proportional,

& my difficulty in the case
of t<u>wo</u> or more Mean
Proportionals. ____
Also, I wished now to return
to the passage, page 29, lines 8
and 9 from the top, (Trigonometry)
which first suggested to me the
necessity of studying the subject
of Incommensurables; in order
that I might see if I could
[72v] n<u>ow</u> demonstrate the Proposition
of (46), for $\theta$ and $\sin\theta$ Incom=
=mensurable quantities. _ But I do
not find that I can. I believe
I understand the example referred
to in (4), the long & short
of which I understand to be
that if in the Right-Angled
Triangle [diagram in original] $A$, $B$, $C$ are
Incommensurables, and $V$ be any
given linear unit, then the
Ratio compounded of $A : V$ and $A : V$
a<u>dded to</u> the Ratio compounded of
$B : V$ & $B : V$, is equal to the
Ratio compounded of $C : V$ and
$C : V$. _
With respect to the Ratio of an
Angle with it's [*sic*] Sine, I began to
[73r] write it out as follows, after the
manner of pages 68 & 69 of the
Number & Magnitude: __
$\theta$ or $\theta : 1$ is the Ratio of $\frac{AB}{AO}$
$\sin\theta$ or $\sin\theta : 1$ is the Ratio of $\frac{BM}{AO}$,
$AB : AO$, $BM : AO$ being
Incommensurable Ratios, what
then does $\frac{\theta}{\sin\theta}$ really mean? _
In the first place we may
consider it to mean
$$\theta \frac{1}{\sin\theta} : 1\text{, or a Ratio}$$
compounded of the Ratio $\theta : 1$ and
$\frac{1}{\sin\theta}$, or compounded of the
Ratios $AB : AO$ and $AO : BM$. _

But further than this I cannot get, nor see my way at all. _

I conclude that in Incommensurable language, a Ratio equal to 1 or a Ratio approximately to 1 can [73v] only mean a Ratio in which the Magnitude constituting the Antecedent is equal to the Magnitude constituting the Consequent, or is constantly approaching an equality to it, and therefore that if we take the above Ratio compounded of $AB : AO$ and $AO : BM$, or the Ratio $AB : BM$, & prove that $AB$ constantly approaches in equality to $BM$, that is the desired Demonstration. __

I can only end by repeating what I have often said before, that I am very troublesome, & only wish I could do you any such service as you are doing me. Yours most truly

A. A. L

[74r]

Ockham

Monday. 4th Jan y ['1841' added by later reader]

Dear Mr De Morgan. We have had company ever since I last wrote to you, so I have been at a Stand-still, & only yesterday was able to read over with attention your replies. I am reluctant to trouble you again with remarks on the Series $1+\frac{x^2}{2}+\frac{x^4}{2.3.4}+$ &c, for it seems as if I was determined to plague you about it. However I feel I must do so. Your added [74v] remarks of last time, about $B$ &c, are quite clear in themselves, but I felt at once that they did not meet my difficulty which was that as long as $\frac{5}{4}$ (which is greater than 1) is to be added to $\frac{1}{2}\sqrt{1000,000+\frac{1}{4}}$, it matters not whether for $\sqrt{1000,000+\frac{1}{4}}$ we substitute the whole number next above it or "the intermediate fraction" alluded to by you in the line I have marked [mark a bit like her '$\surd$'], but we never can bring out $n=$ to anything less than 502, whence $n+1$ the required term must [75r] be 503. This, after reading over & over, remains in my mind a most obstinate fact, and I believe I have found out the real source of the discrepancy between the result at the bottom of the first page & the top of the Second

one. I am presumptuous
enough to think there is
certainly an error in your
writing out, in the line I
have marked X, & it is one
which is very likely to have
occurred in writing ['it' crossed out] in
a hurry. __
The $(n+1)^{\text{th}}$ term divided by
the $n^{\text{th}}$ is I believe not
[75v] $\frac{Z}{(2n-2)(2n-3)}$, but $\frac{Z}{2n(2n-1)}$,
and I have re-written & now
enclose the rest of the demonstration
(exactly like yours) with this
correction. The result comes
out as I expected, owing to
$\frac{1}{4}$ taking the place of $\frac{5}{4}$,
& everything appears to me
consistent. __ the $n^{\text{th}}$ term
divided by the $(n-1)^{\text{th}}$ term
would be, (as you have
written) $\frac{Z}{(2n-2)(2n-3)}$, & this
correction would do instead of
the others, & be perhaps ['a' inserted] more
simple mode of making it,
as your demonstration would
[76r] then remain correct, the $\underline{n}^{\text{th}}$
term being in that case the
required unknown one instead
of the $(n+1)^{\text{th}}$. _
I am afraid all this is a
little complicated to explain
in letters, & perhaps I have
still not succeeded very
perfectly in doing so; but
I feel it now all very
clear in my own mind,
& am only anxious to receive
confirmation as to my being
right, both as satisfactory
to me in the present instance,
& as tending to give me

[76v] confidence in future in
my own conclusions, or, (if
I am in this case puzzle-
-headed), a due diffidence
of them. ____
I therefore beg your indulgence
for being so teasing. ________
Believe me

Yours most truly

A. A. Lovelace

[77r]
Ockham
Sunday. 10th Jan y

Dear Mr De Morgan. I send
you the ['Series' crossed out] Analysis of the
new Series I received in your
letter yesterday morning. I
believe I have made it out
quite correctly. In fact, the
Verification at the end proves
this. But, owing to a
carelessness in my ['first' inserted] inspection
of it, I have had the
trouble & advantage of analysing
two Series. I glanced too
hastily at it, & did not
observe that the factors of the
[77v] Denominators (of the Co-efficients),
are not powers of 2, but
simply multiples of 2. __
If you will open my Sheet,
you will find on the inside
my analysis of the Series I at
first mistook your's [*sic*] for;
and I am not sorry this
has happened. I believe
both are correctly made out. __
You kindly request me
if I do not understand the
erasure in the former ['small' inserted] paper,
again to return it &c. Now
I do not agree to it; & ['I' inserted] still
fancy that we are in fact
meaning exactly the same
thing, only that you are
[78r] speaking of the $\underline{n}^{\text{th}}$ Term, & I
of the $\underline{n+1}^{\text{th}}$. __ For
convenience of reference I again
return the former large paper
(& at any rate H. M's
Post-Office will benefit). ____
I quite understand that

$\frac{1}{4} + \frac{1}{2}\sqrt{10^6 + \frac{1}{4} + a}$ is less than
501. Therefore as $n$ is
the next whole number above
this fractional expression, $x =$
$= 501$. But $n$ is not the
Term sought; the unknown
term to be determined being
by the conditions of the
Hypothesis & Demonstrations,
$n + 1$, & therefore $= 502$.
And if you will examine
[78v] your own ['former' inserted] Verification, you
will see that you there
determine the Term at which
Convergence begins, to be
$A_{502}$, or the 502nd Term,
which agrees with my
result $n + 1 = 502$. __
I think it is quite clear
that we are both agreed,
but that you were not aware
at the moment you made
the erasure that I was
not speaking of the next
whole number above $\frac{1}{4} + \frac{1}{2}\sqrt{10^6 + \frac{1}{4} + a}$
but of the next but one above
it. __
So much for the three Series: _
Now I must go to other
[79r] matters. I am indeed sending
you a Budget. __
I have been working hard
at the Differential Calculus,
& am putting together some
remarks upon Differential
Co-efficients (which in due
time will travel up to
Town for your approbation),
but in the progress of which
I am interrupted by a
slight objection to an old

matter of Demonstration,
which did not occur to me
at the time I was studying
it before, & sent you a
paper upon it ['from Ashley' inserted]. In the
course of the observations I
[79v] am now writing, I have ['had' inserted]
occasion to refer to the old
['general' inserted] Demonstration, (pages 46 & 47
of your Differential Calculus),
as to the finite existence of
a Differential Co-efficient
for all Functions of $x$; &
a slight flaw, or rather what
<u>appears</u> to <u>me</u> a flaw, in the
conclusions drawn, has occurred
to me. It is most clearly
proved that, $\theta$ being supposed
to diminish without limit,
the Fractions $Q_1$, $Q_2$ &c
<u>must</u> have finite limit, for
<u>some</u> value or other at all
events of $n\theta$ or $h$. But the
fractions in question do not
[80r] appear to me to be strictly
speaking analogous to $\frac{\Delta u}{\Delta x}$,
except the fi<u>rs</u>t of them $\frac{\varphi(a+\theta)-\varphi a}{\theta}$
and the la<u>st</u> of them $\frac{\varphi(a+n\theta)-\varphi(a+\overline{n-1}\theta)}{\theta}$,
and for this reason.
In the expansion $\frac{\Delta u}{\Delta x}$ or
$\frac{\varphi(x+\theta)-\varphi x}{\theta}$, as $\theta$ alters
<u>$x$</u> does n<u>ot</u> alter, but remains
the same. In these fractions
on the contrary, which all
have the form $\frac{\varphi(a+k\theta)-\varphi(a+\overline{k-1}\theta)}{\theta}$
and in which $a+\overline{k-1}\theta$ [bar over $k-1$ should have little downward-pointing hooks at the ends]
stands for the $x$ of the
expression $\frac{\Delta u}{\Delta x}$ or $\frac{\varphi(x+\theta)-\varphi x}{\theta}$,
not only does $\theta$ alter, but
from the conditions of the

Hypothesis & Demonstrations, $\overline{k-1}\theta$
[80v] & consequently $a + \overline{k-1}\ \theta$ must
likewise alter along with $\theta$.
There is therefore a double
alteration in value going on
simultaneously, which appears
to me to make the Case quite
a different one from that
of $\frac{\Delta u}{\Delta x}$, & consequently to
invalidate all conclusions
deduced from the former with
respect to the latter. __
The validity of the Conclusions
with respect to the fractions
$Q_1$, $Q_2$ &c, you understand
I do not question. What I
question is the analogy between
these Fractions & the Fraction
$\frac{\Delta u}{\Delta x}$ or $\frac{\varphi(x+\theta)-\varphi x}{\theta}$ ['of' inserted] which
[81r] latter it is required to
investigate the Limits. __
I also have another slight
objection to make, not to the
extent of Conclusions established
respecting the Fractions $Q_1$, $Q_2$
&c having finite limits,
but to the Conclusions on that
point not going far enough, _
not going as far as they
might :__ "either these are
"finite limits, or some increase
"without limit and the rest
"diminish without limit ; if the
"latter, we shall have two
"contiguous fractions, one of which
"is as small as we please, and
"the other as great as we please,
"&c, &c, _ a phenomenon which
"which [*sic*] can only be true when
[81v] "$Q_k$ is the fraction which is
"near to some singular value
"of the Fraction, & cannot be

“true of ordinary & calculable
“values of it &c.” _ Now it
appears to me that in no
possible case could such a
phenomenon as this be true,
when we consider how the
fractions are successively
formed one out of the others
by the substitution of $a + \theta$ for
$a$, $\theta$ too being as small as
we please. I therefore think
it might have been concluded
at once that there must
always be finite limits to
the Fractions $Q_1$, $Q_2$ &c,
[82r] and this whatever $k$ or $n\theta$
may be. I suppose it is not
so, but I cannot conceive
the Case in which it could
be otherwise. __
I do not know if in writing
upon my two difficulties in
these pages 46, 47, 48, I have
expressed my objections (especially
in the former case of the
fractions $Q_1$, $Q_2$ not being
similar to $\frac{\Delta u}{\Delta x}$) with the
clearness necessary to enable
you to answer them, or indeed
to apprehend the precise points
which I dispute. It is not
always easy to write upon
these things, & at best one
must be lengthy. _ I shall be
[82v] exceedingly obliged if you will
also tell [‘me’ inserted] whether a little
Demonstration I enclose as to
the Differential Co-efficient
of $x^n$, is valid. It appears
to me perfectly so; & if it is,
I think I prefer it to your’s [*sic*]
in page 55. It strikes me

as having the advantage in
simplicity, & in referring to
fewer ['requisite' inserted] previous Propositions.
I have another
enquiry to make, respecting
something that has lately
occurred to me as to the
Demonstration of the Logarithmic
& Exponential Series in
your Algebra, but the real
truth is I am quite ashamed
[83r] to send any more; so will
at least defer this. ____
I am afraid you will indeed
say that the office of my
mathematical Counsellor or
Prime-Minister, is no joke.
I am much pleased to
find how very well I stand
work, & how my powers of
attention & continued effort
increase. I am never so
happy as when I am
really engaged in good
earnest; & it makes me most
wonderfully cheerful & merry
at other times, which is
curious & very satisfactory. __
What will you say when
[83v] you open this packet?____
Pray do not be very angry,
& exclaim that it really is
too bad. __
Yours most truly
A. A. Lovelace

[84r]
Ockham
Sunday. 17$^{\text{th}}$ Jan $^{\text{y}}$

Dear M[r] De Morgan. Many thanks for your reply to my enquiries. I believe I now understand about the limit of $\frac{\varphi(x+n\theta+\theta)-\varphi(x+n\theta)}{\theta}$ not being affected by $n\theta$ being a gradually varying quantity. I think your explanation of it amounts to this : that provided [something crossed out] $(x+n\theta)$ varies only towards a fixed limit, either of increase or diminution ; then [84v] the result of the Subtraction of $\varphi(x+n\theta)$ from $\varphi(x+n\theta+\theta)$ remains just the same as if, (calling $(x+n\theta)=Z$), $Z$ were a fixed quantity. ____ Now by the conditions of the Demonstration in question, (in your pages 46 & 47), when a decrease takes place in $\theta$, a certain simultaneous increase takes place in $n$. That is to say, suppose $\theta$ has at any one moment a certain value corresponding to which $n$ has the value $k$. If I alter $\theta$ to a lesser value $\chi$, then say that the corresponding [85r] value of $n$, necessary to fulfil the constant condition $n\theta=h$, is not $k$, but $k+m=p$. What happens now? _ Why as follows, I believe : there were, before $\theta$ became $\chi$, $k$ fractions ; there are now $k+m$, or $p$ fractions. _ In ['each of' inserted] the $k$ former fractions, [something crossed out] $Z$ will

have diminished, towards a
fixed limit ['of diminution' inserted] $x$ ; in ['each of' inserted] the $m$
new fractions introduced, $Z$
will be greater than in the
old $k$ fractions ; but there
is a fixed limit of increase,
$h$, which it can never pass,
[85v] up to the very last Term
of the Series of Fractions. _
Therefore tho' the quantity
$x + n\theta$ or $Z$ varies necessarily
with a variation in the value
of $\theta$, yet it varies within
fixed limits either of
diminution or increase, & thus
the result of the subtraction
$\varphi(Z+\theta) - \varphi(Z)$ is not
affected. _
I hope I have made
myself clear. I think it is
now distinct & consistent in
my head. _

I see that my proof of
the limit for the function $x^n$
is a piece of circular argument,
[86r] containing the enquiry which
I was in fact aiming at
in the former paper, but
which required to be
separated from the confusion
attendant on my erroneous
statements on other points.
I merely return the old
paper with the present one,
because it might perhaps be
convenient to compare them.

On the other side
of the sheet containing the
remarks on $\frac{a^\theta - 1}{\theta}$, you
will find an enquiry
which struck me lately
quite by accident in

[86v] referring to some old
matters. ____
I ought to make many
apologies I am sure for
this most abundant
budget. _ I am very
anxious about the matter
of the successive Differential
Co-efficients, & their
finiteness & continuity. I
think it troubles my
mind more than any
obstacles generally do. I
have a sort of feeling
that I ought to have
understood it before, &
[87r] that it is not a legitimate
difficulty. __
With many thanks,

Yours most truly

A. A. Lovelace

[88r]
Ockham
Friday 22$^{nd}$ Jan$^{y}$
['1841' added by later reader]

Dear M$^{r}$ De Morgan. Then
I shall be in Gower S$^{t}$ on
Monday Evening about 8
o'clock. Lord L is not
afraid of Algebra, but he
goes up on business that
will occupy all his evenings,
viz : Lord Melbourne's
dinner, & the House of
Lords. I feel as if I
[88v] had a great deal to
say & to talk about; __
& by the bye one thing,
unmathematical, is about a
visit sometime from M$^{rs}$ De
Morgan ; because it strikes
me that your lectures need
not always tie her down
too. __
Lord L desires me to say
he regrets not being able to
accompany me. ____

Yours very truly
A. A. Lovelace

[88r]
Ockham
Friday 22$^{nd}$ Jan$^{y}$
['1841' added by later reader]

Dear M$^{r}$ De Morgan. Then
I shall be in Gower S$^{t}$ on
Monday Evening about 8
o'clock. Lord L is not
afraid of Algebra, but he
goes up on business that
will occupy all his evenings,
viz : Lord Melbourne's
dinner, & the House of
Lords. I feel as if I
[88v] had a gr<u>eat</u> deal to
say & to talk about; __
& by the bye one thing,
unmathematical, is about a
visit sometime from <u>M$^{rs}$</u> De
Morgan ; because it strikes
me that yo<u>ur</u> lectures need
not always tie h<u>er</u> down
too. __
Lord L desires me to say
he regrets not being able to
accompany me. ____
Yours very truly
A. A. Lovelace

[89r]

[Signature written sideways at the top of this page — belongs at end of letter so transcribed there]

Ockham
Wed$^{dy}$ 3$^{d}$ Feb$^{y}$

Dear M$^{r}$ De Morgan. I have a question to put respecting a <u>condition</u> in the establishment of the conclusion $\frac{\varphi(a+h)}{\psi(a+h)} = \frac{\varphi^{(n+1)}(a+\theta h)}{\psi^{(n+1)}(a+\theta h)}$ in page 69 of the Differential Calculus. _ I have written down, & enclose, my notions on the steps of the reasoning used to establish that [89v] conclusion. So that you may judge if I take in the objects & methods of it.

The point I do not understand, is why the distinction is made, (& evidently considered so important a one), of "$\psi x$ "being a function which has "the property of <u>always</u> "<u>increasing or always decreasing</u>, "<u>from $x = a$ to $x = a + h$</u>, "in other respects fulfilling the "conditions of continuity in "the same manner as $\varphi x$". [90r] For this, see page 68, lines 9, 10, 11, 12 from the top ; _ page 68, line 12 from the bottom ;
page 69, lines 7, 8 from the bottom ; &c
I see perfectly that this condition must exist, & that without it we could not secure the denominators

(alluded to in page 68, line
13 from the bottom), being
all of one sign. __
But what I do <u>not</u>
understand, is [something crossed out] why the
condition is not made
[90v] for $\varphi\ x$ also. It appears
to me to be equally requisite
for this latter ; because if
we do not suppose it,
how can we secure the
numerators $\varphi(x + k\Delta x) -$
$-\varphi(x + \overline{k-1}\Delta x)$ being <u>all</u>
<u>of one sign</u> ; & unless they
<u>are</u> all of one sign, we
cannot be sure that they
will [something crossed out] when added,
so destroy one another as to
give us $\varphi(a + h) - \varphi a$ ; _
an expression essential to
obtain. ____ I think I have
explained my difficulty, &

[something missing here?]

[the following written vertically on 89r]

believe me
Yours most truly
A. A. Lovelace

[beginning of letter seems to be missing]

[91r] (unless the limit for $\frac{v^n - w^n}{v - w}$
is dispensed with in the
demonstration for the Binomial
Theorem, which it is not
in your Algebra, nor am
I aware that it c<u>an</u> be
dispensed with in any of the
<u>elementary</u> proofs of that
Theorem). __ It had not
struck me that, calling
$(x + \theta) = v$, the form
$\frac{(x+\theta)^n - x^n}{\theta}$ becomes $\frac{v^n - x^n}{v - x}$.
And by the bye, I may
here remark that the curious
<u>transformations</u> many formulae
can undergo, the unexpected
& to a beginner apparently
[91v] <u>impossible</u> <u>identity</u> of forms
exceedingly <u>dissimilar</u> at first
sight, is I think one of
the chief difficulties in the
early part of mathematical
studies. I am often reminded
of certain sprites & fairies
one reads of, who are at
one's elbow in o<u>ne</u> shape
now, & the next minute in
a form the most dissimilar,
and uncommonly deceptive,
troublesome & tantalizing are
the mathematical sprites &
fairies sometimes ; like the
types I have found for them
in the world of Fiction. __
[92r] I will now go to the question
I delayed asking before :
In the development of the
Exponential Series
$a^x = 1 + (\log a)x + \frac{(\log a)^2 x^2}{2} +$ &c,
and the Logarithmic Series
$\log a = (a - 1) - \frac{1}{2}(a - 1)^2 +$ &c

deduced from it ; I object
to the necessity involved of
supposing $x$ to be diminished
without limit, _ a supposition
['obviously' inserted] quite necessary to the completion
of the Demonstration. It has
struck me that though this
supposition leaves the Demonstration
& Conclusions perfect for the
cases in which $x$ is supposed
to diminish without limit, yet
[92v] it makes it valueless for the
many in which $x$ may be
anything else which does not
diminish. _ No _ by the bye,
I think I begin to see it now ;
I am sure I do. It is as
follows : _ the supposition of
$x$ diminishing without limit
is merely a parenthetical
one, by means of which a
limit for a certain expression
$\frac{a^x-1}{x}$ is deduced under those
circumstances ; & then the
argument proceeds, that having
already obtained in another
place, a ['different' inserted] limit for this same
expression under the same
[93r] circumstances, we at once
deduce the equality of these two
limits, from whence follows
&c, &c. Thus this supposition
of $x$ diminishing without limit,
is not a portion of the main
argument, but only a totally
independent & parenthetical
hypothesis introduced in order
to prove something else which
is a part of the main
argument. _ Yes _ this is
it, I am sure. I had
had the same objection to

the Demonstration in Bourdon,
to which I have had the
curiosity to refer. I am
[93v] sometimes very much interested
in seeing how the same
conclusions are arrived at
in different ways by different
people ; and I happen to
have been inclined to compare
you & Bourdon in this
case of developing Exponential
& Logarithmic Series ; and
very amusing has it been to
me to see him begin exactly
where you end, &c. Your
demonstration is much the
best for practical purposes.
His is exceedingly general, &
the vast number of substitutions
[94r] of one thing for another make
it lengthy, & by no means very
simple to follow. __ But it
is very well occasionally to
make these comparisons. __

We are going to
Town on Monday the 25$^{\text{th}}$,
for two or three nights, &
I will ask M$^{\text{rs}}$ De Morgan's
& your permission to spend
Monday Evening with you,
going towards 8 o'clock,
as I did before. It would
give me great pleasure, &
may perhaps be not only
agreeable to me, but of use
[94v] too, as there are one or two
points [something crossed out] relating to my future plans
which I rather think of
speaking to you upon. __

By the bye, Lord
Lovelace & I are both of us
much vexed, at our own

negligence in letting the Xmas
Vacation go bye [*sic*], without
proposing to you & your
lady & children to visit us
here, which you might
perhaps have been able to do
during Holiday-time. I fear
you <u>may</u> <u>now</u> be unable to
think of it ; but pray consider
[95r] the question with her ; if not
for any immediate use, at
any rate for the <u>next</u> occasion.
The fact is, that we have
so much the habit of thinking
of you <u>only</u> in connexion
with Town & engagements there,
that it only suddenly occurred
to us whether you <u>migh</u>t not
be able to breathe country
air like other people. __
You would come by Railway,
& we would send the
carriage to the Station for
you.

Yours most truly
A. A. L

[96r]
Ockham
Sat[dy] 6[th] Feb[y]
['1841' added by later reader]

Dear M[r] De Morgan. Had I
waited a day or two longer,
I need not have troubled
you with my letter of Wed[dy],
& I can only reproach
myself now with having been
a little too <u>hasty</u> in my
examination of the Theorem in
pages 68, 69, and having
sent you an enquiry which
certainly indicates some
negligence. I fear this letter
[96v] may not be in time to
stop one from you. [something crossed out]
However I will try to
send it by an opportunity this
afternoon. __
But, to show you that I
now understand the matter
completely :_
In the first place the question
of the Denominator, or the
Numerator, being all of the
<u>same</u> sign, in such [something crossed out] collection of
expressions as
$\frac{a-b}{m-n}, \frac{c-a}{p-m}, \frac{d-c}{r-p}, \frac{e-d}{q-r}$ &c
has <u>nothing whatever to do</u>
<u>with the letters effacing each</u>
<u>other</u> when the above are
[97r] put into the form,
$\frac{(a-b)+(c-a)+(d-c)+(e-d)}{(m-n)+(p-m)+(r-p)+(q-r)}$ &c ;
whether $(a-b)$, &c be positive
or negative, or some one &
some the other, still
$\frac{a-b+c-a+d-c+e-d}{m-n+p-m+r-p+q-r}$ &c
must $= \frac{e-b}{q-n}$
In the second place, the

Denominator must be all of
the same sign, in order
to fulfil the conditions of
the Lemma in page 48 ;
& this is the reason why
the condition is made respectively
$\psi\ x$ always increasing or
[97v] always decreasing &c. _
For $\varphi\ x$, it matters not
whether it alternately increases
& decreases (provided always
that it be continuous). _
I believe I now
have the whole quite clear;
& I shall be more careful
in future.
I enclose a paper upon
pages 70, 71, 72, 73.
It is merely the general
argument, put into my own
order & from ; & I send
it in order to know if
you think I understand as
much about the matter as
[98r] I am intended to do.
You know I always have
so many metaphysical
enquiries & speculations which
intrude themselves, that I
never am really satisfied
that I understand anything ;
because, understand it as
well as I may, my
comprehension can only be
an infinitesimal fraction of
all I want to understand
about the many connexions
& relations which occur to
me, how the matter in
question was first thought of
[98v] or arrived at, &c, &c.
I am particularly curious

about this wonderful Theorem.
However I try to keep
my metaphysical head in
order, & to remember Locke's
two axioms. __
You should receive this about
6 o'clock this evening, if not
before. I fear you will
have written to me today
however. _ Believe me
Yours most truly
A. A. Lovelace

[100r]

Ockham Park
Friday. 19th Feb[y]

Dear M[r] De Morgan. I have one or two queries to make respecting the "Calculus of Finite Differences" up to page 82.
Page 80, line 4 from the top, "remembering ....
".... that in $\varphi''(x+\theta\omega)$, $\theta$ itself <u>is a function</u>
"<u>of $x$ and $\omega$</u>, &c" ; Now, neither on examining $\theta$ as here used & introduced, nor on referring to the first <u>rise & origin</u> of $\theta$ in this capacity, (see page 69), can I discover that it is a function of $x$ and $\omega$ here, or a function of the analogous $a$ and $h$ in page 69. I neither see the truth of this assertion, nor do I perceive the importance of it (supposing it <u>is</u> true) to the rest of the argument & demonstration in page 80.

There is also a point of doubt I have relating to the conclusion in lines 15, 16 from [100v] the top of page 79 :
It is very clear that the law for the Co-efficients being proved for $u_n$, and for $\Delta u_n$, follows immediately & easily for $u_{n+1}$, or $u_n + \Delta u_n$.

But if we now wish to establish it for $u_{n+2}$, we must prove it true not <u>only</u> for $u_{n+1}$, but also for $\Delta u_{n+1}$ :_

To retrace from the beginning : the object in the first half of page 79 evidently is to prove firstly, that <u>any</u> order of $u$, say $u_n$ <u>can</u> be expressed in term ['of,' inserted] or in a Series of all the Differences of $u$ ; $\Delta u$, $\Delta^2 u$, $\Delta^3 u$, ........... $\Delta^n u$ ; _
Secondly, that the Co-efficients for this Series follow the law of those in the Binomial Theorem.

Now the first part is evident from the law of formulation of the Table of Differences ; Since all the Differences $\Delta u$, $\Delta^2 u$, $\Delta^3 u$ &c are made out of $u$, $u_1$, $u_2$ &c, it is obvious that by exactly retracing & reversing the process, we can make $u$, $u_1$, $u_2$ &c [101r] out of $\Delta u$, $\Delta^2 u$, $\Delta^3 u$ &c.

For the second part of the above ; if we
can [something crossed out] show that the law for the Co-efficients
holds good up to a certain point, say $u_4$ ;
and also that being true for any one
value, it must be true [something crossed out] for the next
value too ; the demonstration is effected for
all values :
Now the fact is shown that it is true up
to $u_4$. (I must not here enquire why the
fact is so. That is I suppose not your
arranging, or any part of your affairs).
It is shown that the two parts $u_3$, $\Delta u_3$ of
which $u_4$ is made up are under this law,
& therefore that $u_4$ is so. And next it is
shown that any other two parts $u_n$, $\Delta u_n$
being under this law, their sum $u_{n+1}$
must be so. But this proves nothing
for a continued succession. $u_{n+1}$ being
under this law does not prove that $\Delta u_{n+1}$
is under it, & therefore that $u_{n+2}$ is under it.
[101v] There seems to me to be a step or condition
omitted.
I am sorry still to be obliged to trouble
you about $f\,x$, $f'x$, $f''x$, I cannot yet
agree to the assertion that the result would
not be affected by discontinuity or singularity
in $f'x$, $f''x$, &c. The result it is true
would not be directly affected ; but it surely
would be ['indirectly' inserted] affected, inasmuch as the conditions
of page 69, necessary to prove that result,
could not be fulfilled unless we suppose
$f'x$, $f''x$ ..... $f^{(n+1)}x$ continuous &
ordinary as well as $f\,x$. To arrive at
the equation $\frac{\varphi(a+h)}{\psi(a+h)} = \frac{\varphi^{(n+1)}(a+\theta h)}{\psi^{(n+1)}(a+\theta h)}$
page 69, it is a necessary condition that
$\varphi x$, $\varphi' x$, $\varphi'' x$ ......... $\varphi^{(n+1)}x$ be all
continuous & without singularity from $x = a$ to
$x = a + h$. And the $\varphi' x$, $\varphi'' x$ ..... $\varphi^{(n)}x$, $\varphi^{(n+1)}x$
of page 71, could not fulfil this condition
unless $f'x$, $f''x$ ..... $f^{(n)}x$, $f^{(n+1)}x$ did so
[102r] also. _ I fear I am very troublesome about
this. _

I have remarks to make respecting some of
the conclusions of the Chapter on Algebraical
Development ; but they will keep, and
therefore I will delay them, as I think
I have send abundance, & I have also
some questions to put on the last 8 pages
of your "Number & Magnitude" on Logarithms. _
On the Differential Calculus I will only
now further say that on the whole I believe
I go on pretty well with it ; and that
I suppose I understand as much about it,
[something crossed out] as I am intended to do ;
possibly more, for I spare no pains to do
so. __
Now for the Logarithms : I had not till now
read the last pages of your Number & Magnitude,
& there are certain points I do not fully
understand. The last line of the whole, on
the natural logarithms is one. I cannot
[102v] identify the constituent quality of the natural
logarithms there given, with the constituent
qualities I am already acquainted with thro'
other relations & means : I know ['for instance' inserted] that the
natural logarithms must have 2.717281828
for their Base ; that is to say that the
line $HL$, or $A$ ($OK$, or $V$ being the linear
unit) should be 2.717281828 $V$ units.
Now I do not see ['but' inserted] that the condition in the
last paragraph of the book is one that
might perfectly consist with any Base whatever.
To prove that I understand
the previous part, at least to a considerable
degree, I enclose a Demonstration I wrote
out of the property to be deduced by the
Student, (see second paragraph of page 79),
& which I believe is quite correct. __
Pray of what use is the Theorem
(page 75, ['& which' inserted] continues in page 76)? I do not see
that it is subservient to anything that
[103r] follows ; and it appears to me, to say the
truth, to be rather a useless & cumbersome
addition to a subject already sufficiently

complicated & cumbersome. _ The passage I mean is from line 13 (from the top) page 75, to the middle of page 76. _

Believe me

Yours very truly

A. A. Lovelace

[104r]

Ockham
Monday. 22nd Feby

Dear Mr De Morgan. The reply to one of my queries to you, dispatched on Friday, has I believe just occurred to me. Probably this letter will cross one from you tonight, but the remaining points continue still unsolved, so that I shall be equally glad if I do receive an answer tomorrow morning.

The difficulty I have solved is the one relating to the law for the Co-efficients of ['the series for' inserted] $\Delta u_n$. I remarked that the law for the Co-efficients of the Series for $u_n$ being ascertained, did not ascertain those for $\Delta u_n$ as a necessary consequence. But I see I am wrong. If a Series is obtained for $u_n$, [104v–105r] we have only in order to obtain one for $\Delta u_n$, to take the Difference of every term ['of the' crossed out] , (that is of the variable part of every term), of the Series for $u_n$. Thus,

$u_n$ being $= u + n\Delta u + n\frac{n-1}{2}\Delta^2 u + \cdots + n\frac{n_1}{2}\Delta^{n-2}u + n\Delta^{n-1}u + \Delta^n u$

$\Delta u_n$ must $= \Delta u + \Delta(n\Delta u) + \Delta\left(n\frac{n-1}{2}\Delta^2 u\right) + \cdots + \Delta\left(n\frac{n-1}{2}\Delta^{n-2}u\right) + \Delta(n\Delta^{n-1}u) + \Delta(\Delta^n u)$

$= \Delta u + n\Delta^2 u + n\frac{n-1}{2}\Delta^3 u + \cdots + n\frac{n-1}{2}\Delta^{n-1}u + n\Delta^n u + \Delta^{n+1}u$

Whence &c, &c. I think this is quite sufficiently obvious. _

But I now have another query to put, in the place of the one I have just disposed of, relating to the development in page 83,

$$\Delta u = amx^{m-1} + Ax^{m-2} + \cdots\cdots + Px + Q$$

and in which I cannot help thinking there is a mistake ['in the first Term' inserted]:_ I make out that

it ought to be

$$\Delta u = am\omega x^{m-1} + Ax^{m-2} + \cdots\cdot$$

But I enclose my developments and observations therefore, on a longer & more convenient sheet. I will only add here, that we move to Town on Thursday; and that I should much like to spend Sunday Evening with Mrs De Morgan & you, if this arrangement is suitable & agreeable to you. I [105v] should arrive as usual, about 8 o'clock

I believe I shall have by the end of this week several papers ready to discuss. __

You see I do not waste my time, at any rate; and I only hope that I am

not the means of wasting <u>yours</u> either.
Believe me
Yours very truly
A. A. Lovelace

[106r]

Ockham Park
Ripley
Surrey
$4^{\text{th}}$ July

Dear M[r] De Morgan. You are perhaps surprised that I have not sooner troubled you again. And you may think it a very <u>bad</u> reason to give, that I have done nothing. We returned here on Tuesday, & <u>now</u> I am working away famously, & hope I have before me 7 or 8 months of <u>ditto</u>. You left me at page 106. I remember your enquiry if I were s<u>ure</u> that I understood $\int_b^{b+k} fx \times \frac{dx}{dt}$ as developped [*sic*] in pages 102, 103. I answered confidently, that I did. I now enclose you my own development of this Integration, that we may be <u>quite</u> certain of my comprehension of [something crossed out] it. On the other page of my sheet, is the <u>application</u> of it to $\int udv = uv - \int vdu$ (page 105); & to $\int_a^x \frac{1}{v}\frac{dv}{dx}dx$ (page 107).

I have now two questions to propose. I differ from y<u>ou</u> in my development of $\int \frac{1}{1-x}dx$ (see page 107)

[106v] I cannot see why the Constant $C$ is omitted in t<u>his</u> more than in $\int \frac{1}{1+x}dx$.

I subjoin <u>my</u> development: Let $v = 1 - x$

$\int \frac{1}{1-x}dx = \int \frac{1}{1-x} \times -(-1)dx$ (which is only another way of writing $\int \frac{1}{1-x}.1.dx$)

And as $\frac{dv}{dx}$ or $\frac{d(1-x)}{dx} = -1$, we may in the above substitute $\int \frac{1}{1-x}dx = \int \frac{1}{1-x} \times -\left(\frac{d(1-x)}{dx}\right) dx$

Or $\int \frac{1}{v}dx = \int \frac{1}{v} \times -\frac{dv}{dx}dx$

$= \int \frac{1}{v}\frac{dv}{dx}.(-1)dx$ which by

$\int budx = b\int udx$ (see page 105) is $= (-1)\int \frac{1}{v}\frac{dv}{dx}dx$

or $= -\int \frac{1}{v}\frac{dv}{dx}dx$

Now since by line 4, $\int \frac{1}{v}\frac{dv}{dx}dx = \int \frac{1}{v}dv =$

$= \log v + C$, it follows that

(— this same expression) must $= -(\log v + C)$

$= -(\log(1-x) + C) = -\log(1-x) - C$

$= \log \frac{1}{1-x} - C$

[107r] Now how do you get rid of $(-C)$? __

My second question is unconnected with any of your books. _ But I think I may venture to trouble you with it. _ In the two equations,

$V = gT \quad (1)$

$S = \frac{1}{2} g.T^2 \quad (2)$

which you will at once recognise, I want to know how (2) is derived from (1). __
Will you refer to Mechanics (in the Useful Knowledge Library), page 10, Note, which is as follows, "Let $S$ be the space described by the "falling body. $V = \frac{dS}{dT} = gT$. Hence $dS = gT\,dT$, "which being integrated gives $S = \frac{1}{2} g.T^2$."
Now can I ['as yet' inserted] understand this application of Differentiation & Integration? __
I conclude that $\frac{dS}{dT}$ here means

Diff. co of $S$ with respect to $T$, $S$ being (by Definition & Hypothesis) a function of $T$, & of $V$
I know that $V = gT$

And that $V = \frac{S}{T}$ But I neither see how $V = \frac{dS}{dT}$, nor how the subsequent Integration applies.
[107v] The object, I need not say, is the solution of $S$.
I mean to work very hard at my Chapter on Integration &c, now. __ And I hope this summer & autumn will see me progressing at no small rate.
How is the Baby? __ And does Mrs De Morgan enjoy Highgate? I ['am' inserted] enjoying the country not a little, I assure you.

Yours most truly

A. A. Lovelace

[108r]

Ockham
Sunday. $6^{th}$ July

Dear Mr De Morgan. It is perhaps unfair of me to write again with a batch of observations & enquiries, before you have had time to reply to the previous one. But I am so anxious to get the present matters off my mind, that I cannot resist dispatching them by this post. __

I have two series of observations to send, one relating to the passage from page 107, (line 8 from the bottom), to the last line of page 108; the other to certain former passages in pages 99, 100 & 103, concerning which some questions have suddenly occurred to me quite recently.

I shall begin with pages 107 & 108: I enclose you my development & explanation of $\int \frac{x^n dx}{\sqrt{a^2-x^2}}$ up to $\int \frac{x^n dx}{\sqrt{a^2-x^2}} = -x^{n-1}\sqrt{a^2-x^2} + (n-1)a^2 \int \frac{x^{n-2}dx}{\sqrt{a^2-x^2}} - (n-1)\int \frac{x^n dx}{\sqrt{a^2-x^2}}$ from which you will judge if I understand it so far. I should tell you that I have not yet begun page 109.

I will now ask two or three questions : 1stly: page 107, [108v] (line 3 from the bottom): "the diff. co of $a^2 - x^2$ being $\underline{(-2xdx)}$ &c". This surely is incorrect ; & you will see that in my development I have written it as I fancy it should be "being $= (-2x)$, &c"

2ndly: page 108, (lines 8, 9, 10 form the top) : "By $\int UdV$ "we mean $\cdots\cdots\cdots\cdots\cdots$ p. 102, where "the values of $\Delta V$ in the several terms are "different, but comminuent." I do not see that this is a case of page 102 rather than of page 100 ; in other words, that the increments in this Integration are "unequal but comminuent". __

3dly: the subtraction in line 15 from the top, of $(n-1)x^{n-2} \times dx$ for $d.(-x^{n-1})$ appears to me quite inconsistent with the inseparable indivisible nature of a diff. co. __

4thly: Lines 9, 10 from the bottom, "We have therefore "&c $\cdots\cdots\cdots\cdots$ that of $\sqrt{a^2-x^2}x^{n-2}dx$".

Admitted, most fully. But $\int \sqrt{a^2-x^2}x^{n-2}dx$ does not answer exactly to $\int vdx$ or $\int \sqrt{v}d2u$, and therefore it appears to me that this Integration is not strictly an example of lines 5, 6, 7 (from the bottom) of page 107. You will remember that $-x^{n-1}$ was $= 2V$, therefore the $x^{n-2}$ of $(\sqrt{a^2 - x^2}x^{n-2})$ is equal to $(-1) \times \frac{2V}{x}$ or $\frac{-1}{x}.2V$. So that another factor $\frac{-1}{x}$ enters into the [109r] expression which was, as I understand it, to answer strictly to $\int vdu$ or $\int \sqrt{v}d2u$

5thly (line 5 from the bottom) page 108: I think there is an Erratum. Surely $\int \left( \frac{a^2x^{n-2}}{\sqrt{a^2-x^2}} - \frac{x^n dx}{\sqrt{a^2-x^2}} \right)$ ought to be $\int \left( \frac{a^2x^{n-2}dx}{\sqrt{a^2-x^2}} - \frac{x^n dx}{\sqrt{a^2-x^2}} \right)$

I don't know if my pencil Sheet enclosed will be very intelligible, for it is as I wrote it down at the time quite roughly, & without any very great amplitude or method. __

I now proceed to my series of observations relating to former pages, beginning with page 102, (line 10 from the bottom)

"+ less than $nC\frac{\Omega^2}{2}$, or $Ch\frac{\Omega}{2}$";

now in order to ['effect' inserted] the substitution of $Ch\frac{\Omega}{2}$ for $nC\frac{\Omega^2}{2}$ the latter is resolved into $C.n\Omega.\frac{\Omega}{2}$, & ['for' inserted] $n\Omega$ is substituted $h$. But by the hypothesis & conditions, $h$ must be less than $n\Omega$. Therefore it does not necessarily follow that that which is proved less than $nC\frac{\Omega^2}{2}$, is also less than $Ch\frac{\Omega}{2}$. _ You see my objection. __

2ndly. See Note to page 102 : If the "completion of the ['first' inserted] Series" [109v] in this page is unnecessary, surely it is equally unnecessarily in the first Series of page 100 ; for the same observation applies to the latter as to the former, viz : that the additional term is comminuent with $w$. __

3dly. See page 99 (line 8 from the bottom) :

"$\int_a^x \varphi x.dx = (x-a)a + \frac{(x-a)^2}{2} = \frac{x^2-a^2}{2}$"

This is another form of $\int_a^{a+h} xdx = ha + \frac{h^2}{2}$ 8 lines above, & of the limit of the summation for $\varphi x = x$ in the previous page. And therefore it appears to me that it ought to be

$$\int_a^x x.dx = (x-a)a + \frac{(x-a)^2}{2} = \frac{x^2-a^2}{2}$$

I do not see what business $\varphi x$ has. __

Now at last, I have done troubling you. —
I am very anxious on all these points. ——
With many apologies, believe me

Yours very truly

A. A. Lovelace

[110r]

Ockham
Monday. 6$^{th}$ July

Dear M$^r$ De Morgan. Since dispatching my letter yesterday, I remember that I have not even quite fully & correctly stated the whole points of difference ['between' inserted] $\int \sqrt{a^2 - x^2} x^{n-2} dx$ and $\int \sqrt{v} d2u$. I think I stated that $\int \sqrt{a^2 - x^2} x^{n-2} dx = \int \sqrt{v} d2u . \frac{-1}{x}$, that in other words the 1$^{st}$ side differs from $\int \sqrt{v} d2u$ in containing a factor $\left(-\frac{1}{x}\right)$. But it differs also in containing $dx$ as well, which in writing yesterday I omitted I believe to notice. So that $\int \sqrt{a^2 - x^2} x^{n-2} dx = \int \sqrt{v} d2u . \frac{(-1)}{x} . dx$ or the 1$^{st}$ side differs from $\int \sqrt{v} d2u$ in containing $-\frac{1}{x} . dx$. Is not this what I ought to have stated? Or is there still any confusion?

I also wish to observe upon what I wrote on Friday as to the application of the Differential & Integral Calculus to $\frac{gt^2}{2}$, [110v] that I am aware this formula ['$e = \frac{gt^2}{2}$' inserted] can be derived from $V = gt$, by the simple Theory of algebraical proportion ; but that I was anxious to know how it is derived in the other way.

I will with your leave ['(which I do not wait for)' inserted], send you my paper making it out on the doctrine of Proportions. __
You must tell me if I presume too much on your kindness to me. I am so engaged at present with my mathematical & scientific plans & pursuits that I can think of little else ; & perhaps may be a plague & bore to my friends about [something crossed out] these subjects ; for after my interruption from Paris & London pursuits & occupations, my whole heart is with my renewed studies ; & every minutia even is a matter of the greatest interest.

Believe me
Yours most truly
A. A. Lovelace

[111r] [something crossed out] You ['will receive' inserted] two papers on $e = \frac{gt^2}{2}$ tomorrow evening, or Wed[dy]. __ One of them is to show the absurdity of the supposition that the spaces might be as the velocities ; ['& that' inserted] on merely abstract grounds it could not be. __

[112r]

Ockham Park
Sunday. 11th July ['1841' added by later reader]

Dear Mr De Morgan. I enclose you a paper (marked No 1) from which I think you will see that I now quite understand the real relationship between $\int \frac{x^n dx}{\sqrt{a^2-x^2}}$ and $\int \sqrt{a^2-x^2}x^{n-2}dx$ ; & that I [something crossed out] am now aware I wanted to apply to the latter what is not intended to be directly applied to it at all ; & that ['my' inserted] getting both $du$ and $dx$ in, was a complete puzzle & blunder. For where a few lines previously $(-\overline{n-1})\int \sqrt{a^2-x^2}x^{n-2}dx$ is substituted for $\int \sqrt{a^2-x^2} \times (-\overline{n-1})x^{n-2}dx$, $du$ ceases of course to enter under the Integrated quantity, [something crossed out] since it has been decomposed & otherwise distributed. __

I am still occupied on pages 108, 109, 110, & hope to complete to page 112 during this week. I find this part requires studying with great care. I think you anticipated this. __
I must now thank you very much for your two letters ; & will proceed to notice one or two points [112v] in your replies to my enquiries.
I see that in objecting to what I called the division of $\frac{dV}{dx}$, when $dV$ is substituted for $\frac{dV}{dx}dx$, I took a completely wrong view of the matter. It does so happen that the expression (derived from a separate & distinct Theorem) which we may substitute for $\frac{dV}{dx}dx$ coincides in form with what we may call the numerator $dV$ of the diff_co. But the $dV$ that is substituted is not therefore derived from $\frac{dV}{dx}$, at least ['not directly or' inserted] from the decomposition of that which is indecomposible [sic]. __

I return again my former paper (marked No 2.) with a clearer explanation of what I intended to convey by the term equivalent ; a term which it seems I had no business to use in the application which I ['there' inserted] meant to make of it. __

I enclose (marked No 3) my answer to your "Try "to prove the following. It is only when $y = ax$ "($a$ being constant) that $\frac{dy}{dx} = \frac{y}{x}$" I do not feel quite sure that my proof is a proof. But I think

it is too. __
Now about $v = gt$ and $s = \frac{1}{2}gt^2$ ; a subject
which troubles me not a little. __
Is the following a correct development of the note in
Useful Knowledge Mechanics? I re-copy the notes first ;
[113r] "$V = \frac{dS}{dT} = gT$. Hence $dS = gT.dT$, which being
"integrated gives $S = \frac{1}{2}g.T^2$"
[something crossed out on two lines]
The Integral of $\frac{dS}{dT}dT$ or of $gT.dT$ will
obviously give us $S$ ; & we know that $\int gT.dT =$
$= \frac{1}{2}gT^2 + C$, (by formula of page 104 of the Calculus).
But it appears to me that the statement
above "Hence $dS = gT.dT$" is an unnecessary
intermediate step :
It is true that $\int \frac{dS}{dT}dT = \int dS$,
that is providing we extend the theorem

$$\int fx\frac{dx}{dt}dt = \int fx.dx$$

to the case when $fx = 1$, which I conclude it is
allowable to do, since 1 may be considered a
function of anything, I imagine ; thro' the formula
$\frac{fx}{fx} = 1$. But tho' true, yet the above ['clause' inserted] appears
to me ['an' inserted] unnecessary introduction. __
I am not sure that I have explained myself well.
With respect to this formula
$\frac{1}{2}gt^2$, & it's [sic] derivation & application ; I have
referred as you desired to pages 27, 28, & have
[113v] fully refreshed my memory upon them. But I
do not feel this helps me much. In this first
place the process is the converse of that I enquired
upon. $S$ is there give, & $V$ is to be derived
from $S$. My position was ; __ $V$ given, & $S$ to
be derived from $V$. _
I understand the process of pages 27, 28, considered
as a distinct & separate thing. But I do not
identify it with Differentiation or Integration. ____
I, (knowing by abstract rules & theorems) that
$2x$ is the diff_co of $x^2$, see that the limit $2t$
which comes out, might be perfectly well expressed
by $\frac{d(t^2)}{dt}$. And that we may put the result
of the Differentiation of $t^2$, and the result of all
the reasoning of pages 27, 28, indifferently one for
the other. But I only see it as I see that

in the processes $12 \div 4 = 3$ $1 + 2 = 3$ we might indifferently put the results (3, in both cases) one for the other. There may, for anything I yet see or understand, be as little connection between the abstract process of Differentiation and the Stone-falling process, as between the above processes of Division & Addition, which latter tho' their results agree, cannot be identified, or one made to represent [114r] the other. __
I apprehend [something crossed out] you will perhaps answer me here, that I must wait patiently for Chapter 8, in which (page 143) I see something very like an explanation of all I want. __ At the same time I think it better to express fully my difficulties.

I am very anxious to see your Comments on my two papers ['sent the other day' inserted] upon $\frac{1}{2}gt^2$. For I do not see where the flaw in them can be ; & yet I suppose there is one. It is some comfort in the confusions & puzzles one makes, that they are always exceedingly amusing to me, after they are cleared away. And this is at least some compensation for the plague of them before. _
With many thanks,

Yours most truly
A. A. Lovelace

[115r]

Ockham Park
Sunday. 15th Augst
['1841' added by later reader]

Dear Mr De Morgan. You must be beginning to think me lost. I have been however hard at work, with the exception of 10 days complete interruption from company. __ I have now many thing to enquire. __ First of all; can I spend an evening with Mrs De Morgan & yourself on Tuesday the 24th? On that day we go to Town to remain till Friday, when we move down to Ashley for 2 months at least. __ I would endeavour to be early in Gower St; before eight or not later than eight. And I feel as if I should have many mathematical things to discuss. ____
Now to my business : ____
1stly: I send you a paper marked 1, containing my development of two Integrals in page 116,
$\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right)$
And $\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x+a+\sqrt{2ax+x^2}) + \log 2$
The former one I think is plain enough, & I and the book are quite agreed upon it. Not so with [115v] the latter one, & I begin to suspect the book. I cannot make it anything but

$$\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x+a+\sqrt{2ax+x^2})$$

or else $\qquad = \log(\frac{x}{2}+a+\frac{\sqrt{2ax+x^2}}{2}) + \log 2$

I have tried various methods ; ____ but the only one which I find hold good [*sic*] at all, is that applied in page 115 to $\int \frac{dx}{\sqrt{a^2+x^2}}$, & which seems clearly to bring out
my result above. By the bye I have a remark to make on the Integration of $\int \frac{dx}{\sqrt{a^2+x^2}}$ as developped [*sic*] in page 115. __
Line 10 (from the bottom), you have $xdx = ydy$:_ This is obvious, & similarly I deduce in my paper No 1, $(2a+x)dx = ydy$. _ But I see no use in what follows, "and $ydx + xdx = ydx + ydy$". _ It is equally obvious with the former equation, but seems to me to have no purpose in bringing out the results, which I deduce as follows :

Since $xdx = ydy$, we have $\frac{dx}{y} = \frac{dy}{x}$
Therefore by the Theorem of page 48, or at least ['by' inserted] a Corollary of it, we have $\frac{dx+dy}{x+y} = \frac{dx}{y}$, whence &c, &c.
And this is the method also which I have used [116r] in developping [*sic*] $\int \frac{dx}{\sqrt{2ax+x^2}}$. ____
2ndly: Page 113, lines 16 ['&c 17' inserted] from the bottom, you say "The "first form becomes impossible when $x$ is greater than "$\sqrt{c}$, for in that case the Integral becomes the "Logarithm of a Negative Quantity". Now there are surely certain cases in which negative quantities may be powers, & therefore may have Logarithms.
All the odd whole numbers may surely be the Logarithms of Negative Quantities. __

$(-a) \times (-a) = a^2$ But $(-a) \times (-a) \times (-a) = -a^3$
or $(+a) \times (+a) = a^2$ $\quad (+a) \times (+a) \times (+a) = a^3$

3 is here surely the Logarithm of a Negative Quantity.
Similarly a negative quantity multiplied into itself any odd number of times will give a negative result.
3dly: In the Paper marked 3, which I return again ['for reference' inserted] ; I perfectly understand the proof by means of the Logarithms (added by you), why $\frac{dy}{dx}$ can only $= \frac{y}{x}$ when $y$ is either $= x$, or $= ax$ ($a$ being Constant)
Your proof is perfect, but still I do not see that mine was not sufficient, tho' derived from much more general grounds. __
My argument was as follows : Given us $\frac{dy}{dx} = \frac{y}{x}$, what conditions must be fulfilled in order to make this equation possible? _ Firstly : I see that [116v] since $\frac{dy}{dx}$ means a Differential Co-efficient, which from it's [*sic*] nature (being a Limit) is a constant & fixed thing, $\frac{y}{x}$ must also be a constant & fixed quantity. That is $y$ must have to $x$ a constant Ratio which we may call $a$. __
This seems to me perfectly valid. And surely a Differential Co-efficient is as fixed & invariable in it's [*sic*] nature as anything under the sun can be.
To be sure you may say that there is a different Differential Co-efficient for every different initial value of $x$ taken to start from, thus :

$$\frac{d(x^2)}{dx} = 2x \qquad \text{if } x = a, \ \frac{d(x^2)}{dx} = 2a$$

$$\text{if } x = b,\ \frac{d(x^2)}{dx} = 2b$$

And this is perhaps what invalidates my argument
above. __
4[thly]: In the two papers folded together & marked 2,
which I also again return for reference, I perfectly see
that tho' mathematically correct. I was completely
wrong in my application. But my proofs do apply
to any two different & independent velocities, whatever
of two different bodies, or of the same body moving
at two different uniform ratio [*sic*] at different epochs. __

Thus my paper (marked upon it 1st Paper) proves
[117r] the following : that the Spaces moved over at two
different times, in virtue of the Velocity acquired at
the end of each of those times, (the impelling cause
being supposed to cease at the end respectively of each
time fixed on), would be to each other as the
squares of the times fixed on. _ But I perfectly
see that this is quite a different & independent
consideration from that of the Space actually moved
over by a body impelled by an accelerating force,
& how wholly inapplicable my ['former' inserted] view of it was. __

I have been especially studying this
subject of [something crossed out] Accelerating Force, & believe that I now understand
it very completely. I found I could not rest upon
it at all, until I made the whole of the subject out
entirely to my satisfaction : _ I enclose you (marked
4) the first of a Series of papers I am making out
in the different parts ['of' inserted] it. This one is the more
general development of the particular case of Gravitation
in pages 27, 28 ; & my more especial object in it
has been the identification of the results arrived at
in this real application, with the Mathematical
Differential Co-efficient. __
I have worked most earnestly & incessantly at the
Application of the Differential & Integral Calculus to the
[117v] subject of Accelerating Force, & Accelerated Motion,
during the last 2 or 3 weeks. It has interested me
beyond everything. After making out (according to
my own notions) the two papers on $v = \frac{ds}{dt}$, and
$s = \int vdt$, (the first of which I now send, & the
Second you will have in a day or two), I attacked
your Chapter 8, pages 144, 145, worked out all the

Formulae there ; & had excessive trouble with my
third paper on $t = \int \frac{ds}{v}$, (now successfully terminated);
and I am now on $f = \frac{dv}{dt}$, page 146. ____
You will perhaps not approve my having thus run
a little riot, & anticipated. But I think it has
done me great good. And I am anxious to know
if I may read the rest of this Chapter 8, before
reading Chapter 7 on Trigonometrical Analysis ; & if
I am likely to understand it all without having
read Chapter 7.
I shall probably write again tomorrow ; or if not,
certainly I shall on Tues^dy. _
We are very anxious to know if there is no time
between the 1st Nov^r & the middle of Feb^y, when you &
M^rs De Morgan (& family) would come & stay here
for as long as you can & would like. We should
be delighted if you would remain 2 or 3 weeks.
[118r] And if this should be impossible for you, perhaps still
you would bring M^rs De Morgan & the children here,
& remain a few days ; having them to stay longer.

We both of us assure you that it would be
no inconvenience whatever to us ; but rather contrary
the greatest pleasure. And I am certain it
would do M^rs De Morgan good to be here for
a time. __ Pray consider my proposal ; __
at any rate for her & the children, if your own
avocations should make it impossible for you even.
Believe me

Yours most truly

A. A. Lovelace

[119r]

Ockham.
Monday Morning
16th ['July' crossed out] Augst
['1841' added by later reader]

Dear Mr De Morgan. I send you today, 1stly:
a paper in which I have proved the Theorem
$\int fx\frac{dx}{dt}dt = \int fx.dx$ (of pages 102, 103), backwards.
That is I have assumed $\int fx.dx$, & deduced
from it that when $x = \psi t$, then $\int fx.dx = \int fx\frac{dx}{dt}dt$;
whereas in the book $\int fx\frac{dx}{dt}dt$ was assumed,
& the process was exactly reversed. __
I do not send this as having any advantage
over the other proof. Merely because it happened
accidentally to strike me, & I wrote it down ;
& I now ['may' crossed out] enclose it for inspection, to see that I
have correctly deduced each step. __
2ndly: I enclose you my Second Paper on ['the' inserted] Accelerating
Force subject. This one is the explanation of

$$S = \int v.dt$$

Tomorrow I hope to send you the one on $t = \int \frac{ds}{v}$.
[119v] I hope I am not plaguing you very much.
I am anxious to read up to a certain point
before moving to Devonshire. __

Believe me

Yours most truly
A. A. Lovelace

[121r]

Ockham.

Friday. [something crossed out] 21st Augst

Dear Mr De Morgan. You have received safely I hope my packet of yesterday, & my packet sent on Tuesday. _

I now re-enclose you the paper marked 1. There is another Integral added at the bottom. Also I have altered one or two little minutiae in the development of $\int \frac{dx}{\sqrt{2ax+x^2}}$ above, which you had omitted to correct. _

I quite understand your observations upon it, & see the mistake I had made ; & which related to the Differential $dy$, and $d(\varphi x.x)$

If $y = \varphi x.x$, then $dy = d(\varphi x.x) = \frac{d(\varphi x.x)}{dx} dx =$

$$= \left\{ x.\frac{d(\varphi x)}{dx} + \varphi x.\frac{dx}{dx} \right\} dx = x.\left( \frac{d(\varphi x)}{dx} dx \right) + \varphi x.dx =$$
$$= x.d\varphi x + \varphi x.dx$$

Or if $y^2 = \varphi x.x$, then $dy^2 = x.d.\varphi x + \varphi x.dx$

$$\text{or } \frac{d(y^2)}{dy} dy = x.d.\varphi x + \varphi x.dx$$
$$\text{or } 2ydy = x.d.\varphi x + \varphi x.dx, \text{ and } ydy = \tfrac{1}{2} x.d.\varphi x + \tfrac{1}{2} \varphi x.dx$$

[121v] This is all now right in my head.

In $\int \frac{dx}{\sqrt{2ax+x^2}}$ we arrive then in my corrected paper, at $\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x + a + \sqrt{2ax + x^2})$

$$= \log\left( \frac{x}{2} + \frac{a}{2} + \frac{\sqrt{2ax+x^2}}{2} \right) + \log 2$$

which, as you observe "again with the book all but "the log .2, which being a Constant, matters nothing". Very true ; but why did you then insist the log 2 in page 116?_ it seems as if put in on purpose to be effaced in the parenthesis (Omit the Constant). And it might just as well have been log 3, log 4, log (anything in the world). ____

As to my two papers marked 2 (& which I again return, merely for the convenience of reference), I see that in order to make them valid, as applying each to two separate & different velocities, they should be re-written (which is not worth while), & the terms of the enunications altered as follows :

"If two quantities $V$, $V'$ be respectively equal to the "Ratios $\frac{S}{T}$, $\frac{S'}{T'}$, and if $V : V' = T : T'$, then the values "$S$, $S'$ must be to each other as the squares of $T$, $T'$

"are to one another" &c, &c
[122r] At last I believe I have it quite correctly. __
As for $\frac{dy}{dx} = \frac{y}{x}$, I see my fallacy about $\frac{y}{x}$ being
a fixed quantity. _
About page 113, "The first form becomes impossible
"when $x$ is greater than $\sqrt{c}$, for &c", I fancy I
[something crossed out] had a little misunderstood
the mathematical meaning of the words impossible
quantity. I have loosely interpreted it as being
equivalent to "an absurdity", or at least to
"an absurdity, unless an extension be made in the
"ordinary meaning of words". _ And in this
instance I perceived that if the Logarithm be
an odd number, there would be no absurdity
even without extension in the meaning of terms ;__
because that it would then merely imply a
negative Base ; which negative Base, would I think
be admitted theoretically (tho' inconvenient practically)
on the common beginner's instruction on the Theory of
Logarithms. __ Am I right? __
By the bye this subject reminds me that I think
I find a mistake in page 117, line 13 (from the top)
"($n$ an integer) $\int_{-a}^{+a} x^n dx = 0$ when $n$ is odd, $= \frac{2a^{n+1}}{n+1}$ when $n$ is even"
[122v] It seems to me just the reverse, thus :
$= 0$ when $n$ is even, $= \frac{2a^{n+1}}{n+1}$ when $n$ is odd
I have it as follows :

$$\int_{-a}^{+a} x^n dx = \frac{a^{n+1}}{n+1} - \frac{(-a)^{n+1}}{n+1} = \frac{a^{n+1}-(-a)^{n+1}}{n+1} =$$

$$= \frac{a^{n+1}-a^{n+1}}{n+1} \text{ or } 0 \text{ if } n+1 \text{ be even}$$

(I now see it while working ; for if
$n+1$ be even, $n$ must be odd.)
and vice versa.

So I need not trouble you upon this ; as I have
solved my difficulty whilst stating it. I had only
looked at this Integral in ['a' inserted] great hurry, this morning.

I hope on Sunday to send you two
remaining papers I have to make out, on the
Accelerating Force subject ;
upon $f = \frac{dv}{dt}$, and $v = \int f dt$
I think I have been encouraged by your great
kindness, so as to give you really no Sinecure
just at this moment.

Yours most truly
A. A. Lovelace

[123r]

Ashley Combe
Thurs$^{dy}$ Mor$^{g}$
9$^{th}$ Sep$^{r}$ ['1841' added by later reader]

Dear M$^{r}$ De Morgan. I have rather a large batch now for you altogether :
1$^{stly}$: I am in the middle of the article on Negative & Impossible Quantities ; & I have a question to put on page 134, (Second Column, lines 1, 2, 3, 4, 5 from the bottom)

$$(a+bk)^{m+nk} = \varepsilon^{A}\cos B + k\,\varepsilon^{A}.\sin B \text{ \&c}$$

I have tried a little to demonstrate this Formula ; but before I proceed further in spending more time upon it, I think I may as well ask if it is <u>intended</u> to be <u>demonstrable by the Student</u>. For you know I sometimes try to do more than anyone means me to attempt. I have as yet only got thus far [something crossed] with the above formula : If in $(a+bk)^{m+nk}$,
$r$ is given $=\sqrt{a^2+b^2}$, ['and' inserted] $\tan\theta = \frac{b}{a}$ ; then $\sin\theta = b$
$\cos\theta = a$
and $(a+bk)^{m+nk} = (\cos\theta + k.\sin\theta)^{m+nk} =$

$$= \left(\varepsilon^{k\theta}\right)^{m+nk} = \varepsilon^{k(m\theta)} \times \left\{\varepsilon^{k(n\theta)}\right\}^{k}$$

or $= (\cos.m\theta + k.\sin.m\theta) \times (\cos.n\theta + k.\sin.m\theta)^{k}$, and
[123v] I dare say that from some of these transformations, the <u>Second Side</u> of the given equation, with the determination of $A$ and $B$, may be deduced. But it appears to me ['it must be' inserted] a very <u>complicated process</u> ; & therefore I should like to know before I undertook it, that I was not <u>wasting time</u> ['in' deleted] doing so. __
2$^{dly}$: I am plagued over page 135 of the Calculus. It is not that there is <u>any one thing</u> in it which I do not clearly see. But it is the <u>depth of the whole argument</u> which I cannot manage to discover. I should say that whole argument from "We now know &c" page 134, to "We can therefore take a function, "which, for a particular value of $x$, &c, &c" page 135.

It seems to me all to be <u>much ado about nothing</u>; and I do not see <u>what is arrived at</u> by means of it [something crossed out]. A very complicated process appears to be used in the 1$^{st}$ Paragraph of page 135, to prove that when <u>$h$ is small</u> then the Increment in $\varphi x$ is <u>very nearly</u> represented by $\varphi' a + h$, which was already shown in page 134. And then suddenly in the <u>Second Paragraph</u> the Formula $\varphi a + \varphi' a(x-a) + {}$ $+\varphi'' a\frac{(x-a)^2}{2}$ is introduced, & I do not understand <u>à quoi bon</u> the closing conclusion drawn from it.

$3^{\text{dly}}$: I am not sure that I agree to what you say in preference (for ascertaining Maxima & Minima) of the <u>direct ascertainment of</u> <u>the value of $\varphi' x$</u>, over [124r] the ordinary method. Because it seems to me in many cases impossible <u>after</u> you have determined 0 or $\alpha$ values of $\varphi' x$, to <u>determine</u> further that the sign d<u>oes</u> change at them & <u>how</u> it changes, <u>unless by means of the ordinary rule</u>. I have written out and enclose an example from Peacock, in which unless I had used the ordinary rule, a<u>fter</u> I had determined 0 values for $\varphi' x$, I should have been at my wits' end how to bring out the conclusion.

$4^{\text{thly}}$: I send you a little Maxima & Minima Theorem of my own, which occurred to me by accident ; It is for $\varphi x = x^2 - mx$. After proving it by the Differential Calculus, I have given a d<u>irect</u> <u>proof</u> of another sort. I merely wrote this ['<u>direct proof</u>' inserted], because it [something crossed out] occurred to me ; but it gave me <u>a great deal of trouble</u>, & I think was rather a <u>work of supererogation</u>; but <u>I believe</u> it is quite correct. You will find enclosed in the same sheet the demonstration of "What is the number whose excess above it's [*sic*] Square "Root is the <u>least possible</u>?" (see page 133 of the Calculus) ; and on the reverse side of this latter [something crossed out] is the "verification round the 4 Right Angles" for the <u>continual increase together</u> of $x$ and it's [*sic*] tangent (See page 132). But here I have something further to add. In this Chapter VIII, we hear of Differential Co-efficient which become $= 0$, or $= \alpha$. In this very [124v] instance, $1 + \tan^2 x$ is alternately $= 1$, and $= \alpha$. __ Now according to my p<u>revious ideas</u>, the terms <u>Differential Co-efficient</u> was only applied to some f<u>init</u>e quantity ; and referring to pages 47, 48, where one acquired one's f<u>irst ideas</u> of a Differential Co-efficient, I think it is there clearly explained that the term is only used with reference to a <u>finite limit</u>. But in this Chapter VIII, there seems to be a considerable <u>extension</u> of <u>meaning</u> on

the subject. __
5thly. In page 132, it is very clearly deduced that
the Ratio of a [something crossed out] Logarithm to it's [*sic*] number is increasing
as long as $x$ is $< \varepsilon$, and afterwards decreases. __

The proof is most obvious. But, unluckily, the
conclusion seems to me to be contrary to the fact ; at
least the first half of the conclusion, not the latter half.

On this principle : from the very nature of a
Logarithm, it is obvious that ($x$ being $> \varepsilon$), for
equal increments to the log, $x$, there will be
larger & larger Increments to $x$. The one being in
arithmetical, the other in geometrical progression.
Therefore clearly the Ratio of the Logarithm to the
number, is a diminishing one. But then the
same thing seems to me to apply [something crossed out] when
$x < \varepsilon$. Surely there is then just the same
[125r] arithmetical & geometrical progression for equal
Increments of the Logarithms. I suppose there is
some link that I have over-looked.

I send you two Integrations worked out. They
are from Peacock. I in vain spent hours over the
one marked 2, of which I could make nothing by
any method that I devised ; until in despair, I
looked thro' your Chapter XIII to see if I could there
find any hints ; & accordingly at page 277, I
found a general formula which included this
case. But I do not believe I should ever have
hit upon it by myself. The Integral marked 1,
might of course be proved also in the same way ;
tho' ['my' crossed out] the method ['I have used' inserted] is sufficient in this instance. __

I have written out no more papers on
Forces. In fact there is only one more that is
left for me, viz: $f = v\frac{dv}{ds}$. And for this I see
no occasion ; for I am sure that I must thoro'ly
understand it, after all I have written. ____
I quite see ['the truth' inserted] your remarks on my having treated
Acceleration of Velocity as being identical with
Force ; whereas, (as I now understand it), it is
simply the measure of Force, & our only way of
getting at expressions for this latter. On the subject
[125v] of $v^2 = 2\int f ds + C$ ; I have considered it a great
deal ; and any direct demonstration of it, after the

manner of my other papers, seems to me to be
quite impracticable. Neither $\int v.dv$, nor $\int f.ds$
[something crossed out] now appear to me to have any actual proto-types in the
real motion. Here then suggests itself to me the
question : "then are there certain truths & conclusions
"which can be arrived at by pure analysis, & in
"no other way?" And also, how far abstract
analytical expressions must express & mean
something real, or not. In short, it has
suggested to me a good deal of enquiry, which
I am desirous of being put in the way of
satisfying. __
By the bye, I fear that one little paper of mine
dropped out of the last packet. It was a little
pencil memorandum on ['the meaning of' inserted] $\int f.ds$ ; & there were
remarks upon it, (if you remember) in my accompanying
letter. It bears upon the above question.
I could write it out again, if it has been lost.__

Is not this a budget indeed? __

Yours most truly

A. A. Lovelace

[127r]

Ashley-Combe
Sunday. 19th Sepr ['1841' added by later reader]

Dear Mr De Morgan. I have more to say to you than ever ; ( beginning with many thanks for your bountiful replies to my last packet). __
I will begin with the Article on Negative & Impossible Quantities, on which I have <u>a good deal</u> to remark. I have finished it ; & I think with on the whole great success. I need scarcely say that I like it <u>parenthetically</u>. __ I enclose you the demonstration of the formula $(a + bk)^{m+nk} = \varepsilon^A. \cos B + k.\varepsilon^A. \sin B$, which I found exceedingly easy, after your observations. ____

I should tell you that the allusions to the <u>Irreducible Case</u> of Cubic Equations in this Article, has so excited my curiosity on the subject, that I have attacked the chapter on Cubic Equations in ['page 47' inserted] of R. Murphy's treatise on the Theory of Equations (Library Useful Knowledge), hoping there to gain some light on the subject. For I know not to w<u>hat</u> exactly this alluded, (my Algebra wits, as you say, not having been quite proportionally stretched with some of my other wits). I have got thro' the first two pages ; and [127v] shall have to write you some remarks upon t<u>hese</u>, either in this letter, or in one as soon as possible. But as yet I meet with nothing about

$$\sqrt[3]{(a + b\sqrt{-1})} + \sqrt[3]{(a - b\sqrt{-1})}$$

I hope I shall be able to <u>understand</u> the rest of the Chapter. __
At the bottom of my demonstration of $(a + bk)^{m+nk}$, you will find a memorandum (simple as to the <u>working out</u>) of the formula $\cos .(a + bk)$, see page 137 of the Cyclopedia. You there say that such a formula <u>may be interpreted by it's [*sic*] identical expression on the Second Side</u>. That is to say I imagine that the meaning of $\cos(a + b\sqrt{-1})$, which (as before pointed out in the case of the line $h$) is a misapplication of symbols, may be got at thro' an examination of the r<u>esults</u> arrived at by ['the application of' inserted] symbolical rules to this unmeaning or mis-meaning expression. That if in a calculation, such an answer as $\cos(a + b\sqrt{-1})$ were worked out, the answer means in fact

[something crossed out]
the remaining side of a parallelogram in which
$\cos\alpha\frac{\varepsilon^b+\varepsilon^{-b}}{2}$ is a diagonal, and $\sin\alpha.\frac{\varepsilon^b-\varepsilon^{-b}}{2}.k$ the
other side : the diagonal itself being a 4th Proportional
to 1, $\cos\alpha$, $\frac{\varepsilon^b+\varepsilon^{-b}}{2}$, inclined to 1 ['(that is to the Unit-Line)' inserted] as the $\cos\alpha$ is ; &
the remaining side being a 4th Proportional to 1, $\sin\alpha$,
$\frac{\varepsilon^b-\varepsilon^{-b}}{2}$ inclined to 1 at an angle equal to the sum of a
[128r] Right-Angle and the angle made by $\sin.\alpha$ with the Unit-Line.

I enclose you an explanation I
have written out (according to the Definition of this
Geometrical Algebra), of the two formulae for the Sine
and Cosine. I am at work now on the Trigonometrical
Chapters of the Differential Calculus.
I do not agree to what is said in page 119 ['(of the Calculus)' inserted] that
results would be the same whether we worked [something crossed out] algebraically
with forms expressive of quantities or not. It is true
that ['in' inserted] the form $a+\sqrt{m}-\sqrt{n}$, if $(-1$ be substituted for
$m$ and $n$, the results come out the same as if we
work with $a$ only. but were the form $a+\sqrt{m}$,
$a-\sqrt{m}$, $a\times\sqrt{m}$, or fifty others one can thin of,
surely the substitution of $(-1)$ for $m$ will not bring
out results the same as if we worked with $a$ only;
and in fact can only do so when the impossible
expression is so introduced as to neutralize itself,
if I may so speak. I think I have explained
myself clearly. __
It cannot help striking me that this extension of
Algebra ought to lead to a further extension
similar in nature, to Geometry in Three-Dimensions ;
& that again perhaps to a further extension into
some unknown region, & so on ad-infinitum possibly.
And that it is especially the consideration of
an angle $=\sqrt{-1}$, which should lead to this ; a symbol,
which when it appears, sees to me in no way more
[128v] satisfactorily accounted for & explained than was
formerly the appearances ['which' inserted] Bombelli in some degree
cleared up by showing that at any rate they
(tho' in themselves unintelligible) led to intelligible
& true results. You do hint in parts of page 136
at the possibility of something of this sort. __

I enclose you also a paper I have written explaining a difficulty of mine in the Definitions of this Geometrical Algebra. __

It appears to me that there is no getting on at all without this Algebra. In the 3d Chapter of your Trigonometry (which I have just been going thro'), though there are no impossible quantities introduced ; yet how unintelligible are such formulae as $2ac.\cos B$, $a.\sin B$, or any in short where lines are multiplied into lines, if one only takes the common notion of a line into a line being a Rectangle. ____

I cannot send more today ; but I have many other matters to write on ; especially the Logarithmic Theory at the end of the Article. I am considering it very carefully ; & studying at the same time the Article on Logarithms in the Cyclopedia. And I believe I shall have much to say on it all.

The passage I wanted to ask you about in Lamé's [129r] 1st Vol, is pages 54, 55, 56, on the Resultant of the pressures of a liquid on a vase. I want to know if I ought to understand these three pages, or if they entail some knowledge of mathematical (especially of trigonometrical) application to Mechanics, which I do not yet possess. __

I hope you receive game regularly.

Yours most truly

A. A. Lovelace

P.S. Did you ever hear of a Science called Descriptive Geometry? I think Monge is the originator of it. ____

[130r] Dear M$^{r}$ de Morgan
I have for the last
fortnight ['been' inserted] daily intending to
write to you on mathematical
matters ; & now I do not
think it worth while because
I have an idea of being in
Town on Tues$^{dy}$ next for a
day or two. And if could
see you on Wed$^{dy}$, either by
your kindly coming to me
[130v] at 12 o'clock that day, or by
my going to spend Wed$^{dy}$
Ev$^{g}$ in Gower S$^{t}$, it
would answer the purpose much
better. Besides having a
list of ['<u>particular</u>,' inserted] little things to ask
you about ; I am now anxious
to consult you again as to
my <u>general</u> progress & way of
going on. I have one or
two little difficulties just now.
I believe we shall
remain on here for some
weeks longer. My intended
journey to Town is only on
particular business. And by the
bye it is <u>not to be known</u>
<u>that I am going</u>. My mother
[131r] even has no idea of it ; & I
do not wish that she should.
So I will thank you & M$^{rs}$
De Morgan to mention nothing
about it to any one.
How is she? And <u>when</u> to
be confined? __
Yours most truly
A. A. Lovelace

Ashley-Combe
Porlock
Somerset
Wed$^{dy}$ 27$^{th}$ Oct$^{r}$ ['1841' added by later reader]

[132r]

Ashley-Combe
Porlock
Somerset
Thurs$^{dy}$ $4^{th}$ Nov$^{r}$ ['1841' inserted by later reader]

Dear M$^{r}$ De Morgan. As I find my journey to Town is extremely uncertain, & may possibly even not take place at all, I will trouble you without further delay on the more important of my present points of difficulty. _

I will begin with those relating to Chapter $9^{th}$ of the Calculus, which I am now studying. I have arrived at page 156.

page 132 : (at the bottom). I make $u = \cos^{-1}\left(\frac{1-\varepsilon^{2(C-x)}}{\varepsilon^{2(C-x)}+1}\right)$

instead of $u = \cos^{-1}\left(\frac{\varepsilon^{2(C-x)}-1}{\varepsilon^{2(C-x)}+1}\right)$

I enclose a paper with my version of it. _

page 153 : "For instance, we should not recommend "the student to write the preceding thus, $d^2.du + d^2x.du = 0$, "tho' is it certainly true that upon the implicit "suppositions with regard to the successive Increments, "$\Delta^2u.\Delta x + \Delta^2x.\Delta u$ diminishes without limit as compared "with $(\Delta x)^3$." Why this comparison with $(\Delta x)^3$?

[132v] Had the expression been $\frac{\Delta^2u.\Delta x+\Delta^2x.\Delta u}{(\Delta x)^3}$ instead of $\Delta^2u.\Delta x + \Delta^2x.\Delta u$, it would then be clear that if the Numerator diminished without limit with respect to the Denominator, the fraction itself would approach without limit to 0. But as it is, I see no purpose answered by a comparison with $(\Delta x)^3$.

Also, I not only do not see the object of this comparison, but I do not perceive the fact itself either.

Where is the proof that $\Delta^2u.\Delta x + \Delta^2x.\Delta u$ does diminish without limit with respect to $(\Delta x)^3$? ____

Page 135 : (at the top) : There is a slight misprint $C = K^2 + K^{12}$ instead of $C = K^2 + K'^2$

Page 156 : (line 9 from the top) : $u = C.\sin\theta + C'.\cos\theta + \frac{1}{2}\theta.\sin\theta$
(Explain this step?)

Now I cannot "explain this step".

In the previous line, we have :

(1)... $u = C\sin\theta + C'\cos\theta + \frac{1}{2}\theta.\sin\theta + \frac{1}{4}\cos\theta$ (quite clear)

(2)... And $u = \cos\theta - \frac{d^2u}{d\theta^2}$ (by hypothesis)

$$= \tfrac{1}{4}\cos\theta + \left(\tfrac{3}{4}\cos\theta - \tfrac{d^2u}{d\theta^2}\right)$$

whence one may conclude that

$$C.\sin\theta + C'\cos\theta + \tfrac{1}{2}\theta.\sin\theta = \tfrac{3}{4}\cos\theta - \tfrac{d^u}{d\theta^2}$$

But how $u = C\sin\theta + C'\cos\theta + \sin\theta.\frac{1}{2}\theta$ is to be deduced [133r] I do not discover : By subtracting $\frac{1}{4}\cos\theta$ from both sides of (1), we get

$$u - \tfrac{1}{4}\cos\theta = C\sin\theta + C'.\cos\theta + \tfrac{1}{2}\theta.\sin\theta$$

But unless $\frac{1}{4}\cos\theta = 0$, (which would only be the case I conceive if $\theta = \frac{w}{2}$), I do not see how to derive the equation in line 9 of the book.

Page 156 : Show that $\frac{d^2u}{dx^2} - u = X$ (a function of $x$)

gives $u = C\varepsilon^x + C'\varepsilon^{-x} + \frac{1}{2}\varepsilon^x\int\varepsilon^{-x}X.du - \frac{1}{2}\varepsilon^{-x}\int\varepsilon^x X.dx$

---

I have, $\frac{d^2u}{dx^2} - u = X \qquad u = K\varepsilon^x + K'\varepsilon^{-x}$

$$\tfrac{du}{dx} = K\varepsilon^x + K'\varepsilon^{-x} + \tfrac{dK}{dx}\varepsilon^x + \tfrac{dK'}{dx}\varepsilon^{-x}$$

Assume $\frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} = 0$

Then $\frac{du}{dx} = K\varepsilon^x + K'\varepsilon^{-x}$, and $\frac{d^2u}{du^2} = K\varepsilon^x + K'\varepsilon^{-x} + \frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x}$

$$\left.\begin{array}{l} X = \frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} \\ 0 = \frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} \end{array}\right\}$$

which tell nothing at all as to the values of $\frac{dK}{dx}$, $\frac{dK'}{dx}$ of $K$, & of $K'$

If we had $$\left.\begin{array}{l} X = \frac{dK}{dx}\varepsilon^x - \frac{dK'}{dx}\varepsilon^{-x} \\ 0 = \frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} \end{array}\right\}$$ the expression in the book will be then at once deduced.

[133v] But I do not see how to get these two latter equations co-existent. __

I enclose an attempt of mine, making the assumed to be $\frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} = x^2$ instead of $= 0$ ; and also ['one' inserted] making this relation to be $K + K' = x^3$, but which latter I found led to such very complicated results that I proceeded but a little way, thinking it a probable loss of time to go on.

With the relationship $\frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} = x^2$, I am as unsuccessful as with $= 0$. _

I defer to another letter some other difficulties of mine not relating to this Chapter, but partly to some remaining points in the 8th Chapter, & partly to miscellaneous matters. __

I hope M$^{rs}$ De Morgan & the "large boy" continue to flourish. So M$^{rs}$ De M has beat the Queen in the race, out & out! ——

Yours most truly

A. A. L.

[134r]

Ashley-Combe
Monday. 8th Nov[r] ['1841' added by later reader]

Dear M[r] De Morgan. I hope you intend to christen the "large boy" by the name of Podge, with which I am particularly pleased.
I am much obliged by your letter. I send a corrected version (now I believe quite right) of $\frac{d^2u}{dx^2} - u = X$; on my assumed supposition $\frac{dK}{dx}\varepsilon^x + \frac{dK'}{dx}\varepsilon^{-x} = 0$. As for my other assumption $K + K' = x^3$, it is so complicated a one that I have not thought it worth while to pursue it's [*sic*] development.
I cannot think how I could be so negligent as to forget that $\varepsilon^{-x}$ is a function of $(-x)$ which is itself a function of $x$. A complete oversight ; as indeed most of the enquiries in my last letter seem to have been.
I should perhaps mention that lately [something crossed out] I have had my mind a good deal distracted by some circumstances of considerable annoyance & anxiety to me ; & I have certainly studied much less well & more negligently in consequence. Indeed the last few weeks I have not at all got on as I wished and intended ; & I find that to force myself, (when [134v] disinclined & distraite), beyond a certain point is very disadvantageous. So on these occasions I just keep gently going, without however attempting very much. I am hoping now to get a good lift again before long ; as I think I am returning to a more settled & concentrated state of mind. I mention all this as an excuse for some errors & over-sights which I conceive are more likely just at present to creep into my performances than would usually be the case. Now to business : Chapter VIII :
1. I send you two Problems on hypotheses of my own, intended as being worked out on the model of those in page 150. There are three different Hypotheses. In the one where I obtain $t = \frac{1}{\sqrt{2}}\int \frac{ds}{\sqrt{s^{-1}-a^{-1}}}$ I have not attempted to develop this Integral further. Perhaps I ought to have done so ; but it was only my object to get quite a general expression.
2. Page 141 ; (lines 9, 10, from the bottom) : Series in page 116 (of Chapter VI), it was shown that $\int \frac{ds}{\sqrt{2kx-x^2}} = \sin^{-1}\frac{x-k}{x} =$

$= \left(v \sin^{-1} \frac{x}{k}\right) + \left(\frac{w}{2}\right)$, I do not see how it can be said (page 141)
that the Constant may have any value $P$.
3. I have never succeeded in properly understanding
the Paragraph beginning page 134, ending page 135, on
which I before applied to you ; & the paragraph of
page 148 – (marked 2) – has only added to my mistiness on
[135r] the subject. There is something or other which I cannot
get at in the argument & it's [*sic*] objects. That of page
135 seems very like another way of arriving at
Taylor's Theorem. The expression taken in line 25 from
the top, I conclude to be arrived at as follows :
Having obtained $\varphi a + \varphi' a.(x-a)$ ; a function agreeing
in value and diff-co with $\varphi x$ when $x = a$, let us now
find a function agreeing not only in these two points
but also in second diff-co with $\varphi x$, when $x = a$ ;
(the same conditions being continued of $\varphi' a$, $\varphi'' a$) ;
We see therefore that $\varphi' a$ must be of the form

$$\varphi'' a.x + m \quad \text{where} \quad m = \varphi' a - \varphi'' a.a$$

Substituting this in $\varphi a + \varphi' a.(x-a)$ we have

$$\begin{aligned} \varphi a + (\varphi'' a.x + m)(x-a) &= \varphi a + (\varphi'' a.\overline{x-a} + \varphi' a)(x-a) \\ &= \varphi a + \varphi' a.(x-a) + \varphi'' a(x-a)^2 \end{aligned}$$

Similarly we may obtain $\varphi a + \varphi' a.(x-a) + \varphi'' a.(x-a)^2 + \varphi''' a.(x-a)^3$
(By the bye I don't see how you get $\frac{(x-a)^2}{2}$ and $\frac{(x-a)^3}{2.3}$, instead
of $(x-a)^2$ and $(x-a)^3$ as I make it).
But I cannot perceive what all this is for ; & (as
I mentioned below), paragraph 2 of page 148 has added
to my blindness. _ I am sorry to plague you again
about it. On receiving your former reply, I felt none
the wiser ; but determined to wait, thinking I
might see it as I went on, which is often the case
[135v] with difficulties.
I now proceed to some miscellaneous matters.
1. I make nothing of the Irreducible Case in the
Penny Cyclopedia. Is it perhaps for want of having
read Involution & Evolution? I am puzzled quite
in the beginning of the Article.
2. Article "Negative & Impossible Quantities" P. Cyclopedia – page 137
"If the logarithm of two Units inclined at angles $\theta$ and $\theta'$ be
added, (the bases being inclined at angles $\varphi$ and $\varphi'$) ; the result
is the logarithm of a Unit inclined &c, &c"
I cannot develop this ; but I enclose some remarks upon it.
3. In the treatise you sent me on the "Foundation of Algebra",

I cannot make out ['in' inserted] the least [something crossed out] (page 5), about the general
solution of $\varphi^2 x = ax$. I suspect I do not understand
the notation $f^{-1}x$. I quite understand $f^2 x$ or $\varphi^2 x$,
$f^n x$ or $\varphi^n x$. Judging by analogy, from page 82 of the
Differential & Integral Calculus, (where $\Delta^{-1}x$ is explained),
I conceive $f^{-1}x$ or $\varphi^{-1}x$ may mean "the quantity
"which having had an operation $f$ or $\varphi$ performed
"with & upon it, is $= x$." But I have considered
much over the last half of this page 5, & I can't
understand it. __
I have one or two other matters still to write about ;
but they do not press ; & this is plenty I think for
today. __ Pray congratulate Mrs De Morgan on the
arrival & prosperity of Podge. __

Yours most truly

A. A. L.

[136r]

Ashley-Combe
Wed$^{dy}$ Mor$^{g}$
10$^{th}$ Nov$^{r}$ ['1841' inserted by later reader]

Dear M$^{r}$ De Morgan.
In consequence of a sudden decision that I must go to Town tomorrow, I write without delay to beg for a line in S$^{t}$ James' Sq$^{re}$, if you will kindly fix any hour on Sat$^{dy}$ between nine & three o'clock, that may be most convenient to you to call on me there. I am [136v] very anxious to see you ; ____ & hope it may not be very inconvenient to you. __ I had not the least idea till an hour ago, that I should go to Town this week. It is in consequence of letters unexpectedly received. ____ I will just again mention that I do not wish my journey to be known ; & I have told no one except those necessarily concerned.
In much haste
Yours most truly
A. A. Lovelace

[138r]

Ashley Combe
Thursday
11th Novr ['1841' inserted by later reader]

Dear Mr De Morgan. What
<u>will</u> you say when I write
word that my intended
journey is <u>again</u> delayed? &
cannot now <u>take</u> place before
Sunday or Monday week. I
fear you will never trust me
again ; but I must beg to
assure you that this uncertainty
has been <u>quite unavoidable</u>.

Everything was arranged
last night for our setting out
at six this morning ; when
[138v] unexpected circumstances obliged
us to delay once more ['fulfilling' inserted] our
intentions. I shall really feel
quite as<u>hamed</u> next time, to
write to you on a similar
appointment ; altho' I do not
think it is possible that
another put-off should occur.

May I hope meanwhile
to receive a reply to my last
letter of Monday.
I am going on very well
indeed I believe now, with
my studies ; __ next week I
expect a few interruptions
however from some engagements
I have made. __
With many apologies, pray
[139r] believe me

Yours most truly
A. A. Lovelace

[140r] My Dear M$^r$ De Morgan.
I am to be in Town
(for as certain as any human
affairs can be), next week.
May I hope to see you on
Tuesday ; _ any hour you like
['to fix' inserted] between 12 & 4 o'clock?
Thanks for yesterday's
packet. I shall probably
send you a Budget in a
day or two again ; as it
will save time when I see
[140v] you. For I have so many
Memoranda down to ask
about, that it will be
best all the principal
things should just be
noted to you first. ______
Then we shall know at once
where & what to begin
upon. __
I am in the middle of
Operation ; indeed nearly
thro' it ; & find no
difficulties scarcely ; at
least none of any moment.
I see I have made a
bother of $u = C\varepsilon^x + C'\varepsilon^{-x} +$
$$+\tfrac{1}{2}\varepsilon^x \int \varepsilon^{-x} X dx - \tfrac{1}{2}\varepsilon^{-x} \int \varepsilon^x X dx$$
[141r] ; not from any mis-comprehension
of it ; but from too much
haste in writing out. As
I will explain in my
promised packet. __

Yours always
Most truly
A. A. L

Ashley-Combe
Wed$^{dy}$

[142r]

Ashley-Combe
Sunday. 21$^{\text{st}}$ Nov$^{\text{r}}$ ['1841' inserted by later reader]

Mr Dear M$^{\text{r}}$ De Morgan. [something crossed out]

I said Wed$^{\text{dy}}$. At least I meant to do so. On Tuesday I have already an engagement in the morning. Perhaps you have written Tuesday by mistake. But of you cannot come on Wed$^{\text{dy}}$, then I must put off my Tuesday's engagement, that I may see you then. If it is the same to you however, I should much prefer Wed$^{\text{dy}}$.

Can you kindly give me one line tomorrow to say which it is to be. I shall get ['it' inserted] in the evening in S$^{\text{t}}$ James' Sq$^{\text{re}}$. _ Now I proceed to business :

1$^{\text{stly}}$: You have mistaken my intentions I think about the formulae of pages 155, 156. My enclosures 1 & 2 will explain. __

2$^{\text{ndly}}$. Enclosure 3 contains the demonstration of "Exercise" page 159

3$^{\text{dly}}$. Enclosure 4 . . . . . . . . . . . "Exercise" page 158

4$^{\text{thly}}$: About the Constant in page 141 : I still am [142v] unsatisfied. I perfectly understand that "any value" consists with everything in page 141. The principle is I conceive exactly the same as that by which in page 149, $y$ is made $= a + \sin .x$ instead of $y = \sin x$.

I only mean that this result seems inconsistent with page 116 when it is shown that the Constant must $= \frac{w}{2}$. _

5$^{\text{thly}}$: page 161, (line 14 from the top):

$$\varphi''(x + \theta h, y + k) - \varphi''(x + \theta h, y) = \varphi_1^{('')}(x + \theta h, y + vk).k \qquad v < 1$$

Why is $v$ introduced at all? __

I have as follows :

$$\frac{\varphi''(x+\theta h,y+k)-\varphi''(x+\theta h,y)}{k} = \varphi_1^{('')}(x + \theta h, y)$$

if $k$ diminishes without limit ; ($k$ being $= \Delta y$)

or $\varphi''(x + \theta h, y + k) - \varphi''(x + \theta h, y) = \varphi_1^{('')}(x + \theta h, y)k$

But I do not see how $v$ comes in. __

6$^{\text{thly}}$ : I have several remarks to make altogether on the Article Operation. I will only now subjoin two. I believe on the whole that I understand the

Article very well.
See page 443, at the top, ($2^{nd}$ Column) :

$$\varphi^2 + 2\varphi\psi + \psi^2, \text{ or } (x^2)^2 + 2(x^3)^2 + (x^3)^3$$

should be <u>it appears to me</u> $\varphi^2 + 2\varphi\psi + \psi^2$, or $(x^2)^2 + 2x^3.x^3 + (x^3)^2$

$$\text{or } (x^2)^2 + 2(x^3)^2 + (x^3)^2$$
$$= (x^2)^2 + 3(x^3)^2$$

[143r] I only allude to $(x^3)^3$, instead of $(x^3)^2$ as <u>I</u> make it.
See page 444, at the bottom, ($2^{nd}$ column) :
"Where $B_0$, $B_1$, &c are the values of $fy$ and its
"successive diff-co's [*sic*] <u>when $y = 0$</u>, &c, &c"
Surely it should be <u>when $y = 1$</u>.
The same as when immediately afterwards, (see page
445, $1^{st}$ column, at the top), in developping [*sic*] $(2 + \Delta)^{-1}\varphi x$;
$B_0$, $B_1$ &c are the values of $fy$ & its Co-efficients
<u>when $y = 2$</u>, &c, &c. ____

I have referred to <u>Numbers of Bernoulli</u>
& to <u>Differences of Nothing</u> ; in consequence of
reading this Article <u>Operation</u>. And find that
I must read that on <u>Series</u> also. __

I left off at page 165 of the Calculus ; &
suppose that I may now resume it ; (when I return
here that is). __
I will not trouble you further in this letter.
But I have a f<u>ormidable list</u> of s<u>mall matters</u>
down, against I see you. __

Yours most sincerely
A. A. Lovelace

[144r]

Ashley-Combe
Porlock
Somerset
Sunday Mor$^{g}$. 28$^{th}$ Aug$^{st}$
['1842' added by later reader]

Dear M$^{r}$ De Morgan. I am going on well ; __ ['quite' inserted] as I
could wish. I have done much since I saw you ;
& you will have all the results of the last few days
in good time. I enclose you now two papers ; one
on $f = \frac{dv}{dt}$, the other on $\int_a^{a'} f.dt$.
You will have next those on $v\frac{dv}{dt} = f$, and
$v^2 = 2\int f.ds + C$. This latter I think I have
succeeded in analysing to my mind.
I have ['now' inserted] two observations to make : [something crossed out]
1$^{stly}$: I think I have detected a slight error in one
of my former papers, that on $t = \int \frac{ds}{v}$. I return
it for reference. In order in the [something crossed out] Summation
[something crossed out]
$\left\{\frac{1}{\varphi s} + \frac{1}{\varphi(s+ds)} + \cdots \frac{1}{\varphi(2s)}\right\} ds$, to end with $\frac{1}{\varphi(2s)}$,
I should have begun with $\frac{1}{\varphi(s+ds)}$ not with $\frac{1}{\varphi s}$.
If the time elapsed during the first fraction of Space
[144v] (starting from $s$) were ['made' inserted] $= \frac{1}{\varphi s}$, then the time for the last
of the Fractions necessary to complete up to $2s$, would
be $\frac{1}{\varphi(2s-ds}$, and not $\frac{1}{\varphi(2s)}$ which it ought to be.
I don't know that this affects the correctness of the
ultimate limit of the Summation. But here, where
the Summation itself is made to represent a
hypothetical movement, it is clearly wrong.
The error is avoided in the former paper I had
written on $s = \int v.dt$, which I likewise return to
refer to this Point. _
2$^{ndly}$: In considering a priori the Integral $\int f.ds$,
I am inclined still to adhere to my original
opinion (expressed in the pencil Memorandum I showed
you & ['which I' inserted] now return). I should premise that I now
mention this merely as a curious subject of investigation,
not because it is concerned in the [something crossed out] papers I
am making out upon $v^2 = 2\int f.ds + C$, in which I
have avoided the direct consideration of $\int_a^{a'} f.ds$. ____

I am disposed to contend that tho' $ds$
does here represent Space, that still the $ds$ fraction
of any one of the terms of the Summation, say $\varphi(a + n.ds)ds$
means the same fraction of $\varphi(a + n.ds)$ which $ds$ is of
[145r] a Unit of Space ; & therefore that since $\varphi(a + n.ds)$
represents Force, (or ['uniform' inserted] Acceleration of Velocity for 1 Second
in operation during the performance of the length $ds$),
the $ds$ fraction of this expression must represent the
['$ds$ part of this Force or the' inserted] actual Acceleration for $\frac{1}{ds}$ of a Second. I treat $ds$ as
an abstract quantity. And so I conceive [something crossed out] $dt$ must
be treated in $s = \int v.dt$, ['$ds$' inserted] in $t = \int \frac{ds}{v}$, $dt$ in $\int f.dt$,
&c, &c. ____
I should tell you that I am much pleased with
the observation you added to my inverse demonstration
of $\int fx.\frac{dx}{dt}dt = \int fx.dx$ , and that I quite
understand ['why' inserted] my proof can only be admissible on
the Infinitesimal Leibnitzian Theory. But this
theory is to my mind the only intelligible or
satisfactory one. In fact, (notwithstanding it's [*sic*] error),
I should call it the only true one. __
By and bye, you will have some observations
of mine upon Differential Co-efficients & Integrals,
abstractly considered. I have been thinking much
upon them. _
I am going on with Chapter VIII.
By the bye, I believe you will receive somehow tomorrow
[145v] a book (the 1st Vol of Lamé's Cours de Physique)
in which there is a passage which I will write
to you about as soon as I find time. __
I forgot to mention it to you on Thursday ; &
so have ordered the Book to be sent to you, that
I might write about it sometime. __

Believe me

Yours very truly

A. A. Lovelace

[146r]

Ashley-Combe
Wed$^{dy}$. 16$^{th}$ Nov$^{r}$

['1842' or '1847' added by later reader]

[I think this might date from earlier than 1842, as it contains material that seems to fit better with the earlier letters — I'll try to slot it in]

Dear M$^{r}$ De Morgan. I am very much obliged for your long letter. The Formula in Peacock comes out quite correct now that I have written diff. co of $(b-x)^4 = 4(b-x)^3 \times (-1) = -4(b-x)^3$. It is odd that notwithstanding the caution you gave me in Town on this very point, I should have fallen into the trap. There is nothing like one's own blunders after all for instruction. __ I do not however understand why example (19) page 4, has not come out wrong also in my working out. I enclose a copy of my solution, and it appears to me it ought to be wrong, because I surely should have had diff. co of $(1-x)^4 = 4(1-x)^3 \times (-1) = -4(1-x)^3$, whereas I have diff. co of $(1-x)^4 = 4(1-x)^3$. __

On looking over my development again very carefully, I am inclined to think that my solution [146v] $\frac{(1+x)^2}{(1-x)^5} \times (7+x)$, comes out right only because I have managed to make another blunder of a sign in the course of the proofs, which has corrected the first blunder. I therefore now write on the other side of the paper, what I think it should be. __ The note in page 2 I do not imagine to be of any consequence. It is on "rendering the Differentiation "of complicated Functions sometimes much easier" by means of three Theorems from Maclaurin's Fluxions.

Certainly had I thought a little more upon what I read some weeks ago, before I wrote my last letter to you, I should not have sent the question about $du = \varphi(x) \times dx$ [flourishes at tops of stems of 'd's, here and after]. I ['must have' inserted] forgot exactly what a Differential Co-efficient means, when I did so. __ But how is it then that in your 1$^{st}$ Chapter of the Differential Calculus" there is no mention of the multiplication by $dx$?"

I conclude that the real Differential Co-efficient
is $\frac{du}{dx} = \varphi(x)$, and that Peacock's solutions are"
not strictly speaking Differential Co-efficients. __"
I think pages 13 to 15 of your Elementary
[147r] Illustrations bear considerably upon the observations
in your letter, do they not? __
Your explanation of Euler's proof of the Binomial
Theorem is perfectly satisfactory to me. Unluckily
I have not any Book here which contains the
Theory of Combinations. I wanted to refer to
this when reading page 215, as I have forgotten
it in it's [*sic*] particulars. However this can very
well wait a short time, & I have only to take
the Formula for Combinations for granted meanwhile.
The necessity of the truth of $(1+x)^n \times (1+x)^m = (1+x)^{n+m}$
for all values of $n$ and $m$, since it is true
when they are whole numbers, I shall probably
see more clearly at some further time. ____

I can explain exactly what my
difficulty is in Chapter X. _ "For instance, if we
"know that $\varphi(xy) = x \times \varphi y$, supposing this always
"true, it is true when $y = 1$, which gives $\varphi(x) =$
"$= x \times \varphi(1)$. But $\varphi(1)$ is an independent quantity,
"made by writing 1 instead of $y$ in $\varphi(y)$. Let us
"call it $c$ &c. _"
It is this substitution of 1 and of $c$, and
consequent ascertainment of the form which will
[147v] satisfy the equation, which is all dark to me.
It is ditto in lines 12, 13, & 14 from the top. __

I understand quite well I believe from
"We have seen that if $\varphi x = c^x$ &c", all through
the next page.
That I do not comprehend at all the means of
deducing from a Functional Equation the form
which will satisfy it, is I think clear from
my being quite unable to solve the example
at the end of the Chapter "Shew that the equation
"$\varphi(x+y) + \varphi(x-y) = 2\varphi x \times \varphi y$ is satisfied
"by $\varphi x = \frac{1}{2}(a^x + a^{-x})$". I have tried
several times, substituting first 1 for $x$, then
1 for $y$. but I can make nothing whatever
of it, and I think it is evident there is

something that has preceded, which I have
not understood. _ The 2nd example given for
practice "Shew that $\varphi(x+y) = \varphi x + \varphi y$ can
"have no other solution than $\varphi x = ax$", I
have not attempted. ____
I have a question to ask upon page 229.
"By extracting a sufficiently high root of $z$, we
[148r] "can bring $z^m$ as near to 1 as we please, or
"make $z^m - 1$ as small as we please ; that is
"(page 187) $z^m - 1$ may be made as nearly equal
"to the sum of the whole series as we please". _
I cannot find what it is that is referred
to in page 187 ; and Secondly, it appears to me
somewhat of a contradiction that a quantity
$z^m - 1$ which can certainly be made as small
as we please by the diminution of $m$, should
become as near as we please to a fixed
limit or sum (the $\log z$ I conclude is the
sum of the series, referred to), since by continued
diminution the quantity $z^m - 1$ may become a
great deal less than the sum of the Series, &
keep receding from it. __
To return to Chapter X, there is one other
thing in it that I do not understand. Page
205, lines 5, 6, 7 from the bottom. It seems
to me fallacious to substitute first one value
0, for a letter ; & then another value, let $y = -x$,
[148v] in the same equation & in a manner at the
same time. How can the two suppositions
consist together at all. ____
I go on well with the Trigonometry, &
have nearly finished the Number & Magnitude.
I think there is another Erratum in page
34 of the Trigonometry, line 13 from the bottom
$= \frac{OM}{ON} \cdot \frac{ON}{OP} - \frac{NR}{NP} \cdot \frac{NP}{NO}$ &c
should be $-\frac{NR}{NP} \cdot \frac{NP}{OP}$
I am really ashamed to send you such
troublesome letters. ____
Believe me
Yours most truly
A. A. Lovelace

[149r]

Ashley-Combe
Nov$^{r}$ 27$^{th}$

Dear M$^{r}$ De Morgan. I have I believe made some little progress towards the comprehension of the Chapter on Notation of Functions, & I enclose you my Demonstration of one of the Exercises at the end of it : "Show that the equation $\varphi(x+y) =$
"$= \varphi x + \varphi y$ can be satisfied by no other solution
"than $\varphi x = ax$." At the same time I am by no means satisfied that I do understand these Functional Equations perfectly well, because I am completely baffled by the other Exercise :
"Shew that the equation $\varphi(x+y) + \varphi(x-y) =$
"$= 2\varphi x \times \varphi y$ is satisfied by $\varphi x = \frac{1}{2}(a^x + a^{-x})$
"for every value of $a$". _
I do not know when I have been so tantalized by anything, & should be ashamed to say <u>how</u> much time I have spent upon it, in vain.
[149v] These Functional Equations are complete Will-o'--the-Wisps to me. The moment I fancy I have really at last got hold of something tangible & substantial, it all recedes further & further & vanishes again into thin air. _
But now for this perplexing $\varphi x = \frac{1}{2}(a^x + a^{-x})$.
I be<u>lieve</u> I have left no method untried ; but I cannot get further than as below, with any certainty :_

$$\varphi(x+y) + \varphi(x-y) = 2\varphi x \times \varphi y$$
$$\because 2\varphi x = \frac{\varphi(x+y)+\varphi(x-y)}{\varphi y}$$

Since $x$ and $y$ may have a<u>ny</u> values whatever, (at least such I conclude is of course intended), let $y = 0$. We have then

$$2\varphi x = \frac{\varphi(x)+\varphi(x)}{\varphi(0)}$$
$$\because 2\varphi x \times \varphi(0) = \varphi(x) + \varphi(x)$$
$$\text{or } 2\varphi x \times \varphi(0) = 2\varphi x$$
$$\varphi(0) \text{ must } = 1, \text{ or } = a^0, \text{ since } a^0 \text{ is}$$

the only function of 0 which can $= 1$
I think so far is correct in itself, but whether
[150r] it be the ['right' inserted] road to the rest is another question.
At any rate, <u>I</u> have not succeeded in proving

it such. To assume that since $\varphi(0) = a^0$,
$\varphi(x+y) = a^{x+y}$, $\varphi(x-y) = a^{x-y}$, $\varphi y = a^y$
appears to me scarcely warrantable ; and
besides in that case it must be equally
assumed that $\varphi x = a^x$, (there being the same
ground for the one assumption as for the others),
and we should then have,

$$a^x = \tfrac{a^{x+y}+a^{x-y}}{a^y}$$
$$\because a^x = \tfrac{a^{x+y}}{a^y} + \tfrac{a^{x-y}}{a^y}$$
$$\because a^x = a^x + a^{x-2y}$$

most clearly absurd, independent of it's [*sic*] being
discordant with the book. __
Once I thought I had hit on something very
clever indeed, and wrote as follows :

$$\varphi(x+y) = \varphi(\overline{x+y}.1)$$
$$= \{\varphi(1)\}^{x+y} \text{ by equation } (\varphi a)^n = \varphi(na)$$

page 205, entirely forgetting that the $\varphi$ of that
equation had nothing whatever to do with the
$\underline{\varphi}$ of any other equations ; (a disagreeable truth
[150v] which did not occur to me until 24 hours
later). I then had $\varphi(x-y) = \varphi(\overline{x-y}.1) = \{\varphi(1)\}^{x-y}$
$\varphi(y) = \varphi(y.1) = \{\varphi(1)\}^y$, and

$$2\varphi x = \tfrac{\{\varphi(1)\}^{x+y}+\{\varphi(1)\}^{x-y}}{\{\varphi(1)\}^y}$$
$$= \tfrac{\{\varphi(1)\}^{x+y}}{\{\varphi(1)\}^y} + \tfrac{\{\varphi(1)\}^{x-y}}{\{\varphi(1)\}^y}$$
$$= \{\varphi(1)\}^x + \{\varphi(1)\}^{x-2y}, \text{ and supposing}$$

$x = y$, then $2\varphi x = \{\varphi(1)\}^x + \frac{1}{2}\{\varphi(1)\}^{-x}$
or calling $\varphi(1) = a$, $\varphi x = \frac{1}{2}(a^x + a^{-x})$
But besides my unwarrantable assumption of
$\varphi(\overline{x+y}.1) = \{\varphi(1)\}^{x+y}$, there was t<u>his</u> in the
result which was unsatisfactory, that it was
necessary to assume $x = y$, and the result seemed
to hold good in t<u>hat</u> case a<u>lone</u>. Also, when to
verify, I tried $x = 1$, $\because \varphi(1) = \frac{1}{2}(a^1 + a^{-1})$,
which ought to have come out $a = a$, I could
make neither head or tail of it. _ Well, I
abandoned this, & tried all sorts of other resources.
[151r] I understand to work out something by means
similar to those in page 205 and in the
Problem I send ; but equally unsuccessfully.
I also in equation (A) page 204, changed
$\varphi(x+y)$ into $\varphi(x-y)$ and investigated this,

thinking I might derive a hint possibly from it.
[something crossed out] In short, many & various are the experiments
I have made, but I will not detail any
more. Indeed I think you may be possibly
heartily sick of what I h<u>ave</u> detailed. But
I wished to show you that I have not failed
from want of t<u>rying</u>, at least ; & also to give
you the chance of smiling at my expence [*sic*]. ____

I shall have to trouble you with another
letter shortly, on other knotty points. Really I
do not give you a Sinecure. Your letters are
however well bestowed, in as far as the use
they are of to me, can make them so, and
the great encouragement that such assistance
is to me to continue my Studies with zeal
& spirit. We are to return to Surrey very
[151v] soon. I expect to have occasion to trouble
you again before we go, & after that I shall
hope to see you & M^rs^ De Morgan in Town,
where I intend to be for two or three days
in about a fortnight. __

Yours very truly
A. A. L

[152r]

Ockham Park
Surrey

Dear Mr De Morgan. I am indeed extremely
obliged to you for all your late communications.
In two or three days more, I shall have
several observations & to send you in reply
to some of them. _
My object in writing today, is to make another
enquiry concerning the substitution of $\varphi_1(a+h) - \varphi_1 a$
for $\varphi(a+h) - \varphi a$ in page 100, for I perceive
on carefully examining the passage, that I do
not quite understand it

$\varphi_1 x = \varphi x + C$, which ['last side' inserted] means the Primitive Function ['of $\varphi' x$' inserted]
and the Primitive Function means the Function
which differentiated gives $\varphi' x$
Therefore $\varphi_1(a+h) = \varphi(a+h) + C$
And $\varphi_1 a = \varphi a + C$
Consequently $\varphi_1(a+h) - \varphi_1 a = \varphi(a+h) + C - (\varphi a + C)$
$= \varphi(a+h) - \varphi a$
[152v] This is my version of it. But you tell me,
$\varphi_1(a+h) = \varphi(a+h) + C$
$\varphi_1 a = \varphi a$, (which ought to be I say
$\varphi_1 a = \varphi a + C$)
From which we should have,
$\varphi_1(a+h) - \varphi_1 a = \varphi(a+h) + C - \varphi a$
Consequently $\varphi_1(a+h) - \varphi_1 a$ is not equal
to $\varphi(a+h) - \varphi a$ as is required
to be proved, but is $= \varphi(a+h) - \varphi a + C$. _
I cannot unravel this at all. __
Secondly: [something crossed out] I do not see why the Indefinite
Integral only is $= \varphi x + C =$ Primitive Function.
of $\varphi' x$
The argument at the top of page 101 seems
to [something crossed out] me to apply equally to the Definite Integral
As follows : It is proved that

$$\varphi_1(a+h) - \varphi_1 a = \int_a^{a+h} \varphi' x.dx$$

$\varphi_1 a$ is just as much here an arbitrary Constant
as it is in $\varphi_1 x - \varphi_1 a = \int_a^x \varphi' x.dx$
Therefore $\int_a^{a+h} \varphi' x.dx = \varphi_1(a+h) + C_1$
$= \varphi(a+h) + C + C_1$
$= \varphi(a+h) +$ an arbitrary Constant
[153r] just as with $\varphi_1 x - \varphi_1 a = \int_a^x \varphi' x.dx$

Thirdly : With respect ['to' inserted] the assumption that when $a$ is arbitrary, then any function of $a$, say $\varphi a$, is also arbitrary or may be anything we please, seems to me not always valid. __
For instance if $\varphi a = a^0$, it must be always = 1. We may assume $a =$ anything we like, but $\varphi a$ will not in this case be arbitrary. _
It is curious how many little things I [something crossed out] discover in this Chapter, which in looking back upon them, I find I have only half-understood.

I shall be exceedingly obliged, if you can answer these points soon ; I think a word almost may explain them, & they rather annoy me. ____

Believe me, with many thanks
Yours very truly
A. A. Lovelace

[154r]
Ockham Park
Ripley
Surrey
14$^{th}$ Jan$^{y}$

Dear M$^{r}$ De Morgan
It would give me
great pleasure to see you
again, & to have a chat
with you. If therefore I
am in Town on Tuesday
week, (which is likely), do
you think you could call
on me at 11 o'clock.
I should propose going to
[154v] Gower S$^{t}$ to see M$^{rs}$ De
Morgan, but on this
occasion I should not have
[something crossed out] time to go so far.
If therefore you
answer me affirmatively, I
shall not write again unless
I should be prevented from
going.
I have been doing little
of late in Mathematics ;
just amusing myself
with Murphy's Theory of
Equations (in the Useful
Knowledge) ; & looking into
one or two other little
matters, rather than
[155r] studying them. So I have
no very diligent reckoning
to make you.
My kind remembrances
to M$^{rs}$ De Morgan.
Yours very truly
A. A. Lovelace

[156r]

S$^{t}$ James' Sq$^{re}$
Tuesday. 5 o'clock

---

P. S. Excuse my materials.

Dear M$^{r}$ De Morgan. In
much haste, I write to tell
you that I must adjourn
the proposed engagement for
Monday next, until Monday
week ; owing to an unforeseen
obstacle for next Monday
afternoon. You have very
probably already written
to me ; & if so, I shall (to
save you trouble) take it
for granted if I do not hear,
[156v] that the hour fixed will
be for the Monday week
instead.
I was obliged to stay in
Town last night, unexpectedly;
but am just setting out..

I was very glad to
find M$^{rs}$ De Morgan so
well & cheerful, & moreover
that she approved of a
proposition I made her ;
(not a proposition of our
particular kind however).

I remain

Yours very truly
A. A. L

[157r] East Horsely Park
Ripley
Surrey

Dear M$^r$ De Morgan
In consequence of
y$^r$ kind reply to my
former note, Lord L__
think I had better
send you the paper, ____
(which has been put
into type, tho' not
published). ____
Should you be at
leisure & disposed to
look at it, you will
at once see how much
[157v] stuff there is in it, &
of what a solid quality.
Some few corrections
are wanting, before it
finally goes to proofs. __
There is also one
single sentence, (I
think in page 22),
about Irish Priests &
Douai, which must
be altered or cut out,
for a Roman Catholic
Editor. __
Much of the Paper, tho'
on so dry a subject, is
amusing enough. _ If you
[158r] look at it, & should
discover my inaccuracies
or anything which might
be made clearer, pray
be kind enough to men=
=tion it. __
I will not now detain
you further.
Believe me
very faithfully y$^{rs}$

[159r] 10. S[t] James' Sq[re]
Tues[dy] Morning
$\frac{1}{2}$ past 12 o'clock

Dear M[r] De Morgan. I
am most particularly vexed
& annoyed at finding
that you have just been
here, & are gone without
['my' inserted] seeing you. Owing to
the message received by
the carriage the other
day, I had reserved
this morning particularly
[159v] for our affairs, & I
cannot but think that
either my footman must
have blundered the
matter & sent you away,
or that there has been
some misunderstanding or
other. __ Your time is
so valuable that it is
without measure vexatious
you should have come all
the way here for nothing,
& when I was especially
waiting for you too. __
[160r] Now it does so happen
quite by chance that I
have this evening at
liberty. _ May I come
to Gower S[t] between eight
& nine therefore, & so
repair the mistake of
this morning? I am
quite in a fuss about
my mathematics, for I
am in much want of a
little lift at this
moment ; __ & you know
how I have my progress
[160v] at heart. ________

I write in a great
hurry ; & I only hope
that such a contrariety
will never happen again.

You must forgive my
writing in such a fuss ;
but I cannot imagine
how the thing has happened.
Believe me

Yours very truly

A. A. Lovelace

[161r] Dear M$^{r}$ De Morgan
Yes __ tomorrow
morning between $\frac{1}{2}$ past
twelve & one o'clock __
will do. (Remember I
write Tuesday Night). _
I am much provoked
at my stupid servants'
stupid blunder, but I
[161v] believe the poor man
meant to be highly
discerning & well-judging.
I was at the moment
treating for an Opera
Box for ['the night of' inserted] Rachel's Benefit,
which I conclude the
footman thought a
much more important
matter than any
spectacles in the world
could [something crossed out] betoken. He is
a new man, & does not
know my people yet.
[162r] In haste
Yours most truly obliged
A. Ada Lovelace

Tuesday Night
S$^{t}$ James' Sq$^{re}$

[162v] Augustus De Morgan Esqre
69. Gower S$^{t}$

[166r] Ockham
Thursday

Many thanks for the packet of yesterday morning. I shall write in reply to it tomorrow. ____ I send today some Integral Developments (see pages 116, 117); & another of the Accelerating Force papers. __ Any hour you may appoint on Thursday Morning [something torn away] I will be ready. Shall I say 10 o'clock?

[166v] [postmark?]

[168r] [In De Morgan's hand] When an equation involves only two variables, it is easy enough to write all differential equations so as to contain nothing but differential coefficients; thus

$y = \log x \qquad \frac{dy}{dx} = \frac{1}{x}$

If we prefer to write $dy = \frac{dx}{x}$, it must be under a new understanding. By $\frac{dy}{dx}$, we mean the limit of $\frac{\Delta y}{\Delta x}$, not any value which $\frac{\Delta y}{\Delta x}$ ever can have, but that which it constantly tends towards, as $\Delta x$ is diminished without limit. But in $dy = \frac{dx}{x}$, we cannot by $dy$ and $dx$ mean limits, for the limits are zeros; and $0 = \frac{0}{x}$, though very true, is unmeaning

What then do we mean by this

When $y = \log x$ , $dy = \frac{dx}{x}$

we mean that $\Delta y = \frac{\Delta x}{x}$, as $\Delta x$ diminishes without limit, not only diminishes without limit, but diminishes without limit as compared with $\Delta x$ or $\Delta y$. So that, if we call it $a$, or if

[168v] $\Delta y - \frac{\Delta x}{x} = a$

Then $a$ is useless, and we might as well write 0. For since the processes of the differential calculus always terminate in taking <u>limits of ratios</u> and since

$\frac{\Delta y}{\Delta x} - \frac{1}{x} = \frac{a}{\Delta a}$

(or some transformation of this sort) must come at last, our limiting equation must be

$$\frac{dy}{dx} - \frac{1}{x} = \text{Limit of } \frac{a}{\Delta x} = 0$$

The truth of every equation differentially written, as $dy = p\,dx$, is always absolutely speaking, only approximate: but the approximation is relatively closer and closer . Understand it as if it were

$$dy = (p + \lambda)dx$$

where $\lambda$ diminishes with $dx$, so that the error made in $dy$ by writing $dy = p\,dx$, namely $\lambda\,dx$, not only diminishes

with $dx$, but becomes a smaller and smaller fraction of $dx$: because $\lambda$ diminishes without limit
[169r] All this is conveniently signified in the language of Leibnitz, namely, that when $dx$ is infinitely small, $dy - p\,dx$ is as nothing (or infinitely small) when compared with $dx$, or $dy$ is (relatively to its own value) infinitely near to $p\,dx$.

---

The differential might easily be avoided when there are only two variables, and even when there are more, provided we only want to use one independent variable at a time . Thus

$$u = \varphi(x,\, y,\, z)$$

may give the equations

$$\tfrac{du}{dx} = P, \quad \tfrac{du}{dy} = Q, \quad \tfrac{du}{dz} = R$$

But when we want to make $x$, $y$, and $z$, all vary together, we have no notion of a differential coefficient attached to this simultaneous variation, unless we suppose some one new variable on which $x$, $y$, and $z$ all depend, and the variation of which sets them all varying together.
[169v] If this new variable be $t$, and if $x$, $y$, and $z$ be severally functions of $t$, we have then
[in margin: 'See chapter on Implicit differentiation']

$$\tfrac{d(u)}{dt} = \tfrac{du}{dx} \cdot \tfrac{dx}{dt} + \tfrac{du}{dy} [\cdot] \tfrac{dy}{dt} + \tfrac{du}{dz} \cdot \tfrac{dz}{dt}$$

Thus if $u = xy^2\varepsilon^2$

$$\tfrac{d(u)}{dt} = y^2\varepsilon^2 \tfrac{dx}{dt} + 2xy\varepsilon^2 \tfrac{dy}{dt} + xy^2\varepsilon^2 \tfrac{dz}{dt}$$

But observe that this makes $x$, $y$, and $z$, (which we want to be independent of one another) really functions of one another: thus if $x = t^2$, $y = \log t$, we must have $y = \log\sqrt{x}$. We might it is true avoid this by the following supposition. Let $x$, $y$, and $z$, instead of being given functions of $t$, be unassigned and arbitrary functions, which we can always make whatever functions we please. We can then really hold $\frac{dx}{dt}$ $\frac{dy}{dt}$ $\frac{dz}{dt}$ to be independent of one another, for it is always in our power to assign them any values we like. But this method would be awkward , and would put continual impediments in our way. It is better therefore to avoid that [170r] notation which while it makes the first step by supposing relations to exist between $x$, $y$, and $z$, immediately contradict that supposing by making these relations mean <u>any</u> relations.

If in $\varphi(x, y\, z)$ we suppose $x$, $y$, and $z$ to be simultaneously altered into $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, then $\varphi(x, y\, z)$ takes the value

$$\varphi(x + \Delta x, y + \Delta y, z + \Delta z)$$

which may be expounded as follows

$$\begin{aligned}&\varphi + \tfrac{d\varphi}{dx}\Delta x + \tfrac{d\varphi}{dy}\Delta y + \tfrac{d\varphi}{dz}\Delta z\\ &\quad + A\Delta x\Delta y + B\Delta y\Delta z + C\Delta z\Delta x\\ &\quad + D\overline{\Delta x}^2 + E\overline{\Delta y}^2 + F\overline{\Delta z}^2\\ &\quad + \text{\&c \&c}\end{aligned}$$

say

$$\varphi + \tfrac{d\varphi}{dx}\Delta x + \tfrac{d\varphi}{dy}\Delta y + \tfrac{d\varphi}{dz}\Delta z + M$$

If it be required that $\varphi =$ constant, or $\varphi = c$ we must have

$$\tfrac{d\varphi}{dx}\Delta x + \tfrac{d\varphi}{dy}\Delta y + \tfrac{d\varphi}{dz}\Delta z + M = 0$$

Now if we were to leave out $M$, and say

$$\tfrac{d\varphi}{dx}\Delta x + \tfrac{d\varphi}{dy}\Delta y + \tfrac{d\varphi}{dz}\Delta z = 0$$

we should of course commit an error: but it is one the magnitude of which relatively to $\Delta x$, for instance, diminishes [170v] without limit as the increments $\Delta x$, $\Delta y$, $\Delta z$, are diminished without limit. The considerations already given apply here again : because all the terms contain [*sic*] in $M$, diminish without limit as compared with those which are not [something crossed out] contained in $M$. This rejection of all terms except those of the first order is always accompanied and

When therefore I say that

$$\varphi = c$$

gives $\frac{d\varphi}{dx}.dx + \frac{d\varphi}{dy}[.]dy + \frac{d\varphi}{dz}.dz = 0$

I should, if asked whether this equation is absolutely true, answer <u>no</u> . If then asked why I write it, I should answer that it <u>leads to truth</u> , and for this reason that it is more and more nearly true as $dx$ &c are diminished : not because $\frac{d\varphi}{dx}dx + \&\text{c}$ <u>diminishes</u> in that case, though undoubtedly it does so ; but because it diminishes as compared with $dx$, &c. Hence, when we form ratios and take their limits, it matters nothing, as to the results we obtain, whether we write

$$\frac{d\varphi}{dx}dx + \&\text{c} \qquad = -M$$

$$\text{or} \quad \frac{d\varphi}{dx}dx + \&\text{c} \qquad = 0$$

marked by writing $dx$ for $\Delta x$, $dy$ for $\Delta y$, &c.

[171r] $\sin(90^\circ+30^\circ+10^\circ)=\sin(50^\circ+70^\circ)$

$$\begin{aligned}
\sin 90 &= \underline{\tfrac14\sqrt{(3+\sqrt5)}}+\cancel{\tfrac14\sqrt{(5-\sqrt5)}}\\
\sin 30 &= \tfrac14\sqrt{(5+\sqrt5)}-\tfrac14\sqrt{(3-\sqrt5)}\\
\sin 10 &= \underline{\tfrac14\sqrt{(3+\sqrt5)}}-\cancel{\tfrac14\sqrt{(5-\sqrt5)}}
\end{aligned}$$

$$\boxed{\begin{aligned}
&\tfrac12\sqrt{(3+\sqrt5)}+\\
&+\underline{\tfrac14\sqrt{(5+\sqrt5)}}-\tfrac14\sqrt{(3-\sqrt5)}=
\end{aligned}}$$

$$\boxed{\begin{aligned}
\sin 50 &= \tfrac12\sqrt2\\
\sin 70 &= \underline{\tfrac14\sqrt{(5+\sqrt5)}}+\tfrac14\sqrt{(3-\sqrt5)}
\end{aligned}}$$

$$\begin{aligned}
&\tfrac12\sqrt2+\tfrac14\sqrt{(3-\sqrt5)}=\\
&=\tfrac12\sqrt{(3+\sqrt5)}-\tfrac14\sqrt{(3-\sqrt5)}\\
&\tfrac12\sqrt2+\tfrac12\sqrt{(3-\sqrt5)}=\tfrac12\sqrt{(3+\sqrt5)}\\
&\sqrt2+\sqrt{(3-\sqrt5)}=\sqrt{(3+\sqrt5)}
\end{aligned}$$

[boxed formulae are circled and joined in original]

[171v] [blank]

[172r] $\sqrt2+\sqrt{(3-\sqrt5)}=\sqrt{(3+\sqrt5)}$

$$(\sqrt2)^2+2\sqrt2\times\sqrt{(3-\sqrt5)}+\left(\sqrt{(3-\sqrt5)}\right)^2=\left(\sqrt{(3+\sqrt5)}\right)^2$$

$$2+2\sqrt{2(3-\sqrt5)}+\underline{3}-\sqrt5=\underline{3}+\sqrt5$$

$$2+\sqrt{2(3-\sqrt5}=-2+2\sqrt5$$

$$\sqrt{2(3-\sqrt5)}=-1+\sqrt5$$

$$\underline{6}-2\sqrt5=+\underline{1}-2\sqrt5+\underline{5}$$

[diagrams to right of above formulae in original]

---

$$\begin{aligned}
&\tfrac12\sqrt2=\tfrac14(+1+\sqrt5)+\tfrac12\\
&2\sqrt2=+1+\sqrt5+2=+3+\sqrt5\\
&4\times2=9+2\sqrt5-5
\end{aligned}$$

[above three lines scribbled out]

$$\begin{aligned}
\sin 60&=\tfrac14(1+\sqrt5)\\
\sin 20&=\tfrac14(-1+\sqrt5)
\end{aligned}$$

$$\begin{aligned}
&\tfrac14(1+\sqrt5)=\tfrac14(-1+\sqrt5)+\tfrac12\\
&\quad\underline{1}+\underline{\sqrt5}=-1+\underline{\sqrt5}+2=\underline{1}+\underline{\sqrt5}
\end{aligned}$$

[172v] $\sin\frac{a}{2}=\sqrt{\frac12R^2-\frac12\cos a=}\sqrt{\frac12R^2\mp\frac12R\sqrt{1-\sin^2a}}$

$$\begin{aligned}
&\sin\tfrac{a}{2}=\pm\sqrt{\tfrac12R^2\mp\tfrac{R}{2}\sqrt{1-\sin^2a}}=\\
&\quad=\pm\tfrac12\sqrt{R^2+R\sin a}\mp\tfrac12\sqrt{R^2-R\sin a}\\
&\tfrac12R^2\mp\tfrac{R}{2}\sqrt{1-\sin^2a}=\tfrac14\left(R^2+R\sin a\right)+\tfrac14\left(R^2-R\sin a\right)
\end{aligned}$$

$$-2 \times \frac{1}{2} \times \frac{1}{2}\sqrt{R^2 + R\sin a} \times \sqrt{R^2 - R\sin a}$$

$$\overline{\frac{1}{2}R^2} \mp \frac{R}{2}\sqrt{1 - \sin^2 a} = \overline{\frac{1}{4}R^2} + \overline{\frac{1}{4}R\sin}\, a + \overline{\frac{1}{4}R^2} - \overline{\frac{1}{4}R\sin}\, a$$

$$-\frac{1}{2}\sqrt{R^2 + R\sin a} \times \sqrt{R^2 - R\sin a}$$

$$-\frac{1}{2}\sqrt{R^4 + R^3\sin a}$$

$$-R^3\sin a - R^2\sin^2 a$$

$$\mp\frac{R}{2}\sqrt{1 - \sin^2 a} = -\frac{1}{2}\sqrt{R^4 - R^2\sin^2 a =}$$

$$\mp\frac{R}{2}\sqrt{R^2 - \sin^2 a}$$

[174r] [(mostly) in AAL's hand]
Theorem. Page 199.
If $N$ be a function of $x$ and $y$, giving $\frac{dN}{dx} = p + q\frac{dy}{dx}$
then the equation $\frac{du}{dx.dy} = V.\frac{dN}{dx.dy}$ is incongruous &
self-contradictory, except upon the assumption
that $u$ is , as to $x$ and $y$ , a function of $N$ ;
or contains $x$ and $y$ only thro' $N$.

Let $N = \psi(x, y)$ give $y = \chi(N, x)$ , and
suppose , if possible , that the substitution of
this value of $y$ in $u$ gives $u = \beta(N, x)$ , $x$
not disappearing with $y$ . Then $x$ and $y$
varying

$$\frac{du}{dx.dy} = \frac{d\beta}{dN}.\frac{dN}{dx} + \frac{d\beta}{dN}.\frac{dN}{dy} + \frac{d\beta}{dx}$$

[in above line, $\frac{du}{dx.dy}$ is crossed through in pencil, and '1' written above;
'$= \frac{du}{dx} + \frac{du}{dy}$' added in pencil at end of line — in ADM's hand?]

$$= \frac{d\beta}{dN} \cdot \left(\frac{dN}{dx} + \frac{dN}{dy}\right) + \frac{d\beta}{dx} = \frac{d\beta}{dN} \cdot \frac{dN}{dx.dy} + \frac{d\beta}{dx} =$$

$= V.\frac{dN}{dx.dy}$ , which equation being
universal , is true on the supposition that $x$
does not vary , or that $\frac{d\beta}{dx} = 0$. This gives $\frac{d\beta}{dN} = V$;

$$\text{or } \frac{du}{dx.dy} = V\frac{dN}{dx.dy} + \frac{d\beta}{dx} = V\frac{dN}{dx.dy}$$

because $\frac{d\beta}{dN}$ and $V$ being independent of the variations
&c, &c. Hence $\frac{d\beta}{dx} = 0$ always ; or $\beta$ does not
contain $x$ directly, &c.

I think the above is correct. I cannot see
[174v] the use (page 200) of introducing $t$ in
the proof <u>there</u> given . Is it possible that
I have committed an error in my original
understanding of the <u>ennunciation</u> [*sic*] of the Theorem;
& that the $du$ ['of the equation' crossed out] and the $dN$
of the equation $du = V.dN$ , do not mean
the $du$ and $dN$ derived from differentiating
<u>with respect to the quantities $x$ and $y$,</u>
<u>already</u> introduced ; but with respect
to ['some' crossed out] <u>other</u> given quantity?__

I suspect so .

[the following appears underneath in pencil — still in Ada's hand]

$$u = \beta(N, x)$$
$$\frac{du}{dx} = \frac{d\beta}{dN} \quad \frac{dN}{dx} + \frac{d\beta}{dx}$$
$$\frac{d^n u}{dx.dy} =$$

[175r] [in De Morgan's hand] This complete differential of $\varphi$, as it is called namely

$$\frac{d\varphi}{dx}.dx + \frac{d\varphi}{dy}.dy + \frac{d\varphi}{dz}.dz$$

is a perfectly distinct thing from

$$\frac{d\varphi}{dx} + \frac{d\varphi}{dy} + \frac{d\varphi}{dz}$$

and also from $\frac{d^3\varphi}{dx\,dy\,dz}$

Read again page 86 when $x$ is changed to ___ end of 87

page 198–199 & the references

$$\left.\begin{array}{lll} \frac{d\varphi}{dx}dx & \text{is} & d\varphi \\ \frac{d\varphi}{dy}dy & \text{is} & d\varphi \end{array}\right\}$$

But the first means the $d\varphi$ which is caused by variation of $x$, and the second has the same reference to $y$.

[176r] [diagram] $(a-x)x \quad ax-x^2$

[further diagrams: one Pythagoras-related]

[176v] [Königsberg bridges diagrams, some with labels in Babbage's hand]

[Pythagoras-related diagrams]

[177r] [further Königsberg bridges diagrams and related jotting]

[in Babbage's hand] 1 No return
3
5

| | | |
|---|---|---|
| If there are an odd N° of Bridges | 2 | No ending except |
| if you do not begin you must end | 4 | in island |
| in it | | |

[written at 90°, spanning bottom of 176v and 177r] $\begin{array}{ccc} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{array}$

[177v]

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab$$
$$(a-b) \times (a+b)$$
$$= a^2 - b^2$$

$$\sin \tfrac{1}{2}a = \sqrt{\tfrac{1}{2}R^2 - \tfrac{1}{2}R\cos a}$$

for $\cos a$ put $\pm\sqrt{R^2 - \sin^2 a}$

$$\sin \tfrac{1}{2}a = \sqrt{\tfrac{1}{2}R^2 \mp \tfrac{1}{2}R\sqrt{R^2 - \sin^2 a}}$$

Let $\sin \frac{1}{2}a = \frac{1}{2}\sqrt{R^2 + R\sin a} \mp \frac{1}{2}\sqrt{R^2 - R\sin a}$

$$\underline{\tfrac{1}{2}R^2} \mp \tfrac{1}{2}R\sqrt{R^2 - \sin^2 a} =$$
$$= \underline{\tfrac{1}{4}R^2} + \underline{\tfrac{1}{4}R\sin a} \quad + \underline{\tfrac{1}{4}R^2} - \underline{\tfrac{1}{4}R\sin a} \mp 2 \times \tfrac{1}{2} \times \tfrac{1}{2}\sqrt{R^2 + R\sin a} \times \sqrt{R^2 - R\sin a}$$
$$\mp\tfrac{1}{2}R\sqrt{R^2 - \sin^2 a} = \mp\tfrac{1}{2}\sqrt{R^2 + R\sin a} \times \sqrt{R^2 - R\sin a}$$
$$= \mp\tfrac{1}{2}\sqrt{R^4 - R^2\sin^2 a} = \mp\tfrac{1}{2}R\sqrt{R^2 - \sin^2 a}$$
$$\mp\tfrac{1}{2}R\sqrt{R^2 - \sin^2 a} = \mp\tfrac{1}{2}R\sqrt{R^2 - \sin^2 a}$$

[178] [in pencil down left-hand side] 25$^{\text{th}}$ Aug$^{\text{st}}$
1843

$$u = R + X$$
$$P = \frac{dR}{dx} + \frac{dX}{dx}$$
$$\frac{dX}{dx} = P - \frac{dR}{dx}$$
$$X = \int \left(P - \frac{dR}{dx}\right) dx$$
$$P = -\frac{y^2}{x^2\sqrt{x^2+y^2}}$$
$$R = \frac{\sqrt{x^2+y^2}}{x}$$

---

$$\frac{1}{\varepsilon^x+1}$$
$$\frac{-\varepsilon^x}{(\varepsilon^x+1)^2}$$
$$\frac{-(\varepsilon^x+1)^{\cancel{2}}\cdot\varepsilon^x+\varepsilon^{2x}\,2\cancel{(\varepsilon^x+1)}}{(\varepsilon^x+1)^3}$$
$$\frac{\varepsilon^{2x}-\varepsilon^x}{(\varepsilon^x+1)^3}$$

[179] [in pencil down left-hand side] 25th Augst
1843

$$\frac{x}{x+\frac{x^2}{2}+\frac{x^3}{2.3}+\cdots}$$
$$\frac{1}{1+\frac{x}{2}+\cdots}$$
$$\frac{0}{0} \quad \frac{\varphi x}{\psi x} \quad \frac{\varphi' x}{\psi' x}$$
$$\frac{x}{\varepsilon^x-1} \quad \frac{1}{\varepsilon^x}$$

---

Co. of $x^{2n}$ in $\frac{\frac{1}{2}x}{\varepsilon^{\frac{x}{2}}+1}$

$= \frac{1}{2^{2n}}$ co. of $x^{2n}$ in $\frac{x}{\varepsilon^x+1}$

$= \frac{1}{2^{2n}}$ co. of $x^{2n-1}$ in $\frac{1}{\varepsilon^x+1}$

$= \frac{1}{2^{2n}} \; \frac{1}{1.2.3...2n-1} \; \overline{\frac{d}{dx}}\Big|^{2n-1} \frac{1}{\varepsilon^x+1}$

when $x=0$