Clay Mathematics Proceedings
Volume 9

The Geometry of Algebraic Cycles

Proceedings of the Conference on Algebraic Cycles,
Columbus, Ohio
March 25–29, 2008

$$E_2^{p,q} = H_M^{p-q}(X, \mathbb{Z}(-q)) => K_{-p-q}(X)$$

Reza Akhtar
Patrick Brosnan
Roy Joshua
Editors

American Mathematical Society
Clay Mathematics Institute

The Geometry of Algebraic Cycles

Clay Mathematics Proceedings
Volume 9

The Geometry of Algebraic Cycles

Proceedings of the Conference on Algebraic Cycles, Columbus, Ohio March 25–29, 2008

Reza Akhtar
Patrick Brosnan
Roy Joshua
Editors

American Mathematical Society
Clay Mathematics Institute

2010 *Mathematics Subject Classification.* Primary 14C15, 14C25, 14C30, 14C35, 14F20, 14F42, 19E15.

Library of Congress Cataloging-in-Publication Data

Conference on Algebraic Cycles (2008 : Columbus, Ohio)

The geometry of algebraic cycles : proceedings of the Conference on Algebraic Cycles, Columbus, Ohio, March 25–29, 2008 / Reza Akhtar, Patrick Brosnan, Roy Joshua, editors.

p. cm. — (Clay mathematics proceedings ; v. 9)

Includes bibliographical references.

ISBN 978-0-8218-5191-3 (alk. paper)

1. Algebraic cycles—Congresses. 2. Geometry, Algebraic—Congresses. I. Akhtar, Reza, 1973– II. Brosnan, Patrick, 1968– III. Joshua, Roy, 1956– IV. Title.

QA564.C65685 2008
516.3′5—dc22

2010010765

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`
Visit the Clay Mathematics Institute home page at `http://www.claymath.org/`

10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

Contents

Preface

The subject of algebraic cycles has its roots in the study of divisors, extending as far back as the nineteenth century; however, the field truly began to blossom in the mid-twentieth century after Grothendieck's formulation of a series of conjectures which now bear his name. Since then, algebraic cycles have made a significant impact on many fields of mathematics, among them number theory, algebraic geometry, and mathematical physics, to name only a few. Spencer Bloch introduced the higher Chow groups in the early 1980s extending the classical Chow-groups and roughly having the same relationship to algebraic K-theory as singular cohomology has to topological K-theory. The subject has risen to prominence in recent years in light of the work of Suslin, Voevodsky, and Friedlander on motivic cohomology, which also identified Bloch's higher Chow groups with the latter. In particular Voevodsky's solution of the Milnor conjecture, and work on its extension to other primes (the so-called Bloch-Kato conjecture) has stimulated plenty of interest and work in this area.

Algebraic Cycles II was conceived a sequel to the *Conference on Algebraic Cycles*, held at the Ohio State University in December 2000. The goal of both these conferences was to stimulate further activity in this area by gathering together experts alongside younger mathematicians beginning work in the field. The scientific program of *Algebraic Cycles II* focused on the study of cycles in the contexts of arithmetic geometry, motivic cohomology, and mathematical physics. The conference was also held at The Ohio State University, from March 25 to March 29, 2008, and was organized by Reza Akhtar (Miami University), Patrick Brosnan (University of British Columbia), Roy Joshua (Ohio State) along with David Ellwood (The Clay Mathematics Institute). The conference featured eighteen 40- or 50- minute talks and was attended by about 80 participants from all over the world.

It was felt that a volume devoted to the conference proceedings would better serve the mathematical community. We are very thankful to the Clay Mathematics Institute for agreeing to publish this volume as part of the Clay Mathematical Proceedings (jointly published by the Clay Mathematics Institute and the American Mathematical Society). Several of the articles in this volume contain research presented at this conference, while others represent separate contributions. The editors are happy to acknowledge the enthusiastic support they received from many mathematicians working in this general area, either by contributing a paper to the volume, by serving as referees or by providing other technical assistance. It is our hope that this volume will be of value both to established researchers working in the field and to graduate students who have interest in it.

The editors wish to extend their gratitude to the National Security Agency, the National Science Foundation, the Clay Mathematics Institute, and The Ohio State University for their financial support of this conference. We recognize in particular David Ellwood of the Clay Mathematics Institute for his enthusiasm and support in agreeing to publish these proceedings. We also wish to thank the excellent technical support we received from Vida Salahi (Clay Mathematics Institute), Marilyn Radcliff (Ohio State University), Roshini Joshua (for the Web-page and Poster) and finally everyone else, who helped make this conference a success.

Reza Akhtar, Patrick Brosnan and Roy Joshua

November, 2009

Transcendental Aspects

Clay Mathematics Proceedings
Volume 9, 2010

The Hodge theoretic fundamental group and its cohomology

Donu Arapura

From the work of Morgan [**M**], we know that the fundamental group of a complex algebraic variety carries a mixed Hodge structure, which really means that a certain linearization of it does. This linearization, called the Malčev or pro-unipotent completion destroys the group completely in some cases; for example if it were perfect. So a natural question is whether can one give a Hodge structure on a larger chunk of the fundamental group. There have been a couple of approaches to this. Work of Simpson [**Si**] continued by Kartzarkov, Pantev, Toen [**KPT1**] has shown that one has a weak Hodge-like structure (essentially an action by $\mathbb{C}^*$ viewed as a discrete group) on the entire pro-algebraic completion of the fundamental group when the variety is smooth projective. Hain [**H2**] refining his earlier work [**H1**], has shown that a Hodge structure of a more conventional sort exists on the so called relative Malčev completions (under appropriate hypotheses). In this paper, I want to propose a third alternative. I define a quotient of the pro-algebraic completion called the Hodge theoretic fundamental group $\pi_1^{hodge}(X,x)$ as the Tannaka dual to the category local systems underlying admissible variations of mixed Hodge structures on X, or in more prosaic terms the inverse limit of Zariski closures of their associated monodromy representations. This carries a nonabelian mixed Hodge structure in a sense that will be explained below. The group $\pi_1^{hodge}(X,x)$ dominates the Malčev completion and the Hodge structures are compatible, and I expect a similar statement for the relative completions.

The basic model here comes from arithmetic. Suppose that X is a variety over a field k with separable closure $\bar{k}$. Let $\bar{X} = X \times_{Spec\, k} Spec\, \bar{k}$. Suppose that $x \in X(k)$ is a rational point, and that $\bar{x} \in X(\bar{k})$ a geometric point lying over it. Then there is an exact sequence of étale fundamental groups

$$1 \to \pi_1^{et}(\bar{X},\bar{x}) \to \pi_1^{et}(X,\bar{x}) \to Gal(\bar{k}/k) \to 1$$

The point x gives a splitting, and so an action of $Gal(\bar{k}/k)$ on $\pi_1^{et}(\bar{X},\bar{x})$. This action will pass to the Galois cohomology of $\pi_1^{et}(\bar{X},\bar{x})$. In the translation into Hodge theory, $\pi_1^{et}(\bar{X},\bar{x})$ is replaced by $\pi_1^{hodge}(X,x)$ and the Galois group by a certain universal Mumford-Tate group MT. The action of MT on $\pi_1^{hodge}(X,x)$ is precisely what I mean by a nonabelian mixed Hodge structure. The cohomology $H^*(\pi_1^{hodge}(X,x),V)$ will carry induced mixed Hodge structures for admissible variations V. In fact there is a canonical morphism $H^*(\pi_1^{hodge}(X,x),V) \to H^*(X,V)$. I

Partially supported by NSF.

call X a Hodge theoretic $K(\pi, 1)$ if this is an isomorphism for all V. Basic examples of such spaces are abelian varieties, and smooth affine curves (modulo [**HMPT**]). I want to add that part of my motivation for this paper is to test out some ideas which could be applied to motivic sheaves. So consequently certain constructions are phrased in more generality than is strictly necessary for the present purposes.

My thanks to Roy Joshua for the invitation to the conference. I would also like to thank Dick Hain, Tony Pantev, Jon Pridham, and the referee for their comments, and also Hain for informing me of his recent work with Matsumoto, Pearlstein and Terasoma.

1. Review of Tannakian categories

In this section, I will summarize standard material from [**DM, D2, D3**]. Let k be a field of characteristic zero. Given an affine group scheme G over k, we can express its coordinate ring as a directed union of finitely generated sub Hopf algebras $\mathcal{O}(G) = \varinjlim A_i$. Thus we can, and will, identify G with the pro-algebraic group $\varprojlim Spec\, A_i$ and conversely (see [**DM**, cor. 2.7] for justification).

By a *tensor category* $\mathcal{T}$ over k, I will mean a k-linear abelian category with bilinear tensor product making it into a symmetric monoidal (also called an ACU) category; we also require that the unit object $\mathbf{1}$ satisfy $End(\mathbf{1}) = k$. $\mathcal{T}$ is *rigid* if it has duals. The category $Vect_k$ of finite dimensional k-vector spaces is the key example of a rigid tensor category. A *neutral Tannakian category* $\mathcal{T}$ over k is a k-linear rigid tensor category, which possesses a faithful functor $F : \mathcal{T} \to Vect_k$, called a fibre functor, preserving all the structure. The Tannaka dual $\Pi(\mathcal{T}, F)$ of such a category (with specified fibre functor F) is the group of tensor preserving automorphisms of F. In more concrete language, an element $g \in \Pi(\mathcal{T}, F)$ consists of a collection $g_V \in GL(F(V))$, for each $V \in Ob\mathcal{T}$, satisfying the following compatibilities:

(C1) $g_{V\otimes U} = g_V \otimes g_U$,
(C2) $g_{V\oplus U} = g_V \oplus g_U$,
(C3) the diagram

$$\begin{tikzcd} F(V) \arrow[r, "g_V"] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r, "g_U"] & F(U) \end{tikzcd}$$

commutes for every morphism $V \to U$.

$\Pi = \Pi(\mathcal{T}, F)$ is (the group of k-points of) an affine group scheme. Suppose that $\mathcal{T}$ is generated, as a tensor category, by an object V i.e. every object of $\mathcal{T}$ is finite sum of subquotients of tensor powers

$$T^{m,n}(V) = V^{\otimes m} \otimes V^{*\otimes n}.$$

Observe that for all $m, n \geq 0$

$$Hom_k(F(\mathbf{1}), F(T^{m,n}(V))) = T^{m,n}(F(V))$$

LEMMA 1.1. *Suppose that $\mathcal{T}$ is generated as a tensor category by V, then*

(1) Π *can be identified with the largest subgroup of $GL(F(V))$ that fixes all tensors in the subspaces*

$$Hom_{\mathcal{T}}(\mathbf{1}, T^{m,n}(V)) \subset T^{m,n}(F(V))$$

(2) Π *leaves invariant each subspace of* $T^{m,n}(F(V))$ *that corresponds to a subobject of* $T^{m,n}(V)$.

In particular, Π *is an algebraic group.*

PROOF. This is standard cf. [**DM**]. $\square$

In general,

$$\Pi(\mathcal{T}, F) = \varprojlim_{\mathcal{T}'\subset\mathcal{T} \text{ finitely generated}} Aut(F|_{\mathcal{T}'})$$

exhibits it as a pro-algebraic group and hence a group scheme.

The following key example will explain our choice of notation.

EXAMPLE 1.2. *Let $Loc(X)$ be the category of local systems (i.e. locally constant sheaves) of finite dimensional $\mathbb{Q}$-vector spaces over a connected topological space X. This is a neutral Tannakian category over $\mathbb{Q}$. For each $x \in X$, $F_x(L) = L_x$ gives a fibre functor. The Tannaka dual $\Pi(Loc(X), F_x)$ is isomorphic to the rational pro-algebraic completion*

$$\pi_1(X,x)^{alg} = \varprojlim_{\rho:\pi_1(X)\to GL_n(\mathbb{Q})} \overline{\rho(\pi_1(X,x))}$$

of $\pi_1(X,x)$.

Given an affine group scheme G, let $Rep_\infty(G)$ (respectively $Rep(G)$) be the category of (respectively finite dimensional) k-vector spaces on which G acts algebraically. When $\mathcal{T}$ is a neutral Tannakian category with a fibre functor F, the basic theorem of Tannaka-Grothendieck is that $\mathcal{T}$ is equivalent to $Rep(\Pi(\mathcal{T}, F))$. The role of a fibre functor is similar to the role of base points for the fundamental group of a connected space. We can compare the groups at two base points by choosing a path between them. In the case of pair of fiber functors F and F', a "path" is given by the tensor isomorphism $p \in Isom(F, F')$ between these functors. An element $p \in Isom(F, F')$ determines an isomorphism $\Pi(\mathcal{T}, F) \cong \Pi(\mathcal{T}, F')$ by $g \mapsto pgp^{-1}$. More canonically, one can define $\Pi(\mathcal{T})$ as an "affine group scheme in $\mathcal{T}$" independent of any choice of F [**D2**, §6]. For our purposes, we can view $\Pi(\mathcal{T})$ as the Hopf algebra object which maps to the coordinate ring $\mathcal{O}(\Pi(\mathcal{T}, F))$ for each F.

Note that Π is contravariant. That is, given a faithful exact tensor functor $E : \mathcal{T}' \to \mathcal{T}$ between Tannakian categories, we have an induced homomorphism $\Pi(\mathcal{T}, F) \to \Pi(\mathcal{T}', F \circ E)$.

2. Enriched local systems

Before giving the definition of the Hodge theoretic fundamental group, it is convenient to start with some generalities. A theory of *enriched local systems* E on the category smooth complex varieties consist of

(E1) an assignment of a neutral $\mathbb{Q}$-linear Tannakian category $E(X)$ to every smooth variety X,

(E2) a contravariant exact tensor pseudo-functor on the category of smooth varieties, i.e., a functor $f^* : E(Y) \to E(X)$ for each morphism $f : X \to Y$ together with natural isomorphisms for compositions,

(E3) faithful exact tensor functors $\phi : E(X) \to Loc(X)$ compatible with base change, i.e., a natural transformation of pseudo-functors $E \to Loc$, (E4) a δ-functor $h^\bullet : E(X) \to E(pt)$ with natural isomorphisms $\phi(h^i(L)) \cong H^i(X, \phi(L))$. We also require there to be a canonical map $p^*h^0(L) \to L$ corresponding the adjunction map on local systems, where $p : X \to pt$ is the projection.

By a weak theory of enriched local systems, we mean something satisfying (E1)-(E3). We have the following key examples.

Example 2.1. *Choosing $E = Loc$ and $h^i = H^i$ gives the tautological example of a theory enriched of local systems.*

Example 2.2. *Let $E(X) = MHS(X)$ be the category of admissible variations of mixed Hodge structure* [**K, SZ**] *on X. This carries a forgetful functor $MHS(X) \to Loc(X)$. Let $h^i(L) = H^i(X, L)$ equipped with the mixed Hodge structure constructed by Saito* [**Sa1, Sa2**]*. Here $p^*h^0(L)$ is the invariant part $L^{\pi_1(X)} \subset L$. This can be seen to give a sub variation of MHS by restricting to embedded curves and applying* [**SZ**, 4.19]*. Thus $MHS(X)$ is a theory of enriched local systems.*

Many more examples of theories of enriched local systems can be obtained by taking subcategories of the ones above.

Example 2.3. *The category $E(X) = HS(X) \subset MHS(X)$ of direct sums of pure variations of Hodge structure of possibly different weights.*

Example 2.4. *The category $E(X) = UMHS(X) \subset MHS(X)$ of unipotent variations of mixed Hodge structure.*

Example 2.5. *Finally the category of tame motivic local systems constructed in* [**A**, sect 5] *is, for the present, only a weak theory of enriched local systems. However, the category defined by systems of realizations [loc. cit.] can be seen to be an enriched theory in the full sense.*

Given a theory of enriched local systems $(E, \phi, h^\bullet)$, let $\phi(E(X))$ denote the Tannakian subcategory of $Loc(X)$ generated by the image of $E(X)$. So $\phi(E(X))$ is the full subcategory whose objects are sums of subquotients of objects in the image of ϕ. Set $\pi_1^E(X, x) = \Pi(\phi(E(X)), F_x)$, where F_x is the fibre functor associated to a base point x. More explicitly, this is the inverse limit of the Zariski closures of monodromy representations of objects of $E(X)$. It follows that the isomorphism class of $\pi_1^E(X, x)$ as a group scheme is independent of x. However, certain additional structure will depend on it.

Let $\kappa : E(pt) \to E(X)$ and $\psi : E(X) \to E(pt)$ be given by p^* and i^* respectively, where $p : X \to pt$ and $i : pt \to X$ are the projection and inclusion of x. We also have $\phi : E(X) \to \phi(E(X))$. These functors yield a diagram

$$\pi_1^E(X, x) \to \Pi(E(X), x) \rightleftarrows \Pi(E(pt))$$

where $\Pi(E(X), x) = \Pi(E(X), F_x)$ and $\Pi(E(pt)) = \Pi(E(pt), pt)$. The diagram is clearly canonical in the sense that a morphism $f : (X, x) \to (Y, y)$ of pointed varieties gives rise to a larger commutative diagram

$$\begin{tikzcd} \pi_1^E(X,x) \arrow[r] \arrow[d] & \Pi(E(X),x) \arrow[r, shift left] \arrow[d] & \Pi(E(pt)) \arrow[l, shift left] \arrow[d, "="] \\ \pi_1^E(Y,y) \arrow[r] & \Pi(E(Y),y) \arrow[r, shift left] & \Pi(E(pt)) \arrow[l, shift left] \end{tikzcd}$$

THEOREM 2.6. *The sequence*

$$1 \to \pi_1^E(X,x) \to \Pi(E(X),x) \rightleftarrows \Pi(E(pt)) \to 1$$

is split exact. Therefore there is a canonical isomorphism

$$\Pi(E(X),x) \cong \Pi(E(pt)) \ltimes \pi_1^E(X,x).$$

PROOF. Since $\psi \circ \kappa = id$, the induced homomorphisms

$$\Pi(E(pt)) \to \Pi(E(X),x) \to \Pi(E(pt))$$

compose to the identity. The injectivity of $\pi_1^E(X,x) \to \Pi(E(X),x)$ follows from [**DM**, 2.21].

Therefore it remains to check exactness in the middle. An element of

$$im[\pi_1^E(X,x) \to \Pi(E(X),x)]$$

is given by a collection of elements $g_V \in GL(\phi(V)_x)$, with $V \in ObE(X)$, satisfying (C1)-(C3) such that

$$g_U = \alpha_x \circ g_V \circ \alpha_x^{-1} \tag{1}$$

holds for any isomorphism $\alpha : \phi(V) \cong \phi(U)$. While an element of $ker[\Pi(E(X),x) \to \Pi(E(pt))]$ is given by a collection $\{g_V\}$ such that $g_V = I$ for any object in the image of κ. If V is in the image of κ, the underlying local system is trivial. This implies that $g_V = \alpha_x \circ \alpha_x^{-1} = I$, if (1) holds. Thus, we have

$$im[\pi_1^E(X,x) \to \Pi(E(X),x)] \subseteq ker[\Pi(E(X),x) \to \Pi(E(pt)]$$

Conversely, suppose that $\{g_V\}$ is an element of the kernel on the right. Given an isomorphism $\alpha : \phi(V) \to \phi(U)$, we have to verify (1). Let $H = \kappa(h^0(V^* \otimes U))$, then $g_H = 1$ by assumption. H gives a subobject of $V^* \otimes U$ which maps to the invariant part $\phi(V^* \otimes U)^{\pi_1(X)}$ of the local system $\phi(V^* \otimes U)$. This follows from our axiom (E4). Therefore, α gives a section of $\phi(H)$. The evaluation morphism $ev : (V^* \otimes U) \otimes V \to U$ restricts to give a morphism $H \otimes V \to U$. We claim that the diagram

$$\begin{tikzcd}
\phi(V)_x \arrow[r, "\alpha\otimes I"'] \arrow[d, "g_V"] \arrow[rr, bend left, "\alpha"] & \phi(H)_x \otimes \phi(V)_x \arrow[r, "ev"'] \arrow[d, "I\otimes g_V"] & \phi(U)_x \arrow[d, "g_U"] \\
\phi(V)_x \arrow[r, "\alpha\otimes I"] \arrow[rr, bend right, "\alpha"'] & \phi(H)_x \otimes \phi(V)_x \arrow[r, "ev"] & \phi(U)_x
\end{tikzcd}$$

commutes. The commutativity of the square on the left is clear. For the commutativity on right, apply (C1) and (C3) and the fact that $g_H = I$. Equation (1) is now proven. □

Since any representation of an affine group scheme is locally finite, it follows that $Rep_\infty(E(pt))$ can be identified with the category of ind-objects $Ind\text{-}E(pt)$. We will often refer to an object of this category as an E-structure. To simplify notation, we generally will not distinguish between V and $\phi(V)$. By a *nonabelian E-structure*, we will mean an affine group scheme G over $\mathbb{Q}$ with an algebraic action of $\Pi(E(pt))$. Equivalently, $\mathcal{O}(G)$ possesses an E-structure compatible with the Hopf algebra operations. A morphism of nonabelian E-structures is a homomorphism of group schemes commuting with the $\Pi(E(pt))$-actions. The previous theorem yields a nonabelian E-structure on $\pi_1^E(X,x)$ which is functorial in the category of pointed varieties. When X is connected, for any two base points x_1, x_2, $\pi_1^E(X, x_1)$ and $\pi_1^E(X, x_2)$ are isomorphic as group schemes although the E-structures need not be the same.

LEMMA 2.7. *An algebraic group G carries a nonabelian E-structure if and only if there exists a finite dimensional E-structure V and an embedding $G \subseteq GL(V)$, such that G is normalized by the action of $\Pi(E(pt))$. A nonabelian E-structure is an inverse limit of algebraic groups with E-structures.*

PROOF. Suppose that G is an algebraic group with an E-structure. Then $\Pi = \Pi(E(pt))$ acts on G; denote this left action by ${}^m g$. We also have a left action of Π on $\mathcal{O}(G)$, written in the usual way, such that

$$m(gf) = ({}^m g)(mf) \tag{2}$$

for $m \in \Pi$, $g \in G$ and $f \in \mathcal{O}(G)$. This gives an action of $\Pi \ltimes G$ on $\mathcal{O}(G)$. Let $V' \subset \mathcal{O}(G)$ be a subspace spanned by a finite set of algebra generators. Let $V \subset \mathcal{O}(G)$ be the smallest $\Pi \ltimes G$-submodule containing V'. By standard arguments, V is finite dimensional and faithful as a G-module. So we get $G \subseteq GL(V)$. Equation (2) implies that G is normalized by Π. This proves one direction. The converse is clear.

For the second statement write $\mathcal{O}(G)$ as direct limit $\mathcal{O}(G) = \varinjlim V_i$ of finite dimensional E-structures. Let $A_i \subset \mathcal{O}(G)$ be the smallest Hopf subalgebra containing V_i. This is an E-substructure. So we have $G = \varprojlim A_i$ which gives the desired conclusion. $\square$

An *E-representation* of a nonabelian E-structure G is a representation on an E-structure V such that (2) holds. The adjoint representation gives a canonical example of an E-representation. The above lemma says that every finite dimensional nonabelian E-structure has a faithful E-representation. The following is straightforward and was used implicitly already.

LEMMA 2.8. *There is an equivalence between the tensor category of E-representations of a nonabelian E-structure G and the category of representations of the semidirect product $\Pi(E(pt)) \ltimes G$.*

COROLLARY 2.9. *In the notation of theorem 2.6, $E(X)$ is equivalent to the category of finite dimensional E-representations of $\pi_1^E(X, x)$.*

3. Nonabelian Hodge structures

The category $MHS = MHS(pt)$ (respectively $HS = HS(pt)$) is just the category of rational graded polarizable rational mixed (respectively pure) Hodge structures [**D1**]; its Tannaka dual will be called the universal (pure) Mumford-Tate group and will be denoted by MT (PMT). The Tannaka dual of the category of real Hodge structures $\mathbb{R}$-HS is just Deligne's torus $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{C}^*$. So the obvious functor $\mathbb{R}$-$HS \to HS \otimes \mathbb{R}$ yields an embedding $\mathbb{S}(\mathbb{R}) \hookrightarrow PMT(\mathbb{R})$. In more concrete terms, $(z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ acts by multiplication by $z_1^p \bar{z}_2^q$ on the (p, q) part of a pure Hodge structure. Since HS is semisimple, PMT is pro-reductive. The inclusion $HS \subset MHS$ gives a homomorphism $MT \to PMT$. We have a section $PMT \to MT$ induced by the functor $V \mapsto Gr^W(V) = \oplus Gr_i^W(V)$. Thus MT is a semidirect product of PMT with ker$[MT \to PMT]$. The kernel is pro-unipotent since it acts trivially on $W_\bullet V$ for any $V \in MHS$.

$Rep_\infty(MT)$ is the category Ind-MHS of direct limits of mixed Hodge structures. Given an object $V = \varinjlim V_i$ in this category, we can extend the Hodge and weight filtrations by $F^pV = \varinjlim F^pV_i$ and $W_kV = \varinjlim W_kV_i$.

Set $\pi_1^{hodge} = \pi_1^E$ for $E = MHS$. So this is the inverse limit of Zariski closures of monodromy representations of variations of mixed Hodge structures. The key definition is:

(NH1) A nonabelian mixed (respectively pure) Hodge structure, or simply an $NMHS$ (or an NHS), is an affine group scheme G over $\mathbb{Q}$ with an algebraic action of MT (respectively PMT).

A morphism of these objects is a homomorphism of group schemes commuting with the MT-actions. A Hodge representation is an MT-equivariant representation; it is the same thing as an E-representation for $E = MHS$. The coordinate ring of an $NMHS$ is a Hopf algebra in Ind-MHS. Let us recapitulate the results of the previous section in the present setting.

- $\pi_1^{hodge}(X, x)$ has an $NMHS$ which is functorial in the category of smooth pointed varieties (and this structure will usually depend on the choice of x).
- An algebraic group G admits an $NMHS$ if and only if it has a faithful representation to the general linear group of a mixed Hodge structure for which MT normalizes G.
- Admissible variations of MHS correspond to Hodge representations of $\pi_1^{hodge}(X, x)$.

The notion of a nonabelian mixed Hodge structure is fairly weak, although sufficient for some of the main results of this paper. At this point it is not clear what the optimal set of axioms should be. We would like to spell out some further conditions which will hold in our basic example π_1^{hodge}.

(NH2) An NMHS G satisfies (NH2) or has nonpositive weights if $W_{-1}\mathcal{O}(G) = 0$.

Remark 3.1. *To see why this is"nonpositive", observe that if V is an MHS with $W_1V = 0$, then $W_{-1}\mathcal{O}(V) = W_{-1}Sym^*(V^*) = 0$. (My thanks to the referee for pointing out that the weights gets flipped.)*

The significance of this condition is explained by the following:

Lemma 3.2. *An NMHS G has nonpositive weights if and only if the left and right actions of G on $\mathcal{O}(G)$ preserves the weight filtration.*

Proof. Note that the left and right G-actions preserves the weight filtration if and only if

$$\mu^*(W_i\mathcal{O}(G)) \subseteq \mathcal{O}(G) \otimes W_i\mathcal{O}(G) \tag{3}$$

$$\mu^*(W_i\mathcal{O}(G)) \subseteq W_i\mathcal{O}(G) \otimes \mathcal{O}(G) \tag{4}$$

Comultiplication $\mu^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ is a morphism of Ind-MHS. If G has nonpositive weights, then

$$\mu^*(W_i\mathcal{O}(G)) \subseteq \sum_{j\geq 0} W_j\mathcal{O}(G) \otimes W_{i-j}\mathcal{O}(G) \subseteq \mathcal{O}(G) \otimes W_i\mathcal{O}(G)$$

(4) follows by symmetry.

Suppose $W_{-1}\mathcal{O}(G) \neq 0$ and that (3), and (4) hold. Let $f \in W_{-1}\mathcal{O}(G)$ be a nonzero element, and let $n > 0$ be the largest integer such that $f \in W_{-n}$. Suppose that $\mu^*(f) = \sum g_i \otimes h_i$. By (4),

$$\sum g_i \otimes h_i \equiv 0 \mod \mathcal{O}(G)/W_{-n} \otimes \mathcal{O}(G)$$

Therefore all $g_i \in W_{-n}$. By a similar argument $h_i \in W_{-n}$. Therefore $\mu^*(f) \in W_{-2n}(\mathcal{O}(G) \otimes \mathcal{O}(G))$. Since morphisms of MHS (and therefore Ind-MHS) strictly preserve weight filtrations [**D1**], $\mu^*(f) = \mu^*(f')$ for some $f' \in W_{-2n}$. But μ^* is injective because μ is dominant. Therefore $f = f' \in W_{-2n}$ which is a contradiction. □

LEMMA 3.3. *If G satisfies (NH2), then there is a unique maximal pure quotient G^{pure}.*

PROOF. Since G satisfies (NH2) it acts on the pure Ind-MHS $Gr^W\, \mathcal{O}(G)$ on the left. As a group we take G^{pure} to be the image of G in $Aut(Gr^W\, \mathcal{O}(G))$. To get the finer structure, we apply lemma 2.7, to write $G = \varprojlim G_i$ with $G_i \subset GL(V_i)$ where $V_i \subset \mathcal{O}(G)$ are mixed Hodge structures such that G_i is normalized by MT. G preserves $W_\bullet V_i$ by assumption. Then

$$G^{pure} = \varprojlim im[G_i \to GL(Gr^W\, V_i)]$$

This group carries a nonabelian pure Hodge structure since each group of the limit does. By construction, there is a surjective morphism $G \to G^{pure}$.

Suppose that $G \to H$ is another pure quotient. Then G will act on $\mathcal{O}(H)$ through this map. The image of G in $Aut(\mathcal{O}(H))$ is precisely H. We have a G-equivariant morphism of Ind-MHS $\mathcal{O}(H) \subset \mathcal{O}(G)$. By purity $\mathcal{O}(H) = Gr^W(\mathcal{O}(H)) \subset Gr^W(\mathcal{O}(G))$, which shows that the G-action on $\mathcal{O}(H)$ factors through G^{pure}. Therefore the homomorphism $G \to H$ also factors through this. □

Before explaining the next condition, we recall that some standard definitions. Fix a real algebraic group G. We have a conjugation $g \mapsto \bar{g}$ on the group of complex points $G(\mathbb{C})$, whose fixed points are exactly $G(\mathbb{R})$. A *Cartan involution* C of G is an algebraic involution of $G(\mathbb{C})$ defined over $\mathbb{R}$, such that the group of fixed points of $g \mapsto C(\bar{g}) = \overline{C(g)}$ is compact for the classical topology and has a point in every component. An involution of a pro-algebraic group is Cartan if it descends to a Cartan involution in the usual sense on a cofinal system of finite dimensional quotients. Recall that Deligne's torus $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{C}^*$ embeds into PMT in such a way that $(z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ acts by multiplication by $z_1^p \bar{z}_2^q$ on the (p, q) part of a pure Hodge structure. It will be convenient to say that a group is reductive (or pro-reductive) when its connected component of the identity is.

(NH3) A NMHS G satisfies (NH3) or is S-polarizable if it has nonpositive weights, and the action C of $(i, i) \in \mathbb{S}(\mathbb{C})$ on G^{pure} gives a Cartan involution.

The "S" stands for Simpson, since this condition is related to (and very much inspired by) the notion of a pure nonabelian Hodge structure introduced by him [**Si**, p. 61]. Simpson has shown that the pro-reductive completion of the fundamental group of a smooth projective variety carries a pure nonabelian Hodge structure in his sense. These ideas have been developed further in [**KPT1, KPT2**]. Although there is no direct relation between these notions of nonabelian Hodge structure, there are a number of close parallels (e.g. lemma 3.7 below holds for both). The meaning of (NH3) is explained by the following:

LEMMA 3.4. *Let G be an algebraic group with an NMHS. Then G is S-polarizable if and only if there exists a pure Hodge structure V and an embedding $G^{pure} \subseteq GL(V)$, such that G^{pure} is normalized by the action of PMT and such that G^{pure} preserves a polarization on V. Under these conditions, G^{pure} is reductive.*

PROOF. After replacing G by G^{pure}, we may assume $G = G^{pure}$ is already pure. Given an embedding $G \subseteq GL(V)$ as above, choose a G-invariant polarization $(\, , \,)$ on V. The image W of $(i, i) \in \mathbb{S}(\mathbb{C})$ in $GL(V)$ is nothing but the Weil operator for the Hodge structure on V. Therefore $\langle u, v\rangle = (u, W\bar{v})$ is positive definite Hermitian. The group of fixed points under $\sigma(g) = W^{-1}\bar{g}W$ is easily seen to preserve $\langle \, , \, \rangle$, so it is compact. In other words, $Cg = W^{-1}gW$ is a Cartan involution. Conversely, if C is a Cartan involution then a G-invariant polarization can be constructed by averaging an existing polarization over the Zariski dense compact group of σ-fixed points. When these conditions are satisfied the reductivity of G follows from the existence of a Zariski dense compact subgroup. □

COROLLARY 3.5. *The underlying group of a pure S-polarizable nonabelian Hodge structure is pro-reductive.*

COROLLARY 3.6. *If G is S-polarizable, then $G^{pure} = G^{red}$ is the maximal pro-reductive quotient of G. In particular, if G is pro-reductive then $G = G^{pure}$ is pure.*

PROOF. We have an exact sequence of pro-algebraic groups

$$1 \to U \to G \to G^{pure} \to 1$$

where U is simply taken to be the kernel. This can also be described as above by

$$U = \varprojlim \ker[G_i \to GL(Gr^W V_i)]$$

We can see from this that U is pro-unipotent. By the previous corollary, G^{pure} is pro-reductive. Thus U is the pro-unipotent radical and $G^{pure} = G^{red}$ is the maximal pro-reductive quotient. This forces $G = G^{pure}$ if G is pro-reductive. □

Recall that Simpson [**Si**, p 46] defined a real algebraic group G to be of *Hodge type* if $\mathbb{C}^*$ acts on $G(\mathbb{C})$ such that $U(1)$ preserves the real form and $-1 \in U(1)$ acts as a Cartan involution. Such groups are reductive, and also subject to a number of other restrictions [loc. cit.]. For example, $SL_n(\mathbb{R})$ is not of Hodge type when $n \geq 3$.

LEMMA 3.7. *If an algebraic group admits a pure S-polarizable nonabelian Hodge structure then it is of Hodge type.*

PROOF. Choose an embedding $G \subset GL(V)$ as in lemma 3.4. The Hodge structure on V determines a representation $\mathbb{S}(\mathbb{C}) \to GL(V_{\mathbb{C}})$. The group $G(\mathbb{C})$ is stable under conjugation by elements of $PMT(\mathbb{C}) \supset \mathbb{S}(\mathbb{C})$. Embed $\mathbb{C}^* \subset \mathbb{S}(\mathbb{C})$ by the diagonal. Then $-1 \in \mathbb{C}^*$ acts trivially on $G(\mathbb{C})$. Therefore the $\mathbb{C}^*$ action factors through $\mathbb{C}^*/\{\pm 1\} \cong \mathbb{C}^*$. To see that $U(1)/\{\pm 1\}$ preserves $G(\mathbb{R}) \subset GL(V_{\mathbb{R}})$, it suffices to note that the image of $e^{i\theta} \in U(1)$ in $GL(V_{\mathbb{C}})$, which acts on V^{pq} by multiplication by $e^{(p-q)i\theta}$, is a real operator. Under the isomorphism $U(1) \cong U(1)/\{\pm 1\}$, -1 on the left corresponds to the image of i on the right. Thus -1 acts by a Cartan involution on $G(\mathbb{C})$.

□

THEOREM 3.8. *Given a smooth variety X, $\pi_1^{hodge}(X,x)$ carries an S-polarizable nonabelian Hodge mixed structure. The category of Hodge representations of $\pi_1^{hodge}(X,x)^{red}$ is equivalent to the category of pure variations of Hodge structure on X.*

PROOF. For $V \in MHS(X)$, let $\pi_1^{hodge}(\langle V\rangle, x)$ denote the Zariski closure of the monodromy representation of $\pi_1(X,x) \to GL(V_x)$. Let $MT(V) \subset GL(V_x)$ denote the Tannaka dual of the sub tensor category of MHS generated by V_x. A slight modification of theorem 2.6 together with lemma 2.7 shows that $\pi_1^{hodge}(\langle V\rangle, x)$ is normalized by $MT(V)$. We can also see this directly. The group $\pi_1^{hodge}(\langle V\rangle, x)$ is characterized as the group of automorphisms that fix all monodromy invariant tensors $T^{m,n}(V_x)^{\pi_1(X,x)}$. While $MT(V)$ leaves all sub MHS of $T^{m,n}(V_x)$ invariant by lemma 1.1. Let $g \in MT(V)$ and let $\gamma \in \pi_1^{hodge}(\langle V\rangle, x)$, then it is enough to see that $g^{-1}\gamma g$ fixes every tensor in $T^{m,n}(V_x)^{\pi_1(X,x)}$. This space is a sub MHS of $T^{m,n}(V_x)$ by [**SZ**, 4.19]. Therefore $T^{m,n}(V_x)^{\pi_1(X,x)}$ is invariant under g (although it need not fix elements pointwise). This shows that $g^{-1}\gamma g$ fixes the elements of this space as claimed. Therefore $\pi_1^{hodge}(\langle V\rangle, x)$ carries a NMHS.

Since the weight filtration of V is a filtration by local systems, $\pi_1^{hodge}(\langle V\rangle, x)$ preserves $W_\bullet V_x$. So it satisfies (NH2) by lemma 3.2. Let $\tilde\pi_1^{hodge}(V)$ be the image of $\pi_1^{hodge}(\langle V\rangle, x)$ in $GL(Gr^W V_x)$. This can be identified with the Zariski closure of the monodromy representation of the pure variation of Hodge structure $Gr^W V$. This pure variation is polarizable by definition of admissibility [**SZ, K**]. Therefore $Gr^W V_x$ possesses a $\tilde\pi_1^{hodge}(V)$ invariant polarization. Consequently $\pi_1^{hodge}(\langle V\rangle, x)$ satisfies (NH3) by lemma 3.4. Moreover $\tilde\pi_1^{hodge}(V) = \pi_1^{hodge}(\langle V\rangle, x)^{red}$ is the Tannaka dual to the subcategory of $Loc(X)$ generated by the local system $Gr^W V$. Putting this all together, we see that

$$\pi_1^{hodge}(X,x) = \varprojlim_V \pi_1^{hodge}(\langle V\rangle, x)$$

satisfies (NH3), and that the Tannaka dual to $HS(X)$ is $PMT \ltimes \pi_1^{hodge}(X,x)^{red}$. Lemma 2.8 implies that $HS(X)$ is equivalent to the category of Hodge representations of $\pi_1^{hodge}(X,x)^{red}$. $\square$

4. Nonabelian variations

The goal of this section is to give a characterization of $\pi_1^{hodge}(X,x)$. To this end, we introduce the category of *nonabelian variation of mixed Hodge structures* over X, by which we mean the opposite of the category of Hopf algebras in Ind-$MHS(X)$. Given such a Hopf algebra A, we denote the corresponding nonabelian variation by the *symbol* $Spec\, A$. For any $x \in X$, $Spec\, A_x$ can be understood in the usual sense, and this is an NMHS. Basic examples are given as follows.

EXAMPLE 4.1. *Any object $V \in MHS(X)$ can be identified with the nonabelian variation $Spec\,(Sym^*(V^*))$.*

By applying the forgetful functor $MHS(X) \to Loc(X)$, we can see that any nonabelian variation $Spec\, A$ carries a monodromy action of $\pi_1(X,x) \to Aut(G_x)$, where $G_x = Spec\, A_x$. A nonabelian variation of mixed Hodge structures will be called *inner* if the monodromy action lifts to a homomorphism $\pi_1(X,x) \to G_x$. The examples of 4.1 are rarely inner. However, an ample supply of such examples is given by the following.

EXAMPLE 4.2. *For $V \in MHS(X)$, we have a nonabelian variation $\pi_1^{hodge}(\langle V\rangle)$ whose fibres are $\pi_1^{hodge}(\langle V\rangle, x)$ by [**D2**, §6]. To construct this directly, note that we can realize the coordinate ring of $\pi_1^{hodge}(\langle V\rangle, x)$ as a quotient of $\mathcal{O}(GL(V_x))$ by a Hopf ideal $\sum f_k\mathcal{O}(GL(V_x))$. The generators f_k can be regarded as sections of*

$$\mathcal{R} = \bigoplus_{i,j\geq 0} Sym^i(V^*) \otimes \det(V^*)^{-j}$$

Thus we can define $\pi_1^{hodge}(\langle V\rangle)$ as Spec of the Ind-$MHS(X)$ Hopf algebra $\mathcal{R}/(\sum_k f_k\mathcal{R})$. This is inner since the monodromy is given by homomorphism

$$\pi_1(X, x) \to \pi_1^{hodge}(\langle V\rangle, x).$$

Let $\pi_1^{hodge}(X)$ be the inverse limit of $\pi_1^{hodge}(\langle V\rangle)$ over $V \in MHS(X)$. The fibres are $\pi_1^{hodge}(X, x)$. This is the universal inner nonabelian variation in the following sense:

PROPOSITION 4.3. *If G is an inner nonabelian variation of mixed Hodge structure over X, with monodromy given by a homomorphism $\rho : \pi_1(X, x) \to G_x$. Then ρ extends to a morphism $\pi_1^{hodge}(X) \to G$ of nonabelian variations.*

PROOF. Let $G = Spec\, A$. Since $A \in$ Ind-$MHS(X)$, $\pi_1^{hodge}(X, x)$ will act on it. By an argument similar to the proof of lemma 2.7, we can write A as a direct limit of finitely generated Hopf algebras with $\pi_1^{hodge}(X, x)$-action. So that G_x becomes an inverse limit of algebraic groups carrying inner nonabelian variations. Thus we can assume that G_x is an algebraic group. By standard techniques we can find a finite dimensional faithful (left) G-submodule $V \subset \mathcal{O}(G)$. After replacing this with the span of the $\pi_1^{hodge}(X, x)$-orbit, we can assume that V is stable under $\pi_1^{hodge}(X, x)$. Therefore V corresponds to a variation of mixed Hodge structure. By assumption, the image of $\pi_1(X, x)$ in $GL(V_x)$ lies in G. This implies that G contains $\pi_1^{hodge}(\langle V\rangle, x)$ and that ρ factors through it. □

5. Unipotent and Relative Completion

Morgan [**M**] and Hain [**H1**] have shown that the pro-unipotent completion of the fundamental group of a smooth variety carries a mixed Hodge structure. We want to compare this with our nonabelian Hodge structure. We start by recalling some standard facts from group theory (c.f. [**HZ**], [**Q**, appendix A]). Fix a finitely generated group π. Then

(a) $\mathbb{Q}[\pi]$, and its quotients by powers of the augmentation ideal J, carry Hopf algebra structures with comultiplication $\Delta(g) = g \otimes g \mod J^r$.
(b) A finite dimensional $\mathbb{Q}[\pi]$-module is unipotent if and only if it factors through some power of J. (The smallest power will be called the index of unipotency).
(c) The set of group-like elements

$$G_r(\pi) = \{f \in \mathbb{Q}[\pi]/J^{r+1} \mid \Delta(f) = f \otimes f,\ f \equiv 1 \mod J\}$$

forms a group under multiplication. This is a unipotent algebraic group.

(d) The Lie algebra of $G_r(\pi)$ can be identified with the Lie algebra of primitive elements

$$\mathcal{G}_r(\pi) = \{f \in \mathbb{Q}[\pi]/J^{r+1} \mid \Delta(f) = f \otimes 1 + 1 \otimes f\}$$

with bracket given by commutator.

(e) The exponential map gives a bijection of sets $\mathcal{G}_r(\pi) \cong G_r(\pi)$. The Lie algebra and group structures determine each other via the Baker-Campbell-Hausdorff formula.

(f) $\mathbb{Q}[\pi]/J^{r+1}$ is isomorphic to a quotient of the universal enveloping algebra of $\mathcal{G}_r(\pi)$ by a power of its augmentation ideal.

Let $ULoc(X)$ ($U_rLoc(X)$) denote the category of local systems with unipotent monodromy (with index of unipotency at most r). The category $U_rLoc(X)$ can be identified with the category of $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$-modules. We note that this category has a tensor product: $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ acts on the usual tensor product of representations $U \otimes_{\mathbb{Q}} V$ through Δ. With this structure, the category of $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$-modules is Tannakian. Its Tannaka dual $\pi_1(X,x)^{un_r}$ is isomorphic to the group $G_r(\pi_1(X,x))$ above, and the Tannaka dual $\pi_1(X,x)^{un}$ of $ULoc(X)$, is the inverse limit of these groups.

We need to impose Hodge structures on these objects.

Lemma 5.1. *There is a bijection between*

(1) *The set of nonabelian mixed Hodge structures on $G_r(\pi)$.*
(2) *The set of mixed Hodge structures on $\mathcal{G}_r(\pi)$ compatible with Lie bracket.*
(3) *The set of mixed Hodge structures on $\mathbb{Q}[\pi]/J^{r+1}$ compatible with the Hopf algebra structure.*

Proof. To go from (1) to (2), observe that a nonabelian mixed Hodge structure always induces a mixed Hodge structure on its Lie algebra compatible with bracket. A Lie compatible mixed Hodge structure on $\mathcal{G}_r(\pi)$ induces an Ind-MHS on its universal enveloping algebra, compatible with the Hopf algebra structure. This descends to $\mathbb{Q}[\pi]/J^{r+1}$ by (f) above. A Hopf compatible mixed Hodge structure on $\mathbb{Q}[\pi]/J^{r+1}$ induces one on $G_r(\pi)$ by restriction. $\square$

Hain [**H1**] constructed a mixed Hodge structure on the Hopf algebra

$$\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$$

which is equivalent (in the sense of the lemma) to the one constructed by Morgan on $\mathcal{G}_r(\pi_1(X,x))$. These fit together to form an inverse system as r increases. In brief outline, Chen had shown that $\mathbb{C} \otimes \varprojlim \mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ can be realized as the zeroth cohomology $H^0(B(x, \mathcal{E}^\bullet(X), x))$ of a complex built from the C^∞ de Rham complex via the bar construction:

$$B^\bullet(x, \mathcal{E}^*(X), x) = (\mathcal{E}^*(X))^{\otimes -\bullet}$$

$$\begin{aligned}
\pm d_B(\alpha_1 \otimes \ldots \otimes \alpha_n) \quad = \quad & i_x(\alpha_1)\alpha_2 \otimes \ldots \otimes \alpha_n \\
& + \sum (-1)^i \alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_{i+1} \otimes \ldots \alpha_n \\
& + (-1)^n i_x(\alpha_n)\alpha_1 \otimes \ldots \otimes \alpha_{n-1} \\
& \pm d\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n \\
& + \ldots
\end{aligned}$$

where $i_x : \mathcal{E}^*(X) \to \mathbb{C}$ is the augmentation given by evaluation at x. Hain showed how to extend $B(x, \mathcal{E}^\bullet(X), x)$ to a cohomological mixed Hodge complex (or more precisely a direct limit of such), and was thus able to deduce the corresponding structure on cohomology. One thing that is more readily apparent in Hain's approach is the dependence on base points. As x varies, $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ forms part of an admissible variation of mixed Hodge structure over X called the tautological variation. This is nontrivial since the monodromy representation is the natural conjugation homomorphism $\pi_1(X,x) \to Aut(\mathbb{Q}[\pi_1(X,x)]/J^{r+1})$.

Wojtkowiak [**W**] gave a more algebro-geometric interpretation of Hain's construction, which will be briefly described. Bousfield and Kan defined the total space functor Tot [**BK**, chap X §3], which is a kind of geometric realization, from the category of cosimplicial spaces to the category of spaces. The image of the map of cosimplicial schemes

$$\begin{tikzcd}
X^{\Delta[1]} \arrow[d, "\pi"] & = & X \times X \arrow[d] \arrow[r, shift left=2] \arrow[r, shift right=2] & X \times X \times X \arrow[l] \arrow[d] & \cdots \\
X^{\partial\Delta[1]} & = & X \times X \arrow[r, shift left=2] \arrow[r, shift right=2] & X \times X \arrow[l] & \cdots
\end{tikzcd}$$

under this functor is the path space fibration $X^{[0,1]} \to X^{\{0,1\}} = X \times X$. The horizontal maps on the top are diagonals (from left to right) or projections (from right to left); on the bottom they are all identities. The total space of the fibre $\pi^{-1}(x,x)$ is the space of loops of X based at x, which is an H-space. Therefore $H^0(Tot(\pi^{-1}(x,x)), \mathbb{Q})$ is naturally a Hopf algebra. This Hopf algebra, which is described more precisely in [**W**], can be identified with the coordinate ring of $\varprojlim G_r(\pi_1(X,x))$, or $G_r(\pi_1(X,x))$ if we truncate the cosimplicial space at the rth stage. This follows from the fact $H^0(Tot(\pi^{-1}(x,x)), \mathbb{C})$ can be computed using the total complex of the de Rham complex of the cosimplicial fibre, which is none other than $B(x, \mathcal{E}^\bullet(X), x)$. Under this identification the filtration by truncations induced on H^0 coincides with the filtration by length of tensors on B. The mixed Hodge structure on $H^0(Tot(\pi^{-1}(x,x)), \mathbb{Q})$ can now be constructed using standard machinery: take compatible multiplicative mixed Hodge complexes on each component of the cosimplicial space and then form the total complex [**W**, §5]. Furthermore the tautological variations are given by the 0th total direct images of $\mathbb{Q}$ under $\pi|_X$ under the diagonal embedding $X \subset X \times X$. A useful consequence of this point of view is that the MHS on $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ can be seen to come from a motive in Nori's sense [**C**].

Let $UMHS(X)$ ($U_rMHS(X)$) denote the subcategory of unipotent admissible variations of mixed Hodge structure (with index of unipotency at most r). The tautological variation associated to $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ lies in $U_rMHS(X)$. Given an object V in $U_rMHS(X)$, the monodromy representation extends to an algebra homomorphism

$$\mathbb{Q}[\pi_1(X,x)]/J^{r+1} \to End(V_x)$$

which is compatible with mixed Hodge structures.

THEOREM 5.2 (Hain-Zucker [**HZ**]). *The above map gives an equivalence between $U_rMHS(X)$ and the category of Hodge representations of $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$.*

The above equivalence respects the tensor structure.

We note that every object of $U_rLoc(X)$ is a sum of subquotients of the local system associated to the tautological representation $\pi_1(X,x) \to Aut(\mathbb{Q}[\pi_1(X,x)]/J^{r+1})$. Therefore $\phi(U_rMHS(X)) = U_rLoc(X)$, where $\phi : U_rMHS(X) \to Loc(X)$ is the forgetful functor. Consequently, we get a split exact sequence

$$1 \longrightarrow \pi_1(X,x)^{un_r} \longrightarrow \Pi(U_rMHS(X)) \rightleftarrows MT \longrightarrow 1$$

by theorem 2.6. In particular, $\pi_1(X,x)^{un_r}$ carries an NMHS which is a quotient of the one on $\pi_1^{hodge}(X,x)$.

PROPOSITION 5.3. *The above NMHS on $\pi_1(X,x)^{un_r}$ is equivalent to the Morgan-Hain structure on $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$.*

PROOF. By lemma 5.1, the Morgan-Hain structure on $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ induces an NMHS on $\pi_1(X,x)^{un_r}$. Let MH denote the semidirect product of MT with this Hodge structure on $\pi_1(X,x)^{un_r}$. By theorem 5.2, we have a commutative diagram

$$\begin{tikzcd}
U_rLoc(X) \arrow[d, "="] & U_rMHS(X) \arrow[l, "\phi"'] \arrow[r, shift left, "\psi"] \arrow[d, "\cong"] & MHS \arrow[l, shift left, "\kappa"] \arrow[d, "="] \\
Rep(\pi_1(X)^{un_r}) & Rep(MH) \arrow[l] \arrow[r, shift left] & Rep(MT) \arrow[l, shift left]
\end{tikzcd}$$

where the functors ψ and κ are given by the fibre and the pullback along the constant map. Therefore MH is isomorphic to $\Pi(U_rMHS(X))$ as a semidirect product.

$\square$

COROLLARY 5.4. *The Morgan-Hain NMHS on $\pi_1(X,x)^{un}$ is a quotient of $\pi_1^{hodge}(X,x)$.*

Hain has extended the above construction in [**H2**]. Given a representation $\rho : \pi_1(X,x) \to S$ to a reductive algebraic group, the relative Malčev completion is the universal extension

$$1 \to U \to \mathcal{G} \to S \to 1$$

of S by a prounipotent group with a homomorphism $\pi_1(X,x) \to \mathcal{G}$ such that

$$\begin{tikzcd}
\pi_1(X,x) \arrow[r] \arrow[dr, "\rho"] & \mathcal{G} \arrow[d] \\
 & S
\end{tikzcd}$$

commutes. When $\rho : \pi_1(X,x) \to S = Aut(\mathcal{V}_x, \langle,\rangle)$ is the monodromy representation of a variation of Hodge structure with Zariski dense image, Hain [**H2**] has shown that the relative Malčev completion carries a NMHS.

CONJECTURE 5.5. *The relative completion should carry an NMHS in general with S equal to the Zariski closure of $\pi_1(X,x) \to Aut(\mathcal{V}_x, \langle,\rangle)$. This should be a quotient of $\pi_1^{hodge}(X,x)$.*

I am quite confident about this. The essential point would construct an inner nonabelian variation of mixed Hodge structure on the family of $\mathcal{G}$ as the base point varies. Then proposition 4.3 would give a homomorphism $\pi_1^{hodge}(X,x) \to \mathcal{G}$. The main step would be to establish an appropriate refinement of [**H2**, cor 13.11], and I understand that Hain, Matsumoto, Pearlstein, and Terasoma [**HMPT**] have done this. J. Pridham has pointed to me that his preprint [**P**] may also have some bearing on this conjecture.

REMARK 5.6. *For the applications given later in section 7, only this weaker statement on the existence of a morphism $\pi_1^{hodge}(X,x) \to \mathcal{G}$ extending ρ is needed. There is one notable case in which this can be deduced immediately. If $\pi_1(X,x)$ is abelian, then $\mathcal{G}$ is necessarily abelian, so it splits into a product $U \times S$. Morphisms $\pi_1^{hodge}(X,x) \to U$ and $\pi_1^{hodge}(X,x) \to S$ can be constructed directly from propositions 5.3 and 4.3.*

6. Cohomology

Fix a theory of enriched local systems E. Let G be a nonabelian E-structure. The category of representations of $\Pi(E(pt)) \ltimes G$ is equivalent to the category of E-representations of G. Given such a representation V, let $H^0(G,V) = V^G$ and $H^0(\Pi(E(pt)) \ltimes G, V) = V^{\Pi(E(pt))\ltimes G}$ be the subspaces of invariants. The action of $\Pi(E(pt))$ on V descends to an action on $H^0(G,V)$. These are left exact functors on the category of representations of $Rep_\infty(\Pi(E(pt)) \ltimes G)$. Since this category has enough injectives (c.f. [**J**, I 3.9]), we can define higher derived functors $H^i(G,V)$ and $H^i(\Pi(E(pt) \ltimes G, V)$. Note that $H^i(G,V)$ is an E-structure since it is derived from a functor from $Rep_\infty(\Pi(E(pt) \ltimes G) \to Rep_\infty(\Pi(E(pt))$. Alternatively, $H^i(G,V)$ can be computed as the cohomology of a bar or Hochschild complex $C^\bullet(G,V)$ [**J**, I 4.14], which is a complex in $Rep_\infty(\Pi(E(pt))$. We can define cohomology of $\Pi(E(pt))$ by taking G trivial. We observe that these functors are covariant in V and contravariant in G.

There are products

$$H^i(G,V) \otimes H^j(G,V') \to H^{i+j}(G, V \otimes V')$$

compatible with E-structures. These can be constructed by either using standard formulas for products on the complexes $C^\bullet(G,-)$, or by identifying $H^i(G,V) = Ext^i(\mathbb{Q},V)$ and using the Yoneda pairing

$$Ext^i(\mathbb{Q},V) \otimes Ext^j(\mathbb{Q},V') \to Ext^i(\mathbb{Q},V) \otimes Ext^j(V, V\otimes V') \to Ext^{i+j}(\mathbb{Q}, V\otimes V')$$

LEMMA 6.1. *Given an E-representation V of G, $H^i(G,V) = \varinjlim H^i(G/H_j, V^{H_j})$ where H_j runs over all normal subgroups stable under $\Pi(E(pt))$ such that G/H_j is finite dimensional.*

PROOF. By lemma 3.4 $G = \varprojlim G/H_j$ is an inverse limit of algebraic groups with nonabelian E-structures. Clearly $V = \varinjlim V^{H_j}$ and so

$$H^0(G,V) = \varinjlim H^0(G/H_j, V^{H_j})$$

The lemma follows from exactness of direct limits. $\square$

LEMMA 6.2. *Fix a Hodge representation V of an NMHS G, and an Ind-MHS U. Then*

(1) *$H^i(MT,U) = 0$ for $i > 1$.*
(2) *We have an exact sequence*

$$0 \to H^1(MT, H^{i-1}(G,V)) \to H^i(MT \ltimes G, V) \to H^0(MT, H^i(G,V)) \to 0$$

Proof. Write U as a direct limit of finite dimensional Hodge structures U_j, then

$$H^i(MT, U) = \varinjlim H^i(MT, U_j) = \varinjlim Ext^i_{MHS}(\mathbb{Q}, U_j)$$

Beilinson [**B**] shows that the higher Ext's vanish, which implies the first statement.

For the second statement, note that we can construct a Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(MT, H^q(G, V)) \Rightarrow H^{p+q}(MT \ltimes G, V)$$

in the usual way. This reduces to the given exact sequence thanks to (1). $\square$

Lemma 6.3. *Let G be a nonabelian E-structure and V an E-representation. If G is pro-reductive then $H^i(G, V) = 0$ for $i > 0$. In general*

$$H^i(G, V) = H^0(G^{red}, H^i(U, V))$$

where U is the pro-unipotent radical and $G^{red} = G/U$.

Proof. When G is pro-reductive, its category of representations is semisimple. Therefore $H^0(G, -)$ is exact. So higher cohomology must vanish.

In general, the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G^{red}, H^q(U, V)) \Rightarrow H^{p+q}(G, V)$$

will collapse to yield the above isomorphism. $\square$

Proposition 6.4. *Let X be a smooth variety. Then for each object $V \in E(X)$:*

1. *$H^i(\pi_1^E(X, x), V)$ carries a canonical E-structure.*
2. *There is natural morphism $H^i(\pi_1^E(X, x), V) \to H^i(X, V)$ of E-structures.*
3. *There are $\mathbb{Q}$-linear maps*

$$H^i(\pi_1^E(X, x), V) \to H^i(\pi_1(X, x), V) \to H^i(X, V)$$

whose composition is the map given in (2).

Proof. $H^i(\pi_1^E(X, x), V)$ carries an E-structure by the discussion preceding lemma 6.1. Note that $H^0(\pi_1^E(X, x), V) = V_x^{\pi_1^E(X)}$ is nothing but the monodromy invariant part of V_x, and $H^i(\pi_1^E(X, x), V)$ is the universal δ-functor extending it, in the sense of [**G**]. By axiom (E4), $H^i(X, V)$ with its E-structure also forms a δ-functor, and there is an isomorphism

$$H^0(\pi_1^E(X, x), V) \cong H^0(X, V)$$

of E-structures. Therefore (2) is a consequence of universality.

After disregarding E-structure, we can view $H^i(\pi_1^E(X, x), V)$ as a universal δ-functor from $E(X) \to Vect_{\mathbb{Q}}$. Therefore, from the isomorphisms

$$H^0(\pi_1^E(X, x), V) \cong H^0(\pi_1(X, x), V) \cong H^0(X, V),$$

we deduce $\mathbb{Q}$-linear maps

$$H^i(X, V) \leftarrow H^i(\pi_1^E(X, x), V) \to H^i(\pi_1(X, x), V)$$

The leftmost map was the one constructed in the previous paragraph. Since group cohomology is also a universal δ-functor from the category of $\mathbb{Q}[\pi_1(X, x)]$-modules to $Vect_{\mathbb{Q}}$, we can complete this to a commutative triangle

$$\begin{tikzcd} H^i(\pi_1^E, V) \arrow[r] \arrow[d] & H^i(\pi, V) \arrow[dl, dashed] \\ H^i(X, V) & \end{tikzcd}$$

$\square$

7. Hodge theoretic $K(\pi,1)$'s

The map

$$H^i(\pi_1^E(X,x), V) \to H^i(X,V) \tag{*}$$

constructed in the previous section is trivially an isomorphism for $i = 0$, but usually not in general. For instance if $i = 2$, $V = \mathbb{Q}$ and $X = \mathbb{P}^1$, the cohomology groups are 0 on the left and $\mathbb{Q}$ on the right.

Let us say that X is a $K(\pi^E, 1)$ (or a Hodge theoretic $K(\pi,1)$ when $E = MHS$) if (*) is an isomorphism for every $V \in E(X)$. When X is a $K(\pi,1)$ in the usual sense, then it is a $K(\pi^E, 1)$ if and only if

$$H^i(\pi_1^E(X,x), V) \to H^i(\pi_1(X,x), V) \tag{**}$$

is an isomorphism for all $V \in E(X)$. The following is a straight forward modification of [**Se**, p 13, ex 1].

LEMMA 7.1. *The following are equivalent*

1. *(*) is an isomorphism for all $i \le n$ and injective for $i = n+1$ for all $V \in$Ind-$E(X)$.*
2. *(*) is an isomorphism for all $i \le n$ and injective for $i = n+1$ for all $V \in E(X)$.*
3. *(*) is surjective for all $i \le n$ and all $V \in E(X)$.*
4. *(*) is surjective for all $i \le n$ and all $V \in$ Ind-$E(X)$.*
5. *For all $V \in$ Ind-$E(X)$, $1 \le i \le n$, and $\alpha \in H^i(X,V)$, there exists $V' \in$ Ind-$E(X)$ containing V such that the image of α in $H^i(X,V')$ vanishes.*

In particular, X is a $K(\pi^E, 1)$ if (*) is surjective for all i and $V \in E(X)$.

PROOF. The implications (1)⇒(2) and (2) ⇒(3) are clear. For (3)⇒(4), we can use the fact that cohomology commutes with filtered direct limits. The implication (4)⇒(5) follows because Ind-$E(X)$ contains enough injectives [**J**]. Any injective object $V' \supset V$ will satisfy the conditions of (5) assuming (4).

Finally, we prove (5)⇒(1). This is the only nontrivial step. An exact sequence

$$0 \to V \to V' \to V'/V \to 0$$

yields a diagram

$$\begin{tikzcd} H^{i-1}(\pi_1^E, V') \arrow[r] \arrow[d, "f"] & H^{i-1}(\pi_1^E, V'/V) \arrow[r] \arrow[d, "g"] & H^i(\pi_1^E, V) \arrow[r] \arrow[d] & H^i(\pi_1^E, V') \arrow[d] \\ H^{i-1}(X, V') \arrow[r] & H^{i-1}(X, V'/V) \arrow[r] & H^i(X, V) \arrow[r] & H^i(X, V') \end{tikzcd}$$

We first prove surjectivity of (*) for $i \le n$ by induction. This is trivially true when $i = 0$, so we may assume $i > 0$. Given $\alpha \in H^i(X,V)$, we may choose V' so that α has trivial image in $H^i(X, V')$. Then α lifts to $H^{i-1}(X, V'/V)$ and hence to some $\beta \in H^{i-1}(\pi_1^E, V'/V)$. The image of β in $H^i(\pi_1^E, V)$ will map to α as required.

Now we prove injectivity of (*) for $i \le n+1$ by induction. We may assume that $i > 0$. Let V' be injective in Ind-$E(X)$. Suppose that

$$\alpha \in \ker[H^i(\pi_1^E(X,x), V) \to H^i(X, V)]$$

Then α can be lifted to $\beta \in H^{i-1}(\pi_1^E, V'/V)$. Since the maps labelled f and g are isomorphisms, a simple diagram chase shows that β lies in the image of $H^{i-1}(\pi_1^E, V')$. Therefore $\alpha = 0$. ☐

PROPOSITION 7.2. *When $E = MHS$, for all $V \in MHS(X)$ the map (*) is an isomorphism for $i \le 1$ and injective for $i = 2$ assuming conjecture 5.5 holds. This is true unconditionally if $\pi_1(X)$ is abelian.*

PROOF. As is well known, for any V the map

$$H^i(\pi_1(X,x), V) \to H^i(X, V)$$

is an isomorphism for $i = 1$ and injective for $i = 2$. Thus, by this remark and the previous lemma, it is enough to prove that the map (**) to group cohomology is surjective for $i = 1$. By the 5-lemma and induction on the length of the weight filtration, it is sufficient to the prove this when V is pure.

Let V be a variation of pure Hodge structure, and let $1 \to U \to \mathcal{G} \to S \to 1$ be the associated relative Malčev completion. By lemma 6.3,

$$H^1(\mathcal{G}, V) \cong H^0(S, H^1(U, V)) \cong Hom_S(U/[U,U], V)$$

By [**H2**, prop 10.3]

$$H^1(\pi_1(X,x), V) \cong Hom_S(U/[U,U], V)$$

Therefore the natural map $H^1(\mathcal{G}, V) \to H^1(\pi_1, V)$ is an isomorphism. As this factors through (**) (by conjecture 5.5 or remark 5.6), (**) must be a surjection in degree 1. ☐

THEOREM 7.3. *A (not necessarily affine) connected commutative algebraic group is a Hodge theoretic $K(\pi, 1)$. Assuming conjecture 5.5, a smooth affine curve is a Hodge theoretic $K(\pi, 1)$.*

PROOF. Suppose that X is a commutative algebraic group. Then it is homotopy equivalent to a torus. In particular, it is a $K(\pi, 1)$. So it suffices to check surjectivity of (**). The group $\pi_1(X)$ is abelian and finitely generated, which implies that the pro-algebraic completion of $\pi_1(X)$ is a commutative algebraic group. Therefore the same is true for $\pi_1^{hodge}(X)$. After extending scalars, it follows that the group $\pi_1^{hodge}(X) \otimes \bar{\mathbb{Q}}$ is a product of $\mathbb{G}_a$'s, $\mathbb{G}_m$'s and a finite abelian group. Consequently, any irreducible representation V of $\pi_1^{hodge}(X) \otimes \bar{\mathbb{Q}}$ is one dimensional. For such a module, the Künneth formula implies that

$$H^i(\pi_1(X), V) = \begin{cases} \wedge^i H^1(\pi_1(X), V) & \text{if } V \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

Therefore (**) is surjective in this case. By applying (an appropriate modification of) lemma 7.1 to the category of semisimple $\pi_1^{hodge}(X) \otimes \bar{\mathbb{Q}}$ representations, we can see that (**) is an isomorphism. We note that

$$H^i(\pi_1^{hodge}(X,x), V \otimes \bar{\mathbb{Q}}) \cong H^i(\pi_1^{hodge}(X,x), V) \otimes \bar{\mathbb{Q}}$$

$$H^i(\pi_1(X,x), V\otimes\bar{\mathbb{Q}}) \cong H^i(\pi_1(X,x), V)\otimes\bar{\mathbb{Q}}$$

and likewise for the map between them. Thus we may extend scalars in order to test bijectivity in (**). After doing so, we see that (**) is an isomorphism by an induction on the length of a Jordan-Holder series.

Let X be a smooth affine curve. This is a $K(\pi,1)$ with a free fundamental group. Since a free group has cohomological dimension one, (**) is surjective in all degrees assuming 5.5, by the previous proposition.

□

I expect that Artin neighbourhoods are also Hodge theoretic $K(\pi,1)$'s. This would give a large supply of such spaces. Katzarkov, Pantev and Toen have established an analogous result in their setting [**KPT2**, rmk 4.17]. Although, their proofs do not translate directly into the present framework, I suspect that an appropriate modification may.

References

[A] D. Arapura, *A category of motivic sheaves*, ArXiv:0801.0261

[B] A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of algebraic K-theory to algebraic geometry and number theory, AMS (1987)

[BK] A. Bousfield, D. Kan, *Homotopy limits, completions and localizations*, LNM 304, Springer-Verlag (1972)

[C] M. Cushman, *Morphisms of curves and the fundamental group*, Contemp Trends in Alg. Geom. and Alg. Top., World Scientific (2002)

[DM] P. Deligne, J. Milne, *Tannakian categories*, in LNM 900, Springer-Verlag (1982)

[D1] P. Deligne, *Theorie de Hodge II*, Inst. Hautes Études Sci. Publ. Math 40 (1971)

[D2] P. Deligne, *Le groupes fondemental de la droit projective moin trois points*, Galois Groups over $\mathbb{Q}$, Springer-Verlag (1989)

[D3] P. Deligne, *Catégories Tannakiennes*, Grothendieck Festschrift, Birkhauser (1990)

[G] A. Grothendieck, *Sur quelques points d algébre homologiques*, Tohoku Math. J. 9 (1957)

[H1] R. Hain, *The de Rham homotopy theory of complex algebraic varieties. I*, K-theory 3 (1987)

[H2] R. Hain, *The Hodge de Rham theory of relative Malčev completion,* Ann. Sci. cole Norm. Sup. 31 (1998), 47–92.

[HZ] R. Hain, S. Zucker, *Unipotent variations of mixed Hodge structure*, Inv. Math 88 (1987)

[HMPT] R. Hain, M. Matsumoto, G. Pearlstein, T. Terasoma, *Tannakian Fundamental Groups of Categories of Variations of Mixed Hodge Structure*, in preperation

[J] J. Jantzen, *Representations of algebraic groups*, 2nd ed., AMS (2003)

[K] M Kashiwara, *A study of a variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. 22 (1986)

[KPT1] L. Katzarkov, T. Pantev, B. Toen, *Schematic homotopy types and non-abelian Hodge theory*, Compositio Math (2008)

[KPT2] L. Katzarkov, T. Pantev, B. Toen, *Algebraic and topological aspects of the schematization functor*, arXiv.math.AG/0503418

[Ko] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton U. Press (1995)

[M] J. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math 48 (1978)

[P] J. Pridham, *Formality and splitting of real non-abelian mixed Hodge structures,* ArXiv:0902.0770

[Q] D. Quillen, *Rational homotopy theory*, Ann. Math 90 (1969)

[Sa1] M. Saito, *Mixed Hodge modules and admissible variations*, CR Acad. Sci. Paris 309 (1989)

[Sa2] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990)

[Se] J.P. Serre, *Cohomologie Galois*, 5th ed., LNM 5, Springer-Verlag (1994)

[Si] C. Simpson, *Higgs bundles and local systems,* Inst. Hautes Études Sci. Publ. Math 75 (1992)

[SZ] J. Steenbrink, S. Zucker, *Variation of mixed Hodge structures I*, Inv. Math. 80 (1985)

[W] Z. Wojtkowiak, *Cosimplicial objects in algebraic geometry*, Alg. K-theory and Alg. Top., Kluwer (1993)

Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.

Clay Mathematics Proceedings
Volume 9, 2010

The Real Regulator for a Self-product of a General Surface

Xi Chen and James D. Lewis

Abstract. In [**C-L3**] it is shown that the real regulator for a general self-product of a $K3$ surface is nontrivial. In this note, we prove a theorem which says that the real regulator for a general self-product of a surface of higher order (in a suitable sense), is essentially trivial.

1. Statement of the theorem

Let Γ be a smooth projective curve, $\{Z_t\}_{t\in\Gamma}$ a family of surfaces, all defined over a subfield $k \subset \mathbb{C}$, and put:

$$Z_\Gamma := \coprod_{t\in\Gamma} Z_t.$$

Let us assume for simplicity that Z_Γ is smooth and that any singular Z_t is nodal. In local analytic coordinates, $Z_\Gamma \times_\Gamma Z_\Gamma$ about each singular point looks like

$$x_1^2 + x_2^2 + x_3^2 - (y_1^2 + y_2^2 + y_3^2) = 0,$$

which is an isolated nodal singularity. Then the projectivized tangent cone is a 4-dimensional smooth quadric Q_0. Let $\widetilde{[Z_\Gamma \times_\Gamma Z_\Gamma]}_0$ be the blow-up of $Z_\Gamma \times_\Gamma Z_\Gamma$ at this isolated singular point 0. We are going to assume that $\mathrm{CH}^2(Q_0, \mathbb{Q}) := \mathrm{CH}^2(Q_0) \otimes \mathbb{Q} \hookrightarrow \mathrm{CH}^3(\widetilde{[Z_\Gamma \times_\Gamma Z_\Gamma]}_0, \mathbb{Q})$ is injective for any such singular point. Further, we assume that $\mathrm{CH}^1(Z_t;\mathbb{Q}) \simeq \mathbb{Q}$ for all $t \in \Gamma$. Note that this (latter) condition alone will fail for a general 1-parameter family of quartic surfaces in $\mathbb{P}^3$ (take for example the locus of quartics containing a line). Let us further assume that for all $t \in \Gamma$:

$$\begin{aligned} \mathrm{CH}^2(Z_t \times Z_t;\mathbb{Q}) = {} & \mathrm{CH}^0(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^2(Z_t;\mathbb{Q}) \\ & + \mathrm{CH}^1(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^1(Z_t;\mathbb{Q}) \\ & + \mathrm{CH}^2(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^0(Z_t;\mathbb{Q}) + \mathbb{Q}\{\Delta_{Z_t}\}, \end{aligned} \tag{1.1}$$

where Δ_{Z_t} is the diagonal image $Z_t \to Z_t \times Z_t$. This is not an unreasonable assumption, given the fact that a similar kind of decomposition (specifically without

---

1991 *Mathematics Subject Classification.* Primary 14C25; Secondary 14C30, 14C35.

Key words and phrases. Regulator, Chow group, Deligne cohomology.

Both authors partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

the $\mathbb{Q}\{\Delta_{Z_t}\}$ term), holds for a general product of $K3$ (and higher order) surfaces, under the assumption of a rather deep conjecture (due to Bloch and Beilinson) - see[**L**] and [**C-L2**].

Let $B \subset \Gamma$ be a finite set for which $Z_U \times_U Z_U \to U$ is smooth and proper, where $U = \Gamma \backslash B$. Finally, for a very general choice of $t \in \Gamma$, assume that with respect to the monodromy representation

$$\pi_1(U) \to \mathrm{Aut}\left(H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{Q})\right),$$

there are no classes in the transcendental cohomology $H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{C})$ whose Hodge (p,q) components displace horizontally with respect to the Gauss-Manin connection (the reader may wish to consult [**C-L2**] for a precise definition of this).

Theorem 1.2. *Given the above setting, assume that $c_2(Z_t) \neq 3$ for $t \in U$. Let $t \in \Gamma(\mathbb{C})$ correspond to an embedding $k(\Gamma) \hookrightarrow \mathbb{C}$. Then the reduced regulator map*

$$\underline{r}_{3,1} : \mathrm{CH}^3(Z_{k(\Gamma)} \times Z_{k(\Gamma)}, 1) \to H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{R}) \bigcap H^{2,2}(Z_t \times Z_t),$$

is zero.

Note that for a general self-product of a $K3$ surface, the reduced regulator is nontrivial ([**C-L3**]). What the theorem says is that for a self-product of a general surface of higher order, the reduced regulator is trivial. If we consider for example surfaces in $\mathbb{P}^3$, then in light of the fact that a smooth surface in $\mathbb{P}^3$ is $K3 \Leftrightarrow$ its degree $d = 4$, a higher order surface in this context should be a general surface in $\mathbb{P}^3$ of degree ≥ 5. Theorem 1.2 however, does not directly apply to surfaces in $\mathbb{P}^3$ of degree $d \geq 5$. The subtle point here is that a Lefschetz pencil of surfaces, after an arbitrary base change, is no longer smooth, while Z_Γ is assumed to be smooth in Theorem 1.2. However the real issue is that the injectivity statement $\mathrm{CH}^2(Q_0, \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([Z_\Gamma \times_\Gamma Z_\Gamma]_0, \mathbb{Q})$ needs to be addressed. But this can be fixed at least for surfaces in $\mathbb{P}^3$. Namely, we have the following.

Theorem 1.3. *For a very general surface $Z_t \subset \mathbb{P}^3$ of degree $d \geq 5$, the reduced regulator map*

$$\underline{r}_{3,1} : \mathrm{CH}^3(Z_t \times Z_t, 1) \to H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{R}) \bigcap H^{2,2}(Z_t \times Z_t),$$

is zero if we assume that (1.1) *holds for all $t \in \mathbb{P}^N \backslash \Sigma$, where $\mathbb{P}^N$ is the parameter space of surfaces of degree d in $\mathbb{P}^3$ and $\Sigma \subset \mathbb{P}^N$ is a countable union of subvarieties of $\mathbb{P}^N$ with codimension ≥ 2.*

Implicit in the statement of this theorem is the expectation that a very general surface $Z_t \subset \mathbb{P}^3$ of degree $d \geq 5$ automatically satisfies the assumption that $\mathrm{codim}_{\mathbb{P}^N} \Sigma \geq 2$. There are good reasons to expect this. Firstly, if one assumes a conjecture of Bloch and Beilinson on the injectivity of the Abel-Jacobi map for smooth quasiprojective varieties defined over number fields, then according to [**C-L2**] and [**L**], such a decomposition in (1.1) will hold for Z_t replaced by a very general X. Further, apart from a number of technical issues, the key issue is requiring (1.1) to hold for all $t \in \Gamma$. Such an X will be a very general member of a general (Lefschetz) pencil $\{Z_t\}_{t \in \mathbb{P}^1}$ of surfaces of degree d in $\mathbb{P}^3$. As explained in [**C-L1**], the work of M. Green ([**G**]) on the Noether-Lefschetz locus implies that $\mathrm{CH}^1(Z_t; \mathbb{Q}) \cong \mathbb{Q}$, for all $t \in \mathbb{P}^1$, provided that $d \geq 5$. Although the proof in [**G**] relies on infinitesimal Hodge theoretic methods, an ad hoc explanation goes as follows. The horizontal displacement of a rational topological 2-cycle in the Noether-Lefschetz locus must pair to zero under integration with holomorphic 2-forms. That $d \geq 5 \Rightarrow \dim H^{2,0}(X) > 1$ suggests that this locus is of codimension ≥ 2 in the universal family of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$ (which is indeed a fact). This very same reasoning suggests the decomposition in (1.1), which holds for very general t under our above assumptions, actually holds for all $t \in \mathbb{P}^1$. Finally, one can show that $\deg(c_2(X)) = d(d^2+6-4d)$.

The ideas presented here are similar to those given in [**C-L2**]. Rather than repeat them, we highlight the main points and introduce the new ingredients.

We are grateful to the referee for suggesting improvements in presentation. The second author would like to express his warm appreciation to the organizers (Reza Akhtar, Patrick Brosnan, Roy Joshua) for inviting him to this splendid conference.

2. Proof of Theorem 1.2

This will be carried out in three steps.

Step I: *The spread cycle ξ.* Put

$$Z_B = \coprod_{t \in B} Z_t \times Z_t \subset Z_\Gamma \times_\Gamma Z_\Gamma.$$

Let $\xi_{k(\Gamma)} \in \mathrm{CH}^3(Z_{k(\Gamma)} \times Z_{k(\Gamma)}, 1)$. After possibly enlarging B, we may assume that:

(i) $\xi_{k(\Gamma)}$ spreads to a cycle $\xi \in \mathrm{CH}^3\big(Z_\Gamma \times_\Gamma Z_\Gamma \backslash Z_B, 1\big)$,

Further, up to a base change $\Gamma' \twoheadrightarrow \Gamma$, we can assume that:

(ii) There is a section $\sigma : \Gamma \to Z_\Gamma$ avoiding the double points of the singular fibers, with $\Gamma \simeq \sigma(\Gamma)$. (Note: Our goal is to complete ξ to a cycle on $Z_{\Gamma'} \times_{\Gamma'} Z_{\Gamma'}$. Once we do that, then we can proper push-forward it to $Z_\Gamma \times_\Gamma Z_\Gamma$. Therefore we may assume for simplicity that $\Gamma = \Gamma'$.) For $t \in U$, let $\xi_t \in \mathrm{CH}^3(Z_t \times Z_t, 1)$ be the corresponding class.

We will refer to the diagram

$$\begin{tikzcd}
Z_\Gamma \arrow[dr, "\Delta"] & & \\
& Z_\Gamma \times_\Gamma Z_\Gamma \arrow[r, "Pr_1"] \arrow[d, "Pr_2"] & Z_\Gamma \arrow[d] \\
& Z_\Gamma \arrow[r] & \Gamma
\end{tikzcd}$$

Step II: *Modifying ξ in such a way that it extends to a cycle $\underline{\xi} \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1)$, and such that $\underline{r}_{3,1}(\xi_t) = \underline{r}_{3,1}(\underline{\xi}_t)$ for general $t \in \Gamma$.* The closure of ξ defines a precycle $\overline{\xi}$ on $Z_\Gamma \times_\Gamma Z_\Gamma$ whose boundary $\partial\overline{\xi}$ is supported on Z_B. Thus according to the decomposition in (1.1), we have

$$\{\partial\overline{\xi}\} = R + S + T + W = 0 \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma; \mathbb{Q}),$$

where

$$R = \sum_{t\in B} Z_t \otimes \xi_t^{(R)}, \qquad S = \sum_{t\in B} n_t H_t \otimes H_t'$$
$$T = \sum_{t\in B} \xi_t^{(T)} \otimes Z_t, \qquad W = \sum_{t\in B} m_t \Delta_{Z_t}$$

Here H_t and H_t' are general hyperplane sections of Z_t, $\xi_t^{(R)}$, $\xi_t^{(T)}$ are 0-cycles, and $n_t,\ m_t \in \mathbb{Q}$. Now put

$$\begin{aligned} D_1 &= \coprod_{t\in\Gamma} \sigma(t) \times Z_t \quad &\simeq Z_\Gamma \\ D_2 &= \coprod_{t\in\Gamma} Z_t \times \sigma(t) \quad &\simeq Z_\Gamma \\ H &= \coprod_{t\in\Gamma} H_t' \otimes H_t \\ \Delta &= \Delta(Z_\Gamma) \end{aligned}$$

Let $d = \deg Z_t = \big(H_t^2\big)_{Z_t}$, and put $\hat{\xi} = R + S + T + W$. Also put $N = \big(\Delta_{Z_t}^2\big)_{Z_t\times Z_t}$ for any $t \in U$. Of course, this number is really independent of $t \in \Gamma$, by defining it as a limit $t \mapsto t_0$ over $t \in U$, for any $t_0 \in \Gamma$; and observe that for $t \in U$, $N = \deg(c_2(Z_t))$.

We compute:

$$\begin{aligned} \hat{\xi} \cap D_1 &= \sum_{t\in B} \sigma(t) \times \xi_t^{(R)} + \sum_{t\in B} m_t(\sigma(t), \sigma(t)) \sim_{\text{rat}} 0 \text{ on } D_1 \\ \hat{\xi} \cap D_2 &= \sum_{t\in B} \xi_t^{(T)} \times \sigma(t) + \sum_{t\in B} m_t(\sigma(t), \sigma(t)) \sim_{\text{rat}} 0 \text{ on } D_2 \end{aligned}$$

These equations yield:

$$\sum_{t\in B} \Big(\big[\deg(\xi_t^{(R)})\big] \cdot t + m_t \cdot t \Big) \sim_{\text{rat}} 0 \text{ on } \Gamma$$
$$\sum_{t\in B} \Big(\big[\deg(\xi_t^{(T)})\big] \cdot t + m_t \cdot t \Big) \sim_{\text{rat}} 0 \text{ on } \Gamma$$

Further,

$$\begin{aligned} \hat{\xi} \cap H \sim_{\text{rat}} 0 &\Rightarrow d^2\big(\sum_{t\in B} n_t \cdot t\big) + d\sum_{t\in B} m_t \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \\ &\Rightarrow d\big(\sum_{t\in B} n_t \cdot t\big) + \sum_{t\in B} m_t \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \\ \hat{\xi} \cap \Delta \sim_{\text{rat}} 0 &\Rightarrow d\big(\sum_{t\in B} n_t \cdot t\big) + N\sum_{t\in B} m_t \cdot t \\ &\quad + \sum_{t\in B} \big[\deg(\xi_t^{(R)})\big] \cdot t + \sum_{t\in B} \big[\deg(\xi_t^{(T)})\big] \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \end{aligned}$$

Now put

$$
\begin{aligned}
x_1 &= \sum_{t\in B}[\deg(\xi_t^{(R)})]\cdot t\\
x_2 &= \sum_{t\in B}[\deg(\xi_t^{(T)})]\cdot t\\
x_3 &= \sum_{t\in B} m_t\cdot t\\
x_4 &= \sum_{t\in B} n_t\cdot t
\end{aligned}
$$

Then in terms of rational equivalence to zero on Γ, we have

$$
\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&1&d\\1&1&N&d\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
$$

A simple computation yields

$$
\det\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&1&d\\1&1&N&d\end{bmatrix}=d(3-N),
$$

which is nonzero by our assumptions. In particular $x_j\sim_{\rm rat}0$ on Γ. As in [**C-L2**], by multiplying H and Δ by the relevant rational functions pulled back from Γ, one can then easily modify $\overline{\xi}$ so that $S=W=0$. On D_1,

$$
\sum_{t\in B}\sigma(t)\times\xi_t^{(R)}\sim_{\rm rat}0,
$$

and on D_2,

$$
\sum_{t\in B}\xi_t^{(T)}\times\sigma(t)\sim_{\rm rat}0.
$$

Each of these involves rational functions on curves $C\subset D_i$ where either C dominates Γ or $C\subset Z_t$ for some $t\in\Gamma$. Via the projections $Pr_j: Z_\Gamma\times_\Gamma Z_\Gamma\to Z_\Gamma\simeq D_j$, this all lifts to $R\sim_{\rm rat}0$ and $T\sim_{\rm rat}0$ on $Z_\Gamma\times_\Gamma Z_\Gamma$. Again, by a further modification of $\overline{\xi}$ we arrive at a sought for class $\underline{\xi}\in{\rm CH}^3(Z_\Gamma\times_\Gamma Z_\Gamma,1)$, such that $\underline{r}_{3,1}(\xi_t)=\underline{r}_{3,1}(\underline{\xi}_t)$ for general $t\in\Gamma$, where we reiterate that $\underline{r}_{3,1}(\underline{\xi}_t)$ is a projection of the real regulator image which kills the image of the decomposable cycles, (and for which we modified by cycles which are fiberwise decomposable).

Step III: *A rigidity argument.* The variety $Z_\Gamma\times_\Gamma Z_\Gamma$ is singular (unless $Z_\Gamma\twoheadrightarrow\Gamma$ is smooth) , and so we observe that there is a cycle map on the level of homology:

$$
{\rm CH}^3(Z_\Gamma\times_\Gamma Z_\Gamma,1;\mathbb{Q})\to H_5^{\mathcal{D}}(Z_\Gamma\times_\Gamma Z_\Gamma,\mathbb{R}(2)),
$$

where $H_5^{\mathcal{D}}(Z_\Gamma\times_\Gamma Z_\Gamma,\mathbb{R}(2))$ is real Deligne homology, and $\mathbb{R}(m)$ is the Tate twist. There is an exact sequence

$$
H_6(Z_\Gamma\times_\Gamma Z_\Gamma,\mathbb{R}(3))\to H_5^{\mathcal{D}}(Z_\Gamma\times_\Gamma Z_\Gamma,\mathbb{R}(2))\twoheadrightarrow \hom_{\mathbb{R}-{\rm MHS}}(\mathbb{R}(0),H_5(Z_\Gamma\times_\Gamma Z_\Gamma,\mathbb{R}(2))).
$$

We will show (below) that the cycle $\underline{\xi} \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1; \mathbb{Q})$ has zero image in $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), H_5(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))\big)$. Hence we can assume that $\underline{\xi} \in H_5^{\mathcal{D}}(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))$ lifts to a class in $H_6(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(3))$. Now consider the composite

$$H_6(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(3)) \to H_4(Z_t \times Z_t, \mathbb{R}(2)) \xrightarrow{\sim} H^4(Z_t \times Z_t, \mathbb{R}(2)) \twoheadrightarrow H^{2,2}(Z_t \times Z_t, \mathbb{R}(2)).$$

Then as in [**C-L1**], we can apply Betti rigidity, together with our monodromy assumptions to deduce that $\underline{r}_{3,1}(\underline{\xi}_t)$ (a fortiori $\underline{r}_{3,1}(\xi_t)$) is zero for sufficiently general $t \in \Gamma$. We now attend to the details of modifying $\underline{\xi}$. In local analytic coordinates the singular set of $Z_\Gamma \times_\Gamma Z_\Gamma$ looks like

$$x_1^2 + x_2^2 + x_3^2 = t^M = y_1^2 + y_2^2 + y_3^2,$$

for some positive integer M. Since we assume Z_Γ to be smooth, we necessarily have $M = 1$. Then locally we are in the situation of

$$x_1^2 + x_2^2 + x_3^2 - (y_1^2 + y_2^2 + y_3^2) = 0,$$

which is an isolated nodal singularity. Then the projectivized tangent cone is a 4-dimensional smooth quadric Q_0 whose generators contribute to the vector space $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), Gr_W^0 H_5(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))\big)$. Let $[\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0$ be the blow-up of $Z_\Gamma \times_\Gamma Z_\Gamma$ at this isolated singular point 0. Now recall by assumption that we have an injection $\mathrm{CH}^2(Q_0; \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0; \mathbb{Q})$. From the localization sequence

$$\cdots \to \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0, 1; \mathbb{Q}) \to \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma \backslash \{0\}, 1; \mathbb{Q})$$

$$\to \mathrm{CH}^2(Q_0; \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0; \mathbb{Q}) \to \cdots,$$

it is clear that $\underline{\xi}$ lifts to a class in $\mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0, 1; \mathbb{Q})$. Repeating this procedure for each nodal singularity, we arrive at $\underline{\xi}$ lying in the image of

$$\mathrm{CH}^3(\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}, 1; \mathbb{Q}) \to \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1; \mathbb{Q}),$$

where $\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}$ is a desingularization of $Z_\Gamma \times_\Gamma Z_\Gamma$. Now use the fact that the space $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), H_5(\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}, \mathbb{R}(2))\big) = 0$. □

Remark 2.1. Let us put

$$\widetilde{Z}_t = \begin{cases} Z_t & \text{if } Z_t \text{ smooth} \\ \mathrm{desing}(Z_t) & \text{if } Z_t \text{ singular} \end{cases}$$

where $\mathrm{desing}(Z_t)$ is the minimal desingularization of Z_t. Suppose that

$$H^{2,2}(\widetilde{Z}_t \times \widetilde{Z}_t) \bigcap H^2_{\mathrm{tr}}(\widetilde{Z}_t, \mathbb{Q}) \otimes H^2_{\mathrm{tr}}(\widetilde{Z}_t, \mathbb{Q}) \simeq \mathbb{Q},$$

and that

$$\mathrm{CH}^1(Z_t; \mathbb{Q}) \simeq \mathbb{Q},$$

for all $t \in \Gamma$. (This last condition implies that $H^1(\widetilde{Z}_t, \mathbb{Q}) = 0 = H^3(\widetilde{Z}_t, \mathbb{Q})$.) Then assuming the existence of the conjectured Bloch-Beilinson filtration, one can show as in [**C-L2**] that the decomposition in (1.1) holds.

3. Proof of Theorem 1.3

Let $Z_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a Lefschetz pencil of surfaces of degree d in $\mathbb{P}^3$ with $\mathbb{P}^1 \subset \mathbb{P}^N$. For a very general choice of $\mathbb{P}^1 \subset \mathbb{P}^N$, we can make it avoid Σ and the locus where $\mathrm{Pic}(Z_t) \neq \mathbb{Z}$. Suppose that we have a class $\xi_t \in \mathrm{CH}^3(Z_t \times Z_t, 1; \mathbb{Q})$ for $t \in \mathbb{P}^1$ general. Then after a base change $\Gamma \to \mathbb{P}^1$, we can extend ξ_t to ξ_U over an open set $U \subset \Gamma$. By the same argument in the previous section, we can further extend ξ to all of $Z_\Gamma \times_\Gamma Z_\Gamma$, where $Z_\Gamma = Z_{\mathbb{P}^1} \times_{\mathbb{P}^1} \Gamma$. As mentioned at very beginning, Z_Γ might be singular. Hence, in order to show the vanishing of $\underline{r}_{3,1}(\xi)$ by the monodromy argument, we need to lift ξ to a desingularization of $Z_\Gamma \times_\Gamma Z_\Gamma$.

We first desingularize Z_Γ. Let Y be the minimal desingularization of Z_Γ. Observe that the singularities of Z_Γ consist of the points $p \in Z_t$, where p is an ordinary double point of the surface Z_t and the finite map $\Gamma \to \mathbb{P}^1$ is ramified at $t \in \Gamma$. Locally at p, Z_Γ is given by

$$x_1^2 + x_2^2 + x_3^2 = t^M \tag{3.1}$$

where M is ramification index $\Gamma \to \mathbb{P}^1$ at t. For simplicity, we may assume that M is even; otherwise, we just replace Γ by Γ' with a further base change $\Gamma' \to \Gamma$. The singularity p as in (3.1) can be resolved by a sequence of blowups and we end up with

$$Y_t = \widetilde{Z}_t \cup Q_1 \cup Q_2 \cup ... \cup Q_m$$

locally over p, where $Q_0 = \widetilde{Z}_t$ is the proper transform of Z_t and $Q_1, Q_2, ..., Q_m$ are a chain of $m = M/2$ rational ruled surfaces satisfying that

- $Q_i \cong \mathbb{F}_2$ for $1 \le i \le m-1$ and $Q_m \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$;
- $Q_i \cap Q_j \neq \emptyset$ iff $|i-j| \le 1$;
- Q_{i-1} and Q_i meet transversely along a curve $C_i \cong \mathbb{P}^1$ for $i = 1, 2, ..., m$.

Let us consider $Y \times_\Gamma Y$. This is only a partial resolution of $Z_\Gamma \times_\Gamma Z_\Gamma$; $Y \times_\Gamma Y$ is singular along $C_i \times C_j$ with local equation $x_1 x_2 = y_1 y_2 = t$. On the other hand, $Y \times_\Gamma Y$ admits a small resolution $\widetilde{Y \times_\Gamma Y}$ and we can lift every class in $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$ to $\mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}, 1; \mathbb{Q})$. This small resolution is obtained by subsequently blowing up $Q_i \times Q_j$; locally we are blowing up along $x_1 = y_1 = 0$. Note that $\widetilde{Y \times_\Gamma Y}$ is projective as it is obtained by blowing up along algebraic subvarieties. The exceptional loci of this resolution consist of threefolds E_{ij} which are $\mathbb{P}^1$ bundles over $C_i \times C_j$ for $1 \le i, j \le m$. So the obstruction to the lifting comes from the map

$$\bigoplus_{i,j} \mathrm{CH}^1(E_{ij}; \mathbb{Q}) \to \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}; \mathbb{Q}) \tag{3.2}$$

by the corresponding localization sequence; a class in $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$ can be lifted to $\mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}, 1; \mathbb{Q})$ if the map (3.2) is injective.

Let $\widetilde{Q_i \times Q_j} \subset \widetilde{Y \times_\Gamma Y}$ be the proper transform $Q_i \times Q_j$. The 3-fold E_{ij} is the intersection of two out of the four 4-folds among

$$\{\widetilde{Q_{i-\alpha} \times Q_{j-\beta}} : \alpha, \beta = 0 \text{ or } 1\}$$

and meets the other two transversely along two disjoint sections of $E_{ij} \to C_i \times C_j$, say G_{ij} and $G'_{ij} \subset E_{ij}$; exactly which two depends on the order in which we blow up $Q_i \times Q_j$. Obviously,

$$\mathrm{CH}^1(E_{ij}) \cong \mathrm{CH}^1(C_i \times C_j) \oplus \mathbb{Z}G_{ij}. \tag{3.3}$$

On Y, we obviously have the injection

$$\bigoplus_i \mathrm{CH}^k(C_i) \hookrightarrow \mathrm{CH}^{k+2}(Y) \tag{3.4}$$

for $k = 0, 1$. This gives us the injection

$$\bigoplus_{i,j} \mathrm{CH}^1(C_i \times C_j) \hookrightarrow \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}). \tag{3.5}$$

On the other hand, it is easy to show the injection

$$\bigoplus_{i,j} (\mathbb{Z}G_{ij} \oplus \mathbb{Z}G'_{ij}) \hookrightarrow \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}) \tag{3.6}$$

and the injectivity of (3.2) follows.

So it remains to lift classes in $\mathrm{CH}^3(Z \times_\Gamma Z, 1; \mathbb{Q})$ to $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$. The obstruction is the map

$$\bigoplus_{i,j} \mathrm{CH}^2(Q_i \times Q_j; \mathbb{Q}) \to \mathrm{CH}^3(Y \times_\Gamma Y; \mathbb{Q}) \tag{3.7}$$

and it suffices to show that (3.7) is injective.

Obviously,

$$\mathrm{CH}^2(Q_i \times Q_j) = \bigoplus_{k=0}^{2} \mathrm{CH}^k(Q_i) \otimes \mathrm{CH}^{2-k}(Q_j) \tag{3.8}$$

and the injections

$$\mathrm{CH}^k(Q_i) \hookrightarrow \mathrm{CH}^{k+1}(Y) \tag{3.9}$$

are trivial for $k = 0, 2$. The hard part is to prove (3.9) for $k = 1$, i.e.,

$$\bigoplus_i \mathrm{CH}^1(Q_i) \hookrightarrow \mathrm{CH}^2(Y). \tag{3.10}$$

This has been done in the appendix of [**C-L2**], where it was proved that (3.10) is an injection if $Z_{\mathbb{P}^1}$ is chosen to be a very general pencil.

References

[B] S. Bloch, Algebraic cycles and higher K-theory, *Advances in Math.* **61** (1986), 267-304.

[C-L1] X. Chen and J. D. Lewis, Noether-Lefschetz for K_1 of a certain class of surfaces, *Bol. Soc. Mat. Mexicana* **10(3)** (2004), 29-41.

[C-L2] X. Chen and J. D. Lewis, The real regulator for a product of $K3$ surfaces, in *Mirror Symmetry V, Proceedings of the BIRS conference in Banff, Alberta* (Edited by S.-T. Yau, N. Yui and J. D. Lewis), AMS/IP Studies in Advanced Mathematics, Volume **38**. (2006), 271-283.

[C-L3] X. Chen and J. D. Lewis, Real Regulators on Self-products of $K3$ Surfaces. To appear in the J. of Algebraic Geometry.

[C-L4] X. Chen and J. D. Lewis, The Hodge-$\mathcal{D}$-conjecture for $K3$ and Abelian surfaces, *J. Alg. Geom.* **14** (2005), 213-240.

[G] M. Green, A new proof of the explicit Noether-Lefschetz theorem, *J. Differential Geometry* **27** (1988), 155-159.

[L] J. D. Lewis, A filtration on the Chow groups of a complex projective variety, *Compositio Math.* **128**, (2001), 299-322.

632 Central Academic Building, University of Alberta, Edmonton, Alberta T6G 2G1, CANADA
E-mail address: xichen@math.ualberta.ca

632 Central Academic Building, University of Alberta, Edmonton, Alberta T6G 2G1, CANADA
E-mail address: lewisjd@ualberta.ca

Clay Mathematics Proceedings
Volume 9, 2010

Lipschitz Cocycles and Poincaré Duality

Eric M. Friedlander* and Christian Haesemeyer**

ABSTRACT. Geometric measure theory enables one to view cohomology as equivalence classes of graphs of multi-valued Lipschitz maps to spheres. This geometric point of view gives a new formulation of cohomology, relative cohomology, and cohomology with supports as homotopy groups of spaces of Lipschitz cocycles. Using the graphing construction of the first author and H. Blaine Lawson, this leads to a formulation and proof of weak equivalences whose associated map on homotopy groups is a form of Alexander duality for the complement of a compact subpolyhedron in a compact, oriented smooth manifold.

0. Introduction

Let A be a compact oriented n-dimensional pseudo-manifold which is smoothable outside a subcomplex of codimension 2 and let $\alpha \in H^j(A, \mathbb{Z})$ be an integral cohomology class of A. In [**6**], the first author and H. Blaine Lawson established that $\alpha \cap [A] \in H_{n-j}(A)$ can be represented by the geometric measure-theoretic slice of the graph of a multi-valued Lipschitz map from A to S^j. The purpose of this paper is to extend the constructions of [**6**] to a compactifiable pseudo-manifold $A - A_\infty$ and then use this extension to prove a form of Alexander duality whenever $A - A_\infty$ is smooth. Indeed, we prove a stronger result which is the geometric measure-theoretic analogue of the Friedlander-Lawson [**5**] and Friedlander-Voevodsky [**7**] duality theorems for smooth complex algebraic varieties. Namely, as seen in Corollary 5.5, the graphing construction for Lipschitz maps determines a weak equivalence relating the topological abelian groups of Lipschitz cocycles on $A - A_\infty$ and an appropriate group of rectifiable currents on A with boundary in a closed tubular neighborhood of A_∞.

Our arguments involve a mixture of elementary simplicial topology and geometric measure theory. The fundamental construction Γ^{top} involves the graphing of a multi-valued Lipschitz map to a sphere with domain a compact oriented n-dimensional pseudo-manifold A equipped with a triangulation. Such a space is a Lipschitz neighborhood retract, admitting a good formulation of currents. Moreover, the triangulation enables us to work cell-by-cell, enabling local arguments and consideration of subcomplexes and their complements.

* partially supported by NSF # 03000525.
** partially supported by NSF # 0924631.

To summarize in more detail, Section 1 introduces the open and closed subsets of polyhedra which we shall employ, and discusses various spaces of Lipschitz maps with target a symmetric product of a sphere. In Section 2, we define the Lipschitz cocyle space $\mathcal{Z}^m(A)$ of codimension m cocycles on a finite polyhedron A, the relative cocyle space $\mathcal{Z}^m(A, C)$ for a closed subset $C \subset A$, the cocycle space $\mathcal{Z}^m(A - A_\infty)$ of codimension m Lipschitz cocycles on the complement of a closed subpolyhedron $A_\infty \subset A$, and the space $\mathcal{Z}^m_{A_\infty}(A)$ of codimension m Lipschitz cocycles on A with support in A_∞. Essentially, a Lipschitz cocycle on A is an element of the group completion of spaces of Lipschitz maps from A into symmetric powers of a sphere. As seen in Section 3, the homotopy groups of these cocycle spaces satisfy the expected properties of singular cohomology.

The first author and H. Blaine Lawson constructed in [**6**] a continuous graph mapping $\Gamma^{top} : \mathcal{Z}^m(A) \to Z_n(A \times S^m)$ for A a compact, oriented pseudo-manifold of dimension n. This graphing construction was shown to yield cap product with the fundamental class of A, so that Poincaré Duality could be interpreted in these terms. In Theorem 4.5, we extend this continuous graphing construction to $\mathcal{Z}^m(A - A_\infty)$. With the formalism established for $A - A_\infty$, we not only can formulate various duality theorems for non-compact spaces but also give proofs which use only the techniques developed in this paper. For example, Corollary 5.5 is a formulation in terms of spaces of cocycles/cycles of Poincaré-Alexander duality.

In the final section, we show how the Thom class and the Thom isomorphism admit a natural formulation in terms of Lipschitz cocycles.

We thank Blaine Lawson for sharing with us his geometric insight into Poincaré duality.

1. Polyhedra, rectifiable currents, and Lipschitz Maps

We consider a compact polyhedron A, a cell complex which is the geometric realization of a finite simplicial complex. We shall typically consider a (piece-wise linear) triangulation Δ on A associated to some choice of structure of a finite simplicial complex, and then consider refinements of such a triangulation. By abuse of notation, we shall refer to a compact polyhedron together with a given (finite, piecewise linear) triangulation as a *simplicial complex* as well. We denote by $\Delta(k)$ the (finite) set of open k-simplices of Δ, each homeomorphic to an open k-disk. Observe that the d-fold symmetric product $SP^d(A)$ of A is again a compact polyhedron.

If A is a compact polyhedron, then a simplicial structure on A determines an embedding of A in a Euclidean space $\mathbb{R}^N$, where N denotes the number of vertices of the simplicial structure. A tubular neighborhood of $A \subset \mathbb{R}^N$ provides A with the structure of a Lipschitz neighborhood retract of $\mathbb{R}^N$. Such an embedding provides A with a piecewise smooth Riemannian metric (compatible with the triangulation on A given by the simplicial structure). The class of Lipschitz functions associated to such a metric on A is in fact independent of the choice of such a metric.

Definition 1.1. Let A be a compact polyhedron, a Lipschitz neighborhood retract with Lipschitz retraction $U \to A$ of some tubular neighborhood U of A in a Euclidean space. A polyhedral k-chain on a Euclidean space is a formal integral sum of "k-prisms" defined as the convex hulls of $(k+1)$-tuples of distinct points, and a Lipschitz polyhedral k-chain on A is the image of such a polyhedral chain under a Lipschitz map from a Euclidean space to A. A rectifiable k-current on A is an element in the closure (with respect to the mass norm) of the space of Lipschitz polyhedral k-chains on A.

We denote by $\mathcal{I}_k(A)$ the space of rectifiable k-currents on A with rectifiable boundary (i.e., **integral k-currents**) equipped with the flat norm topology. We denote by $\mathcal{Z}_k(A) \subset \mathcal{I}_k(A)$ the subspace of rectifiable k-currents with 0-boundary (i.e. **integral k-cycles**).

We recall the following theorem of F. Almgren [**1**] (and restated in [**6**, 1.2]).

THEOREM 1.2. *Let $C \subset A$ be a closed subspace, with both A, C compact, local Lipschitz neighborhood retracts in Euclidean spaces. Then there is a natural isomorphism*

$$\mathcal{A} : \pi_j\{\mathcal{Z}_r(A,C)/\mathcal{I}_r(C)\} \xrightarrow{\sim} H_{r+j}(A,C).$$

Here, $\mathcal{I}_r(C)$ denotes the integral r-currents on C with the flat norm topology and $\mathcal{Z}_r(A,C)$ denotes the integral r-currents on A whose boundary has support in C, also provided with the flat norm topology.

Moreover, $\mathcal{Z}_r(A)/\mathcal{Z}_r(C)$ is a closed subspace of $\mathcal{Z}_r(A,C)/\mathcal{I}_r(C)$ with discrete quotient, thereby determining the short exact sequence

$$0 \to \mathcal{Z}_r(A)/\mathcal{Z}_r(C) \ \to \ \mathcal{Z}_r(A,C)/\mathcal{I}_r(C) \ \to \ \ker\{H_{r-1}(C) \to H_{r-1}(A)\} \to 0.$$

We shall work with non-compact spaces of the form $A - A_\infty$, where A is a compact polyhedron and $A_\infty \subset A$ is a (closed) subcomplex with respect to some finite triangulation of A. We shall refer to such a space $A - A_\infty$ as a **compactifiable polyhedron**.

DEFINITION 1.3. Let A be a finite polyhedron. Equip A with a (finite, piecewise linear) triangulation Δ and let $A_\infty \subset A$ be a closed subpolyhedron that is a subcomplex for the triangulation Δ. Embed A in Euclidean space $\mathbb{R}^s$, where s is the number of vertices of A, so that each vertex is a distance 1 along the corresponding axis of $\mathbb{R}^s$.

We define

$$\begin{aligned} D_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) \leq \frac{1}{4}\} \\ O_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) < \frac{1}{4}\} \\ S_\Delta(A_\infty) &\equiv \{a \in A : d_A(a, A_\infty) = \frac{1}{4}\} \end{aligned}$$

Note that $D_\Delta(A_\infty)$, $S_\Delta(A_\infty)$ and $A - O_\Delta(A_\infty)$ are all closed subcomplexes of A for some suitable subdivision of the triangulation Δ. It is useful to observe that if Δ' refines Δ then $O_{\Delta'}(A_\infty) \subset O_\Delta(A_\infty)$.

The following proposition constructs a flow from $A - A_\infty$ to the closed subpolyhedron $A - O_\Delta(A_\infty)$.

PROPOSITION 1.4. *Let $U = A - A_\infty$ be a compactifiable polyhedron.*

1. *There is a homotopy $H : U \times I \to U$ relating the identity of U to a retraction $U \to A - O_\Delta(A_\infty)$ and restricting to $H_| : (A - O_\Delta(A_\infty)) \times I \to A - O_\Delta(A_\infty)$.*
2. *There is a deformation retraction $F : D_\Delta(A_\infty) \times I \to D_\Delta(A_\infty)$ which is a deformation retraction to the subpolyhedron A_∞.*

Proof. Observe that the closure $\overline{\sigma}$ of an open simplex of A meets $D_\Delta(A_\infty)$ if and only if $\overline{\sigma}$ meets A_∞. Let $Y \subset A$ denote the union of those closed simplices of A meeting A_∞, and triangulate Y using the first barycentric subdivision of the given triangulation of A. We define the "link" $L \subset A$ of $A_\infty \subset A$ to be the sub-simplicial complex of Y with vertices the barycenters of simplices of A whose closures do not intersect A_∞. Let $\tilde{L} \subset Y$ consist of those points $y \in Y;\ d(y, L) \leq \frac{1}{4}$. Then we employ a continuous map

$$F : Y \times I \ \to \ Y$$

satisfying

- $F(y, 0) = y,\ y \in Y$
- $F(x, t) = x,\ x \in A_\infty$ *and* $t \in I$
- $F(y, t) = y,\ y \in \tilde{L}$ *and* $t \in I$
- $F(-, t)$ is a homeomorphism for $t \neq 1$
- $F(D_\Delta(A_\infty) \times \{1\})) \subset A_\infty$
- $F(D_\Delta(A_\infty) \times I) \subset D_\Delta(A_\infty)$.

Any such map F restricted to $D_\Delta(A_\infty)$ is a deformation retraction to A_∞. Moreover, , we obtain a deformation retraction $F' : (Y - A_\infty) \times I \to Y - A_\infty$ of $Y - A_\infty$ to $Y - D_\Delta(A_\infty)$ by setting F'_t equal to the inverse of the restriction to $Y - A_\infty$ of F_{1-t}. We define $H : U \times I \to I$ to be this retraction on $(Y - A_\infty) \times I$ and the identity flow on $(A - Y) \times I$. $\square$

We shall use the following elementary lemma from homotopy theory.

Lemma 1.5. *Let $Y \subset X$ be an inclusion of a subspace of a topological space X. Suppose there is a homotopy $H : X \times I \to X$ such that*

- $H(-, 0) = id_X$
- $H(X \times \{1\}) \subset Y$
- $H(Y \times \{t\}) \subset Y$ *for all* t.

Then the inclusion $Y \to X$ is a homotopy equivalence with homotopy inverse $H(-, 1)$.

Applying Lemma 1.5 to the homotopy of Proposition 1.4 (1), we immediately obtain the following corollary.

Corollary 1.6. *Let $U = A - A_\infty$ be a compactifiable polyhedron and let Δ be some finite piece-wise linear triangulation of A such that A_∞ is a subcomplex. Then the embedding*

$$A' = (A - O_\Delta(A_\infty)) \ \subset \ U$$

is a weak equivalence and A' is a compact polyhedron.

We recall that a continuous map $f : A \to B$ of metric spaces is said to be Lipschitz with Lipschitz constant K if for all pairs of points a, a' in A the following inequality is satisfied:

$$d_B(f(a), f(a')) \ \leq \ K \cdot d_A(a, a').$$

Definition 1.7. If $A,\ B$ are metric spaces, then we define

$$Map_{Lip}(A, B)$$

to be the set of Lipschitz maps from A to B with topology of convergence with bounded Lipschitz constant. In other words, the sequence $\{f_n\}$ converges to f :

$A \to B$ in this topology if it is uniformly convergent and there is a $K > 0$ that serves as Lipschitz constant for all the f_n.

REMARK 1.8. If A and B are compact polyhedra equipped with a piecewise smooth metric via embeddings as Lipschitz neighborhood retracts, then the subset

$$Map_{Lip}(A, B) \hookrightarrow Map_{cont}(A, B)$$

together with its topology is independent of the choice of embedding. On the strength of this observation we will refer to its elements as *Lipschitz maps* from A to B without reference to the specific piecewise smooth metric chosen.

Observe that if B is a compact polyhedron, then so is its d-fold symmetric power $SP^d(B)$ for any $d > 0$.

In [**6**, 1.5], the embedding $Map_{Lip}(A, SP^d(S^m)) \hookrightarrow Map_{cont}(A, SP^d(S^m))$ is shown to be a weak homotopy equivalence. We extend this result by allowing A to be a compactifiable polyhedron.

PROPOSITION 1.9. *Let A, B be compact polyhedra, and $d > 0$. Retain the hypotheses and notation of Definition 1.3. Then each of the maps of the following chain is a weak homotopy equivalence*

$$\begin{aligned} Map_{cont}(A - A_\infty, SP^d(B)) \ \to \ & Map_{cont}(A - O_\Delta(A_\infty), SP^d(B)) \leftarrow \\ & Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B)). \end{aligned}$$

PROOF. The homotopy $H : U \times I \to U$ of Proposition 1.4 implies that the first map is a homotopy equivalence by Lemma 1.5; [**6**, 1.5] verifies that the second map is a weak equivalence. $\square$

LEMMA 1.10. *If Δ' is a refinement of the triangulation Δ of A, then for any $d > 0$ the natural restriction map*

$$Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(B)) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B))$$

is a Serre fibration and a weak equivalence.

PROOF. Let $\Lambda^j[n] \subset \Delta[n]$ denote the inclusion of the union of all faces of the n-simplex $\Delta[n]$ except the j-th face into $\Delta[n]$. For each $n \geq 0$ and each j, $0 \leq j \leq n$, we use the structure of $A - O_\Delta(A_\infty) \subset A - O_{\Delta'}(A_\infty)$ (as a simplicial embedding of finite complexes) to exhibit a strong deformation retraction of $(A - O_{\Delta'}(A_\infty)) \times \Delta[n]$ to

$$((A - O_{\Delta'}(A_\infty)) \times \Lambda^j[n]) \cup_{((A - O_\Delta(A_\infty)) \times \Lambda^j[n])} ((A - O_\Delta(A_\infty)) \times \Delta[n])$$

which is a Lipschitz map with Lipschitz constant 1. This implies the Serre lifting property for $Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(S^m)) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$. The fact that this map is a weak equivalence follows from Proposition 1.9 and the fact that both $A - O_\Delta(A_\infty)$, $A - O_{\Delta'}(A_\infty)$ are homotopy equivalent to $A - A_\infty$. $\square$

DEFINITION 1.11. Let $\mathcal{T}(A)$ denote the category of finite, piece-wise linear triangulations of the polyhedron A, with one triangulation Δ' mapping to another Δ provided that Δ' is a refinement of Δ. For any closed subpolyehdron $A_\infty \subset A$ which is a subcomplex with respect to a triangulation Δ, any finite polyhedron B and any $d > 0$, we define

$$(1) \quad Map^b_{Lip}(A - A_\infty, SP^d(B)) \ \equiv \ \varinjlim_{\Delta' \in \mathcal{T}(A)/\Delta} Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(B)).$$

Remark 1.12. Let A, A', B be finite polyhedra and let $A_\infty \subset A$, $A'_\infty \subset A'$ be subpolyhedra. Then a Lipschitz map $f : A \to A'$ with the property that $f^{-1}(A'_\infty) \subset A_\infty$ induces a continuous map

$$f^* : Map^b_{Lip}(A' - A'_\infty, SP^d(B)) \longrightarrow Map^b_{Lip}(A - A_\infty, SP^d(B)). \tag{2}$$

In particular, there is a natural restriction map

$$Map^b_{Lip}(A - A'_\infty, SP^d(B)) \to Map^b_{Lip}(A - A_\infty, SP^d(B))$$

whenever $A'_\infty \subset A_\infty$ is an inclusion of closed subcomplexes for some finite triangulation. This implies that the assignment

$$U \mapsto Map^b_{Lip}(U, SP^d(B))$$

is a contravariant functor on the category of open subsets of A whose complement is a closed subcomplex for some sufficiently fine finite triangulation of A.

Corollary 1.13. *The natural projection and inclusion maps*

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(B)) \;\leftarrow\; Map^b_{Lip}(A - A_\infty, SP^d(B))$$

and

$$Map^b_{Lip}(A - A_\infty, SP^d(B)) \hookrightarrow \; Map_{cont}(A - A_\infty, SP^d(B))$$

are weak equivalences for any finite, piece-wise linear triangulation Δ of A such that A_∞ inherits the structure of a subcomplex.

Proof. The fact that the projection is a weak equivalence follows from Lemma 1.10 and the standard fact that the inverse limit of a tower of maps each of which is a Serre fibration and a weak equivalence projects via a Serre fibration and a weak equivalence to each term in the tower. The fact that the inclusion is a weak equivalence follows from Proposition 1.9. □

2. Lipschitz cocycle spaces

Recall that the group completion $(\coprod_d SP^d(S^m))^+$ is a model for the generalized Eilenberg-MacLane space $K(\mathbb{Z}, m) \times K(\mathbb{Z}, 0)$, so that

$$\pi_i(Map_{cont}(A, (\coprod_d SP^d(S^m))^+) \;=\; \begin{cases} H^m(A) \oplus H^0(A) & \text{if } \; i = 0 \\ H^{m-i}(A) & \text{if } \; i > 0. \end{cases}$$

where $H^i(A) = H^i(A, \mathbb{Z})$ denotes singular cohomology with $\mathbb{Z}$ coefficients. Since A is compact,

$$(\coprod_d Map_{cont}(A, SP^d(S^m))^+ \;\cong\; Map_{cont}(A, (\coprod_d SP^d(S^m))^+).$$

Thus,

$$\pi_i \ker\{(\coprod_d Map_{cont}(A, SP^d(S^m))^+ \to H^0(A)\} \;\cong\; H^i(A), \quad i \geq 0.$$

This motivates the following definition of Lipschitz cocycle spaces. We set

$$Map_{Lip}(A, SP^\infty(S^m)^+) = \ker\{(\coprod_{d \geq 0} Map_{Lip}(A, SP^d(S^m)))^+ \to H^0(A)\}. \tag{3}$$

DEFINITION 2.1. Let A be a compact polyhedron and $C \subset A$ a closed subset that is a subcomplex with respect to some finite triangulation of A. Following [**6**] we define the topological abelian group $\mathcal{Z}^m(A)$ of **Lipschitz** m**-cocycles** on A (topological Abelian group $\mathcal{Z}^m(A, C)$ of relative Lipschitz cocycles, respectively) as

$$\mathcal{Z}^m(A) = Map_{Lip}(A, SP^\infty(S^m)^+)$$

and

$$\mathcal{Z}^m(A, C) = \ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(C)\}$$

Note that by Remark 1.8, these groups are well-defined independent of the choice of a realizations of A and C as Lipschitz neighborhood retracts.

PROPOSITION 2.2. *Let A be a compact polyhedron and $C \subset A$ a closed subcomplex with respect to some finite triangulation. Then there are isomorphisms*

$$\begin{aligned} \pi_i \mathcal{Z}^m(A) \ &\cong \ H^{m-i}(A) \\ \pi_i \mathcal{Z}^m(A, C) \ &\cong \ H^{m-i}(A, C) \end{aligned}$$

where $H^*(A)$ denotes the singular cohomology of A with $\mathbb{Z}$ coefficients.

These isomorphisms are natural for Lipschitz maps of (pairs of) compact polyhedra. Since every continuous map of compact polyhedra is homotopic to a Lipschitz map, these isomorphisms are, in fact, natural on the homotopy category of compact polyhedra.

PROOF. The proposition follows from the special case of $A_\infty = \emptyset$ of Proposition 1.9 (i.e., from [**6**, 1.5]) and the above representation of cohomology of A in terms of $Map_{cont}(A, SP^d(S^m))$. $\square$

A key theorem which enables us to consider Lipschitz cocycles on compactifiable finite polyhedra (rather than relative groups as we do for geometric cycle spaces) is the following important theorem of Kirszbraun.

THEOREM 2.3. *(Kirszbraun's theorem; cf.* [**2**, 2.10.43]*) Let $S \subset \mathbb{R}^m$ be an arbitrary subset of $\mathbb{R}^m$ and consider $f : S \to \mathbb{R}^n$, a Lipschitz map with Lipschitz constant K. Then there exists an extension $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^n$ which is also a Lipschitz map with Lipschitz constant K.*

In particular, $Map_{Lip}(A, SP^\infty(S^m)^+) \to Map_{Lip}(A - O_\Delta(A_\infty), SP^\infty(S^m)^+)$ is surjective by Kirszbraun's Theorem.

We now introduce the topological abelian group of Lipschitz cocycles of codimension m on $A - A_\infty$, a space which will play a central role in the remainder of this paper.

DEFINITION 2.4. Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a (closed) subcomplex with respect to some finite, piece-wise linear triangulation of A. We set

$$\mathcal{Z}^m(A - A_\infty) \ \equiv \ Map^b_{Lip}(A - A_\infty, SP^\infty(S^m)^+),$$

where the right-hand side of the above defining equality is defined to be

$$\ker\{(\coprod_{d \geq 0} \varprojlim_{\Delta'/\Delta} Map_{Lip}(A - O_{\Delta'}(A_\infty), SP^d(S^m)))^+ \to H^0(A - A_\infty)\}.$$

Corollary 1.13 has the following reassuring corollary.

Corollary 2.5. *Let U be provided with open embeddings $U \subset A$, $U \subset A'$. Assume that A, A' admit finite triangulations such that $A - U \subset A$, $A' - U \subset A'$ are subpolyhedra. Then $\mathcal{Z}^m(U)$ determined by the compactification $U \subset A$ is weakly equivalent to the corresponding topological group determined by the compactification $U \subset A'$.*

Moreover, for any finite triangulation Δ of A such that $A_\infty \subset A$ is a subcomplex, there is a natural homomorphism

$$\mathcal{Z}^m(U) \longrightarrow \mathcal{Z}^m(A - O_\Delta(A_\infty))$$

which is a weak equivalence. In particular,

$$\pi_i \mathcal{Z}^m(A - A_\infty) \cong H^{m-i}(A - A_\infty).$$

Remark 2.6. Although $\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_\Delta(A_\infty))$ is surjective for a given triangulation Δ of A, it would appear that $\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - A_\infty)$ is not surjective.

Definition 2.7. Let A be a compact polyhedron equipped with a finite triangulation Δ and let A_∞, $C \subset A$ be (closed) subcomplexes. Set $C_\infty = C \cap A_\infty$. Then we define the space of **Lipschitz cocycles** on A with support in C to be the topological abelian group

$$\mathcal{Z}^m_C(A) \equiv \varprojlim_{\Delta'/\Delta} \ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(C))\}$$

where the inverse limit is taken over all triangulations Δ' refining Δ as above.

More generally, we define

$$\mathcal{Z}^m_{C - C_\infty}(A - A_\infty) \equiv \varprojlim_{\Delta'/\Delta} \ker\{\mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(A - O_\Delta(A_\infty \cup C))\}. \tag{4}$$

Because all the transition maps in the inverse system defining $\mathcal{Z}^m_{C-C_\infty}(A - A_\infty)$ are Serre fibrations and weak equivalences, this inverse limit is weakly equivalent to $\ker\{\mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(A - O_\Delta(A_\infty \cup C))\}$ for any finite, piece-wise linear triangulation Δ of A for which $A_\infty, C \subset A$ are subcomplexes. Thus,

$$\pi_i \mathcal{Z}^m_{C-C_\infty}(A - A_\infty) \cong H^{m-i}_{C-C_\infty}(A - A_\infty),$$

the cohomology of $A - A_\infty$ with supports in $C - C_\infty$.

3. Properties of Cocycle Spaces

In this section, we verify a few of the expected properties of cocycle spaces: multiplicative structure in Proposition 3.1, localization in Proposition 3.2, Mayer-Vietoris in Proposition 3.3, excision in Proposition 3.6, and transfer in Proposition 3.7.

Proposition 3.1. *(Multiplicative structure): Smash product of spheres, $S^m \times S^{m'} \longrightarrow S^{m+m'}$ induces a natural multiplicative structure on the (graded) integral cocycle spaces $\mathcal{Z}^*(A - A_\infty)$, $\mathcal{Z}^*_C(A - A_\infty)$ leading to graded commutative, associative product structures on their homotopy groups:*

$$\mathcal{Z}^m(A - A_\infty) \wedge \mathcal{Z}^{m'}(A - A_\infty) \xrightarrow{\cup} \mathcal{Z}^{m+m'}(A - A_\infty),$$

$$\mathcal{Z}^m(A - A_\infty) \wedge \mathcal{Z}^{m'}_C(A - A_\infty) \xrightarrow{\cup} \mathcal{Z}^{m+m'}_C(A - A_\infty).$$

Proof. First, observe that the smash product $S^m \times S^{m\prime} \longrightarrow S^{m+m'}$ induces Lipschitz maps

$$SP^d(S^m) \times SP^e(S^{m'}) \longrightarrow SP^{de}(S^{m+m'}). \tag{5}$$

Thus, given Lipschitz maps $f : A \to SP^d(S^m),\ g : A \to SP^e(S^{m'})$, we obtain the Lipschitz map $f \wedge g : A \to SP^{de}(S^{m+m'})$. This determines a pairing of monoids

$$\coprod_{d\geq 0} Map_{Lip}(A, SP^d(S^m)) \times \coprod_{e\geq 0} Map_{Lip}(A, SP^e(S^{m'})) \longrightarrow$$
$$\longrightarrow \coprod_{f\geq 0} Map_{Lip}(A, SP^f(S^{m+m\prime})).$$

The pairings (5) induce the usual product structure $K(\mathbb{Z}, m) \times K(\mathbb{Z}, m') \to K(\mathbb{Z}, m+m')$ which in turn induces the cup product in cohomology. Thus, Proposition 2.2 implies that the pairings on homotopy groups of Lipschitz cocycles spaces are graded commutative and associative. $\square$

Proposition 3.2. *(Localization) Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a (closed) subcomplex. Then the natural triple of topological abelian groups*

$$\mathcal{Z}^m_{A_\infty}(A) \ \to\ \mathcal{Z}^m(A) \ \to\ \mathcal{Z}^m(A - A_\infty) \tag{6}$$

is a fibration sequence.

Proof. For each Δ for which A_∞ is a subcomplex of A, the short exact sequence

$$\ker(\mathcal{Z}^m(A) \ \to\ \mathcal{Z}^m(A - O_\Delta(A))) \ \to\ \mathcal{Z}^m(A) \ \to\ \mathcal{Z}^m(A - O_\Delta(A))$$

is a fibration sequence by [**8**]. As argued in the proof of Corollary 1.13, this fibration sequence is weakly homotopy equivalent to (6). $\square$

Proposition 3.3. *(Mayer-Vietoris) Let A be a compact polyhedron equipped with a finite triangulation Δ and let $A_\infty \subset A$ be a closed subcomplex. Let $D \subset A$ be a compact subpolyhedron containing $O_\Delta(A_\infty)$. Then there is a natural short exact sequence of topological abelian groups*

$$\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_\Delta(A_\infty)) \times \mathcal{Z}^m(D) \to \mathcal{Z}^m(D - O_\Delta(A_\infty))$$

which determines the following homotopy Cartesian square

$$\begin{tikzcd} \mathcal{Z}^m(A) \arrow[r] \arrow[d] & \mathcal{Z}^m(D) \arrow[d] \\ \mathcal{Z}^m(A - A_\infty) \arrow[r] & \mathcal{Z}^m(D - A_\infty) \end{tikzcd} . \tag{7}$$

Proof. Observe that for each Δ for which $A_\infty,\ D$ are subcomplexes of A, the short exact sequence

$$\mathcal{Z}^m(A) \ \to\ \mathcal{Z}^m(A - O_\Delta(A_\infty)) \times \mathcal{Z}^m(D) \ \to\ \mathcal{Z}^m(D - O_\Delta(A_\infty))$$

is a fibration sequence by [**8**]. Arguing once again as in the proof of Corollary 1.13 in order to pass to the limit over open tubular neighborhoods of A_∞, we conclude that

$$\mathcal{Z}^m(A) \ \to\ \mathcal{Z}^m(A - A_\infty) \times \mathcal{Z}^m(D) \ \to\ \mathcal{Z}^m(D - A_\infty)$$

is a fibration sequence. This implies that (7) is homotopy Cartesian. $\square$

COROLLARY 3.4. *Let A be a compact polyhedron and let $A_\infty \subset A$ and $B_\infty \subset A$ be closed subcomplexes with respect to some finite triangulation Δ of A. Assume there is a finite subdivision Δ' of Δ such that $O_{\Delta'}(A_\infty) \cap O_{\Delta'}(B_\infty) = \emptyset$. Then there is a natural fibration sequence of topological abelian groups*

$$\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - A_\infty) \times \mathcal{Z}^m(A - B_\infty) \to \mathcal{Z}^m(A - (A_\infty \cup B_\infty))$$

which determines the following homotopy Cartesian square

$$\begin{tikzcd}
\mathcal{Z}^m(A) \arrow[r] \arrow[d] & \mathcal{Z}^m(A - B_\infty) \arrow[d] \\
\mathcal{Z}^m(A - A_\infty) \arrow[r] & \mathcal{Z}^m(A - (A_\infty \cup B_\infty))
\end{tikzcd}.$$

Here the cocycle spaces are taken with respect to the common compactification A, as indicated by the notation.

PROOF. This is an immediate consequence of Theorem 3.3 and Corollary 2.5, once we know the various restriction maps are well-defined. This necessary functoriality property is supplied by the discussion in Remark 1.12. $\square$

COROLLARY 3.5. *Let A be a compact polyhedron and let $A_\infty \subset A$ and $B_\infty \subset A$ be closed subcomplexes with respect to some finite triangulation Δ of A. Then there is a fibration sequence of topological abelian groups*

$$\mathcal{Z}^m_{A_\infty \cap B_\infty}(A) \to \mathcal{Z}^m_{A_\infty}(A) \times \mathcal{Z}^m_{B_\infty}(A) \to \mathcal{Z}^m_{A_\infty \cup B_\infty}(A).$$

PROOF. For any refinement Δ' of Δ, the sequence

$$\begin{tikzcd}
\ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty \cap B_\infty))\} \arrow[d] \\
\ker\{\mathcal{Z}^m(A) \times \mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty)) \times \mathcal{Z}^m(A - O_{\Delta'}(B_\infty))\} \arrow[d] \\
\ker\{\mathcal{Z}^m(A) \to \mathcal{Z}^m(A - O_{\Delta'}(A_\infty \cup B_\infty))\}
\end{tikzcd} \tag{8}$$

is a fibration sequence by the 3×3 lemma and Mayer-Vietoris 3.3 for the closed cover $A - O_{\Delta'}(A_\infty \cap B_\infty) = (A - O_{\Delta'}(A_\infty)) \cup (A - O_{\Delta'}(B_\infty))$. $\square$

PROPOSITION 3.6. *(Excision) Let Δ be a finite triangulation of A and let A_∞, D be closed subpolyhedra such that $O_\Delta(A_\infty) \subset D$. Then the restriction map*

$$\mathcal{Z}^m_{A_\infty}(A) \longrightarrow \mathcal{Z}^m_{A_\infty}(D) \tag{9}$$

is a weak equivalence.

PROOF. This follows immediately from Proposition 3.2 applied to the vertical maps of the homotopy Cartesian square (7). $\square$

PROPOSITION 3.7. *(Transfer) Let A, B be finite polyhedra related by a Lipschitz continuous map $g : A \to SP^e(B)$ with associated ramified covering map $p : B \to A$. Assume that $A_\infty \subset A$ is a nowhere dense closed subpolyhedron with the property that $p : B - B_\infty \to A - A_\infty$ is a covering space map, where $B_\infty = p^{-1}(A_\infty)$. Then p induces a transfer map $p_! : \mathcal{Z}^m(B) \to \mathcal{Z}^m(A)$.*

Moreover, the restriction of $p_!$ *to* $\mathcal{Z}^m(B - B_\infty)$ *has image in* $\mathcal{Z}^m(A - A_\infty)$ *and satisfies*

$$p^* \circ p_! = e(-) : \ \mathcal{Z}^m(A - A_\infty) \to \mathcal{Z}^m(B - B_\infty) \to \mathcal{Z}^m(A - A_\infty),$$

where $e(-)$ *is the* e*-th power map of the topological abelian group* $\mathcal{Z}^m(A - A_\infty)$.

Proof. The map g induces maps $g^{(d)} : SP^d(A) \to SP^{de}(B)$ in the obvious manner, and each of these is Lipschitz. These maps determine a map of abelian monoids

$$\coprod_f Map_{Lip}(B, SP^f(S^m)) \to \coprod_f Map_{Lip}(A, SP^d(S^m)$$

whose group completion is the asserted map $p_! : \mathcal{Z}^m(B) \to \mathcal{Z}^m(A)$.

Choose a triangulation $\tilde{\Delta}$ of B with the property that $p(\tilde{\Delta}) = \Delta$ is a triangulation of A such that with respect to $\tilde{\Delta}$ (respectively, Δ) $B_\infty \subset B$ (resp, $A_\infty \subset A$) is a subpolyhedron. Then the restriction of p to $B - O_{\tilde{\Delta}}(B_\infty)$ is a covering space map to $A - O_\Delta(A_\infty)$. We see by inspection that the composition

$$\coprod_f Map_{Lip}(B - O_{\tilde{\Delta}}(B_\infty), SP^f(S^m)) \to \coprod_f Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m)$$

is multiplication by e. Thus, the second assertion of the proposition follows from Corollary 1.13. □

4. The Graphing Construction Γ^{top}

The purpose of this section is to establish a continuous graphing map

$$\Gamma^{top} : \mathcal{Z}^m(A - A_\infty) \ \longrightarrow \widetilde{\mathcal{Z}_n(A_+ \wedge S^m)} / \varinjlim_{\Delta} \ \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m),$$

where the finite polyhedron A is a compact pseudo-manifold of dimension n. (For notational convenience, we employ the abbreviation

$$\widetilde{Z_r(A)}/Z_r(B) \ \equiv \ \mathcal{Z}_r(A, B)/\mathcal{I}_r(B), \tag{10}$$

the extension of $\ker\{H_{r-1}(B) \to H_{r-1}(A)\}$ by $\mathcal{Z}_r(A)/\mathcal{Z}_r(B)$ given in Theorem 1.2.) Our construction extends that of [**6**] in the case $A_\infty = \emptyset$, and refines the construction there by avoiding the use of the not-everywhere-defined Federer slice construction.

The condition imposed on a compact polyhedron A to be a compact oriented pseudo-manifold of dimension n implies that A has an orientation given by a fundamental class $0 \neq [A] \in H_n(A)$.

We repeat the definition of pseudo-manifold given in [**6**]. Since our definition requires a "resolution of the singularities, all of which are in codimension ≥ 2", this is somewhat stronger than that found elsewhere in the literature.

Definition 4.1. Let A be a compact connected polyhedron. A is said to be a **compact oriented pseudo-manifold** of dimension n if A admits a triangulation Δ satisfying:

- Every simplex of Δ is contained in the closure of some n-simplex $\tau \in \Delta(n)$.
- For some smooth closed oriented n-manifold M equipped with a smooth triangulation, there exists a polyhedral map $p : M \to A$ restricting to a homeomorphism $M - M' \to A - sk_{n-2}A$, where $M' \subset M$ is a subcomplex of dimension $\leq n - 2$.

If A_∞ is a nowhere dense subpolyhedron of the compact oriented pseudo-manifold A of dimension n, then its complement $A - A_\infty$ is said to be a **compactifiable oriented pseudo-manifold** of dimension n.

EXAMPLE 4.2. The underlying analytic space $A = X^{an}$ of any connected complex quasi-projective variety X of complex dimension k is a compactifiable oriented pseudo-manifold of dimension $2k$.

If X is a projective variety of dimension n over $\mathbb{R}$ whose underlying analytic space X^{an} is oriented and connected, then X^{an} is an compact oriented pseudo -manifold provided that X is smooth in codimension 1.

We proceed to construct the graph of a Lipschitz cocycle $f : A - O_\Delta(A_\infty) \to SP^d(S^m)$. As constructed in [**6**, 2.4], the geometric graph of f is the rectifiable current

$$\Gamma(f) \equiv \sum_{\sigma \in \Delta'} \Gamma_\sigma \in \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m), \tag{11}$$

where the sum is indexed by (open) simplices of $A - O_\Delta(A_\infty)$ in a triangulation Δ' refining Δ with the property that $A - O_\Delta(A_\infty)$ is a subcomplex, and Γ_σ is the push-forward of the simplex σ (viewed as a rectifiable current on $A - O_\Delta(A_\infty)$) to the graph. Observe that this construction requires A to be provided with an orientation which is then inherited in a compatible way by each open simplex $\sigma \in \Delta'$.

There are two awkward aspects of this definition: even for A compact, $\Gamma(f)$ might not be a cycle; even if $\Gamma(f)$ is a cycle, the function $f \mapsto \Gamma(f)$ might not be continuous from Lipschitz cocycles to Lipschitz cycles. These difficulties are overcome in [**6**] by restricting attention to the dense subset of "good" Lipschitz cocycles as we now recall.

DEFINITION 4.3. [**6**, 3.2] Let B be a finite polyhedron. Choose a compact neighborhood U of $SP^d(S^m) \subset \mathbb{R}^N$ and a Lipschitz retraction $\pi : U \to SP^d(S^m)$ such that $\pi^{-1}(\Sigma)$ is a subcomplex of codimension ≥ 1, where $\Sigma \subset SP^d(S^m)$ is the singular set. A Lipschitz map $f : B \to SP^d(S^m)$ is said to be **good** if f is of the form $f = \pi \circ \tilde{f}$ where $\tilde{f}$ when restricted to each open simplex of B is smooth and transverse to every open simplex of $\pi^{-1}(\Sigma) \subset U$.

The following lemma is an immediate consequence of [**6**, 3.4] and Kirszbraun's Theorem (Theorem 2.3).

LEMMA 4.4. *Let A be a compact oriented pseudo-manifold of dimension n, Δ a triangulation of A, and $A_\infty \subset A$ a subpolyhedron, and Δ' a refinement of Δ such that $A - O_\Delta(A_\infty)$ is a subcomplex with respect to Δ'. Then the subspace*

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \subset Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$$

of maps $f : A - O_\Delta(A_\infty) \to SP^d(S^m)$ which admit an extension to a good (with respect to Δ') Lipschitz map $\tilde{f} : A \to SP^d(S^m)$ is dense.

Arguing as in [**6**, 3.6], we obtain the following graphing construction. We remind the reader that the smash product $T_+ \wedge S^m$ of a (non-pointed) space T and the pointed m-sphere (with base point chosen to be ∞ when we view S^m as the 1-point compactification of $\mathbb{R}^m$ for $m > 0$ is given by

$$T_+ \wedge S^m \equiv (T \times S^m)/(T \times \{\infty\}).$$

THEOREM 4.5. *Let A be a compact, oriented pseudo-manifold of dimension n, Δ a (piece-wise linear) triangulation of A, and $A_\infty \subset A$ a subpolyhedron. There is a uniquely defined continuous extension*

$$\Gamma^{top} : \mathcal{Z}^m(A - O_\Delta(A_\infty)) \longrightarrow \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m), \tag{12}$$

of the geometric graph construction (11) sending

$$f \in Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good}$$

to the projection of $\Gamma(f) \in \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m)$.

As Δ varies over finer triangulations of A, these maps determine the continuous graphing map

$$\Gamma^{top} : \mathcal{Z}^m(A - A_\infty) \longrightarrow \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\varinjlim_{\Delta} \mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m). \tag{13}$$

PROOF. As constructed in (11), $\Gamma(f)$ is a rectifiable current on $(A - O_\Delta(A_\infty)) \times S^m$ for any $f \in Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good}$. The continuity of this graphing construction

$$\Gamma : Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \longrightarrow \mathcal{R}_n((A - O_\Delta(A_\infty)) \times S^m)$$

is given by [**6**, 3.5].

Let $\tilde{f} : A \to SP^d(S^m)$ be a good Lipschitz map extending f. As verified in [**6**, 3.6], $\Gamma(\tilde{f})$ is an integral n-cycle on $A \times S^m$. Since the restrictions of $\Gamma(f)$ and $\Gamma(\tilde{f})$ agree on any open inside $A - O_\Delta(A_\infty)$, we see that the boundary of $\Gamma(f)$ is supported on $D_\Delta(A_\infty) \times S^m$.

The push-forward of (rectifiable) currents via the proper Lipschitz map $(A - O_\Delta(A_\infty)) \times S^m \to (A - O_\Delta(A_\infty))_+ \wedge S^m$ is that given by Federer in [**2**, 4.1.7]. Thus, for each $d \geq 0$, we obtain continuous graphing maps

$$Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))^{good} \to \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m)$$

sending f to the equivalence class of the push-forward of $\Gamma(\tilde{f})$.

Using Lemma 4.4, we extend this graphing construction to Γ^{top} on $Map_{Lip}(A - O_\Delta(A_\infty), SP^d(S^m))$ exactly as in the proof of [**6**, 3.6], and then use the universal property of group completion to obtain the asserted map (12).

Finally, the graphing map Γ^{top} on $\mathcal{Z}^m(A - A_\infty)$ of (13) is the (inverse) limit indexed by triangulations Δ of these maps. □

REMARK 4.6. Observe that the map in homotopy induced by (13) has the form

$$H^{m-j}(A - A_\infty) \longrightarrow H^{BM}_{n-m+j}(A - A_\infty),$$

where $H^{BM}_*(-)$ denotes Borel-Moore (singular) homology.

As we see in the following proposition, Γ^{top} is compatible with localization. Recall the definition of Lipschitz cocycles $\mathcal{Z}^m_{C-C_\infty}(A - A_\infty)$ given in Definition 4.

PROPOSITION 4.7. *(compatibility with localization) Choose a finite triangulation Δ of A such that the compact subpolyhedra A_∞, $C \subset A$ are subcomplexes such that $D_\Delta(A_\infty) \cap C = \emptyset$. Then Γ^{top} of (12) restricts to a continuous homomorphism on relative Lipschitz cocycles*

$$\Gamma^{top} : \mathcal{Z}^m_C(A - O_\Delta(A_\infty)) \longrightarrow \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m) \tag{14}$$

which fits in a map of fibration sequences

$$\begin{tikzcd}
\mathcal{Z}^m_{C-C_\infty}(A - O_\Delta(A_\infty)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_\infty)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_\infty)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_\infty \coprod C)) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_A(A_\infty \coprod C)_+ \wedge S^m).
\end{tikzcd} \tag{15}$$

Proof. The fact that Γ^{top} restricts to (14) on relative Lipschitz cocycles follows from the observation that if $\tilde{f},\ \tilde{f}' \in Map^{good}_{Lip}(A, SP^d(S^m))$ have equal restrictions to $A - O_\Delta(A_\infty \coprod C)$, then $\Gamma(f) - \Gamma(g)$ has support on $D_\Delta(A_\infty \coprod C) \times S^m$. Thus, the upper square commutes by construction. The naturality of (13) implies the commutativity of the lower square of (15). Both columns are fibration sequences because they are short exact sequences of topological groups (cf. [**8**]). □

Similarly, we see that Γ^{top} is compatible with Mayer-Vietoris.

Proposition 4.8. *(compatibility with Mayer-Vietoris) Let Δ be a finite triangulation of A, and consider closed subpolyhedra $A_1,\ A_2 \subset A$ with $A_1 \cap A_2 = A_{1,2}$ and $A_{1+2} = A_1 \cup A_2$. Then Γ^{top} determines a map of* **Mayer-Vietoris** *fibration sequences*

$$\begin{tikzcd}
\mathcal{Z}^m(A - O_\Delta(A_{1,2})) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(\widetilde{A_+ \times S^m})/\mathcal{Z}_n(D_\Delta(A_{1,2})_+ \wedge S^m) \arrow[d] \\
\times_{i=1,2}\mathcal{Z}^m(A - O_\Delta(A_i)) \arrow[r, "\Gamma^{top}"] \arrow[d] & \times_{i=1,2}\mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_i)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m(A - O_\Delta(A_{1+2})) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(\widetilde{A_+ \wedge S^m})/\mathcal{Z}_n(D_\Delta(A_{1+2})_+ \wedge S^m)
\end{tikzcd}$$

Proof. The vertical columns are short exact sequences of topological groups. The squares commute by construction. □

Corollary 4.9. *Let A be a compact pseudomanifold with finite triangulation Δ, and consider closed subpolyhedra $A_1,\ A_2 \subset A$ with $A_1 \cap A_2 = A_{1,2}$ and $A_{1+2} = A_1 \cup A_2$. Then Γ^{top} determines a map of* **Mayer-Vietoris** *fibration sequences*

$$\begin{tikzcd}
\mathcal{Z}^m_{A_{12}}(A) \arrow[r, "\Gamma^{top}"] \arrow[d] & \mathcal{Z}_n(D_\Delta(A_{12})_+ \wedge S^m) \arrow[d] \\
\times_{i=1,2}\mathcal{Z}^m_{A_i}(A) \arrow[r, "\Gamma^{top}"] \arrow[d] & \times_{i=1,2}\mathcal{Z}_n(D_\Delta(A_i)_+ \wedge S^m) \arrow[d] \\
\mathcal{Z}^m_{A_{1+2}}(A) \arrow[r, "\Gamma^{top}"] & \mathcal{Z}_n(D_\Delta(A_{1+2})_+ \wedge S^m)
\end{tikzcd}$$

Proof. The left column is a fibration sequence by Corollary 3.5. The right column is a short exact sequence of topological groups. The diagram commutes by construction. □

5. Poincaré duality

We will prove a version of Alexander (or, Poincaré-Lefschetz) duality, showing that the graphing construction of Section 4 provides a weak equivalence

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \;\to\; \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

where A is a (compact, or compactifiable) oriented manifold of dimension n and $C \subset A$ is a closed subpolyhedron that is a subcomplex with respect to some finite triangulation of A. As in other proofs of duality, the basic case that needs to be checked by hand occurs when C is a point, or more generally, a (sufficiently small) simplex.

Example 5.1. Let A be a smooth compact oriented manifold of dimension n with a triangulation Δ with the property that for every closed simplex B of Δ, $D_\Delta(B)$ is contained in a Euclidean neighborhood, and let $C \subset A$ be a (closed) simplex. Let $m \geq 0$. Then the graphing map

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \;\to\; \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Indeed, using Proposition 2.2 to identify $\pi_i(\mathcal{Z}^m_C(A))$ with $H^{m-i}(A, A-C)$ and Theorem 1.2 to identify $\pi_i(\mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m))$ with $H_{n+i}(D_\Delta(C) \times S^m, D_\Delta(C) \times \infty))$, we conclude that both groups are zero unless $m = n$ and $i = 0$, when the homomorphism $\Gamma^{top}_\#$ induced by Γ^{top} is of the form $\mathbb{Z} \to \mathbb{Z}$. The generator of the source here is the collapsing map

$$A \to A/(A - O_\Delta(C)) \equiv S^n$$

defining the orientation, minus the constant map to ∞. This is a difference of good Lipschitz maps, and the resulting class in homology is the class given by the difference of the collapsing map and the constant map on the boundary, i.e., it is a generator. This shows that $\Gamma^{top}_\#$ is onto, and hence an isomorphism, as asserted.

Now a bootstrap argument allows to realize Poincaré - Lefschetz duality as the map in homotopy of a map of topological abelian groups.

Theorem 5.2. *Let A be a smooth compact oriented manifold and let $C \subset A$ be a compact subspace that is also a subcomplex for some finite triangulation of A. Then for a sufficiently fine triangulation Δ of A such that C is a subcomplex the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m_C(A) \;\to\; \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Proof. We proceed by induction of the dimension d of C and the number r of simplices of C with respect to Δ. If $d < 0$, then there is nothing to prove. If $r = 1$, then C is a simplex, and we are done by Example 5.1. Now suppose $d \geq 0$, $r > 1$, and we have proved the assertion for all closed subcomplexes C' with at most $r-1$ simplices or of dimension less than d. Let K be a subcomplex of C with $r-1$ simplices, and let B be the closed simplex such that $C = K \cup B$. There are two cases.

First Case: B is disjoint from K. Then the inductive step follows from the obvious fact that $\mathcal{Z}^m_C(A) \equiv \mathcal{Z}^m_K(A) \times \mathcal{Z}^m_B(A)$.
Second Case: B is not disjoint from K. Then $B \cap K = \partial B$. Since the dimension of ∂B is less than d, the inductive hypothesis implies that our assertion holds for the subcomplexes B, K and ∂B. Now the Mayer-Vietoris sequence of Corollary 3.5 and the compatibility of graphing and Mayer-Vietoris given in Corollary 4.9 complete the inductive step. □

Remark 5.3. While it may look as if neither orientability nor smoothness is ever used in the proof of Theorem 5.2, note that the assumptions are needed for the graphing construction to work, as discussed in Section 4.

As corollaries, we obtain Poincaré Duality and Alexander duality.

Corollary 5.4. *Let A be a smooth compact oriented manifold of dimension n. Then the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m(A) \ \to \ \mathcal{Z}_n(A_+ \wedge S^m)$$

is a weak equivalence.

Proof. Choose a sufficiently fine triangulation, and let $C = A$ in Theorem 5.2. □

Corollary 5.5. *Let A be a smooth oriented compact manifold of dimension n, and let $C \subset A$ be a compact subspace that is a subcomplex with respect to some finite triangulation. Then the graphing map*

$$\Gamma^{top} : \mathcal{Z}^m(A - C) \longrightarrow \mathcal{Z}_n(\widetilde{A_+ \wedge S^m}) / \varinjlim_{\Delta} \mathcal{Z}_n(D_\Delta(C)_+ \wedge S^m)$$

is a weak equivalence.

Proof. This follows easily from the preceding results and the compatibility of graphing with localization proved in Proposition 4.7 (taking $A_\infty = \emptyset$ in (15)). □

Definition 5.6. Assume A is a compact, oriented n-manifold and $i : A_\infty \ \subset A$ is a closed, compact, oriented submanifold of codimension e. For any $m \geq e$, we refer to the homotopy class of maps

$$i^! = (\Gamma^{top})^{-1} \circ (i_* \wedge \Sigma^e) \circ \Gamma^{top} : \mathcal{Z}^{m-e}(A_\infty) \to \mathcal{Z}^m(A) \tag{16}$$

as the **Gysin map**.

Remark 5.7. The map on homotopy groups induced by the Gysin map,

$$i^! : H^{m-j-e}(A_\infty) = \pi_j(\mathcal{Z}^{m-e}(A_\infty)) \ \to \ \pi_j(\mathcal{Z}^m(A)) = H^{m-j}(A)$$

(isomorphic to $H_{n+j-m}(A_\infty) \to H_{n+j-m}(A)$), is the Poincaré dual of $i^* : H^\bullet(A) \to H^\bullet(A_\infty)$.

6. Thom Classes and Thom Isomorphism

In this last section we formulate the Thom isomorphism in our context.

PROPOSITION 6.1. *Let A be a compact, oriented pseudo-manifold of dimension n and let $C \subset A$ be a closed submanifold of codimension e, smoothly embedded in the smooth locus of A. Let Δ be a triangulation of A such that C is a subpolyhedron. Then the retraction $p : D_\Delta(C) \to C$ is an oriented disk bundle, called the* **oriented normal disk bundle** *of $C \subset A$. In other words, C admits an open covering $\{U_i\}$ such that each restriction $p_{|U_i} : p^{-1}(U_i) \to U_i$ is homeomorphic to the product projection $D^e \times U_i \to U_i$ and such that $p_{|U_i}, p_{|U_j}$ are related by a continuous maps from $U_i \cap U_j$ to the group of continuous, orientation-preserving, origin-preserving homeomorphisms of the disk D^e.*

Let B be a compact polyhedron and suppose $\mathbf{D} \to B$ is an oriented disk bundle of rank e over B, given a structure of a compact polyhedron such that B is embedded as a subpolyhedron via the zero section and such that the associated sphere bundle $\mathbf{S} \to B$ is also a subpolyhedron.

DEFINITION 6.2. A (geometric) *Thom class* of $\mathbf{D} \to B$ of an oriented disk bundle over a compact polyhedron B is an element $t \in \mathcal{Z}^e(\mathbf{D}, \mathbf{S})$ such that the restriction t_x of t to any fiber $(\mathbf{D}_x, \mathbf{S}_x)$ over a point $x \in B$ defines the given orientation of the disk $\mathbf{D}_x$, that is, t_x is in the same connected component of $\mathcal{Z}^e(\{x\})$ as the difference of the collapsing map $(\mathbf{D}_x, \mathbf{S}_x) \to (S^e, \infty)$ and the constant map to ∞. We write $T(\mathbf{D})$ for the subspace in $\mathcal{Z}^e(\mathbf{D}, \mathbf{S})$ of all geometric Thom classes.

REMARK 6.3. Let B be a contractible polyhedron and $(\mathbf{D}, \mathbf{S}) = B \times (D^e, S^{e-1})$ be the trivial disk and sphere bundles over B, with a choice of orientation. Then there is an obvious geometric Thom class defined as the difference of the projection to (D^e, S^{e-1}) followed by the collapsing map to S^e and the constant map to $\infty \in S^e$. Since $\mathcal{Z}^e(B \times D^e, B \times S^{e-1})$ is homotopy equivalent to $\mathbb{Z}$, the space $T(\mathbf{D})$ is contractible.

THEOREM 6.4. *For any compact polyhedron B and oriented disk bundle $(\mathbf{D}, \mathbf{S})$ as above, the space of geometric Thom classes $T(\mathbf{D})$ is contractible, and in particular, non-empty.*

PROOF. By induction of the dimension of B. It follows immediately from Remark 6.3 that the assertion is true if the dimension of B is zero.

Suppose now that $\dim(B) = n$, and the assertion has been proved for all polyhedra of dimension less than n. Let $B^{(n-1)}$ be the $(n-1)$-skeleton of B, B_n the disjoint union of the closed n-simplices of B, and let ∂B_n be the union of the boundaries. Moreover, let $\mathbf{D}^{(n-1)}$, $\mathbf{D}_n$ and $\partial \mathbf{D}_n$ be the respective restrictions of the oriented disk bundle $\mathbf{D}$, and similarly for the sphere bundle $\mathbf{S}$. We may choose embeddings of $\mathbf{D}$, and of the union $\coprod_n \mathbf{D}_n$ as neighborhood retracts such that all the attaching maps and characteristic maps are Lipschitz continuous. In this way, we obtain a commutative square

$$\begin{tikzcd} T(\mathbf{D}) \arrow[r] \arrow[d] & T(\mathbf{D}^{(n-1)}) \arrow[d] \\ T(\mathbf{D}_n) \arrow[r] & T(\partial \mathbf{D}_n) \end{tikzcd} \tag{17}$$

and the assertion follows from:

Claim: The square (17) is homotopy cartesian.

Indeed, it is clear that this is a cartesian square; therefore it suffices to show that one of the maps, say, $T(\mathbf{D}_n) \to T(\partial \mathbf{D}_n)$, is a (Serre) fibration. Since T obviously transforms finite disjoint unions into finite products, we may assume that there is only one n-simplex. Further, we can choose a Lipschitz continuous trivialization of the disk bundle $\mathbf{D}_n \to B_n$. Hence, we only need to prove that the map $T(\Delta^n \times D^e) \to T(\partial\Delta^n \times D^e)$ induced by restriction is a Serre fibration. Take a commutative diagram

$$\begin{tikzcd} D^k \arrow[r, "f"] \arrow[d] & T(\Delta^n \times D^e) \arrow[d] \\ D^k \times I \arrow[r, "g"] & T(\partial\Delta^n \times D^e) \end{tikzcd} \tag{18}$$

Recall that an element $\phi \in T(\Delta^n \times D^e)$ is simply a difference of two Lipschitz continuous maps ϕ_+ and ϕ_- from $\Delta^n \times D^e$ to some symmetric powers of S^e that become equal on the sphere bundle $\Delta^n \times S^{e-1}$. The maps f and g in the square (18) have compact source and can therefore be taken to correspond to continuous families of Lipschitz maps $G_+ - G_-$ and $F_+ - F_-$ where the $+$-maps have target in some symmetric power $SP^r(S^e)$ and the $-$-maps have target in some symmetric power $SP^s(S^e)$, and such that the restrictions of G_+ and F_+ (respectively, G_- and F_-) to $D^k \times \partial\Delta^n \times D^e$ coincide.

Now we can glue G_+ and F_+ (respectively, G_- and F_-) along $D^k \times \partial\Delta^n \times D^e$ to obtain a continuous family of Lipschitz maps

$$\Phi_+ = G_+ \cup F_+ : D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \to SP^r(S^e)$$

and similarly a continuous family of Lipschitz maps

$$\Phi_- = G_- \cup F_- : D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \to SP^s(S^e)$$

with the property that $\Phi_+ - \Phi_-$ is everywhere locally on $D^k \times I \times \partial\Delta^n \cup D^k \times \Delta^n$ the orientation of the trivial e-disk bundle.

The inclusion (which is the identity on the fibers of the e-disk bundle)

$$D^k \times I \times \partial\Delta^n \times D^e \cup_{D^k \times \partial\Delta^n \times D^e} D^k \times \Delta^n \times D^e \hookrightarrow D^k \times I \times \Delta^n \times D^e$$

is Lipschitz homeomorphic to an inclusion $D^{n+k} \times D^e \hookrightarrow D^{n+k} \times I \times D^e$ that is the identity on the fibers of the (trivial) e-disk bundle. Along the latter inclusion, we can extend Φ_+ and Φ_- constantly along the I-direction to families of Lipschitz maps $\tilde{\Phi}_+$ and $\tilde{\Phi}_-$. Clearly, on each fiber of the e-disk bundle, $\tilde{\Phi}_+ - \tilde{\Phi}_-$ defines the orientation of the disk D^e. That is, we have constructed a lift $\tilde{\Phi} : D^k \times I \to T(\Delta^n \times D^e)$ of the diagram (18), as needed. □

Now that we have defined the Thom classes, we can easily prove the Thom isomorphism theorem.

THEOREM 6.5. *Let B be a compact polyhedron, $\mathbf{D} \to B$ an oriented disk bundle and $t \in T(\mathbf{D})$ a geometric Thom class. Then multiplication by t defines a weak equivalence, independent up to homotopy of the choice of t.*

$$t : \mathcal{Z}^i(B) \to \mathcal{Z}^{e+i}(\mathbf{D}, \mathbf{S})$$

PROOF. In the case that the disk bundle is trivial, multiplication by t is the suspension isomorphism. The general case follows using Mayer-Vietoris and the 5-Lemma. □

The following Thom Isomorphism theorem follows easily from Theorem 6.5.

THEOREM 6.6. *(Thom isomorphism) Let A be a compact polyhedron and $C \subset A$ a closed subpolyhedron of constant codimension $e > 0$ admitting an oriented normal disk bundle in A (as in Proposition 6.1, for example). Then any geometric Thom class $t(C \subset A, \Delta)$ determines a class $\tau_C \in \mathcal{Z}^e_C(A)$. Moreover, multiplication by such a class*

$$\tau_C : \mathcal{Z}^i(C) \to \mathcal{Z}^{e+i}_C(A)$$

is a weak equivalence, independent up to homotopy of the choice of τ_C.

PROOF. The class $t_C = t(C \subset A, \Delta)$ lies in the kernel of $\mathcal{Z}^e(D_\Delta(C)) \to \mathcal{Z}^e(S_\Delta(C))$. Hence, we may extend t_C by 0 on $A - O_\Delta(C)$, obtaining the class τ_C in $\mathcal{Z}^e(A)$ which vanishes off $O_\Delta(C)$; in other words,

$$\tau_C \ \in \ \ker\{\mathcal{Z}^e(A) \to \mathcal{Z}^e(A - O_\Delta(C)\} \ = \ \mathcal{Z}_C(A).$$

We now apply the excision property of Proposition 3.6 to identify the Thom isomorphism of Theorem 6.5 with multiplication by τ_C. □

References

[1] F. Almgren, Jr., *Homotopy groups of the integral cycle groups.* Topology **1** (1962), 257-299.
[2] H. Federer, Geometric Measure Theory, Springer-Verlag, 1969.
[3] E. Friedlander, *Algebraic cocycles on normal, quasi-projective varieties*, Compositio Math 110 (1998) 127-162.
[4] E. Friedlander, *Bloch-Ogus properties for topological cycle theory*, Annales Ec. Norm sup. **33** (2000), 57-65.
[5] E. Friedlander and H.B. Lawson, *Duality relating spaces of algebraic cocycles and cycles*, Topology **36** (1997), 535 - 565.
[6] E. Friedlander and H. B. Lawson, *Graph mappings and Poincaré Duality*, Math. Ann. (2009), 431-461.
[7] E. Friedlander and V. Voevodsky, *Bivariant cycle homology* in Cycles, Transfers, and Motivic Homotopy Theories. Annals of Math Studies, pp. 138-183 (2000).
[8] J.-H. Teh, *Complexification of real cycles and the Lawson suspension theorem.*, J. Lond. Math. Soc. **75** (2007), 463–478.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: ericmf@usc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095
E-mail address: chh@math.ucla.edu

Clay Mathematics Proceedings
Volume 9, 2010

On the motive of a K3 surface

Claudio Pedrini

1. Introduction

The existence of a suitable filtration for the Chow ring of every smooth projective variety over a field k, as conjectured by Bloch and Beilinson has many important consequences both in arithmetic and in geometry. Apart from the trivial case of curves and some other particular cases, this conjecture is still wide open. Jannsen in [J1] has shown that the existence of a Bloch-Beilinson filtration $F^\bullet$ for every smooth projective variety is in turn equivalent to the existence of a Chow-Künneth decomposition as conjectured by Murre (see section 2).

In the case of surfaces a consequence of this Conjecture is *Bloch's Conjecture for surfaces* which asserts for a complex surface X that the action of a correspondence Γ in $X \times X$ on the graded group of 0-cycles of X only depends on the cohomology class of Γ. If X has geometric genus 0 this conjecture implies the converse to Mumford's famous necessary condition for the finite-dimensionality of the Chow group of 0-cycles (see [J1]). It is known for surfaces not of general type, for certain generalized Godeaux surfaces and in a few scattered cases. These conjectures are of motivic nature: in particular for a surface X with $p_g = 0$, Bloch's Conjecture is equivalent to the finite dimensionality of the motive of X (see [G-P]).
The existence of a suitable Chow-Künneth decomposition for the motive of a surface X shows that the information necessary to study the above Conjectures is concentrated in the transcendental part of the motive $t_2(X)$ (see [KMP]).

According to Murre's Conjecture (or equivalently to Bloch-Beilinson's Conjecture) and to Kimura's Conjecture on the finite dimensionality of motives (see [Ki]) the following results should hold for a surface X:

(a) The motive $t_2(X)$ is evenly finite dimensional;

(b) $t_2(X)$ satisfies the Nilpotency Conjecture (see 1.4);

(c) Every homologically trival correspondence in $CH^2(X \times X)_{\mathbb{Q}}$ acts trivially on the Albanese kernel $T(X)$ (see 1.2);

(d) The endomorphism group of $t_2(X)$ (tensored with $\mathbb{Q}$) has finite rank (over a field of characteristic 0).

The motive $t_2(X)$ being a birational invariant vanishes for a rational surface. Also (a) is known to hold for a *Kummer surface* X because $t_2(X)$ is isomorphic to $t_2(A)$, where A is an abelian surface (see [KMP 6.13]). Therefore, by the results in [Ki], also (b) holds for a Kummer surface.

A case where the above properties are still unknown is that of a K3 surface which is not birational to a Kummer surface, in particular over $\mathbb{C}$.

In this paper we analyze the relations between the statements in (a), (b) and (c) and prove some results for K3 surfaces. In particular in Theorem 5.10 we show that, for a complex K3 surface X with a large Picard number ρ, the finite dimensionality of $t_2(X)$ implies the isomorphism of $t_2(X)$ with $t_2(Y)$, where Y is a Kummer surface. On the other hand, if X and Y are complex distinct K3 surfaces, which are general members of smooth projective families $\{X_t\}$ and $\{Y_s\}$ over the disk Δ (hence $\rho(X) = \rho(Y) = 1$), then Murre's Conjecture implies that $Hom(t_2(X), t_2(Y) = 0$ in the category of Chow Motives (Theorem 5.12).
We also relate the Nilpotency Conjecture with Bloch's Conjecture for complex surfaces and extend a result by M. Saito in [MS] on the degree of nilpotency of a homologically trivial correspondence (Theorem 3.6).

In section 2 we recall some known results on the category of Chow Motives and on the different conjectures.
Section 3 contains some properties about the Chow-Künneth decomposition for the motive of a surface and the result on the degree of nilpotency.
In section 4 we show how the transcendental motive of a surface varies in a smooth projective family over a quasiprojective base. This is relevant in the case of K3 surfaces because every complex projective K3 surface can be put in a smooth 1-dimensional family of K3 surfaces.
Section 5 contains the results for complex K3 surfaces.
In section 6 we consider the isomorphisms between the graded Chow ring of a complex variety with the Deligne-Beilinson cohomology ring and collect some results on the regulator map for complex K3 surfaces. We thank B. Kahn, L. Barbieri-Viale, J. Murre, and C. Weibel for useful comments on a earlier draft of this paper. We also thank the Referee who suggested many improvements for the presentation of the paper.

2. Chow Motives, Bloch-Beilinson filtration and Murre's Conjecture

Let X be a smooth projective variety over a field k. We will denote by $A^i(X) = CH^i(X) \otimes \mathbb{Q}$ the group of codimension i cycles on X modulo rational equivalence with $\mathbb{Q}$ coefficients. Let $\mathcal{V}_k$ be the category of smooth projective varieties over k; by $\mathcal{M}_{rat}$ we will denote the (covariant) category of Chow motives over k. Objects in $\mathcal{M}_{rat}$ are triples (X, p, m) where $X = X_d$ is a d-dimensional smooth projective variety, $p \in A^d(X \times X)$ is a projector, i.e.$p^2 = p$, and $m \in \mathbb{Z}$. Morphisms in $\mathcal{M}_{rat}$ are defined as follows:

$$Hom_{\mathcal{M}_{rat}}((X, p, m), (Y, q, n)) = qCorr_{m-n}(X, Y)p$$

where $Corr_i(X, Y) = Corr^{-i}(Y, X)$ and

$$Corr^i(X, Y) = \bigoplus_{\alpha} A^{d_\alpha + i}(X_\alpha \times Y)$$

if $X = \coprod_\alpha X_\alpha$ with X_α equidimensional of dimension d_α. The covariant motive functor $h : \mathcal{V}_k \to \mathcal{M}_{rat}$ is defined as follows: $h(X) = (X, 1_X, 0)$ and $h(f) = [\Gamma_f]$ if $f : X \to Y$. We give ourselves a Weil cohomology theory H^* as defined in [Kl]: we shall denote its field of coefficients by K (by convention it is of characteristic 0). For an element $\alpha \in A^i(X)$ we denote by $cl\alpha$ its image under the cycle map $A^i(X) \to$

$H^{2i}(X)$; $A^i(X)_{hom}$ denotes the kernel of cl, i.e. the subgroup of codimension i cycles which are homologically trivial.
A similar definition of the category of motives can be given by replacing rational equivalence with any other adequate equivalence relation between algebraic cycles. In particular we will consider the (covariant) categories $\mathcal{M}_{hom}$ and $\mathcal{M}_{num}$ of motives modulo homological equivalence and modulo numerical equivalence. By h_{hom} we will denote the functor which associates to every $X \in \mathcal{V}_k$ its motive in $\mathcal{M}_{hom}$. Then one defines a functor $H^i : \mathcal{M}_{hom} \to \mathcal{V}ect_K$ for every $i \in \mathbb{Z}$ by $H^i((X,p,m)) = p^* H^{i-2m}(X)$.
A Weil cohomology theory H^* on the category of smooth projective varieties over k is called *classical* (see [A 3.4]) in the following cases:

(i) $char\ k = 0$ and H^* is algebraic De Rham cohomology or l-adic cohomology for some prime number l or Betti cohomology relative to a complex embedding;

(ii) $char\ k = p > 0$ and H^* is crystalline cohomology or l-adic cohomology for some prime number $l \neq p$.

If $char\ k = 0$ then homological equivalence on algebraic cycles does not depend on the choice of the classical Weil cohomology. Moreover, in any characteristic, the dimension of $H^i(X)$, for $X \in \mathcal{V}_k$ is independent of the choice of the classical Weil cohomology H^*: it is denoted by $b_i(X)$.

In the following we will always consider a classical Weil cohomology theory H^*. Let $X \in \mathcal{V}_k$, $X = X_d$. We say that X has a *Chow-Künneth decomposition* (C-K for short) over k if there exist orthogonal projectors $\pi_i = \pi_i(X) \in A^d(X \times X)$, for $0 \leq i \leq 2d$, such that $cl\pi_i$ is the $(i, 2d-i)$-component of the diagonal Δ_X in $H^{2d}(X \times X)$ and $[\Delta_X] = \sum_{0 \leq i \leq 2d} \pi_i$.

This implies that in $\mathcal{M}_{rat}$ the motive $h(X)$ decomposes as follows:

$$h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X)$$

where $h_i(X) = (X, \pi_i, 0)$.
Next we recall (in our covariant set-up) Murre's Conjecture for a purely d-dimensional, smooth projective variety $X = X_d \in \mathcal{V}_k$.
We will assume that X satisfies the the standard Conjecture $C(X)$ i.e. that the Künneth components of the diagonal Δ_X in $H^{2d}(X \times X)$ are algebraic.

2.1. Murre's Conjecture. [Mu]

(A) X has a Chow -Künneth decomposition $h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X)$.

(B) The correspondences $\pi_i(X)$ act as 0 on $A^j(X)$ for $i < j$ and for $i > 2j$.

(C) Assuming A) and B) we may define a decreasing filtration $F^\bullet$ on $A^j(X)$ as follows:
$F^1 A^j(X) = Ker\ \pi_{2j}$, $F^2 A^j(X) = Ker\ \pi_{2j} \cap Ker\ \pi_{2j-1}, \cdots, F^\nu A^j(X) =$
$= Ker \pi_{2j} \cap Ker \pi_{2j-1}, \cap \cdots \cap Ker \pi_{2j-\nu+1}$.
Then the filtration $F^\bullet$ is independent of the choice of the $\pi_i(X)$

(D) $F^1 A^j(X) = A^j(X)_{hom}$, for all j.

Note that, with the above definitions: $F^{j+1} A^j(X) = 0$.

It easily follows from the definition of $F^\bullet$ (see [Mu 1.4.4]) that:

$$F^1 A^j(X) \subset A^j(X)_{hom}.$$

(A) is known to be true for curves, surfaces, abelian varieties, uniruled 3-folds and Calabi-Yau 3-folds.
If X and Y have a C-K decomposition, with projectors $\pi_i = \pi_i(X)$ and $\pi'_j = \pi_j(Y)$ $0 \le i \le 2d,\ d = dim\ X$, $0 \le j \le 2e, e = dim\ Y$), then $Z = X \times Y$ has also a C-K decomposition with projectors $\Pi_m = \pi_m(Z)$ given by $\Pi_m = \sum_{r+s=m} \pi_r \times \pi'_s$ with $0 \le m \le 2(d+e)$.
If the motive $h(X)$ is finite-dimensional, in the sense of Kimura [Ki], and the Künneth components of the diagonal are algebraic, then it has a C-K decomposition $h(X) = \bigoplus_{0 \le i \le 2d} h_i(X)$ and the motives $h_i(X)$ are unique, up to isomorphism. Moreover the π_i's may be chosen so that $\pi_i^t = \pi_{2d-i}$, see [KMP 6.9].

Up to now it is only known that the Chow motive of a smooth projective variety is finite dimensional if it lies in the tensor subcategory $\mathcal{A} \subset \mathcal{M}_{rat}$ generated by the motives of abelian varieties. A recent result by S. Gorchinskiy and V. Guletskii [G-G] shows that the motive of a non-singular projective 3-fold X over a field k admitting resolution of singularities can be decomposed into sums of tensor powers of Lefschetz motives and the twisted Picard motive of a certain abelian variety iff the subgroup of $A_0(X)$ generated by cycles algebraically equivalent to 0, is representable (in the sense of Mumford). In this case $h(X)$ is finite dimensional and $h(X) \in \mathcal{A}$. This result notably applies to Fano 3-folds. Jannsen [J1 2.1] has shown that the conjectures $(A), \cdots, (D)$ hold for every smooth projective variety X over k iff for every such X there exists a filtration $F^\bullet$ on $A^j(X)$ satisfying the following *Bloch-Beilinson's Conjecture*:

(a) $F^0 A^j(X) = A^j(X)$; $F^1 A^j(X) = (A^j(X))_{hom}$;
(b) $F^\bullet$ is compatible with the intersection product of cycles;
(c) $F^\bullet$ is compatible with f^* and f_* if $f : X \to Y$ is a morphism;
(d) The associated graded group $Gr_F^* A^j(X)$, where

$$Gr_F^\nu A^j(X) = F^\nu A^j(X)/F^{\nu+1} A^j(X),$$

depends only on the motive $h^{2j-\nu}(X)$ of X in $\mathcal{M}_{hom}$;
(e) $F^{j+1} A^j(X) = 0$ for all j.
If such a filtration exists then it is unique.
If we assume that the Künneth components of the diagonal are algebraic, then condition (d) is equivalent to the following (see [J2 4.3]):
(d') Let Y be a smooth projective variety and let $\Gamma \in A^{d+j-i}(X \times Y)$, where $d = dimX$. If the induced map $(\Gamma)_*$ between $H^{2i-\nu}(X)$ and $H^{2j-\nu}(Y)$ is zero then so is the map:

$$Gr_F^\nu \Gamma : Gr_F^\nu A^i(X) \to Gr_F^\nu A^j(Y)$$

In the case of surfaces there is a three-step filtration on $A^2(X)$, with

$$F^0 A^2(X) = A^2(X) \quad F^1 A^2(X) = A^2(X)_0 = Ker[A^2(X) \xrightarrow{deg} \mathbb{Q}] \quad F^2 A^2(X) = T(X)$$

where $T(X)$ is the Albanese Kernel, i.e.$T(X) = Ker[\alpha_X : A^2(X)_0 \to Alb_X(k)]$ and α_X is the Abel-Jacobi map.
Then Bloch made the following (see [B 1.8]):

2.2. Bloch's Conjecture for surfaces. Let X be a smooth projective surface over $\mathbb{C}$ and let $\Gamma \in A^2(X \times X)$. Then the action of Γ on the graded group $A^2(X)$ depends only on the cohomology class $[\Gamma] \in H^4(X \times X, \mathbb{Q})$.

Now let X and Y be smooth projective varieties: the following result in [J1 Prop. 5.8], relates Murre's Conjecture for $Z = X \times Y$, with the mophism groups $Hom_{\mathcal{M}_{rat}}(h_i(X), h_j(Y))$.

PROPOSITION 2.1. *Let X and Y be smooth projective varieties of dimensions respectively d and e, having a C-K decomposition and let $Z = X \times Y$ be provided with the product C-K decomposition.*

(1) If Z satisfies **(B)**, *then:*

$$Hom_{\mathcal{M}_{rat}}(h_i(X), h_j(Y)) = 0 \ \textit{if} \ j < i; \ 0 \le i \le 2d; \ 0 \le j \le 2e;$$

(2) If Z satisfies **(D)** *then:*

$$Hom_{\mathcal{M}_{rat}}(h_i(X), h_i(Y)) \simeq Hom_{\mathcal{M}_{hom}}(h_{i,hom}(X), h_{i,hom}(Y)),$$

where $h_{hom}(X) = \sum_{0 \le i \le 2d} h_{i,hom}(X)$ and $h_{hom}(Y) = \sum_{0 \le j \le 2e} h_{j,hom}(Y)$ in $\mathcal{M}_{hom}$. In particular if $X \times X$ satisfies (D) and the field of coefficients of H^* is $K = \mathbb{Q}$ then (2) implies that the $\mathbb{Q}$-vector space $End_{\mathcal{M}_{rat}}(h_i(X))$, for $0 \le i \le 2d$, has finite dimension, being isomorphic to a sub-vector space of $H^*(X \times X, \mathbb{Q})$.

Note that, because of our covariant definition of the functor $h : \mathcal{V}_k \to \mathcal{M}_{rat}$, in (1) we have $j < i$, while in the contravariant setting (as in [J1 5.8]) one has $i < j$.

REMARK 2.2. In the case dim X =dim Y =2 in [KMP 3.10] it has been proved that (1) in Proposition 2.1 holds unconditionally for all $i < j$ and $0 \le i \le 4$, while (2) holds for all $i \ne 2$. It follows that in Bloch's Conjecture 1.2, the action of a homologically trivial correspondence Γ on the components $\mathbb{Q}$ and $AlbX$ of the Chow group $A_0(X)$ vanishes. Hence the conjecture essentially reduces to the case of the action of Γ on the Albanese kernel $T(X)$.

Another conjecture, related to the previous ones is the following *Nilpotency conjecture* $N(X)$.

2.3. Conjecture N(X). Let $X = X_d$ be a smooth projetive variety over k of dimension d and let $\mathcal{N}(X)$ be the ideal in $A_d(X \times X)$ of correspondences which are numerically equivalent to 0. Then $\mathcal{N}(X)$ is a nilpotent ideal. In particular if $f \in End_{\mathcal{M}_{rat}}(h(X)$ is homologically trivial then f is nilpotent.

If the motive $h(X)$ is finite dimensional then $N(X^m)$ holds for all $m \ge 1$ (see [Ki]). Also Murre's Conjecture for X, $X \times X$ and $X \times X \times X$ implies $N(X)$ by [J1 p. 294]. It follows that Murre's Conjecture for all sufficiently high powers X^m implies the finite dimensionality of $h(X)$ (having assumed $C(X)$).
Kimura and O'Sullivan have conjectured (see [Ki]) that all smooth projective varieties have finite dimensional motives.
The conjecture is known for curves and for some surfaces: rational surfaces, Godeaux surfaces, Kummer surfaces, surfaces with $p_g = 0$ which are not of general type, surfaces isomorphic to a quotient $(C \times D)/G$, where C and D are curves and G is a finite group. It is also known for Fano 3-folds. In all these known cases the motive $h(X)$ lies in the tensor subcategory of $\mathcal{M}_{rat}$ generated by abelian varieties.
By a result of V. Voevodsky [Voe1] (see also [Vois2]) if $\Gamma \in A_d(X \times X)$ is algebraically equivalent to 0 then it is smash nilpotent as an endomorphism in $End_{\mathcal{M}_{rat}}(h(X))$. Therefore Γ is nilpotent (see [Ki 2.16]).

3. The refined Chow-Künneth decomposition for a surface

If X is a smooth projective surface then there exists a *refined Chow-Künneth decomposition* of the motive $h(X)$ with $h_i(X) = (X, \pi_i, 0)$ in $\mathcal{M}_{rat}$, as defined in [KMP 2.3], such that $h_i(X)$ are finite dimensional for $i \neq 2$ and

$$End_{\mathcal{M}_{rat}}(h_i(X)) \simeq End_{\mathcal{M}_{hom}}(h_i^{hom}(X))$$

for $i \neq 2$. Moreover $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$ and $h_2(X) = h_2^{alg}(X) + t_2(X)$, where $t_2(X) = (X, \pi_2^{tr}, 0)$ and $h_2^{alg}(X) \simeq h(\underline{NS_X})(1)$. Here $h(\underline{NS_X})$is the Artin Motive associated to $\underline{NS_X} = NS(X \otimes_k k_s)_{\mathbb{Q}}$ for a separable closure k_s of k. Then: $A^*(h_2^{alg}(X)) = A^1(h_2(X) = NS(X)_{\mathbb{Q}}$ and $H^2(X) = H^2_{alg}(X) \oplus H^2_{tr}(X) = (NS(X) \otimes K) \oplus H^2(t_2(X)$, where K is the field of coefficients of H^* and $H^2_{tr}(X)$ is the transcendental part of the cohomology. We also have the following equalities:

$$H^i(t_2(X)) = 0 \ for \ i \neq 2; \ H^2(t_2(X)) = \pi_2^{tr} H^2(X, \mathbb{Q}) = H^2_{tr}(X, \mathbb{Q}),$$

$$A_i(t_2(X)) = \pi_2^{tr} A_i(X) = 0 \ for \ i \neq 2; \ A_0(t_2(X)) = T(X),$$

where $T(X)$ is the Albanese Kernel of X.
From [KMP 4.3] we get, for X and Y surfaces

$$Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) \simeq \frac{A_2(X \times Y)}{\mathcal{J}(X, Y)}$$

where $\mathcal{J}(X, Y)$ is the subgroup of $A_2(X \times Y)$ generated by the classes of correspondences which are not dominant over X and Y by either the first or the second projection.

The following theorem in [KMP 6.12] explains the relations between the transcendental part $t_2(X)$ of the motive of a surface and the different conjectures for $h(X)$.

THEOREM 3.1. *Let X be a smooth projective surface and let*

$$h(X) = \bigoplus_{0 \leq i \leq 4} h_i(X) = \bigoplus_{0 \leq i \leq 4} (X, \pi_i, 0)$$

be a refined Chow-Künneth decomposition. Let us consider the following conditions:

(1) the motive $h(X)$ is finite-dimensional;

(2) the motive $t_2(S)$ is evenly finite-dimensional;

(3) every endomorphism $f \in End_{\mathcal{M}_{rat}}(h(S))$ which is homologically trival is nilpotent, i.e. the conjecture $N(X)$ holds;

(4) for every correspondence $\Gamma \in A_2(X \times X)_{hom}$, $\alpha_{i,i} = \pi_i \circ \Gamma \circ \pi_i = 0$, for $0 \leq i \leq 4$;

(5) for all i, the map $End_{\mathcal{M}_{rat}}(h_i(X)) \to End_{\mathcal{M}_{hom}}(h_{i,hom}(X))$ is an isomorphism (hence $End_{\mathcal{M}_{rat}}(h_i(X))$ has finite rank in characteristic 0);

(6) the map $End_{\mathcal{M}_{rat}}(t_2(X)) \to End_{\mathcal{M}_{hom}}(t_2^{hom}(X))$ is an isomorphism;

(7) $A_2(X \times X)_{hom} \subset \mathcal{J}(X)$.

Then $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7)$.

Note that, on the base of some conjectures on mixed motives, Beilinson has predicted that for any smooth n-dimensional projective variety X, the $\mathbb{Q}$-algebra $A_n(X \times X)/\mathcal{J}(X, X)$ is semisimple finite dimensional and $A_n(X \times X)_{hom} \subset \mathcal{J}(X, X)$ (see [A 22.3.3]).

In the case of a complex surface X we also have the following result (see [KMP 6.11]).

THEOREM 3.2. *Let X be a smooth projective surface over $\mathbb{C}$. Then the following conditions are equivalent:*
(i) $p_g(X) = 0$ and $h(X)$ is finite dimensional;
(ii) The Albanese Kernel $T(X)$ vanishes;
(iii) $t_2(X) = 0$.

The following conjecture appears in [Vois1]. We will see how it is related with N(X).

3.1. Conjecture V(X). (see [Vois1 Conj. 6]). Let X be a surface: for every $\Gamma \in A^2(X \times X)$

$$Im\ F^2\Gamma_* = Im\ F^2(\Gamma \circ \Gamma^t)_* \subset T(X)$$

where $T(X)$ is the Albanese Kernel, and $F^2\Gamma_*$ denotes the action induced by Γ on $T(X)$

REMARK 3.3. Let X, Y be complex varieties of dimension d and let $\Gamma \in A_d(X \times Y)$. Then, by [Vois1 Lemma 5] there exists a morphism of Hodge structures

$$\psi : H^d_{tr}(X, \mathbb{Q}) \to H^d_{tr}(Y, \mathbb{Q})$$

such that the action of Γ on $H^d_{tr}(X)$ is given by

$$[\Gamma]_* = [\Gamma]_* \circ [\Gamma^t]_* \circ \psi$$

Moreover, if the Hodge conjecture is true, then $\psi = Z_*$, for some $Z \in A_d(X \times Y)$. In particular, if $dim\ X = dim\ Y = 2$ then

$$Im\ [\Gamma \circ \Gamma^t]_* = Im[\Gamma]_* \subset H^2_{tr}(Y)$$

and $Ker[\Gamma^t]_* = Ker[\Gamma \circ \Gamma^t]_*$.

PROPOSITION 3.4. *Let X be a complex surface. Assume Conjectures $N(X)$ and $V(X)$. Then X satisfies Bloch's Conjecture 1.2.*

PROOF. Let $\Gamma \in End_{\mathcal{M}_{rat}}(t_2(X))$ be homologous to 0. By $V(X)$, in order to compute the action of Γ on $T(X)$ we may replace Γ by $\Gamma \circ \Gamma^t$, hence we may assume Γ is self-adjoint in $A_2(X \times X)$. By $N(X)$ there exists n such that $\Gamma^n = 0$ in $A_2(X \times X)$, hence $(F^2\Gamma_*)^n$ is 0 on $T(X)$. Since Γ is self-adjoint, from $V(X)$ it follows [Vois1 Remark 2] that $Ker\ F^2\Gamma_* = Ker F^2(\Gamma \circ \Gamma)_*$ on $T(X)$. Therefore $F^2\Gamma_*$ is 0 on $T(X)$. □

EXAMPLE 3.5. [Vois1] Let X be a smooth projective surface which is fibered over a 1-dimensional smooth basis S. Then every $\Gamma \in A^2(X \times_S X)$ satisfies Conjecture $V(X)$.

By a result of M. Saito in [MS] the vanishing of $T(X)$ for a complex surface X implies that the cube of the ideal:

$$A_2(X \times X)_{hom} = Ker(A_2(X \times X) \to H^4(X \times X))$$

is 0. Note that, due to a result by Bloch and Srinivas in [B-Sr] if the Albanese kernel $T(X)$ vanishes then there exist a divisor D on X and a subvariety $V \subset X$ of dimension ≤ 1 such that

$$\Delta_X = \Gamma_1 + \Gamma_2$$

where Γ_1 is supported on $D \times X$ and Γ_2 is supported on $X \times V$. Therefore, if $T(X) = 0$ every correspondence Z in $A_2(X \times X)$ splits in $Z_1 + Z_2$, where Z_1 is supported on $D \times X$ and Z_2 is supported on $X \times V$.

Next Theorem generalizes M. Saito's result.

THEOREM 3.6. *Let X be a smooth (irreducible) projective surface over a field k and let Ω be a universal domain containing k. Let Z be a homologically trivial correspondence in $A^2(X \times X)$ such that F^2Z acts trivially on $T(X_\Omega)$. Then $Z^3 = 0$.*

PROOF. We have $F^2 Z = \pi_2^{tr} \circ Z \circ \pi_2^{tr}$. Let $k(X) = k(\xi)$ be the field of fractions of X, where ξ is the generic point of X. Choose an embedding $k(X) \subset \Omega$. Then F^2Z acts trivially on $A_0(X_{k(X)})$ so that $(\pi_2^{tr} \circ Z \circ \pi_2^{tr})(\xi) = 0$. By [KMP 4.3] it follows that $Z = Z_1 + Z_2$ where $Z_1, Z_2 \in A_2(X \times X)$, Z_1 is supported on $D \times X$ and Z_2 is supported on $X \times C$ with $dim\ D \leq 1$ and $dim\ C \leq 1$. By eventually considering the normalizations $\tilde{D}$ and $\tilde{C}$ we may also assume that both D and C are smooth.

Let $h(X) = \sum_{0 \leq i \leq 4} h_i(X)$ be a refined Chow-Künneth decomposition as in [KMP 2.3], where $h_i(X) = (X, \pi_i, 0)$ are such that $\pi_0 = [X \times P]$, $\pi_4 = [P \times X]$, and $\pi_i = \pi_{4-i}^t$. Here P is a rational point on X such that $P \notin D$, $P \notin C$. In the case X has no such rational point then one takes $h_0(X) = (X, \pi_0, 0)$, where $\pi_0 = \frac{1}{n}[X \times P]$, P a closed point on X with separable residue field k' of degree n over k (and similarly for π_4). Moreover $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$, $h_2(X) = h_2^{alg}(X) + t_2(X)$ where $h_2^{alg}(X) = (X, \pi_2^{alg}, 0) \simeq h(\underline{NS_X}(1))$, $t_2(X) = (X, \pi_2^{tr}, 0)$ and $h(\underline{NS_X})$ is the Artin motive associated to the Neron-Severi group NS_X. Then $A_0(h_2^{alg}(X)) = 0$ and $A_0(t_2(X)) = T(X)$.
We have $Z = \Delta_X \circ Z \circ \Delta_X = \sum_{i,j=0,\dots,4} \pi_i \circ Z \circ \pi_j$. Since $\pi_i \circ \pi_j = 0$ when $i \neq j$ and $\pi_i \circ \pi_i = \pi_i$ we get

$$Z^3 = \sum_{h,i,j,k=0,\dots,4} \pi_h \circ Z \circ \pi_i \circ Z \circ \pi_j \circ Z \circ \pi_k$$

Since $Z \in J(X) \cap A_2(X \times X)_{hom}$ from [KMP 3.10] and [KMP 4.5] we get

$$\pi_j \circ Z \circ \pi_i = 0$$

for $j \leq i$ and all $i, j = 0, \cdots, 4$. Hence the sum above reduces to the strictly decreasing sequences

$$Z^3 = \pi_4 \circ Z \circ \pi_3 \circ Z \circ \pi_2 \circ Z \circ \pi_1 + \pi_3 \circ Z \circ \pi_2 \circ Z \circ \pi_1 \circ Z \circ \pi_0$$

To prove the claim it is sufficient to show that $Z \circ \pi_0 = \pi_4 \circ Z = 0$. By the choice of π_0 we have $Z_1 \circ \pi_0 = 0$, hence $Z \circ \pi_0 = Z_2 \circ \pi_0$. Similarly $\pi_4 \circ Z = \pi_4 \circ Z_1$ and so, passing to the transpose it suffices to show that $Z_2 \circ \pi_0 = 0$. But

$$Z_2 \in A^1(X \times C) = Hom(h(X), h(C)) = Hom(h(X) \otimes h(C), \mathbb{L})$$

Hence

$$Z_2 \circ \pi_0 \in Hom(1, h(C)) = Hom(h(C), \mathbb{L}) = A^1(C)$$

The cycle $Z \circ \pi_0 = Z_2 \circ \pi_0$ is homologically trivial in $A^1(C)$, therefore it must be 0. Hence $Z^3 = 0$. $\square$

Remarks 3.7. : 1) According to Bloch's Conjecture for surfaces the action of every homologically trivial correspondence on $T(X_\Omega)$ should be trivial, if Ω is a universal domain containing the field k. This conjecture implies the converse of Mumford's Theorem i.e. if $H^2(\bar{X}, \mathbb{Q}_l(1))$ is algebraic for some prime $l \neq char k$ then the Abel-Jacobi map

$$\alpha_\Omega : A^2(X_\Omega)_0 \to Alb_X(\Omega)$$

is an isomorphism (see [J1 1.12]).
2) From the proof of theorem 2.7 it follows that, if $T(X) \neq 0$ and $H^1(X, \mathbb{Q}) \neq 0$, then the correspondence $(\pi_2 \circ Z) \circ (\pi_1 \circ Z)$ may be non zero. Hence $Z^2 \neq 0$. A similar result appears in [MS 2.4], where an example is given of a complex surface X, with $H^1(X) \neq 0$, such that the square of the ideal $A_2(X \times X)_{hom}$ is nonzero.

4. The Chow specialization map

Fix a ground field k and let S be a smooth irreducible quasi-projective variety over k. For simplicity, we will assume that k is algebraically closed of characteristic 0 and $dim\ S = 1$. Let η be the generic point of S, and let 0 be a distinguished closed point on S. By considering the localizations of S at η and at 0 we may restrict to the case where $S = Spec\ R$, R a regular local ring with residue field k and function field $K = k(S)$.

For a scheme X smooth and proper over S there is a specialization map (see [Fu p. 399])

$$\sigma : A_*(X_\eta) \to A_*(X_0)$$

which is a ring homomorphism, i.e. it preserves the intersection product. Here X_η and X_0 are the fibers of X respectively over the generic point and the closed point of S, and A_* is the graded Chow ring of algebraic cycles modulo rational equivalence with $\mathbb{Q}$ coefficients.

If $X \to S$ is a smooth projective morphism, then the specialization map σ on Chow groups is compatible with the specialization isomorphism in cohomology and with the cycle map cl, thus yielding a commutative diagram (see [C-H p. 465])

$$\begin{array}{ccc} A^i(X_\eta) & \xrightarrow{\sigma} & A^i(X_0) \\ {\scriptstyle cl}\downarrow & & \downarrow{\scriptstyle cl} \\ H^{2i}(X_\eta) & \xrightarrow{\simeq} & H^{2i}(X_0) \end{array}$$

We also recall the 'spreading out' argument for a cycle $Z_\eta \in A^i(X_\eta \times X_\eta)$ (see [C-H p. 489]): if Z_η is a cycle on $X_\eta \times X_\eta$, where η is the generic point of S, then one takes a neighborhood $U \subset S$ of η and a cycle Z_U on $X_U \times_U X_U$ such that $Z_U/\eta = Z_\eta$ and let Z on $X \times_S X$ be its Zariski closure. There is a commutative diagram:

$$\begin{array}{ccccccc} A^{i-1}(X_0 \times X_0) & \xrightarrow{i_*} & A^i(X \times_S X) & \xrightarrow{j^*} & A^i(X_\eta \times X_\eta) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i^!} & & & & \\ & & A^i(X_0 \times X_0) & & & & \end{array}$$

where the top row is exact, i and j are the inclusion maps of $X_0 \times X_0$ and $X_\eta \times X_\eta$ in $X \times_S X$ and $i^!$ is the Gysin homomorphism as defined in [Fu 6.2]. For every cycle $Z_\eta \in A^i(X_\eta \times X_\eta)$ the specialization map

$$\sigma : A^i(X_\eta \times X_\eta) \to A^i(X_0 \times X_0)$$

can be defined as follows: $\sigma(Z_\eta) = i^!(Z)$ where $j^*(Z) = Z_\eta$ and the result does not depend on the choice of the spread Z. The ambiguity in spreading a cycle in $A^i(X_\eta \times X_\eta)$ is given by cycles in in $A^{i-1}(X_0 \times X_0)$.

PROPOSITION 4.1. *If $X \to S$ is a smooth projective morphism of relative dimension d and $\{\pi_i\}$, with $0 \le i \le 2d$, is a set of orthogonal idempotents in $A^d(X_\eta \times X_\eta)$ lifting the Künneth components of the diagonal Δ_{X_η}, then $\{\sigma(\pi_i)\}$ have the same property in $A^d(X_0 \times X_0)$.*

PROOF. From the commutativity of j^* and $i^!$ with proper pushforwards, flat pullbacks and intersection products ([Fu 20.3]) it follows that the specialization map σ in (3) is compatible with the composition of correspondences. Using the fact that σ induces isomorphism in cohomology we get our result. $\square$

THEOREM 4.2. *Let $f : \mathcal{X} \to S$ be a smooth, projective family of surfaces over an algebraically closed field k of characteristic 0, with S smooth and dim S =1. Let $X = X_\eta$ be the fiber of f over the generic point η of S and X_0 the fiber over a closed point 0 of S. Let $h(X) = \sum_{0 \le i \le 4} h_i(X)$ be a refined Chow-Künneth decomposition with $h_i(X) = (X, \pi_i(X), 0)$ and let $h(X_0) = \sum_{0 \le i \le 4} h_i(X_0)$ be the corresponding decomposition in $\mathcal{M}_{rat}(k)$, with $h_i(X_0) = (X_0, \sigma(\pi_i(X)), 0)$, induced by the specialization map σ. Let $\pi_2(X) = \pi_2^{alg}(X) + \pi_2^{tr}(X)$: then*

$$\sigma(\pi_2^{tr}(X)) = \pi_2^{tr}(X_0) \oplus \tau(X_0) \quad \pi^{alg}(X_0) = \sigma(\pi_2^{alg}(X)) \oplus \tau(X_0)$$

Here $\tau(X_0) \in A_2(X_0 \times X_0)$ is a projector such that $(X_0, \tau(X_0), 0) \simeq A$ and A is the direct sum of $r = \rho(X_0) - \rho(X)$ copies of the Lefschetz motive $\mathbb{L}$, with $\rho(X_0)$ and $\rho(X)$ the ranks of the Neron- Severi groups respectively of X_0 and X.

PROOF. Let $h_2(X) = h_2^{alg}(X) \oplus t_2(X)$ where $h_2^{alg}(X) = (X, \pi_2^{alg}(X), 0)$ and

$$h_2^{alg}(X) \simeq \mathbb{L}^{\oplus \rho(X)}.$$

Let $[D_i]$ be an orthogonal basis for $NS(X)_{\mathbb{Q}}$ and let $(D_1)_0, \dots, (D_n)_0$ be the Chow specialization of the divisors D_i in $A^1(X_0)$. Since the Chow-specialization is an isomorphism on Weil cohomology, and since the intersection matrix is determined on the cohomological level, we can complete the system $\{(D_1)_0, \dots, (D_n)_0\}$ to a basis in the Neron-Severi group of X_0

$$(D_1)_0, \dots, (D_n)_0 \quad H_{n+1}, \dots, H_m$$

such that $m = \rho(X_0) \ge n$ and

$$h_2^{alg}(X_0) = A \oplus B,$$

where the motive B is defined by the projector

$$\pi_2^{alg}(X_0) = \sum_i \frac{1}{< [(D_0))_i], [(D_0)_i] >} [(D_0)_i] \otimes [(D_0)_i] \in A^2(X_0 \times X_0)$$

and the motive A is defined by the projector

$$\tau(X_0) = \sum_j \frac{[H_j \times H_j]}{< [H_j], [H_j] >}$$

for $n+1 \leq j \leq m$. Then, $\sigma(\pi_2(X)) = \pi_2(X_0)$, by Prop. 3. 1, $\sigma(h_2^{alg}(X)) = B$ and $A \simeq \mathbb{L}^{\oplus r}$ with $r = m - n$. Therefore

$$\pi_2^{alg}(X_0) = \sigma(\pi_2^{alg}(X)) + \tau(X_0);\ \sigma(\pi_2^{tr}(X)) = \tau(X_0) + \pi_2^{tr}(X_0)$$

□

Proposition 4.3. *Let $f : \mathcal{X} \to S$ be a smooth, projective family of surfaces over a field k of characteristic 0, where S is a geometric DVR of equicharacteristic 0 with generic point η and closed point s. If $t_2(X_\eta)$ is finite dimensional then also $t_2(X_s)$ is finite dimensional.*

Proof. Let K be any field and let $DM_{\mathbb{Q}}(K)$ be the triangulated category of motives as defined by V. Voevodsky in [Voev 2]. There is a fully faithful embedding

$$j : \mathcal{M}_{rat}(K) \to DM_{\mathbb{Q}}(K).$$

Let Ψ be the "vanishing cycle functor" defined by J. Ayoub in [Ay 1.5]

$$\Psi : DM^{ct}_{\mathbb{Q}}(k(\eta)) \to DM^{ct}_{\mathbb{Q}}(k(s))$$

where, for a field K, $DM^{ct}_{\mathbb{Q}}(K)$ denotes the subcategory generated (up to twists and taking direct factors) by the images, under the functor j, of the motives $h(X) \in \mathcal{M}_{rat}(K)$. The functor Ψ is triangulated and tensor, hence $\Psi(\wedge^n h(X)) = \wedge^n \Psi(h(X)$, for all $h(X) \in \mathcal{M}_{rat}(k(\eta))$. If $t_2(X_\eta)$ is finite dimensional, then it is evenly finite dimensional (see Theorem 3.1) so that $\wedge^n t_2(X_\eta) = 0$ for some n. Therefore $\wedge^n t_2(X_s) = \Psi(\wedge^n t_2(X_\eta)) = 0$. □

5. Complex K3 surfaces

In this paragraph we prove some results on the motive $t_2(X)$ for a complex K3 surface X. X is a regular surface (i.e. $q(X) = 0$), therefore it has refined Chow-Künneth decomposition of the form $h(X) = \sum_{0 \leq i \leq 4} h_i(X)$ with $h_1(X) = h_3(X) = 0$. Moreover $h_2(X) = h_2^{alg}(X) + t_2(X)$, where $t_2(X) = (X, \pi_2^{tr}, 0)$ and $h_2^{alg}(X) \simeq \mathbb{L}^{\oplus \rho}$. Here ρ is the rank of the $NS(X)_{\mathbb{Q}} = (PicX)_{\mathbb{Q}}$ so that $1 \leq \rho \leq 20$. Moreover

$$H^i(t_2(X)) = 0\ for\ i \neq 2;\ H^2(t_2(X)) = \pi_2^{tr} H^2(X, \mathbb{Q}) = H^2_{tr}(X, \mathbb{Q}),$$

$$A_i(t_2(X)) = \pi_2^{tr} A_i(X) = 0\ for\ i \neq 2;\ A_0(t_2(X)) = T(X),$$

where $T(X)$ is the Albanese Kernel. Since $q(X) = 0$, we also have $T(X) = A_0(X)_0$ (0-cycles of degree 0) and

$$dim H^2(X) = b_2(X) = 22\ \ dim H^2_{tr}(X) = b_2(X) - \rho$$

The family of K3 surfaces with Picard number $\geq \rho$ form a dense countable union of subvarieties of dimension $20 - \rho$ in the family of all K3 surfaces. On a general K3 surface all divisors are linearly equivalent to some rational multiple of the hyperplane class, hence $\rho = 1$.

A K3 surface is called *exceptional* (or singular) if its Picard number $\rho = dim\ Pic(X)_{\mathbb{Q}}$ $= dim\ NS(X)_{\mathbb{Q}} = 20$.
For a Kummer surface, obtained by resolving the singularities of the quotient A/G, with A an abelian surface and $G = \{1, -1\}$, we have $\rho \geq 17$.

Definition 5.1. A *polarized* complex K3 surface X is a K3 surface equipped with an element $\alpha \in Pic\ X$ which is the class of an ample invertible sheaf. A *smooth family* of K3 surfaces parametrized by a scheme S (over $\mathbb{C}$) is a proper smooth morphism $f : \mathcal{X} \to S$ whose fibers are K3 surfaces. A polarization of f is a section $\theta \in Pic_S\ X$ such that for any $s \in S$ it is a polarization of $\mathcal{X}_s = f^{-1}(s)$. Then $R^2 f_* \mathbb{Z}$ is a variation of Hodge structures on $\mathbb{C}$; θ is at every point of type $(1, 1)$ and its orthogonal $P^2 f_* \mathbb{Z}$ is again a variation of Hodge structures (see [D 6.3]).
By a result of Deligne [D 6.4] for every complex polarized K3 surface X there exists a smooth family of polarized K3 surfaces $\{X_t\}_{t\in\Delta}$, where Δ is the unitary disk, such that the central fiber X_0 is isomorphic to X.
A complex K3 surface X will be said to be *general* if there exists a family $\{X_t\}_{t\in\Delta}$ such that $X = X_t$, where t belongs to the complement of a countable union of analytic subvarieties of Δ.

Remark 5.2. In [Ay 1.5.1] it has been conjectured that the functor Ψ, which appears in Prop. 3. 3, is conservative, i.e. $\Psi(A) = 0$ implies $A = 0$. If this is the case then every general K3 surface X will have finite dimensional motive. In fact there exists a smooth projective family of $\mathcal{X} \to S$ such that $X = X_t$ and the special fiber $Y = X_0$ is a Kummer surface. Therefore $t_2(Y)$ is evenly finite dimensional i.e. $\wedge^n t_2(Y) = 0$, for some n. Then $\Psi(\wedge^n(t_2(X)) = \wedge^n t_2(Y) = 0$, hence $\wedge^n t_2(X) = 0$.

We will now show, using results in [P-S] and [K], that for a general K3 surface the image $t_2^{hom}(X)$ of the transcendental motive $t_2(X)$ in the category $\mathcal{M}_{hom}(\mathbb{C})$ of homological motive is absolutely simple. It will follow that, for two distinct general K3 surfaces X and Y

$$Hom_{\mathcal{M}_{hom}}(t_2^{hom}(X), t_2^{hom}(Y)) = 0$$

In Theorem 5.11 we will show that Murre's Conjecture (D) implies

$$Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) = 0.$$

Let's consider the categories $\mathcal{M}_{rat}(\mathbb{C})$ and $\mathcal{M}_{hom}(\mathbb{C})$ and let $HS_{\mathbb{Q}}$ be the Tannakian category of $\mathbb{Q}$-Hodge structures. By $HSGr_{\mathbb{Q}}$ we will denote the rigid tensor category of $\mathbb{Z}$-graded objects of HS_Q (see [A 7.1.2.1]). There is a Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbb{C}) \to HSGr_{\mathbb{Q}}$$

which factorizes trough $\mathcal{M}_{hom}(\mathbb{C})$. Let $\tilde{\mathcal{M}}_{hom}(\mathbb{C})$ be the subcategory generated by the motives of all smooth projective varieties X satisfying the standard conjecture $C(X)$, which amounts to the algebraicity of the Künneth components of the diagonal in $H^*(X \times X)$. $\tilde{\mathcal{M}}_{hom}(\mathbb{C})$ contains all the motives of curves, surfaces and abelian varieties. H_{hodge} induces a functor

$$H_{Hodge} : \tilde{\mathcal{M}}_{hom}(\mathbb{C}) \to HS_{\mathbb{Q}}$$

which is faithful.

Let X be a smooth complex projective surface, and let

$$h(X) = \sum_{0 \le i \le 4} h_i(X)$$

be a refined Chow-Künneth decomposition in $\mathcal{M}_{rat}(\mathbb{C})$, with $h_2(X) = h_2^{alg}(X) \oplus t_2(X)$. Let

$$h_{hom}(X) = \sum_{0 \le i \le 4} h_{i,hom}(X)$$

be the corresponding decomposition in $\mathcal{M}_{hom}(\mathbb{C})$. The maps

$$End_{\mathcal{M}_{rat}}(h_i(X)) \to End_{\mathcal{M}_{hom}}(h_{i,hom}(X)) \subset End_{Vec(\mathbb{Q})}(H^i(X))$$

are isomorphisms for $i \neq 2$. Also $h_2^{alg}(X) \simeq \mathbb{L}^{\oplus \rho}$, where ρ is the rank of the Neron-Severi group, and $End_{\mathcal{M}_{rat}}(\mathbb{L}^{\oplus \rho}) \simeq End_{\mathcal{M}_{hom}}(\mathbb{L}^{\oplus \rho})$. Therefore one has an isomorphism

$$h_{2,hom}(X) = t_2^{hom}(X) \oplus \bigoplus_{\alpha} S_\alpha$$

where $S_\alpha \simeq \mathbb{L}$ and $1 \le \alpha \le \rho(X)$.
Let $f : X \to S$ be a proper and smooth morphism of complex algebraic varieties. The Betti cohomology groups $H^i(X_s)$ of the fibers X_s fit together into a local system which underlines a variation of Hodge structure $\mathbb{V}$ on S such that the Hodge structure at s is just the Hodge structure on $H^i(X_s)$. Therefore for every point $s \in S(\mathbb{C})$ there is a canonical action, i.e. the monodromy of $\pi_1(S(\mathbb{C}), s)$ on $H^*(X_s)$. Then we have the following

Proposition 5.3. *Let S be a smooth complex variety. For a general $s \in S(\mathbb{C})$ a finite index subgroup of the monodromy group is contained in the Mumford-Tate group of the Hodge structure on $\mathbb{V}_s$.*

Proof. See [P-S Prop. 14]. □

From the result above we get (see [K 6.3]):

Theorem 5.4. *Let X be a general complex K3 surface. Then the motive $t_2^{hom}(X)$ is absolutely simple.*

Proof. From [P-S Ex. 5 and Cor. 18] it follows that the Hodge realization of the motive $t_2^{hom}(X)$ is absolutely simple. The functor H_{Hodge} being faithful we get that $t_2^{hom}(X)$ is also absolutely simple.

We recall from [AK 7] that, for an object M in a $\mathbb{Q}$-linear, monoidal, rigid, symmetric, tensor category $\mathcal{C}$ such that $End(1) = \mathbb{Q}$ the trace $tr(f) \in \mathbb{Q}$ of a morphism $f \in End_{\mathcal{C}}(M)$ is defined by the composition

$$1 \xrightarrow{\eta} M^* \otimes M \xrightarrow{1 \otimes f} M^* \otimes M \xrightarrow{R} M \otimes M^* \xrightarrow{\epsilon} 1$$

where ϵ is the evaluation map, η its dual and R is the switch.
If $f = id_M$ then $tr(id_M) = \chi(M)$ is called the the Euler characteristic (or the dimension) of M.

The following example shows that for a Kummer surface the motive $t_2^{hom}(X)$ may be not absolutely simple.

□

Example 5.5. Let A be a simple abelian surface over $\mathbb{C}$ "having many endomorphisms " (or "having complex multplication", see [Mi]) i.e. such that

$$\delta(M)d(M) = 4$$

where $M = EndA \otimes \mathbb{Q}$ and δ, d are defined as follows:

$$\delta(M) = [Z(M) : \mathbb{Q}],\ d(M) = [M : Z(M)]^{1/2}$$

with $Z(M)$ the center of M (see [K 1.1]). Let X be the Kummer surface obtained from A/G, where $G =< 1, -1 >$, by resolving singularities. Then $t_2(X) \simeq t_2(A)$ (KMP 7. 6. 13]), so that also $t_2^{hom}(X) \simeq t_2^{hom}(A)$. According to [K 5.2] one has $t_2^{hom}(X) \otimes \bar{\mathbb{Q}} = \oplus_\alpha S_\alpha$ with S_α simple, so that $\chi(S_\alpha) = 1$ and

$$\sum_\alpha \chi(S_\alpha) = b_2 - \rho(X)$$

with $b_2 = 22$ and $\rho(X) \leq 20$. Hence the number of S_α is greater than 1. Let n_τ be the copies of S_τ appearing in the sum. Then

$$End(t_2^{hom}(X)) \otimes \bar{\mathbb{Q}} \simeq End(\bigoplus_\tau n_\tau S_\tau) = \prod_\tau M_{n_\tau}(\bar{\mathbb{Q}})$$

where M_{n_τ} is the ring of n_τ by n_τ matrices.

Definition 5.6. Let X and Y be two complex K3 surfaces. A *Hodge cycle* $Z \in H^4(X \times Y, \mathbb{Q})$ is the sum of $Z_\nu \in (NS(X) \otimes NS(Y))_{\mathbb{Q}}$ and $Z_\tau \in (T_X \otimes T_Y)$, where T_X is the transcendental lattice $H^2_{tr}(X, \mathbb{Q})$ of X, and similarly for Y. A Hodge cycle is algebraic iff so is Z_τ. A Hodge cycle $Z \in H^4(X \times Y, \mathbb{Q})$ is called a *Hodge isometry* if it induces an isometry

$$Z_* : H^2_{tr}(X, \mathbb{Q}) \to H^2_{tr}(Y, \mathbb{Q})$$

which preserves the Hodge structures of $H^2_{tr}(X) \otimes \mathbb{C}$ and $H^2_{tr}(Y) \otimes \mathbb{C}$.

Proposition 5.7. *Let X and Y be complex K3 surfaces and $\phi : H^2_{tr}(X) \to H^2_{tr}(Y)$ a Hodge isometry. Then there exists an algebraic cycle $\Gamma \in H^4(X \times Y, \mathbb{Q}))$ such that $\phi = \Gamma_*$, where Γ_* denotes the action of Γ on $H^*(X)$. Moreover, if $\rho(X) > 11$ then there exists an isomorphism $f : Y \to X$ such that $f^* = \phi$ on $H_2^{tr}(X)$.*

Proof. See [Muk Prop. 6.2] □

The following result can be viewed as an extension of Prop. 4. 7 to the case of motives.

Proposition 5.8. *Let X and Y be complex K3 surfaces such that there exists an algebraic correspondence $\Gamma \in A_2(X \times Y)$ which induces an isomorphism $\Gamma_* : H^2_{tr}(X) \to H^2_{tr}(Y)$. Assume that $t_2(Y)$ is finite dimensional. Then $t_2(Y)$ is a direct summand of $t_2(X)$.*

Proof. By Remark 2. 4 $Im\ [\Gamma \circ \Gamma^t]_* = Im\ [\Gamma]_*$ and $Ker\ [\Gamma^t]_* = Ker\ [\Gamma \circ \Gamma^t]_* = 0$. Hence

$$[\Gamma \circ \Gamma^t]_* : H^2_{tr}(Y) \to H^2_{tr}(Y)$$

is an isomorphism because $H^2_{tr}(X)$ and $H^2_{tr}(Y)$ are $\mathbb{Q}$-vector spaces of the same dimension. The correspondence $Z = \Gamma \circ \Gamma^t$ yields a map between the transcendental motives

$$\bar{Z} : t_2(Y) \to t_2(Y)$$

such that $\bar{Z}_{hom} : t_2^{hom}(Y) \to t_2^{hom}(Y)$ is an isomorphism in $\mathcal{M}_{hom}(\mathbb{C})$. Let $\mathcal{A}$ be the full subcategory of $\mathcal{M}_{rat}(\mathbb{C})$ generated by evenly finite dimensional motives. Then $t_2(Y) \in \mathcal{A}$ and by [A-K 8.2.4] the functor $\Phi : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ is conservative, i.e. it preserves isomorphisms. Here $\mathcal{N}$ is the largest monoidal ideal in $\mathcal{A}$ distinct from $\mathcal{A}$. The ideal $\mathcal{N}$ corresponds to numerical equivalence, hence $\Phi(\bar{Z})$ is an isomorphism in $\mathcal{A}/\mathcal{N}$. Therefore $\bar{Z}$ is an isomorphism in $\mathcal{A}$. Let Z' be such that $Z' \circ \bar{Z} = id_{t_2(Y)}$. Then $(Z' \circ \Gamma) \circ \Gamma^t = id_{t_2(Y)}$ so that the map

$$\Gamma^t : t_2(Y) \to t_2(X)$$

is injective. Therefore $t_2(Y)$ is a direct summand of $t_2(X)$. □

DEFINITION 5.9. A *Nikulin involution* i on a K3 surface X is an involution such that $i^*(\omega) = \omega$ for all $\omega \in H^{2,0}(X)$.

THEOREM 5.10. *Let X be a complex K3 surface and suppose that either $\rho(X) = 19, 20$ or that $\rho(X) = 17, 18$ and there exists an embedding*

$$T_X \otimes \mathbb{Q} \subset (U^3 \otimes \mathbb{Q})$$

where U is a free $\mathbb{Z}$-module of rank 2, whose bilinear form has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then there exists a Kummer surface Y such that $t_2(Y)$ is direct summand of $t_2(X)$. If $t_2(X)$ is finite dimensional then $t_2(X) \simeq t_2(Y)$.

PROOF. By a result in [Mor pag. 121], which extends, using [Muk], a previous one by [S-I] for $\rho = 20$, there exist an abelian surface A and an algebraic correspondence between X and A which induces a Hodge isometry between the transcendental lattices T_X and T_A of X and A. Let Y be the Kummer surface associated to A. Then there exists is a Nikulin involution i on X such that $X/ < i >$ is birational to the Kummer surface Y. As in [Mor §3], by blowing up the 8 fixed points on X of the involution i, we have a diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ \tilde{Y} & \longrightarrow & X/<i> \end{array}$$

where $\tilde{Y} \simeq \tilde{X}/G$, G is the group generated by i and $h(\tilde{Y})$ is finite dimensional, because $\tilde{Y}$ is birational to the Kummer surface Y. Moreover $t_2(\tilde{X}) = t_2(X)$ because $t_2(-)$ is a birational invariant for surfaces. Also

$$H^2_{tr}(X) \simeq H^2_{tr}(\tilde{X}) \simeq H^2_{tr}(A) \simeq H^2_{tr}(\tilde{Y}).$$

Let $p = 1/2(\Gamma^t_g \circ \Gamma_g) \in A^2(\tilde{X} \times \tilde{X})$: then p is a projector and

$$h(\tilde{X}) = h((\tilde{X}, p)) \oplus h((\tilde{X}, \Delta_{\tilde{X}} - p)) \simeq h(\tilde{Y}) \oplus N$$

where $N = h((\tilde{X}, \Delta_{\tilde{X}} - p)$. Hence $t_2(\tilde{Y})$ is a direct summand of $t_2(\tilde{X})$.

If $t_2(X)$ is finite dimensional so it is $h(\tilde{X})$ and we are left to show that $N = 0$. The map g induces an isomorphism on the cohomology groups $H^*(h(\tilde{X}))$ and $H^*(h(\tilde{Y}))$, because $b_2(\tilde{X}) = b_2(\tilde{Y}) = 22$ and $\rho(\tilde{X}) = \rho(\tilde{Y})$. Therefore the correspondence $\Delta_{\tilde{X}} - p$ is homologically trivial. From [Ki] it follows that the projector $\Delta_{\tilde{X}} - p$ is nilpotent, hence it is 0. Therefore we get $N = 0$. $\square$

REMARK 5.11. The following result by Nikulin in [N] has extended that of Mukai in Prop. 4. 7. The lattice of algebraic cycles S_X of a K3 surface X is said to represent 0 if there exists an element $x \in S_X$ such that $x \neq 0$ and $x^2 = 0$. If $\phi : T_X \otimes Q \simeq T_Y \otimes \mathbb{Q}$ is an isomorphism of the lattices of transcendental cycles of two K3 surfaces X and Y preserving the Hodge structure, then the cycle

$$\Gamma_\phi \in (T_X \otimes T_Y) \otimes \mathbb{Q} \subset H^4(X \times Y, \mathbb{Q})$$

is algebraic iff S_X represents 0, i.e. the surface X has a pencil of elliptic curves (in particular this is so if $rk\ S_X \geq 5$).

THEOREM 5.12. *Let X and Y be distinct complex K3 surfaces which are general members of smooth projective families $\mathcal{X} = \{X_s\}_{s \in S}$ and $\mathcal{Y} = \{Y_t\}_{t \in T}$, where S and T are smooth quasiprojective varieties. Assume Murre's Conjecture (D) for $X \times Y$. Then*

$$Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) = 0$$

Consequently there exists no Hodge isometry between T_X and T_Y.

PROOF. By Proposition 2.1 the functor $h : \mathcal{M}_{rat}(\mathbb{C}) \to \mathcal{M}_{hom}(\mathbb{C})$ induces an isomorphism

$$Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) \simeq Hom_{\mathcal{M}_{hom}}(t_2^{hom}(X), t_2^{hom}(Y))$$

The variations of Hodge structures $R^2 f_* \mathbb{Q}$ and $R^2 g_* \mathbb{Q}$ corresponding to the families $f : \mathcal{X} \to S$ and $g : \mathcal{Y} \to T$ are distinct and simple. Since the Hodge realizations H_{Hodge} is faithful it follows that

$$Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) = 0.$$

Assume that there exists an isometry ϕ between the transcendental lattices T_X and T_Y. Then, by Prop 4. 7, ϕ is induced by an algebraic correspondence $\Gamma \in A^2(X \times Y)$. The associated map

$$F^2\Gamma = \pi_2^{tr}(Y) \circ \Gamma \circ \pi_2^{tr}(X) \in Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y))$$

is 0. The map

$$\phi = \Gamma_* : T_X = H^2_{tr}(X) \to T_Y = H^2_{tr}(Y)$$

also vanishes because $\pi^2_{tr}(X)$ and $\pi^2_{tr}(Y)$ act as the identity on H^2_{tr}. Therefore we get a contradiction. $\square$

REMARK 5.13. In [C-L 2 3.2] it is proved that the existence of a Bloch-Beilinson filtration for $X \times Y$, where X, Y are general K3 surfaces, implies

$$A^2(X \times Y) = \sum_{0 \leq i \leq 2} A^i(X) \otimes A^{2-i}(Y)$$

Therefore $A^2(X \times Y) = \mathcal{J}(X, Y)$, and this, by (1), implies $Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(Y)) = 0$.

6. Deligne-Beilinson's Cohomology and the regulator map

Let $A \subset \mathbb{C}$ be a subring, which for us will be $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$: for every smooth algebraic manifold Y over $\mathbb{C}$, of dimension n, we will denote by $H^i_{\mathcal{D}}(Y, A(r))$ the Deligne-Beilinson cohomology groups as defined in [E-V], where $A(r) = (2\pi i)^r A \subset \mathbb{C}$. For $A = \mathbb{Q}$ and $i = 2k-1$ and Y smooth and projective there is an isomorphism

$$H^{2k-1}_{\mathcal{D}}(Y, \mathbb{Q}(k)) \simeq \frac{H^{2k-2}(Y, \mathbb{C})}{F^k H^{2k-2}(Y, \mathbb{C}) + H^{2k-2}(Y, \mathbb{Q}(k))}.$$

If $A = \mathbb{R}$ and we set

$$\pi_{k-1} : \mathbb{C} = \mathbb{R}(k) \oplus \mathbb{R}(k-1) \to \mathbb{R}(k-1)$$

then

$$\begin{aligned} H^{2k-1}_{\mathcal{D}}(Y, \mathbb{R}(k)) &\simeq \frac{H^{2k-2}(Y, \mathbb{C})}{F^k H^{2k-2}(Y, \mathbb{C}) + H^{2k-2}(Y, \mathbb{R}(k))} \\ &\xrightarrow[\pi_{k-1}]{\simeq} H^{k-1,k-1}(Y, \mathbb{R}) \otimes \mathbb{R}(1) = H^{k-1,k-1}(Y, \mathbb{R}(1)) \end{aligned}$$

where $H^{k-1,k-1}(Y, \mathbb{R}(1)) \simeq H^{n-k+1,n-k+1}(Y, \mathbb{R}(n-k+1))^{\vee}$(see [C-L1]).
Let $\mathcal{H}^p_{\mathcal{D}}(A(r))$ be the Zariski sheaves associated to Deligne-Beilinson cohomology, i.e. the sheaves on Y_{Zar} associated to the presheaves

$$U \to H^p_{\mathcal{D}}(U, A(r)).$$

Similarly we will denote by $\mathcal{H}^p(A(r))$ the Zariski sheaves associated to the singular cohomology groups with A coefficients. The sheaf $\mathcal{H}^p_{\mathcal{D}}(A(r))$ has a Gersten resolution

$$0 \to \mathcal{H}^p_{\mathcal{D}}(A(r)) \to \coprod_{y \in Y^0} (i_y)_* H^p_{\mathcal{D}}(\mathbb{C}(y), A(r)) \to \coprod_{y \in Y^1} (i_y)_* H^{p-1}_{\mathcal{D}}(\mathbb{C}(y), A(r-1)) \to \cdots$$

where Y^i is the set of points of codimension i in Y and

$$H^i_{\mathcal{D}}(\mathbb{C}(y), A(r)) = \lim_{U \subset \overline{\{y\}}} H^i_{\mathcal{D}}(U, A(r)).$$

There is a coniveau spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{H}^q_{\mathcal{D}}(A(r))) \Longrightarrow H^{p+q}_{\mathcal{D}}(Y, A(r))$$

which gives an exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^1(Y, \mathcal{H}^2_{\mathcal{D}}(A(2))) &\xrightarrow{i} H^3_{\mathcal{D}}(Y, A(2))) \xrightarrow{\rho} H^0(Y, \mathcal{H}^3_{\mathcal{D}}(A(2))) \\ &\xrightarrow{\delta} H^2(Y, \mathcal{H}^2_{\mathcal{D}}(A(2))) \to H^4_{\mathcal{D}}(Y, A(4)) \end{aligned}$$

For every U there is a natural map

$$\rho_U : \mathcal{O}(U)^*_{alg} \to H^1_{\mathcal{D}}(U, A(1))$$

which for $A = \mathbb{Z}$ is an isomorphism by [E-V 2.12]. From the Gersten resolution of the sheaf $\mathcal{H}^p_{\mathcal{D}}(\mathbb{Q}(r))$ one gets a Bloch-Quillen isomorphism $A^p(Y) \simeq H^p(Y, \mathcal{H}^p_{\mathcal{D}}(\mathbb{Q}(p)))$ (see [Gi]). This isomorphism yields an isomorphism of graded rings

$$\eta : \oplus A^p(Y) \simeq \oplus H^p(Y, \mathcal{H}^p_{\mathcal{D}}(\mathbb{Q}(p)))$$

where the intersection product of algebraic cycles corresponds to the cup product in Deligne-Beilinson cohomology.

There are Chern class maps:

$$c_i : \mathcal{K}_p \otimes \mathbb{Q} \to \mathcal{H}^{2i-p}_{\mathcal{D}}(\mathbb{Q}(i))$$

from the Zariski sheaf associated to K-theory to the Deligne-Beilinson cohomology sheaf. The kernel and cokernel of the Chern class c_2 are constant sheaves (see [Ped]), hence c_2 induces a surjective map $H^1(c_2) : H^1(Y, \mathcal{K}_2) \otimes Q \to H^1(Y, \mathcal{H}^2_{\mathcal{D}}(\mathbb{Q}(2))$. From the exact sequence in (4) we get for $A = \mathbb{Q}$ an exact sequence (see [BV 2])

$$0 \to H^1(Y, \mathcal{H}^2_{\mathcal{D}}(\mathbb{Q}(2))) \xrightarrow{i} H^3_{\mathcal{D}}(Y, \mathbb{Q}(2)) \xrightarrow{\rho} H^0(Y, \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2))) \xrightarrow{\delta} H^2(Y, \mathcal{H}^2_{\mathcal{D}}(2)))$$

By composing the map $H^1(c_2)$ with the inclusion $H^1(Y, \mathcal{H}^2_{\mathcal{D}}(\mathbb{Q}(2))) \subset H^3_{\mathcal{D}}(Y, \mathbb{Q}(2))$ one gets a map $H^1(Y, \mathcal{K}_2) \otimes \mathbb{Q} \to H^3_{\mathcal{D}}(Y, \mathbb{Q}(2)))$. The *regulator map*

$$c_{2,1} : CH^2(Y, 1)_{\mathbb{Q}} \to H^3_{\mathcal{D}}(Y, \mathbb{Q}(2))$$

is then defined using the isomorphism: $H^1(Y, \mathcal{K}_2) \simeq CH^2(Y, 1)$ where $CH^k(Y, m)$ are Bloch's Higher Chow groups. The map $c_{2,1}$ can be explicitly described as follows. A class ξ in $H^1(Y, \mathcal{K}_2) \simeq CH^2(Y, 1)$ is represented by an element of the form $\xi = \sum_i (f_i, Z_i)$ where $f_i \in \mathbb{C}(Z_i)^*$, *codim* $Z_i = 1$ and $\sum_i div\ f_i = 0$. Choose a branch of the log function on $\mathbb{C} - [0, \infty)$ and put $\gamma_i = f_i^{-1}([0, \infty))$, $\gamma = \sum \gamma_i$. Then $\gamma = \delta\zeta$ is a boundary. For $\omega \in F^1H^2(Y, \mathbb{C}) \subset H^2(Y, \mathbb{C})$ one defines

$$c_{2,1}(\xi)(\omega) = \frac{1}{(2\pi i)^{n-1}}\Big(\sum_i \int_{Z_i - \gamma_i} \omega log f_i + 2\pi i \int_\zeta \omega\Big)$$

The definition above gives a unique element in the quotient group

$$H^3_{\mathcal{D}}(Y, \mathbb{Q}(2)) \simeq \frac{H^2(Y, \mathbb{C})}{F^2H^2(Y, \mathbb{C}) + H^2(Y, \mathbb{Q}(2))}$$

For any form $\omega \in H^{n-1,n-1}(Y, \mathbb{R}(n-1))^\vee$ the real part of $c_{2,1}(\xi)(\omega)$ is given by

$$\frac{1}{(2\pi i)^{n-1}}\Big(\sum_i \int_{Z_i} \omega log|f_i|\Big)$$

(see [C-L 1 p. 220]). The real regulator map

$$r_{2,1} : CH^2(Y, 1) \to H^3_{\mathcal{D}}(Y, \mathbb{R}(2)) \simeq H^{1,1}(Y, \mathbb{R}(1)) \simeq H^{n-1,n-1}(Y, \mathbb{R}(n-1))^\vee$$

is defined by

$$\xi = \sum_i (f_i, Z_i) \to r_{2,1}(\xi)(\omega) = \frac{1}{(2\pi i)^{n-1}}\Big(\sum_i \int_{Z_i} \omega log|f_i|\Big)$$

The following result has been proved by X. Chen and J. Lewis. Recall that, for a variety Y, a real analytic Zariski open set U in Y is the complement of a real analytic subvariety of Y. If $\{X_t\}_{t \in Y}$ is a family of projective algebraic manifolds then $X = X_t$ is said to be general if $t \in U$ for some real analytic Zariski open set U.

Proposition 6.1. *Let X be a smooth projective K3 surface over $\mathbb{C}$ which is general. Then the real regulator $r_{2,1} \otimes \mathbb{R}$ is surjective.*

Proof. See [C-L1 1.1]. □

Corollary 6.2. *Let X be a K3 surface as in Proposition 6.1. Then the exact sequence in (4) gives an isomorphism*

$$H^0(X, \mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))) = Ker\ [H^2(X, \mathcal{H}^2_{\mathcal{D}}(\mathbb{R}(2)) \xrightarrow{cl} H^4_{\mathcal{D}}(X, \mathbb{R}(2))]$$

PROOF. By Proposition 6.1 the real regulator $r_{2,1}$ is surjective, hence the group

$$H^3_{\mathcal{D}}(X,\mathbb{R}(2)) \simeq H^{1,1}(X,\mathbb{R}(1)) \simeq (H^{1,1}(X,\mathbb{R}(1)))^{\vee}$$

is generated by elements of the form

$$\frac{1}{(2\pi i)^{n-1}}(\sum_i \int_{Z_i} \omega log|f_i|)$$

with codimension Z_i =1. Therefore, for every $\alpha \in H^3_{\mathcal{D}}(X,\mathbb{R}(2))$, there is Zariski open subset $U \subset X$ such that α vanishes on U. The sheaf $\mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))$ has the following resolution

$$0 \to \mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2)) \to i_*(H^3_{\mathcal{D}}(\mathbb{C}(X),\mathbb{R}(2)) \to \coprod_{x \in X^1} H^2_{\mathcal{D}}(\mathbb{C}(x),\mathbb{R}(1)) \to \cdots$$

where

$$H^2_{\mathcal{D}}(\mathbb{C}(x),\mathbb{R}(1)) = \lim_{U \subset \overline{x}} H^2_{\mathcal{D}}(U,\mathbb{R}(1)) = lim_{U \subset \overline{x}} \frac{H^1(U,\mathbb{C})}{F^1H^1 + H^1(U,\mathbb{R}(1))} = 0$$

for $x \in X^1$. The second equality comes from the vanishing of $H^2(U,\mathbb{R}(1))$ for U an affine curve. The vanishing

$$lim_{U \subset \overline{x}} \frac{H^1(U,\mathbb{C})}{F^1H^1 + H^1(U,\mathbb{Q}(1))} = 0$$

has been proved in[BV-S]. Therefore the resolution of the sheaf $\mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))$ gives an isomorphism

$$H^0(X,\mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))) \simeq H^3_{\mathcal{D}}(\mathbb{C}(X),\mathbb{R}(2)) = lim_{U \subset X} H^3_{\mathcal{D}}(U,\mathbb{R}(2)) \simeq$$
$$\simeq lim_{U \subset X} \frac{H^2(U,\mathbb{C})}{F^2H^2(U,\mathbb{C}) + H^2(U,\mathbb{R}(2))}$$

As we have seen for every $\alpha \in H^3_{\mathcal{D}}(X,\mathbb{R}(2))$ there exists an open $U \subset X$ such that α vanishes in $H^3_{\mathcal{D}}(U,\mathbb{R}(2))$. It follows that the map ρ in the exact sequence coming from (4)

$$H^3_{\mathcal{D}}(X,\mathbb{R}(2)) \xrightarrow{\rho} H^0(X,\mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))) \xrightarrow{\delta} H^2(X,\mathcal{H}^2_{\mathcal{D}}(\mathbb{R}(2))) \xrightarrow{cl} H^4_{\mathcal{D}}(X,\mathbb{R}(2))$$

is 0, so that

$$H^0(X,\mathcal{H}^3_{\mathcal{D}}(\mathbb{R}(2))) = Ker\ [H^2(X,\mathcal{H}^2_{\mathcal{D}}(\mathbb{R}(2))) \xrightarrow{cl} H^4_{\mathcal{D}}(X,\mathbb{R}(2))].$$

□

The following result, which appears in [Ros 6.1], shows that for a smooth projective surface X over $\mathbb{C}$ the map

$$H^0(X,\mathcal{H}^3_{\mathcal{D}}(X,\mathbb{Q}(2)) \to T(X) \to 0$$

is an isomorphism iff $p_g(X) = 0$. Here

$$T(X) = Ker\ [H^2(X,\mathcal{H}^2_{\mathcal{D}}(\mathbb{Q}(2))) \xrightarrow{cl} H^4_{\mathcal{D}}(X,\mathbb{Q}(2))]$$

is the Albanese Kernel. Therefore for a K3 surface the regulator

$$c_{2,1} : CH^2(X,1)_{\mathbb{Q}} \to H^3_{\mathcal{D}}(X,\mathbb{Q}(2))$$

is not surjective because the map ρ in (6) is not 0.

THEOREM 6.3. *Let X be a smooth projective surface over $\mathbb{C}$. Then there is an exact sequence*

$$0 \to \frac{H^2(X,\mathcal{O}_X)}{H^2_{tr}(X,\mathbb{Q})+G} \to H^0(X,\mathcal{H}^3_{\mathcal{D}}(X,\mathbb{Q}(2)) \to T(X) \to 0$$

The group G is at most countable.

PROOF. By [Ros 6.1] there is an exact sequence

$$0 \to H^0(X,\mathcal{H}^2/\mathcal{F}^{2,2}_{\mathbb{Q}}) \to H^0(X,\mathcal{H}^2(\mathbb{C})/\mathcal{F}^2\mathcal{H}^2) \xrightarrow{H^0(k^{22})} H^0(X,\mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2))) \to T(X) \to 0$$

where $\mathcal{F}^{2,2}_{\mathbb{Q}} = Image\ [\mathcal{H}^2_{\mathcal{D}}(\mathbb{Q}(2)) \to \mathcal{H}^2(\mathbb{Q}(2))]$ is the discrete part of the Deligne-Beilinson cohomology sheaf. k^{22} is the map in the following exact sequence of sheaves

$$0 \to \mathcal{H}^2(\mathbb{Q})/\mathcal{F}^{2,2}_{\mathbb{Q}} \to \mathcal{H}^2(\mathbb{C})/\mathcal{F}^2\mathcal{H}^2 \xrightarrow{k^{22}} \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2)) \to 0$$

and $\mathcal{F}^i\mathcal{H}^j$ are the Zariski sheaves associated to the presheaves $U \to F^iH^j(U,\mathbb{C})$.

We also have

$$H^0(X,\mathcal{H}^2/\mathcal{F}^{2,2}_{\mathbb{Q}}) \simeq H^1(X,\mathcal{F}^{2,2}_{\mathbb{Q}}) \oplus H^0(X,\mathcal{H}^2(\mathbb{Q}))$$

where $H^0(X,\mathcal{H}^2(\mathbb{Q})) \simeq H^2_{tr}(X,\mathbb{Q})$ and the group $G = H^1(X,\mathcal{F}^{2,2}_{\mathbb{Q}})$ is at most countable (see [Ped 1.7]). Finally, from [BV 2 §1], there is an isomorphism

$$H^0(X,\mathcal{H}^2/\mathcal{F}^2\mathcal{H}^2) \simeq H^2(X,\mathcal{O}_X).$$

□

REMARKS 6.4. 1) For every smooth projective variety X over $\mathbb{C}$ there are exact sequences as in [Ped Th. 1.7]

$$0 \to NS(X) \otimes \mathbb{C}^* \to R(X) \to H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}}) \to 0$$

$$0 \to H^0(X,\mathcal{H}^2(\mathbb{Q}/\mathbb{Z})) \to H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}}) \to H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}}) \otimes \mathbb{Q} \to 0$$

where $R(X)$ is the image of the regulator map $c_{2,1} : H^1(X,\mathcal{K}_2) \to H^3_{\mathcal{D}}(X,\mathbb{Z}(2))$, $\mathcal{F}^{2,2}_{\mathbb{Z}}$ is the image of $\mathcal{H}^2_{\mathcal{D}}\mathbb{Z}(2))$ in $\mathcal{H}^2(\mathbb{Z}(2))$. The group $(H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}}))_{tors}$ is isomorphic to $H^0(X,\mathcal{H}^2(\mathbb{Q}/\mathbb{Z}))$ (see [Ros 5.1]) and $H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}}) \otimes \mathbb{Q}/Z = 0$. Also $H^1(X,\mathcal{F}^{2,2}) \otimes \mathbb{Q}$ is at most countable. The group $H^1(X,\mathcal{K}_2) \otimes \mathbb{Q}$, which is isomorphic to $CH^2(X,1)_{\mathbb{Q}}$, decomposes as follows ([Ped 1.10]):

$$H^1(X,\mathcal{K}_2) \otimes \mathbb{Q} \simeq H^1(X,\mathcal{F}^{2,2}_{\mathbb{Q}}) \oplus (NS(X) \otimes \mathbb{C}^*)_{\mathbb{Q}} \oplus Ker(R(X))_{\mathbb{Q}}$$

For a surface X with $p_g(X) = 0$ the regulator map $c_{2,1}$ is surjective, because the map ρ in (6) is 0 by [BV 2 3.1]). Hence $R(X)_{\mathbb{Q}} = H^3_{\mathcal{D}}(X,\mathbb{Q}(2))$ and $T(X) = H^0(X,\mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2))$. Moreover $H^1(X,\mathcal{F}^{2,2}) \otimes \mathbb{Q} = 0$, so that $H^3_{\mathcal{D}}(X,\mathbb{Q}(2)) = (NS(X) \otimes \mathbb{C}^*)_{\mathbb{Q}} = NS(X) \otimes \mathbb{C}/\mathbb{Q}$.

2) If $X \subset \mathbb{P}^3$ is a general hypersurface of degree $d \geq 5$ then the regulator map is not surjective because $R(X)_{\mathbb{Q}}$ is contained in the image of $H^3_{\mathcal{D}}(\mathbb{P}^3, \mathbb{Q}(2)) \to H^3_{\mathcal{D}}(X, \mathbb{Q}(2))$. More precisely: $R(X)_{\mathbb{Q}} = NS(X) \otimes \mathbb{C}/\mathbb{Q} \simeq \mathbb{C}/Q$ and $(H^1(X, \mathcal{F}^{2,2}_{\mathbb{Z}})_{\mathbb{Q}} = 0$.

3) Let X be a surface of general type with $p_g(X) = 0$. Then also $q(X) = 0$ and Bloch's Conjecture in 1.2 is equivalent to the vanishing of the Albanese Kernel $T(X)$. In fact the idempotent $\pi_2^{tr}(X)$ which defines the summand $t_2(X)$ of the motive $h(X)$ is in this case homologically trivial because $H^2_{tr}(X) = 0$. From Theorem 6.2 and the vanishing of $H^2(X, \mathcal{O}_X)$ we get the isomorphism

$$H^0(X, \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2))) \simeq T(X)$$

(see also [BV-Sr]). We also have $H^2(X \times X, \mathcal{O}_{X \times X}) = 0$, hence by the results in [BV 2]

$$A^2(X \times X)_{hom} \simeq H^0(X \times X, \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2)).$$

Therefore Bloch's Conjecture is equivalent to show that every element in $H^0(X \times X, \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2)))$ acts trivially on $H^0(X, \mathcal{H}^3_{\mathcal{D}}(\mathbb{Q}(2))$.

References

[A] Yves André pp. 1 - 260 *Une introduction aux Motives* 2004 Soc. Math. de France Panoramas et Syntheses 17

[AK] Yves André and B. Kahn pp. 106-291 *Nilpotence radicaux et structures monoidales (with an Appendix by P. O'sullivan)* 2003 Rend. Sem. Mat Univ. Padova 108

[Ay] J. Ayoub pp. 3-54 *The Motivic vanishing Cycles and the Conservation conjecture* 2007 "Algebraic Cycles and Motives" Vol 2, London Math. Soc. LNS 344, Cambridge University Press

[BV1] L. Barbieri-Viale pp. 298-312 *Balanced varieties* 1999 Algebraic K-theory and its Applications, World Scientific

[BV2] L. Barbieri-Viale pp. 298-312 *On the Deligne -Beilinson cohomology sheaves* 1994 Alg. Geom /9412006

[BV-S] L. Barbieri-Viale and V. Srinivas pp. 211-214 *A reformulation of Bloch's Conjecture* 1995 C. R. Acad. Sc. Paris 321 Serie I

[B] S. Bloch: *Lectures on Algebraic Cycles* Duke University Mathematics Series IV, Duke University, Durham U.S.A. 1980

[B-Sr] S. Bloch and V. Srinivas pp. 1235-1253 *Remarks on correspondences and algebraic cycles* 1983 Amer. J. of Math. 105

[C-H] A. Corti and M. Hanamura pp. 459-522 *Motivic decomposition and intersection Chow groups I* 2000 Duke Math. J. 103, no. 3

[C-L1] X. Chen and J. Lewis pp. 213-240 *The Hodge- $\mathcal{D}$-Conjecture for K3 and Abelian Surfaces* 2005 Journal of Alg. Geom. 14

[C-L2] Xi Chen and J. Lewis pp. 271-283 *The Real regulator for a product of K3 surfaces* 2006 AMS Studies in Adv. Math. Vol. 38

[D] P. Deligne pp. 137-252 *La Conjecture de Weil pour les surfaces K3* 1972 Inv. Math. 15

[E-V] H. Esnault and E. Viehweg pp. 43-92 *Deligne -Beilinson cohomology* 1988 "Beilinson's Conjectures on special values of L-functions", Persp. Math. Vol. 4, Academic Press

[FU] W. Fulton *Intersection Theory* Springer-Verlag Heidelberg- New-York 1984

[Gi] H. Gillet pp. 285-288 *Deligne homology and the Abel-Jacobi map* 1983 Bull. AMS 10

[G-G] S. Gorchinskiy and V. Guletskii pp. 1-21 *Motives and representability of algebraic cycles on threefolds over a field* 2008 preprint

[G-P] V. Guletskii and C. Pedrini pp. 243-263 *Finite dimensional motives and the Conjectures of Beilinson and Murre* 2003 K-Theory 30

[J1] U. Jannsen pp. 245-302 *Motivic Sheaves and Filtrations on Chow groups* 1994 Proceedings of Symposia in Pure mathematics Vol 55 Part. 1

[J2] U. Jannsen pp. 305-372 *Deligne cohomology, Hodge $\mathcal{D}$ conjecture and motives* 1988 "Beilinson's Conjectures on special values of L-functions", Persp. Math. Vol. 4, Academic Press

[K] B. Kahn pp. 1-69 *Zeta functions and motives* 2009 Pure and Applied Mthematics Quarterly Vol 5, N. 1

[KMP] B. Kahn, J Murre and C. Pedrini pp. 1-58 *On the transcendental part of the motive of a surface* 2007 "Algebraic Cycles and Motives" Vol 2, London Math. Soc. LNS 344, Cambridge University Press

[Ki] S. I. Kimura pp. 173-201 *Chow groups can be finite dimensional in some sense* 2005 Math. Ann. 331

[Mi] J. Milne pp. 135-166 *The Tate Conjecture for certain abelian varieties over finite fields* 2001 Acta Arithmetica 100

[Mor] D. R. Morrison pp. 105-121 *On K3 surfaces with large Picard number* 1984 Inv. Math. 75

[Mu] J. Murre pp. 177-188 *On a conjectural filtration on the Chow group of an algebraic variety* 1993 Indag. Math. (N. S.) 4 no. 2

[Muk] S. Mukai pp. 341-413 *On the moduli space of bundles of K3 surfaces 1* 1987 Tata Inst. of Fund. research Stud. math. 11 4 no. 2

[MS] Morihiko Saito pp. 1-7 *Bloch's Conjecture and Chow Motives* 2001 preprint

[N] V. Nikulin pp. 375-383 *On correspondences between K3 surfaces* 1988 Math. USSR Izvestyia Vol. 30 n. 2

[Ped] C. Pedrini pp. 195-213 *Bloch's conjecture and the K-theory of projective surfaces* 2000 CIRM Proceedings and Lecture Notes, Vol 24

[P-S] C. Peters and M. Steenbrick pp. 183 -194 *Monodromy of Variation of Hodge structures* 2003 Acta Appl. Mathematicae 75

[Ros] A. Rosenschon pp. 185-199 *Indecomposable elements in K_1 of a smooth projective variety* 1999 K-Theory 16

[S-I] T. Shioda and H. Inose pp. 119-136 *On singular K3 Surfaces* 1977 Complex Analysis and Algebraic Geometry, Iwanami-Shoten, Tokyo

[Voev1] V. Voevodsky pp. 187-198 *A nilpotence theorem for cycles algebraically equivalent to 0* 1995 Internat. Math. Res. Notices 4

[Voev2] V. Voevodsky pp. 188-238 *Triangulated categories of motives over a field* 2000 "Cycles transfers and motivic homology theories", Ann. of Math. Stud. 143 Princeton Univ. Press

[Vois1] C. Voisin pp. 1-18 *Remarks on filtrations on Chow groups and the Bloch Conjecture* 2004 Annali di Matematica

[Vois2] C. Voisin pp. 265-285 *Remarks on zero cycles on self products of varieties* 1996 Moduli of Vector Bundles, Lect. Notes in pure and apllied Math. 179, Dekker

Università Degli Studi Di Genova, Dipartmento Di Mathematica, Via Dodecanesco, 35, 16146 Genova, Italy

E-mail address: pedrini@dima.unige.it

URL: http://www.dima.unige.it/~pedrini

Clay Mathematics Proceedings
Volume 9, 2010

Two observations about normal functions

Christian Schnell

Abstract. Two simple observations are made: (1) If the normal function associated to a Hodge class has a zero locus of positive dimension, then it has a singularity. (2) The intersection cohomology of the dual variety contains the cohomology of the original variety, if the degree of the embedding is large.

This brief note contains two elementary observations about normal functions and their singularities that arose from a conversation with G. Pearlstein. Throughout, X will be a smooth projective variety of dimension $2n$, and ζ a primitive Hodge class of weight $2n$ on X, say with integer coefficients. We shall assume that X is embedded into projective space by a very ample divisor H, and let $\pi\colon \mathfrak{X} \to P$ be the family of hyperplane sections for the embedding. The discriminant locus, which parametrizes the singular hyperplane sections, will be denoted by $X^\vee \subseteq P$; on its complement, the map π is smooth.

1. The zero locus of a normal function

Here we show that if the zero locus of the normal function associated to a Hodge class ζ contains an *algebraic* curve, then the normal function must be singular at one of the points of intersection between $X^\vee$ and the closure of the curve.

Proposition 1. *Let ν_ζ be the normal function on $P \setminus X^\vee$, associated to a non-torsion primitive Hodge class $\zeta \in H^{2n}(X,\mathbb{Z}) \cap H^{n,n}(X)$. Assume that the zero locus of ν_ζ contains an algebraic curve, and that $H = dA$ for A very ample and $d \geq 3$. Then ν_ζ is singular at one of the points where the closure of the curve meets $X^\vee$.*

Before giving the proof, we briefly recall some definitions. In general, a normal function for a variation of Hodge structure of odd weight on a complex manifold Y_0 has an associated *cohomology class*. If $H_\mathbb{Z}$ is the local system underlying the variation, then a normal function ν determines an extension of local systems

$$0 \longrightarrow H_\mathbb{Z} \longrightarrow H'_\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{1}$$

The cohomology class $[\nu] \in H^1\big(Y_0, H_\mathbb{Z}\big)$ of the normal function is the image of $1 \in H^0\big(Y_0, \mathbb{Z}\big)$ under the connecting homomorphism for the extension.

2000 *Mathematics Subject Classification.* 14D05.
Key words and phrases. Normal function, Singularity of normal function, Hodge conjecture, Intersection cohomology, Dual variety.

In particular, the normal function ν_ζ associated to a Hodge class ζ determines a cohomology class $[\nu_\zeta] \in H^1\big(P \setminus X^\vee, R^{2n-1}\pi_*\mathbb{Z}\big)$. With rational coefficients, that class can also be obtained directly from ζ through the Leray spectral sequence

$$E_2^{p,q} = H^p\big(P \setminus X^\vee, R^q\pi_*\mathbb{Q}\big) \Longrightarrow H^{p+q}\big(P \times X \setminus \pi^{-1}(X^\vee), \mathbb{Q}\big).$$

That is to say, the pullback of ζ to $P \times X \setminus \pi^{-1}(X^\vee)$ goes to zero in $E_2^{0,2n}$ because ζ is primitive, and thus gives an element of $E_2^{1,2n-1}$; this element is precisely $[\nu_\zeta]$. (Details can be found, for instance, in [6, Section 4].)

Lemma 2. *Let $C_0 \to P \setminus X^\vee$ be a smooth affine curve mapping into the zero locus of ν_ζ, and let $\psi\colon W_0 \to C_0$ be the pullback of the family $\pi\colon \mathfrak{X} \to P$. Then the image of the Hodge class ζ in $H^{2n}(W_0, \mathbb{Q})$ is zero.*

Proof. By topological base change, the pullback of $R^{2n-1}\pi_*\mathbb{Q}$ to C_0 is naturally isomorphic to $R^{2n-1}\psi_*\mathbb{Q}$; moreover, when ν_ζ is restricted to C_0, its class is simply the image of $[\nu_\zeta]$ in the group $H^1\big(C_0, R^{2n-1}\psi_*\mathbb{Q}\big)$. That image has to be zero, because C_0 maps into the zero locus of ν_ζ.

Now let $\zeta_0 \in H^{2n}\big(W_0, \mathbb{Q}\big)$ be the image of the Hodge class ζ. The Leray spectral sequence for the map ψ gives a short exact sequence

$$0 \longrightarrow H^1\big(C_0, R^{2n-1}\psi_*\mathbb{Q}\big) \longrightarrow H^{2n}\big(W_0, \mathbb{Q}\big) \longrightarrow H^0\big(C_0, R^{2n}\psi_*\mathbb{Q}\big) \longrightarrow 0,$$

and as before, ζ_0 actually lies in $H^1\big(C_0, R^{2n-1}\psi_*\mathbb{Q}\big)$. Because the spectral sequences for ψ and π are compatible, ζ_0 is equal to the image of $[\nu_\zeta]$; but we have already seen that this is zero. ☐

Returning to our review of general definitions, let ν be a normal function on a complex manifold Y_0. When $Y_0 \subseteq Y$ is an open subset of a bigger complex manifold, one can look at the behavior of ν near points of $Y \setminus Y_0$. The *singularity* of ν at a point $y \in Y \setminus Y_0$ is by definition the image of $[\nu]$ in the group

$$\lim_{U \ni y} H^1\big(U \cap Y_0, H_{\mathbb{Z}}\big),$$

the limit being over all analytic open neighborhoods of the point. If the singularity is non-torsion, ν is said to be *singular* at the point y; this definition from [1] is a generalization of the one by M. Green and P. Griffiths [5].

When ν_ζ is the normal function associated to a non-torsion primitive Hodge class $\zeta \in H^{2n}(X, \mathbb{Z})$, P. Brosnan, H. Fang, Z. Nie, and G. Pearlstein [1, Theorem 1.3], and independently M. de Cataldo and L. Migliorini [2, Proposition 3.7], have proved the following result: Provided the vanishing cohomology of the smooth fibers of π is nontrivial, ν_ζ is singular at a point $p \in X^\vee$ if, and only if, the image of ζ in $H^{2n}\big(\pi^{-1}(p), \mathbb{Q}\big)$ is nonzero. By recent work of A. Dimca and M. Saito [3, Theorem 6], it suffices to take $H = dA$, with A very ample and $d \geq 3$.

Proof of Proposition 1. Let C be the normalization of the closure of the curve in the zero locus. Pulling back the universal family $\pi\colon \mathfrak{X} \to P$ to C and resolving singularities, we obtain a smooth projective $2n$-fold W, together with the two maps shown in the following diagram:

$$\begin{tikzcd} W \arrow[r, "\lambda"] \arrow[d, "\psi"'] & X \\ C & \end{tikzcd}$$

This may be done in such a way that the general fiber of ψ is a smooth hyperplane section of X; let $C_0 \subseteq C$ be the open subset where this holds, and $W_0 = \psi^{-1}(C_0)$ its preimage. Assume in addition that, for each $t \in C \setminus C_0$, the fiber $E_t = \psi^{-1}(t)$ is a divisor with simple normal crossing support. The map λ is generically finite, and we let d be its degree.

Let $\zeta_W = \lambda^*(\zeta)$ be the pullback of the Hodge class to W. By Lemma 2, the restriction of ζ_W to W_0 is zero. Consider now the exact sequence

$$H^{2n}(W, W_0, \mathbb{Q}) \xrightarrow{i} H^{2n}(W, \mathbb{Q}) \longrightarrow H^{2n}(W_0, \mathbb{Q}).$$

By what we have just observed, ζ_W belongs to the image of the map i, say $\zeta_W = i(\alpha)$. Under the nondegenerate pairing (given by Poincaré duality)

$$H^{2n}(W, W_0, \mathbb{Q}) \otimes \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t, \mathbb{Q}) \to \mathbb{Q},$$

and the intersection pairing on W, the map i is dual to the restriction map

$$H^{2n}(W, \mathbb{Q}) \to \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t \mathbb{Q}),$$

and so we get that

$$d \cdot \int_X \zeta \cup \zeta = \int_W \zeta_W \cup \zeta_W = \langle i(\alpha), \zeta_W \rangle = \sum_{t \in C \setminus C_0} \langle \alpha, i_t^*(\zeta_W) \rangle$$

where $i_t \colon E_t \to W$ is the inclusion. But the first integral is nonzero, because the intersection pairing on X is definite on the subspace of primitive (n,n)-classes. We conclude that the pullback of ζ to at least one of the E_t has to be nonzero.

By construction, E_t maps into one of the singular fibers of π, say to $\pi^{-1}(p)$, where p belongs to the intersection of $X^\vee$ with the closure of the curve. Thus ζ has nonzero image in $H^{2n}\bigl(\pi^{-1}(p), \mathbb{Q}\bigr)$; by the result of [1, 2] mentioned above, ν_ζ has to be singular at p, concluding the proof. □

I do not know whether a "converse" to Proposition 1 is true; that is to say, whether the normal function associated to an algebraic cycle on X has to have a zero locus of positive dimension for sufficiently ample H. If it was, this would give one more equivalent formulation of the Hodge conjecture.

2. Cohomology of the discriminant locus

G. Pearlstein pointed out that the singularities of the discriminant locus should be complicated enough to capture all the primitive cohomology of the original variety, once $H = dA$ is a sufficiently big multiple of a very ample class. In this section, we give an elementary proof of this fact for $d \geq 3$.

To do this, we need a simple lemma, used to estimate the codimension of loci in $X^\vee$ where the fibers of π have a singular set of positive dimension. Let

$$V_d = H^0\bigl(X, \mathscr{O}_X(dA)\bigr)$$

be the space of sections of dA, for A very ample.

Lemma 3. *Let $Z \subseteq X$ be a closed subvariety of positive dimension $k > 0$. Write $V_d(Z)$ for the subspace of sections that vanish along Z. Then*

$$\operatorname{codim}\bigl(V_d(Z), V_d\bigr) \geq \binom{d+k}{k}.$$

PROOF. Since A is very ample, we may find $(k+1)$ points $P_0, P_1, \dots, P_k$ on Z, together with $(k+1)$ sections $s_0, s_1, \dots, s_k \in V_1$, such that each s_i vanishes at all points P_j with $j \neq i$, but does not vanish at P_i. Then all the sections

$$s_0^{\otimes i_0} \otimes s_1^{\otimes i_1} \otimes \cdots \otimes s_k^{\otimes i_k} \in V_d,$$

for $i_0 + i_1 + \cdots + i_k = d$, are easily seen to be linearly independent on Z. The lower bound on the codimension follows immediately. ☐

We now use this estimate to make the above idea about the cohomology of $X^\vee$ precise. As one further bit of notation, let $\mathfrak{X}_{sing} \subseteq \mathfrak{X}$ stand for the union of all the singular points in the fibers of π. It is well-known that $\mathfrak{X}_{sing}$ is a projective space bundle over X, and in particular smooth.

PROPOSITION 4. *Let $H = dA$ for a very ample class A. If $d \geq 3$, then the map $\phi \colon \mathfrak{X}_{sing} \to X^\vee$ is a small resolution of singularities, and therefore*

$$IH^*(X^\vee, \mathbb{Q}) \simeq H^*(\mathfrak{X}_{sing}, \mathbb{Q}).$$

In particular, $H^*(X, \mathbb{Q})$ is a direct summand of $IH^*(X^\vee, \mathbb{Q})$ once $d \geq 3$.

PROOF. By [3, p. Theorem 6], the discriminant locus is a divisor in P once $d \geq 3$. This means that there are hyperplane sections of X with exactly one ordinary double point [8, p. 317]. The map ϕ is then birational, and therefore a resolution of singularities of $X^\vee$. To prove that it is a small resolution, take a stratification of $X^\vee$ with smooth strata, and such the fibers of the map ϕ have constant dimension over each stratum; this is easily done, using the constructibility of the higher direct image sheaves $R^k\phi_*\mathbb{Q}$.

Let $S \subseteq X^\vee$ be an arbitrary stratum along which the singular set of the fiber has dimension $k > 0$. At a general point $t \in S$, there then has to be an irreducible component Z in the singular locus of $pi^{-1}(t)$ that remains singular to first order along S. Now a tangent vector to S may be represented by a section s of $\mathscr{O}_X(dA)$, and a simple calculation in local coordinates shows that, in order for Z to remain singular to first order, the section s has to vanish along Z. By Lemma 3, the space of such sections has codimension at least $\binom{d+k}{k}$, and a moment's thought shows that, therefore,

$$\operatorname{codim}(S, X^\vee) \geq \binom{d+k}{k} - 1.$$

This quantity is evidently a lower bound for the codimension of the locus where the fibers of ϕ have dimension k. In order for ϕ to be a small resolution, it is thus sufficient that

$$\binom{d+k}{k} - 1 > 2k$$

for all $k > 0$. Now one easily sees that this condition is satisfied provided that $d \geq 3$. This proves the first assertion; the second one is a general fact about intersection cohomology [4, pp. 120–1]. Finally, $H^*(X, \mathbb{Q})$ is a direct summand of $H^*(\mathfrak{X}_{sing}, \mathbb{Q})$ because $\mathfrak{X}_{sing}$ is a projective space bundle over X, and the third assertion follows. ☐

The proof shows that, as in the theorem by A. Dimca and M. Saito, $d \geq 2$ is sufficient in most cases. A result related to Proposition 4, and also showing the effect of taking H sufficiently ample, was pointed out to me by H. Clemens;

he noticed that, as a consequence of M. Nori's connectivity theorem, one has an isomorphism

$$H^{2n}(X,\mathbb{Q})_{\mathrm{prim}} \simeq H^1\bigl(P \setminus X^\vee, (R^{2n-1}\pi_*\mathbb{Q})_{\mathrm{van}}\bigr),$$

once H is sufficiently ample [7, Corollary 4.4 on p. 364].

References

[1] P. Brosnan, H. Fang, Z. Nie, and G. Pearlstein, *Singularities of admissible normal functions* (2007), available at `arXiv:0711.0964`. To appear in *Inventiones Mathematicae.*

[2] M. A. A. de Cataldo and L. Migliorini, *A remark on singularities of primitive cohomology classes* (2007), available at `arXiv:0711.1307v1`. To appear in *Proceedings of the A.M.S.*

[3] A. Dimca and M. Saito, *Vanishing cycle sheaves of one-parameter smoothings and quasi-semistable degenerations* (2009), available at `arXiv:0810.4896v2`.

[4] M. Goresky and R. MacPherson, *Intersection homology. II*, Inventiones Mathematicae **72** (1983), no. 1, 77–129.

[5] M. L. Green and P. Griffiths, *Algebraic cycles and singularities of normal functions*, Algebraic cycles and motives (Grenoble, 2007), London Mathematical Society Lecture Note Series, vol. 343, Cambridge University Press, 2007, pp. 206–263.

[6] M. Saito, *Hodge conjecture and mixed motives. I*, Complex geometry and Lie theory (Sundance, UT, 1989), Proceedings of Symposia in Pure Mathematics, vol. 53, American Mathematical Society, Providence, RI, 1991, pp. 283–303.

[7] M. V. Nori, *Algebraic cycles and Hodge-theoretic connectivity*, Inventiones Mathematicae **111** (1993), no. 2, 349–373.

[8] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés, vol. 10, Société Mathématique de France, Paris, 2002.

Department of Mathematics, Statistics & Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607

E-mail address: `cschnell@math.uic.edu`

Positive Characteristics and Arithmetic

Clay Mathematics Proceedings
Volume 9, 2010

Autour de la conjecture de Tate à coefficients $\mathbf{Z}_\ell$ pour les variétés sur les corps finis

Jean-Louis Colliot-Thélène et Tamás Szamuely

1. Introduction

Soient k un corps fini, $\bar{k}$ une clôture algébrique de k, G le groupe de Galois $\mathrm{Gal}\,(\bar{k}|k)$ et ℓ un nombre premier inversible dans k. Considérons une variété projective, lisse, géométriquement intègre X, de dimension d. D'après la conjecture de Tate, l'application cycle à valeurs dans la cohomologie étale ℓ-adique induit une *surjection*

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \twoheadrightarrow H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^G. \tag{1}$$

Une forme équivalente de la conjecture est la surjectivité du morphisme

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^U \tag{2}$$

où $\overline{X} := X \times_k \bar{k}$ et U parcourt le système des sous-groupes ouverts de G. La forme plus forte ci-dessus en résulte par un argument de restriction-corestriction.

On peut également considérer des formes entières de ces énoncés, et se demander si les morphismes

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^G \tag{3}$$

ou

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^U \tag{4}$$

induits par l'application cycle sont surjectifs. Ici le deuxième énoncé de surjectivité est a priori plus faible.

Comme nous allons le rappeler dans la section 2, on ne s'attend pas à ce que les formes entières de la conjecture ci-dessus soient vraies. Néanmoins, il est raisonnable d'espérer la surjectivité de (3) et (4) pour $i = d - 1$, i.e. pour les 1-cycles.

Dans ce cas, la surjectivité de (4) a été conditionnellement démontrée par Chad Schoen :

Théorème 1.1. (Schoen [**15**]) *Soient k, G et X comme ci-dessus. Supposons la conjecture de Tate connue pour les diviseurs sur une surface projective et lisse sur un corps fini. Alors le morphisme*

$$CH_1(\overline{X}) \otimes \mathbf{Z}_\ell \to \bigcup_U H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(d-1))^U$$

est surjectif, où U parcourt le système des sous-groupes ouverts de G.

Notons que la conjecture de Tate pour les diviseurs sur une surface au-dessus d'un corps fini peut être perçue comme un analogue de la finitude hypothétique du groupe de Tate-Shafarevich de la jacobienne d'une courbe sur un corps de nombres.

Nous expliquons la démonstration de Schoen (avec quelques modifications) dans les sections 3, 4 et 5.

Au paragraphe 6, on voit que le théorème de Schoen a des conséquences sur l'existence de zéro-cycles sur certaines variétés définies sur un corps de fonctions d'une variable sur la clôture algébrique d'un corps fini. Voici un cas particulier concret :

Corollaire 1.2. *Soient $\bar{k}$ et X comme ci-dessus. Supposons qu'il existe un $\bar{k}$-morphisme propre surjectif $f : \overline{X} \to \overline{C}$, avec $\overline{C}$ une $\bar{k}$-courbe propre lisse. Supposons en outre que la fibre générique de f est une intersection complète lisse de dimension ≥ 3 et de degré premier à $\mathrm{car}(k)$ dans un espace projectif, et que chacune des fibres de f possède une composante de multiplicité 1. Si la conjecture de Tate pour les diviseurs sur les surfaces projectives lisses sur un corps fini est vraie, alors le pgcd des degrés des multisections de $\overline{X} \to \overline{C}$ est égal à 1.*

2. Généralités sur la conjecture de Tate à coefficients entiers

On entend souvent dire : la conjecture de Hodge à coefficients entiers est fausse, il n'est pas raisonnable d'énoncer la conjecture de Tate avec des coefficients entiers. Quelle est la situation ?

Chacune de ces conjectures porte sur l'image d'une application cycle émanant du groupe de Chow $CH^r(X)$ des cycles de codimension r sur une variété projective et lisse X de dimension d, à valeurs dans un groupe de cohomologie. Il s'agit de $H^{2r}(X, \mathbf{Z})$ pour Hodge et de $H^{2r}_{\text{ét}}(X \times_k \overline{k}, \mathbf{Z}_\ell(r))$ pour Tate (dans cette section on va distinguer les groupes de cohomologie étale des groupes de cohomologie singulière par des indices pour ne pas induire une confusion). On trouvera dans le survol [**23**] de Voisin un état des lieux pour la conjecture de Hodge.

Si l'on croit à ces conjectures à coefficients rationnels, la variante entière peut être mise en défaut de deux façons :

(a) on trouve une classe de cohomologie de torsion qui n'est pas la classe d'un cycle ;

(b) on trouve une classe de cohomologie d'ordre infini qui n'est pas dans l'image de l'application cycle, mais qui donne un élément de torsion dans son conoyau.

Pour la conjecture de Hodge entière, il y a des contre-exemples de type (a) dus à Atiyah et Hirzebruch [**1**], reconsidérés plus récemment par Totaro ([**20**], [**21**]), pour les groupes $H^{2r}(X, \mathbf{Z})$ avec $r \geq 2$. L'exemple de dimension minimale chez eux est une variété de dimension 7, avec une classe de torsion dans $H^4(X, \mathbf{Z})$. Dans la littérature (par exemple dans Milne [**10**], Aside 1.4) il est affirmé que l'on peut adapter ces exemples pour donner des contre-exemples à la conjecture de Tate entière sous la forme (4), mais à notre connaissance aucune démonstration n'a été écrite. Voici donc une esquisse de démonstration qui met en relief les modifications à faire par rapport au cas analytique discuté dans [**1**]. Il s'agit de prouver le théorème suivant :

Théorème 2.1.

(1) *Soit V une variété projective et lisse sur un corps algébriquement clos. Pour tout $\ell \geq i$ inversible sur V les opérations de Steenrod de degré impair s'annulent sur la classe de tout cycle algébrique dans le groupe $H^{2i}_{ét}(V, \mathbf{Z}/\ell\mathbf{Z}(i))$.*

(2) *Au-dessus de tout corps algébriquement clos, pour tout premier ℓ différent de la caractéristique, il existe une intersection complète lisse $Y \subset \mathbf{P}^N$, un groupe fini G agissant librement sur Y, une classe c de ℓ-torsion dans $H^4_{ét}(Y/G, \mathbf{Z}_\ell(2))$ et une opération de Steenrod de degré impair qui n'annule pas l'image de c dans $H^4_{ét}(Y/G, \mathbf{Z}/\ell\mathbf{Z}(2))$.*

Ici pour $\ell > 2$ premier « opération de Steenrod de degré impair » veut dire un composé d'opérations de Steenrod $\mathcal{P}^j$ et d'une opération de Bockstein. Les opérations $\mathcal{P}^j : H^i_{\text{ét}}(V, \mathbf{Z}/\ell\mathbf{Z}) \to H^{i+2j(\ell-1)}_{\text{ét}}(V, \mathbf{Z}/\ell\mathbf{Z})$ en cohomologie étale ont été définies par Mme Raynaud dans [**14**]. Pour $\ell = 2$ on utilise des opérations $Sq^j : H^i_{\text{ét}}(V, \mathbf{Z}/2\mathbf{Z}) \to H^{i+j}_{\text{ét}}(V, \mathbf{Z}/2\mathbf{Z})$, également définies dans [**14**].

Si le corps de base est une clôture algébrique $\overline{F}$ d'un sous-corps F, toute classe de torsion dans $H^{2i}_{\text{ét}}(V, \mathbf{Z}_\ell(i))$ est invariante par un sous-groupe ouvert de $\mathrm{Gal}\,(\overline{F}|F)$, donc pour F fini le théorème nous fournit un contre-exemple du type (a) à la surjectivité des applications (3) et (4).

Esquissons une démonstration du théorème qui nous a été généreusement communiquée par Burt Totaro. Pour démontrer (1), la première observation est que par le théorème de Riemann-Roch sans dénominateurs de Jouanolou ([**5**], Example 15.3.6) pour ℓ premier à $(i-1)!$ toute classe de cycle dans $H^{2i}_{\text{ét}}(V, \mathbf{Z}/\ell\mathbf{Z}(i))$ est combinaison linéaire de classes de Chern $c_i(E)$ de fibrés vectoriels E sur X. Il suffit donc de démontrer l'énoncé d'annulation pour les $c_i(E)$. Un calcul d'opérations de Steenrod montre que l'annulation vaut pour E si et seulement si elle vaut pour $E \otimes L$ avec L très ample de rang un. Ainsi, on peut supposer que E est engendré par ses sections globales, et a fortiori qu'il est la tirette du fibré tautologique d'une grassmannienne. L'énoncé résulte alors de la fonctorialité contravariante des $\mathcal{P}^j$ et de la trivialité de la cohomologie d'une grassmannienne en degrés impairs ([**8**], exposé VII, proposition 5.2).

Un point clef de l'argument d'Atiyah–Hirzebruch [**1**] était l'identification de la cohomologie en bas degrés d'une variété de Godeaux–Serre Y/G comme dans (2) à celle du produit d'espaces classifiants $BG \times B\mathbf{G}_m$. On peut algébriser leur méthode en utilisant l'approximation algébrique de BG introduite par Totaro. En effet, d'après ([**21**], Remark 1.4) pour tout $s \geq 0$ il existe une représentation k-linéaire W de G telle que l'action de G soit libre en dehors d'un fermé S de codimension s dans W. La cohomologie de $BG := (W \setminus S)/G$ est égale à celle de G jusqu'en degré s ; en particulier, elle ne dépend ni du choix de W ni du choix de S. Le quotient $\mathbf{P}(W)//G := (\mathbf{P}(W) \times (W \setminus S))/G$ est un fibré projectif sur BG, donc son anneau de cohomologie est un anneau de polynômes sur celui de BG. En particulier, la cohomologie de BG est facteur direct dans celle de $\mathbf{P}(W)//G$.

Si maintenant $Y \subset \mathbf{P}(W)$ est une intersection complète lisse sur laquelle l'action de G est libre, la cohomologie de Y/G est isomorphe à celle de $\mathbf{P}(W)//G$ en bas degrés. En effet, la cohomologie de Y est isomorphe à celle de $\mathbf{P}(W)$ jusqu'en degré $\dim(Y) - 1$ par le théorème de Lefschetz faible. On en déduit un isomorphisme entre les cohomologies de $(Y \times (W \setminus S))/G$ et de $\mathbf{P}(W)//G$ jusqu'en degré $\dim(Y)$ en appliquant la suite spectrale de Hochschild–Serre aux G-revêtements $Y \times (W \setminus S) \to (Y \times (W \setminus S))/G$ et $(\mathbf{P}(W) \times (W \setminus S)) \to \mathbf{P}(W)//G$. Or la cohomologie de $(Y \times (W \setminus S))/G$ s'identifie à celle de $(Y \times W)/G$ jusqu'en degré s, et finalement à celle de Y/G dans le même intervalle, puisque W est un espace affine.

En somme, en bas degrés la cohomologie de BG (donc celle de G) s'identifie à un facteur direct de celle de Y/G ci-dessus. La fin de la démonstration de (2) est alors similaire à celle de ([**1**], Proposition 6.7). Prenons $G = (\mathbf{Z}/\ell\mathbf{Z})^3$. Comme G est d'exposant ℓ, la suite exacte longue associée à $0 \to \mathbf{Z}_\ell \to \mathbf{Z}_\ell \to \mathbf{Z}/\ell\mathbf{Z} \to 0$ montre que $H^i(G, \mathbf{Z}_\ell)$ s'identifie au noyau du morphisme de Bockstein $\beta : H^i(G, \mathbf{Z}/\ell\mathbf{Z}) \to H^{i+1}(G, \mathbf{Z}/\ell\mathbf{Z})$. Le cup-produit des éléments d'une base du $(\mathbf{Z}/\ell\mathbf{Z})$-espace vectoriel $H^1(G, \mathbf{Z}/\ell\mathbf{Z}) \cong (\mathbf{Z}/\ell\mathbf{Z})^3$ donne une classe dans $H^3(G, \mathbf{Z}/\ell\mathbf{Z})$. Essentiellement le même calcul que dans [**1**] montre que pour $\ell > 2$ l'image de cette classe dans $H^4(G, \mathbf{Z}/\ell\mathbf{Z})$ par le Bockstein β n'est pas annulée par l'opération $\beta\mathcal{P}^1$, dont le degré est $2\ell - 1$. Pour $\ell = 2$ la même conclusion vaut pour Sq^3.

Terminons cette section par une brève discussion des contre-exemples de type (b). Un célèbre contre-exemple de ce type à la conjecture de Hodge a été fabriqué par J. Kollár [**9**] ; voir aussi [**17**]. Il s'agit d'une hypersurface « très générale » dans $\mathbf{P}^4_{\mathbf{C}}$ de degré m un multiple de ℓ^3 avec ℓ entier premier à 6, et de l'application cycle $CH^2(X) \to H^4(X, \mathbf{Z})$. Comme il s'agit d'une hypersurface de degré m, ici on a $H^4(X, \mathbf{Z}) \cong \mathbf{Z}$, et l'image de l'application cycle contient $m\mathbf{Z}$. Mais Kollár montre par un argument de déformation astucieux que toute courbe sur X a un degré divisible par ℓ. En d'autres mots, l'image de $CH^2(X) \to H^4(X, \mathbf{Z})$ est contenue dans $\ell H^4(X, \mathbf{Z})$ et ne peut être surjective. Comme le note C. Voisin ([**17**], [**23**]), on peut à partir de cet exemple fabriquer des contre-exemples à la conjecture de Hodge entière en d'autres degrés aussi, par éclatement ou par produit direct avec une autre variété.

L'énoncé de Kollár en induit un au niveau de la cohomologie ℓ-adique étale. En effet, si on travaille sur un corps algébriquement clos non dénombrable et on choisit ℓ premier à la caractéristique et à 6, sa méthode fournit toujours une hypersurface $X \subset \mathbf{P}^4$ sur laquelle toute courbe a un degré divisible par ℓ. (Le corps non dénombrable sert ici pour pouvoir choisir le point correspondant à X d'un schéma de Hilbert convenable en dehors de la réunion d'une famille dénombrable de fermés propres.) Ensuite, comme pour toute variété digne de ce nom, on trouve un corps K de type fini sur le corps premier sur lequel X est définie. Notant $\overline{K}$ une clôture algébrique de K, l'image de l'application cycle

$$CH^2(X \times_K \overline{K}) \to H^4_{\text{ét}}(X \times_K \overline{K}, \mathbf{Z}_\ell(2)) \cong \mathbf{Z}_\ell$$

est alors contenue dans $\ell\mathbf{Z}_\ell$; noter qu'ici l'action de Galois sur la cohomologie induit l'action triviale sur $\mathbf{Z}_\ell$.

Remarques 2.2.

1. La méthode ci-dessus ne permet pas de trouver un tel exemple avec K un corps de nombres.

2. Par le théorème 1.1, si on croit à la conjecture de Tate rationnelle pour les diviseurs sur les surfaces, en caractéristique positive le corps K ci-dessus ne peut être un corps fini.

3. Nous ne savons pas s'il existe des contre-exemples du type (b) à la surjectivité de (3) sur un corps fini. En d'autres mots, nous ne savons pas si pour X projective, lisse, géométriquement connexe sur un corps fini k, l'application

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}_{\text{ét}}(\overline{X}, \mathbf{Z}_\ell(i))^G/\text{torsion}$$

induite par l'application cycle est toujours surjective.

Cette question est équivalente à la question suivante, fort intéressante du point de vue de [**2**] : pour tout $i \geq 0$, l'application

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}_{\text{ét}}(X, \mathbf{Z}_\ell(i))/\text{torsion}$$

induite par l'application cycle

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}_{\text{ét}}(X, \mathbf{Z}_\ell(i)) \tag{5}$$

est-elle surjective ? Le lien entre les deux questions est fourni par les suites exactes

$$0 \to H^1(k, H^{2i-1}_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(i)))) \to H^{2i}_{\text{ét}}(X, \mathbf{Z}_\ell(i)) \to H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(i))^G \to 0,$$

où les groupes $H^1(k, H^{2i-1}_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(i))))$ sont des groupes finis (ceci est une conséquence du théorème de Deligne établissant les conjectures de Weil).

Notons ici pour usage ultérieur que par un argument bien connu utilisant la suite de Kummer et le groupe de Brauer, pour $i = 1$ la surjectivité de (5) est équivalente à la conjecture de Tate à coefficients $\mathbf{Q}_\ell$, et même à la bijectivité du morphisme (1) (voir [**19**], Proposition 4.3). En vertu de la suite exacte ci-dessus, dans le cas des diviseurs la conjecture de Tate à coefficients $\mathbf{Q}_\ell$ implique donc la version entière sous toutes ses formes possibles.

3. Le théorème de Schoen, I : un argument de type Lefschetz

Nous commençons maintenant l'exposition de la démonstration du théorème 1.1, suivant [**15**].

Au cours de la preuve nous ferons à plusieurs reprises des extensions du corps de base de degré premier à ℓ. Un argument de restriction-corestriction fournit alors le résultat au-dessus du corps de base initial.

Lemme 3.1. *Il suffit d'établir le théorème 1.1 pour $d = 3$.*

Démonstration. D'après le théorème de Bertini sur un corps fini [**6, 13**], on peut trouver un plongement projectif de X et une hypersurface H tels que le k-schéma $Y = X \cap H$ soit de codimension 1 dans X, lisse et géométriquement connexe. Comme $X \setminus Y$ est affine, pour $d > 3$, les théorèmes sur la dimension cohomologique des schémas affines ([**12**], §VI.7) donnent

$$H^{2d-3}(\overline{X} \setminus Y_{\bar{k}}, \mathbf{Z}_\ell(d-1)) = 0, \quad H^{2d-2}(\overline{X} \setminus Y_{\bar{k}}, \mathbf{Z}_\ell(d-1)) = 0.$$

Donc le morphisme composé

$$H^{2d-4}(Y_{\bar{k}}, \mathbf{Z}_\ell(d-2)) \xrightarrow{\sim} H^{2d-2}_{Y_{\bar{k}}}(\overline{X}, \mathbf{Z}_\ell(d-1)) \to H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(d-1))$$

de l'isomorphisme de pureté et de la flèche provenant de la suite de localisation est un isomorphisme (théorème de Lefschetz faible, *ibidem*). Comme l'application cycle est compatible aux morphismes de Gysin ([**12**], Proposition VI.9.3), par récurrence sur d on se ramène donc au cas $d = 3$. □

Jusqu'à la fin du paragraphe 4, on suppose donc $d = \dim(X) = 3$.

Remarquons ensuite qu'il suffit d'établir le théorème après avoir éclaté un point de X, puisque la cohomologie de X s'identifie à un facteur direct de celle de l'éclaté ([**4**], exposé XVIII, 2.2.2) et l'application cycle est compatible aux morphismes propres de variétés propres et lisses ([**11**], théorème 6.1 et remarque 6.4). Ainsi, après avoir fait un éclatement convenable, on peut tranquillement supposer que le deuxième nombre de Betti ℓ-adique $b_2(\overline{X})$ de $\overline{X}$ est *impair*. En effet, si par malheur ce nombre est pair, on éclate un point fermé de degré f impair (d'après un argument de type Lang–Weil, un tel point existe puisque la variété X est géométriquement intègre), ce qui donne pour l'éclaté X^* la formule $b_2(X^*) = b_2(X) + f$ d'après [**4**], exposé XVIII, (2.3.1). La raison pour cette hypothèse supplémentaire se dévoilera lors de la preuve de la proposition 3.3 ci-dessous.

Un calcul simple de classes de Chern (voir [**16**], 9.2.1) montre que, quitte à composer le plongement projectif donné de X avec un plongement de Veronese de degré *pair*, on peut supposer que le deuxième nombre de Betti ℓ-adique de toute section hyperplane lisse de X est *pair*. Cette information de parité sera également importante pour la suite.

Ayant fait une extension de degré premier à ℓ si nécessaire, on trouve un éclaté $V \to X$ muni d'un pinceau de Lefschetz $V \to D \cong \mathbf{P}^1$ de sections hyperplanes ([**4**], exposé XVII, théorème 2.5). Notons $\dot{D} \subset D$ le lieu au-dessus duquel le morphisme $V \to D$ est lisse, et choisissons un point générique géométrique ε de D. D'après ce qui précède, le deuxième nombre de Betti de V_ε est pair.

Introduisons les notations π (resp. $\overline{\pi}$) pour le groupe fondamental arithmétique (resp. géométrique) de $\dot{D}$ ayant ε pour point base.

Proposition 3.2. *Quitte à faire une extension de k de degré premier à ℓ, on peut choisir le pinceau V de sorte que l'image de $\overline{\pi}$ dans $\mathrm{Aut}_{\mathbf{Z}_\ell}(H^2(V_\epsilon, \mathbf{Z}_\ell(1)))$ via la représentation de monodromie soit infinie.*

Démonstration. C'est la Proposition 1.1 de [**15**]. On ne donne que l'idée de l'argument. Soit $\mathbf{P}$ l'espace projectif paramétrisant les intersections de X avec les hypersurfaces de degré fixe suffisamment grand dans un plongement projectif fixé. Soient $\mathbf{V} \subset \mathbf{P} \times X$ l'hypersurface universelle, et $\dot{\mathbf{P}} \subset \mathbf{P}$ l'ouvert de lissité de la fibration $\mathbf{V} \to \mathbf{P}$. Le choix d'un pinceau de Lefschetz correspond au choix d'une droite $D \subset \mathbf{P}$, et on a $V = \mathbf{V} \times_{\mathbf{P}} D$. Par un argument de type Bertini (voir par exemple [**18**], Lemma 5.7.2) après une extension finie de k de degré premier à ℓ on trouve D assez générale pour laquelle l'homomorphisme $\pi_1(\dot{D}_{\bar{k}}, \epsilon) \to \pi_1(\dot{\mathbf{P}}_{\bar{k}}, \epsilon)$ est surjectif. Il suffit donc de montrer que l'image du deuxième groupe dans $\mathrm{Aut}_{\mathbf{Z}_\ell}(H^2(V_\epsilon, \mathbf{Z}_\ell(1)))$ est infinie. Schoen montre par une construction de géométrie algébrique classique qu'il existe un autre espace projectif P et un morphisme $P \to \mathbf{P}$ tels que le morphisme $\mathbf{V} \times_{\mathbf{P}} P \to P$ se factorise en $\mathbf{V} \times_{\mathbf{P}} P \to W \to P$, où W est une hypersurface projective lisse, et la dimension relative de $W \to P$ est 2. Comme W est une hypersurface, elle se relève en caractéristique 0, et un théorème de Deligne ([**22**], Théorème B) montre que la monodromie de tout pinceau de Lefschetz balayant W est infinie (sous l'hypothèse $\mathrm{car}\,(k) \neq 0$; en caractéristique 2 un petit argument supplémentaire est donné dans [**15**]). Ceci implique que la monodromie doit être infinie pour la fibration $\mathbf{V} \times_{\mathbf{P}} P \to P$, et finalement pour $\mathbf{V} \to \mathbf{P}$. □

Expliquons maintenant l'idée de la preuve du théorème 1.1. Tout d'abord, il suffit de montrer que toute classe dans $H^4(\overline{X}, \mathbf{Z}_\ell(2))^G$ est la classe d'un cycle algébrique sur $\overline{X}$ (ensuite, pour un sous-groupe ouvert $U \subset G$ on peut appliquer ce résultat après changement de base de X au sous-corps fixé par U). Etant donc donné $w \in H^4(\overline{X}, \mathbf{Z}_\ell(2))$ fixé par G, on montre qu'il est la poussette d'un élément de $H^2(V_{\bar{x}}, \mathbf{Z}_\ell(1))^{\mathrm{Gal}\,(\bar{k}|k(x))}$, où $\bar{x}$ est un point géométrique au-dessus d'un point fermé $x \in \dot{D}$. La conjecture de Tate pour les diviseurs sur la surface V_x (qui est valable à coefficients $\mathbf{Z}_\ell$ si elle est valable à coefficients $\mathbf{Q}_\ell$ d'après ce qu'on a dit dans la remarque 2.2 (3) ci-dessus) montre alors que cet élément est la classe d'un cycle algébrique.

Notons qu'à coefficients $\mathbf{Q}_\ell$ l'énoncé voulu est une conséquence directe du théorème de Lefschetz difficile (*cf.* la preuve du lemme 5.1 *infra*) ; toute la finesse de l'argument consiste à en tirer un énoncé à coefficients entiers.

Voici une reformulation. Écrivons X_ε pour le changement de base $X \times_{\mathrm{Spec}\,k} \varepsilon$. Par définition, il est muni de l'action triviale de $\overline{\pi}$ (celui-ci agissant sur les fibres de la fibration triviale $X \times D \to D$), et par conséquent

$$H^i(\overline{X}, \mathbf{Z}_\ell(2))^{\mathrm{Gal}\,(\bar{k}|k)} \cong H^i(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$$

pour tout $i > 0$. Notant $D_{\bar{x}}$ le groupe de décomposition d'un point $\bar{x}$ du revêtement universel profini de $\dot{D}$ au-dessus de x, on obtient

$$H^i(V_{\bar{x}}, \mathbf{Z}_\ell(2))^{\mathrm{Gal}\,(\bar{k}|k(x))} \cong H^i(V_\varepsilon, \mathbf{Z}_\ell(2))^{D_{\bar{x}}}$$

pour tout $i > 0$ par le théorème de changement de base propre et l'isomorphisme $D_{\bar{x}} \cong \mathrm{Gal}\,(\bar{k}|k(x))$.

Notons i l'inclusion de la surface V_ε dans la variété X_ε (qui est de dimension 3). Elle induit un morphisme de restriction

$$i^* : H^2(X_\varepsilon, \mathbf{Z}_\ell(1)) \to H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$$

ainsi qu'une poussette

$$i_* : H^2(V_\varepsilon, \mathbf{Z}_\ell(1)) \to H^4(X_\varepsilon, \mathbf{Z}_\ell(2)).$$

D'après la discussion ci-dessus, il suffit donc de montrer :

Proposition 3.3. *Chaque élément de $H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$ est de la forme $i_*(\beta)$, avec un $\beta \in H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ invariant sous l'action d'un sous-groupe de décomposition $D_{\bar{x}}$ dans π, pour un point $\bar{x}$ convenable.*

Interrompons-nous pour quelques considérations d'algèbre $\mathbf{Z}_\ell$-linéaire.

4. Le théorème de Schoen, II : Lemmes d'algèbre linéaire

Etant donnés un $\mathbf{Z}_\ell$-module B et un sous-ensemble $A \subset B$, on définit *le saturé* A_s de A dans B comme l'ensemble des $b \in B$ avec $\ell^n b \in A$ pour un $n \geq 0$ convenable. On dit que A est saturé dans B si $A_s = A$.

Lemme 4.1. *Pour un module B de type fini sur $\mathbf{Z}_\ell$ il existe un sous-groupe ouvert $\Gamma \subset Aut_{\mathbf{Z}_l}(B)$ tel que B^g soit saturé dans B pour tout $g \in \Gamma$.*

Démonstration. Ecrire $B = F \oplus T$ avec F libre et T de torsion, et prendre $\Gamma = Aut_{\mathbf{Z}_\ell}(F) \times \{\mathrm{id}_T\}$. □

PROPOSITION 4.2. *Soient F un $\mathbf{Z}_\ell$-module libre de rang fini impair, $S \subset F$ un sous-ensemble ouvert pour la topologie ℓ-adique, et $\Phi : F \times F \to \mathbf{Z}_\ell$ une forme bilinéaire symétrique non dégénérée sur $\mathbf{Q}_\ell$. Il existe alors un sous-ensemble ouvert $\mathcal{S} \subset O(F, \Phi)$ tel que :*

(a) chaque élément de $\mathcal{S}$ admet un vecteur fixe non nul dans S ;

(b) l'ouvert $\mathcal{S}$ contient des éléments arbitrairement proches de 1 pour la topologie ℓ-adique.

Ici $O(F, \Phi)$ désigne le groupe des automorphismes $\mathbf{Z}_\ell$-linéaires de F préservant Φ. Pour la preuve nous avons besoin d'un résultat ancillaire.

LEMME 4.3. *Soient K un corps de caractéristique différente de 2, V un K-espace vectoriel de dimension finie impaire, et Φ une forme quadratique non dégénérée sur V. Tout élément de $SO(\Phi)$ admet 1 comme valeur propre.*

Démonstration. Notons A la matrice de Φ dans une base fixée de V. Pour un élément de $O(\Phi)$ dont la matrice est M et la matrice transposée M^t, on a $M^t.A.M = A$. D'où

$$M^t.A.(M - I) = A - M^t.A = (I - M^t).A.$$

En prenant le déterminant on obtient :

$$\det(M).\det(A).\det(M - I) = \det(I - M).\det(A).$$

Ici $\det(A) \neq 0$ et $\det(M) = 1$ (comme $M \in SO(\Phi)$), donc $\det(M-I) = \det(I-M)$. Comme V est de dimension impaire, ceci n'est possible que si $\det(M - I) = 0$. □

Démonstration de la proposition 4.2. Soit $U \subset SO(F_{\mathbf{Q}_\ell}, \Phi)$ l'ouvert de Zariski formé des éléments ayant des valeurs propres distinctes. C'est aussi un ouvert de Zariski de $O(F_{\mathbf{Q}_\ell}, \Phi)$. Comme F est de rang impair, tout élément de $SO(F_{\mathbf{Q}_\ell}, \Phi)$ admet 1 comme valeur propre par le lemme 4.3. Donc tout $u \in U$ stabilise un sous-espace L_u de dimension 1 correspondant à la valeur propre 1. Envoyant u sur L_u on obtient une application continue $\lambda : U \to \mathbf{P}(F_{\mathbf{Q}_\ell})$. L'image de $S \setminus 0$ dans $\mathbf{P}(F_{\mathbf{Q}_\ell})$ par la projection naturelle $F_{\mathbf{Q}_\ell} \setminus 0 \to \mathbf{P}(F_{\mathbf{Q}_\ell})$ est ouverte, tout comme son image inverse $\mathcal{S}$ dans $U \subset SO(F_{\mathbf{Q}_\ell}, \Phi)$.

Reste à voir que l'ensemble $\mathcal{S}$ est non vide, et qu'il contient des éléments arbitrairement proches de 1. Soit $v \in S$ un vecteur non isotrope. Ecrivant $F_{\mathbf{Q}_\ell} = \langle v \rangle \perp M$ avec un $\mathbf{Q}_\ell$-vectoriel M, on commence par montrer qu'il existe un élément de $SO(M, \Phi|_M)$ à valeurs propres distinctes, et toutes différentes de 1. Pour cela, on décompose l'espace quadratique M en une somme orthogonale d'espaces quadratiques V_i de dimension 2. Chaque $SO(V_i)$ est un tore $T_i = R^1_{k_i/k}\mathbf{G}_m$ de dimension 1, où k_i/k est une algèbre étale de degré 2 sur $\mathbf{Q}_\ell$. Si $k_i \simeq \mathbf{Q}_\ell \times \mathbf{Q}_\ell$, alors $T_i \simeq \mathbf{G}_{m,k}$, et $\mathbf{G}_{m,k}$ agit sur $V_i = \mathbf{Q}_\ell \oplus \mathbf{Q}_\ell$ par $\lambda.(u, v) = (\lambda.u, \lambda^{-1}.v)$. Si k_i est une extension quadratique de $\mathbf{Q}_\ell$, alors $SO(V_i)(\mathbf{Q}_\ell)$ est le groupe des éléments de norme 1 dans k_i, et l'action de ce groupe sur $V_i \simeq k_i$ est donnée par la multiplication dans k_i. Les deux valeurs propres d'un élément $\alpha \in SO(V_i)(\mathbf{Q}_\ell) \subset SO(\Phi) \subset GL(V)$ sont les conjugués de α (qui sont inverses l'un à l'autre). On trouve donc une famille d'éléments $\alpha_i \in SO(V_i)$ dont la somme définit un élément de $SO(M, \Phi|_M)$ qui a des valeurs propres distinctes et différentes de 1. De plus, on peut choisir les matrices des α_i de sorte qu'elles aient des coefficients dans $\mathbf{Z}_\ell$ et qu'elles soient arbitrairement proches de la matrice 1 pour la topologie ℓ-adique. Si elles sont suffisamment proches de 1, leur somme directe doit préserver la trace du réseau F sur M. □

5. Le théorème de Schoen, III : fin de la démonstration

Il nous reste à démontrer la proposition 3.3.

Lemme 5.1. *Il existe une inclusion*

$$H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi \subset i_*((\ker i_* + H^\pi)_s),$$

où

$$H := \mathrm{Im}\,(i^*) \subset H^2(V_\varepsilon, \mathbf{Z}_\ell(1)).$$

Démonstration. Le morphisme composé

$$H^2(X_\varepsilon, \mathbf{Z}_\ell(1)) \xrightarrow{i^*} H^2(V_\varepsilon, \mathbf{Z}_\ell(1)) \xrightarrow{i_*} H^4(X_\varepsilon, \mathbf{Z}_\ell(2))$$

est un isomorphisme après tensorisation par $\mathbf{Q}_\ell$ selon le théorème de Lefschetz difficile, car c'est le cup-produit par la classe de la section hyperplane V_ε. Il est équivariant pour l'action de π, car V_ε provient par changement de corps de base d'une $k(\mathbf{P}^1)$-variété.

Donc par définition de H

$$i_*(H^\pi) \otimes \mathbf{Q}_\ell \cong H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi \otimes \mathbf{Q}_\ell,$$

d'où

$$H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi \subset (i_*(H^\pi))_s. \tag{6}$$

Remarquons maintenant que le morphisme

$$i_* : H^2(V_\epsilon, \mathbf{Z}_\ell(1)) \to H^4(X_\varepsilon, \mathbf{Z}_\ell(2))$$

est surjectif. Ceci résulte du théorème de Lefschetz faible : dans la suite de localisation

$$H^4_{V_\varepsilon}(X_\varepsilon, \mathbf{Z}_\ell(2)) \to H^4(X_\varepsilon, \mathbf{Z}_\ell(2)) \to H^4(X_\varepsilon \setminus V_\varepsilon, \mathbf{Z}_\ell(2))$$

le dernier terme est 0, car la variété $X_\varepsilon \setminus V_\varepsilon$ est affine de dimension 3, et le premier terme est isomorphe à $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ par pureté.

En particulier, étant donné $w \in H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$, on trouve $\beta \in H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ avec

$$w = i_*(\beta).$$

D'autre part, (6) implique

$$\ell^n w = i_*(\gamma)$$

pour un $\gamma \in H^\pi$ convenable et $n \geq 0$. Mais comme $\ell^n w = i_*\ell^n\beta$, on obtient $i_*(\gamma - \ell^n\beta) = 0$, i.e. $\ell^n\beta \in \ker i_* + H^\pi$, d'où le lemme. □

Corollaire 5.2. *Pour $w \in H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$ fixé, le sous-ensemble*

$$H_w := \{v \in \ker i_* : w \in i_*((v + H^\pi)_s)\}$$

de $\ker i_* \subset H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ est un ouvert non vide de $\ker i_*$, stable par multiplication par ℓ.

Démonstration. Le lemme donne $H_w \neq \emptyset$; plus précisément, la preuve du lemme montre que $v_0 := \ell^n\beta - \gamma \in H_w$. Ce choix de n donne $(v_0 + \ell^n \ker i_*) \subset H_w$, car pour $\delta \in \ker i_*$ et $v = v_0 + \ell^n\delta$ on a $i_*(\beta + \delta) = w$ et $\ell^n(\beta + \delta) = v_0 + \ell^n\delta + \gamma = v + \gamma \in (v + H^\pi)$. Enfin, la stabilité de H_w par multiplication par ℓ résulte de la définition. □

Considérons maintenant la forme $\mathbf{Z}_\ell$-bilinéaire sur $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ induite par le cup-produit (i.e. la forme d'intersection) sur la cohomologie de la surface V_ε, et notons $H^\perp$ l'orthogonal de H. On a alors $\ker i_* \subset H^\perp \subset H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$, et l'inclusion $\ker i_* \subset H^\perp$ devient égalité après tensorisation par $\mathbf{Q}_\ell$. En effet, les accouplements de cup-produit satisfont à la compatibilité

$$\alpha \cup i_*(\beta) = i^*(\alpha) \cup \beta$$

pour $\alpha \in H^2(X_\varepsilon, \mathbf{Z}_\ell(2))$ et $\beta \in H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$, et ils sont non dégénérés à coefficients $\mathbf{Q}_\ell$.

Soit $F \subset H^\perp$ un $\mathbf{Z}_\ell$-module libre, complément direct au sous-module de torsion T. Alors $F \cap \ker i_*$ est un sous-module ouvert dans F, et d'indice fini dans $\ker i_*$. Ainsi le corollaire précédent implique :

Corollaire 5.3. *Pour $w \in H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$ fixé le sous-ensemble*

$$S_w := \{v \in \ker i_* \cap F : \, w \in i_*((v + H^\pi)_s)\}$$

est un ouvert non vide de F. □

Remarque 5.4. Quand X est une hypersurface dans $\mathbf{P}^4$, tous les $\mathbf{Z}_\ell$-modules considérés sont sans torsion, et l'on a $\ker i_* = H^\perp = F$, d'où $H_w = S_w$.

Le lemme suivant distille la stratégie de la démonstration de la proposition 3.3.

Lemme 5.5. *Fixons $w \in H^4(X_\varepsilon, \mathbf{Z}_\ell(2))^\pi$. Supposons qu'il existe $g \in \pi$ satisfaisant aux trois hypothèses suivantes :*

(1) *$H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^g$ est saturé dans $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$;*

(2) *g engendre topologiquement le sous-groupe de décomposition $D_{\bar{x}}$ dans π ;*

(3) *g fixe un élément $v \in S_w$.*

Alors il existe $\beta \in H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^{D_{\bar{x}}}$ avec $w = i_*(\beta)$.

Démonstration. Pour un v comme dans (3) on a $(v + H^\pi) \subset H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^g$. Comme $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^g$ est saturé dans $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$, on a de plus $(v + H^\pi)_s \subset H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^g = H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^{D_{\bar{x}}}$. Mais par le corollaire précédent on a $w = i_*(\beta)$ pour un $\beta \in (v + H^\pi)_s$. □

Démonstration de la proposition 3.3. On cherche un $g \in \pi$ satisfaisant aux conditions du lemme.

Par le théorème de Lefschetz difficile, la restriction de la forme d'intersection sur $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ à $H \otimes \mathbf{Q}_\ell$ est non dégénérée (voir [**3**], Lemme 4.1.2). Sa restriction à $H^\perp \otimes \mathbf{Q}_\ell$ est donc non dégénérée. Ecrivant $H^\perp = F \oplus T$ comme ci-dessus, on peut identifier $O(F)$ avec le stabilisateur (point par point) de T, qui est un sous-groupe ouvert d'indice fini de $O(H^\perp)$. Comme l'image de $\bar{\pi}$ par la représentation de monodromie ρ est infinie par construction (Proposition 3.2), un théorème de Deligne ([**3**], Théorème 4.4.1) assure que c'est un sous-groupe ouvert de $O(H^\perp \otimes \mathbf{Q}_\ell)$. A fortiori $\rho(\pi)\cap O(F)$ est ouvert dans $O(F)$. Par le lemme 4.1 il existe un sous-groupe ouvert $G_0 \subset \rho(\pi)\cap O(F)$ tel que $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))^g$ est saturé dans $H^2(V_\varepsilon, \mathbf{Z}_\ell(1))$ pour tout $g \in G_0$.

On applique maintenant la proposition 4.2 à F. Pour ce faire, on doit d'abord vérifier que le rang de F est impair, i.e. que la dimension de $H^\perp \otimes \mathbf{Q}_\ell$ est impair. Ceci résulte de nos hypothèses initiales que la dimension de $H^2(V_\varepsilon, \mathbf{Q}_\ell(1))$ est paire et celle de $H^2(X_\varepsilon, \mathbf{Q}_\ell(1))$ impaire ; on conclut par l'injectivité de $i^* \otimes \mathbf{Q}_\ell$ (voir le début de la preuve du lemme 5.1). La proposition 4.2 (a) fournit donc un ouvert $\mathcal{S}$ de $O(F)$ dont tout élément a un vecteur fixe dans l'ouvert non vide S_w donné par le corollaire 5.3. De plus, la proposition 4.2 (b) assure que $\mathcal{S}$ contient des éléments arbitrairement proches de 1, donc son intersection avec le sous-groupe ouvert G_0 est un ouvert non vide.

L'image inverse de $\mathcal{S}\cap G_0$ dans π est un ouvert dont les éléments satisfont aux propriétés (1) et (3) du lemme ci-dessus. En outre, par définition de la topologie de π elle contient une cosette hV d'un sous-groupe normal ouvert $V \subset \pi$. Appliquant le théorème de densité de Tchebotarev au revêtement galoisien $Z \to \dot{D}$ correspondant à V on obtient un point fermé $z \in Z$ dont le Frobenius associé dans π/V est $\bar{h}$, la classe de hV. Prenons alors un point $\bar{x}$ du revêtement universel profini de $\dot{D}$ au-dessus de z. Le sous-groupe de décomposition $D_{\bar{x}}$ est engendré par un élément g d'image $\bar{h}$ dans π/V. Ceci veut dire que g est un élément de hV, et en tant que tel satisfait aux hypothèses (1) et (3) du lemme. Par construction, il satisfait également à (2). □

Remarque 5.6. Si l'on pouvait choisir V de telle sorte que le conoyau du morphisme $\bar{\pi}/(V\cap\bar{\pi}) \to \pi/V$ soit d'ordre premier à ℓ, alors une variante plus précise de l'argument de Tchebotarev ci-dessus donnerait un point fermé x de degré premier à ℓ. L'existence d'un tel V impliquerait donc la conjecture de Tate à coefficients $\mathbf{Z}_\ell$ pour les 1-cycles sur X (en supposant la conjecture connue pour les surfaces).

6. Conséquences du théorème de Schoen

Nous donnons ici des applications du théorème 1.1 à l'existence de zéro-cycles de degré premier à la caractéristique sur certaines variétés définies sur le corps des fonctions d'une courbe au-dessus de la clôture algébrique d'un corps fini. Il s'agit de deux énoncés apparentés mais non équivalents dont chacun implique le corollaire 1.2.

Théorème 6.1. *Soit $\bar{k}$ la clôture algébrique d'un corps fini k de caractéristique p, et soit $\overline{C}$ une $\bar{k}$-courbe propre lisse connexe, de corps des fonctions $F = \bar{k}(\overline{C})$. Fixons une clôture séparable $\overline{F}$ de F.*

Soit $\overline{X}$ une $\bar{k}$-variété projective, lisse, connexe, admettant un $\bar{k}$-morphisme projectif et dominant $f : \overline{X} \to \overline{C}$ dont la fibre générique $\overline{X}_F$ est lisse et géométriquement intègre.

Supposons :

(i) Le groupe de Picard $\operatorname{Pic} \overline{X}_{\overline{F}}$ est sans torsion.

(ii) Pour tout premier ℓ différent de p, la partie ℓ-primaire du groupe de Brauer $\operatorname{Br} \overline{X} \subset \operatorname{Br} \overline{X}_F$ est finie.

(iii) La F-variété $\overline{X}_F$ a des points dans tous les complétés de F aux points de C.

(iv) Pour tout premier ℓ différent de p, la conjecture de Tate ℓ-adique vaut pour les diviseurs sur les surfaces projectives et lisses sur un corps fini de caractéristique p.

Alors $\overline{X}_F$ possède un zéro-cycle de degré une puissance de p.

Démonstration. Soit $d+1$ la dimension de $\overline{X}$, et donc d la dimension de la F-variété $\overline{X}_F$. Fixons une clôture séparable $\overline{F}$ de F. Considérons la suite d'applications

$$CH^d(\overline{X}) \otimes \mathbf{Z}_\ell \to H^{2d}(\overline{X}, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X}_F, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X}_{\overline{F}}, \mathbf{Z}_\ell(d)) \cong \mathbf{Z}_\ell$$

La flèche $H^{2d}(\overline{X}, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X}_F, \mathbf{Z}_\ell(d))$ est obtenue par passage à la limite sur les applications de restriction

$$H^{2d}(\overline{X}, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X} \times_{\overline{C}} U, \mathbf{Z}_\ell(d))$$

pour U parcourant les ouverts non vides de la courbe $\overline{C}$. La compatibilité de l'application cycle à la restriction à un ouvert ([**12**], §VI, Prop. 9.2) montre que l'application composée ci-dessus se factorise à travers le groupe $CH^d(\overline{X}_F) \otimes \mathbf{Z}_\ell$. Il suffit donc d'établir la surjectivité de l'application composée en question, pour tout ℓ premier à p. On le fait en plusieurs étapes.

Surjectivité de $CH^d(\overline{X}) \otimes \mathbf{Z}_\ell \to H^{2d}(\overline{X}, \mathbf{Z}_\ell(d))$. Il existe un corps fini $k \subset \bar{k}$ et une k-variété X telle que $\overline{X} = X \times_k \bar{k}$. La surjectivité requise résulte du théorème 1.1, pourvu qu'on démontre que tout élément de $H^{2d}(\overline{X}, \mathbf{Z}_\ell(d))$ est fixé par un sous-groupe ouvert de $\mathrm{Gal}\,(\bar{k}|k)$.

Or pour tout $n > 0$ la dualité de Poincaré

$$H^{2d}(\overline{X}, \mu_{\ell^n}^{\otimes d})) \times H^2(\overline{X}, \mu_{\ell^n}) \to \mathbf{Z}/\ell^n\mathbf{Z}$$

est un accouplement parfait équivariant pour l'action de Galois. D'autre part, on a la suite exacte de Kummer

$$0 \to \mathrm{Pic}\,\overline{X}/\ell^n \mathrm{Pic}\,\overline{X} \to H^2(\overline{X}, \mu_{\ell^n}) \to {}_{\ell^n}\mathrm{Br}\,\overline{X} \to 0.$$

Le groupe $\mathrm{Pic}\,\overline{X}$ est extension du groupe de Néron-Severi $NS(\overline{X})$ par le groupe ℓ-divisible $\mathrm{Pic}^0\,\overline{X}$. Le groupe $NS(\overline{X})$ est de type fini, donc $NS(\overline{X})/\ell^n \cong \mathrm{Pic}\,\overline{X}/\ell^n$ est fixé par un sous-groupe ouvert de $\mathrm{Gal}\,(\bar{k}|k)$ indépendant de n. Par l'hypothèse (ii) le groupe ${}_{\ell^n}\mathrm{Br}\,\overline{X}$ a également cette propriété. Il en est donc de même pour le groupe fini $H^{2d}(\overline{X}, \mu_{\ell^n}^{\otimes d})$, et a fortiori pour $H^{2d}(\overline{X}, \mathbf{Z}_\ell(d))$.

Surjectivité de $H^{2d}(\overline{X}, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X}_F, \mathbf{Z}_\ell(d))$. On va démontrer la surjectivité de $H^{2d}(\overline{X}, \mu_{\ell^n}^{\otimes d}) \to H^{2d}(\overline{X}_F, \mu_{\ell^n}^{\otimes d})$ pour tout $n > 0$. Ceci donnera un morphisme surjectif de systèmes projectifs de groupes abéliens finis, d'où une surjection après passage à la limite projective suivant n.

On a la suite exacte de localisation

$$H^{2d}(\overline{X}, \mu_{l^n}^{\otimes d}) \to H^{2d}(\overline{X}_F, \mu_{l^n}^{\otimes d}) \to \bigoplus_{P \in \overline{C}_0} H^{2d+1}_{\overline{X}_P}(\overline{X}, \mu_{l^n}^{\otimes d}),$$

où P parcourt les points fermés de $\overline{C}$. Montrons que chaque flèche $H^{2d}(\overline{X}_F, \mu_{l^n}^{\otimes d}) \to H^{2d+1}_{\overline{X}_P}(\overline{X}, \mu_{l^n}^{\otimes d})$ est nulle. Pour ce faire, par excision on peut se restreindre au-dessus du hensélisé $R = O^h_{\overline{C},P}$ de $\overline{C}$ en P, dont on note L le corps des fractions. On dispose de la suite exacte de localisation

$$H^{2d}(\overline{X}_R, \mu_{l^n}^{\otimes d}) \to H^{2d}(\overline{X}_L, \mu_{l^n}^{\otimes d}) \to H^{2d+1}_{X_P}(\overline{X}_R, \mu_{l^n}^{\otimes d}).$$

Il suffit donc d'établir la surjectivité du morphisme $H^{2d}(\overline{X}_R, \mu_{\ell^n}^{\otimes d}) \to H^{2d}(\overline{X}_L, \mu_{\ell^n}^{\otimes d})$.

Le corps L est de dimension cohomologique 1 (c'est un corps C_1). La suite spectrale de Hochschild–Serre donne donc naissance à la suite exacte courte

$$0 \to H^1(L, H^{2d-1}(\overline{X}_{\overline{L}}, \mu_{\ell^n}^{\otimes d})) \to H^{2d}(\overline{X}_L, \mu_{\ell^n}^{\otimes d}) \to H^{2d}(\overline{X}_{\overline{L}}, \mu_{\ell^n}^{\otimes d})^{\mathrm{Gal}\,(\overline{L}|L)} \to 0.$$

D'autre part, le groupe $H^{2d-1}(\overline{X}_{\overline{L}}, \mu_{\ell^n}^{\otimes d})$ est dual de $H^1(\overline{X}_{\overline{L}}, \mu_{\ell^n}) \cong {}_{\ell^n}\mathrm{Pic}\,\overline{X}_{\overline{L}}$ par dualité de Poincaré. La torsion dans le groupe de Picard ne change pas par extension de corps algébriquement clos, donc ce dernier groupe est nul par l'hypothèse (i). Ceci donne des isomorphismes de modules galoisiens

$$H^{2d}(\overline{X}_L, \mu_{\ell^n}^{\otimes d})) \cong H^{2d}(\overline{X}_{\overline{L}}, \mu_{\ell^n}^{\otimes d}) \cong \mathbf{Z}/\ell^n\mathbf{Z}.$$

L'hypothèse que $\overline{X}_F$ possède des points dans tous les complétés de F est équivalente à la même hypothèse avec les hensélisés, d'où en particulier une section du morphisme $\overline{X}_R \to \mathrm{Spec}\,R$ par propreté de $\overline{X}$. Elle donne naissance à un 1-cycle sur $\overline{X}_R$ dont la classe de cohomologie dans $H^{2d}(\overline{X}_R, \mu_{\ell^n}^{\otimes d})$ s'envoie sur le générateur de $H^{2d}(\overline{X}_L, \mu_{\ell^n}^{\otimes d})$.

Surjectivité de $H^{2d}(\overline{X}_F, \mathbf{Z}_\ell(d)) \to H^{2d}(\overline{X}_{\overline{F}}, \mathbf{Z}_\ell(d))$. Ici encore, il suffit d'établir la surjectivité à niveau fini, car il résulte de l'étape précédente que les groupes $H^{2d}(\overline{X}_F, \mu_{\ell^n}^{\otimes d})$ sont finis. Le corps F est de dimension cohomologique 1, donc le morphisme

$$H^{2d}(\overline{X}_F, \mu_{\ell^n}^{\otimes d}) \to H^{2d}(\overline{X}_{\overline{F}}, \mu_{\ell^n}^{\otimes d})^{\mathrm{Gal}\,(\overline{F}|F)}$$

dans la suite spectrale de Hochschild–Serre est une surjection. On a $H^{2d}(\overline{X}_{\overline{F}}, \mu_{\ell^n}^{\otimes d}) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ avec action triviale de Galois, d'où la surjectivité requise. □

Une conséquence facile du théorème est le corollaire 1.2 de l'introduction.

Démonstration du corollaire 1.2. Comme le degré de la fibre générique du morphisme $\overline{X} \to \overline{C}$ est supposé premier à p, il suffit de montrer que les hypothèses (i) et (ii) du théorème 6.1 sont satisfaites. En d'autre termes, on doit vérifier que pour $\overline{X}_{\overline{F}}$ une intersection complète lisse de dimension ≥ 3 dans $\mathbf{P}^n_{\overline{F}}$ le groupe de Picard est sans torsion ℓ-primaire et la partie ℓ-primaire du groupe de Brauer est finie.

Le premier énoncé résulte du théorème de Noether–Lefschetz ([**7**], exposé XII, corollaire 3.7) : la flèche de restriction $\mathbf{Z} = \mathrm{Pic}\,\mathbf{P}^n_{\overline{F}} \to \mathrm{Pic}\,\overline{X}_{\overline{F}}$ est un isomorphisme. D'autre part, la suite exacte de Kummer en cohomologie étale donne une suite exacte

$$0 \to \mathrm{Pic}\,\overline{X}_{\overline{F}}/\ell\,\mathrm{Pic}\,\overline{X}_{\overline{F}} \to H^2(\overline{X}_{\overline{F}}, \mathbf{Z}/\ell\mathbf{Z}) \to {}_\ell\mathrm{Br}\,\overline{X}_{\overline{F}} \to 0.$$

On vient de voir que le premier terme est isomorphe à $\mathbf{Z}/\ell\mathbf{Z}$. Mais ceci vaut également pour le deuxième, car il est isomorphe à $H^2(\mathbf{P}^n_{\overline{F}}, \mathbf{Z}/\ell\mathbf{Z})$ par le théorème de Lefschetz faible en cohomologie étale. On constate donc avec satisfaction que le dernier terme disparait, ce qui montre que la partie ℓ-primaire de $\mathrm{Br}\,\overline{X}_{\overline{F}}$ est en fait triviale. □

Le corollaire 1.2 peut également se déduire du théorème 6.2 ci-après. Il s'agit d'une variante du théorème 6.1, avec la différence sensible qu'ici on fait une hypothèse au-dessus du corps de fonctions d'une courbe définie sur un corps fini, et non pas sur $\overline{\mathbf{F}}_p$.

Soient donc C une courbe propre, lisse, géométriquement connexe définie sur un corps fini k, et Y une variété lisse sur le corps des fonctions $k(C)$ de C. Pour un point fermé P de C notons K_P le complété de $k(C)$ pour la valuation discrète associée à P. Une famille $\{z_P\}$ de zéro-cycles de degré 1 sur $Y \times_{k(C)} K_P$ pour tout P définit un homomorphisme $\mathrm{Br}\, Y \to \mathbf{Q}/\mathbf{Z}$ donné par $A \mapsto \Sigma_P \mathrm{inv}_P(A[z_P])$. Ici $\mathrm{Br}\, Y$ est le groupe de Brauer de Y, le morphisme inv_P est l'invariant de Hasse du corps local K_P, et $A[z_P]$ est l'évaluation de l'élément $A \in \mathrm{Br}\, Y$ en z_P définie via la fonctorialité contravariante du groupe de Brauer.

On dit qu'il n'y a pas d'obstruction de Brauer–Manin à l'existence d'un zéro-cycle de degré 1 sur Y s'il existe une famille $\{z_P\}$ pour laquelle l'homomorphisme ci-dessus est nul. Notons que cette condition est automatique si la flèche naturelle $\mathrm{Br}\, k(C) \to \mathrm{Br}\, Y$ est surjective.

Théorème 6.2. *Soient k un corps fini de caractéristique p, et $X \to C$ un morphisme projectif et dominant de k-variétés projectives lisses géométriquement connexes, où C est une courbe et la fibre générique $X_{k(C)}$ est lisse et géométriquement intègre.*

Supposons :

(i) Il n'y a pas d'obstruction de Brauer–Manin à l'existence d'un zéro-cycle de degré 1 sur la $k(C)$-variété $X_{k(C)}$.

(ii) La conjecture de Tate sur les diviseurs vaut sur les surfaces projectives et lisses sur un corps fini.

Alors la $\overline{k}(C)$-variété $X \times_{k(C)} \overline{k}(C)$ possède un zéro-cycle de degré une puissance de p.

Démonstration. Soit $d+1$ la dimension de X, et donc d la dimension de la $k(C)$-variété $X_{k(C)}$. Soit $\ell \neq p$ un nombre premier, et soit $\{z_P\}$ une famille de zéro-cycles de degré 1 sur les $X \times_{k(C)} K_P$ pour laquelle l'application $\mathrm{Br}\, X_{k(C)} \to \mathbf{Q}/\mathbf{Z}$ définie ci-dessus est nulle. La proposition 3.1 de [**2**] fournit alors un élément z de $H^{2d}(X, \mathbf{Z}_\ell(d))$ dont la restriction dans le groupe $H^{2d}(X \times_{k(C)} \overline{k(C)}, \mathbf{Z}_\ell(d)) \simeq \mathbf{Z}_\ell$ est égale à $1 \in \mathbf{Z}_\ell$. D'après le théorème 1.1, l'image de z dans $H^{2d}(X \times_k \bar{k}, \mathbf{Z}_\ell(d))$ provient d'un élément Z de $CH_1(X \times_k \bar{k}) \otimes \mathbf{Z}_\ell$. Prenant la trace de Z dans le groupe $CH_0(X \times_{k(C)} \overline{k}(C)) \otimes \mathbf{Z}_\ell$ on voit que le morphisme $CH_0(X \times_{k(C)} \overline{k}(C)) \to \mathbf{Z}/\ell\mathbf{Z}$ induit par le degré est surjectif. □

Remarque 6.3. La démonstration de la proposition 3.1 de [**2**] invoquée ci-dessus utilise des arguments voisins de ceux rencontrés dans la preuve du théorème 6.1. Une différence notable est que, dans la notation de ladite preuve, l'existence de classes de cohomologie de degré 1 sur les schémas X_R implique directement l'existence d'une classe globale de degré 1 sur X grâce à l'hypothèse « arithmétique »de type Brauer–Manin, sans imposer une hypothèse géométrique comme l'hypothèse (i) du théorème 6.1.

Remerciements. Nous remercions chaleureusement Burt Totaro pour nous avoir communiqué la démonstration de 2.1, et Bruno Kahn pour plusieurs discussions édifiantes. Une grande partie de ce travail a été fait lors d'une visite du premier auteur à Budapest dans le cadre du programme intra-européen BUDALGGEO de l'Institut Rényi. Le second auteur remercie également OTKA (projet no. K 61116) pour son soutien.

Bibliographie

[1] M. F. Atiyah, F. Hirzebruch, Analytic cycles on complex manifolds. Topology 1 (1962), 25–45.

[2] J.-L. Colliot-Thélène, Conjectures de type local-global sur l'image des groupes de Chow dans la cohomologie étale, in *Algebraic K-theory* (Seattle, WA, 1997), W. Raskind and C. Weibel, ed., Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999, 1–12.

[3] P. Deligne, La conjecture de Weil, II, Publ. Math. IHES **52** (1980), 137–252.

[4] P. Deligne, N. M. Katz, *Groupes de monodromie en géométrie algébrique* (SGA 7 II), Lecture Notes in Math. **340**, Springer-Verlag, Berlin-New York, 1973.

[5] W. Fulton, *Intersection Theory,* 2nd edition, Springer-Verlag, Berlin–Heidelberg, 1998.

[6] O. Gabber, On space filling curves and Albanese varieties. Geom. Funct. Anal. **11** (2001), 1192–1200.

[7] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (SGA 2), North-Holland, Amsterdam, 1968. Nouvelle édition : Société Mathématique de France, Paris, 2005.

[8] A. Grothendieck et al., *Cohomologie ℓ-adique et fonctions L* (SGA 5), Lecture Notes in Math. **589**, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[9] J. Kollár, Lemma p. 134 in *Classification of irregular varieties*, Lecture Notes in Math. **1515**, Springer-Verlag 1990.

[10] J. S. Milne, The Tate conjecture over finite fields (AIM talk), prépublication `arXiv:0709.3040.`

[11] G. Laumon, Homologie étale, in *Séminaire de Géométrie Analytique*, A. Douady, J-L. Verdier, eds., Astérisque 36–37 (1976), 163–188.

[12] ______, *Étale Cohomology*, Princeton University Press, 1980.

[13] B. Poonen, Bertini theorems over finite fields. Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.

[14] M. Raynaud, Modules projectifs universels, *Invent. Math.* 6 (1968), 1–26.

[15] C. Schoen, An integral analog of the Tate conjecture for one-dimensional cycles on varieties over finite fields. Math. Ann. **311** (1998), no. 3, 493–500.

[16] ______, On the image of the l-adic Abel-Jacobi map for a variety over the algebraic closure of a finite field. J. Amer. Math. Soc. **12** (1999), no. 3, 795–838.

[17] C. Soulé et C. Voisin, Torsion cohomology classes and algebraic cycles on complex projective manifolds, Adv. Math. **198** (2005) 107–127.

[18] T. Szamuely, *Galois Groups and Fundamental Groups*, Cambridge Studies in Advanced Mathematics, vol. 117, Cambridge University Press, 2009.

[19] J. Tate, Conjectures on algebraic cycles in ℓ-adic cohomology, in *Motives*, Proceedings of Symposia in Pure Mathematics, vol. 55/1, U. Jannsen et al., eds., Amer. Math. Soc., Providence, 1994, 71–83.

[20] B. Totaro, Torsion algebraic cycles and complex cobordism, J. Amer. Math. Soc. 10 (1997), 467–493.

[21] ______, The Chow ring of a classifying space, in *Algebraic K-theory (Seattle, 1997),* Proc. Sympos. Pure Math., vol. 67, W. Raskind and C. Weibel, ed., Amer. Math. Soc., Providence, 1999, 249–281.

[22] J-L. Verdier, Indépendance par rapport à ℓ des polynômes caractéristiques des endomorphismes de Frobenius de la cohomologie ℓ-adique, Séminaire Bourbaki, exp. 423, Lecture Notes in Math. **383**, Springer-Verlag, Berlin-New York, 1974, pp. 98–115.

[23] C. Voisin, Some aspects of the Hodge conjecture, Japan J. Math **2** (2007) 261–296.

C.N.R.S., U.M.R. 8628, Université de Paris-Sud, Mathématique, Bâtiment 425, 91405 Orsay, France
E-mail address: jlct@math.u-psud.fr

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13–15, H-1053 Budapest, Hungary
E-mail address: szamuely@renyi.hu

Clay Mathematics Proceedings
Volume 9, 2010

Regulators via iterated integrals (numerical computations)

Herbert Gangl

1. Motivation

Polylogarithms are known to give regulator values of elements in algebraic K-groups of number fields. In the case of the dilogarithm, Bloch found a criterion for elements in the free abelian group $\mathbb{Z}[F]$ for a number field F to produce such elements (cf. [**3**]), and for higher order polylogarithms an analogous criterion was proposed by Zagier which gave rise to his polylogarithm conjecture [**15**]. Beilinson and Deligne (cf. e.g. [**1**]) reinterpreted that criterion in terms of extension classes of mixed Tate motives over F, and realizations of the latter, given in terms of polylogarithms, provide real mixed Hodge–Tate structures; in a preprint [**2**] that unfortunately never made it into print they gave a proof of that reinterpretation, and a corresponding K-theoretic statement was independently shown by de Jeu (cf. [**10**]). As a consequence, given a natural number n, there are criteria for a formal linear combination $\sum_i \lambda_i[z_i]$ in $\mathbb{Z}[F]$ which guarantee that an appropriate single-valued version of the n-logarithm function (e.g. the function P_n in [**15**]) maps the image of $\sum_i \lambda_i[\sigma z_i]$ under a given embedding $\sigma : F \hookrightarrow \mathbb{C}$ to the regulator value of a suitable extension class.

Since polylogarithms can be expressed as iterated integrals, using a single 1-form of the kind $\frac{dt}{t-1}$ as well as further 1-forms of the type $\frac{dt}{t}$ only, one can ask whether more general iterated integrals also produce—possibly new—extension classes. Promising candidates are iterated integrals where we allow at least *two* 1-forms of the kind $\frac{dt}{t-1}$.

In his work on mixed Hodge structures and iterated integrals [**13**], Wojtkowiak generalizes the setup of the paper by Beilinson and Deligne [**1**] on the motivic interpretation of Zagier's conjecture to arbitrary iterated integrals involving only 1-forms with a linear form in the denominator. In this more general framework there arise new conditions on linear combinations in $\mathbb{Z}[F]$ (for a number field F) to represent an extension class in the category MTM$/F$ of mixed Tate motives over a field F (for the setting, see e.g. [**7**]), which then give rise to extensions of mixed Hodge-Tate structures after applying the associated iterated integral.

The aim of this note is to give examples representing such classes and having non-vanishing regulator values. For this we provide elements which satisfy the conditions mentioned above and evaluate them via some single-valued version for the associated iterated integrals. Finally we compare the result with the corresponding value of the Dedekind zeta function of F (the latter is motivated by Borel's theorem on regulators for the algebraic K-groups of F, combined with Zagier's polylogarithm conjecture). The output confirms numerically what the theory predicts, namely for the functions we consider (iterated integrals of type $\int \frac{dt}{t-1} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t} \circ \frac{dt}{t-1}$ of weight ≤ 5) we encounter the same regulator values (rationally) as for the classical polylogarithms, although for the most interesting case which we have investigated (the one of depth 5) the corresponding function is not expressible (cf. [**13**], §10.3) in terms of classical polylogarithms.

We want to emphasize that the tremendously useful software package GP-PARI [**9**] played an integral part for the experiments in this note.

2. Conditions to produce regulator values

2.1. Conditions from Zagier's polylogarithm conjecture. Let F be a number field, with r_1 real and $2r_2$ complex embeddings. Due to a famous result of A. Borel [**4**], we know that, using a suitable regulator map, its higher K-groups of odd order $K_{2n-1}F$ ($n \geq 2$) can be mapped isomorphically, up to torsion, to a lattice of rank r_2 or $r_1 + r_2$, depending on whether n is even or odd; we will refer to such a lattice as a "higher regulator lattice". Bloch (unconditionally in the case of the dilogarithm, [**3**]) and Zagier (conjecturally for the higher cases, [**15**]) gave conditions for an element $\xi = \sum_i \lambda_i [z_i]$ in $\mathbb{Z}[F]$ to provide explicit entries in such a "higher regulator lattice" for F, at least up to a rational multiple. If those conditions are satisfied then any such entry takes the form $\mathcal{L}_{n,\sigma}(\xi) := \sum_i \lambda_i \mathcal{L}_n(\sigma\, z_i)$ for some embedding $\sigma : F \hookrightarrow \mathbb{C}$, where $\mathcal{L}_n(z)$ denotes a single-valued cousin (e.g. one can take the functions denoted by $\widetilde{\mathcal{D}}_n(z)$ or $P_n(z)$ in [**15**]) of the classical n-logarithm $Li_n(z) = \sum_{r\geq 1} z^r/r^n$, analytically continued to $\mathbb{C} \setminus \{0,1\}$ via an iterated integral of the form $-\displaystyle\int_0^z \frac{dt}{t-1} \circ \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n-1 \text{ factors}}$.

For the dilogarithm the corresponding condition can be described using the second exterior power $\bigwedge^2 F^\times$ of the multiplicative group $F^\times$ of F: the condition for ξ alluded to above is simply to lie in $\ker(\beta_2)$ where the map $\beta_2 : \mathbb{Z}[F] \to \bigwedge^2 F^\times$ is given on generators as $[z] \mapsto z \wedge (1-z)$ (and [0], [1] are mapped to 0).

For the higher polylogarithms Zagier gave a similar "main condition", i.e. a good combination has to lie in $\ker(\beta_n)$, with $\beta_n : \mathbb{Z}[F] \to \bigotimes^{n-2} F^\times \otimes \bigwedge^2 F^\times$ which is defined on generators as $[z] \mapsto z^{\otimes(n-2)} \otimes z \wedge (1-z)$ for $n \geq 2$ (due to the symmetry of the situation we can replace $\bigotimes^{n-2}$ by Sym^{n-2}).

In addition to that main condition, though, he had to impose further "side conditions", coming from homomorphisms $\alpha_i : F^\times \to \mathbb{Z}$ ($i \in I$ for some index set I) and more generally from $\bigotimes^j F^\times$ to $\bigotimes^{j-1} F^\times$ ($1 \leq j \leq n-2$) by applying these α_i to any one of the tensor factors on the left (we interpret $\bigotimes^0 F^\times$ on the right as $\mathbb{Z}$). By composing several of the α_i, one can map $\bigotimes^{n-2} F^\times$ to $\bigotimes^{k-2} F^\times$ for any $k = 2, \ldots, n-1$, and it turns out that the resulting (composed) induced homomorphisms $\alpha_{i_1} \circ \cdots \circ \alpha_{i_{n-k}} : \mathbb{Z}[F] \longrightarrow \mathbb{Z}[F]$ sending a generator $[z]$ to $\alpha_{i_1}(z) \cdots \alpha_{i_{n-k}}(z)\,[z]$, map elements from $\ker(\beta_n)$ to $\ker(\beta_k)$ for the corresponding k. Now the side conditions alluded to above amount to imposing that the image of ξ under any of those homomorphisms for any $k = 2, \ldots, n-1$ is not only in $\ker(\beta_k)$, but moreover lies in a certain subgroup of "universal" elements coming from functional equations for $\mathcal{L}_k$ (some single-valued version of the k-logarithm as above). For many examples illustrating the above process we refer to [**15**] and [**6**].

The condition for an element $\xi \in \ker(\beta_n)$ to be a consequence of functional equations for the n-logarithm is very difficult to analyze (already for the simplest case of the dilogarithm there is no algorithm known for that). For this reason, Zagier has given a slightly different—and conjecturally equivalent—formulation where the above "lies in a certain subgroup of universal elements" is replaced by "vanishes under $\mathcal{L}_{k,\sigma}$ for all embeddings σ". This provides an effective check for conjectural triviality of Bloch elements.

Then one builds an inductive procedure: first we take linear combinations in $\ker(\beta_n)$ whose images in $\ker(\beta_2)$ under any composition of $n-2$ homomorphisms α_i as above vanish (numerically) when evaluated by the dilogarithm, then restrict to those linear combinations among them all of whose homomorphic images in $\ker(\beta_3)$ vanish (numerically) under the trilogarithm, and work our way up successively to $k = n-1$. Zagier's conjecture then implies that a combination ξ satisfying all those inductive conditions should map, up to multiplying by a rational number, to a vector $(\mathcal{L}_{n,\sigma})_\sigma$ inside the corresponding higher regulator lattice of F (cf. [**15**]).

In the framework of the paper by Beilinson and Deligne, the above conditions on ξ (in the non-numerical formulation) imply that it represents an extension class of mixed Tate motives in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0),\mathbb{Q}(n))$.

2.2. The conditions in the case $\Lambda_{10001}(z)$. In [**13**], Wojtkowiak has suggested a way to generalize the picture by invoking iterated integrals different from the ones for polylogarithms as candidates for regulator functions. We treat in this note (a single-valued version of) iterated integrals of the form

$$\Lambda_{\varepsilon_1,\dots,\varepsilon_n}(z) = \int_0^z \frac{dt}{t-\varepsilon_1} \circ \frac{dt}{t-\varepsilon_2} \circ \cdots \circ \frac{dt}{t-\varepsilon_n}\,,$$

where $\varepsilon_j \in \{0,1\}$ $(j = 1,\dots,n)$, and in particular the case where $n=5$ and $(\varepsilon_1,\dots,\varepsilon_5) = (1,0,0,0,1)$. The case ε_0 needs to be treated separately, cf. §3.2 below.

In this case, the "side conditions" on the corresponding linear combinations (in analogy to the above set-up) are somewhat more complicated, as there are now more different types of homomorphism for an element in the kernel $\ker(\tilde{\beta}_n)$ where

$$\tilde{\beta}_n : \mathbb{Z}[F] \longrightarrow \bigotimes{}^{n-2} F^\times \otimes \bigwedge{}^2 F^\times$$

is given on generators $(z \neq 0,1)$ as

$$[z] \mapsto (1-z) \otimes z^{\otimes(n-3)} \otimes \big(z \wedge (1-z)\big)\,.$$

As usual, $[0]$ and $[1]$ are mapped to 0.

An example of a new type of homomorphism that we encounter here is obtained if we factor through (from tensors to antisymmetric tensors in the first two factors)

$$\bigwedge{}^2 F^\times \otimes \bigotimes{}^{n-4} F^\times \otimes \bigwedge{}^2 F^\times\,,$$

where $\xi \in \ker(\tilde{\beta}_n)$ is mapped to an element in $\ker(\beta_2) \otimes (F^\times)^{\otimes(n-4)} \otimes \ker(\beta_2)$, which in turn is mapped homomorphically to $\mathbb{R}$, using the following function $\mathcal{L}_2 \otimes \log|\cdot|^{\otimes(n-4)} \otimes \mathcal{L}_2$ (more precisely, we first need to apply individual embeddings $F \hookrightarrow \mathbb{C}$ for each tensor factor).

For the case in question the conditions for a $\xi = \sum \lambda_i [z_i] \in \mathbb{Z}[F]$ to provide a regulator value of an extension class in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0), \mathbb{Q}(5))$ via the single-valued version $D_{10001}(z)$ (defined in §3 below) attached to $\Lambda_{10001}(z)$ are given by Wojtkowiak ([**13**], in §10.3). We try to formulate his result in down-to-earth terms:

PROPOSITION 2.1. *Let F be a number field and let $\sum \lambda_i[z_i] \in \mathbb{Z}[F]$ satisfy the following conditions (M), (X1-3), (Y1-2):*

(M) *The main condition is*

$$\textstyle\sum_i \lambda_i \ (1 - z_i) \wedge z_i \otimes z_i \otimes z_i \otimes (1 - z_i) = 0 \qquad \text{in } \bigwedge^2 F^\times \otimes \big(F^\times\big)^{\otimes 3} ,$$

(X) *Conditions of the first kind. For any embedding $\sigma : F \hookrightarrow \mathbb{C}$ we have*

$$\begin{array}{ll} 1) & \sum_i \lambda_i \ \mathcal{L}_2(\sigma z_i) \otimes z_i \otimes z_i \otimes (1 - z_i) = 0 \qquad \text{in } \mathbb{C} \otimes \big(F^\times\big)^{\otimes 3} , \\ 2) & \sum_i \lambda_i \ \mathcal{L}_3(\sigma z_i) \otimes z_i \otimes (1 - z_i) = 0 \qquad \text{in } \mathbb{C} \otimes \big(F^\times\big)^{\otimes 2} , \\ 3) & \sum_i \lambda_i \ \mathcal{L}_4(\sigma z_i) \otimes (1 - z_i) = 0 \qquad \text{in } \mathbb{C} \otimes F^\times . \end{array}$$

(Y) *Conditions of the second kind. For any embeddings $\sigma, \sigma' : F \hookrightarrow \mathbb{C}$ we have*

$$\begin{array}{ll} 1) & \sum_i \lambda_i \big(z_i \otimes \mathcal{L}_2(\sigma z_i) \mathcal{L}_2(\sigma' z_i)\big) = 0 \quad \text{in } F^\times \otimes \mathbb{R} . \\ 2) & \sum_i \lambda_i \ \mathcal{L}_3(\sigma z_i) \mathcal{L}_2(\sigma' z_i) = 0 . \end{array}$$

Then the combination $\sum \lambda_i[z_i]$ gives an extension of $\mathbb{Q}$ by $(2\pi i)^5\mathbb{Q}$ in the category of mixed Tate motives over F.

We can view Proposition 2.1 as a generalization of Zagier's criteria (for elements in $\mathbb{Z}[F]$ representing elements in the algebraic K-theory of F which are mapped to a lattice under an appropriate single-valued polylogarithm function). In the spirit of Zagier's conjecture we now expect that the vectors $(\sum_i D_{10001}(\sigma\, z_i))_\sigma$ generate a full lattice in $\mathbb{R}^{r_1+r_2}$ when applied to elements satisfying the six criteria from that proposition.

Moreover, combining the above with Borel's Theorems (cf. [**4**]) we expect that the covolume of the (conjecturally) ensuing lattice is rationally, up to well-known factors, given by $\zeta_F(5)$.

CONJECTURE 2.2. *Let F be a number field of discriminant d_F. Then there are elements in $\mathbb{Z}[F]$, satisfying conditions (M), (X1–3) and (Y1–2) whose images under D_{10001} generate a lattice of full rank in $\mathbb{R}^{r_1+r_2}$, of covolume a rational number times $|d_F|^{9/2}\pi^{-5r_2}\zeta_F(5)$.*

The conditions above, with the exception of (Y2), can be rephrased in terms of homomorphisms, e.g. for (X1): $\sum_i \lambda_i \ \mathcal{L}_2(\sigma z_i)\, \alpha(z_i)\, \alpha'(z_i)\, \alpha''(1 - z_i) = 0$ for all homomorphisms $\alpha, \alpha', \alpha'' : F^\times \to \mathbb{Z}$. Note that for simplicity we ignore torsion in $F^\times$ here (our computer program does in fact treat it, but in most of our example it is 2-torsion only, anyway). We will use the statement in this form for the description of the verification in §4 and in our examples in §5 below.

3. One-valued functions attached to $\Lambda_{10\ldots01}(z)$

3.1. The symbolic part of the calculation. In the notation of [**13**], the single-valued functions $D_{10^r1}(z)$ associated to $\Lambda_{10^r1}(z)$ are obtained using the Drinfel'd associator $\Lambda_{\overrightarrow{01}}(z) := \sum_w c_w(z) w$ where w runs through all the words on the alphabet $\{X, Y\}$, and $c_w(z)$ is the corresponding iterated integral $\int_0^z$ over the composition of 1-forms of type $\frac{dt}{t}$ in place of X and $\frac{dt}{t-1}$ in place of Y. More precisely, $D_{10^r1}(z)$ is obtained as the coefficient of YX^rY in a power series associated to a particular automorphism of a certain Lie algebra (this Lie algebra, denoted $L(V)$ in [**13**], §8.0, with $V = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, is obtained as a quotient of (a completion of) the free Lie algebra on two generators, and the automorphism mentioned arises from left multiplication by the above $\Lambda_{\overrightarrow{01}}(z)$, denoted $L_{\Lambda_{\overrightarrow{01}}(z)}$ in loc.cit., as $\frac{1}{2}\log(L_{\Lambda_{\overrightarrow{01}}(z)} \circ \overline{\overline{L}}^{-1}_{\Lambda_{\overrightarrow{01}}(z)})$); for details and notation, we refer to [**13**].

Specifically, $D_{10001}(z)$ is also denoted $\mathcal{D}^{f_5}(z, \overrightarrow{01})$ in [**13**] §10.3. For symbolic manipulations (which were performed in Mathematica) we can restrict ourselves to consider only the terms where at most two Y's and three X's appear.
Eventually, by taking appropriate real part $\Re$ or imaginary part $\Im$ and interpreting $\Lambda_0(z)$ as $\log(z)$, one finds the following one-valued function for $\Lambda_{101}(z)$:

$$\begin{aligned} D_{101}(z) = \;& \Re\Lambda_{101}(z) + \Im\Lambda_1(z)\Im\Lambda_{01}(z) - \Re\Lambda_{10}(z)\Re\Lambda_1(z) \\ & - \Im\Lambda_1(z)\Im\Lambda_0(z)\Re\Lambda_1(z) + \tfrac{1}{3}\Re\Lambda_1(z)\Re\Lambda_0(z)\Re\Lambda_1(z)\,. \end{aligned}$$

Similarly, the one for $\Lambda_{1001}(z)$ is given by:

$$\begin{aligned} D_{1001}(z) \;=\;& \Im\Lambda_{1001}(z) - \Im\Lambda_1(z)\Re\Lambda_{001}(z) - \Re\Lambda_{10}(z)\Im\Lambda_{01}(z) - \Im\Lambda_{100}(z)\Re\Lambda_1(z) \\ & -\Im\Lambda_1(z)\Im\Lambda_0(z)\Im\Lambda_{01}(z) + \frac{1}{3}\Re\Lambda_1(z)\Re\Lambda_0(z)\Im\Lambda_{01}(z) \\ & + \frac{1}{3}\Im\Lambda_{10}(z)\Re\Lambda_0(z)\Re\Lambda_1(z) + \Re\Lambda_{10}(z)\Im\Lambda_0(z)\Re\Lambda_1(z) \\ & - \frac{1}{3}\Re\Lambda_1(z)\Im\Lambda_0(z)\Re\Lambda_0(z)\Re\Lambda_1(z) + \frac{1}{6}\Im\Lambda_1(z)\Re\Lambda_0(z)\Re\Lambda_0(z)\Re\Lambda_1(z) \\ & + \frac{1}{2}\Im\Lambda_1(z)\Im\Lambda_0(z)\Im\Lambda_0(z)\Re\Lambda_1(z)\,. \end{aligned}$$

In fact, this function turns out to be identically zero (as had been predicted by Wojtkowiak).

In the above notation we have already tried to indicate some combinatorial structure of the terms involved. For any $r = 1, \ldots, n$ we partition the string $1\,0^{n-2}\,1 := 1\underbrace{0\cdots0}_{n-2}1$ into r substrings $B_1, \ldots, B_r$, which will be referred to as *blocks*, and we attach to each block B_j either the imaginary part or the real part of the associated functions $\Lambda_{B_j}(z)$. The blocks are separated by vertical bars, so e.g. the partition $1\,0|0|1$ has three blocks $1\,0$, 0 and 1. We introduce the following shorthand: we write $\overline{B}$ and B for $i \cdot \Im\Lambda_B(z)$ and $\Re\Lambda_B(z)$, respectively, and separate blocks by a $|$.

Then e.g. the function $D_{101}(z)$ above is represented as

$$[101] - [\overline{1}|\overline{01}] - [10|1] + [\overline{1}|\overline{0}|1] + \frac{1}{3}[1|0|1]\,.$$

So a priori we find 2^r terms for each partition, many of which come with a zero coefficient, though. We find in particular:

- the coefficient of any partition containing blocks of the form 0^k with $k > 1$ vanishes;
- if the number of "imaginary" blocks of a given partition and n have the same parity then the term has zero coefficient;

- if the final block in a partition does not have the form $\overline{0^{2k+1}1}$ or $0^{2k}1$ for some $k \geq 0$ (in the shorthand defined above), then the corresponding term has coefficient zero.

PROPOSITION 3.1. *In the above shorthand, the function $D_{10001}(z)$ is written as*

$$\begin{aligned}
&[10001] - [\overline{1}|\overline{0001}] - [10|001] - [\overline{100}|\overline{01}] - [1000|1] \\
&+[\overline{1}|\overline{0}|001] + \tfrac{1}{3}[1|0|001] + \tfrac{1}{3}[\overline{10}|0|\overline{01}] + [10|\overline{0}|\overline{01}] + [\overline{100}|\overline{0}|1] + \tfrac{1}{3}[100|0|1] \\
&- \tfrac{1}{3}[1|\overline{0}|0|\overline{01}] + \tfrac{1}{6}[\overline{1}|0|0|\overline{01}] - \tfrac{1}{2}[\overline{1}|\overline{0}|\overline{0}|\overline{01}] \\
&- \tfrac{1}{3}[\overline{10}|\overline{0}|0|1] - \tfrac{1}{2}[10|\overline{0}|\overline{0}|1] + \tfrac{1}{6}[10|0|0|1] \\
&+ \tfrac{1}{6}[\overline{1}|\overline{0}|\overline{0}|\overline{0}|1] - \tfrac{1}{6}[\overline{1}|\overline{0}|0|0|1] + \tfrac{1}{6}[1|\overline{0}|\overline{0}|0|1] - \tfrac{7}{90}[1|0|0|0|1]\,.
\end{aligned}$$

More generally, we expect the following single-valued functions as the respective coefficient of $YX^{n-2}Y$ in the above power series (we have checked this symbolically up to $n = 12$):

$$\begin{aligned}
i^{\varepsilon}\, D_{10^{n-2}1}(z) \quad &= \quad \Re_n \Lambda_{10^{n-2}1}(z) - \\
- \sum_{r,s\geq 0} \sum_{1\leq b\leq n-1} (-1)^r \frac{\alpha_s}{r!} &\Big[\Re_{r+b+1-n}(1\,0^{n-r-s-b-1})|\underbrace{\overline{0}|\ldots|\overline{0}}_{r}|\underbrace{0|\ldots|0}_{s}|\Re_b(0^{b-1}1)\Big]
\end{aligned}$$

with α_s denoting the coefficient of x^s in the power series $\frac{x}{\sinh(x)} + \big(\frac{x}{\sinh(x)}\big)' = 1 - \frac{1}{3}x - \frac{1}{6}x^2 + \frac{7}{90}x^3 + \frac{7}{360}x^4 - \frac{31}{2520}x^5 \pm \ldots$, and where $\Re_j = \Re$ or $= i\,\Im$, depending on whether j is odd or even, and $\varepsilon = 0$ or 1 depending on whether n is odd or even (and the first block requires $n > r + s + b$, of course).

REMARK 3.2. *Somewhat different candidates for single-valued versions of the above functions (and many more) have in the meantime been given by F. Brown in* [**5**].

3.2. The computational aspect of Λ_{10001}. Note that $\Lambda_{\varepsilon_1\ldots\varepsilon_r}(z)$ ($\varepsilon_i \in \{0,1\}$, for $z \neq 0$) does not converge if $\varepsilon_1 = 0$. Therefore, in order to arrive at some computable (i.e. programmable) object, we treat $\Lambda_{0\ldots 0}(z)$ as $\frac{1}{k!}\log^k(z)$ and produce the functions $\Lambda_{0\ldots 01}$ from (the convergent) $\Lambda_{10\ldots 0}$ via shuffle relations ("shuffle regularization"), and the latter ones are (up to sign) standard polylogarithms. At least inside the unit circle one has a rapidly convergent power series for computing $\Lambda_{10\ldots 01}$, while outside the unit circle it can be given using an inversion relation. For the latter functions, such inversion relations have been provided by Wojtkowiak (cf. [**13**], §9 and [**14**], §10 (3)).
The main problems of evaluating the function arise close to the unit circle itself. A procedure given by Cohen, Lewin and Zagier [**6**] in the case of classical polylogarithms can be adapted to our situation, though: develop $\Lambda_{10\ldots 01}(e^x)$ in a power series in x, of which most of the coefficients are expressed in terms of ζ-values (possibly evaluated at negative integers). The resulting expansion turns out to converge reasonably fast for an annulus $1/\rho < |x| < \rho$ for $\rho = 3$, say.

4. Description of the successive steps in the program

Let F be a number field of discriminant d_F and choose a set S of primes in $\mathbb{Q}$. We take a system $\mathcal{F}$ of fundamental S-units in F, i.e. a basis $\mathcal{F}_0$ for the S-units in $F^\times/\text{tors}$, together with a root ζ of 1 generating the torsion in $F^\times$, as provided by GP/PARI [**9**]. Note that for simplicity we ignore this torsion in the following (the actual program actually does respect the torsion, typically resulting in multiplying any linear combination by a factor of the order of ζ), hence we can disregard ζ for the following. Any S-unit has a unique representation in terms of $\mathcal{F}$. For each element $f_\nu \in \mathcal{F}_0$ we get a natural homomorphism $\alpha_\nu : \langle\mathcal{F}\rangle \to \mathbb{Z}$ picking the exponent of f_ν.

Moreover, we number the embeddings by first listing the pairs of complex conjugate embeddings $\sigma_1, \overline{\sigma}_1, \ldots, \sigma_{r_2}, \overline{\sigma}_{r_2}$ and then appending successively the r_1 real embeddings. For the conditions involving $\mathcal{L}_2$ or $\mathcal{L}_4$ we only need to consider $\sigma_1, \ldots, \sigma_{r_2}$, while for conditions involving $\mathcal{L}_3$ we invoke both real and complex embeddings $\sigma_1, \ldots, \sigma_{r_1+r_2}$ (i.e. for each pair of complex embeddings we choose one).

4.1. Strategy for invoking the conditions. Initialize the procedure by searching for "many" exceptional S-units z_ν ($\nu \in \mathcal{V}$, an index set of size $N := |\mathcal{V}|$), i.e. S-units $z_\nu \in F^\times$ such that $1 - z_\nu$ is also an S-unit.

(M) For any $\nu \in \mathcal{V}$, represent z_ν and $1 - z_\nu$ in terms of $\mathcal{F}$, and associate to it the vector of integer entries arising from the different choices for the homomorphisms $\alpha_* : \langle\mathcal{F}\rangle \to \mathbb{Z}$ as above:

$$\alpha_i(z_\nu)\,\alpha_j(z_\nu)\,\alpha_k(1-z_\nu)\big[\alpha_l(z_\nu)\,\alpha_m(1-z_\nu) - \alpha_l(1-z_\nu)\,\alpha_m(z_\nu)\big]\,, \tag{1}$$

$1 \le i \le j \le s$, $1 \le k \le s$, $1 \le l < m \le s$. This provides a row m_ν of some integer matrix M_0 and the main obstruction (M) for giving an element $\sum_\nu \lambda_\nu[z_\nu]$ as in Proposition 2.1 is that the corresponding vector $(\lambda_\nu)_\nu$ has to lie in the kernel of M_0. Find the (integer) kernel I_0 of M_0, these form the first conditions on the linear combination of the rows (corresponding to the conditions on the z_ν).

(X1): Invoke further conditions using dilogarithmic conditions by computing the matrix M_2 of size $N \times \binom{s+1}{2}\binom{s}{2}s$

$$(M_2)_{\nu,\iota} = \big(\alpha_i(z_\nu)\,\alpha_j(z_\nu)\,\alpha_k(1-z_\nu)\,\mathcal{L}_2(\sigma_\ell z_\nu)\big)_{\nu,\iota}\,, \tag{2}$$

$1 \le i \le j \le s$, $1 \le k \le s$, $\ell = 1, \ldots, r_2$, where $\sigma_\ell : F \hookrightarrow \mathbb{C}$. Numerically, the columns of the matrix $I_0 \cdot M_2^t$ span a lattice, of covolume a rational number times $|d_F|^{3/2}\pi^{-2(r_1+r_2)}\zeta_F(2)$, and we compute its integer kernel K_0. Then $I_1 := K_0 \cdot I_0$ will annihilate both M_0 and M_2.

(Y1): Then take the "products of dilogs"

$$(M_{2,2})_{\nu,k} = \big(\alpha_i(z_\nu)\,\mathcal{L}_2(\sigma_\ell z_\nu)\,\mathcal{L}_2(\sigma_m z_\nu)\big)_{\nu,k}\,, \tag{3}$$

$1 \le i \le s$, $1 \le \ell \le m \le r_2$, into a matrix $M_{2,2}$ (the index k runs through the $\binom{r_1+r_2}{2}$ pairs (ℓ, m)). Similarly, $I_1 \cdot M_{2,2}$ gives rise (numerically) to a lattice, more specifically of covolume a rational number times $\big(|d_F|^{3/2}\pi^{-2(r_1+r_2)}\zeta_F(2)\big)^2$, and we can find its (integer) kernel K_1. We define $I_2 := K_1 \cdot I_1$, which annihilates M_0, M_2 and $M_{2,2}$.

(X2): The next step consists in taking the trilogarithmic conditions

$$(M_3)_{\nu,\ell} = \big(\alpha_i(z_\nu)\,\alpha_j(1-z_\nu)\,\mathcal{L}_3(\sigma_\ell z_\nu)\big)_{\nu,\ell}\,, \tag{4}$$

$1 \le i, j \le s$, this time $\ell = 1, \ldots, r_1 + r_2$, generating a matrix M_3, and we find the kernel K_2 of the matrix $I_2 \cdot M_3$ (whose rows give rise to a lattice of covolume a rational number times $|d_F|^{5/2}\pi^{-3r_2}\zeta_F(3)\big)$ and then form $I_3 := K_2 \cdot I_2$, which annihilates M_0, M_2, $M_{2,2}$ and M_3.

(Y2): Compute further the expressions

$$(M_{3,2})_{\nu,k} = \big(\mathcal{L}_2(\sigma_\ell z_\nu)\,\mathcal{L}_3(\sigma_m z_\nu)\big)_{\nu,k}\,, \tag{5}$$

$1 \le \ell \le r_2$, $1 \le m \le r_1 + r_2$ (the index k runs through the $r_1 r_2$ pairs (ℓ, m)). The resulting matrix $M_{3,2}$ should give an integer kernel K_3 for the lattice generated by the columns of $I_3 \cdot M_{3,2}$. The matrix $I_4 := K_3 \cdot I_3$ annihilates M_0, M_2, $M_{2,2}$, M_3 and $M_{3,2}$.

(X3): As a final preliminary step, consider the expressions in

$$(M_4)_{\nu,\ell} = \big(\alpha_i(z_\nu)\,\mathcal{L}_4(\sigma_l z_\nu)\big)_{\nu,\ell}\,, \tag{6}$$

$1 \le i \le s$, $1 \le l \le r_2$, into a matrix M_4 and compute the integral kernel K_4 of $I_4 \cdot M_4$ and put $I_5 := K_4 \cdot I_4$ which annihilates all of the above as well as M_4.

$(D_{10^3 1})$ Now everything is in place to apply the function D_{10001}, namely we consider

$$(M_5)_{\nu,\ell} = \big(D_{10001}(\sigma_\ell z_\nu)\big)_{\nu,\ell}$$

$1 \le \ell \le r_1 + r_2$, the entries of which form the matrix M_5.

$\zeta_F(5)$ We finally find that the columns of $R_5 := I_5 \cdot M_5$ generate (numerically) a lattice; moreover, we can determine its covolume and check whether it is a rational number (of small height) times

$$\frac{|d_F|^{9/2}}{\pi^{5r_2}} \cdot \zeta_F(5)\,.$$

In our examples, the corresponding ratio indeed looks rational, at least within the (40-digit) precision typically used.

5. A detailed example for a cubic field

Let $F = \mathbb{Q}(\theta)$ with $\theta^3 - \theta - 1 = 0$ ($d_F = -23$, signature $(r_1, r_2) = (1,1)$). Then θ is a generator for the group of units modulo torsion; the latter is generated by -1 and will be ignored in the following. We put $\mathcal{F} = \mathcal{F}_0 = \{\theta\}$.

Notation: In the following we indicate by $\doteq$ an "approximate equality", i.e. an equality which holds up to a given precision. Typically the computer performed the calculations up to 100 digits precision, except for the calculations for D_{10001} where we typically used 40 digits.

0. *Finding sufficiently many S-units.* The exceptional S-units z_ν that the computer found are the following twelve ones:

$$\begin{aligned}(z_\nu)_{\nu=1,\dots,12} \quad = \quad &\big(-\theta^2+2,\ -\theta+1,\ \theta^2-\theta,\ -\theta^2+\theta+1,\ \theta^2-1,\\ &-\theta^2+1,\ \theta,\ -\theta,\ \theta^2,\ \theta+1,\ -\theta^2-\theta,\ \theta^2+\theta+1\big)\,.\end{aligned}$$

We encounter the very special case (cf. Lewin's ladders as explained in [**15**], §9C) that there is essentially only one homomorphism $\langle\mathcal{F}\rangle \to \mathbb{Z}$

involved, namely the one with respect to the fundamental unit θ. The respective exponents are given as

$$\begin{aligned}\big(\alpha(z_\nu)\big)_\nu &= (-5,-4,-3,-2,-1,-1,1,1,2,3,4,5) \ \text{ and}\\ \big(\alpha(1-z_\nu)\big)_\nu &= (-1,1,-2,-3,-5,2,-4,3,-1,1,5,4).\end{aligned}$$

(M) This condition (the only one which is purely algebraic, as it does not involve any polylogarithms) is trivially satisfied, since $\langle\mathcal{F}\rangle$ has rank 1, hence I_0 is the 12×12 identity matrix.

(X2) *Dilogarithm conditions.* The dilogarithm values are (only the non-real embedding σ_2 for each z_ν has to be considered since $r_2 = 1$, so we suppress it from the notation)

$$M_2 := \big(\alpha(1-z_\nu)\,\alpha(z_\nu)\,\alpha(z_\nu)\,\mathcal{L}_2(z_\nu)\big)_\nu \doteq$$

$$\doteq \begin{pmatrix} -11.78384203\\ -7.541658902\\ 16.96873253\\ -11.31248835\\ 2.356768406\\ 1.885414725\\ -1.885414725\\ -2.828122088\\ 3.770829451\\ 8.484366265\\ 37.70829451\\ -47.13536814 \end{pmatrix} \doteq 1.885414725 \begin{pmatrix} -6.250000000\\ -4.000000000\\ 9.000000000\\ -6.000000000\\ 1.250000000\\ 1.000000000\\ -1.000000000\\ -1.500000000\\ 2.000000000\\ 4.500000000\\ 20.00000000\\ -25.00000000 \end{pmatrix}$$

and the covolume $c_2 \doteq 1.885414725/4 \doteq 0.4713536814$ of the associated lattice is found to be[1]

$$c_2 \doteq \frac{3}{8}\,\frac{|d_F|^{3/2}}{\pi^{2(r_1+r_2)}}\zeta_F(2)\,.$$

We can find the integer kernel K_0 of M_2 and put $I_1 = K_0 \cdot I_0 = K_0$. The latter is given in terms of the transpose of the following matrix I_1^t the entries of which are very close (within the given precision) to integers, therefore we round them off and are left with

[1]Note that we tend to display several zeros after the decimal point, which is more of a psychological feature (it reflects somewhat the satisfaction of the programmer verifying that numbers match in a computer calculation).

$$I_1^t = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $I_1 \cdot M_2 \doteq 0$.

(Y1) *Products of dilogarithms.* After those preparations, we can expect condition (Y1) to give us a lattice from the matrix with entries products of dilogarithms

$$M_{2,2} := \big(\alpha(z_\nu)\,\mathcal{L}_2(z_\nu)\,\mathcal{L}_2(z_\nu)\big)_\nu \doteq$$

$$\doteq \begin{pmatrix} -1.110871464 \\ -0.8886971718 \\ -2.666091515 \\ -1.777394343 \\ -0.2221742929 \\ -0.8886971718 \\ 0.2221742929 \\ 0.8886971718 \\ 1.777394343 \\ 2.666091515 \\ 0.8886971718 \\ 1.110871464 \end{pmatrix} \doteq -\,0.2221742929 \begin{pmatrix} 5.000000000 \\ 4.000000000 \\ 12.00000000 \\ 8.000000000 \\ 1.000000000 \\ 4.000000000 \\ -1.000000000 \\ -4.000000000 \\ -8.000000000 \\ -12.00000000 \\ -4.000000000 \\ -5.000000000 \end{pmatrix}$$

with the covolume $c_{2,2} \doteq 0.2221742929$ being equal to c_2^2.

Note that $I_1 \cdot M_{2,2} \neq 0$, which shows that both conditions (X2) and (Y1) are needed. Instead, $I_1 \cdot M_{2,2}$ gives an 11×1-matrix with commensurable entries (obvious since $M_{2,2}$ already does), and one can give an integer kernel K_1 of it and multiply the result by I_1. Call the resulting matrix $I_2 = K_1 \cdot I_1$; it annihilates both M_2 and $M_{2,2}$. Its transpose is given by

$$I_2^t = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 & 3 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & -3 \\ 0 & -3 & 3 & 0 & 3 & 0 & 3 & -3 & 3 & 3 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 10 & -4 & -3 & 12 & -25 & 11 & -27 & 0 & 0 & 5 \\ 16 & -4 & -6 & 9 & -28 & 8 & -24 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(X2) *Trilogarithmic conditions.* The next step is to satisfy condition (X2). To this end, we compute the vector of trilogarithms multiplied by the corresponding homomorphisms (this time there are $r_1 + r_2 = 2$ embeddings σ_i of F into $\mathbb{C}$ to consider, which is reflected in the notation):

$$M_3 = \big(\alpha(1-z_\nu)\alpha(z_\nu)\,\mathcal{L}_3(\sigma_\ell\, z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 4.034886241 & 4.677163445 \\ 3.081869446 & -3.987905178 \\ 6.013387652 & 2.498989698 \\ 6.553788692 & 0.2649913900 \\ 5.827735082 & -3.651760402 \\ 1.784944975 & -1.482786776 \\ -4.662188065 & 2.921408322 \\ -2.677417462 & 2.224180165 \\ -2.184596230 & -0.08833046334 \\ 3.006693826 & 1.249494849 \\ -15.40934723 & 19.93952589 \\ 16.13954496 & 18.70865378 \end{pmatrix}.$$

We find numerically that the rows of $I_2 \cdot M_3$ generate a lattice:

$$I_2 \cdot M_3 \doteq \begin{pmatrix} -21.09978266 & 10.54989133 \\ 2.549001106 & -6.082728165 \\ 4.230598524 & -5.120441520 \\ -9.198456008 & 45.46916271 \\ 44.28825961 & -16.13384529 \\ -2.476557456 & 1.839307179 \\ 35.98477232 & -7.173874034 \\ -12.23789998 & 0.7096939263 \\ 5.669220069 & -5.839752292 \\ -21.24466996 & 19.03673330 \end{pmatrix}.$$

The lattice property becomes more apparent if we multiply by the inverse matrix of the first 2×2-submatrix of the above and multiply by the common denominator 88, the result being:

$$\begin{pmatrix} 88.00000000 & 0.00000000 \\ 0.00000000 & 88.00000000 \\ -11.00000000 & 55.00000000 \\ -52.00000000 & -748.0000000 \\ -198.0000000 & -110.0000000 \\ 9.000000000 & -11.00000000 \\ -174.0000000 & -198.0000000 \\ 63.00000000 & 99.00000000 \\ -17.00000000 & 55.00000000 \\ 70.00000000 & -154.0000000 \end{pmatrix}.$$

The determinant c_3 of the above 2×2-submatrix is $\doteq 101.452557625925282$ and it is expressed in terms of Dedekind zeta values via

$$c_3 \doteq \frac{11}{9} \cdot \frac{|d_F|^{5/2}}{\pi^{3r_2}} \zeta_F(3)\,.$$

Again, we can find some matrix K_2 which represents the integer kernel of the above, and $I_3 = K_2 \cdot I_2$ annihilates M_2, $M_{2,2}$ and M_3, where its transpose has the form

$$I_3^t = \begin{pmatrix} 2 & 0 & 1 & 3 & 0 & -1 & 0 & -3 \\ -1 & 5 & -1 & -2 & 3 & 0 & 0 & -3 \\ 0 & -1 & 2 & 0 & -2 & 2 & 1 & 0 \\ -3 & -1 & -2 & -4 & 5 & 2 & 0 & 3 \\ -2 & 0 & 3 & 1 & 0 & -3 & 0 & -1 \\ -7 & 3 & -2 & 7 & -10 & 3 & -2 & -7 \\ -8 & 0 & 4 & 4 & -7 & 4 & -4 & -4 \\ -2 & 2 & 2 & 0 & 0 & 0 & 4 & 0 \\ -1 & -3 & 1 & -1 & -1 & 4 & 2 & -2 \\ -2 & 2 & -1 & 1 & 0 & 1 & -2 & -1 \\ 0 & 1 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

(Y2) *Products of dilogarithm and trilogarithm.* I_3 does not annihilate the expressions arising from condition (Y2) which are displayed in the next equation

$$M_{3,2} = \big(\mathcal{L}_2(\sigma_1\, z_\nu)\, \mathcal{L}_3(\sigma_\ell\, z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 0.3803716968 & 0.4409196416 \\ 0.3631626273 & -0.4699284467 \\ -0.9448108025 & -0.3926359979 \\ 1.029717475 & 0.04163488907 \\ -0.5493848770 & 0.3442541418 \\ -0.8413403851 & 0.6989170060 \\ 0.5493848770 & -0.3442541418 \\ 0.8413403851 & -0.6989170060 \\ -1.029717475 & -0.04163488907 \\ 0.9448108025 & 0.3926359979 \\ -0.3631626273 & 0.4699284467 \\ -0.3803716968 & -0.4409196416 \end{pmatrix}$$

(note that for $\mathcal{L}_2$ we only use the first embedding) but it gives a lattice generated by the rows of

$$I_3 \cdot M_{3,2} \doteq \begin{pmatrix} -2.641083126 & -0.9458342228 \\ 5.505177483 & 0.08038099116 \\ -1.628466720 & -4.001815185 \\ -5.607617226 & 5.920075319 \\ 13.37917049 & -3.573318541 \\ -1.699781839 & -1.699781839 \\ -2.043364556 & -4.077663240 \\ 6.214401668 & -3.957091754 \end{pmatrix},$$

the span of which can be recognized with the naked eye using the same procedure as above (this time we multiply by 26):

$$\begin{pmatrix} 26.00000000 & 0.00000000 \\ 0.00000000 & 26.00000000 \\ 114.0000000 & 47.00000000 \\ -172.0000000 & -109.0000000 \\ 108.0000000 & 115.0000000 \\ 48.00000000 & 15.00000000 \\ 116.0000000 & 46.00000000 \\ 116.0000000 & 85.00000000 \end{pmatrix}.$$

We take the integer kernel K_4 of $I_3 \cdot M_{3,2}$ and keep $I_4 = K_3 \cdot I_3$ which annihilates all the above (M_0, M_2, $M_{2,2}$, M_3) as well as $M_{3,2}$. We give its transpose as

$$I_4^t = \begin{pmatrix} -202 & 422 & -216 & -122 & -232 & -310 \\ -147 & 321 & -389 & -27 & -114 & -387 \\ 99 & -109 & 63 & 67 & 72 & 85 \\ 337 & -729 & 569 & 211 & 394 & 511 \\ 306 & -318 & 216 & 18 & 232 & 206 \\ 605 & -695 & 151 & 369 & 622 & 375 \\ 1016 & -1272 & 682 & 488 & 824 & 824 \\ 186 & -126 & -14 & 66 & 244 & 62 \\ 281 & -525 & 427 & 197 & 306 & 319 \\ 108 & -100 & -14 & 92 & 88 & 36 \\ -73 & 83 & -63 & -41 & -46 & -85 \\ 0 & 0 & -26 & 0 & 0 & 0 \end{pmatrix}.$$

This time the covolume $c_{2,3} \doteq 0.1921035533$ of $I_3 \cdot M_{3,2}$ is a rational multiple of $c_2^2 \cdot c_3$. (From the conjectural framework, we expect this covolume to be a rational number times $c_2^{r_1+r_2} c_3^{r_2}$.)

(X3) *Tetralogarithmic conditions.* Continuing in this way, we determine the tetralogarithmic expressions

$$M_4 \;=\; \big(\alpha(z_\nu)\,\mathcal{L}_4(z_\nu)\big)_\nu \;\doteq\; \begin{pmatrix} -0.3814538586 \\ -0.3456381997 \\ 1.759643788 \\ -2.947003507 \\ 2.965401097 \\ 1.431744064 \\ -2.372320878 \\ -2.147616097 \\ 0.9823345025 \\ 0.8798218942 \\ 1.728190998 \\ -1.525815434 \end{pmatrix}.$$

Multiplying I_4 by M_4, we obtain a single column

$$I_4 \cdot M_4 \;\doteq\; \begin{pmatrix} -57.01118640 \\ 99.01942902 \\ -67.01314893 \\ -37.00726135 \\ -58.01138266 \\ -71.01393394 \end{pmatrix} \doteq 1.0001962525 \begin{pmatrix} -57.00000000 \\ 99.00000000 \\ -67.00000000 \\ -37.00000000 \\ -58.00000000 \\ -71.00000000 \end{pmatrix}.$$

The covolume $c_4 \doteq 1.0001962525$ is found to be

$$c_4 \doteq \frac{135}{832} \cdot \frac{|d_F|^{7/2}}{\pi^{4(r_1+r_2)}} \cdot \zeta_F(4)\,.$$

The corresponding kernel I_5 (simultaneously satisfying all the conditions (X1-3) and (Y1-2)) can be written as the transpose of

$$I_5^t = \begin{pmatrix} -1 & -8 & 5 & 0 & -3 \\ 7 & 5 & 14 & 0 & -28 \\ 2 & -1 & 0 & 0 & -4 \\ -7 & -2 & 1 & 1 & 12 \\ -3 & 0 & -9 & -16 & -9 \\ 20 & 16 & 6 & -16 & -50 \\ 7 & 2 & -7 & -16 & -39 \\ 10 & 8 & 2 & -16 & -22 \\ -5 & -7 & 8 & 1 & 0 \\ 5 & 2 & 3 & 0 & -13 \\ -1 & 1 & 2 & 0 & 0 \\ 1 & 2 & -1 & 0 & -1 \end{pmatrix}.$$

(D_{10^31}) In the last step we compute the values under the "exotic Bloch-Wigner function" D_{10001} as given above, and arrive at

$$M_5 := \big(D_{10001}(\sigma_\ell\, z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 0.3798787865 & 4.822987148 \\ 0.3906525699 & 4.688219297 \\ 0.8300465454 & 3.469259751 \\ 1.231915999 & 2.696057038 \\ 1.859803935 & 1.458356504 \\ 0.8801812017 & 3.243252430 \\ 4.330778718 & 1.182477899 \\ 1.346885380 & 2.427729773 \\ 4.871430134 & 1.713967038 \\ 5.141619571 & 1.645178851 \\ 2.153722031 & 1.301307423 \\ 5.247265750 & 1.073768725 \end{pmatrix}.$$

The product of I_5 and M_5 yields the matrix

$$I_5 \cdot M_5 \doteq \begin{pmatrix} 55.64437827 & 108.5338106 \\ 17.97137546 & 44.41609166 \\ 22.97809554 & 115.5346022 \\ -128.5790416 & -128.5790416 \\ -331.9825665 & -424.5390732 \end{pmatrix}$$

which can be decomposed as

$$\doteq \begin{pmatrix} 38.00000000 & 0.00000000 \\ 0.00000000 & 38.00000000 \\ -77.00000000 & 287.0000000 \\ -248.0000000 & 496.0000000 \\ -519.0000000 & 905.0000000 \end{pmatrix} \begin{pmatrix} 1.464325743 & 2.856152912 \\ 0.4729309331 & 1.168844517 \end{pmatrix}.$$

$\zeta_F(5)$: *The special value.* We can compute the covolume $R_{10001}(F)$ of $I_5 \cdot M_5$ as

$$R_{10001}(F) \doteq 13.71063010$$

(which is 38 times 0.3608060553, the latter number being the determinant of the 2×2-matrix on the right). Guided by Borel's theorems, we compare this number $R_{10001}(F)$ with the special value

$$\zeta_F(5) \doteq 1.00041799247384495$$

and we observe (denoting d_F the discriminant of F and r_2 the number of complex embeddings which are in our case -23 and 1, respectively):

Experimental Evidence: For the field F of degree 3 over $\mathbb{Q}$ and of discriminant $d_F = -23$, the five columns $(a_{j,\nu})_\nu$, $1 \le j \le 5$, of I_5^t give rise to non-trivial extension classes $\sum_{\nu=1}^{12} a_{j,\nu}[z_\nu]$ in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0),\mathbb{Q}(5))$, and moreover their images under $(D_{10001,\sigma})_\sigma$ generate a lattice of (full) rank 2 and of covolume $R_{10001}(F)$ with

$$\frac{|d_F|^{9/2}}{\pi^{5r_2}} \cdot \zeta_F(5) \;\doteq\; 320 \cdot R_{10001}(F)\,.$$

5.1. Remarks.

(1) The example above is one of the simplest cases that worked, i.e. that gave a non-zero regulator. Usually the number of z_ν which are needed to achieve such a non-zero regulator is considerably larger, and more often than not the program does not find sufficiently many of them.

(2) The procedure of taking the kernel rationally is numerically highly unstable, and since we want to be able to recognize a lattice, it is crucial that we find generators of the kernels which form a $\mathbb{Z}$-basis for the lattice (or which are at least not far away from this property, i.e. the denominator should be bounded). The problem is that, for the matrix entries we encounter, it may take very long to find such a $\mathbb{Z}$-basis already for, say, a lattice given by a matrix of size 300×600. The number of conditions grows very fast with the order of S and it is typically impractical to include more than 3 or 4 primes into S.

(3) One can view the above weight 5 function as a multiple polylogarithm as introduced by Goncharov (see e.g. [**8**]), but specialized to one variable only (then also called “generalized polylogarithm” in the literature), and those multiple polylogarithms in turn have appeared early as “hyperlogarithms”, in particular in work of Lappo-Danilevsky [**11**]. In this setting the function above is denoted $Li_{4,1}(1,z)$, and it might be tempting to think of the resulting special value not as a Dedekind zeta value $\zeta_F(5)$ but as a kind of “multiple Dedekind zeta value” $\zeta_F(4,1)$ (for some candidates see Wojtkowiak’s original article [**13**]) which then would seem, modulo the product $\zeta_F(2)\zeta_F(3)$, to be a rational multiple of the former, similar to what is known to be true for $F=\mathbb{Q}$ where one has $\zeta(4,1)=5\zeta(5)-\zeta(2)\zeta(3)$. Alas, we were unable to give a sensible evaluation of such a multiple Dedekind zeta value which might have corroborated such a statement.

6. Further results

6.1. Totally real fields. The simplest cases to consider seem to be the ones which are totally real, as the conditions involving dilogarithms and tetralogarithms all are trivially satisfied, since the function $\mathcal{L}_{2n}(z)$ vanishes on the real line. Nevertheless, they turn out to be hard, due to complexity reasons (the integer kernels involved tend to have large coefficients).

6.1.1. *The case* $F=\mathbb{Q}$. In the case of the rational numbers we also find a non-trivial result; using the set $S=\langle 2,3,5\rangle$, we find 98 exceptional S-units, and a similar calculation as above, but where we only need to satisfy conditions (M), followed by (X2) which produces a (numerical) lattice of covolume $\zeta(3)/96$, gives us a lattice generated by the image under D_{10001} of 50 linear combinations in $\mathbb{Z}[F]$, of covolume $\frac{1}{480}\zeta(5)$ (and we get the same lattice if we add the prime 7 to S, yielding 178 exceptional S–units and 87 generators of the lattice).

6.2. Other number fields. We have obtained similar results for number fields of degree ≤ 6, typically we were lucky to find a full regulator lattice for some small discriminants. We list signatures and corresponding discriminants for which we have obtained a (conjectural) lattice of full rank r_1+r_2:

signature	discriminants
$(2,0)$	5, 8, 13
$(0,1)$	-3, -7, -8, -15, -20
$(1,1)$	-31, -44, -59, -76, -83, -104, -108, -116, -139, -152
$(3,0)$	49, 148, 229
$(2,1)$	-275, -283, -331, -400, -448
$(4,0)$	725, 1125
$(0,3)$	-9747 .

Moreover, if we divide the covolume of the (conjectural) lattice of D_{10001}–values by $|d_F|^{9/2}\pi^{-5r_2}\zeta_F(5)$, the result in each case is numerically close to a rational number of small height.

For many other number fields, the height (or complexity) of the rational numbers in the inductive steps explodes quickly and gets out of control.

Acknowledgements. We are very grateful to Zdzisław Wojtkowiak for not only drawing our attention to the questions surrounding his formalism, but also for communicating the appropriate tools in a down-to-earth language which made it possible to have it implemented. We express our gratitude to the University of Nice for supporting several research visits, as well as to the Institute for Experimental Mathematics in Essen and the Max–Planck–Institute in Bonn and the Isaac Newton Institute in Cambridge for providing an inspiring environment, and to the DFG for long-term financial support. Finally we want to thank Hiroaki Nakamura for several clarifications and in particular the referee for many very helpful remarks which improved the exposition considerably.

References

[1] **Beilinson, A. and Deligne, P.;** *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs*, Motives (Seattle, WA, 1991), 97–121, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[2] **Beilinson, A. and Deligne, P.;** *Motivic polylogarithms and Zagier's conjecture*, Preprint 1992.

[3] **Bloch, S.;** *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves,* CRM Monograph Series, 11. American Mathematical Society, Providence, RI, 2000 (known also as "Irvine Lecture Notes", 1978).

[4] **Borel, A.;** *Cohomologie de* SL_n *et valeurs de fonctions zêta aux points entiers,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 4 (1977), no. 4, 613–636.

[5] **Brown, F.;** *Polylogarithmes multiples uniformes en une variable,* C. R. Math. Acad. Sci. Paris **338** (2004), no. 7, 527–532.

[6] **Cohen, H., Lewin, L. and Zagier, D.;** *A sixteenth order polylogarithmic ladder*, Experiment. Math. vol. **1**, 1992, 25–34.

[7] **Deligne, P. and Goncharov, A.B.;** *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 1–56.

[8] **Goncharov, A.B.;** *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5** (1998), no. 4, 497–516.

[9] **The PARI–Group;** PARI/GP, versions `2.1-2.4`, Bordeaux, `http://pari.math.u-bordeaux.fr/`.

[10] **de Jeu, R.;** *Zagier's conjecture and wedge complexes in algebraic K-theory,* Compositio Math. 96 (1995), no. 2, 197–247.

[11] **Lappo-Danilevsky, J. A.;** *Mémoires sur la théorie des systèmes des équations différentielles linéaires*, Chelsea Publishing Co., New York, N. Y., 1953.

[12] **Lewin, L.;** *Polylogarithms and associated functions*, North-Holland, New York-Amsterdam, 1981.

[13] **Wojtkowiak, Z.;** *Mixed Hodge Structures and Iterated Integrals I*, in "Motives, polylogarithms and Hodge theory, Part I" (Irvine, CA, 1998), 121–208, Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, 2002.

[14] **Wojtkowiak, Z.;** *Monodromy of iterated integrals and non-abelian unipotent periods,* in "Geometric Galois actions", 2, 219–289, London Math. Soc. Lecture Note Ser., 243, Cambridge Univ. Press, Cambridge, 1997.

[15] **Zagier, D.;** *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields*, Proc. Texel Conf. on Arithm. Alg. Geometry 1989, Birkhäuser, Boston (1991) 391–430.

Department of Mathematical Sciences, South Road, University of Durham, DH1 3LE, United Kingdom

E-mail address: herbert.gangl@durham.ac.uk

Clay Mathematics Proceedings
Volume 9, 2010

Zero-cycles on algebraic tori

Alexander S. Merkurjev

1. The map φ_T

Let T be an algebraic torus over a field F and X a smooth *compactification of* T, i.e., a geometrically irreducible smooth complete variety containing T as an open set. The Chow group $\mathrm{CH}_0(X)$ of classes of zero dimensional cycles on X does not depend (up to canonical isomorphism) on the choice of X (cf. [**7**, 16.1.11], [**4**, Prop. 6.3], [**8**]).

Recall that two F-points $t, t' \in T(F)$ are called *R-equivalent* if there is rational morphism $f : \mathbb{A}^1 \dashrightarrow T$ defined at 0 and 1 satisfying $f(0) = t$ and $f(1) = t'$ (cf. [**2**, §4]). We write $T(F)/R$ for the group of R-equivalence classes in $T(F)$.

For an F-point $t \in T(F)$ let $[t]$ denote its class in $\mathrm{CH}_0(X)$. Consider the map from $T(F)$ to $\mathrm{CH}_0(X)$ taking a point t to the class $[t] - [1]$. This map does not depend on the choice of X (up to canonical isomorphism) and it factors through R-equivalence. Indeed, a map f as above extends to a morphism $g : \mathbb{P}^1 \to X$ and $[t] = g_*([0]) = g_*([1]) = [t']$, where $g_* : \mathrm{CH}_0(\mathbb{P}^1) \to \mathrm{CH}_0(X)$ is the push-forward homomorphism (cf. [**7**, 1.4]).

We denote the resulting map by

$$\varphi_T : T(F)/R \to \mathrm{CH}_0(X).$$

Note that there is a homomorphism $\psi_T : A_0(X) \to T(F)/R$ such that $\psi_T \circ \varphi_T$ is the identity (cf. [**2**, Prop. 12]). It follows that the map φ_T is injective.

One can ask whether φ_T is a homomorphism. It is known that φ_T is a homomorphism for all tori T of dimension at most 3 (cf. [**10**]). In this note we shall give an example of a torus T such that φ_T is not a homomorphism although it has left inverse map ψ_T that is a homomorphism. It follows that φ_T is not surjective.

The map φ_T is a homomorphism if and only if for any two points t_1 and t_2 in $T(F)$ one has

$$[t_1 t_2] - [t_1] - [t_2] + [1] = 0 \qquad (1)$$

in $\mathrm{CH}_0(X)$.

2010 *Mathematics Subject Classification.* 14C25.
Key words and phrases. Algebraic cycles, Algebraic tori.
The work has been supported by the NSF grant DMS #0652316.

Let T' be another torus with a compactification X'. Then $X \times X'$ is a compactification of $T \times T'$. Let $t \in T(F)$ and $t' \in T'(F)$. The condition (1) for the elements $t_1 = (t, 1)$ and $t_2 = (1, t')$ of $(T \times T')(F)$ amounts to

$$([t] - [1]) \times ([t'] - [1]) = 0 \tag{2}$$

in $\mathrm{CH}_0(X \times X')$, where $\times$ denotes the external product for Chow groups (cf. [**7**, 1.10]). In the next section we shall give examples of tori T and T' such that the condition (2) fails for some t and t'. It would follows that $\varphi_{T \times T'}$ is not a homomorphism.

2. The tori $\mathrm{R}^1_{L/F}(\mathbf{G}_\mathrm{m})$

Let F be a field with $\operatorname{char} F \neq 2$. For an element $a \in F^\times$, let F_a denote the quadratic (étale) F-algebra $F[t]/(t^2 - a)$.

Let $a, b \in F^\times$. Consider the biquadratic F-algebra $L = F_a \otimes F_b$ and let G be the Galois group $\mathrm{Gal}(L/F)$. Write $\sigma \in G$ for the generator of $\mathrm{Gal}(L/F_a)$ and $\tau \in G$ for the generator of $\mathrm{Gal}(L/F_b)$.

Let T be the torus $\mathrm{R}^1_{L/F}(\mathbf{G}_\mathrm{m})$ of norm 1 elements of the extension L/F. For a field extension K/F, a point t of $T(K)$ is an element $t \in (K \otimes L)^\times$ satisfying $N_{(K\otimes L)/K}(t) := t \cdot \sigma(t) \cdot \tau(t) \cdot \sigma\tau(t) = 1$, where $N_{(K\otimes L)/K} : (K \otimes L)^\times \to K^\times$ is the norm homomorphism. The element $N_{(K\otimes L)/(K\otimes F_a)} = t \cdot \sigma(t)$ in $K \otimes F_a$ has norm 1 in K. By Hilbert Theorem 90, applied to the quadratic extension $(K \otimes F_a)/K$, there is an element $z \in (K \otimes F_a)^\times$ with $t \cdot \sigma(t) = z \cdot \tau(z)^{-1}$. Note that z is unique up to a multiple from $K^\times$. Hence the norm $N_{(K\otimes F_a)/K}(z) = z \cdot \tau(z)$ is unique up to a multiple from $K^{\times 2}$. It follows that the class $q_K(t)$ quaternion algebra $\big(z \cdot \tau(z), b\big)_K$ in the Brauer group $\mathrm{Br}(K)$ is well defined. Thus, we get a group homomorphism

$$q_K : T(K) \to \mathrm{Br}(K), \qquad t \mapsto q_K(t).$$

The collection of the homomorphisms q_K over all field extensions K of F form a morphism q of functors T and Br from the category of all field extensions of F to the category of groups. In other words, q is an invariant of the algebraic torus T with values in the Brauer group (cf. [**9**]).

Remark 2.1. It is shown in [**11**, p. 427] that q_F induces an isomorphism between $T(F)/R$ and the subgroup of $\mathrm{Br}(F)$ consisting of classes of algebras that are split over all three quadratic subalgebras of L.

Example 2.2. Assume that F contains a square root i of -1. Then we can view i as an element of $T(F)$. We have $i \cdot \sigma(i) = -1 = z \cdot \tau(z)^{-1}$ with $z = \sqrt{a}$ in F_a. Hence $q_F(i)$ is the class of the quaternion algebra $\big(z \cdot \tau(z), b\big)_F \simeq \big(-a, b\big)_F \simeq \big(a, b\big)_F$.

Let $F(T)$ be the function field of T over F and let v be a discrete valuation on $F(T)$ over F. The residue field $F(v)$ is a field extension of F. By [**5**, §5], there is the residue homomorphism

$$\partial_v : \mathrm{Br}\big(L(T)/F(T)\big) \to G^*,$$

where G^* is the character group of G. An element α in $\mathrm{Br}\big(L(T)/F(T)\big)$ is called *unramified with respect to* v if $\partial_v(\alpha) = 0$ and *(totally) unramified* if α is unramified with respect to every discrete valuation of $F(T)$ over F.

Proposition 2.3. *For any $t \in T\big(F(T)\big)$, the element $q_{F(T)}(t)$ in $\mathrm{Br}\big(L(T)/F(T)\big)$ is unramified.*

PROOF. Write K for $F(T)$, so $L(T) = K \otimes L = KL$. As the character group G^* is of exponent 2, it suffices to show that $q_K(t)$ is divisible by 2 in $\mathrm{Br}(KL/K)$. By Hilbert Theorem 90, there are elements $z \in K_a^\times$ and $w \in K_b^\times$ such that $t \cdot \sigma(t) = z \cdot \tau(z)^{-1}$ and $t \cdot \tau(t) = w^{-1} \cdot \sigma(w)$. Consider the cross product central simple K-algebra (cf. [**6**, §12]):

$$A = KL1 \oplus KLu_\sigma \oplus KLu_\tau \oplus KLu_\tau u_\sigma$$

with multiplication table:

$$u_\sigma^2 = z, \quad u_\tau^2 = w, \quad u_\sigma u_\tau = t u_\tau u_\sigma.$$

As KL is a maximal subalgebra of A, the Brauer class of A belongs to $\mathrm{Br}(KL/K)$.

The centralizer C of the quadratic subalgebra $K_a \subset KL \subset A$ in A is generated by KL and u_σ and hence is isomorphic to the quaternion algebra $(z, b)_{K_a}$. It follows from [**6**, §7] that

$$[A \otimes_K K_a] = [\big(z, b\big)_{K_a}] \quad \text{in } \mathrm{Br}(KL/K_a),$$

hence

$$q_K(t) = [\big(z \cdot \sigma(z), b\big)_K] = \mathrm{cor}_{K_a/K}[\big(z, b\big)_{K_a}] = \mathrm{cor}_{K_a/K}[A \otimes_K K_a] = 2[A]. \quad \square$$

We write α_T for the element $q_{F(T)}(t)$ in $\mathrm{Br}\big(L(T)/F(T)\big)$, where t is the generic point in $T\big(F(T)\big)$. As $2\alpha_T = 0$, we can view α_T as an element of the group $H^2\big(F(T), \mathbb{Z}/2\mathbb{Z}\big) = \mathrm{Br}_2\big(F(T)\big)$. By Proposition 2.3, α_T is an unramified element of $H^2\big(F(T), \mathbb{Z}/2\mathbb{Z}\big)$ in the sense of [**1**] (cf. [**10**, 2.2]).

Remark 2.4. If L/F is a field extension, by [**3**, Prop. 9.5], the factor group of the group of unramified elements in $\mathrm{Br}\big(F(T)\big)$ modulo $\mathrm{Br}(F)$ is canonically isomorphic to $H^2(G, \widehat{T}) \simeq H^3(G, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, where $\widehat{T}$ is the Galois module of characters of T. The class α_T corresponds to the only nontrivial element of the group $H^2(G, \widehat{T})$.

Choose a smooth compactifications X of T, so we can view α as an unramified element of $H^2\big(F(X), \mathbb{Z}/2\mathbb{Z}\big)$. Let $x \in X(F)$ be any point over F. We write $\alpha(x) \in H^2\big(F, \mathbb{Z}/2\mathbb{Z}\big)$ for the *value of* α *at* x (cf. [**10**, 2.1]). If $x \in T(F)$, then $\alpha(x) = q_F(x)$. In particular, we have $\alpha(1) = 0$ and $\alpha(i) = (a) \cup (b)$ by Example 2.2 if F contains a square root i of -1.

Let $L' = F_{a'} \otimes F_{b'}$ be another biquadratic F-algebra and $T' := \mathrm{R}^1_{L'/F}(\mathbf{G}_\mathrm{m})$ and let $\alpha_{T'} \in H^2\big(F(T'), \mathbb{Z}/2\mathbb{Z}\big)$ be the element as above. Choose also a smooth compactification X' of T'. Restricting α and α' to $F(X \times X')$ and taking the cup-product, we get the unramified element

$$\beta = \alpha \times \alpha' \in H^4\big(F(X \times X'), \mathbb{Z}/2\mathbb{Z}\big).$$

Let $\mathrm{Z}_0(X \times X')$ be the group of zero-dimensional cycles on $X \times X'$. The map $\mathrm{Z}_0(X \times X') \to H^4(F, \mathbb{Z}/2\mathbb{Z})$ taking the class of a closed point $z \in X \times X'$ to $N_{F(z)/F}\big(\beta(z)\big)$ factors through a homomorphism

$$\rho : \mathrm{CH}_0(X \times X') \to H^4(F, \mathbb{Z}/2\mathbb{Z})$$

(cf. [**10**, 2.4]). Note that for every $t \in T(F)$ and $t' \in T'(F)$ we have

$$\rho([t] \times [t']) = \beta(t, t') = \alpha(t) \cup \alpha'(t') \in H^4(F, \mathbb{Z}/2\mathbb{Z}).$$

It follows that

$$\rho(([t] - [1]) \times ([t'] - [1])) = \big(\alpha(t) - \alpha(1)\big) \cup \big(\alpha'(t') - \alpha'(1)\big) = \alpha(t) \cup \alpha'(t')$$

in $H^4(F, \mathbb{Z}/2\mathbb{Z})$.

Assume that F contains a square root i of -1, so $i \in T(F)$. We then have

$$\rho\big(([i] - [1]) \times ([i] - [1])\big) = (a) \cup (b) \cup (a') \cup (b') \in H^4(F, \mathbb{Z}/2\mathbb{Z}).$$

One can easily find a field F and elements a, b, a', b' with $(a) \cup (b) \cup (a') \cup (b') \neq 0$ in $H^4(F, \mathbb{Z}/2\mathbb{Z})$. For example, one can take $F = k(a, b, a', b')$, where a, b, a', b' are variables over a field k. This contradicts (2). Hence $\varphi_{T \times T'}$ is not a homomorphism.

References

[1] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.

[2] J.-L. Colliot-Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229.

[3] J.-L. Colliot-Thélène and J.-J. Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), no. 1, 148–205.

[4] J.-L. Colliot-Thélène and D. Coray, *L'équivalence rationnelle sur les points fermés des surfaces rationnelles fibrées en coniques*, Compositio Math. **39** (1979), no. 3, 301–332.

[5] J.-L. Colliot-Thélène and J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 113–186.

[6] P. K. Draxl, *Skew fields*, London Mathematical Society Lecture Note Series, vol. 81, Cambridge University Press, Cambridge, 1983.

[7] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[8] A. Merkurjev, *Rational correspondences*, http://www.math.ucla.edu/ merkurev/publicat.htm (2004).

[9] A. Merkurjev, *Invariants of algebraic groups*, J. Reine Angew. Math. **508** (1999), 127–156.

[10] A. Merkurjev, *Unramified elements in cycle modules*, J. London Math. Soc. **78** (2008), no. 1, 51–64.

[11] J.-P. Tignol, *Produits croisés abéliens*, J. Algebra **70** (1981), no. 2, 420–436.

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA

E-mail address: merkurev@math.ucla.edu

Clay Mathematics Proceedings
Volume 9, 2010

Chow-Künneth projectors and ℓ-adic cohomology

Andrea Miller

1. Introduction

This article deals with motivic decompositions of Chow motives of universal families over certain Shimura varieties of PEL-type. "PEL-type" means that the universal family is given by a family of abelian varieties with polarization, endomorphism and level structure. Absolute Chow-Künneth decompositions of such families (meaning Chow-Künneth decompositions of the total space of such a family) were proven to exist by the author in [**25**] in the case of families over compact unitary Shimura varieties of a certain type. These results were the content of the author's talk at the conference. They are the motivation and starting point for the present article and we will give a very short account on them in Section 3. For more details see [**25**] and [**26**] directly.

The main objective of the present article is to relate our Chow-Künneth projectors obtained in [**25**] to certain projectors in the ring of endomorphisms of ℓ-adic cohomology. As base variety we take the compact unitary Shimura varieties used in [**16**] and studied in [**20**] and [**9**]. We then relate the ℓ-adic realizations of Chow motives, cut out by suitable projectors in the Chow ring, with homological motives which are cut out by projectors in the ℓ-adic cohomology of these Shimura varieties. The homological projectors we use are the Künneth projectors used in [**16**]. The Chow-Künneth projectors we relate them to are the ones constructed by the author in [**25**]. For the fibres of the universal abelian scheme over the Shimura variety this is done in Section 5.

As simple as the definition of a Chow motive is, we know very little about them. Thus working out problems usually means working with a specific realization of a Chow motive. Yet this is often not fully satisfying for obtaining a purely **algebraic** geometric understanding of how motives decompose. For example in the cohomology of Shimura varieties Hecke correspondences are **the** essential tool to decompose these Grothendieck motives. Yet, if and how these "Hecke decompositions" lift to decompositions of corresponding Chow motives is widely unknown.

The contents of this article are as follows. In section 2 we recall the terminology of Chow motives and explain Murre's conjecture. In section 3 we give a short introduction to our work [**25**]. In section 4 we first recall the definition of Fourier-Mukai transform. Fourier-Mukai transforms were the essential tool in constructing motivic decompositions of abelian schemes (see [**10**]) and we then explain the results we need. In an upcoming paper we treat Fourier-Mukai transforms for compactifications of 1-motives. This, we hope, will be useful to treat Murre's Conjecture in a more general context of Shimura varieties which are neither compact nor related to Gl_2 (as are modular curves or Hilbert modular varieties, which were treated in [**15**], [**12**]).
Finally in Section 5 we relate our Chow-Künneth projectors to projectors of ℓ-adic cohomology for universal families over the above mentioned compact unitary Shimura varieties.

It is a pleasure to thank the organizers of the conference for organizing such an interesting conference. I especially want to thank Roy Joshua for his patience. I thank the referee for useful comments. This work was partly supported by a research grant of the "Deutsche Forschungsgemeinschaft".

2. Standard Conjecture "C" and Murre's Conjecture

2.1. In this paragraph, we recall the basic theory of Chow motives, especially as related to Murre's Conjecture. Further details may be found in [**27**].
For a smooth projective variety Y over a field k let $\mathrm{CH}^j(Y)$ denote the Chow group of algebraic cycles of codimension j on Y modulo rational equivalence, and let $\mathrm{CH}^j(Y)_{\mathbb{Q}} := \mathrm{CH}^j(Y) \otimes \mathbb{Q}$. For a cycle Z on Y we write $[Z]$ for its class in $\mathrm{CH}^j(Y)$. We will be working with relative Chow motives as well, so let us fix a smooth connected, quasi-projective base scheme $S \to \operatorname{Spec} k$. If $S = \operatorname{Spec} k$, we typically write $X \times_k Y$ for $X \times_{\operatorname{Spec} k} Y$. Let Y, Y' be smooth projective varieties over S, i.e., all fibers are smooth. For our purposes we may assume that Y is irreducible and of relative dimension g over S. The group of relative correspondences from Y to Y' of degree r is defined as

$$\mathrm{Corr}^r(Y \times_S Y') := \mathrm{CH}^{r+g}(Y \times_S Y')_{\mathbb{Q}}.$$

Every S-morphism $Y' \to Y$ defines an element in $\mathrm{Corr}^0(Y \times_S Y')$ via the class of the transpose of its graph. In particular one has the class $[\Delta_{Y/S}] \in \mathrm{Corr}^0(Y \times_S Y)$ of the relative diagonal. The self correspondences of degree 0 form a ring, see [**27**, p. 127]. Furthermore there is a ring homomorphism

$$(*) \qquad \mathrm{CH}^*(Y \times Y) \longrightarrow \mathrm{End}(\mathrm{CH}^*(Y))$$

([**11**], Corollary 16.1.2). This yields an action of correspondences on Chow groups. Using the relative correspondences one proceeds as usual to define the category $CH\mathcal{M}(S)$ of (pure) Chow motives over S. The objects of this pseudoabelian $\mathbb{Q}$-linear tensor category are triples (Y, p, n) where Y is as above, p is a projector, i.e. an idempotent element in $\mathrm{Corr}^0(Y \times_S Y)$, and $n \in \mathbb{Z}$. The morphisms are

$$\mathrm{Hom}_{\mathcal{M}_S}((Y,p,n),(Y',p',n')) := p' \circ \mathrm{Corr}^{n'-n}(Y \times_S Y') \circ p.$$

When $n = 0$ we write (Y, p) instead of $(Y, p, 0)$, and $h(Y) := (Y, [\Delta_Y])$.

2.2. Murre's conjectures. Let k be a field. Fix a Weil cohomology theory H^* and a smooth projective variety Y over k with $\dim Y = d$. Then by the Künneth formula we have

$$H^{2d}(Y \times Y)(d) = \oplus_{n=0}^{2d} H^n(Y) \otimes H^{2d-n}(Y)(d).$$

Since Y is projective, by Poincaré duality, we have

$$H^{2d-n}(Y)(d) = H^n(Y)^\vee.$$

We then get

$$\begin{aligned} H^{2d}(Y \times Y)(d) &= \oplus_{n=0}^{2d} H^n(Y) \otimes H^n(Y)^\vee \\ &= \oplus_{n=0}^{2d} \operatorname{Hom}(H^n(Y), H^n(Y)). \end{aligned}$$

We can thus identify $H^{2d}(Y \times Y)(d)$ with the vector space of graded k-linear maps $f : H^*(Y) \longrightarrow H^*(Y)$. In particular we can write

$$id_{H^*(Y)} = \sum_{n=0}^{2d} \overline{\pi}_Y^n \quad \text{where } \overline{\pi}_Y^n \in H^n(Y) \otimes H^n(Y)^\vee.$$

The projector

$$\overline{\pi}_Y^n : H^*(Y) \longrightarrow H^*(Y)$$

is called the *n-th Künneth projector*.
Denote the cycle class map by

$$cl : CH^*(Y) \longrightarrow H^{2*}(Y)$$

for a chosen Weil cohomology $H^*(\cdot)$. Recall that a class in $H^*(Y)$ is called *algebraic* if it lies in the image of the cycle class map cl.

CONJECTURE 2.1 (Grothendieck). *The Künneth projectors*

$$\overline{\pi}_Y^n : H^*(Y) \longrightarrow H^*(Y), n = 0 \ldots 2d$$

are algebraic.

As the diagonal $[\Delta_{Y/k}] \in \mathrm{CH}^d(Y \times Y)$ is mapped to $id_{H^*(Y)}$ by the cycle class map we can ask further if the $\overline{\pi}_Y^n$, $(n = 0, \ldots 2d)$ lift to orthogonal projectors $\pi_0, \ldots, \pi_{2d} \in \mathrm{CH}^d(Y \times Y)$.

DEFINITION 2.2. For a smooth projective variety Y/k of dimension d a *Chow-Künneth-decomposition* of Y consists of a collection of pairwise orthogonal projectors $\pi_0, \ldots, \pi_{2d}$ in $\mathrm{Corr}^0(Y \times Y)$ satisfying

(1) $\pi_0 + \ldots + \pi_{2d} = [\Delta_Y]$ and
(2) for some Weil cohomology theory H^* one has $\overline{\pi}_i(H^*(Y)) = H^i(Y)$.

If one has a Chow-Künneth decomposition for Y one writes $h^i(Y) = (Y, \pi_i)$. A similar notion of a *relative Chow-Künneth-decomposition* over S can be defined in a straightforward manner, see [**27**].

The existence of such a decomposition for every smooth projective variety is part of the following conjecture of Murre:

Conjecture 2.3 (Murre). *Let Y be a smooth projective variety of dimension d over some field k.*

(1) *There exists a Chow-Künneth decomposition for Y.*
(2) *For all $i < j$ and $i > 2j$ the action of π_i on $CH^j(Y)_{\mathbb{Q}}$ (see (*) in section 2.1) is trivial, i.e. $\pi_i \cdot CH^j(Y)_{\mathbb{Q}} = 0$.*
(3) *The induced j step filtration on*
$$F^{\nu} CH^j(Y)_{\mathbb{Q}} := \mathrm{Ker}\pi_{2j} \cap \cdots \cap \mathrm{Ker}\pi_{2j-\nu+1}$$
is independent of the choice of the Chow–Künneth projectors.
(4) *The first step of this filtration should give exactly the subgroup of homological trivial cycles in $CH^j(Y)_{\mathbb{Q}}$.*

Remark 2.4. Jannsen showed in [**19**] that this conjecture is equivalent to the Bloch-Beilinson conjecture.

Convention 2.5. We will follow the general convention of calling part (1) of the above conjecture of Murre " Murre's Conjecture ". It can be seen as a strengthening of Grothendieck's conjecture "C" and this is the part of the above conjecture we will be concerned with in this article.

There are not many cases for which Murre's conjecture (e.g. as by the above convention the existence of a Chow-Künneth decomposition) has been proved. It is known to be true for curves and surfaces [**27**]. See also the above mentioned work of Jannsen [**19**] over finite fields (where a lot more is proved). Furthermore for abelian schemes over a smooth projective base, Deninger and Murre have constructed a relative Chow-Künneth decomposition in [**10**], generalizing work of Shermenev [**29**] and Beauville ([**3**], [**4**]). Apart from the work already mentioned and the work on families over Shimura varieties, which will be the content of the next section, special cases have been treated among others by Akhtar-Joshua [**1**], del Angel-Müller-Stach [**2**], Brosnan [**5**], Iyer ([**17**],[**18**]). We will now proceed to the results which are known for universal families over Shimura varieties.

3. Murre's Conjecture and mixed Shimura varieties

As for mixed Shimura varieties, the basic method of approach was established in the cases of families over modular curves and Hilbert modular varieties by Gordon, Hanamura and Murre, see [**12**], [**13**], [**14**], [**15**]. Certain families over Picard modular surfaces were incompletely treated in [**24**]. A complete Chow-Künneth decomposition for some mixed Kottwitz Shimura varieties was constructed by the author in [**25**]. In [**25**] the approach of Gordon, Hanamura and Murre was modified to cover more general Shimura varieties. There we also proved that necessary conditions assumed in this approach were fulfilled in all the cases mentioned above. So, at the present moment, the known cases involving Shimura varieties are restricted first to some Shimura varieties arising from Gl_2 (Gordon, Hanamura, Murre) and secondly to the author's work on compact Shimura surfaces arising from certain unitary groups associated to division algebras [**25**] . We will now indicate how the results of [**25**] were obtained. Proofs for all the results we present here can be found in [**25**].

Consider the following situation. For simplicity let our ground field be $\mathbb{C}$. Let X be a compact Shimura variety of Kottwitz type (i.e. smooth projective over $\mathrm{Spec}\,\mathbb{C}$ of a certain unitary type, see [**25**]) and let $\mathcal{A}/X$ in

$$\begin{tikzcd} \mathcal{A} \arrow[d, "p"'] \arrow[dd, bend right=60, "\pi"'] \\ X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

be an abelian scheme over X. Let $\mathcal{A}_t$ denote the fibre of $\mathcal{A}$ over a point $t \in X$. Then the first Betti homology group of $\mathcal{A}_t$ carries a Hodge structure of type $(-1,0),(0,-1)$ and we denote its Hodge group by $\mathrm{Hod}(\mathcal{A}_t)$.
In [**25**] we prove the following theorem. A weaker version of this theorem goes back to work of Gordon, Hanamura and Murre in [**15**]. See (ii) of the Remarks 3.2 below.

THEOREM 3.1. *Let $p : \mathcal{A} \to X$ as above be surjective, smooth and projective satisfying the following conditions:*

(1) *The scheme $\mathcal{A}/X$ has a relative Chow-Künneth decomposition.*
(2) *X has a Chow-Künneth decomposition over $\mathbb{C}$.*
(3) *If t is a point of X the natural map.*

$$CH^r(\mathcal{A}) \to H_B^{2r}(\mathcal{A}_t(\mathbb{C}),\mathbb{Q})^{\mathrm{Hod}(\mathcal{A}_t)}$$

is surjective for $0 \le r \le d = \dim\mathcal{A} - \dim X$.

(4) *For i odd, $H_B^i(\mathcal{A}_t(\mathbb{C}),\mathbb{Q})^{\mathrm{Hod}(\mathcal{A}_t)} = 0$ (redundant condition).*
(5) *Let ρ be an irreducible, non-constant representation of $\mathrm{Hod}(\mathcal{A}_t)$ and $\mathcal{V}$ the corresponding local system on X.*
Then the cohomology $H^q(X,\mathcal{V})$ vanishes if $q \neq \dim X$.

Under these assumptions $\mathcal{A}$ has a Chow-Künneth decomposition over $\mathbb{C}$.

□

The proof of this theorem is rather technical and we will not comment on it. The interested reader should consult our paper [**25**] directly. In [**25**] we then proceed to verify assumptions (1) - (5) of Theorem 3.1 in the case of a compact unitary Shimura surface S of Kottwitz type. By the conclusion of Theorem 3.1, this then yields Chow-Künneth decompositions for the corresponding total space.

REMARKS 3.2. (i) The fundamental strategy of the proof goes back to [**15**]. The main new insight for the proof in our case is the fact, that we need to weaken the assumptions on monodromy of [**15**] since these assumptions are false in more general cases. Generally the hypotheses of Theorem 1.3. of [**15**] are too strong even in the compact case, so there is very little hope of being able to use it for non-compact Shimura varieties beyond the ones studied in [**15**]. See [**24**] for a non-trivial low-dimensional case where the prerequisites of Theorem 1.3. of [**15**] fail.

(ii) To remedy the problems described in (ii), the main observation is to systematically work with Mumford-Tate groups instead of fundamental groups as was done in all of the previous work on other cases given by Shimura varieties ([**12**], [**15**]).

4. Fourier-Mukai transforms for Abelian schemes

Let us shortly recall the definition of Fourier-Mukai transform. For a general reference on Fourier-Mukai transforms we refer to [**6**].

Let $X \longrightarrow S$, $Y \longrightarrow S$ be smooth projective schemes X, Y over a smooth quasi-projective scheme S. Denote by p (resp. q) the projectors from $X \times_S Y$ to X/S (resp. Y/S).
Let cohomology be singular cohomology with coefficients in $\mathbb{Q}$. Recall that $H^*(X, \mathbb{Q})$ comes equipped with a ring structure. For $f : X \longrightarrow Y$ denote by f_* and f^* the induced morphisms on H^*. Fourier-Mukai transformation is defined as follows:

Definition 4.1. Let X, Y, p, q be as above. Then for any $a \in H^*(X \times Y)$ ("kernel"), define the cohomological Fourier-Mukai transform associated to a to be

$$\begin{array}{cccc} \Phi_a^H : & H^*(X, \mathbb{Q}) & \longrightarrow & H^*(Y, \mathbb{Q}) \\ & b & \mapsto & q_*(a \cdot p^*(b)). \end{array}$$

A similar definition holds for Fourier-Mukai transforms between Chow groups, the kernel then being an element $a \in CH^*(X \times Y)$.

We will now specialize the situation to our purpose, namely to abelian schemes A/S. Fourier-Mukai transforms between abelian varieties and their duals were at the origins of Mukai's work on Fourier-Mukai transforms. He used them to show that the bounded derived category of an abelian variety is equivalent to the bounded derived category of its dual.
They were further used by Beauville ([**3**],[**4**]), generalizing work of Shermenev [**29**], to give a motivic decomposition of abelian varieties. This was generalized by Deninger and Murre [**10**] to give a motivic decomposition of abelian schemes over smooth quasi-projective bases.
We will now briefly recall the results from Deninger and Murre (see [**10**]), which we will need later.
Denote by A/S an abelian scheme, by $\hat{A}/S$ its dual, by $n : A \longrightarrow A$ multiplication by $n \in \mathbb{Z}$ on A and by $\overline{n} = id_A \times n : A \times_S \hat{A} \longrightarrow A \times_S \hat{A}$.
Fourier-Mukai transform from $CH^*(A)$ to $CH^*(\hat{A})$ are constructed by using a Poincaré line bundle P on $A \times \hat{A}$ as kernel. Pick such a P and rigidify it along the zero section. It is then known [**10**] that

$$\overline{n}^*(x) = nx \quad \text{in} \quad CH^1(A \times_S \hat{A}).$$

Following Beauville ([**3**], [**4**]), Deninger and Murre [**10**] defined the following eigenspaces of $CH^i(A, \mathbb{Q})$ with respect to n^*:

$$CH^i_s(A, \mathbb{Q}) := \{y \in CH^i(A, \mathbb{Q}) \mid n^* y = n^{2i-s} y \quad \text{for all} \quad n \in \mathbb{Z}\}.$$

Generalizing the above mentioned results of Beauville, Deninger and Murre [**10**] prove (their Theorem 2.19)

THEOREM 4.2. *Let A/S an abelian scheme of fibre dimension g and let $d = dimS$. Then for all i we have*

$$CH^i(A,\mathbb{Q}) = \bigoplus_{s=k}^{l} CH^i_s(A,\mathbb{Q})$$

with $k = \mathrm{Max}(i-g, 2i-2g)$ and $l = \mathrm{Min}(2i, i+d)$.

□

For the abelian scheme $A \times_S A$ over A/S (via projection on the first factor) one derives

COROLLARY 4.3. *Let A/S an abelian scheme of fibre dimension g. Then for all i we have*

$$CH^i(A \times_S A,\mathbb{Q}) = \bigoplus_{s=k}^{l} CH^i_s(A \times_S A,\mathbb{Q})$$

with $k = \mathrm{Max}(i-g, 2i-2g)$ and $l = \mathrm{Min}(2i, i+g)$.

PROOF. Apply theorem 4.2 to the abelian scheme $A \times_S A$ over A/S. The relative dimension of $A \times_S A$ over A/S is again g. □

If we now apply this corollary to the degree $i = g$ component of the Chow group we get the following "eigenspace" decomposition of $CH^g(A \times_S A)$:

$$CH^g(A \times_S A) = \bigoplus_{s=0}^{2g} CH^g_s(A \times_S A,\mathbb{Q}).$$

As explained in section 2, we are particularly interested in one very specific cycle in $CH^g(A \times_S A,\mathbb{Q})$, namely the diagonal $\Delta_{A/S}$, which will lead to Chow-Künneth decompositions.

4.1. Projectors for Abelian Varieties. Let S be a fixed base scheme. We recall the Chow-Künneth decomposition of an abelian scheme A of fibre dimension g over S (see [**10**]). For $n \in \mathbb{N}$ let $[n]$ denote the cycle class induced from multiplication by n on A/S. (Yet, we will stick with $[\Delta_{A/S}]$ as notation for the Chow class of $\Delta_{A/S}$.)
Firstly we have a functorial decomposition of the relative diagonal $\Delta_{A/S}$.

THEOREM 4.4. *There is a unique decomposition*

$$[\Delta_{A/S}] = \sum_{s=0}^{2g} \pi_i \qquad in \qquad \mathrm{CH}^g(A \times_S A)_{\mathbb{Q}}$$

such that $(id_A \times [n])^*\pi_i = n^i\pi_i$ for all $n \in \mathbb{Z}$. Moreover the π_i are mutually orthogonal idempotents, and $[{}^t\Gamma_{[n]}] \circ \pi_i = n^i\pi_i = \pi_i \circ [{}^t\Gamma_{[n]}]$, where $[n]$ denotes the multiplication by n on A.

PROOF. This is Theorem 3.1 of [**10**].

□

As a corollary they obtain

COROLLARY 4.5. *Let $R(A/S)$ be the relative Chow motive of the relative abelian scheme A/S and denote by $R^i(A/S) := (R(A/S), \pi_i)$ the relative Chow motive determined by π_i. Then we have*

$$R(A/S) = \bigoplus_{i=0}^{2g} R^i(A/S)$$

and n^* acts on $R^i(A/S)$ by multiplication with n^i.

REMARK 4.6. One of the many interesting properties of Fourier-Mukai transforms for abelian schemes is their relation to Poincaré-duality: Poincaré duality can be obtained as the composition of two Fourier-Mukai transforms.
Set $h^i(A/S) = (A/S, \pi_i)$ with π_i as in theorem 4.4.

THEOREM 4.7. (Poincaré-duality)

$$h^{2g-i}(A/S)^\vee \simeq h^i(A/S)(g)$$

PROOF. This was proved by Künnemann in [**22**, 3.1.2]. □

5. Comparing projectors

We start with the following question.

QUESTION 5.1. How do the projectors of Theorem 4.4, which are used to cut out eigenspaces of Chow groups (Chow-Künneth projectors) of an abelian scheme relate to the (Künneth-) projectors which cut out the different degrees of ℓ-adic étale cohomology of an abelian scheme (Künneth decomposition for a Weil cohomology) as used in [**16**] ?

In the following, m denotes the power in the m-th Kuga-Sato variety $A^m/S = A \times_S \cdots \times_S A \longrightarrow S$ of $A \longrightarrow S$.

5.1. m=1.

Chow-Künneth projectors π_i, if they exist, are by definition lifts (under the cycle class map *cl*) of Künneth projectors $\overline{\pi}_i$. (From now on we change notation from $\overline{\pi}^i_A$ to $\overline{\pi}_i$ for Künneth projectors for an abelian scheme A).
We will now give a finer description of the Künneth projectors $\overline{\pi}_i$ and relate the motivic decomposition of abelian schemes to certain ℓ-adic Galois representations.
Denote by $Pol_{\leq 2g,\mathbb{Q}}$ the space of rational polynomials of degree $\leq 2g$. We need the following lemma. It is an easy and well known fact from linear algebra, but we will give a proof which introduces a specific base of polynomials of $Pol_{\leq 2g,\mathbb{Q}}$ as we will use them later.

LEMMA 5.2. *The map*

$$\phi : Pol_{\leq 2g,\mathbb{Q}} \xrightarrow{\simeq} \mathbb{Q}^{2g+1}, \quad f \mapsto \phi(f) = (f(1), f(2), \dots f(2^{2g})).$$

is an isomorphism of vector spaces.

PROOF. As we are dealing with finite dimensional vector spaces of the same dimension, it is enough to show the surjectivity of ϕ. Consider the following polynomials

$$f_i(t) = \prod_{\substack{j=0 \\ j\neq i}}^{2g} \frac{t - 2^j}{2^i - 2^j} \quad \in Pol_{\leq 2g,\mathbb{Q}} \tag{1}$$

Then it is straightforward to check that

$$f_i(2^j) = 1 \ \text{ if } i = j \quad \text{ and } \quad f_i(2^j) = 0 \ \text{ if } i \neq j.$$

Hence

$$\phi(f_i) = (0, \ldots, 0, 1, 0 \ldots, 0),$$

for $i = 0, \ldots, 2g$ and the 1 at the i-th position.
This proves surjectivity, hence ϕ is an isomorphism. □

Recall that we denoted by A/S an abelian scheme of relative dimension g.
We obtain the following diagram [**6**]

$$\begin{tikzcd} CH^*(A) \otimes \mathbb{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & CH^*(A) \otimes \mathbb{Q} \arrow[d] \\ H^*(A, \mathbb{Q}) \arrow[r] & H^*(A, \mathbb{Q}). \end{tikzcd} \tag{2}$$

Here the horizontal arrows are Chow-theoretic, respectively homological, Fourier-Mukai transforms. The vertical arrows are given in terms of Chern classes and Todd classes (see [**6**]). This diagram commutes [**6**]. We will now give a further description of the above diagram in the situation described in Section 4.1. I.e. we consider Fourier-Mukai transforms on Chow groups of abelian schemes.

For $n \in \mathbb{Z}$, let $[n]$ denote the element in $\mathrm{End}(H^*(A, \mathbb{Q}))$ generated by the multiplication-by-n morphism $n : A \longrightarrow A$. For $m \in \mathbb{Z}$ denote by m the endomorphism given by $x \to mx$ for a cohomology class $x \in H^*(A, \mathbb{Q})$. Thus "m" is a short way of writing "$m \cdot [1]$". To formally distinguish this from scalar multiplication by integers, we will put the latter in parenthesis "( )". We denote the composition of two endomorphisms by $\circ$.

Theorem 5.3. *Let g denote the relative dimension of the abelian scheme A/S. The expressions*

$$\overline{\pi}_i = \prod_{\substack{j=0 \\ j \neq i}}^{2g} \frac{[2] - 2^j}{(2^i - 2^j)}. \tag{3}$$

are Künneth projectors. Furthermore the i-th Chow-Künneth projector π_i for A/S, as constructed in Theorem 4.4, maps to $\overline{\pi}_i$ under the cycle class map $cl : CH^*(A \times_S A, \mathbb{Q}) \longrightarrow H^{2*}(A \times_S A, \mathbb{Q})$.

Proof. We first show that the $\overline{\pi}_i$ are Künneth projectors.
Recall that for $n \in \mathbb{N}, n \neq 0, 1$, the $H^i(A, \mathbb{Q})$ is the n^i-eigenspace of $n^* = [n] \in \mathrm{End}(H^*(A, \mathbb{Q}))$. This is a standard argument using the fact that $H^i(A, \mathbb{Q}) = \bigwedge^i H^1(A, \mathbb{Q})$ ([**8**], Theorem 2A8) and that $n^*(x) = n \cdot x$ for $x \in H^1(A, \mathbb{Q})$ ([**8**],

Lemma 2A3). For an element $x \in H^k(A, \mathbb{Q})$ we have

$$\overline{\pi}_i \cdot x = \prod_{\substack{j=0 \\ j \neq i}}^{2g} \frac{[2] - 2^j}{(2^i - 2^j)} \cdot x$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^{2g} \frac{2^k - 2^j}{(2^i - 2^j)} \cdot x \tag{4}$$

$$= \begin{cases} 0, & \text{if } k \neq i; \\ x, & \text{if } k = i. \end{cases} \tag{5}$$

Clearly

$$\overline{\pi}_i \circ \overline{\pi}_k = \begin{cases} 0, & \text{if } k \neq i; \\ \overline{\pi}_i, & \text{if } k = i. \end{cases}$$

and thus the $\overline{\pi}_i$ are Künneth projectors.

Let W denote the $(2g+1)$-dimensional subspace of $\mathrm{End}(H^*(A, \mathbb{Q}))$, generated by the Künneth projectors $\overline{\pi}_i, i = 0 \dots 2g$ and recall the map ϕ from Lemma 5.2

$$Pol_{\leq 2g, \mathbb{Q}} \longrightarrow \mathbb{Q}^{2g+1}, \quad f(t) \mapsto (f(1), f(2), f(2^2), \dots, f(2^{2g})).$$

Then $Pol_{\leq 2g, \mathbb{Q}}$ is isomorphic to W via

$$f_i(t) \mapsto \overline{\pi}_i = \prod_{\substack{j=0 \\ j \neq i}}^{2g} \frac{[2] - 2^j}{(2^i - 2^j)}, \quad i = 0 \dots 2g.$$

Here the f_i are the base elements of $Pol_{\leq 2g, \mathbb{Q}}$ chosen in the proof of Lemma 5.2. We then get the following diagram

$$\begin{tikzcd} CH^g(A \times_S A) \arrow[d, "cl"'] & Pol_{\leq 2g, \mathbb{Q}} \arrow[l] \arrow[d, "\simeq"] \\ \mathrm{End}(H^*(A, \mathbb{Q})) & W \arrow[l, "inj"'] \end{tikzcd} \tag{6}$$

Here the upper map in the diagram is a map of modules given by mapping the multiplicative generator t of $Pol_{\leq 2g, \mathbb{Q}}$ to $2^* \in \mathrm{Corr}(A/S, A/S) = CH^g(A \times_S A)$ (notation of [**10**]).
As $n^* \in \mathrm{End}(CH^*(A/S))$ maps to $[n] \in \mathrm{End}(H^*(A, \mathbb{Q}))$, the Künneth projectors

$$\overline{\pi}_i = \prod_{\substack{j=0 \\ j \neq i}}^{2g} \frac{[2] - 2^j}{(2^i - 2^j)}, \quad i = 0 \dots g,$$

lift to Chow-Künneth projectors π_i under the vertical maps of diagram (2). □

COROLLARY 5.4. *The ℓ-adic Grothendieck motives constructed on page 98 of* [**16**] *for $m = 1$ can be lifted to Chow motives.*

PROOF. This is now clear for an ℓ-adic realization of the $R^i(A/S)$ as introduced above. □

5.2. $\mathbf{m > 1}$. The Künneth projectors

$$\overline{\pi}_i = \prod_{\substack{j=0 \\ j\neq i}}^{2g} \frac{[2] - 2^j}{(2^i - 2^j)}, \quad i = 0 \ldots g,$$

used above, are actually special cases of projectors (when $m = 1$) constructed in [**16**] for Kuga-Sato varieties
$(A/S)^m \longrightarrow X$. The generalized projectors of [**16**] are given by

$$\overline{\pi}_{i,x} = \prod_{x=1}^{m} \prod_{\substack{j=0 \\ j\neq i}}^{2g} \frac{[2]_x - 2^j}{(2^i - 2^j)}.$$

Here $[2]_x$ denotes multiplication by 2 on the x-th factor of $A^m = A \times_S \cdots \times_S A \longrightarrow X$ and g is as before. These projectors are used in [**16**] to construct ℓ-adic Galois representations associated to certain automorphic representations π of $G(\mathbb{A}^\infty)$. Here G denotes Kottwitz' unitary groups introduced in [**16**] and $\mathbb{A}^\infty$ denote the adèles away from the archimedian primes.
It is straight forward to generalize the arguments of [**10**] and of the present paper from A/S to A^m/S.
One of the key ingredients in the proof of a Chow-Künneth decomposition for abelian schemes in [**10**] (for $m = 1$) is an argument of Kleiman (see 2A11 of [**8**]) which constructs a linear system by taking $k_j^*\ (\Delta_A)$ for enough $k_j \in \mathbb{N}$. The exact same argument goes through for $A^m = A \times_S \cdots \times_S A \longrightarrow X$ by singling out multiplication by k_j for one factor of $A^m/S = A \times_S \cdots \times_S A \longrightarrow S$ at a time.

References

[1] R. Akhtar, R. Joshua: *Künneth decompositions for quotient varieties*, Indag. Math. (N.S.) 17 (2006), no. 3, 319–344.

[2] P. del Angel and S. Müller-Stach: *On Chow motives of 3–folds*, Transactions of the AMS **352**, 1623-1633 (2000).

[3] A. Beauville: *Quelques remarques sur la transformation de Fourier dans* , Algebraic Geometry, Springer Lecture Notes **1016** (1983), 238-260.

[4] A. Beauville: *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. **273** (1986), 647-651.

[5] P. Brosnan: *On motivic decompositions arising from the method of Bialynicki-Birula.*, Invent. Math. 161 (2005), no. 1, 91–111.

[6] D. Huybrechts: *Fourier-Mukai transforms in Algebraic Geometry*, Oxford Math. Monographs, (2006).

[7] E. Friedlander et al. (editors): *Some recent developments in algebraic K-theory*, ICTP Lecture Notes **23** (2008).

[8] S.L. Kleiman: *Algebraic Cycles and the Weil Conjectures* , in *Dix exposés sur la cohomologie des schemas*, Amsterdam (1968).

[9] L. Clozel: *On the cohomology of Kottwitzs arithmetic varieties* Duke Math. J. **72** (1993), no. 3, 757795.

[10] C. Deninger and J. Murre: *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math., **422** (1991), pp. 201–219.

[11] W. Fulton: *Intersection Theory*, Ergebnisse der Mathematik, Vol. 2, Springer (1998).

[12] B. B. Gordon, J. P. Murre: *Chow motives of elliptic modular threefolds*, J. Reine Angew. Math. **514** (1999), 145–164.

[13] B. B. Gordon, M. Hanamura, and J. P. Murre: *Relative Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math., **558** (2003), 1–14.

[14] B. B. Gordon, M. Hanamura, and J. P. Murre: *Chow-Künneth projectors for modular varieties*, C. R. Acad. Sci. Paris, Ser. I, **335**, pp. 745–750 (2002).

[15] B. B. Gordon, M. Hanamura, and J. P. Murre: *Absolute Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math. **580**, 139–155 (2005).

[16] M. Harris, R. Taylor: *The geometry and cohomology of some simple Shimura varieties*, Annals of Maths Studies **151**, Princeton University Press (2001).

[17] JN.Iyer: *Murre's conjectures and explicit Chow-Knneth projections for varieties with a NEF tangent bundle.* Trans. Amer. Math. Soc. 361 (2009), no. 3, 1667–1681.

[18] JN.Iyer: *Murre's conjectures for a rational homogenous bundle over a variety*, arXiv:0805.2048

[19] U. Jannsen: *Motivic sheaves and filtrations on Chowgroups*, in Motives (Seattle, WA, 1991), Providence, RI, 1994, Amer. Math. Soc., pp. 245–302.

[20] R. Kottwitz: *On the λ-adic representations associated to some simple Shimura varieties*, Invent. Math. **108** (1992), pp. 653–665.

[21] R. Kottwitz: *Points on some Shimura varieties over finite fields*, Journal of the AMS **5** (1992), pp. 373–444.

[22] K. Künnemann: *On the Chow motive of an abelian scheme*, in Motives (Seattle, WA, 1991), Providence, RI, 1994, Amer. Math. Soc., pp. 189–205.

[23] M. Levine: *Lectures given at the ICTP summer school on K-theory*, in [**7**].

[24] A. Miller, S. Müller-Stach, S. Wortmann, Y.-H. Yang, K.Zuo: *Chow-Künneth decomposition for universal families over Picard modular surfaces*, Algebraic Cycles and Motives, vol.2, Ed: J.Nagel, C.Peters, Cambridge University Press (2007).

[25] A. Miller: *Chow motives of universal families over some Shimura surfaces*, Preprint, `arXiv:math/0710.4209`

[26] A. Miller: *Chow motives of mixed Shimura varieties*, to appear in the Conference Proceedings "Cycles, Motives and Shimura Varieties", TIFR Mumbai, (2008).

[27] J. P. Murre: *Lectures on Motives*, in Proceedings Transcendental Aspects of Algebraic Cycles, Grenoble 2001, LMS Lectures Note Series **313**, Cambridge Univ. Press (2004), 123–170.

[28] J. P. Murre: *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), pp. 190–204.

[29] A. Shermenev: *The motive of an abelian variety*, Funct. Anal. **8** (1974), 55-61.

Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138

E-mail address: `millerae@math.harvard.edu`

Connections with Mathematical Physics

Clay Mathematics Proceedings
Volume 9, 2010

Motives Associated to Sums of Graphs

Spencer Bloch

1. Introduction

In quantum field theory, the path integral is interpreted perturbatively as a sum indexed by graphs. The coefficient (Feynman amplitude) associated to a graph Γ is a period associated to the motive given by the complement of a certain hypersurface X_Γ in projective space. Based on considerable numerical evidence, Broadhurst and Kreimer suggested [**4**] that the Feynman amplitudes should be sums of multi-zeta numbers. On the other hand, Belkale and Brosnan [**2**] showed that the motives of the X_Γ were not in general mixed Tate.

A recent paper of Aluffi and Marcolli [**1**] studied the images $[X_\Gamma]$ of graph hypersurfaces in the Grothendieck ring $K_0(Var_k)$ of varieties over a field k. Let $\mathbb{Z}[\mathbb{A}^1_k] \subset K_0(Var_k)$ be the subring generated by $1 = [\operatorname{Spec} k]$ and $[\mathbb{A}^1_k]$. It follows from [**2**] that $[X_\Gamma] \notin \mathbb{Z}[\mathbb{A}^1_k]$ for many graphs Γ.

Let $n \geq 3$ be an integer. In this note we consider a sum $S_n \in K_0(Var_k)$ of $[X_\Gamma]$ over all connected graphs Γ with n vertices, no multiple edges, and no tadpoles (edges with just one vertex). (There are some subtleties here. Each graph Γ appears with multiplicity $n!/|Aut(\Gamma)|$. For a precise definition of S_n see (5.1) below.) Our main result is

THEOREM 1.1. *$S_n \in \mathbb{Z}[\mathbb{A}^1_k]$.*

For applications to physics, one would like a formula for sums over all graphs with a given loop order. I do not know if such a formula could be proven by these methods. Even if it could, the referee points out it is not obvious how the Feynman amplitude could be interpreted as a "motivic measure", i.e. as a functional on a Grothendieck group like $K_0(Var_k)$.

Dirk Kreimer explained to me the physical interest in considering sums of graph motives, and I learned about $K_0(Var_k)$ from correspondence with H. Esnault. Finally, the recently paper of Aluffi and Marcolli [**1**] provides a nice exposition of the general program.

2. Basic Definitions

Let E be a finite set, and let

$$0 \to H \to \mathbb{Q}^E \to W \to 0; \quad 0 \to W^\vee \to \mathbb{Q}^E \to H^\vee \to 0 \tag{2.1}$$

be dual exact sequences of vector spaces. For $e \in E$, let $e^\vee : \mathbb{Q}^E \to \mathbb{Q}$ be the dual functional, and let $(e^\vee)^2$ be the square, viewed as a quadratic function. By restriction, we can view this as a quadratic function either on H or on $W^\vee$. Choosing bases, we get symmetric matrices M_e and N_e. Let $A_e, e \in E$ be variables, and consider the homogeneous polynomials

$$\Psi(A) = \det(\sum A_e M_e); \quad \Psi^\vee(A) = \det(\sum A_e N_e). \tag{2.2}$$

LEMMA 2.1. *$\Psi(\ldots A_e, \ldots) = c\prod_{e\in E} A_e \Psi^\vee(\ldots A_e^{-1}, \ldots)$, where $c \in k^\times$.*

PROOF. This is proposition 1.6 in [**3**]. □

Let Γ be a graph. Write E, V for the edges and vertices of Γ. We have an exact sequence

$$0 \to H_1(\Gamma, \mathbb{Q}) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^V \to H_0(\Gamma, \mathbb{Q}) \to 0. \tag{2.3}$$

We take $H = H_1(\Gamma)$ and $W = \mathrm{Image}(\partial)$ in (2.1). The resulting polynomials $\Psi = \Psi_\Gamma$, $\Psi^\vee = \Psi^\vee_\Gamma$ as in (2.2) are given by [**3**]

$$\Psi_\Gamma = \sum_{t\in T}\prod_{e\notin t} A_e; \quad \Psi^\vee_\Gamma = \sum_{t\in T}\prod_{e\in t} A_e. \tag{2.4}$$

Here T is the set of *spanning trees* in Γ.

LEMMA 2.2. *Let $e \in \Gamma$ be an edge. Let Γ/e be the graph obtained from Γ by shrinking e to a point and identifying the two vertices. We do not consider Γ/e in the degenerate case when e is a loop, i.e. if the two vertices coincide. Let $\Gamma - e$ be the graph obtained from Γ by cutting e. We do not consider $\Gamma - e$ in the degenerate case when cutting e disconnects Γ or leaves an isolated vertex. Then*

$$\Psi_{\Gamma/e} = \Psi_\Gamma|_{A_e=0}; \quad \Psi_{\Gamma-e} = \frac{\partial}{\partial A_e}\Psi_\Gamma. \tag{2.5}$$

$$\Psi^\vee_{\Gamma/e} = \frac{\partial}{\partial A_e}\Psi^\vee_\Gamma; \quad \Psi^\vee_{\Gamma-e} = \Psi^\vee_\Gamma|_{A_e=0}. \tag{2.6}$$

(In the degenerate cases, the polynomials on the right in (2.5) *and* (2.6) *are zero.)*

PROOF. The formulas in (2.5) are standard [**3**]. The formulas (2.6) follow easily using lemma 2.1. (In the case of graphs, the constant c in the lemma is 1.) □

More generally, we can consider strings of edges $e_1, \ldots, e_p \in \Gamma$. If at every stage we have a nondegenerate situation we can conclude inductively

$$\Psi^\vee_{\Gamma-e_1-\cdots-e_p} = \Psi^\vee_\Gamma|_{A_{e_1}=\cdots=A_{e_p}=0} \tag{2.7}$$

In the degenerate situation, the polynomial on the right will vanish, i.e. X_Γ will contain the linear space $A_{e_1} = \cdots = A_{e_p} = 0$.

For example, let $\Gamma = e_1 \cup e_2 \cup e_3$ be a triangle, with one loop and three vertices. We get the following polynomials

$$\Psi_\Gamma = A_{e_1} + A_{e_2} + A_{e_3}; \quad \Psi^\vee_\Gamma = A_{e_1}A_{e_2} + A_{e_2}A_{e_3} + A_{e_1}A_{e_3} \tag{2.8}$$

$$\Psi_{\Gamma-e_i} = 1; \quad \Psi^\vee_{\Gamma-e_i} = A_{e_j}A_{e_k} = \Psi^\vee_\Gamma|_{A_{e_i}=0} \tag{2.9}$$

The sets $\{e_i, e_j\}$ are degenerate because cutting two edges will leave an isolated vertex.

3. The Grothendieck Group and Duality

Recall $K_0(Var_k)$ is the free abelian group on generators isomorphism classes $[X]$ of quasi-projective k-varieties and relations

$$[X] = [U] + [Y]; \quad U \overset{\text{open}}{\hookrightarrow} X,\ Y = X - U. \tag{3.1}$$

In fact, $K_0(Var_k)$ is a commutative ring with multiplication given by cartesian product of k-varieties. Let $\mathbb{Z}[\mathbb{A}^1_k] \subset K_0(Var_k)$ be the subring generated by $1 = [\operatorname{Spec} k]$ and $[\mathbb{A}^1_k]$. Let $\mathbb{P}_\Gamma$ be the projective space with homogeneous coordinates $A_e, e \in E$. We write $X_\Gamma : \Psi_\Gamma = 0,\ X^\vee_\Gamma : \Psi^\vee_\Gamma = 0$ for the corresponding hypersurfaces in $\mathbb{P}_\Gamma$. We are interested in the classes $[X_\Gamma], [X^\vee_\Gamma] \in K_0(Var_k)$.

Let $\Delta : \prod_{e\in E} A_e = 0$ in $\mathbb{P}_\Gamma$, and let $\mathbb{T} = \mathbb{T}_\Gamma = \mathbb{P}_\Gamma - \Delta$ be the torus. Define

$$X^0_\Gamma = X_\Gamma \cap \mathbb{T}_\Gamma; \quad X^{\vee,0}_\Gamma = X^\vee_\Gamma \cap \mathbb{T}_\Gamma. \tag{3.2}$$

Lemma 2.1 translates into an isomorphism (Cremona transformation)

$$X^0_\Gamma \cong X^{\vee,0}_\Gamma. \tag{3.3}$$

(In fact, this is valid more generally for the setup of (2.1) and (2.2).) We can stratify $X^\vee_\Gamma$ by intersecting with the toric stratification of $\mathbb{P}_\Gamma$ and write

$$[X^\vee_\Gamma] = \sum_{\{e_1,\dots,e_p\}\subset E} [(X^\vee_\Gamma \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] \in K_0(Var_k) \tag{3.4}$$

where the sum is over all subsets of E, and superscript 0 means the open torus orbit where $A_e \neq 0, e \notin \{e_1,\dots,e_p\}$. We call a subset $\{e_1,\dots,e_p\} \subset E$ degenerate if $\{A_{e_1} = \cdots = A_{e_p} = 0\} \subset X^\vee_\Gamma$. Since $[\mathbb{G}_m] = [\mathbb{A}^1] - [pt] \in K_0(Var_k)$ we can rewrite (3.4)

$$[X^\vee_\Gamma] = \sum_{\substack{\{e_1,\dots,e_p\}\subset E\\ \text{nondegenerate}}} [(X^\vee_\Gamma \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] + t \tag{3.5}$$

where $t \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var_k)$. Now using (2.7) and (3.3) we conclude

$$[X^\vee_\Gamma] = \sum_{\substack{\{e_1,\dots,e_p\}\subset E\\ \text{nondegenerate}}} [X^0_{\Gamma-\{e_1,\dots,e_p\}}] + t. \tag{3.6}$$

4. Complete Graphs

Let Γ_n be the complete graph with $n \geq 3$ vertices. Vertices of Γ_n are written $(j),\ 1 \leq j \leq n$, and edges e_{ij} with $1 \leq i < j \leq n$. We have $\partial e_{ij} = (j) - (i)$.

A more precise version of the following lemma is proven in [**5**], theorem (4.1).

PROPOSITION 4.1. *We have* $[X^\vee_{\Gamma_n}] \in \mathbb{Z}[\mathbb{A}^1_k]$.

PROOF. Let $\mathbb{Q}^{n,0} \subset \mathbb{Q}^n$ be row vectors with entries which sum to 0. We have

$$0 \to H_1(\Gamma_n) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^{n,0} \to 0. \tag{4.1}$$

In a natural way, $(\mathbb{Q}^{n,0})^\vee = \mathbb{Q}^n/\mathbb{Q}$. Take as basis of $\mathbb{Q}^n/\mathbb{Q}$ the elements $(1), \dots, (n-1)$. As usual, we interpret the $(e^\vee_{ij})^2$ as quadratic functions on $\mathbb{Q}^n/\mathbb{Q}$. We write N_e for the corresponding symmetric matrix.

LEMMA 4.2. *The $N_{e_{ij}}$ form a basis for the space of all $(n-1)\times(n-1)$ symmetric matrices.*

Proof of lemma. The dual map $\mathbb{Q}^n/\mathbb{Q} \to \mathbb{Q}^E$ carries

$$(k) \mapsto \sum_{\mu>k} -e_{k\mu} + \sum_{\nu<k} e_{\nu k}; \quad k \le n-1. \tag{4.2}$$

We have

$$(e_{ij}^\vee)^2(\sum_{k=1}^{n-1} a_k \cdot (k)) = \begin{cases} a_i^2 - 2a_ia_j + a_j^2 & i<j<n \\ a_i^2 & j = n. \end{cases} \tag{4.3}$$

It follows that if $j < n$, $N_{e_{ij}}$ has -1 in positions (ij) and (ji) and $+1$ in positions $(ii), (jj)$ (resp. N_{in} has 1 in position (ii) and zeroes elsewhere). These form a basis for the symmetric $(n-1)\times(n-1)$ matrices. $\square$

It follows from the lemma that $X_{\Gamma_n}^\vee$ is identified with the projectivized space of $(n-1)\times(n-1)$ matrices of rank $\le n-2$. In order to compute the class in the Grothendieck group we detour momentarily into classical algebraic geometry. For a finite dimensional k-vector space U, let $\mathbb{P}(U)$ be the variety whose k-points are the lines in U. For a k-algebra R, the R-points $\operatorname{Spec} R \to \mathbb{P}(U)$ are given by pairs (L,ϕ) where L on *Spec* R is a line bundle and $\phi: L \hookrightarrow U \otimes_k R$ is a locally split embedding.

Suppoose now $U = Hom(V,W)$. We can stratify $\mathbb{P}(Hom(V,W)) = \coprod_{p>0} \mathbb{P}(Hom(V,W))^p$ according to the rank of the homomorphism. Looking at determinants of minors makes it clear that $\mathbb{P}(Hom(V,W))^{\le p}$ is closed. Let R be a local ring which is a localization of a k-algebra of finite type, and let a be an R-point of $\mathbb{P}(Hom(V,W))^p$. Choosing a lifting b of the projective point a, we have

$$0 \to \ker(b) \to V \otimes R \xrightarrow{b} W \otimes R \to \operatorname{coker}(b) \to 0, \tag{4.4}$$

and $\operatorname{coker}(b)$ is a finitely generated R-module of constant rank $\dim W - p$ which is therefore necessarily free.

Let $Gr(\dim V - p, V)$ and $Gr(p,W)$ denote the Grassmann varieties of subspaces of the indicated dimension in V (resp. W). On $Gr(\dim V - p, V) \times Gr(p,W)$ we have rank p bundles E, F given respectively by the pullbacks of the universal quotient on $Gr(\dim V - p, V)$ and the universal subbundle on $Gr(p,W)$. It follows from the above discussion that

$$\mathbb{P}(Hom(V,W))^p = \mathbb{P}(\mathrm{Isom}(E,F)) \subset \mathbb{P}(Hom(E,F)). \tag{4.5}$$

Suppose now that $W = V^\vee$. Write $\langle\ ,\ \rangle : V \otimes V^\vee \to k$ for the canonical bilinear form. We can identify $Hom(V,V^\vee)$ with bilinear forms on V

$$\rho : V \to V^\vee \leftrightarrow (v_1, v_2) \mapsto \langle v_1, \rho(v_2)\rangle. \tag{4.6}$$

Let $SHom(V,V^\vee) \subset Hom(V,V^\vee)$ be the subspace of ρ such that the corresponding bilinear form on V is symmetric. Equivalently, $Hom(V,V^\vee) = V^{\vee,\otimes 2}$ and $SHom(V,V^\vee) = Sym^2(V^\vee) \subset V^{\vee,\otimes 2}$.

For ρ symmetric as above, one seees easily that $\rho(V) = \ker(V)^\perp$ so there is a factorization

$$V \to V/\ker(\rho) \xrightarrow{\cong} (V/\ker(\rho))^\vee = \ker(\rho)^\perp \hookrightarrow V^\vee. \tag{4.7}$$

The isomorphism in (4.7) is also symmetric.

Fix an identification $V = k^n$ and hence $V = V^\vee$. A symmetric map is then given by a symmetric $n \times n$ matrix. On $Gr(n-p,n)$ we have the universal rank

p quotient $Q = k^n \otimes \mathcal{O}_{Gr}/K$, and also the rank p perpendicular space $K^\perp$ to the universal subbundle K. Note $K^\perp \cong Q^\vee$. It follows that

$$\mathbb{P}(SHom(k^n,k^n))^p \cong \mathbb{P}(SHom(Q,Q^\vee))^p \subset \mathbb{P}(SHom(Q,Q^\vee)). \tag{4.8}$$

This is a fibre bundle over $Gr(n-p,n)$ with fibre $\mathbb{P}(Hom(k^p,k^p))^p$, the projectivized space of symmetric $p\times p$ invertible matrices.

We can now compute $[X^\vee_{\Gamma_n}]$ as follows. Write $c(n,p) = [\mathbb{P}(SHom(k^n,k^n))^p]$. We have the following relations:

$$c(n,1) = [\mathbb{P}^{n-1}]; \quad \sum_{p=1}^{n} c(n,p) = [\mathbb{P}^{\binom{n+1}{2}-1}]; \tag{4.9}$$

$$c(n,p) = [Gr(n-p,n)]\cdot c(p,p) \tag{4.10}$$

$$[X^\vee_{\Gamma_n}] = \sum_{p=1}^{n-2} c(n-1,p) \tag{4.11}$$

Here (4.10) follows from (4.8). It is easy to see that these formulas lead to an expression for $[X^\vee_{\Gamma_n}]$ as a polynomial in the $[\mathbb{P}^N]$ and $[Gr(n-p-1,n-1)]$ (though the precise form of the polynomial seems complicated). To finish the proof of the proposition, we have to show that $[Gr(a,b)] \in \mathbb{Z}[\mathbb{A}^1_k]$. Fix a splitting $k^b = k^{b-a}\oplus k^a$. Stratify $Gr(a,b) = \coprod_{p=0}^{a} Gr(a,b)^p$ where

$$\begin{aligned}(4.12)\quad Gr(a,b)^p = &\\ &\{V\subset k^{b-a}\oplus k^a \mid \dim(V)=a,\ \mathrm{Image}(V\to k^a)\text{ has rank }p\} = \\ &\qquad\qquad \{(X,Y,f)\mid X\subset k^{b-a},\ Y\subset k^a,\ f:Y\to X\}\end{aligned}$$

where $\dim X = a-p$, $\dim(Y)=p$. This is a fibration over $Gr(b-a-p,b-a)\times Gr(p,a)$ with fibre $\mathbb{A}^{p(b-a-p)}$. By induction, we may assume $[Gr(b-a-p,b-a)\times Gr(p,a)]\in\mathbb{Z}[\mathbb{A}^1_k]$. Since the class in the Grothendieck group of a Zariski locally trivial fibration is the class of the base times the class of the fibre, we conclude $[Gr(a,b)^p]\in\mathbb{Z}[\mathbb{A}^1_k]$, completing the proof. □

In fact, we will need somewhat more.

LEMMA 4.3. *Let Γ be a graph.*
(i) Let $e_0\in\Gamma$ be an edge. Define $\Gamma' = \Gamma\cup\varepsilon$, the graph obtained from Γ by adding an edge ε with $\partial\varepsilon = \partial e_0$. Then $X^\vee_{\Gamma'}$ is a cone over $X^\vee_\Gamma$.
(ii) Define $\Gamma'=\Gamma\cup\varepsilon$ where ε is a tadpole, i.e. $\partial\varepsilon = 0$. Then $X^\vee_{\Gamma'}$ is a cone over $X^\vee_\Gamma$.

PROOF. We prove (i). The proof of (ii) is similar and is left for the reader.

Let E,V be the edges and vertices of Γ. We have a diagram

$$\begin{tikzcd} \mathbb{Q}^E \arrow[r, "\partial"] \arrow[d] & \mathbb{Q}^V \arrow[d, equal] \\ \mathbb{Q}^E\oplus\mathbb{Q}\cdot\varepsilon \arrow[r, "\partial"] & \mathbb{Q}^V \end{tikzcd} \tag{4.13}$$

Dualizing and playing our usual game of interpreting edges as functionals on $\mathrm{Image}(\partial)^\vee \cong \mathbb{Q}^V/\mathbb{Q}$, we see that $\varepsilon^\vee = e_0^\vee$. Fix a basis for $\mathbb{Q}^V/\mathbb{Q}$ so the $(e^\vee)^2$

correspond to symmetric matrices M_e. We have

$$X_\Gamma^\vee : \det(\sum_E A_e M_e) = 0; \quad X_{\Gamma'}^\vee : \det(A_\varepsilon M_{e_0} + \sum_E A_e M_e) = 0. \tag{4.14}$$

The second polynomial is obtained from the first by the substitution $A_{e_0} \mapsto A_{e_0} + A_\varepsilon$. Geometrically, this is a cone as claimed. □

Let Γ_N be the complete graph on $N \geq 3$ vertices. Let $\Gamma \supset \Gamma_N$ be obtained by adding r new edges (but no new vertices) to Γ_N.

Proposition 4.4. *$[X_\Gamma^\vee] \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var_k)$.*

Proof. Note that every pair of distinct vertices in Γ_N are connected by an edge, so the r new edges e either duplicate existing edges or are tadpoles ($\partial e = 0$). It follows from lemma 4.3 that $X_\Gamma^\vee$ is an iterated cone over $\mathbb{X}_{\Gamma_N}^\vee$. In the Grothendieck ring, the class of a cone is the sum of the vertex point with a product of the base times an affine space, so we conclude from proposition 4.1. □

5. The Main Theorem

Fix $n \geq 3$. Let Γ_n be the complete graph on n vertices. It has $\binom{n}{2}$ edges. Recall (lemma 2.2) a set $\{e_1, \ldots, e_p\} \subset \text{edge}(\Gamma_n)$ is nondegenerate if cutting these edges (but leaving all vertices) does not disconnect Γ_n. (For the case $n = 3$ see (2.8) and (2.9).) Define

$$S_n := \sum_{\substack{\{e_1,\ldots,e_p\} \\ \text{nondegenerate}}} [X_{\Gamma_n - \{e_1,\ldots,e_p\}}] \in K_0(Var_k). \tag{5.1}$$

Let Γ be a connected graph with n vertices and no multiple edges or tadpoles. Let $G \subset Sym(\text{vert}(\Gamma))$ be the subgroup of the symmetric group on the vertices which acts on the set of edges. Then $[X_\Gamma]$ appears in S_n with multiplicity $n!/|G|$.

Theorem 5.1. *$S_n \in \mathbb{Z}[\mathbb{A}_k^1] \subset K_0(Var_k)$.*

Proof. It follows from (3.6) and proposition 4.1 that

$$\sum_{\substack{\{e_1,\ldots,e_p\} \\ \text{nondegenerate}}} [X^0_{\Gamma_n - \{e_1,\ldots,e_p\}}] \in \mathbb{Z}[\mathbb{A}_k^1]. \tag{5.2}$$

Write $\vec{e} = \{e_1, \ldots, e_p\}$ and let $\vec{f} = \{f_1, \ldots, f_q\}$ be another subset of edges. We will say the pair $\{\vec{e}, \vec{f}\}$ is nondegenerate if $\vec{e}$ is nondegenerate in the above sense, and if further $\vec{e} \cap \vec{f} = \emptyset$ and the edges of $\vec{f}$ do not support a loop. For $\{\vec{e}, \vec{f}\}$ nondegenerate, write $(\Gamma_n - \vec{e})/\vec{f}$ for the graph obtained from Γ_n by removing the edges in $\vec{e}$ and then contracting the edges in $\vec{f}$. If we fix a nondegenerate $\vec{e}$, we have (by an argument dual to that in section 3)

$$\sum_{\substack{\vec{f} \\ \{\vec{e},\vec{f}\} \text{ nondeg.}}} [X^0_{(\Gamma_n - \vec{e})/\vec{f}}] + t = [X_{\Gamma_n - \vec{e}}]. \tag{5.3}$$

Here $t \in \mathbb{Z}[\mathbb{A}^1]$ accounts for the $\vec{f}$ which support a loop. These give rise to degenerate edges in $X_{\Gamma_n - \vec{e}}$ which are linear spaces and hence have classes in $\mathbb{Z}[\mathbb{A}^1]$.

Summing now over both $\vec{e}$ and $\vec{f}$, we conclude

$$S_n \equiv \sum_{\substack{\{\vec{e},\vec{f}\} \\ \text{nondegen.}}} [X^0_{(\Gamma_n - \vec{e})/\vec{f}}] \mod \mathbb{Z}[\mathbb{A}^1]. \tag{5.4}$$

Note that if $\vec{e}, \vec{f}$ are disjoint and $\vec{f}$ does not support a loop, then $\vec{e}$ is nondegenerate in Γ_n if and only if it is nondegenerate in $\Gamma_n/\vec{f}$. This means we can rewrite (5.4)

$$S_n \equiv \sum_{\vec{f}} \sum_{\substack{\vec{e} \subset \Gamma_n/\vec{f} \\ \text{nondegen.}}} [X^0_{(\Gamma_n/\vec{f}) - \vec{e}}]. \tag{5.5}$$

Let $\vec{f} = \{f_1, \ldots, f_q\}$ and assume it does not support a loop. Then $\Gamma_n/\vec{f}$ has $n-q$ vertices, and every pair of distinct vertices is connected by at least one edge. This means we may embed $\Gamma_{n-q} \subset \Gamma_n/\vec{f}$ and think of $\Gamma_n/\vec{f}$ as obtained from Γ_{n-q} by adding duplicate edges and tadpoles. We then apply proposition 4.4 to conclude that $[X^\vee_{\Gamma_n/\vec{f}}] \in \mathbb{Z}[\mathbb{A}^1_k]$. Now arguing as in (3.6) we conclude

$$\sum_{\substack{\vec{e} \subset \Gamma_n/\vec{f} \\ \text{nondegen.}}} [X^0_{(\Gamma_n/\vec{f}) - \vec{e}}] \in \mathbb{Z}[\mathbb{A}^1_k] \tag{5.6}$$

Finally, plugging into (5.5) we get $S_n \in \mathbb{Z}[\mathbb{A}^1]$ as claimed. □

References

[1] Aluffi, P., and Marcolli, M., Feynman Motives of Banana Graphs, arXiv:0807.1690v2 [hep-th]

[2] Belkale, P., and Brosnan, P., Matroids, motives and a conjecture on Kontsevich, Duke Math. Journal, Vol. 116 (2003) 147-188.

[3] Bloch, S, Esnault, H., and Kreimer, D., On Motives Associated to Graph Polynomials, Comm. Math. Phys. **267** (2006), 181-225.

[4] Broadhurst, D.J., and Kreimer, D., Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Phys. Lett.**B 393** (1997) 403.

[5] Stanley, R., Spanning Trees and a Conjecture of Kontsevich, Annals of Combinatorics 2(1999), 351-363.

Dept. of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: bloch@math.uchicago.edu

Clay Mathematics Proceedings

Double shuffle relations and renormalization of multiple zeta values

Li Guo, Sylvie Paycha, Bingyong Xie, and Bin Zhang

Abstract. In this paper we present some of the recent progresses in multiple zeta values (MZVs). We review the double shuffle relations for convergent MZVs and summarize generalizations of the sum formula and the decomposition formula of Euler for MZVs. We then discuss how to apply methods borrowed from renormalization in quantum field theory and from pseudodifferential calculus to partially extend the double shuffle relations to divergent MZVs.

1. Introduction

The purpose of this paper is to give a survey of recent developments in multiple zeta values (MZVs). We emphasize on the double shuffle relations which underlie the algebraic relations among the convergent MZVs, and on renormalization methods that aim to extend the double shuffle relations to MZVs outside of the convergent range of the nested sums defining MZVs. We also provide background on double shuffle relations and renormalization, as well as the closely related Rota-Baxter algebras and some analytic tools in pseudodifferential calculus in view of renormalization.

1.1. Double shuffle relations and Euler's formulas. A **multiple zeta value (MZV)** is the special value of the complex valued function

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

at positive integers $s_1, \cdots, s_k$ with $s_1 \geqslant 2$ to insure the convergence of the nested sum. MZVs are natural generalizations of the Riemann zeta values $\zeta(s)$ to multiple variables. The two variable case (double zeta values) was already studied by Euler.

MZVs in the general case were introduced in the 1990s with motivations from number theory [**70**], combinatorics [**40**] and quantum field theory [**11**]. Since then the subject has turned into an active area of research that involves many areas of mathematics and mathematical physics [**13**]. Its number theoretic significance can be seen from the fact that all MZVs are periods of mixed Tate motives over $\mathbb{Z}$ and the conjecture that all periods of mixed Tate motives are rational combinations of MZVs [**25, 27, 69**].

It has been discovered that the analytically defined MZVs satisfy many algebraic relations. Further it is conjectured that these algebraic relations all follow from the combination of two algebra structures: the shuffle relation and the stuffle (harmonic shuffle or quasi-shuffle) relation [**46**]. This remarkable conjecture not only links the analytic study of MZVs to the algebraic study of double shuffle relations, but also implies the more well-known conjecture on the algebraic independence of $\zeta(2), \zeta(2k+1), k \geqslant 1$, over $\mathbb{Q}$.

Many results on algebraic relations among MZVs can be regarded as generalizations of Euler's sum formula and decomposition formula on double zeta values which preceded the general developments of multiple zeta values by over two hundred years. We summarize these results in Section 3. With the non-experts in mind, we first give in Section 2 preliminary concepts and results on double shuffle relations for MZVs and the related Rota-Baxter algebras.

1.2. Renormalization. Values of the Riemann zeta function at negative integers are defined by analytic continuation and possess significant number theory properties, such as Bernoulli numbers, Kummer congruences and p-adic L-functions. Thus it would be interesting to similarly study MZVs outside the convergent domain of the corresponding nested sums. However, most of the MZVs remain undefined even after the analytic continuation. To bring new ideas into the study, we introduce the method of renormalization from quantum field theory.

Renormalization is a process motivated by physical insight to extract finite values from divergent Feynman integrals in quantum field theory, after adding in a so-called counter-term. Despite its great success in physics, this process was well-known for its lack of a solid mathematical foundation until the seminal work of Connes and Kreimer [**14, 15, 16, 50**]. They obtained a Hopf algebra structure on Feynman graphs and showed that the separation of Feynman integrals into the renormalized values and the counter-terms comes from their algebraic Birkhoff decomposition similar to the Birkhoff decomposition of a loop map.

The work of Connes and Kreimer establishes a bridge that allows an exchange of ideas between physics and mathematics. In one direction, their work provides the renormalization of quantum field theory with a mathematical foundation which was previously missing, opening the door to further mathematical understanding of renormalization. For example, the related Riemann-Hilbert correspondence and motivic Galois groups were studied by Connes and Marcolli [**17**], and motivic properties of Feynman graphs and integrals were studied by Bloch, Esnault and Kreimer [**5**]. See [**2, 11, 56**] for more recent studies on the motivic aspect of Feynman rules and renormalization.

In the other direction, the mathematical formulation of renormalization provided by the algebraic Birkhoff decomposition allows the method of renormalization dealing with divergent Feynman integrals in physics to be applied to divergent problems in mathematics that could not be dealt with in the past, such as the divergence in MZVs [**36, 37, 55, 73**] and Chen symbol integrals [**54, 55**]. We survey these studies on renormalization in mathematics in Sections 5 and 6 after reviewing in Section 4 the general framework of algebraic Birkhoff decomposition in the context of Rota-Baxter algebras. We further present an alternative renormalization method using Speer's generalized evaluators [**67**] and show it leads to the same renormalized double zeta values as the algebraic Birkhoff decomposition method.

We hope our paper will expose this active area to a wide range of audience and promote its further study, to gain a more thorough understanding of the double shuffle relations for convergent MZVs and to establish a systematical renormalization theory for the divergent MZVs. One topic that we find of interest is to compare the various renormalization methods presented in this paper from an abstract point of view in terms of a renormalization group yet to be described in this context, again motivated by the study in quantum field theory. With implications back to physics in mind, we note that MZVs offer a relatively handy and tractable field of experiment for such issues when compared with the very complicated Feynman integral computations.

Acknowledgements: L. Guo acknowledges the support from NSF grant DMS-0505643 and thanks JAMI at Johns Hopkins University for its hospitality. S. Paycha is grateful to D. Manchon for his comments on a preliminary version of part of this paper. B. Xie is supported by postdoctoral grant 533149-087 provided by Peking University. B. Zhang acknowledges the support from NSFC grant 10631050. The authors also thank the referee for detailed comments.

2. Double shuffle relations for convergent multiple zeta values

All rings and algebras in this paper are assumed to be unitary unless otherwise specified. Let $\mathbf{k}$ be a commutative ring whose identity is denoted by 1.

2.1. Rota-Baxter algebras. Let $\lambda \in \mathbf{k}$ be fixed. A unitary (resp. nonunitary) **Rota–Baxter k-algebra (RBA) of weight** λ is a pair (R, P) in which R is a unitary (resp. nonunitary) $\mathbf{k}$-algebra and $P : R \to R$ is a $\mathbf{k}$-linear map such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, \ y \in R. \tag{1}$$

In some references such as [**55**], the notation $\theta = -\lambda$ is used.

We will mainly consider the following Rota-Baxter operators in this paper. See [**19, 30, 64**] for other examples.

Example 2.1. **(The integration operator)** Define the integration operator

$$I(f)(x) = \int_0^x f(t)dt \tag{2}$$

on the algebra $C[0, \infty)$ of continuous functions $f(x)$ on $[0, \infty)$. Then it follows from the integration by parts formula that I is a Rota-Baxter operator of weight 0 [**4**].

Example 2.2. **(The summation operator)** Consider the summation operator [**75**]

$$P(f)(x) := \sum_{n \geqslant 1} f(x + n).$$

Under certain convergency conditions, such as $f(x) = O(x^{-2})$ and $g(x) = O(x^{-2})$, $P(f)(x)$ and $P(g)(x)$ define absolutely convergent series and we have

$$\begin{aligned} P(f)(x)P(g)(x) &= \sum_{m \geqslant 1} f(x + m) \sum_{n \geqslant 1} g(x + n) \\ &= \sum_{n > m \geqslant 1} f(x + m)g(x + n) + \sum_{m > n \geqslant 1} f(x + m)g(x + n) \end{aligned} \tag{3}$$

$$+\sum_{m\geqslant 1} f(x+m)g(x+m)$$
$$= P(fP(g))(x) + P(gP(f))(x) + P(fg)(x).$$

Thus P is a Rota-Baxter operator of weight 1.

EXAMPLE 2.3. **(The partial sum operator)** The operator P defined on sequences $\sigma : \mathbb{N} \to \mathbb{C}$ by:

$$P(\sigma)(n) = \sum_{k=0}^{n} \sigma(k) \tag{4}$$

satisfies the Rota-Baxter relation with weight -1. Similarly, the operator $Q = P - Id$ which acts on sequences $\sigma : \mathbb{N} \to \mathbb{C}$ by:

$$Q(\sigma)(n) = \sum_{k=0}^{n-1} \sigma(k) \tag{5}$$

satisfies the Rota-Baxter relation with weight 1.

EXAMPLE 2.4. **(Laurent series)** Let $A = \mathbf{k}[\varepsilon^{-1}, \varepsilon]]$ be the algebra of Laurent series. Define $\Pi : A \to A$ by

$$\Pi\big(\sum_n a_n \varepsilon^n\big) = \sum_{n<0} a_n \varepsilon^n.$$

Then Π is a Rota-Baxter operator of weight -1.

2.2. Shuffles, quasi-shuffles and mixable shuffles. We briefly recall the construction of shuffle, stuffle and quasi-shuffle products in the framework of mixable shuffle algebras [**32, 33**].

Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra *that is not necessarily unitary.* For a given $\lambda \in \mathbf{k}$, the **mixable shuffle algebra of weight** λ **generated by** A (with coefficients in $\mathbf{k}$) is $\mathrm{MS}(A) = \mathrm{MS}_{\mathbf{k},\lambda}(A)$ whose underlying $\mathbf{k}$-module is that of the tensor algebra

$$T(A) = \bigoplus_{k\geq 0} A^{\otimes k} = \mathbf{k} \oplus A \oplus A^{\otimes 2} \oplus \cdots \tag{6}$$

equipped with the **mixable shuffle product** $\diamond_\lambda$ **of weight** λ defined as follows.

For pure tensors $\mathfrak{a} = a_1 \otimes \ldots \otimes a_k \in A^{\otimes k}$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_\ell \in A^{\otimes \ell}$, a **shuffle** of $\mathfrak{a}$ and $\mathfrak{b}$ is a tensor list of a_i and b_j without change the natural orders of the a_is and the b_js. More precisely, for $\sigma \in \Sigma_{k,\ell} := \{\tau \in S_{k+\ell} \mid \tau^{-1}(1) < \cdots < \tau^{-1}(k), \tau^{-1}(k+1) < \cdots < \tau^{-1}(k+\ell)\}$, the shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ by σ is

$$\mathfrak{a}ш_\sigma\mathfrak{b} := c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(k+\ell)}, \qquad \text{where } c_i = \begin{cases} a_i, & 1 \leqslant i \leqslant k, \\ b_{i-k}, & k+1 \leqslant i \leqslant k+\ell \end{cases}$$

The **shuffle product** of $\mathfrak{a}$ and $\mathfrak{b}$ is

$$\mathfrak{a}ш\mathfrak{b} := \sum_{\sigma\in\Sigma_{k,\ell}} \mathfrak{a}ш_\sigma\mathfrak{b}.$$

More generally, for a fixed $\lambda \in \mathbf{k}$, a **mixable shuffle** (of weight λ) of $\mathfrak{a}$ and $\mathfrak{b}$ is a shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ in which some (or *none*) of the pairs $a_i \otimes b_j$ are merged into $\lambda\, a_i b_j$. Then the **mixable shuffle product of weight** λ is defined by

$$\mathfrak{a}\diamond_\lambda\mathfrak{b} = \sum \text{ mixable shuffles of } \mathfrak{a} \text{ and } \mathfrak{b} \tag{7}$$

where the subscript λ is often suppressed when there is no danger of confusion. For example,

$$a_1 \diamond_\lambda (b_1 \otimes b_2) := \underbrace{a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1}_{\text{shuffles}} + \underbrace{\lambda(a_1 b_1) \otimes b_2 + \lambda b_1 \otimes (a_1 b_2)}_{\text{merged shuffles}}.$$

With $\mathbf{1} \in \mathbf{k}$ as the unit, this product makes $T(A)$ into a commutative $\mathbf{k}$-algebra that we denote by $\mathrm{MS}_{\mathbf{k},\lambda}(A)$. See [**32**] for further details on the mixable shuffle product. When $\lambda = 0$, we simply have the shuffle product which is also defined when A is only a $\mathbf{k}$-module, treated as an algebra with the zero multiplication.

We have the following relation between mixable shuffle product and free commutative Rota-Baxter algebras. A Rota-Baxter algebra homomorphism f between Rota-Baxter $\mathbf{k}$-algebras (R, P) and (R', P') is a $\mathbf{k}$-algebra homomorphism $f : R \to R'$ such that $f \circ P = P' \circ f$.

THEOREM 2.5. ([**32**]) *The tensor product algebra* $\text{Ш}(A) := \text{Ш}_{\mathbf{k},\lambda}(A) = A \otimes \mathrm{MS}_{\mathbf{k},\lambda}(A)$*, with the linear operator* $P_A : \text{Ш}(A) \to \text{Ш}(A)$ *sending* $\mathfrak{a} \to 1 \otimes \mathfrak{a}$*, is the free commutative Rota-Baxter algebra generated by* A *in the following sense. Let* $j_A : A \to \text{Ш}(A)$ *be the canonical inclusion map. Then for any Rota-Baxter* $\mathbf{k}$*-algebra* (R, P) *and any* $\mathbf{k}$*-algebra homomorphism* $\varphi : A \to R$*, there exists a unique Rota-Baxter* $\mathbf{k}$*-algebra homomorphism* $\tilde{\varphi} : (\text{Ш}(A), P_A) \to (R, P)$ *such that* $\varphi = \tilde{\varphi} \circ j_A$ *as* $\mathbf{k}$*-algebra homomorphisms.*

The product $\diamond_\lambda$ can also be defined by the following recursion [**18, 37, 55**] which provides the connection between mixable shuffle algebras and quasi-shuffle algebras of Hoffman [**42**]. First we define the multiplication by $A^{\otimes 0} = \mathbf{k}$ to be the scalar product. In particular, $\mathbf{1}$ is the identity. For any $k, \ell \geqslant 1$ and $\mathfrak{a} := a_1 \otimes \cdots \otimes a_k \in A^{\otimes k}$, $\mathfrak{b} := b_1 \otimes \cdots \otimes b_\ell \in A^{\otimes \ell}$, define $a \diamond_\lambda b$ by induction on the sum $k + \ell \geqslant 2$. When $k + \ell = 2$, we have $a = a_1$ and $b = b_1$. We define

$$a \diamond_\lambda b = a_1 \otimes b_1 + b_1 \otimes a_1 + \lambda a_1 b_1.$$

Assume that $\mathfrak{a} \diamond_\lambda \mathfrak{b}$ has been defined for $k + \ell \geqslant n \geqslant 2$ and consider $\mathfrak{a}$ and $\mathfrak{b}$ with $k + \ell = n + 1$. Then $k + \ell \geqslant 3$ and so at least one of k and ℓ is greater than 1. We define

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = \begin{cases} a_1 \otimes b_1 \otimes \cdots \otimes b_\ell + b_1 \otimes \big(a_1 \diamond_\lambda (b_2 \otimes \cdots \otimes b_\ell)\big) \\ \qquad + \lambda(a_1 b_1) \otimes b_2 \otimes \cdots \otimes b_\ell, \text{ when } k = 1, \ell \geqslant 2, \\ a_1 \otimes \big((a_2 \otimes \cdots \otimes a_k) \diamond_\lambda b_1\big) + b_1 \otimes a_1 \otimes \cdots \otimes a_k \\ \qquad + \lambda(a_1 b_1) \otimes a_2 \otimes \cdots \otimes a_k, \text{ when } k \geqslant 2, \ell = 1, \\ a_1 \otimes \big((a_2 \otimes \cdots \otimes a_k) \diamond_\lambda (b_1 \otimes \cdots \otimes b_\ell)\big) \\ \qquad + b_1 \otimes \big((a_1 \otimes \cdots \otimes a_k) \diamond_\lambda (b_2 \otimes \cdots \otimes b_\ell)\big) \\ \qquad + \lambda(a_1 b_1)\big((a_2 \otimes \cdots \otimes a_k) \diamond_\lambda (b_2 \otimes \cdots \otimes b_\ell)\big), \text{ when } k, \ell \geqslant 2. \end{cases}$$

Here the products by $\diamond_\lambda$ on the right hand side of the equation are well-defined by the induction hypothesis.

Let S be a semigroup and let $\mathbf{k}\, S = \sum_{s \in S} \mathbf{k}\, s$ be the semigroup nonunitary $\mathbf{k}$-algebra. A canonical $\mathbf{k}$-basis of $(\mathbf{k}\, S)^{\otimes k}, k \geqslant 0$, is the set $S^{\otimes k} := \{s_1 \otimes \cdots \otimes s_k \mid s_i \in S, 1 \leqslant i \leqslant k\}$. Let S be a graded semigroup $S = \coprod_{i \geqslant 0} S_i$, $S_i S_j \subseteq S_{i+j}$ such that $|S_i| < \infty$, $i \geqslant 0$. Then the mixable shuffle product $\diamond_1$ of weight 1 is identified with the **quasi-shuffle product** $*$ defined by Hoffman [**42, 18, 37**].

NOTATION 2.6. To simplify the notation and to be consistent with the usual notations in the literature on multiple zeta values, we will identify $s_1\otimes\cdots\otimes s_k$ with the concatenation $s_1\cdots s_k$ unless there is a danger of confusion. We also denote the weight 1 mixable shuffle product $\diamond_1$ by $*$ and denote the corresponding mixable algebra $\mathrm{MS}_{\mathbf{k},1}(A)$ by $\mathcal{H}^*_A$. Similarly, when A is taken to be a $\mathbf{k}$-module, we denote the weight zero mixable shuffle algebra $\mathrm{MS}_{\mathbf{k},0}(A)$ by $\mathcal{H}^{ш}_A$.

Yet another interpretation of the mixable shuffle or quasi-shuffle product can be given in terms of order preserving maps that are called **stuffle** in the study of MZVs but could be traced back to Cartier's work [**12**] on free commutative Rota-Baxter algebras.

For positive integers k and ℓ, denote $[k]=\{1,\cdots,k\}$ and $[k+1,k+\ell]=\{k+1,\cdots,k+\ell\}$. Define

$$\mathfrak{J}_{k,\ell}=\left\{(\varphi,\psi)\;\middle|\;\begin{array}{l}\varphi:[k]\to[k+\ell],\psi:[\ell]\to[k+\ell]\text{ are order preserving}\\ \text{injective maps and }\operatorname{im}(\varphi)\sqcup\operatorname{im}(\psi)=[k+\ell]\end{array}\right\}\tag{8}$$

Let $\mathfrak{a}\in A^{\otimes k}$, $\mathfrak{b}\in A^{\otimes \ell}$ and $(\varphi,\psi)\in\mathfrak{J}_{k,\ell}$. We define $\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}$ to be the tensor whose i-th factor is

$$(\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b})_i=\begin{cases}a_j & \text{if } i=\varphi(j)\\ b_j & \text{if } i=\psi(j)\end{cases}=a_{\varphi^{-1}(i)}b_{\psi^{-1}(i)},\quad 1\leqslant i\leqslant k+\ell,\tag{9}$$

with the convention that $a_\emptyset=b_\emptyset=1$. Then we have

$$\mathfrak{a}ш\mathfrak{b}=\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell}}\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}.\tag{10}$$

More generally, for $0\leqslant r\leqslant\min(k,\ell)$, define

$$\mathfrak{J}_{k,\ell,r}=\left\{(\varphi,\psi)\;\middle|\;\begin{array}{l}\varphi:[k]\to[k+\ell-r],\psi:[\ell]\to[k+\ell-r]\text{ are order preserving}\\ \text{injective maps and }\operatorname{im}(\varphi)\cup\operatorname{im}(\psi)=[k+\ell-r]\end{array}\right\}$$

Clearly, $\mathfrak{J}_{k,\ell,0}=\mathfrak{J}_{k,\ell}$. Let $\mathfrak{a}\in A^{\otimes k}$, $\mathfrak{b}\in A^{\otimes \ell}$ and $(\varphi,\psi)\in\mathfrak{J}_{k,\ell,r}$. We define $\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}$ to be the tensor whose i-th factor is

$$(\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b})_i=\left\{\begin{array}{ll}a_j & \text{if } i=\varphi(j), i\notin\operatorname{im}\psi\\ b_j & \text{if } i=\psi(j), i\notin\operatorname{im}\varphi\\ a_jb_{j'} & \text{if } i=\varphi(j), i=\psi(j')\end{array}\right\}=a_{\varphi^{-1}(i)}b_{\psi^{-1}(i)}, 1\leqslant i\leqslant k+\ell-r,$$

with the convention that $a_\emptyset=b_\emptyset=1$. Then we have [**32, 36**]

$$\mathfrak{a}\diamond_\lambda\mathfrak{b}=\sum_{r=0}^{\min(k,\ell)}\lambda^r\Big(\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell,r}}\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}\Big).\tag{11}$$

In particular,

$$\mathfrak{a}*\mathfrak{b}=\sum_{r=0}^{\min(k,\ell)}\Big(\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell,r}}\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}\Big)=\sum_{(\varphi,\psi)\in\bar{\mathfrak{J}}_{k,\ell}}\mathfrak{a}ш_{(\varphi,\psi)}\mathfrak{b}$$

where $\bar{\mathfrak{J}}_{k,\ell}=\cup_{r=0}^{\min(k,\ell)}\mathfrak{J}_{k,\ell,r}$.

Equivalently, let $\mathrm{stfl}(k,\ell,r)$ denote the set of surjective maps from $[k+\ell]$ to $[k+\ell-r]$ that preserve the natural orders of $[k]$ and $\{k+1,\cdots,k+\ell\}$. Let

$$\mathrm{stfl}(k,\ell)=\bigcup_{r=0}^{\min(k,\ell)}\mathrm{stfl}(k,\ell,r).$$

Then

$$(a_1\otimes\cdots\otimes a_k)*(a_{k+1}\otimes\cdots\otimes a_{k+\ell})=\sum_{\pi\in\mathrm{stfl}(k,\ell)} c_1^\pi\otimes\cdots\otimes c_{k+\ell-r}^\pi, \tag{12}$$

where $c_i^\pi=\prod_{j\in\pi^{-1}(i)}a_j$. A **connected filtered Hopf algebra** is a Hopf algebra (H,Δ) with $\mathbf{k}$-submodules $H^{(n)},\ n\geqslant 0$ of H such that

$$H^{(n)}\subseteq H^{(n+1)},\quad \cup_{n\geqslant 0}H^{(n)}=H,\quad H^{(p)}H^{(q)}\subseteq H^{(p+q)},$$
$$\Delta(H^{(n)})\subseteq \textstyle\sum_{p+q=n}H^{(p)}\otimes H^{(q)},\quad H^{(0)}=\mathbf{k}\ \text{(connectedness)}.$$

On the algebra $\mathrm{MS}_{\mathbf{k},\lambda}(A)$ further define

$$\begin{aligned}\Delta(a_1\otimes\cdots\otimes a_n)&=1\bigotimes(a_1\otimes\cdots\otimes a_n)+a_1\bigotimes(a_2\otimes\cdots\otimes a_n)\\ &\quad+\cdots+(a_1\otimes\cdots a_{n-1})\bigotimes a_n+(a_1\otimes\cdots\otimes a_n)\otimes 1.\end{aligned} \tag{13}$$

Then Δ extends by linearity to a linear map $\mathrm{MS}_{\mathbf{k},\lambda}(A)\to\mathrm{MS}_{\mathbf{k},\lambda}(A)\bigotimes\mathrm{MS}_{\mathbf{k},\lambda}(A)$.

THEOREM 2.7. *(*[**37, 42, 55**]*) The triple* $(\mathrm{MS}_{\mathbf{k},\lambda}(A),\diamond_\lambda,\Delta)$*, together with the unit* $u:\mathbf{k}\hookrightarrow\mathrm{MS}_{\mathbf{k},\lambda}(A)$ *and the counit* $\varepsilon:\mathrm{MS}_{\mathbf{k},\lambda}(A)\to\mathbf{k}$ *projecting onto the direct summand* $\mathbf{k}\subseteq\mathrm{MS}_{\mathbf{k},\lambda}(A)$*, equips* $\mathrm{MS}_{\mathbf{k},\lambda}(A)$ *with the structure of a connected filtered Hopf algebra with the filtration* $\mathrm{MS}(A)^{(n)}:=\sum_{i\leqslant n}A^{\otimes i}$.

We also have the following easy extension of Hoffman's isomorphism between the shuffle Hopf algebra and the quasi-shuffle Hopf algebra (see also [**18**]). Recall the notation $\mathcal{H}_A^*=\mathrm{MS}_{\mathbf{k},1}(A)$ and $\mathcal{H}_A^{ш}=\mathrm{MS}_{\mathbf{k},0}(A)$.

THEOREM 2.8. ([**42, 55**]) *Let* $\mathbf{k}$ *be a* $\mathbb{Q}$*-algebra. There is an isomorphism of Hopf algebras :*

$$\exp:\mathcal{H}_A^{ш}\xrightarrow{\sim}\mathcal{H}_A^*. \tag{14}$$

Hoffman's isomorphism (14) is built explicitly as follows. Let $\mathcal{P}(n)$ be the set of compositions of the integer n, i.e. the set of sequences $I=(i_1,\ldots,i_k)$ of positive integers such that $i_1+\cdots+i_k=n$. For any $u=v_1\otimes\cdots\otimes v_n\in T(A)$ and any composition $I=(i_1,\ldots,i_k)$ of n we set:

$$I[u]:=(v_1\cdot\cdots\cdot v_{i_1})\otimes(v_{i_1+1}\cdot\cdots\cdot v_{i_1+i_2})\otimes\cdots\otimes(v_{i_1+\cdots+i_{k-1}+1}\cdot\cdots\cdot v_n).$$

Then the isomorphism exp is defined by

$$\exp u=\sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{1}{i_1!\cdots i_k!}I[u].$$

Moreover ([**42**], Lemma 2.4), the inverse log of exp is given by :

$$\log u=\sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{(-1)^{n-k}}{i_1\cdots i_k}I[u].$$

2.3. Double shuffle of MZVs and related conjectures. A multiple zeta value (MZV) is defined to be

$$\zeta(s_1,\cdots,s_k):=\sum_{n_1>\cdots>n_k\geqslant 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}} \tag{15}$$

where $s_i \geqslant 1$ and $s_1 > 1$ are integers. As is well-known, an MZV has an integral representation due to Kontsevich [**51**]

$$\zeta(s_1, \cdots, s_k) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{|\vec{s}|-1}} \frac{dt_1}{f_1(t_1)} \cdots \frac{dt_{|\vec{s}|}}{f_{|\vec{s}|}(t_{|\vec{s}|})} \tag{16}$$

Here $|\vec{s}| = s_1 + \cdots + s_k$ and

$$f_j(t) = \begin{cases} 1 - t_j, \; j = s_1, s_1 + s_2, \cdots, s_1 + \cdots + s_k, \\ t_j, \qquad \text{otherwise.} \end{cases}$$

The MZVs spanned the following $\mathbb{Q}$-subspace of $\mathbb{R}$

$$\mathbf{MZV} := \mathbb{Q}\{\zeta(s_1, \cdots, s_k) \mid s_i \geqslant 1, s_1 \geqslant 2\} \subseteq \mathbb{R}.$$

Since the summation operator in Eq. (15) and the integral operator in Eq. (2) are both Rota-Baxter operators (of weight 1 and 0 respectively) by Example 2.2 and Example 2.1, it can be expected that the multiplication of two MZVs follows the multiplication rule in a free Rota-Baxter algebra and thus in a mixable shuffle algebra. This viewpoint naturally leads to the following double shuffle relations of MZVs.

For the sum representation of MZVs in Eq. (15), consider the semigroup

$$Z := \{z_s \mid s \in \mathbb{Z}_{\geqslant 1}, \quad z_s \cdot z_t = z_{s+t}, s, t \geqslant 1.\}$$

With the convention in Notation 2.6, we denote the quasi-shuffle algebra $\mathcal{H}^* := \mathcal{H}^*_{\mathbb{Q}\, Z}$ which contains the subalgebra

$$\mathcal{H}^*_0 := \mathbb{Q} \oplus \Big(\bigoplus_{s_1 > 1} \mathbb{Q} z_{s_1} \cdots z_{s_k} \Big).$$

Then the multiplication rule of MZVs according to their summation representation follows from the fact that the linear map

$$\zeta^* : \mathcal{H}^*_0 \to \mathbf{MZV}, \quad z_{s_1, \cdots, s_k} \mapsto \zeta(s_1, \cdots, s_k) \tag{17}$$

is an algebra homomorphism [**41, 46**].

For the integral representation of MZVs in Eq. (16), consider the set $X = \{x_0, x_1\}$. With the convention in Notation 2.6, we denote the shuffle algebra $\mathcal{H}^{\text{Ш}} := \mathcal{H}^{\text{Ш}}_{\mathbb{Q}\, X}$ which contains subalgebras

$$\mathcal{H}^{\text{Ш}}_0 := \mathbb{Q} \oplus x_0 \mathcal{H}^{\text{Ш}} x_1 \subseteq \mathcal{H}^{\text{Ш}}_1 := \mathbb{Q} \oplus \mathcal{H}^{\text{Ш}} x_1 \subseteq \mathcal{H}^{\text{Ш}}.$$

Then the multiplication rule of MZVs according to their integral representations follows from the statement that the linear map

$$\zeta^{\text{Ш}} : \mathcal{H}^{\text{Ш}}_0 \to \mathbf{MZV}, \quad x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1 \mapsto \zeta(s_1, \cdots, s_k)$$

is an algebra homomorphism [**41, 46**].

There is a natural bijection of $\mathbb{Q}$-vector spaces (but *not* algebras)

$$\eta : \mathcal{H}^{\text{Ш}}_1 \to \mathcal{H}^*, \quad 1 \leftrightarrow 1, \; x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1 \leftrightarrow z_{s_1, \cdots, s_k}.$$

that restricts to a bijection of vector spaces $\eta : \mathcal{H}^{\text{Ш}}_0 \to \mathcal{H}^*_0$. Then the fact that MZVs can be multiplied in two ways is reflected by the commutative diagram

$$\begin{tikzcd} \mathcal{H}^{\text{Ш}}_0 \arrow[rr, "\eta"] \arrow[dr, "\zeta^{\text{Ш}}"'] & & \mathcal{H}^*_0 \arrow[dl, "\zeta^*"] \\ & \mathbf{MZV} & \end{tikzcd}$$

Through η, the shuffle product ш on $\mathcal{H}_1^{\text{ш}}$ and $\mathcal{H}_0^{\text{ш}}$ transports to a product ш_* on $\mathcal{H}^*$ and $\mathcal{H}_0^*$. That is, for $w_1, w_2 \in \mathcal{H}_0^*$, define

$$w_1 \text{ш}_* w_2 := \eta(\eta^{-1}(w_1)\text{ш}\eta^{-1}(w_2)). \tag{18}$$

Then the **double shuffle relation** is simply the set

$$\{w_1 \text{ш}_* w_2 - w_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^*\}$$

and the **extended double shuffle relation** [**46, 63, 75**] is the set

$$\{w_1 \text{ш}_* w_2 - w_1 * w_2,\ z_1 \text{ш}_* w_2 - z_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^*\}. \tag{19}$$

THEOREM 2.9. **([41, 46, 63])** *Let $I_{\mathbf{EDS}}$ be the ideal of $\mathcal{H}_0^*$ generated by the extended double shuffle relation in Eq. (19). Then $I_{\mathbf{EDS}}$ is in the kernel of ζ^*.*

It is conjectured that $I_{\mathbf{EDS}}$ is in fact the kernel of ζ^*. A consequence of this conjecture is the irrationality of $\zeta(2n+1), n \geqslant 1$.

3. Generalizations of Euler's formulas

We begin with stating Euler's sum and decomposition formulas in Section 3.1. Generalizations of Euler's sum formula are presented in Section 3.2 and generalizations of Euler's decomposition formula are presented in Section 3.3.

3.1. Euler's sum and decomposition formulas. Over two hundred years before the general study of multiple zeta values was started in the 1990s, Goldbach and Euler had already considered the two variable case, the double zeta values [**23, 66**]

$$\zeta(s_1, s_2) := \sum_{n_1 > n_2 \geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2}}.$$

Among Euler's major discoveries on double zeta values are his **sum formula**

$$\sum_{i=2}^{n-1} \zeta(i, n-i) = \zeta(n)$$

expressing one Riemann zeta values as a sum of double zeta values and the **decomposition formula**

$$\zeta(r)\zeta(s) = \sum_{k=0}^{s-1} \binom{r+k-1}{k} \zeta(r+k, s-k) + \sum_{k=0}^{r-1} \binom{s+k-1}{k} \zeta(s+k, r-k), \quad r, s \geqslant 2, \tag{20}$$

expressing the product of two Riemann zeta values as a sum of double zeta values.

A major aspect of the study of MZVs is to find algebraic and linear relations among MZVs, such as Euler's formulas. Indeed a large part of this study can be viewed as generalizations of Euler's formulas.

3.2. Generalizations of Euler's sum formula. Soon after MZVs were introduced, Euler's sum formula was generalized to MZVs [**40, 28, 71**] as the well-known sum formula, followed by quite a few other generalizations that we will next summarize.

3.2.1. *Sum formula.* The first generalization of Euler's sum formula is the sum formula conjectured in [**40**]. Let

$$I(n,k)=\{(s_1,\cdots,s_k)\mid s_1+\cdots+s_k=n, s_i\geqslant 1, s_1\geqslant 2\}. \tag{21}$$

For $\vec{s}=(s_1,\cdots,s_k)\in I(n,k)$, define the **multiple zeta star value (or non-strict MZV)**

$$\zeta^\star(s_1,\cdots,s_k)=\sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}. \tag{22}$$

Note the subtle difference between the notations ζ^* in Eq. (17) and $\zeta^\star$ in Eq. (22).

THEOREM 3.1. (**Sum formula**) *For positive integers $k<n$ we have*

$$\sum_{\vec{s}\in I(n,k)}\zeta(\vec{s})=\zeta(n),\qquad \sum_{\mathbf{k}\in I(n,k)}\zeta^\star(\vec{s})=\binom{n-1}{k-1}\zeta(n). \tag{23}$$

The case of $k=3$ was proved by M. Hoffman and C. Moen [**43**] and the general case was proved by Zagier [**71**] with another proof given by Granville [**28**]. Later S. Kanemitsu, Y. Tanigawa, M. Yoshimoto [**48**] gave a proof for the case of $k=2$ using Mellin transformation.

J.-I. Okuda and K. Ueno [**62**] gave the following version of the sum formula

$$\sum_{k=r}^{n}\binom{k-1}{r-1}\Big(\sum_{\vec{s}\in I(n,k)}\zeta(\vec{s})\Big)=\binom{n-1}{r}\zeta(n)$$

for $n>r\geqslant 1$ from which they deduced the sum formula Eq. (23).

3.2.2. *Ohno's generalized duality theorem.* Another formula conjectured in [**40**] is the **duality formula**. To state the duality formula, we need an involution τ on the set of finite sequences of positive integers whose first element is greater than 1. If

$$\vec{s}=(1+b_1,\underbrace{1,\cdots,1}_{a_1-1},\cdots,1+b_k,\underbrace{1,\cdots,1}_{a_k-1}),$$

then

$$\tau(\vec{s})=(1+a_k,\underbrace{1,\cdots,1}_{b_k-1},\cdots,1+a_1,\underbrace{1,\cdots,1}_{b_1-1}).$$

THEOREM 3.2. **(Duality formula)**

$$\zeta(\vec{s})=\zeta(\tau(\vec{s})).$$

This formula is an immediate consequence of the integral representation in Eq. (16).

Y. Ohno [**57**] provided a generalization of both the sum formula and the duality formula.

THEOREM 3.3. (**Generalized Duality Formula** [**57**]) *For any index set $\vec{s}=(s_1,\cdots,s_k)$ with $s_1\geqslant 2, s_2\geqslant 1,\cdots,s_k\geqslant 1$, and a nonnegative integer ℓ, set*

$$Z(s_1,\cdots,s_k;\ell)=\sum_{\substack{c_1+\cdots+c_k=\ell\\ c_i\geqslant 0}}\zeta(s_1+c_1,\cdots,s_k+c_k).$$

Then

$$Z(\vec{s};\ell) = Z(\tau(\vec{s});\ell).$$

When $\ell = 0$, this is just the duality formula. When $\vec{s} = (k+1)$ and $\ell = n-k-1$, this becomes the sum formula.

3.2.3. *Sum formulas with further conditions on the variables.* M. Hoffman and Y. Ohno [**44**] gave a cyclic generalization of the sum formula.

THEOREM 3.4. **(Cyclic sum formula)** *For any positive integers $s_1, \cdots, s_k$ with some $s_i \geqslant 2$,*

$$\begin{aligned}&\sum_{i=1}^{k} \zeta(s_i+1, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}) \\ &= \sum_{\{i \mid s_i \geqslant 2\}} \sum_{j=0}^{s_i-2} \zeta(s_i - j, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}, j+1).\end{aligned}$$

Y. Ohno and N. Wakabayashi [**59**] gave a cyclic sum formula for non-strict MZVs and used it to prove the sum formula Eq. (23).

THEOREM 3.5. (**Cyclic sum formula in the non-strict case**) *For positive integers $k < n$ and $(s_1, \cdots, s_k) \in I(n,k)$ we have*

$$\sum_{i=1}^{k} \sum_{j=0}^{s_i-2} \zeta^\star(s_i - j, s_{i+1}, \cdots, s_k, s_1, \cdots, s_{i-1}, j+1) = n\zeta(n+1), \tag{24}$$

where the empty sums are zero.

M. Eie, W.-C. Liaw and Y. L. Ong [**22**] gave a generalization of the sum formula by allowing a more general form in the arguments in the MZVs.

THEOREM 3.6. *For all positive integers n, k with $n > k$, and a nonnegative integer p,*

$$\sum_{\substack{s_1+\cdots+s_k=n \\ s_1 \geqslant 2}} \zeta(s_1, \cdots, s_k, \{1\}^p) = \sum_{\substack{c_1+\cdots+c_{p+1}=n+p \\ c_1 \geqslant n-k+1}} \zeta(c_1, \cdots, c_{p+1}).$$

When $p = 0$, this becomes the sum formula.

Y. Ohno and D. Zagier [**60**] studied another kind of sum with certain restrictive conditions. Let

$$I(n,k,r) = \{(s_1, \cdots, s_k) \mid s_i \in \mathbb{Z}_{\geqslant 1}, s_1 + \cdots + s_k = n, \#\{s_i \mid s_i \geqslant 2\} = r\}$$

and put

$$G(n,k,r) = \sum_{\vec{s} \in I(n,k,r)} \zeta(\vec{s}).$$

They studied the associated generating function

$$\Phi(x,y,z) = \sum_{r \geqslant 1, k \geqslant r, n \geqslant k+r} G(n,k,r) x^{n-k-r} y^{k-r} z^{r-1} \in \mathbb{R}[x,y,z]$$

and proved the following

THEOREM 3.7. *We have*

$$\Phi(x,y,z) = \frac{1}{xy-z}\left(1-\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}S_n(x,y,z)\right)\right),$$

where $S_n(x,y,z)$ *are given by the identity*

$$\log\left(1-\frac{xy-z}{(1-x)(1-y)}\right) = \sum_{n=2}^{\infty}\frac{S_n(x,y,z)}{n}$$

and the requirement that $S_n(x,y,z^2)$ *is a homogeneous polynomial of degree* n. *In particular, all of the coefficients* $G(n,k,r)$ *can be expressed as polynomials in* $\zeta(2),\zeta(3),\ldots$ *with rational coefficients.*

3.2.4. *Sum formulas for q-MZVs.* The concept of q-multiple zeta values (q-MZVs, or multiple q-zeta values) was introduced as a "quantumization" of MZVs that recovers MZVs when $q \mapsto 1$ [**9, 72**].

For positive integers $s_1,\cdots,s_k$ with $s_1 \geqslant 2$, define the q-MZV

$$\zeta_q(s_1,\cdots,s_k) = \sum_{n_1>\cdots>n_k\geqslant 1}\frac{q^{n_1(s_1-1)+\cdots+n_k(s_k-1)}}{[n_1]^{s_1}\cdots[n_k]^{s_k}}$$

and the non-strict q-MZV

$$\zeta_q^{\star}(s_1,\cdots,s_k) = \sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 1}\frac{q^{n_1(s_1-1)+\cdots+n_k(s_k-1)}}{[n_1]^{s_1}\cdots[n_k]^{s_k}},$$

where $[n] = \frac{1-q^n}{1-q}$.

D. M. Bradley [**9**] proved the q-analogue of the sum formula for ζ_q.

THEOREM 3.8. (q**-analogue of the sum formula**) *For positive integers* $0 < k < n$ *we have*

$$\sum_{\substack{s_i\geqslant 1, s_1\geqslant 2\\ s_1+\cdots+s_k=n}}\zeta_q(s_1,\cdots,s_k) = \zeta_q(n). \tag{25}$$

Y. Ohno and J.-I. Okuda [**58**] gave the following q-analogue of the cyclic sum formula (24) and then used it to prove a q-analogue of the sum formula for $\zeta_q^{\star}$.

THEOREM 3.9. (q**-analogue of the cyclic sum formula**) *For positive integers* $0 < k < n$ *and* $(s_1,\cdots,s_k) \in I(n,k)$ *we have*

$$\sum_{i=1}^{k}\sum_{j=0}^{s_i-2}\zeta_q^{\star}(s_i-j,s_{i+1},\cdots,s_k,s_1,\cdots,s_{i-1},j+1) = \sum_{\ell=0}^{k}(n-\ell)\binom{k}{\ell}(1-q)^{\ell}\zeta_q(n-\ell+1),$$

where the empty sums are zero.

THEOREM 3.10. (q**-analogue of the sum formula in the non-strict case**) *For positive integers* $0 < k < n$ *we have*

$$\sum_{\substack{s_i\geqslant 1, s_1\geqslant 2\\ s_1+\cdots+s_k=n}}\zeta_q^{\star}(s_1,\cdots,s_k) = \frac{1}{n-1}\binom{n-1}{k-1}\sum_{\ell=0}^{k-1}(n-1-\ell)(1-q)^{\ell}\zeta_q(n-\ell). \tag{26}$$

3.2.5. *Weighted sum formulas.* Among other directions to generalize Euler's sum formula, there is the weighted version of Euler's sum formula recently obtained by Ohno and Zudilin [**61**].

THEOREM 3.11. (**Weighted Euler's sum formula** [**61**]) *For any integer $n \geqslant 2$, we have*

$$\sum_{i=2}^{n-1} 2^i \zeta(i, n-i) = (n+1)\zeta(n). \tag{27}$$

They applied it to study multiple zeta star values. By the sum formula, Eq. (27) is equivalent to the following equation

$$\sum_{i=2}^{n-1} (2^i - 1)\zeta(i, n-i) = n\zeta(n). \tag{28}$$

As a generalization of Eq. (28), two of the authors proved the following

THEOREM 3.12. (**Weighted sum formula** [**35**]) *For integers $k \geqslant 2$ and $n \geqslant k+1$, we have*

$$n\zeta(n) = \sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1 + \cdots + s_k = n}} \Big[2^{s_1 - 1} + (2^{s_1-1} - 1)\Big(\big(\sum_{i=2}^{k-1} 2^{S_i - s_1 - (i-1)}\big) + 2^{S_{k-1} - s_1 - (k-2)} \Big)\Big] \zeta(s_1, \cdots, s_k),$$

where $S_i = s_1 + \cdots + s_i$ for $i = 1, \cdots, k-1$.

3.3. Generalizations of Euler's decomposition formula. Unlike the numerous generalizations of Euler's sum formula, no generalization of Euler's decomposition formula to MZVs, neither proved nor conjectured, had been given until [**34**] even though Euler's decomposition formula was recently revisited in connection with modular forms [**24**] and weighted sum formula [**61**] on weighted sum formula of double zeta values, and was generalized to the product of two q-zeta values [**10, 72**].

3.3.1. *Euler's decomposition formula and double shuffle.* The first step in generalizing Euler's decomposition formula is to place it as a special case in a suitable broader context. In [**34**], Euler's decomposition formula was shown to be a special case of the double shuffle relation. We give a proof of Euler's formula in this context before presenting its generalization in the next subsection.

We recall that the extended double shuffle relation is the set

$$\{ w_1 ш_* w_2 - w_1 * w_2,\ z_1 ш_* w_2 - z_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^* \}.$$

Thus the determination of the double shuffle relation amounts to computing the two products $*$ and $ш_*$.

It is straightforward to compute the product $*$, either from its recursive definition in Eq. (8) or its explicit interpretation as mixable shuffles in Eq. (7) and stuffles in Eq. (11) or (12). For example, to determine the double shuffle relation from multiplying two Riemann zeta values $\zeta(r)$ and $\zeta(s)$, $r, s \geqslant 2$, one uses their sum representations and easily gets the quasi-shuffle relation

$$\zeta(r)\zeta(s) = \zeta(r,s) + \zeta(s,r) + \zeta(r+s).$$

On the other hand, computing the product $ш_*$ is more involved as can already be seen from its definition in Eq. (18). One first needs to use their integral representations to express $\zeta(r)$ and $\zeta(s)$ as iterated integrals of dimensions r and s, respectively. One then uses the shuffle relation to express the product of these two iterated integrals as a sum of $\binom{r+s}{r}$ iterated integrals of dimension $r+s$. Finally, these last iterated integrals are translated back to MZVs and give the shuffle relation of $\zeta(r)\zeta(s)$. As an illustrating example, consider $\zeta(100)\zeta(200)$. The quasi-shuffle relation is simply $\zeta(100)\zeta(200) = \zeta(100, 200) + \zeta(200, 100) + \zeta(300)$, but the shuffle relation is a large sum of $\binom{300}{100}$ shuffles of length (dimension) 300. As we will show below, an explicit formula for this is precisely Euler's decomposition formula (20).

THEOREM 3.13. *For $r, s \geqslant 2$, we have*

$$z_r ш_* z_s = \sum_{k=0}^{s-1} \binom{r+k-1}{k} z_{r+k} z_{s-k} + \sum_{k=0}^{r-1} \binom{s+k-1}{k} z_{s+k} z_{r-k}.$$

Via the algebra homomorphism ζ^* in Eq. (17) this theorem immediately gives Euler's decomposition formula. Applying to the above example, we have

$$\zeta(100)\zeta(200) = \sum_{k=0}^{199} \binom{100+k-1}{k} \zeta(100+k, 200-k) + \sum_{k=0}^{99} \binom{200+k-1}{k} \zeta(200+k, 100-k).$$

PROOF. Following the definition of $ш_*$ in Eq. (18), we have

$$z_r ш_* z_s = \eta(x_0^{r-1} x_1 ш x_0^{s-1} x_1).$$

So we just need to prove

$$x_0^{r-1} x_1 ш x_0^{s-1} x_1 = \sum_{k=0}^{s-1} \binom{r+k-1}{k} x_0^{r+k-1} x_1 x_0^{s-k-1} x_1 + \sum_{k=0}^{r-1} \binom{s+k-1}{k} x_0^{s+k-1} x_1 x_0^{r-k-1} x_1$$

since $ш_* (x_0^{r+k-1} x_1 x_0^{s-k-1} x_1) = z_{r+k} z_{s-k}$ and $ш_* (x_0^{s+k-1} x_1 x_0^{r-k-1} x_1) = z_{s+k} z_{r-k}$. This has a direct shuffle proof [**8**]. But we use the description of order preserving maps of shuffles in order to motivate the general case.

By Eq. (10), we have

$$x_0^{r-1} x_1 ш x_0^{s-1} x_1 = \sum_{(\varphi,\psi)\in \mathcal{I}(r,s)} x_0^{r-1} x_1 ш_{(\varphi,\psi)} x_0^{s-1} x_0.$$

Since φ and ψ are order preserving, we have the disjoint union $\mathcal{I}(r, s) = \mathcal{I}(r, s)' \sqcup \mathcal{I}(r, s)''$ where

$$\mathcal{I}(r, s)' = \{(\varphi, \psi) \in \mathcal{I}(r, s) \mid \psi(s) = r + s\}$$

and

$$\mathcal{I}(r, s)'' = \{(\varphi, \psi) \in \mathcal{I}(r, s) \mid \varphi(r) = r + s\}.$$

Again by the order preserving property, for $(\varphi, \psi) \in \mathcal{I}(r, s)'$, we must have $\varphi(r) = r + k$ where $k \geqslant 0$. Thus for such (φ, ψ), we have

$$x_0^{r-1} x_1 ш_{(\varphi,\psi)} x_0^{s-1} x_1 = x_0^{r-1+k} x_1 x_0^{s-1-k} x_1$$

since $\operatorname{im}\varphi \sqcup \operatorname{im}\psi = [r+s]$. For fixed $k \geqslant 0$, $\varphi(r) = r + k$ means that there are k elements $i_1, \cdots, i_k$ from $[s-1]$ such that $\psi(i_j) \in [r+k-1]$ since $\psi(s) = r+s$. Thus $k \geqslant s-1$ and, since ψ is order preserving, we have $\{i_1, \cdots, i_k\} = [k]$. Further there are $\binom{r+k-1}{k}$ such ψ's since $\psi([k])$ can take any k places in $[r+k-1]$ in increasing order and then $\phi([r])$ takes the rest places in increase order. Thus

$$\sum_{(\varphi,\psi)\in\mathcal{I}(r,s)'} x_0^{r-1}x_1 ш_{(\varphi,\psi)} x_0^{s-1}x_1 = \sum_{k=0}^{s-1}\binom{r+k-1}{k} x_0^{r+k-1}x_1x_0^{s-k-1}x_1.$$

By a similar argument, we have

$$\sum_{(\varphi,\psi)\in\mathcal{I}(r,s)''} x_0^{r-1}x_1 ш_{(\varphi,\psi)} x_0^{s-1}x_1 = \sum_{k=0}^{r-1}\binom{s+k-1}{k} x_0^{s+k-1}x_1x_0^{r-k-1}x_1.$$

This completes the proof. $\square$

3.3.2. *Generalizations of Euler's decomposition formula.* In a recent work [**34**], two of the authors generalized Euler's decomposition formula in two directions, from the product of one variable functions to that of multiple variables and from multiple zeta values to multiple polylogarithms.

A **multiple polylogarithm value** [**7, 25, 26**] is defined by

$$\mathrm{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) := \sum_{n_1>\cdots>n_k\geqslant 1}\frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

where $|z_i|\leqslant 1$, $s_i\in\mathbb{Z}_{\geqslant 1}$, $1\leqslant i\leqslant k$, and $(s_1,z_1)\neq(1,1)$. When $z_i=1, 1\leqslant i\leqslant k$, we obtain the multiple zeta values $\zeta(s_1,\cdots,s_k)$. With the notation of [**7**], we have

$$\mathrm{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) = \lambda\binom{s_1,\cdots,s_k}{b_1,\cdots,b_k} := \sum_{n_1>n_2\cdots>n_k\geqslant 1}\frac{\left(\frac{1}{b_1}\right)^{n_1}\left(\frac{b_1}{b_2}\right)^{n_2}\cdots\left(\frac{b_{k-1}}{b_k}\right)^{n_k}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}},$$

where $(b_1,\cdots,b_k)=(z_1^{-1},(z_1z_2)^{-1},\cdots,(z_1\cdots z_k)^{-1})$.

To state the result, let k and ℓ be positive integers and let $\mathcal{I}_{k,\ell}$ be as defined in Eq. (8). Let $\vec{r}=(r_1,\cdots,r_k)\in\mathbb{Z}_{\geqslant 1}^k$, $\vec{s}=(s_1,\cdots,s_\ell)\in\mathbb{Z}_{\geqslant 1}^\ell$ and $\vec{t}=(t_1,\cdots,t_{k+\ell})\in\mathbb{Z}_{\geqslant 1}^{k+\ell}$ with $|\vec{r}|+|\vec{s}|=|\vec{t}|$. Here $|\vec{r}|=r_1+\cdots+r_k$ and similarly for $|\vec{s}|$ and $|\vec{t}|$. Denote $R_i=r_1+\cdots+r_i$ for $i\in[k]$, $S_i=s_1+\cdots+s_i$ for $i\in[\ell]$ and $T_i=t_1+\cdots+t_i$ for $i\in[k+\ell]$. For $(\varphi,\psi)\in\mathcal{I}_{k,\ell}$ and $i\in[k+\ell]$, define

$$h_{(\varphi,\psi),i}=h_{(\varphi,\psi),(\vec{r},\vec{s}),i}=\begin{cases} r_j & \text{if } i=\varphi(j)\\ s_j & \text{if } i=\psi(j)\end{cases} = r_{\varphi^{-1}(i)}s_{\psi^{-1}(i)},$$

with the convention that $r_\emptyset=s_\emptyset=1$.

With these notations, we define

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)=\begin{cases}\binom{t_i-1}{h_{(\varphi,\psi),i}-1} & \text{if } i=1, \text{if } i-1,i\in\mathrm{im}(\varphi)\\ & \text{or if } i-1,i\in\mathrm{im}(\psi),\\ \binom{t_i-1}{T_i-R_{|\varphi^{-1}([i])|}-S_{|\psi^{-1}([i])|}} & \\ =\binom{t_i-1}{\sum_{j=1}^{i}t_j-\sum_{j=1}^{i}h_{(\varphi,\psi),j}} & \text{otherwise.}\end{cases}\tag{29}$$

For $\vec{a}\in(S^1)^k$ and $\vec{b}\in(S^1)^\ell$, as in Eq. (9), define

$$\vec{a}ш_{(\varphi,\psi)}\vec{b}=(a_{\varphi^{-1}(1)}b_{\psi^{-1}(1)},\cdots,a_{\varphi^{-1}(k+\ell)}b_{\psi^{-1}(k+\ell)}).\tag{30}$$

THEOREM 3.14. ([**34**]) *Let k,ℓ be positive integers. Let $\vec{r}\in\mathbb{Z}^k_{\geqslant 1}$ and $\vec{s}\in\mathbb{Z}^\ell_{\geqslant 1}$. Let $\vec{a}=(a_1,\cdots,a_k)\in(S^1)^k$ and $\vec{b}=(b_1,\cdots,b_\ell)\in(S^1)^\ell$ such that $\left[{r_1\atop a_1}\right]\neq\left[{1\atop 1}\right]$ and $\left[{s_1\atop b_1}\right]\neq\left[{1\atop 1}\right]$. Then*

$$\lambda\Big({\vec{r}\atop\vec{a}}\Big)\,\lambda\Big({\vec{s}\atop\vec{b}}\Big)=\sum_{\vec{t}\in\mathbb{Z}^{k+\ell}_{\geqslant 1},|\vec{t}|=|\vec{r}|+|\vec{s}|}\ \sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell}}\Big(\prod_{i=1}^{k+\ell}c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}(i)\Big)\lambda\Big({\vec{t}\atop \vec{a}\,ш_{(\varphi,\psi)}\vec{b}}\Big).$$

where $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}(i)$ is given in Eq. (29) and $\vec{a}\,ш_{(\varphi,\psi)}\vec{b}$ is given in Eq. (30).

COROLLARY 3.15. *Let $\vec{r}\in\mathbb{Z}^k_{\geqslant 1}$ and $\vec{s}\in\mathbb{Z}^\ell_{\geqslant 1}$ with $r_1,s_1\geqslant 2$. Then*

$$\zeta(\vec{r})\,\zeta(\vec{s})=\sum_{\vec{t}\in\mathbb{Z}^{k+\ell}_{\geqslant 1},|\vec{t}|=|\vec{r}|+|\vec{s}|}\Big(\sum_{(\varphi,\psi)\in\mathfrak{J}_{k,\ell}}\prod_{i=1}^{k+\ell}c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}(i)\Big)\zeta(\vec{t})$$

where $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}(i)$ is given in Eq. (29).

4. The algebraic framework of Connes and Kreimer on renormalization

The Algebraic Birkhoff Decomposition of Connes and Kreimer is a fundamental result in their ground breaking work [**15**] on Hopf algebra approach to renormalization of perturbative quantum field theory (pQFT). This decomposition also links the physics theory of renormalization to Rota-Baxter algebra that has evolved in parallel to the development of QFT renormalization for several decades.

The introduction of Rota-Baxter algebra by G. Baxter [**4**] in 1960 was motivated by Spitzer's identity [**68**] that appeared in 1956 and was regarded as a remarkable formula in the fluctuation theory of probability. Soon Atkinson [**3**] proved a simple yet useful factorization theorem in Rota-Baxter algebras. The identity of Spitzer took its algebraic form through the work of Cartier, Rota and Smith [**12, 65**] in 1972.

It was during the same period when the renormalization theory of pQFT was developed, through the the work of Bogoliubov and Parasiuk [**6**] in 1957, Hepp [**39**] in 1966 and Zimmermann [**74**] in 1969, later known as the BPHZ prescription.

Recently QFT renormalization and Rota-Baxter algebra were tied together through the algebraic formulation of Connes and Kreimer for the former and a generalization of classical results on Rota-Baxter algebras in the latter [**20, 21**]. More precisely, generalizations of Spitzer's identity and Atkinson factorization give the twisted antipode formula and the algebraic Birkhoff decomposition in the work of Connes and Kreimer.

We recall the algebraic Birkhoff decomposition in Section 4.1, prove the Atkinson factorization in Section 4.2 and derive the algebraic Birkhoff decomposition from the Atkinson factorization in Section 4.3.

4.1. Algebraic Birkhoff decomposition. Let A be a $\mathbf{k}$-algebra A and let C be a $\mathbf{k}$-coalgebra. We define the convolution of two linear maps f,g in $\mathrm{Hom}(C,A)$ to be the map $f\star g\in\mathrm{Hom}(C,A)$ given by the composition

$$C\xrightarrow{\Delta}C\otimes C\xrightarrow{f\otimes g}A\otimes A\xrightarrow{m}A.$$

THEOREM 4.1. **(Algebraic Birkhoff Decomposition)** *Let H be a connected filtered Hopf algebra over $\mathbb{C}$. Let (A, Π) be a commutative Rota-Baxter algebra of weight -1 with $\Pi^2 = \Pi$.*

(a) *For any algebra homomorphism $\phi : H \to A$, there are unique linear maps $\phi_- : H \to \mathbf{k} + \Pi(A)$ and $\phi_+ : H \to \mathbf{k} + (\mathrm{id} - \Pi)(A)$ such that*

$$\phi = \phi_-^{\star(-1)} \star \phi_+. \tag{31}$$

(b) *The elements ϕ_- and ϕ_+ take the following forms on* $\ker \varepsilon$.

$$\phi_-(x) = -\Pi(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')), \tag{32}$$

$$\phi_+(x) = \tilde{\Pi}(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')), \tag{33}$$

where we have used the notation $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{(x)} x' \otimes x''$ with $x', x'' \in \ker \varepsilon$.

(c) *The linear maps ϕ_- and ϕ_+ are also algebra homomorphisms.*

We call ϕ_+ the **renormalization** of ϕ and call ϕ_- the **counter-term**. Here is roughly how the renormalization method can be applied through the Algebraic Birkhoff Decomposition. See the tutorial article [**31**] for further details, examples and references.

Theorem 4.1 can be applied to renormalization as follows. Suppose there is a set of divergent formal expressions, such as MZVs with not necessarily positive arguments, that carries a certain algebraic combinatorial structure and from which we would like to extract finite values. On the one hand, we first apply a suitable regularization (deformation) to each of these formal expressions so that the formal expression can be viewed as a singular value of the deformation function. Expanding around the singular point gives a Laurent series in $\mathbf{k}[\varepsilon^{-1}, \varepsilon]]$. On the other hand, the algebraic combinatorial structure of the formal expressions, inherited by the deformation functions, can be abstracted to a free object in a suitable category. This free object parameterizes the deformation functions and often gives a Hopf algebra H. Thus the parametrization gives a morphism $\phi : H \to \mathbf{k}[\varepsilon^{-1}, \varepsilon]]$ in the suitable category. Upon applying the Algebraic Birkhoff Decomposition, we obtain $\phi_+ : H \to \mathbf{k}[[\varepsilon]]$ which, composed with $\varepsilon \mapsto 0$, gives us well-defined values in $\mathbf{k}$.

4.2. Atkinson factorization. The following is the classical result of Atkinson.

THEOREM 4.2. **(Atkinson Factorization)** *Let (R, P) be a Rota–Baxter algebra of weight $\lambda \neq 0$. Let $a \in R$. Assume that b_ℓ and b_r are solutions of the fixed point equations*

$$b_\ell = 1 + P(b_\ell a), \qquad b_r = 1 + (\mathrm{id}_R - P)(a b_r). \tag{34}$$

Then

$$b_\ell(1 + \lambda a)b_r = 1.$$

Thus

$$1 + \lambda a = b_\ell^{-1} b_r^{-1} \tag{35}$$

if b_ℓ and b_r are invertible.

We note that the factorization (35) depends on the existence of invertible solutions of Eq. (34) that we will address next.

DEFINITION 4.3. *A* **filtered k-algebra** *is a* **k**-*algebra R together with a decreasing filtration R_n, $n \geqslant 0$, of nonunitary subalgebras such that*

$$\bigcup_{n\geqslant 0} R_n = R, \quad R_n R_m \subseteq R_{n+m}.$$

It immediately follows that $R_0 = R$ and each R_n is an ideal of R. A filtered algebra is called **complete** if R is a complete metric space with respect to the metric defined by the subsets $\{R_n\}$. Equivalently, a filtered **k**-algebra R with $\{R_n\}$ is complete if $\cap_n R_n = 0$ and if the resulting embedding

$$R \to \bar{R} := \varprojlim R/R_n$$

of R into the inverse limit is an isomorphism.

A Rota-Baxter algebra (R, P) is called **complete** if there are submodules $R_n \subseteq R, n \geqslant 0$, such that (R, R_n) is a complete algebra and $P(R_n) \subseteq R_n$.

THEOREM 4.4. **(Existence and uniqueness of Atkinson factorization)** *Let (R, P, R_n) be a complete Rota-Baxter algebra. Let a be in R_1.*

(a) *The equations in (34) have unique solutions b_ℓ and b_r. Further b_ℓ and b_r are invertible. Hence Atkinson Factorization (35) exists.*
(b) *If λ has no non-zero divisors in R_1 and $P^2 = -\lambda P$ (in particular if $P^2 = -\lambda P$ on R), then there are unique $c_\ell \in 1+P(R)$ and $c_r \in 1+(\mathrm{id}_R - P)(R)$ such that*

$$1 + \lambda a = c_\ell c_r.$$

4.3. From Atkinson factorization to algebraic Birkhoff decomposition. We now derive the Algebraic Birkhoff Decomposition of Connes and Kreimer in Theorem 4.1 from Atkinson Factorization in Theorem 4.4. Adapting the notations in Theorem 4.1, let H be a connected filtered Hopf algebra and let (A, Q) be a commutative Rota-Baxter algebra of weight $\lambda = -1$ with $Q^2 = Q$, such as the pair (A, Q) in Theorem 4.1 (see also Example 2.4). The increasing filtration on H induces a decreasing filtration $R_n = \{f \in \mathrm{Hom}(H, A) \mid f(H^{n-1}) = 0\}, n \geqslant 0$ on $R := \mathrm{Hom}(H, A)$, making it a complete algebra. Further define

$$P : R \to R, \quad P(f)(x) = Q(f(x)), \quad f \in \mathrm{Hom}(H, A), x \in H.$$

Then it is easily checked that P is a Rota-Baxter operator of weight -1 and $P^2 = P$. Thus (R, R_n, P) is a complete Rota-Baxter algebra.

Now let $\phi : H \to A$ be a character (that is, an algebra homomorphism). Consider $e - \phi : H \to A$. Then

$$(e - \phi)(1_H) = e(1_H) - \phi(1_H) = 1_H - 1_H = 0.$$

Thus $e - \phi$ is in R_1. Take $e - \phi$ to be our a in Theorem 4.4, we see that there are unique $c_\ell \in P(R_1)$ and $c_r \in P(R_1)$ such that

$$\phi = c_\ell c_r.$$

Further, by Theorem 4.2, for $b_\ell = c_\ell^{-1}$, $b_\ell = e + P(b_\ell \star (e-\phi))$. Thus for $x \in \ker \varepsilon = \ker e$, we have

$$\begin{aligned}
b_\ell(x) &= P(b_\ell \star (e-\phi))(x) \\
&= \sum_{(x)} Q(b_\ell(a_{(1)})(e-\phi)(a_{(2)})) \\
&= Q\big(b_\ell(1_H)(e-\phi)(x) + \sum_{(a)} b_\ell(x')(e-\phi)(x'') + b_\ell(x)(e-\phi)(1_H)\big) \\
&= -Q\big(\phi(x) + \sum_{(x)} b_\ell(x')\phi(x'')\big).
\end{aligned}$$

In the last equation we have used $e(a) = 0, e(a'') = 0$ by definition. Since $b_\ell(1_H) = 1_H$, we see that $b_\ell = \phi_-$ in Eq. (32).

Further, we have

$$c_r = c_\ell^{-1}\phi = b_\ell\phi = -b_\ell(e-\phi) + b_\ell = -b_\ell(e-\phi) + e + P(b_\ell(e-\phi)) = e - (\mathrm{id}-P)(b_\ell(e-\phi)).$$

With the same computation as for b_ℓ above, we see that $c_r = \phi_+$ in Eq. (33).

5. Heat-kernel type regularization approach to the renormalization of MZVs

To extend the double shuffle relations to MZVs with non-positive arguments, we have to make sense of the divergent sums defining these MZVs. For this purpose, we adapt the renormalization method from quantum field theory in the algebraic framework of Connes-Kreimer recalled in the last section. We will give three approaches including the approach in this section using a heat-kernel type regularization, named after a similar process in physics. Since examples and motivations of this approach can already be found elsewhere [**30, 36, 37**], we will be quite sketchy in this section. More details will be given to the two other approaches in Section 6.

5.1. Renormalization of MZVs. Consider the abelian semigroup

(36) $$\mathfrak{M} = \{ \left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \mid (s,r) \in \mathbb{Z} \times \mathbb{R}_{>0} \}$$

with the multiplication

$$\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} s' \\ r' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s+s' \\ r+r' \end{smallmatrix}\right].$$

With the notation in Section 2.2, we define the Hopf algebra

$$\mathcal{H}_{\mathfrak{M}} := \mathrm{MS}_{\mathbb{C},1}(\mathbb{C}\mathfrak{M})$$

with the quasi-shuffle product $*$ and the deconcatenation coproduct Δ in Section 2.2. For $w_i = \left[\begin{smallmatrix} s_i \\ r_i \end{smallmatrix}\right] \in \mathfrak{M},\ i = 1, \cdots, k$, we use the notations

$$\vec{w} = (w_1, \ldots, w_k) = \left[\begin{smallmatrix} s_1, \ldots, s_n \\ r_1, \ldots, r_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right], \text{ where } \vec{s} = (s_1, \ldots, s_k), \vec{r} = (r_1, \ldots, r_k).$$

For $\vec{w} = \begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix} \in \mathfrak{M}^k$ and $\varepsilon \in \mathbb{C}$ with $\mathrm{Re}(\varepsilon) < 0$, define the **directional regularized MZV**:

$$Z(\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix}; \varepsilon) = \sum_{n_1 > \cdots > n_k > 0} \frac{e^{n_1 r_1 \varepsilon} \cdots e^{n_k r_k \varepsilon}}{n_1^{s_1} \cdots n_k^{s_k}} \tag{37}$$

It converges for any $\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix}$ and is regarded as the regularization of the **formal MZV**

$$\zeta(\vec{s}) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \tag{38}$$

which converges only when $s_i > 0$ and $s_1 > 1$. It is related to the multiple polylogarithm

$$\mathrm{Li}_{s_1, \ldots, s_k}(z_1, \ldots, z_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

by a change of variables $z_i = e^{r_i \varepsilon}, 1 \leqslant i \leqslant k$.

This regularization defines an algebra homomorphism [**36**]:

$$\tilde{Z} : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[T][[\varepsilon, \varepsilon^{-1}], \tag{39}$$

In the same way, for

$$\mathfrak{M}^- = \left\{ \begin{bmatrix}s\\ r\end{bmatrix} \;\middle|\; (s, r) \in \mathbb{Z}_{\leq 0} \times \mathbb{R}_{>0} \right\}, \tag{40}$$

$\tilde{Z}$ restricts to an algebra homomorphism

$$\tilde{Z} : \mathcal{H}_{\mathfrak{M}^-} \to R := \mathbb{C}[[\varepsilon, \varepsilon^{-1}]. \tag{41}$$

Since both $(\mathbb{C}[T][\varepsilon^{-1}, \varepsilon]], \Pi)$ and $(\mathbb{C}[\varepsilon^{-1}, \varepsilon]], \Pi)$, with Π defined in Example 2.4, are commutative Rota-Baxter algebras with $\Pi^2 = \Pi$, we have the decomposition

$$\tilde{Z} = \tilde{Z}_-^{-1} \star \tilde{Z}_+$$

by the algebraic Birkhoff decomposition in Theorem 4.1 and obtain

THEOREM 5.1. ([**36, 37**]) *The map $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[T][[\varepsilon]]$ is an algebra homomorphism which restricts to an algebra homomorphism $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}^-} \to \mathbb{C}[[\varepsilon]]$.*

Because of Theorem 5.1, the following definition is valid.

DEFINITION 5.2. *For $\vec{s} = (s_1, \ldots, s_k) \in \mathbb{Z}^k$ and $\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}_{>0}^k$, define the* **renormalized directional MZV** *by*

$$\zeta(\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix}) = \lim_{\varepsilon \to 0} \tilde{Z}_+(\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix}; \varepsilon). \tag{42}$$

Here $\vec{r}$ is called the **direction vector**.

As a consequence of Theorem 5.1, we have

COROLLARY 5.3. *The renormalized directional MZVs satisfy the quasi-shuffle relation*

$$\zeta(\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix})\zeta(\begin{bmatrix}\vec{s}'\\ \vec{r}'\end{bmatrix}) = \zeta(\begin{bmatrix}\vec{s}\\ \vec{r}\end{bmatrix} * \begin{bmatrix}\vec{s}'\\ \vec{r}'\end{bmatrix}). \tag{43}$$

Here the right hand side is defined in the same way as in Eq. (8).

DEFINITION 5.4. *For $\vec{s} \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leqslant 0}$, define*

$$\zeta(\vec{s}) = \lim_{\delta \to 0^+} \zeta\left(\left[\begin{smallmatrix} \vec{s} \\ |\vec{s}| + \delta \end{smallmatrix}\right]\right), \tag{44}$$

where, for $\vec{s} = (s_1, \cdots, s_k)$ and $\delta \in \mathbb{R}_{>0}$, we denote $|\vec{s}| = (|s_1|, \cdots, |s_k|)$ and $|\vec{s}| + \delta = (|s_1| + \delta, \cdots, |s_k| + \delta)$. These $\zeta(\vec{s})$ are called the **renormalized MZVs** *of the multiple zeta function $\zeta(u_1, \cdots, u_k)$ at $\vec{s}$.*

THEOREM 5.5. ([**36**])

(a) *The limit in Eq. (44) exists for any $\vec{s} = (s_1, \cdots, s_k) \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leqslant 0}$.*

(b) *When s_i are all positive with $s_1 > 1$, we have $\zeta\left(\left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right]\right) = \zeta(\vec{s})$ independent of $\vec{r} \in \mathbb{Z}^k_{>0}$. In particular, we have $\bar{\zeta}(\vec{s}) = \zeta(\vec{s})$.*

(c) *When s_i are all positive, we have $\bar{\zeta}(\vec{s}) = \zeta\left(\left[\begin{smallmatrix} \vec{s} \\ \vec{s} \end{smallmatrix}\right]\right)$. Further, $\bar{\zeta}(\vec{s})$ agrees with the regularized MZV $Z^*_{\vec{s}}(T)$ defined by Ihara-Kaneko-Zagier [**46**].*

(d) *When s_i are all negative, we have $\bar{\zeta}(\vec{s}) = \zeta\left(\left[\begin{smallmatrix} \vec{s} \\ -\vec{s} \end{smallmatrix}\right]\right) = \lim_{\vec{r} \to -\vec{s}} \zeta\left(\left[\begin{smallmatrix} \vec{s} \\ \vec{r} \end{smallmatrix}\right]\right)$. Further, these values are rational numbers.*

(e) *The value $\bar{\zeta}(\vec{s})$ agrees with $\zeta(\vec{s})$ whenever the latter is defined by analytic continuation.*

Furthermore,

(f) *the set $\{\bar{\zeta}(\vec{s}) \big| \vec{s} \in \mathbb{Z}^k_{>0}\}$ satisfies the quasi-shuffle relation;*

(g) *the set $\{\bar{\zeta}(\vec{s}) \big| \vec{s} \in \mathbb{Z}^k_{\leqslant 0}\}$ satisfies the quasi-shuffle relation.*

Table 1 lists $\bar{\zeta}(-a_1, -a_2)$ for $0 \leqslant a_1, a_2 \leqslant 6$.

5.2. The differential structure. The shuffle relation for convergent MZVs from their integral representations does not directly generalize to renormalized MZVs due to the lack of a suitable integral representation. However a differential variation of the shuffle relation might exist for renormalized MZVs. One evidence is the following differential version of the algebraic Birkhoff decomposition [**37**] for renormalized MZVs and further progress will be discussed in a paper under preparation. We first recall some concepts.

(a) A **differential algebra** is a pair (A, d) where A is an algebra and d is a **differential operator**, that is, such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. A differential algebra homomorphism $f : (A_1, d_1) \to (A_2, d_2)$ between two differential algebras (A_1, d_1) and (A_2, d_2) is an algebra homomorphism $f : A_1 \to A_2$ such that $f \circ d_1 = d_2 \circ f$.

(b) A **differential Hopf algebra** is a pair (H, d) where H is a Hopf algebra and $d : H \to H$ is a differential operator such that

$$\Delta(d(x)) = \sum_{(x)} \left(d(x_{(1)}) \bigotimes x_{(2)} + x_{(1)} \bigotimes d(x_{(2)})\right). \tag{45}$$

(c) A **differential Rota-Baxter algebra** is a triple (A, Π, d) where (A, Π) is a Rota-Baxter algebra and $d : R \to R$ is a differential operator such that $P \circ d = d \circ P$.

Table 1. Values of renormalized MZVs, Part I

$\bar{\zeta}(-a_1,-a_2)$	$a_1=0$	$a_1=1$	$a_1=2$	$a_1=3$	$a_1=4$	$a_1=5$	$a_1=6$
$a_2=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{1}{120}$	$-\frac{1}{120}$	$-\frac{1}{252}$	$\frac{1}{252}$	$\frac{1}{240}$
$a_2=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$\frac{83}{64512}$	$\frac{1}{504}$	$-\frac{3925}{2239488}$	$-\frac{1}{480}$
$a_2=2$	$-\frac{1}{120}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$-\frac{319}{437400}$	$-\frac{1}{480}$	$\frac{2494519}{1362493440}$
$a_2=3$	$-\frac{1}{240}$	$-\frac{71}{35840}$	$\frac{1}{504}$	$\frac{1}{28800}$	$-\frac{1}{480}$	$\frac{114139507}{139519328256}$	$\frac{1}{264}$
$a_2=4$	$\frac{1}{252}$	$\frac{1}{504}$	$\frac{319}{437400}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$-\frac{41796929201}{26873437500000}$
$a_2=5$	$\frac{1}{504}$	$\frac{32659}{15676416}$	$-\frac{1}{480}$	$-\frac{21991341}{25836912640}$	$\frac{1}{264}$	$\frac{1}{127008}$	$-\frac{691}{65520}$
$a_2=6$	$-\frac{1}{240}$	$-\frac{1}{480}$	$-\frac{2494519}{1362493440}$	$\frac{1}{264}$	$\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$	0

Theorem 5.6. **(Differential Algebraic Birkhoff Decomposition)** [**37**] *Under the same assumption as in Theorem* 4.1*, if in addition* (H, d) *is a differential Hopf algebra,* (A, Π, ∂) *is a commutative differential Rota-Baxter algebra, and* $\phi : H \to A$ *is a differential algebra homomorphism, then the maps* ϕ_- *and* ϕ_+ *in Theorem* 4.1 *are also differential algebra homomorphisms.*

Theorem 5.7. ([**37**])

(a) *For* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \in \mathfrak{M}$*, define* $d(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]) = r\left[\begin{smallmatrix} s-1 \\ r \end{smallmatrix}\right]$*. Extend* d *to* $\mathcal{H}_{\mathfrak{M}} = \oplus_{k\geqslant 0}(\mathbf{k}\mathfrak{M})^{\otimes k}$ *by defining, for* $\mathfrak{a} := a_1 \otimes \cdots \otimes a_k \in (\mathbf{k}\mathfrak{M})^{\otimes k}$*,*

$$d(\mathfrak{a}) = \sum_{i=1}^{k} a_{i,1} \otimes \cdots \otimes a_{i,k}, \quad a_{i,j} = \begin{cases} a_j, & j \neq i, \\ d(a_j), & j = i. \end{cases} \tag{46}$$

Then $(\mathcal{H}_A, d)$ *is a differential Hopf algebra.*

(b) *The triple* $(\mathbb{C}[\varepsilon^{-1}, \varepsilon]], \Pi, \frac{d}{d\varepsilon})$ *is a commutative differential Rota-Baxter algebra.*

(c) *The map* $\tilde{Z} : \mathcal{H}_{\mathfrak{M}} \to \mathbb{C}[[\varepsilon, \varepsilon^{-1}]$ *defined in Eq. (*41*) is a differential algebra homomorphism.*

(d) *The algebra homomorphism* $\tilde{Z}_+ : \mathcal{H}_{\mathfrak{M}^-} \to \mathbb{C}[[\varepsilon]]$ *in Theorem* 5.1 *is a differential algebra homomorphism.*

6. Renormalization of multiple zeta values seen as nested sums of symbols

We present two more approaches to renormalize multiple zeta functions at nonpositive integers, both of which lead to MZVs which obey stuffle relations. Like the renormalization method described in the previous section, they both give rise to rational multiple zeta values at non-positive integers and we check that the two methods yield the same double multiple zeta values at non-positive integer arguments. This presentation is based on joint work of one of the authors with D. Manchon [**55**] in which multiple zeta functions are viewed as particular instances of nested sums of symbols and where the algebraic Birkhoff decomposition approach is used to renormalize multiple zeta functions at poles. Here, we furthermore present an alternative renormalization method based on generalized evaluators used in physics [**67**].

6.1. A class of symbols. For a complex number b, a smooth function $f: \mathbb{R}-\{0\} \to \mathbb{C}$ is called **positively homogeneous of degree** b if $f(t\,\xi) = t^b f(\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}$.

The symbols which were originally defined on $\mathbb{R}^n$ are now defined on $\mathbb{R}$ which is sufficient for our needs in this paper. We call a smooth function $\sigma : \mathbb{R} \to \mathbb{C}$ a **symbol** if there is a real number a such that for any non-negative integer γ, there is a positive constant C_γ with

$$|\partial^\gamma \sigma(\xi)| \leq C_\gamma (1+|\xi|)^{a-\gamma}, \quad \forall \xi \in \mathbb{R}.$$

For a complex number a and a non-negative integer j, let $\sigma_{a-j} : \mathbb{R}-\{0\} \to \mathbb{C}$ be a smooth and positively homogeneous function of degree $a-j$. We write $\sigma \sim \sum_{j=0}^\infty \sigma_{a-j}$ if, for any non-negative integer N and non-negative integer γ, there is a positive constant $C_{\gamma,N}$ such that

$$\left|\partial^\gamma \left(\sigma(\xi) - \sum_{j=0}^N \sigma_{a-j}(\xi)\right)\right| \leq C_{\gamma,N}(1+|\xi|)^{\mathrm{Re}\ (a)-N-1-\gamma}, \quad \forall \xi \in \mathbb{R}-\{0\},$$

where $\mathrm{Re}(a)$ stands for the real part of a.

For any complex number a and any non-negative integer k, a symbol $\sigma : \mathbb{R} \to \mathbb{C}$ is called a **log-polyhomogeneous of log-type k and order a** if

$$\sigma(\xi) = \sum_{l=0}^k \sigma_l(\xi) \log^l |\xi|, \quad \sigma_l(\xi) \sim \sum_{j=0}^\infty \sigma_{a-j,l}(\xi) \tag{47}$$

with $\sigma_{a-j,l}(\xi)$ positively homogeneous of degree $a-j$.

Let $\mathcal{S}^{a,k}$ denote the linear space over $\mathbb{C}$ of log-polyhomogeneous symbols on $\mathbb{R}$ of log-type k and order a. Then we have $\mathcal{S}^{a,k} \subseteq \mathcal{S}^{a,k+1}$. Let $\mathcal{S}^{*,k}$ denote the linear span over $\mathbb{C}$ of all $\mathcal{S}^{a,k}$ for $a \in \mathbb{C}$. Then $\mathcal{S}^{*,0}$ corresponds to the algebra of classical symbols on $\mathbb{R}$. We also define

$$\mathcal{S}^{*,*} := \bigcup_{k=0}^\infty \mathcal{S}^{*,k}$$

which is an algebra for the ordinary product of functions filtered by the log-type [**52**] since the product of two symbols of log-types k and k' respectively is of log-type $k+k'$. The union $\bigcup_{a\in\mathbb{Z}} \bigcup_{k=0}^\infty \mathcal{S}^{a,k}$ is a subalgebra of $\mathcal{S}^{*,*}$, and $\bigcup_{a\in\mathbb{Z}} \mathcal{S}^{a,0}$ is a subalgebra of $\mathcal{S}^{*,0}$.

Let $\mathcal{P}^{\alpha,k}$ be the algebra of **positively supported** symbols, i.e. symbols in $\mathcal{S}^{\alpha,k}$ with support in $(0,+\infty)$ so that they are non-zero only at positive arguments. We keep *mutatis mutandis* the above notations; in particular $\mathcal{P}^{*,0}$ is a subalgebra of the filtered algebra $\mathcal{P}^{*,*}$.

For $\sigma \in \mathcal{P}^{\alpha,k}$ we call $\mathrm{fp}_{\xi\to\infty}\, \sigma(\xi) := \sigma_{0,0}(\xi)$ the **finite part** at zero (so named since it it reminiscent of Hadamard's finite parts) of such a symbol σ which corresponds to the constant term in the expansion.

The following rather elementary statement is our main motivation here for introducing log-polyhomogeneous symbols.

PROPOSITION 6.1. [**54**] *The operator I defined in (2) on the algebra $C[0,\infty)$ by*

$$f \mapsto \left(\xi \mapsto I(f)(\xi) = \int_0^\xi f(t)dt\right)$$

maps $\mathcal{P}^{*,k-1}$ to $\mathcal{P}^{*,k}$ for any positive integer k.

By Proposition 6.1, for any σ in $\mathcal{P}^{*,k}$, the primitive $I(\sigma)(\xi)$ has an asymptotic behavior as $\xi \to \infty$ of the type (47) with k replaced by $k+1$. The constant term defines the cut-off regularized integral (see e.g. [**52**]):

$$-\!\!\!\!\!\!\int_0^\infty \sigma(t)\, dt := \mathrm{fp}_{\xi\to\infty} \int_0^\xi \sigma(t)\, dt.$$

6.2. Nested sums of symbols and their pole structures.

6.2.1. *Nested sums.* Recall that the operator I on $\mathcal{P}^{*,*}$ defined by Eq. (2) satisfies the weight zero Rota-Baxter relation (1). On the other hand the operator P defined by Eq. (4) satisfies the Rota-Baxter relation with weight $\lambda = -1$ and the operator $Q = P - Id$ in Eq. (5) satisfies the Rota-Baxter relation with weight $\lambda = 1$.

The two Rota-Baxter operators P and I are related by means of the Euler-MacLaurin formula which compares discrete sums with integrals. For $\sigma \in \mathcal{P}^{*,*}$ the Euler-MacLaurin formula (see e.g. [**38**]) reads:

$$\begin{aligned} P(\sigma)(N) - I(\sigma)(N) &= \frac{1}{2}\sigma(N) + \sum_{k=2}^{2K} \frac{B_k}{k!}\sigma^{(k-1)}(N) \\ &\quad + \frac{1}{(2K+1)!}\int_0^N \overline{B_{2K+1}}(x)\sigma^{(2K+1)}(x)\, dx. \end{aligned} \tag{48}$$

with $\overline{B_k}(x) = B_k\,(x - [x])$. Here $B_k(x) = \sum_{i=0}^k \binom{k}{i}\, B_{k-i}\, x^i$ are the Bernoulli polynomials of degree k, the B_i being the Bernoulli numbers, defined by the generating series:

$$\frac{t}{e^t - 1} = \sum_i \frac{B_i}{i!} t^i.$$

Since $B_k(1) = B_k$ for any $k \geqslant 2$, setting $x = 1$ we have

$$B_k = \sum_{i=0}^k \binom{k}{i}\, B_{k-i} = \sum_{i=0}^k \binom{k}{i}\, B_i, \quad \forall k \geqslant 2. \tag{49}$$

The Euler-MacLaurin formula therefore provides an interpolation of $P(\sigma)$ by a symbol.

PROPOSITION 6.2. [**55**] *For any $\sigma \in \mathcal{P}^{a,k}$, the discrete sum $P(\sigma)$ can be interpolated by a symbol $\overline{P}(\sigma)$ in $\mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ (i.e. $\overline{P}(\sigma)(n) = P(\sigma)(n) = \sum_{k=0}^{n} \sigma(k), \forall n \in \mathbb{N}$) such that*

$$\overline{P}(\sigma) - I(\sigma) \in \mathcal{P}^{a,k}.$$

The operator $\overline{Q} := \overline{P} - Id : \mathcal{P}^{a,k} \to \mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ interpolates Q.

By Proposition 6.2, given a symbol σ in $\mathcal{P}^{a,k}$, the interpolating symbol $\overline{P}(\sigma)$ lies in $\mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$. It follows that the discrete sum $P(\sigma)(N) = \overline{P}(\sigma)(N)$ has an asymptotic behavior for large N given by finite linear combinations of expressions of the type (47) with k replaced by $k+1$ and a by $a+1$ or 0. Picking the finite part, for any $\sigma \in \mathcal{P}^{*,*}$ we define the following cut-off sum:

$$\sum_{0}^{\infty} \sigma := \operatorname*{fp}_{N\to\infty} P(\sigma)(N) = \operatorname*{fp}_{N\to\infty} \sum_{k=0}^{N} \sigma(k), \tag{50}$$

which extends the ordinary discrete sum $\sum_0^\infty$ on L^1-symbols. If σ has non-integer order, we have $\sum\limits_{0}^{\infty} \sigma = \operatorname*{fp}\limits_{N\to\infty} \sum_{k=0}^{N+K} \sigma(k)$ for any integer K, so that in particular $\sum\limits_{0}^{\infty} \sigma = \operatorname*{fp}\limits_{N\to\infty} Q(\sigma)(N)$ since the operators P and Q only differ by an integer in the upper bound of the sum.

With the help of the interpolation map described in Proposition 6.2, we can assign to a tensor product $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$ of (positively supported) classical symbols, two log-polyhomogeneous symbols defined inductively in the degree k of the tensor product, which interpolate the nested iterated sum

$$\sum_{0 \leqslant n_k \leqslant n_{k-1} \leqslant \cdots \leqslant n_2 \leqslant n_1} \sigma_1(n_1) \cdots \sigma_k(n_k) = \sigma_1 \, P\Big(\cdots \sigma_{k-2} \, P\big(\sigma_{k-1} \, P(\sigma_k)\big) \ldots \Big),$$

$$\sum_{0 \leqslant n_k < n_{k-1} < \cdots < n_2 < n_1} \sigma_1(n_1) \cdots \sigma_k(n_k) = \sigma_1 \, Q\Big(\cdots \sigma_{k-2} \, Q\big(\sigma_{k-1} \, P(\sigma_k)\big) \ldots \Big).$$

In the following we will only consider the second class of symbols, including their regularization, renormalization and application to multiple zeta values. A parallel approach applies to the first class of symbols with application to non-strict multiple zeta values in Eq. (22) [**61, 75**].

THEOREM 6.3. [**55**] *Given $\sigma_i \in \mathcal{P}^{\alpha_i,0}$, $i = 1, \ldots, k$, setting $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$, the function $\widetilde{\sigma}$ defined by:*

$$\widetilde{\sigma} := \sigma_1 \, \overline{Q}\Big(\cdots \sigma_{k-2} \, \overline{Q}\big(\sigma_{k-1} \, \overline{Q}(\sigma_k)\big) \ldots \Big) \tag{51}$$

which interpolates nested sums in the following way:

$$\widetilde{\sigma}(n_1) = \sum_{0 \leqslant n_k < n_{k-1} < \cdots < n_2 < n_1} \sigma_1(n_1) \cdots \sigma_k(n_k), \quad \forall n_1 \in \mathbb{N},$$

lies in $\mathcal{P}^{*,k-1}$ that is expressed as linear combinations of (positively supported) symbols in $\mathcal{P}^{\alpha_1+\cdots+\alpha_j+j-1,j-1}$, $j \in \{1, \ldots, k\}$.

On the ground of this result, we define the cut-off nested discrete sum of a tensor product of (positively supported) classical symbols.

DEFINITION 6.4. For $\sigma_1, \ldots, \sigma_k \in \mathcal{P}^{*,0}$ and $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$ we call

$$\sum_{<}^{\text{Chen}} \sigma := \sum_{0}^{\infty} \widetilde{\sigma} = \sum_{0<n_k<\cdots<n_1} \sigma_1(n_1)\cdots\sigma_k(n_k)$$

the **cut-off nested sum** of $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$.

6.2.2. *The pole structure of nested sums of symbols.* To build meromorphic extensions, we combine the cut-off sum $\sum_0^\infty$ introduced in (50) with holomorphic deformations of the symbol in the integrand.

A family $\{a(z)\}_{z\in\Omega}$ in a topological vector space $\mathcal{A}$ which is parameterized by a complex domain Ω, is holomorphic at $z_0 \in \Omega$ if the corresponding function $f : \Omega \to \mathcal{A}$ admits a Taylor expansion in a neighborhood N_{z_0} of z_0

$$a(z) = \sum_{k=0}^{\infty} a^{(k)}(z_0)\,\frac{(z-z_0)^k}{k!}$$

which is convergent, uniformly on compact subsets of N_{z_0} (i.e. locally uniformly), with respect to the topology on $\mathcal{A}$. The vector spaces of functions we consider here are $C(\mathbb{R},\mathbb{C})$ and $C^\infty(\mathbb{R},\mathbb{C})$ equipped with their usual topologies, namely uniform convergence on compact subsets, and uniform convergence of all derivatives on compact subsets respectively.

DEFINITION 6.5. Let k be a non-negative integer, and let Ω be a domain in $\mathbb{C}$. A **simple holomorphic family of log-polyhomogeneous symbols** $\sigma(z) \in \mathcal{S}^{*,k}$ parameterized by Ω is a holomorphic family $\sigma(z)(\xi) := \sigma(z,\xi)$ of smooth functions on $\mathbb{R}$ such that:

(a) the order $\alpha : \Omega \to \mathbb{C}$ is holomorphic on Ω,

(b) $\sigma(z)(\xi) = \sum_{l=0}^{k} \sigma_l(z)(\xi) \log^l |\xi|$ with

$$\sigma_l(z)(\xi) \sim \sum_{j\geqslant 0} \sigma(z)_{\alpha(z)-j,l}(\xi).$$

Here $\sigma(z)_{\alpha(z)-j,l}$ is positively homogeneous of degree $\alpha(z)-j$,

(c) for any positive integer N there is some positive integer K_N such that the remainder term

$$\sigma_{(N)}(z)(\xi) := \sigma(z)(\xi) - \sum_{l=0}^{k}\sum_{j=0}^{K_N} \sigma(z)_{\alpha(z)-j,l}(\xi)\log^l|\xi| = o(|\xi|^{-N})$$

is holomorphic in $z \in \Omega$ as a function of ξ and verifies for any $\epsilon > 0$ the following estimates:

$$\partial_\xi^\beta \partial_z^k \sigma_{(N)}(z)(\xi) = o(|\xi|^{-N-|\beta|+\epsilon})$$

locally uniformly in $z \in \Omega$ for $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

A **holomorphic family of log-polyhomogeneous symbols** is a finite linear combination (over $\mathbb{C}$) of simple holomorphic families.

It follows from the Euler-MacLaurin formula (see e.g. [**38, 55**]) that for any holomorphic family $\sigma(z)$ of symbols in $\mathcal{P}^{*,*}$, we have

$$\sum_{n=0}^{\infty} \sigma(z)(n) = -\!\!\!\!\!\!\int_0^\infty \sigma(z)(\xi)\,d\xi + C(\sigma(z))$$

with $z \mapsto C(\sigma(z))$ a holomorphic function at zero. Hence, $z \mapsto \sum_{n=0}^{\infty} \sigma(z)(n)$ and $z \mapsto -\!\!\!\!\!\!\int_0^\infty \sigma(z)(\xi)\, d\xi$ have the same pole structure. Results by Kontsevich and Vishik [**49**] for classical symbols and their generalization by Lesch [**52**] to log-polyhomogeneous symbols, and relative to the pole structure of cut-off integrals of holomorphic families of symbols, therefore carry out to discrete cut-off sums of holomorphic (positively supported) log-polyhomogeneous symbols. Let us briefly recall the notion of holomorphic regularization inspired by [**49**].

Definition 6.6. A **holomorphic regularization procedure** on $\mathcal{S}^{*,*}$ is a map

$$\begin{aligned}\mathcal{R} : \mathcal{S}^{*,*} &\to \mathrm{Hol}_\Omega\ (\mathcal{S}^{*,*})\\ f &\mapsto \{\sigma(z) = \sigma_f(z)\}_{z\in\Omega}\end{aligned}$$

where Ω is an open subset of $\mathbb{C}$ containing 0, and $\mathrm{Hol}_\Omega\,(\mathcal{S}^{*,*})$ is the algebra of holomorphic families in $\mathcal{S}^{*,*}$, such that for any $f \in \mathcal{S}^{*,*}$,

(a) $\sigma(0) = f$,
(b) the holomorphic family $\sigma(z)$ can be written as a linear combination of simple ones:

$$\sigma(z) = \sum_{j=1}^{k} \sigma_j(z),$$

the holomorphic order $\alpha_j(z)$ of which verifies $\mathrm{Re}(\alpha_j'(z)) < 0$ for any $z \in \Omega$ and any $j \in \{1, \ldots, k\}$.

A holomorphic regularization $\mathcal{R}$ is **simple** if, for any log-polyhomogeneous symbol $\sigma \in \mathcal{S}^{\alpha,k}$, the holomorphic family $\mathcal{R}(\sigma)$ is simple. Since we only consider simple holomorphic regularizations, we drop the explicit mention of simplicity.

A similar definition holds with suitable subalgebras of $\mathcal{S}^{*,*}$, e.g. classical symbols $\mathcal{S}^{*,0}$ instead of log-polyhomogeneous. Holomorphic regularization procedures naturally arise in physics:

Example 6.7. Let $z \mapsto \tau(z) \in \mathcal{S}^{*,0}$ be a holomorphic family of classical symbols such that $\tau(0) = 1$ and $\tau(z)$ has holomorphic order $\alpha(z)$ with $\mathrm{Re}(\alpha'(z)) < 0$. Then

$$\mathcal{R} : \sigma \mapsto \sigma(z) := \sigma\, \tau(z)$$

yields a holomorphic regularization on $\mathcal{S}^{*,*}$ as well as on $\mathcal{S}^{*,0}$. Choosing $\tau(z)(\xi) := \chi(\xi) + \big(1 - \chi(\xi)\big)\big(H(z)\, |\xi|^{-z}\big)$ where H is a scalar valued holomorphic map such that $H(0) = 1$, and where χ is a smooth cut-off function which is identically one outside the unit interval and zero in a small neighborhood of zero, we get

$$\mathcal{R}(\sigma)(z)(\xi) = \chi(\xi)\sigma(\xi) + \big(1 - \chi(\xi)\big)\big(H(z)\, \sigma(\xi)\, |\xi|^{-z}\big).$$

Dimensional regularization commonly used in physics is of this type, where H is expressed in terms of Gamma functions which account for a "complexified" volume of the unit sphere. When $H \equiv 1$, such a regularization $\mathcal{R}$ is called Riesz regularization.

Proposition 6.8. *Given a holomorphic regularization* $\mathcal{R} : \sigma \mapsto \sigma(z)$ *on* $\mathcal{P}^{*,k}$, *for any* $\sigma \in \mathcal{P}^{*,k}$, *the map* $z \mapsto \sum_0^\infty \sigma(z)$ *is meromorphic with poles of order* $\leqslant k+1$ *in the discrete set* $\alpha^{-1}(\{-1, 0, 1, 2, \cdots\})$ *whenever* $\sigma(z)$ *is a holomorphic family with order* $\alpha(z)$ *such that* $\mathrm{Re}(\alpha'(z)) \neq 0$ *for any* z *in* Ω.

Let $\Omega \subset \mathbb{C}$ be an open neighborhood of 0. Given symbols $\sigma_1, \cdots, \sigma_k \in \mathcal{P}^{*,0}$, and a holomorphic regularization $\mathcal{R}$ which sends σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$, $z \in \Omega$, we build holomorphic perturbations in the complex multivariable $\underline{z} := (z_1, \cdots, z_k) \in \Omega^k$ of the symbols $\widetilde{\sigma}$ introduced in (51):

$$\widetilde{\sigma}(\underline{z}) := \sigma_1(z_1)\,\overline{Q}\Big(\cdots \sigma_{k-2}(z_{k-2})\,\overline{Q}\big(\sigma_{k-1}(z_{k-1})\,\overline{Q}(\sigma_k(z_k))\big)\ldots\Big).$$

By Theorem 6.3, these are linear combinations of log-polyhomogeneous symbols of log-type $j-1$ and order $\alpha_1(z_1)+\cdots+\alpha_j(z_j)+j-1$, $j \in \{1,\ldots,k\}$. Applying Proposition 6.8 to each of these symbols provides information on the pole structure of nested sums of (positively supported) classical symbols reminiscent of the pole structure of multiple zeta functions [**1, 25, 73**].

THEOREM 6.9. *Fix symbols $\sigma_1, \cdots, \sigma_k \in \mathcal{P}^{*,0}$ and a holomorphic regularization $\mathcal{R}$ which sends σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$.*

(a) *The map*

$$(z_1, \cdots, z_k) \mapsto \sum_{<}^{Chen} \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)$$

is meromorphic with poles on a countable number of hypersurfaces

$$\sum_{i=1}^{j} \alpha_i(z_i) \in -j + \mathbb{N}_0,$$

of multiplicity j varying in $\{1, \cdots, k\}$. Here $\mathbb{N}_0$ stands for the set of non-negative integers.

(b) *Let $\sigma(z) := \sigma_1(z) \otimes \cdots \otimes \sigma_k(z)$ with $z \in \Omega$. Assume that the orders $\alpha_i(z)$ of the σ_i's are nonconstant affine with $\alpha_j'(0) = -q$ for any j in $\{1, \cdots, k\}$ and some positive real number q. The map $z \mapsto \sum_{<}^{Chen} \sigma(z)$ is meromorphic on Ω with poles $z \in (\sum_{i=1}^{j} \alpha_i(0) + j - \mathbb{N}_0)/(q\,j)$ of order $\leqslant j$.*

(c) *If $\mathrm{Re}(\alpha_1(z_1)+\cdots+\alpha_j(z_j)) < -j$ for any $j \in \{1,\ldots,k\}$, the nested sums converge and boil down to ordinary nested sums (independently of the perturbation). Setting $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ we have:*

$$\sum_{<}^{Chen,\mathcal{R}} \sigma := \lim_{z \to 0} \sum_{<}^{Chen} \sigma(z) = \sum_{<}^{Chen} \sigma.$$

6.3. A twisted holomorphic regularization. We now take $\mathcal{A}$ to be a subalgebra of $\mathcal{P}^{*,0}$ equipped with the ordinary product on functions. Any holomorphic regularization $\mathcal{R}$ on $\mathcal{A}$ with parameter space $\Omega \subset \mathbb{C}$ induces one on the tensor algebra $T(\mathcal{A})$:

$$\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) := \mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k).$$

It is compatible with the shuffle product

$$\widetilde{\mathcal{R}}\left((\sigma_1 \otimes \cdots \otimes \sigma_k) ш (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) = \widetilde{\mathcal{R}}\left(\sigma_1 \otimes \cdots \otimes \sigma_k\right) ш \widetilde{\mathcal{R}}\left(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right)$$

for any $\sigma_i \in \mathcal{A}$, $i \in \{1, \cdots, k+l\}$.

REMARK 6.10. Note that $\widetilde{\mathcal{R}}(\sigma_1 ш \sigma_2)(z_1, z_2) \neq \mathcal{R}(\sigma_1)(z_1) ш \mathcal{R}(\sigma_2)(z_2)$ even though $\widetilde{\mathcal{R}}(\sigma_1 ш \sigma_2)(z_1, z_2) = (\mathcal{R}(\sigma_1) ш \mathcal{R}(\sigma_2))(z_1, z_2)$.

Let

$$\delta_k : \mathbb{C} \to \mathbb{C}^{\otimes k}, \quad z \mapsto z \cdot 1^{\otimes k},$$

be the diagonal map $\delta : \mathbb{C} \mapsto T(\mathbb{C})$ and δ^* the induced map on tensor products of holomorphic symbols

$$\delta_k^* : T\left(\mathrm{Hol}_\Omega\left(\mathcal{A}\right)\right) \to \mathrm{Hol}_\Omega\left(T(\mathcal{A})\right), \quad \sigma \mapsto \sigma \circ \delta_k.$$

The regularization $\widetilde{\mathcal{R}}$ induces a one parameter holomorphic regularization:

$$\left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) = \mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z)$$

compatible with the shuffle product:

$$\begin{aligned} &\left(\delta^* \circ \widetilde{\mathcal{R}}\right)\left((\sigma_1 \otimes \cdots \otimes \sigma_k) ш (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ &= \left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k) ш \left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}), \end{aligned} \tag{52}$$

for any $\sigma_i \in \mathcal{A}$, $i \in \{1, \cdots, k+l\}$.

Twisting it by Hoffman's isomorphism in Theorem 2.8 yields a holomorphic regularization $\left(\delta^* \circ \tilde{\mathcal{R}}\right)^*$ (denoted by $\mathcal{R}^*$ in [**55**]) on $T(\mathcal{A})$:

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)^* := \exp \circ \left(\delta^* \circ \tilde{\mathcal{R}}\right) \circ \log,$$

which is compatible with the stuffle product:

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)^*(\sigma * \tau) = \left(\delta^* \circ \tilde{\mathcal{R}}\right)^*(\sigma) * \left(\delta^* \circ \tilde{\mathcal{R}}\right)^*(\tau), \quad \forall \sigma, \tau \in T\left(\mathcal{A}\right). \tag{53}$$

Consequently, the following regularization

$$\widetilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) = \exp \circ \tilde{\mathcal{R}} \circ \log(\sigma_1 \otimes \cdots \otimes \sigma_k)\left(z_1, \cdots, z_k\right) \tag{54}$$

is compatible with stuffle relations after symmetrization in the complex variables z_i

$$\left(\widetilde{\mathcal{R}}^*(\sigma * \tau)\right)_{\mathrm{sym}} = \left(\tilde{\mathcal{R}}^*(\sigma) * \tilde{\mathcal{R}}^*(\tau)\right)_{\mathrm{sym}}, \quad \forall \sigma, \tau \in \mathcal{T}\left(\mathcal{A}\right), \tag{55}$$

where the subscript sym stands for symmetrization

$$f_{\mathrm{sym}}(z_1, \cdots, z_k) := \frac{1}{k!} \sum_{\tau \in \Sigma_k} f(z_{\tau(1)}, \cdots, z_{\tau(k)}),$$

over all the complex variables $z_1, \cdots, z_{k+l}$ if σ is a tensor of degree k and τ a tensor of degree l. Setting $z_1 = \cdots = z_{k+l} = z$ in (55) yields back (53) so that (55) can be seen as a polarization of (53). Given symbols $\sigma_1, \cdots, \sigma_k$ in $\mathcal{P}^{*,0}$, and a holomorphic regularization $\mathcal{R} : \sigma \mapsto \sigma(z)$, sending σ_i to $\sigma_i(z)$ with order $\alpha_i(z)$, we are now ready to build a map

$$(z_1, \cdots, z_k) \mapsto \sum_{<}^{\mathrm{Chen}} \widetilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k),$$

which, by Theorem 6.9, is meromorphic with poles on a countable number of hypersurfaces

$$\sum_{i=1}^{j} \alpha_i(z_i) \in -j + \mathbb{N}_0,$$

with multiplicity j varying in $\{1, \cdots, k\}$. In particular, if the holomorphic regularization $\mathcal{R}$ sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q, the hypersurfaces of poles are given by

$$\sum_{i=1}^{j} z_i \in \frac{\sum_{i=1}^{j} \alpha_i(0) + j - n}{q}, \quad n \in \mathbb{N}_0,$$

so that hyperplanes of poles containing the origin correspond to $\sum_{i=1}^{j} z_i = 0$ each of which with multiplicity j varying in $\{1, \cdots, k\}$.

6.4. Meromorphic nested sums of symbols. Let $\mathrm{Mer}_0(\mathbb{C})$ denote the germ of meromorphic functions in a neighborhood of zero in the complex plane and let $\mathrm{Hol}_0(\mathbb{C})$ be the germ of holomorphic functions at zero. We consider the (Grothendieck closure of the) tensor algebra

$$T\left(\mathrm{Mer}_0(\mathbb{C})\right) = \bigoplus_{k=0}^{\infty} T^k(\mathrm{Mer}_0(\mathbb{C}))$$

over $\mathrm{Mer}_0(\mathbb{C})$ and its subalgebra $T(\mathrm{Hol}_0(\mathbb{C})) := \oplus_{k=0}^{\infty} T^k(\mathrm{Hol}_0(\mathbb{C}))$ where we have set

$$T^k(\mathrm{Mer}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Mer}_0(\mathbb{C}), \quad T^k(\mathrm{Hol}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Hol}_0(\mathbb{C})$$

and where $\hat{\otimes}$ stands for the Grothendieck closure. They come equipped with the product:

$$(f_1 \otimes \cdots \otimes f_k) \bigotimes (f_{k+1} \otimes \cdots \otimes f_{k+l}) = f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} \otimes \cdots \otimes f_{k+l}.$$

We consider the following linear extension of $T^k(\mathrm{Mer}_0(\mathbb{C}))$ which corresponds to germs at zero of meromorphic maps in severable variables with linear poles. Let $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty) := \oplus_{k=1}^{\infty} \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ where

$$\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k) := \left\{ \prod_{i=1}^{m} f_i \circ L_i \;\middle|\; f_i \in \mathrm{Mer}_0(\mathbb{C}), \quad L_i \in \left(\mathbb{C}^k\right)^* \right\}$$

or equivalently,

$$\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k) := \left\{ (z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in (\mathbb{C}^k)^*} \left(L(z_1, \cdots, z_k)\right)^{m_L}} \;\middle|\; h \in \mathrm{Hol}_0(\mathbb{C}^k), m_L \in \mathbb{N} \right\}.$$

Setting $m = k$ and $L_i(z_1, \cdots, z_k) = z_i$ yields a canonical injection

$$\begin{aligned} i : T^k(\mathrm{Mer}_0(\mathbb{C})) &\to \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k), \\ f_1 \otimes \cdots \otimes f_k &\mapsto \left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{k} f_i \circ L_i(z_1, \cdots, z_k) \right), \end{aligned}$$

and the tensor product on $T(\mathrm{Mer}_0(\mathbb{C}))$ extends to $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$, by

$$\begin{aligned} (56)\quad &\Big((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{m} f_i \circ L_i(z_1, \cdots, z_k)\Big) \bullet \Big((z_1, \cdots, z_l) \mapsto \prod_{j=1}^{n} f_{m+j} \circ L_{i+j}(z_1, \cdots, z_l)\Big) \\ &= \Big((z_1, \cdots, z_k, \cdots, z_{k+l}) \mapsto \prod_{i=1}^{m} f_i \circ L_i(z_1, \cdots, z_k) \prod_{j=1}^{n} f_{m+j} \circ L_{m+j}(z_{k+1}, \cdots, z_{k+l})\Big) \end{aligned}$$

which makes it a graded algebra.

Specializing to linear forms

$$\mathcal{L}_k := \left\{ L \in \left(\mathbb{C}^k\right)^* \mid \exists\, J \subset \{1, \cdots, k\}, \quad L(z_1, \cdots, z_k) = \sum_{j \in J} z_j \right\}$$

gives rise to a subalgebra $\mathcal{LM}_0(\mathbb{C}^\infty) := \oplus_{k=1}^\infty \mathcal{LM}_0(\mathbb{C}^k) \subset \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ defined by

$$\mathcal{LM}_0(\mathbb{C}^k) := \left\{ (z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in \mathcal{L}_k} \left(L(z_1, \cdots, z_k)\right)^{m_L}} \,\Big|\, h \in \mathrm{Hol}_0(\mathbb{C}^k), m_L \in \mathbb{N} \right\}.$$

For future use, we consider the map $\delta^* : \mathcal{LM}_0(\mathbb{C}^k) \to \mathrm{Mer}_0(\mathbb{C})$ defined by

$$\delta_k^* : \mathcal{LM}_0(\mathbb{C}^k) \to \mathrm{Mer}_0(\mathbb{C}), \quad f \mapsto f \circ \delta_k,$$

induced by the diagonal map $\delta : \mathbb{C} \mapsto T(\mathbb{C})$ previously defined.

By definition of the twisted regularization $\tilde{\mathcal{R}}^*$, the expressions $\sum\limits_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)$ are linear combinations of expressions of the type $\sum\limits_{<}^{\mathrm{Chen}} \tau_1(u_1) \otimes \cdots \otimes \tau_l(u_k)$ with symbols $\tau_j(u_j)$ built from products of the $\sigma_i(z_i)$'s. It therefore follows from Theorem 6.9, that the functions $(z_1, \cdots, z_k) \mapsto \sum\limits_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)$ lie in $\mathcal{LM}_0(\mathbb{C}^k)$. Since the stuffle relations are satisfied for convergent nested sums, given two tensor products $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ and $\tau = \tau_1 \otimes \cdots \otimes \tau_l$ of symbols in $\mathcal{P}$, setting $\sigma_i(z_i) := \mathcal{R}(\sigma_i)(z_i)$, for $\mathrm{Re}(z_i)$ sufficiently large we have:

$$\begin{aligned}
&\sum_{<}^{\mathrm{Chen}} \left(\tilde{\mathcal{R}}^* \left(\sigma_1 \otimes \cdots \otimes \sigma_k\right) (z_1, \cdots, z_k)\right) * \left(\tilde{\mathcal{R}}^* \left(\tau_1 \otimes \cdots \otimes \tau_l\right) (z_{k+1}, \cdots, z_{k+l})\right) \\
=&\left(\sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)\right) \left(\sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\tau_1 \otimes \cdots \otimes \tau_l)(z_{k+1}, \cdots, z_{k+l})\right).
\end{aligned}$$

By analytic continuation (see for example [**29**], in particular the Identity Theorem in Chapter 1, Section A, or [**45**]), this holds as an identity of meromorphic functions. Since

$$\begin{aligned}
&\left(\tilde{\mathcal{R}}^* \left((\sigma_1 \otimes \cdots \otimes \sigma_k) * (\tau_1 \otimes \cdots \otimes \tau_l)\right) (z_1, \cdots, z_{k+l})\right)_{\mathrm{sym}} \\
&= \left(\left(\tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k) * \tilde{\mathcal{R}}^*(\tau_1 \otimes \cdots \otimes \tau_l)\right) (z_1, \cdots, z_{k+l})\right)_{\mathrm{sym}},
\end{aligned}$$

symmetrization in the variables z_i yields

$$\begin{aligned}
&\left(\sum_{<}^{\mathrm{Chen}} \left(\tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k) * \tilde{\mathcal{R}}^*(\tau_1 \otimes \cdots \otimes \tau_l)\right) (z_1, \cdots, z_{k+l}) \right)_{\mathrm{sym}} \\
=&\left(\left(\sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)\right) \left(\sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^*(\tau_1 \otimes \cdots \otimes \tau_l)(z_{k+1}, \cdots, z_{k+l})\right)\right)_{\mathrm{sym}}.
\end{aligned}$$

This can be reformulated as follows.

THEOREM 6.11. [**55**] *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\, z$ for some positive real number q.*

(a) *The map*

$$\Psi^{\mathcal{R}} : (T(\mathcal{A}), *) \to (\mathcal{L}\mathcal{M}_0(\mathbb{C}^\infty), \bullet)$$
$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \left((z_1, \cdots, z_k) \mapsto \sum_{<}^{Chen} \tilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) \right),$$

satisfies the following relation:

$$\left(\Psi^{\mathcal{R}}(\sigma * \tau)\right)_{\mathrm{sym}} = \left(\Psi^{\mathcal{R}}(\sigma) \bullet \Psi^{\mathcal{R}}(\tau)\right)_{\mathrm{sym}},$$

which holds as an equality of meromorphic functions in several variables. Here, as before the subscript sym stands for the symmetrization in the complex variables z_i.

(b) *After composition with δ^* this in turn gives rise to a map*

$$\psi^{\mathcal{R}} : (T(\mathcal{A}), *) \to \mathrm{Mer}_0(\mathbb{C})$$
$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \left(z \mapsto \delta^* \circ \sum_{<}^{Chen} \tilde{\mathcal{R}}^\star(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \right), \tag{57}$$

which is an algebra morphism. In other words, $\psi^{\mathcal{R}}$ satisfies the relation:

$$\psi^{\mathcal{R}}(\sigma * \tau) = \psi^{\mathcal{R}}(\sigma) \cdot \psi^{\mathcal{R}}(\tau),$$

which holds as an equality of meromorphic functions in one variable.

6.5. Renormalized nested sums of symbols. We want to extract finite parts from the meromorphic functions in Theorem 6.11 while preserving the stuffle relations using a renormalization procedure. Renormalized evaluators inspired from generalized evaluators used in physics provide a first renormalization procedure.

6.5.1. *Renormalized nested sums via renormalized evaluators.* We call regularized evaluator at zero on the germ $\mathrm{Mer}_0(\mathbb{C})$ of meromorphic functions around zero, any linear form on $\mathrm{Mer}_0(\mathbb{C})$ which extends the evaluation at zero $\mathrm{ev}_0 : h \mapsto h(0)$ on holomorphic germs at zero. The map $\mathrm{ev}_0^{\mathrm{reg}}$ defined by

$$\mathrm{ev}_0^{\mathrm{reg}} := \mathrm{ev}_0 \circ (I - \Pi),$$

where $\Pi : \mathrm{Mer}_0(\mathbb{C}) \to \mathrm{Mer}_0(\mathbb{C})$ as defined in Example 2.4 corresponds to the projection onto the pole part of the Laurent expansion at zero, is such a regularized evaluator at zero. When we need to specify the complex variable z we also write $\mathrm{ev}_{z=0}^{\mathrm{reg}}$. Following Speer [**67**] we introduce renormalized evaluators which correspond to his generalized evaluators.

Definition 6.12. A renormalized evaluator Λ on a graded subalgebra $\mathcal{B} = \oplus_{k=0}^\infty \mathcal{B}_k$ of $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty) = \oplus_{k=0}^\infty \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ equipped with the product $\bullet$ introduced in (56), is a character on $\mathcal{B}$ which is compatible with the filtration induced by the grading and extends the ordinary evaluation at zero on holomorphic maps. Equivalently, Λ satisfies all of the following:

(a) It is compatible with the filtration: Let $\mathcal{B}^K := \oplus_{k=0}^K \mathcal{B}_k$ and $\Lambda_K := \Lambda|_{\mathcal{B}^K}$. Then $\Lambda_{K+1}|_{\mathcal{B}^K} = \Lambda_K$.

(b) It coincides with the evaluation map at zero on holomorphic maps:

$$\Lambda|_{T(\mathrm{Hol}_0(\mathbb{C}))} = \mathrm{ev}_0.$$

(c) It fulfills a multiplicativity property:

$$\Lambda(f \bullet g) = \Lambda(f)\,\Lambda(g), \quad \forall f, g \in \mathcal{B}.$$

We call the evaluator symmetric if moreover for any f in $\mathcal{B}_k$ and τ in Σ_k, we have

$$\Lambda(f_\tau) = \Lambda(f),$$

where we have set $f_\tau(z_1, \cdots, z_k) := f(z_{\tau(1)}, \cdots, z_{\tau(k)})$.

Example 6.13. Any regularized evaluator at zero λ on $\mathrm{Mer}_0(\mathbb{C})$ uniquely extends to a renormalized evaluator $\tilde{\lambda}$ on the tensor algebra $(T(\mathrm{Mer}_0(\mathbb{C})), \otimes)$ defined by

$$\tilde{\lambda}(f_1 \otimes \cdots \otimes f_k) = \prod_{i=1}^{k} \lambda(f_i).$$

Example 6.14. Any regularized evaluator λ on $\mathrm{Mer}_0(\mathbb{C})$ extends to renormalized evaluators Λ and Λ' on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ defined on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ by

$$\Lambda := \lambda_{z_1} \circ \cdots \circ \lambda_{z_k}, \quad \Lambda' := \lambda_{z_k} \circ \cdots \circ \lambda_{z_1}$$

and to a symmetrized evaluator defined on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ by

$$\Lambda^{\mathrm{sym}} := \frac{1}{k!} \sum_{\tau \in \Sigma_k} \lambda_{z_{\tau(1)}} \circ \cdots \circ \lambda_{z_{\tau(k)}},$$

where λ_{z_i} stands for the evaluator λ implemented in the sole variable z_i, the others being kept fixed. Their restrictions to $T(\mathrm{Mer}_0(\mathbb{C}))$ all coincide with $\tilde{\lambda}$.

Example 6.15. Take $\lambda := \mathrm{ev}_0^{\mathrm{reg}}$, and set with the above notations

$$\mathrm{ev}_0^{\mathrm{ren}} := \Lambda; \quad \mathrm{ev}_0^{\mathrm{ren}'} := \Lambda', \quad \mathrm{ev}_0^{\mathrm{ren,sym}} := \Lambda^{\mathrm{sym}},$$

then given a holomorphic function $h(z_1, z_2)$ in a neighborhood of 0 and setting $f(z_1, z_2) := \frac{h(z_1,z_2)}{z_1+z_2}$, we have

$$\mathrm{ev}_0^{\mathrm{ren}}(f) = \partial_1 h(0,0), \ \mathrm{ev}_0^{\mathrm{ren}'}(f) = \partial_2 h(0,0),$$

$$\mathrm{ev}_0^{\mathrm{ren,sym}}(f) = \frac{\partial_1 h(0,0) + \partial_2 h(0,0)}{2} = \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*(f),$$

though in general,

$$\mathrm{ev}_0^{\mathrm{ren,sym}} \neq \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*.$$

Proposition 6.16. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q. Let $\mathcal{E}$ be a* symmetrized *renormalized evaluator on $\mathcal{L}\mathcal{M}_0$. The map*

$$\begin{aligned} \Psi^{\mathcal{R},\mathcal{E}} &: (T(\mathcal{A}), *) \to \mathbb{C} \\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \mathcal{E} \circ \Psi^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k) \end{aligned}$$

defines a character. In other words, $\Psi^{\mathcal{R},\mathcal{E}}$ satisfies the stuffle relation:

$$\Psi^{\mathcal{R},\mathcal{E}}(\sigma * \tau) = \Psi^{\mathcal{R},\mathcal{E}}(\sigma) \cdot \Psi^{\mathcal{R},\mathcal{E}}(\tau).$$

Remark 6.17. Here, we use the fact that for a symmetrized evaluator Λ we have $\Lambda(f) = \Lambda(f_{\mathrm{sym}})$ where as before the subscript "sym" stands for the symmetrization in the complex variables z_i.

This proposition gives rise to renormalized nested sums of symbols

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \sigma_1\otimes\cdots\otimes\sigma_k := \Psi^{\mathcal{R},\mathcal{E}}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

which obey stuffle relations:

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} (\sigma * \tau) = \left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \sigma\right)\left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathcal{E}} \tau\right).$$

6.5.2. *Renormalized nested sums via algebraic Birkhoff decomposition.* On the other hand, the tensor algebra $T(\mathcal{A})$ can be equipped with the deconcatenation coproduct:

$$\Delta\left(\sigma_1\otimes\cdots\otimes\sigma_k\right) := \sum_{j=0}^{k}\left(\sigma_1\otimes\cdots\otimes\sigma_j\right)\bigotimes\left(\sigma_{j+1}\otimes\cdots\otimes\sigma_k\right)$$

which then inherits a structure of connected graded commutative Hopf algebra [**42**]. Using the convolution product $\star$ associated with the product and coproduct on $T(\mathcal{A})$ and since $\mathrm{Mer}_0(\mathbb{C})$ embeds into the Rota-Baxter algebra $\mathbb{C}[\varepsilon^{-1},\varepsilon]]$ we can implement an algebraic Birkhoff decomposition as in (31) to the map $\psi^{\mathcal{R}}$ in Eq. (57):

$$\psi^{\mathcal{R}} = \left(\psi_-^{\mathcal{R}}\right)^{\star(-1)}\star\psi_+^{\mathcal{R}}$$

associated with the minimal substraction scheme to build characters

$$\psi_+^{\mathcal{R}}(0) : \left(T\left(\mathcal{A}\right), *\right)\to\mathbb{C}.$$

Proposition 6.18. [**55**] *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}^{*,0}$ and let $\mathcal{R}$ be a holomorphic regularization which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q. The map*

$$\begin{aligned}\psi^{\mathcal{R},\mathrm{Birk}} &: \left(T\left(\mathcal{A}\right), *\right)\to\mathbb{C}\\ &\sigma_1\otimes\cdots\otimes\sigma_k \mapsto \psi_+^{\mathcal{R}}(0)(\sigma_1\otimes\cdots\otimes\sigma_k)\end{aligned}$$

defines a character

$$\psi^{\mathcal{R},\mathrm{Birk}}(\sigma*\tau) = \psi^{\mathcal{R},\mathrm{Birk}}(\sigma)\,\cdot\,\psi^{\mathcal{R},\mathrm{Birk}}(\tau).$$

The map yields an alternative set of renormalized nested sums of symbols

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \sigma_1\otimes\cdots\otimes\sigma_k := \psi^{\mathcal{R},\mathrm{Birk}}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

which obey stuffle relations:

$$\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} (\sigma * \tau) = \left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \sigma\right)\left(\sum_{<}^{\mathrm{Chen},\mathcal{R},\mathrm{Birk}} \tau\right).$$

6.6. Renormalized (Hurwitz) multiple zeta values at non-positive integers.

6.6.1. *An algebra of symbols.* Since we consider both zeta and Hurwitz zeta functions, let us first observe that for any non-negative number v and any σ in $\mathcal{P}^{*,k}$, the map $\xi \mapsto t_v^*\sigma(\xi) := \sigma(\xi + v)$ defines a symbol in $\mathcal{P}^{*,k}$.

Let $\widetilde{\mathcal{A}}$ be the subalgebra of $\mathcal{P}^{*,0}$ generated by the continuous functions with support inside the interval $(0, 1)$ and the set

$$\{\sigma \in \mathcal{P}^{*,0} \,\big|\, \exists v \in [0, +\infty), \exists s \in \mathbb{C},\, \sigma(\xi) = (\xi + v)^{-s} \text{ when } \xi \geq 1\}.$$

Consider the ideal $\mathcal{N}$ of $\widetilde{\mathcal{A}}$ of continuous functions with support inside the interval $(0, 1)$. The quotient algebra $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{N}$ is then generated by elements $\sigma_{s,v} \in \mathcal{P}^{*,0}$ with $\sigma_{s,v}(\xi) = (\xi+v)^{-s}$ for $|\xi| \geq 1$. For any $v \in \mathbb{R}_+$ the subspace $\mathcal{A}_v$ of $\mathcal{A}$ generated by $\{\sigma_{s,v} \mid s \in \mathbb{C}\}$ is a subalgebra of $\mathcal{A}$. We equip $\mathcal{A}_v$ with the following holomorphic regularization on an open neighborhood Ω of 0 in $\mathbb{C}$:

$$\begin{aligned} \mathcal{R} : \mathcal{A}_v &\to \mathrm{Hol}_\Omega\left(\mathcal{A}_v\right) \\ \sigma_{s,v} &\mapsto \left(z \mapsto (1 - \chi)\, \sigma_{s,v} + \chi\; \sigma_{s+z,v}\right) \end{aligned}$$

where χ is any smooth cut-off function which is identically one outside the unit ball and vanishes in a small neighborhood of 0.

Let $\mathcal{W}$ be the $\mathbb{C}$-vector space freely spanned symbols indexed by sequences $(u_1, \ldots, u_k)$ of real numbers. In other words, $\mathcal{W}$ is $T(W)$ where $W = \oplus_{u\in\mathbb{R}}\mathbb{R}x_u$ where we identify x_u with u for simplicity and set $x_u \cdot x_v = x_{u+v}, u, v \in \mathbb{R}$. We then define the stuffle product on $\mathcal{W}$ as usual in Eq. (8) or Eq. (11) with $\lambda = 1$. The map

$$\sigma : \mathcal{W} \to T(\mathcal{A}_v), \quad u = (u_1, \cdots, u_k) \mapsto \sigma_{u;v} := \sigma_{(u_1,\ldots,u_k;\, v)} := \sigma_{u_1;v} \otimes \cdots \otimes \sigma_{u_k;v}$$

induces a stuffle product on $\mathcal{T}(\mathcal{A}_v)$:

$$\sigma_{u;v} * \sigma_{u';v} := \sigma_{u*u';v}.$$

As before, we twist the regularization $\widetilde{\mathcal{R}}$ induced by $\mathcal{R}$ on $T(\mathcal{A}_v)$ by the Hoffman isomorphism (14) to build a twisted holomorphic regularization $\widetilde{\mathcal{R}}^*$ in several variables which satisfies

$$\left(\widetilde{\mathcal{R}}^*(\sigma_{u;v}) * \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right)_{\text{sym}} = \left(\widetilde{\mathcal{R}}^*(\sigma_{u*u';v})\right)_{\text{sym}}$$

and a twisted holomorphic regularization $\delta^* \circ \widetilde{\mathcal{R}}^*$ in one variable compatible with the stuffle product:

$$\left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u;v})\right) * \left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right) = \delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u*u';v}).$$

6.6.2. *Multiple zeta values renormalized via renormalized evaluators.* Let Ω be an open neighborhood of 0 in $\mathbb{C}$ and let $\mathcal{R} : \sigma \mapsto \{\sigma(z)\}_{z\in\Omega}$ be the holomorphic regularization procedure on $\widetilde{\mathcal{A}}$ previously introduced. The multiple Hurwitz zeta functions defined by:

$$\zeta(s_1, \ldots, s_k;\, v_1, \ldots, v_k) := \Psi^{\mathcal{R}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k})$$

are meromorphic in all variables with poles[1] on a countable family of hyperplanes $s_1 + \cdots + s_j \in]-\infty, j] \cap \mathbb{Z}$, j varying from 1 to k. When $v_1 = \cdots = v_k = v$, we set

$$\zeta(s_1, \ldots, s_k;\, v) := \zeta(s_1, \ldots, s_k;\, v_1, \ldots, v_k)$$

[1]When $k = 2$ and $v_1 = \cdots = v_l = v$ a more refined analysis actually shows that for some any negative real number v, poles actually only arise for $s_1 = -1$ or $s_1 + s_2 \in \{-2, -1, 0, 2, 4, 6, \cdots\}$.

in which case they satisfy the following relations:

$$\left(\zeta^{\mathcal{E}}(u * u'; v)\right)_{\rm sym} = \left(\zeta^{\mathcal{E}}(u; v)\, \zeta^{\mathcal{E}}(u'; v)\right)_{\rm sym}.$$

The renormalized multiple Hurwitz zeta values derived from a *symmetrized* renormalized evaluator $\mathcal{E}$ on $\mathcal{LM}_0(\mathbb{C}^\infty)$:

$$\zeta^{\mathcal{E}}(s_1, \dots, s_k;\, v_1, \dots, v_k) := \Psi^{\mathcal{R},\mathcal{E}}(\sigma_{s_1, v_1} \otimes \cdots \otimes \sigma_{s_k, v_k})$$

denoted by $\zeta^{\mathcal{R},\mathcal{E}}(s_1, \dots, s_k;\, v)$ when $v_1 = \cdots = v_k = v$, in which case they satisfy stuffle relations:

$$\zeta^{\mathcal{E}}(u * u'; v) = \zeta^{\mathcal{E}}(u; v)\, \zeta^{\mathcal{E}}(u'; v).$$

When $k = 1$, the symmetrized renormalized evaluator $\mathcal{E}_1$ coincides with the regularized evaluator at zero $\mathrm{ev}_0^{\mathrm{reg}}$ and

$$\zeta(s_1;\, v) := \zeta^{\mathcal{E}}(s_1; v)$$

is independent of $\mathcal{E}$. Note that the meromorphic map $\zeta(s;\, v) := \sum_{n=0}^{\infty} \sigma_{s,v}(n)$ differs from the usual Hurwitz function $\zeta(u, v)(s) := \sum_{n=0}^{\infty} (n+v)^{-s}$ by the first term v^{-s}. At a negative integer $-a$, using the Euler MacLaurin formula, one can check that

$$\zeta(-a,\, v) = -\frac{B_{a+1}(v)}{a+1}; \quad \zeta(-a;\, v) = -\frac{B_{a+1}(v)}{a+1} - v^a$$

which are rational numbers whenever v is rational.

Let us now compute renormalized values in the case $k = 2$ using a renormalized evaluator. For any $a \in \mathbb{R}$ and $m \in \mathbb{N} - \{0\}$ we introduce the notation:

$$[a]_j := a(a-1)\cdots(a-j+1).$$

We extend this to $j = 0$ and $j = -1$ by setting: $[a]_0 := 1, [a]_{-1} := \frac{1}{a+1}$. Combining Definition (54)

$$\widetilde{\mathcal{R}}^*(\sigma_1 \otimes \sigma_2)(z_1, z_2) = \sigma_1(z_1) \otimes \sigma_2(z_2) - \frac{1}{2}(\sigma_1 \bullet \sigma_2)(z_1) + \frac{1}{2}\sigma_1(z_1) \bullet \sigma_2(z_2)$$

applied to the regularization

$$\mathcal{R}(\sigma_i)(z)(x) = (x+v)^{-s_i - z} \quad \text{of} \quad \text{order} \quad \alpha_i(z) = -s_i - z_i,$$

with the Euler-MacLaurin formula (48) and following [**55**] (see the proof of Theorem 9 applied to the symbols $\sigma_{s_1, v}, \sigma_{s_2, v}$), we compute

$$\begin{aligned}
&\zeta(s_1, s_2; v)(z_1, z_2)\\
= &\Psi^{\mathcal{R}}(\sigma_{s_1, v} \otimes \sigma_{s_2, v})(z_1, z_2)\\
= &\sum_{<}^{\text{Chen}} \sigma_{s_1, v}(z_1) \otimes \sigma_{s_2, v}(z_2) + \frac{1}{2}\sigma_{s_1, v}(z_1)\, \sigma_{s_2, v}(z_2) - \frac{1}{2}\,(\sigma_{s_1, v}\, \sigma_{s_2, v})(z_1)\\
= &\sum_{j=0}^{2J_2} B_j\, \frac{[-s_2 - z_2]_{j-1}}{j!}\, \left(\zeta(s_1 + s_2 + z_1 + z_2 + j - 1;\, v) - \zeta(s_1 + z_1;\, v)\right)\\
&+ \frac{1}{2}\zeta(s_1 + s_2 + z_1 + z_2;\, v) - \frac{1}{2}\,\zeta(s_1 + s_2 + z_1;\, v)\\
&+ \frac{[-s_2 - z_2]_{2J_2+1}}{(2J_2+1)!} \sum_{0}^{\infty} \left((n+v)^{-s_1 - z_1} \int_1^n \overline{B_{2J_2+1}}(y)\, (y+v)^{-s_2 - z_2 - 2J_2 - 1}\, dy \right).
\end{aligned}$$

Hence, for non-positive integers $s_1 = -a_1, s_2 = -a_2$ and $2J_2 = a_1 + a_2 + 2$ we have:

$$
\begin{aligned}
&\zeta(-a_1, -a_2;\, v)(z_1, z_2) \\
(58) \qquad &= \sum_{j=0}^{a_1+a_2+2} B_j \,\frac{[a_2 - z_2]_{j-1}}{j!}\big(\zeta(-a_1 - a_2 + z_1 + z_2 + j - 1;\, v) \\
&\quad -\zeta(-a_1 + z_1;\, v)\big) \\
&\quad +\frac{1}{2}\zeta(-a_1 - a_2 + z_1 + z_2;\, v) - \frac{1}{2}\,\zeta(-a_1 - a_2 + z_1;\, v) \\
&\quad +\frac{[a_2 - z_2]_{a_2+2}}{(a_2+2)!}\sum_0^{\infty}\left((n+v)^{a_1 - z_1}\int_1^n \overline{B_{a_1+a_2+3}}(y)\,(y+v)^{-2}\,dy\right).
\end{aligned}
$$

The last line on the right hand side is a holomorphic expression at zero on which all renormalized evaluators at zero vanish. The second line on the right hand side is a linear combination of ordinary zeta functions at negative integers which are holomorphic at zero. Any evaluator Λ at zero vanishes on these terms; indeed, we have

$$
\begin{aligned}
&\Lambda\left(\zeta(-a_1 - a_2 + z_1 + z_2;\, v) - \zeta(-a_1 - a_2 + z_1;\, v)\right) \\
&= \zeta(-a_1 - a_2;\, v) - \zeta(-a_1 - a_2;\, v) = 0.
\end{aligned}
$$

Only when evaluated on the expression on the first line of the right hand side can various evaluators differ.

We want to implement the symmetrized evaluator at zero

$$
\mathrm{ev}_0^{\mathrm{ren,sym}} := \frac{1}{2}\left(\mathrm{ev}^{\mathrm{reg}}_{z_2=0} \circ \mathrm{ev}^{\mathrm{reg}}_{z_1=0} + \mathrm{ev}^{\mathrm{reg}}_{z_1=0} \circ \mathrm{ev}^{\mathrm{reg}}_{z_2=0}\right)
$$

introduced in Example 6.15. We first compute

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\zeta(-a_1, -a_2; v)(z_1, z_2)\right)\right) \\
&= \mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Bigg(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Bigg(\sum_{j=0}^{a_1+a_2+2} B_j \,\frac{[a_2 - z_2]_{j-1}}{j!}\big(\zeta(-a_1 - a_2 + z_1 + z_2 + j - 1;\, v) \\
&\qquad -\zeta(-a_1 + z_1;\, v)\big)\Bigg)\Bigg) \\
&= \mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\sum_{j=0}^{a_2+1} B_j \,\frac{[a_2]_{j-1}}{j!}\,\left(\zeta(-a_1 - a_2 + z_1 + j - 1;\, v) - \zeta(-a_1 + z_1;\, v)\right)\right) \\
&= \frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j}\left(\zeta(-a_1 - a_2 + j - 1;\, v) - \zeta(-a_1;\, v)\right).
\end{aligned}
$$

This leads to:

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\zeta(-a_1, -a_2; v)(z_1, z_2)\right)\right) \\
(59) \qquad &:= \mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\zeta(-a_1, -a_2; v)(z_1, z_2; 0)\right)\right) \\
&= \frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j}\Big(-\frac{B_{a_1+a_2-j+2}(v)}{a_1+a_2-j+2} - v^{a_1+a_2-j+2} \\
&\qquad +\frac{B_{a_1+1}(v)}{a_1+1} + v^{a_1+1}\Big).
\end{aligned}
$$

We next compute

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\zeta(-a_1,-a_2;\,v)(z_1,z_2)\right)\right)\\
=&\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\frac{B_0}{a_2-z_2+1}\left(\zeta(-a_1-a_2+z_1+z_2-1;v)-\zeta(-a_1+z_1;v)\right)\Big)\Big)\\
&+\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\sum_{j=1}^{a_1+1}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1+z_2+j-1;\,v)\\
&\quad-\zeta(-a_1+z_1;\,v)\big)\Big)\Big)\\
&+\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\sum_{j=a_1+2}^{a_1+a_2+2}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1+z_2+j-1;\,v)\\
&\quad-\zeta(-a_1+z_1;\,v)\big)\Big)\Big)\\
=&\ \mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\frac{B_0}{a_2+1}\left(\zeta(-a_1-a_2+z_2-1;\,v)-\zeta(-a_1;\,v)\right)\right)\\
&+\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\sum_{j=1}^{a_1+1}B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\left(\zeta(-a_1-a_2+z_2+j-1;\,v)-\zeta(-a_1;\,v)\right)\Big)\\
&+\sum_{j=1}^{a_2+1}B_{j+a_1+1}\,\partial_{z_2}\left(\frac{[a_2-z_2]_{j+a_1}}{(j+a_1+1)!}\right)_{|_{z_2=0}}\mathrm{Res}_{z_2=0}\left(\zeta(-a_2+z_2+j;\,v)\right)\\
=&\ \frac{1}{a_2+1}\sum_{j=0}^{a_2+1}B_j\,\binom{a_2+1}{j}\left(\zeta(-a_1-a_2+j-1;\,v)-\zeta(-a_1;\,v)\right)\\
&+(-1)^{a_1+1}a_1!a_2!\,\frac{B_{a_1+a_2+2}}{(a_1+a_2+2)!},
\end{aligned}
$$

since the only contribution to the residue comes from the term $j=a_1+a_2+2$. This combined with (49) applied to $k=a_2+1$ yields

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\left(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\left(\zeta(-a_1,-a_2;\,v)(z_1,z_2)\right)\right)\\
=&-\frac{1}{a_2+1}\sum_{j=0}^{a_2+1}B_j\,\left(\binom{a_2+1}{j}\frac{B_{a_1+a_2-j+2}(v)}{a_1+a_2-j+2}+v^{a_1+a_2-j+2}\right)\\
&+\frac{B_{a_1+1}}{a_1+1}\left(\frac{B_{a_2+1}(v)}{a_2+1}+v^{a_2+1}\right)+(-1)^{a_1+1}a_1!a_2!\,\frac{B_{a_1+a_2+2}}{(a_1+a_2+2)!}.
\end{aligned}\tag{60}
$$

Combining (59) and (60) yields

$$
\begin{aligned}
&\zeta^{\mathrm{ev}}(-a_1,-a_2;v):=\mathrm{ev}^{\mathrm{ren,sym}}_0\left(\zeta(-a_1,-a_2;v)(z_1,z_2)\right)\\
=&-\frac{1}{a_2+1}\sum_{j=0}^{a_2+1}B_j\,\binom{a_2+1}{j}\left(\frac{B_{a_1+a_2-j+2}(v)}{a_1+a_2-j+2}+v^{a_1+a_2-j+2}\right)\\
&+\frac{B_{a_1+1}}{a_1+1}\left(\frac{B_{a_2+1}(v)}{a_2+1}+v^{a_2+1}\right)+(-1)^{a_1+1}a_1!a_2!\,\frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}.
\end{aligned}\tag{61}
$$

Since Bernoulli polynomials $B_k(x)$ are of degree k in x renormalized multiple zeta values of depth 2 at non-positive arguments $(-a_1,-a_2)$ derived this way are polynomials of degree a_1+a_2+2 in the parameter v with rational coefficients given by rational linear combinations of Bernoulli numbers, and hence give rise to rational numbers for any rational value of v. More generally, an inductive procedure on k carried out in the same spirit as the proof of Theorem 10 in [**55**] would show that the renormalized multiple zeta values $\zeta^{\mathcal{E}}(-a_1, \cdots, -a_k; v)$ of depth k at non-positive integer arguments $(-a_1, \cdots, -a_k)$ are polynomials in the parameter v with rational coefficients so that they lead to rational numbers for rational values of v.

6.6.3. *Multiple zeta values renormalized via Birkhoff decomposition.* The renormalized multiple Hurwitz zeta values derived from a Birkhoff decomposition:

$$\zeta^{\mathrm{Birk}}(s_1, \ldots, s_k;\, v_1, \ldots, v_k) := \Psi^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k})$$

denoted by $\zeta^{\mathrm{Birk}}(s_1, \ldots, s_k;\, v)$ when $v_1 = \cdots = v_k = v$, satisfy stuffle relations

$$\zeta^{\mathrm{Birk}}(u * u'; v) = \zeta^{\mathrm{Birk}}(u; v)\, \zeta^{\mathrm{Birk}}(u'; v).$$

A striking holomorphy property arises at non-positive integer arguments [**55**] after implementing the diagonal map δ.

Proposition 6.19. *At non-positive integer arguments s_i and for a rational parameter v, the map*

$$z \mapsto \psi^{\mathcal{R}}\left(\sigma_{s_1,v} \otimes \cdots \otimes \sigma_{s_k,v}\right)(z)$$

defined in (57) is holomorphic at zero.

Consequently,

$$\zeta^{\mathrm{Birk}}(s_1, \ldots, s_k;\, v) = \lim_{z\to 0} \psi^{\mathcal{R}}(\sigma_{s_1,v} \otimes \cdots \otimes \sigma_{s_k,v}).$$

Let us compute double zeta values at non-positive integer arguments using Birkhoff decomposition. Setting $z_1 = z_2 = z$ in (58) leads to

$$\begin{aligned}
&\zeta(-a_1, -a_2; v)(z)\\
&= \sum_{j=0}^{a_2+1} B_j\, \frac{[a_2 - z]_{j-1}}{j!} \left(\zeta(-a_1 - a_2 + 2z + j - 1;\, v) - \zeta(-a_1 + z;\, v)\right)\\
&\quad + \frac{[a_2 - z]_{a_2+2}}{(a_2+2)!} \sum_0^{\infty} \left((n+v)^{a_1 - z} \int_1^n \overline{B_{a_1+a_2+3}}(y)\, (y+v)^{-2}\, dy \right).
\end{aligned}$$

Evaluating this expression at $z = 0$ in a similar manner to the previous computation, yields:

$$\begin{aligned}
&\zeta^{\mathrm{Birk}}(-a_1, -a_2; v)\\
&= \lim_{z\to 0} \left(\sum_{j=0}^{a_2+1} B_j\, \frac{[a_2 - z]_{j-1}}{j!} \left(\zeta(-a_1 - a_2 + 2z + j - 1;\, v) - \zeta(-a_1 + z;\, v)\right) \right)\\
&= \sum_{j=0}^{a_2+1} B_j\, \frac{[a_2]_{j-1}}{j!} \left(\zeta(-a_1 - a_2 + j - 1;\, v) - \zeta(-a_1;\, v)\right)\\
&\quad + (-1)^{a_1+1} a_1! a_2!\, \frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}.
\end{aligned}$$

TABLE 2. Values of renormalized MZVs, Part II

$\zeta(-a_1,-a_2)$	$a_1=0$	$a_1=1$	$a_1=2$	$a_1=3$	$a_1=4$	$a_1=5$	$a_1=6$
$a_2=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{7}{720}$	$-\frac{1}{120}$	$-\frac{11}{2\,520}$	$\frac{1}{252}$	$\frac{1}{224}$
$a_2=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$-\frac{19}{10\,080}$	$\frac{1}{504}$	$\frac{41}{20\,160}$	$-\frac{1}{480}$
$a_2=2$	$-\frac{7}{720}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$\frac{113}{151\,200}$	$-\frac{1}{480}$	$-\frac{307}{166\,320}$
$a_2=3$	$-\frac{1}{240}$	$\frac{1}{840}$	$\frac{1}{504}$	$\frac{1}{28\,800}$	$-\frac{1}{480}$	$-\frac{281}{332\,640}$	$\frac{1}{264}$
$a_2=4$	$\frac{11}{2\,520}$	$\frac{1}{504}$	$-\frac{113}{151\,200}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$\frac{117\,977}{75\,675\,600}$
$a_2=5$	$\frac{1}{504}$	$-\frac{103}{60\,480}$	$-\frac{1}{480}$	$\frac{1}{1232}$	$\frac{1}{264}$	$\frac{1}{127\,008}$	$-\frac{691}{65\,520}$
$a_2=6$	$-\frac{1}{224}$	$-\frac{1}{480}$	$\frac{307}{166\,320}$	$\frac{1}{264}$	$-\frac{117\,977}{75\,675\,600}$	$-\frac{691}{65\,520}$	0

For $v=0$ this yields:

$$\begin{aligned}\zeta^{\mathrm{Birk}}(-a_1,-a_2) &:= \zeta^{\mathrm{Birk}}(-a_1,-a_2;0)\\ &= -\frac{1}{a_2+1}\sum_{j=0}^{a_2+1} B_j \binom{a_2+1}{j} \frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}\\ &+ \frac{B_{a_1+1}}{a_1+1}\frac{B_{a_2+1}}{a_2+1} + (-1)^{a_1+1} a_1! a_2! \frac{B_{a_1+a_2+2}}{2\,(a_1+a_2+2)!}\end{aligned}$$

which coincides with (61).

Thus, renormalized double zeta values at non-positive integers obtained by two different methods – using the symmetrized renormalized evaluator $\mathrm{ev}_0^{\mathrm{ren,sym}}$ or a Birkhoff decomposition– coincide.

Formula (61) yields Table 2 of values $\zeta(-a_1,-a_2)$ for $a_1,a_2\in\{0,\dots,6\}$ derived in [**55**]:

This table of values differs from the one derived in [**36**] (see Table 1) with which it however matches for arguments (a,b) with $a+b$ odd, and $b\neq 0$ and for diagonal arguments $(-a,-a)$. It would be interesting to see whether the two tables could have a better match when the second table is obtained from $\zeta^{\mathrm{Birk}}(-a_1,-a_2;\nu)$ with a different value of ν.

References

[1] S. Akiyama, S. Egami, Y. Tanigawa, Analytic continuation of multiple zeta functions and their values at non-positive integers, *Acta Arith.* **98**,(2001), 107-116 .

[2] P. Aluffi and M. Marcolli, Algebro-geometric Feynman rules, arXiv:0811.2514[hep-th].

[3] F. V. Atkinson, Some aspects of Baxter's functional equation, *J. Math. Anal. Appl.* **7** (1963), 1–30.

[4] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960), 731-742.

[5] S. Bloch, H. Esnault and D. Kreimer, Motives associated to graph polynomials, *Comm. Math. Phys.* **267** (2006) 181-225, arXiv:math.ag/0510011.

[6] N. N. Bogoliubov and O. S. Parasiuk, On the multiplication of causal functions in the quantum theory of fields, *Acta Math.*, **97** (1957) 227–266.

[7] J. M. Borwein, D. J. Broadhurst, D. M. Bradley, and P. Lisoněk, Special values of multiple polylogarithms, *Trans. Amer. Math. Soc.*, **353**, (2001), no. 3, 907–941.

[8] J. M. Borwein, D. J. Broadhurst, D. M. Bradley, and P. Lisoněk, Combinatorial aspects of multiple zeta values, Electron. J. Combin. 5 (1998), no. 1, Research Paper 38, 12.

[9] D. M. Bradley, Multiple q-zeta values, *J. Algebra*, **283**, (2005), no. 2, 752–798.

[10] D. M. Bradley, A q-analog of Euler's decomposition formula for the double zeta function. *Int. J. Math. Math. Sci.* **21** (2005), 3453–3458.

[11] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett. B*, **393**, (1997), no. 3-4, 403–412.

[12] P. Cartier, On the structure of free Baxter algebras, *Adv. in Math.*, **9** (1972), 253-265.

[13] P. Cartier, Fonctions polylogarithmes, nombres polyzetas et groupes pro-unipotents, *Astérisque*, **282**, (2002), 137–173, (Sém. Bourbaki no. 885).

[14] A. Connes and D. Kreimer, Hopf algebras, Renormalization and Noncommutative Geometry, *Comm. Math. Phys.* **199** (1988) 203-242.

[15] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem., *Comm. Math. Phys.*, **210** (2000), no. 1, 249-273.

[16] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group., *Comm. Math. Phys.*, **216**, (2001), no. 1, 215-241.

[17] A. Connes and M. Marcolli, From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory, in "Frontiers in Number Theory, Physics, and Geometry, II", Springer-Verlag, 2006, pp.617–713, arXiv:hep-th/0411114.

[18] K. Ebrahimi-Fard and L. Guo, Quasi-shuffles, mixable shuffles and Hopf algebras, *J. Algebraic Combin.* **24** (2006), 83-101, arXiv:math.RA/0506418.

[19] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras in renormalization of perturbative quantum field theory, in: *Universality and Renormalization*, I. Binder and D. Kreimer, editors, Fields Institute Communications v. 50, AMS, 2007, 47-105, arXiv:hep-th/0604116.

[20] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Integrable Renormalization II: the General case, *Annales Henri Poincare* **6** (2005), 369-395.

[21] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Spitzer's Identity and the Algebraic Birkhoff Decomposition in pQFT, *J. Phys. A: Math. Gen.*, **37** (2004), 11037-11052.

[22] M. Eie, W.-C. Liaw and Y. L. Ong, A restricted sum formula among multiple zeta values, *J. Number Theory*, **129 (4)** (2009), 908-921

[23] L. Euler, Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol* **20** (1775), 140-186; Reprinted in: Opera Omnia, Ser. I, vol. 15, Teubner, Berlin (1927), 217-267.

[24] H. Gangl, M. Kaneko, and D. Zagier, Double zeta values and modular forms, Automorphic forms and zeta functions, Proceedings of the conference in memory of Tsuneo Arakawa,World Sci. Publ., Hackensack, NJ (2006), 71-106.

[25] A.B. G. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, *Math. Res. Lett.* **5** (1998), 497-516.

[26] A. B. Goncharov, The Dihedral Lie Algebras and Galois Symmetries of $\pi_1^{(l)}(\mathbf{P} - (\{0,\infty\} \cup \mu_N))$, *Duke Math. J.* **110** (2001), 397-487.

[27] A. Goncharov and Y. Manin, Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$, *Comp. Math.* **140** (2004), 1 - 14.

[28] A. Granville, A decomposition of Riemann's zeta-function, in: Analytic Number Theory, London Math. Soc. Lecture Note Ser., Vol. 247, Cambridge University Press, Cambridge, 1997, pp. 95-101.

[29] R. Gunning, H. Rossi, "Analytic functions of several complex variables", Prentice Hall Series in Modern Analysis (1965)

[30] L. Guo, Baxter algebra and differential algebra, in: "Differential Algebra and Related Topics", World Scientific Publishing Company, (2002), 281–305.

[31] L. Guo, Algebraic Birkhoff decomposition and its applications, in: "Automorphic Forms and Langlands Program", International Press, 2008, 283-323.
[32] L. Guo and W. Keigher, Free Baxter algebras and shuffle products, *Adv. in Math.,* **150**, (2000), 117–149.
[33] L. Guo and W. Keigher, On free Baxter algebras: completions and the internal construction, *Adv. Math.* **151** (2000), 101–127, arXiv:math/0407156.
[34] L. Guo and B. Xie, Explicit double shuffle relations and a generalization of Euler's decomposition formula, arXiv:0808.2618[math.NT].
[35] L. Guo and B. Xie, Weighted sum formula for multiple zeta values, *J. Number Theory.* **129 (11)** (2009), 2747-2765.
[36] L. Guo and B. Zhang, Renormalization of multiple zeta values, *J. Algebra*, **319** (2008), 3770-3809, arXiv:math.NT/0606076.
[37] L. Guo and B. Zhang, Differential Algebraic Birkhoff Decomposition and renormalization of multiple zeta values, *J. Number Theory*, **128** (2008), 2318-2339, arXiv:0710.0432(math.NT).
[38] G. Hardy, "Divergent series", Oxford University Press, 1967.
[39] K. Hepp, Proof of the Bogoliubov-Parasiuk theorem on renormalization, *Comm. in Math. Phys.*, **2** (1966), 301–326.
[40] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.*, **152** (1992), no. 2, 275–290.
[41] M. E. Hoffman, The algebra of multiple harmonic series, *J. Algebra*, **194**, no. 2, (1997), 477–495.
[42] M. E. Hoffman, Quasi-shuffle products, *J. Algebraic Combin.*, **11**, no. 1, (2000), 49–68.
[43] M. E. Hoffman and C. Moen, Sums of triple harmonic series, *J. Number Theory* **60** (1996), 329-331.
[44] M. E. Hoffman and Y. Ohno, Relations of multiple zeta values and their algebraic expression. *J. Algebra* **262** (2003), 332–347.
[45] L. Hörmander, An Introduction to Complex Analysis in Several Variables. The university series in higher mathematics, D. Van Nostrand Company (1966)
[46] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compos. Math. **142** (2006), 307–338.
[47] A. P. Juskevic, and E. Winter, eds., Leonhard Euler und Christian Goldbach: Briefwechsel 1729-1764, Akademie-Verlag, Berlin, 1965.
[48] S. Kanemitsu, Y. Tanigawa, M. Yoshimoto, Convolution of Riemann zeta-values, *J. Math. Soc. Japan* **57** (2005), 1167–1177.
[49] M. Kontsevich, S. Vishik, Determinants of elliptic pseudo-differential operators, Max Planck Institut preprint, 1994.
[50] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, *Adv. Theor. Math. Phys.*, **2** (1998), 303-334.
[51] T.Q.T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of the multiple zeta functions, *Topology Appl.* **62** (1995) 193-206.
[52] M. Lesch, On the non-commutative residue for pseudo-differential operators with log-polyhomoge-neous symbols, *Ann. of Global Anal. and Geom.* **17** (1998) 151–187.
[53] D. Manchon, Hopf algebras in renormalization, *Handbook of Algebra*, Vol. 5 (M. Hazewinkel ed.), 365-427 (2008). See also arXiv:math. QA/0408405.
[54] D. Manchon and S. Paycha, Shuffle relations for regularized integrals of symbols, *Comm. Math. Phys.* **270** (2007), 13–51, arXiv:math-ph/0510067.
[55] D. Manchon and S. Paycha, Renormalized Chen integrals for symbols on $\mathbb{R}^n$ and renormlized polyzeta functions, arXiv:math.NT/0702135.
[56] M. Marcolli, Motivic renormalization and singularities, arXiv:0804.4824[math.ph].
[57] Y. Ohno, A generalization of the duality and sum formulas on the multiple zeta values, *J. Number Theory* **74** (1999), 39–43.
[58] Y. Ohno and J.-I. Okuda, On the sum formula for the q-analogue of non-strict multiple zeta values, *Proc. Amer. Math. Soc.* **135** (2007), 3029–3037.
[59] Y. Ohno and N. Wakabayashi, Cyclic sum of multiple zeta values. *Acta Arith.* **123** (2006), 289–295.
[60] Y. Ohno and D. Zagier, Multiple zeta values of fixed weight, depth, and height, *Indag. Math. (N.S.)* **12** (2001), 483–487.

[61] Y. Ohno and W. Zudilin, Zeta stars, *Comm. Number Theory and Phys.* **2** (2008) 327-349.

[62] J.-I. Okuda and K. Ueno, The sum formula for multiple zeta values and connection problem of the formal Knizhnik-Zamolodchikov equation. In: Zeta functions, Topology and Quantum Physics, *Dev. Math.*, **14**, Springer, New York, 2005, 145–170.

[63] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l'unité, *Pub. Math. IHES*, **95** (2002), 185-231.

[64] G.-C. Rota, Baxter operators, an introduction, In: "Gian-Carlo Rota on Combinatorics, Introductory Papers and Commentaries", Joseph P.S. Kung, Editor, Birkhäuser, Boston, 1995.

[65] G.-C. Rota, D. A. Smith, Fluctuation theory and Baxter algebras, *Istituto Nazionale di Alta Matematica,* **IX** (1972), 179–201.

[66] E. Sandifer, Multi-zeta functions, *MAA Online: How Euler did it*, January 2008. http://www.maa.org/ editorial/euler/ How%20Euler%20Did%20It%2051%20multizeta %20functions.pdf

[67] E. Speer, Analytic renormalization, *Journ. Math. Phys.* **9** (1967), 1404-1410.

[68] F. Spitzer, A combinatorial lemma and its application to probability theory, *Trans. Amer. Math. Soc.* **82** (1956), 323–339.

[69] T. Terasoma, Mixed Tate motives and multiple zeta values, *Invent. Math.*, **149**, (2002), no. 2, 339–369. `math.AG/0104231`

[70] D. Zagier, Values of zeta functions and their applications, First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, *Progr. Math.*, **120**, Birkhäuser, Basel, 1994

[71] D. Zagier, Multiple zeta values, unpublished manuscript, Bonn 1995.

[72] J. Zhao, Multiple q-zeta functions and multiple q-polylogarithms. *Ramanujan J.* **14** (2007), 189–221

[73] J. Zhao, Analytic continuation of multiple zeta functions. *Proc. Amer. Math. Soc.* **128** (2000), 1275-1283.

[74] W. Zimmermann, Convergence of Bogoliubov's method of renormalization in momentum space, *Comm. Math. Phys.* **15** (1969), 208-234.

[75] W. Zudilin, Algebraic relations for multiple zeta values, (Russian), *Uspekhi Mat. Nauk*, **58**, no.1, 2003, 3–32, translation in *Russian Math. Survey*, **58**, (2003) 1–29.

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA

E-mail address: `liguo@rutgers.edu`

Laboratoire de Mathématiques Appliquées, Université Blaise Pascal (Clermont II), Complexe Universitaire des Cézeaux, 63177 Aubière Cedex, France

E-mail address: `sylvie.paycha@math.univ-bpclermont.fr`

Department of Mathematics, Peking University, Beijing, 100871, P. R. China

E-mail address: `byhsie@math.pku.edu.cn`

Yangtze Center of Mathematics, Sichuan University, Chengdu, 610064, P. R. China

E-mail address: `zhangbin@scu.edu.cn`

The subject of algebraic cycles has its roots in the study of divisors, extending as far back as the nineteenth century. Since then, and in particular in recent years, algebraic cycles have made a significant impact on many fields of mathematics, among them number theory, algebraic geometry, and mathematical physics. The present volume contains articles on all of the above aspects of algebraic cycles. It also contains a mixture of both research papers and expository articles, so that it would be of interest to both experts and beginners in the field.

ISBN 978-0-8218-5191-3

9 780821 851913

CMIP/9

www.ams.org

www.claymath.org