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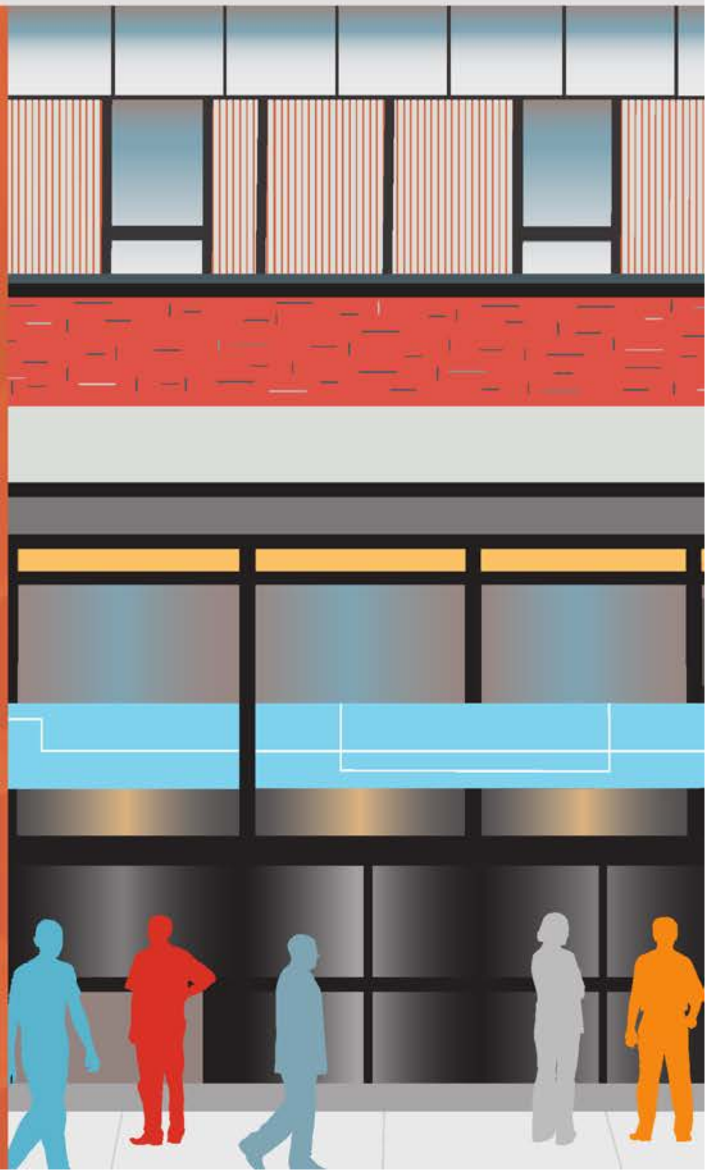
Topics in Noncommutative Geometry

Third Luis Santaló Winter School -
CIMPA Research School
Topics in Noncommutative Geometry
Universidad de Buenos Aires, Buenos Aires, Argentina
July 26–August 6, 2010

Guillermo Cortiñas
Editor



American Mathematical Society
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Preface

This volume contains the proceedings of the third Luis Santaló Winter School, organized by the Mathematics Department and the Santaló Mathematical Research Institute of the School of Exact and Natural Sciences of the University of Buenos Aires (FCEN). This series of schools is named after the geometer Luis Santaló. Born in Spain, the celebrated founder of Integral Geometry was a Professor in our department where he carried out most of his distinguished professional career.

This edition of the Santaló School took place in the FCEN from July 26 to August 6 of 2010. On this occasion the school was devoted to Noncommutative Geometry, and was supported by several institutions; the Clay Mathematics Institute was one of its main sponsors.

The topics of the school and the contents of this volume concern Noncommutative Geometry in a broad sense: it encompasses the various mathematical and physical theories that incorporate geometric ideas to study noncommutative phenomena.

One of those theories is that of deformation quantization. A main result in this area is Kontsevich's formality theorem. It implies that a Poisson structure on a manifold can always be formally quantized. More precisely it shows that there is an isomorphism (although not canonical) between the moduli space of formal deformations of Poisson structures on a manifold and the moduli space of star products on the manifold. The question of understanding Morita equivalence of star products under (a specific choice of) this isomorphism was solved by Henrique Bursztyn and Stefan Waldmann; their article in the present volume gives a survey of their work. They start by discussing how deformation quantization arises from the quantization problem in physics. Then they review the basics on star products, the main results on deformation quantization and the notion of Morita equivalence of associative algebras. After this introductory material, they present a description of Morita equivalence for star products as orbits of a suitable group action on the one hand, and the B-field action on (formal) Poisson structures on the other. Finally, they arrive at their main results on the classification of Morita equivalent star products.

The next article, by Boris Tsygan, reviews Tamarkin's proof of Kontsevich's formality theorem using the theory of operads. It also explains applications of the formality theorem to noncommutative calculus and index theory. Noncommutative calculus defines classical algebraic structures arising from the usual calculus on manifolds in terms of the algebra of functions on this manifold, in a way that is valid for any associative algebra, commutative or not. It turns out that noncommutative analogs of the basic spaces arising in calculus are well-known complexes from homological algebra. These complexes turn out to carry a very rich algebraic structure, similar to the one carried by their classical counterparts. It follows from Kontsevich's formality theorem that when the algebra in question is the algebra of functions, those noncommutative geometry structures are equivalent to the classical ones. Another consequence is the algebraic index theorem for deformation quantizations. This is a statement about a trace of a compactly supported difference of projections in the algebra of matrices over a deformed algebra. It turns out that all the data entering into this problem (namely, a deformed algebra, a trace on it, and projections in it) can be classified using formal Poisson structures on the manifold. The algebraic index theorem implies the celebrated index theorem of Atiyah-Singer and its various generalizations.

Operad theory, mentioned before as related to Kontsevich's formality theorem, is the subject of Eduardo Hoefel's paper. It follows from general theory that the Fulton-MacPherson operad $\mathcal{F}_n$ and the little discs operad $\mathcal{D}_n$ are equivalent. Hoefel gives an elementary proof of this fact by exhibiting a rather explicit homotopy equivalence between them.

Many important examples in noncommutative geometry appear as crossed products. Ralf Meyer's lecture notes deal with several crossed-products of C^*-algebras, (e.g. crossed-products by actions of locally compact groups, twisted crossed-products, crossed products by C^*correspondences) and discuss notions of equivalence of these constructions. The author shows how these examples lead naturally to the concept of a strict 2-category and to the unifying approach of a functor from a group to strict 2-categories whose objects are C^*-algebras. In addition, this approach enables the author to define in all generality the crossed product of a C^*-algebra by a group acting by Morita-Rieffel bimodules.

Jonathan Rosenberg's article is concerned with two related but distinct topics: noncommutative tori and Kasparov's KK-theory. Noncommutative tori are certain crossed products of the algebra of continuous functions on the unit circle by an action of $\mathbb{Z}$. They turn out to be noncommutative deformations of the algebra of continuous functions on the 2-torus. The article reviews the classification of noncommutative tori up to Morita equivalence, and of bundles (i.e. projective modules) over them, as well as some applications of noncommutative tori to number theory and physics. Kasparov's K-theory is one of the main homological invariants in C^*-algebra theory. It can be presented in several equivalent manners; the article reviews Kasparov's original definition in terms of Kasparov bimodules (or generalized elliptic pseudodifferential operators), Cuntz's picture in terms of quasi-homomorphisms, and Higson's description of KK as a universal homology theory for separable C^*-algebras. It assigns groups $KK_*(A, B)$ to any two separable C^*-algebras A and B; usual operator K-theory and K-homology are recovered by setting A (respectively B) equal to the complex numbers. The article also considers the equivariant version of KK for C^*-algebras equipped with a group action, and considers its applications to the K-theory of crossed products, including the Pimsner-Voiculescu sequence for crossed products with $\mathbb{Z}$ (used for example to compute the K-theory of noncommutative tori), Connes' Thom isomorphism for crossed products with $\mathbb{R}$ and the Baum-Connes conjecture for crossed products with general locally compact groups.

The Baum-Connes conjecture predicts that the K-theory of the reduced C^*-algebra $C_r(G)$ of a group G is the G-equivariant K-homology $KK_*^G(\underline{EG}, \mathbb{C})$ of the classifying space for proper actions. The conjecture is known to hold in many cases. The point of the conjecture is that the K-homology is in principle easier to compute than the K-theory of $C_r(G)$, but concrete computations are often very hard, especially if the group contains torsion. The article by Jean-François Lafont, Ivonne Ortiz and Rubén Sánchez-García is concerned with computing the groups $KK_*^G(\underline{EG}, \mathbb{C}) \otimes \mathbb{Q}$ in the case where G admits a 3-dimensional manifold model M^3 for $\underline{EG}$. In this case the authors provide an explicit formula in terms of the combinatorics of the model M^3.

As noted above, groups play a key role in noncommutative geometry; group theory is therefore an important tool in the area. Andrzej Zuk's article presents an introduction to the theory of groups generated by finite automata. This class contains several remarkable countable groups, which provide solutions to long standing questions in the field. For instance, Aleshin's automata gives a group which is infintely generated yet torsion; this answers affirmatively a question raised by Burnside in 1902 of whether such groups could exist. Aleshin's group is also an example of a group which is not of polynomial growth yet it is of subexponential growth. Other remarkable examples of automata groups are also presented, including a group without uniform exponential growth, and exotic amenable groups.

In Connes' theory of noncommutative geometry, spectral triples play the role of noncommutative Riemannian manifolds; they are also the source of elements in Kasparov's KK-theory. Bram Mesland's article deals with the construction of a category of spectral triples that is compatible with the Kasparov product in KK-theory. The theory described shows that by introducing a notion of smoothness on unbounded KK-cycles, the Kasparov product of such cycles can be defined directly, by an algebraic formula. This allows one to view such cycles as morphisms in a category whose objects are spectral triples.

The article by Roberto Trinchero is also concerned with spectral triples. It provides a link between Connes' noncommutative geometry and quantum field theory. It displays, in detail, a toy model that aims to shed some light in the dimensional renormalization framework of Quantum Field Theory, by comparison with the results obtained under the paradigm of such theory, where a Grasmannian algebra is the starting point.

Max Karoubi's article concerns twisted K-theory, a theory of very active current research, which was originally defined by Karoubi and Donovan in the late 1960s. The approach presented here is based on the notion of *twisted vector bundles*. Such bundles may be interpreted as modules over suitable algebra bundles. Roughly speaking, twisted K-theory appears as the Grothendieck group of the category of twisted vector bundles. Thus a geometric description of the theory is obtained. The usual operations on vector bundles are extended to twisted vector bundles. The article also contains a section on cup-products, where it is shown that the various ways to define them coincide up to isomorphism. An analogue of the Chern character is then defined, going from twisted K-theory to twisted cohomology. The approach is based on a version of Chern-Weil theory for connections on twisted vector bundles in the finite and infinite dimensional cases.

Noncommutative algebraic geometry is represented by Gonçalo Tabuada's article. It surveys the theory of noncommutative motives, a fast developing area of current research, led by Tabuada among others. In commutative algebraic geometry, Grothendieck envisioned a theory of motives as a universal cohomological invariant for schemes, through which all the Weil cohomology theories should factor. The theory is currently one of the most active areas of research in commutative algebraic geometry. In the noncommutative world, schemes are replaced by DG-categories. Roughly speaking, a DG-category should be regarded as the category of complexes of modules over the structure sheaf of the underlying noncommutative scheme. There is a noncommutative motivic category whose objects are the DG-categories, through which classical invariants such as K-theory and (topological) Hochschild and cyclic homology all factor. The morphisms in the motivic category form a bivariant theory of DG-categories with formal properties analogous to those Kasparov's KK-theory has in the C^*-algebra setting.

In conclusion, this volume presents a good sample of the wide range of aspects of current research in noncommutative geometry and its applications.

Guillermo Cortiñas
Buenos Aires, June 2012

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Classifying Morita Equivalent Star Products

Henrique Bursztyn and Stefan Waldmann

ABSTRACT. These notes are based on lectures given at CIMPA's school *Topics in noncommutative geometry*, held in Buenos Aires in 2010. The main goal is to expound the classification of deformation quantization algebras up to Morita equivalence.

CONTENTS

1. Introduction

Deformation quantization [**1**] (see e.g. [**15**] for a survey) is a quantization scheme in which algebras of quantum observables are obtained as formal deformations of classical observable algebras. For a smooth manifold M, let $C^\infty(M)$ denote the algebra of complex-valued smooth functions on M, and let $C^\infty(M)[[\hbar]]$ be the space of formal power series in a parameter $\hbar$ with coefficients in $C^\infty(M)$; deformation quantization concerns the study of associative products $\star$ on $C^\infty(M)[[\hbar]]$, known as *star products*, deforming the pointwise product on $C^\infty(M)$,

$$f \star g = fg + O(\hbar),$$

in the sense of Gerstenhaber [**13**]. The noncommutativity of a star product $\star$ is controlled, in first order, by a Poisson structure $\{\cdot, \cdot\}$ on M, in the sense that

$$f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + O(\hbar^2).$$

Two fundamental issues in deformation quantization are the existence and isomorphism classification of star products on a given Poisson manifold, and the most general results in these directions follow from Kontsevich's formality theorem [**18**]. In these notes we treat another kind of classification problem in deformation quantization, namely that of describing when two star products define *Morita equivalent* algebras. This study started in [**2, 5**] (see also [**17**]), and here we will mostly review the results obtained in [**3**] (where detailed proofs can be found), though from a less technical perspective.

Morita equivalence [**21**] is an equivalence relation for algebras, which is based on comparing their categories of representations. This type of equivalence is weaker than the usual notion of algebra isomorphism, but strong enough to capture essential algebraic properties. The notion of Morita equivalence plays a central role in noncommutative geometry and has also proven relevant at the interface of noncommutative geometry and physics, see e.g. [**17, 20, 27**]. Although there are more analytical versions of deformation quantization and Morita equivalence used in noncommutative geometry (especially in the context of C^*-algebras, see e.g. [**24, 25**] and [**8**, Chp. II, App. A]), our focus in these notes is on deformation quantization and Morita equivalence in the purely algebraic setting.

The classification of Morita equivalent star products on a manifold M [**3**] builds on Kontsevich's classification result [**18**], which establishes a bijective correspondence between the moduli space of star products on M, denoted by $\mathrm{Def}(M)$, and the set $\mathrm{FPois}(M)$ of equivalence classes of formal families of Poisson structures on M,

$$\mathcal{K}_* : \mathrm{FPois}(M) \overset{\sim}{\to} \mathrm{Def}(M). \tag{1.1}$$

Morita equivalence of star products on M defines an equivalence relation on $\mathrm{Def}(M)$, and these notes explain how one recognizes Morita equivalent star products in terms of their classes in $\mathrm{FPois}(M)$, through Kontsevich's correspondence (1.1). We divide the discussion into two steps: first, we identify a canonical group action on $\mathrm{Def}(M)$ whose orbit relation coincides with Morita equivalence of star products (Thm. 5.2); second, we find the expression for the corresponding action on $\mathrm{FPois}(M)$, making the quantization map (1.1) equivariant (Thm. 7.1).

This paper is structured much in the same way as the lectures presented at the school. In Section 2, we briefly discuss how deformation quantization arises from the quantization problem is physics; Section 3 reviews the basics on star products and the main results on deformation quantization; Morita equivalence is recalled in Section 4, while Section 5 presents a description of Morita equivalence for star products as orbits of a suitable group action. Section 6 discusses the B-field action on (formal) Poisson structures, and Section 7 presents the main results on the classification of Morita equivalent star products.

Notation and conventions: For a smooth manifold M, $C^\infty(M)$ denotes its algebra of smooth complex-valued functions. Vector bundles $E \to M$ are taken to be complex, unless stated otherwise. $\mathcal{X}^\bullet(M)$ denotes the graded algebra of (complex) multivector fields on M, $\Omega^\bullet(M)$ is the graded algebra of (complex) differential forms, while $\Omega^p_{cl}(M)$ denotes the space of closed p-forms on M. We use the notation $H^\bullet_{dR}(M)$ for de Rham cohomology. For any vector space V over $k = \mathbb{R}$ or $\mathbb{C}$, $V[[\hbar]]$ denotes the space of formal power series with coefficients in V in a formal parameter $\hbar$, naturally seen as a module over $k[[\hbar]]$. We will use Einstein's summation convention whenever there is no risk of confusion.

Acknowledgements: We are grateful to V. Dolgushev for his collaboration on [**3**], and G. Cortiñas for his invitation to the school, hospitality in Buenos Aires, and encouragement to have these notes written up. H.B. thanks CNPq, Faperj and IMPA for financial support.

2. A word on quantization

Quantization is usually understood as a map assigning quantum observables to classical ones. In general, classical observables are represented by smooth functions on a symplectic or Poisson manifold (the classical "phase space"), whereas quantum observables are given by (possibly unbounded) operators acting on some (pre-)Hilbert space. A "quantization map" is expected to satisfy further compatibility properties (see e.g. [**20**] for a discussion), roughly saying that the algebraic features of the space of classical observables (e.g. pointwise multiplication and Poisson bracket of functions) should be obtained from those of quantum observables (e.g. operator products and commutators) in an appropriate limit "$\hbar \to 0$". As we will see in Section 3, deformation quantization offers a purely algebraic formulation of quantization. In order to motivate it, we now briefly recall the simplest quantization procedure in physics, known as *canonical quantization*.

Let us consider the classical phase space $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, equipped with global coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$, and the canonical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j}, \quad f, g \in C^\infty(\mathbb{R}^{2n}), \tag{2.1}$$

so that the brackets of canonical coordinates are

$$\{q^k, p_\ell\} = \delta^k_\ell,$$

for $k, \ell = 1, \dots, n$. Quantum mechanics tells us that the corresponding Hilbert space in this case is $\mathrm{L}^2(\mathbb{R}^n)$, the space of wave functions on the configuration space $\mathbb{R}^n = \{(q^1, \dots, q^n)\}$. To simplify matters when dealing with unbounded operators, we will instead consider the subspace $C_0^\infty(\mathbb{R}^n)$ of compactly supported functions on $\mathbb{R}^n$. In canonical quantization, the classical observable $q^k \in C^\infty(T^*\mathbb{R}^n)$ is taken to the multiplication operator $Q^k : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, $\psi \mapsto Q^k(\psi)$, where

$$Q^k(\psi)(q) := q^k\psi(q), \quad \text{for } q = (q^1, \dots, q^n) \in \mathbb{R}^n, \tag{2.2}$$

while the classical observable $p_\ell \in C^\infty(T^*\mathbb{R}^n)$ is mapped to the differentiation operator $P_\ell : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ given by

$$\psi \overset{P_\ell}{\mapsto} -\mathrm{i}\hbar\frac{\partial \psi}{\partial q^\ell}. \tag{2.3}$$

Here $\hbar$ is Planck's constant. The requirements $q^k \mapsto Q^k$, $p_\ell \mapsto P_\ell$, together with the condition that the constant function 1 is taken to the identity operator $\mathrm{Id} : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, constitute the core of canonical quantization.

A natural issue is whether one can extend the canonical quantization procedure to assign operators to more general functions on $T^*\mathbb{R}^n$, including higher order monomials of q^k and p_ℓ. Since on the classical side $q^k p_\ell = p_\ell q^k$, but on the quantum side we have the canonical commutation relations

$$[Q^k, P_\ell] = Q^k P_\ell - P_\ell Q^k = \mathrm{i}\hbar\delta^k_\ell, \tag{2.4}$$

any such extension relies on the choice of an *ordering prescription*, for which one has some freedom. As a concrete example, we consider the *standard ordering*, defined by writing, for a given monomial in q^k and p_ℓ, all momentum variables p_ℓ to the right, and then replacing q^k by Q^k and p_ℓ by P_ℓ; explicitly, this means that $q^{k_1}\cdots q^{k_r}p_{\ell_1}\cdots p_{\ell_s}$ is quantized by the operator $Q^{k_1}\cdots Q^{k_r}P_{\ell_1}\cdots P_{\ell_s}$. If f is a polynomial in q^k and p_ℓ, $k,\ell = 1,\dots,n$, we can explicitly write this *standard-ordered quantization map* as

$$f \;\mapsto\; \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p_{k_1}\cdots\partial p_{k_r}}\Big|_{p=0} \frac{\partial^r}{\partial q^{k_1}\cdots \partial q^{k_r}}. \tag{2.5}$$

One may verify that formula (2.5) in fact defines a linear bijection

$$\varrho_{\mathrm{Std}} \colon \mathrm{Pol}(T^*\mathbb{R}^n) \longrightarrow \mathrm{DiffOp}(\mathbb{R}^n) \tag{2.6}$$

between the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of smooth functions on $T^*\mathbb{R}^n$ that are polynomial in the momentum variables $p_1,\dots,p_n$, and the space $\mathrm{DiffOp}(\mathbb{R}^n)$ of differential operators with smooth coefficients on $\mathbb{R}^n$. In order to compare the pointwise product and Poisson bracket of classical observables with the operator product and commutator of quantum observables, one may use the bijection (2.6) to pull back the operator product to $\mathrm{Pol}(T^*\mathbb{R}^n)$,

$$f \star_{\mathrm{Std}} g := \varrho_{\mathrm{Std}}^{-1}\left(\varrho_{\mathrm{Std}}(f)\varrho_{\mathrm{Std}}(g)\right), \tag{2.7}$$

so as to have all structures defined on the same space. A direct computation yields the explicit formula for the new product $\star_{\mathrm{Std}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$:

$$f \star_{\mathrm{Std}} g = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p_{k_1}\cdots\partial p_{k_r}} \frac{\partial^r g}{\partial q^{k_1}\cdots \partial q^{k_r}}. \tag{2.8}$$

With this formula at hand, one may directly check the following properties:

1. $f \star_{\mathrm{Std}} g = fg + O(\hbar)$;
2. $f \star_{\mathrm{Std}} g - g \star_{\mathrm{Std}} f = \mathrm{i}\hbar\{f,g\} + O(\hbar^2)$;
3. The constant function 1 satisfies $1 \star_{\mathrm{Std}} f = f = f \star_{\mathrm{Std}} 1$, for all $f \in \mathrm{Pol}(T^*\mathbb{R}^n)$;
4. $\star_{\mathrm{Std}}$ is an *associative* product.

The associativity property is evident from construction, since $\star_{\mathrm{Std}}$ is isomorphic to the composition product of differential operators. As we will see in the next section, these properties of $\star_{\mathrm{Std}}$ underlie the general notion of a *star product*.

Before presenting the precise formulation of deformation quantization, we have two final observations.

- First, we note that there are alternatives to the standard-ordering quantization (2.5). From a physical perspective, one is also interested in comparing the involutions of the algebras at the classical and quantum levels, i.e., complex conjugation of functions and adjoints of operators. Regarding the standard-ordering quantization, the (formal) adjoint of $\varrho_{\mathrm{Std}}(f)$ is not given by $\varrho_{\mathrm{Std}}(\overline{f})$. Instead, an integration by parts shows that $\varrho_{\mathrm{Std}}(f)^* = \varrho_{\mathrm{Std}}(N^2\overline{f})$, for the operator $N : \mathrm{Pol}(T^*\mathbb{R}^n) \to \mathrm{Pol}(T^*\mathbb{R}^n)$,

$$N = \exp\left(\frac{\hbar}{2\mathrm{i}} \frac{\partial^2}{\partial q^k \partial p_k}\right), \tag{2.9}$$

  where exp is defined by its power series. If we pass to the *Weyl-ordering quantization map*,

$$\varrho_{\mathrm{Weyl}} : \mathrm{Pol}(T^*\mathbb{R}^n) \to \mathrm{DiffOp}(\mathbb{R}^n), \quad \varrho_{\mathrm{Weyl}}(f) := \varrho_{\mathrm{Std}}(Nf), \tag{2.10}$$

we have $\varrho_{\mathrm{Weyl}}(f)^* = \varrho_{\mathrm{Weyl}}(\overline{f})$. This quantization, when restricted to monomials in q^k, p_ℓ, agrees with the ordering prescribed by total symmetrization. Just as (2.7), the map (2.10) is a bijection, and it defines the *Weyl product* $\star_{\mathrm{Weyl}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$ by

$$f \star_{\mathrm{Weyl}} g = \varrho_{\mathrm{Weyl}}^{-1}(\varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g)). \tag{2.11}$$

The two products $\star_{\mathrm{Std}}$ and $\star_{\mathrm{Weyl}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$ are related by

$$f \star_{\mathrm{Weyl}} g := N^{-1}(Nf \star_{\mathrm{Std}} Ng); \tag{2.12}$$

since $N = \mathrm{Id} + O(\hbar)$, one may directly check that $\star_{\mathrm{Weyl}}$ satisfies the same properties (1)–(4) listed above for $\star_{\mathrm{Std}}$. But $\star_{\mathrm{Weyl}}$ satisfies an additional compatibility condition relative to complex conjugation:

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f}. \tag{2.13}$$

The are other possible orderings leading to products satisfying (2.13), such as the so-called *Wick ordering*, see e.g. [**28**, Sec. 5.2.3].

- The second observation concerns the difficulties in extending the quantization procedures discussed so far to manifolds other than $T^*\mathbb{R}^n$. The quantizations ϱ_{Std} and ϱ_{Weyl} are only defined for functions in $\mathrm{Pol}(T^*\mathbb{R}^n)$, i.e., polynomial in the momentum variables. On an arbitrary manifold M, however, there is no analog of this class of functions, and generally there are no natural subalgebras of $C^\infty(M)$ to be considered. From another viewpoint, one sees that the expression for $\star_{\mathrm{Std}}$ in (2.8) does not make sense for arbitrary smooth functions, as the radius of convergence in $\hbar$ is typically 0, so $\star_{\mathrm{Std}}$ does not extend to a product on $C^\infty(T^*\mathbb{R}^n)$ (and the same holds for $\star_{\mathrm{Weyl}}$). One can however interpret (2.8) as a *formal power series* in the parameter $\hbar$, i.e., as a product on $C^\infty(T^*\mathbb{R}^n)[[\hbar]]$. This viewpoint now carries over to arbitrary manifolds and leads to the general concept of deformation quantization, in which quantization is formulated in purely algebraic terms by means of associative product structures $\star$ on $C^\infty(M)[[\hbar]]$ rather than operator representations [1].

3. Deformation quantization

Let M be a smooth manifold, and let $C^\infty(M)$ denote its algebra of *complex-valued* smooth functions. We consider $C^\infty(M)[[\hbar]]$, the set of formal power series in $\hbar$ with coefficients in $C^\infty(M)$, as a module over the ring $\mathbb{C}[[\hbar]]$.

3.1. Star products. A *star product* [**1**] on M is an associative product $\star$ on the $\mathbb{C}[[\hbar]]$-module $C^\infty(M)[[\hbar]]$ given as follows: for $f, g \in C^\infty(M)$,

$$f \star g = fg + \sum_{r=1}^{\infty} \hbar^r C_r(f,g), \tag{3.1}$$

where $C_r : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $r = 1, 2, \ldots$, are bidifferential operators, and this product operation is extended to $C^\infty(M)[[\hbar]]$ by $\hbar$-linearity (and $\hbar$-adic

[1]Deformation quantization, in its most general form, completely avoids analytical issues (such as convergence properties in $\hbar$ and related operator representations); these aspects are mostly considered in particular classes of examples, see e.g. [**15**, Sec. 4] for a discussion and further references.

continuity). Additionally, we require that the constant function $1 \in C^\infty(M)$ is still a unit for $\star$:

$$1 \star f = f \star 1 = f, \quad \forall f \in C^\infty(M).$$

Since

$$f \star g = fg \mod \hbar, \quad \forall f, g \in C^\infty(M),$$

one views star products as associative, but not necessarily commutative, deformations (in the sense of [**13**]) of the pointwise product of functions on M. The $\mathbb{C}[[\hbar]]$-algebra $(C^\infty(M)[[\hbar]], \star)$ is called a *deformation quantization* of M.

Two star products $\star$ and $\star'$ on M are said to be *equivalent* if there are differential operators $T_r : C^\infty(M) \to C^\infty(M)$, $r = 1, 2, \ldots$, such that

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r T_r \tag{3.2}$$

satisfies

$$T(f \star g) = T(f) \star' T(g). \tag{3.3}$$

We define the *moduli space of star products* on M as the set of equivalence classes of star products, and we denote it by $\mathrm{Def}(M)$.

EXAMPLE 3.1. *Formula* (2.8) *for* $\star_{\mathrm{Std}}$ *defines a star product on* $M = T^*\mathbb{R}^n$*, and the same holds for the product* $\star_{\mathrm{Weyl}}$ *given in* (2.11)*; by* (2.12)*, the operator* N *in* (2.9) *defines an equivalence between the star products* $\star_{\mathrm{Std}}$ *and* $\star_{\mathrm{Weyl}}$*.*

3.2. Noncommutativity in first order: Poisson structures. Given a star product $\star$ on M, its noncommutativity is measured, in first order, by the bilinear operation $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$,

$$\{f, g\} := \frac{1}{\mathrm{i}\hbar}(f \star g - g \star f)\Big|_{\hbar=0} = \frac{1}{\mathrm{i}}(C_1(f, g) - C_1(g, f)), \quad f, g \in C^\infty(M). \tag{3.4}$$

It follows from the associativity of $\star$ that $\{\cdot, \cdot\}$ is a *Poisson structure* on M (see e.g. [**7**, Sec.19]); recall that this means that $\{\cdot, \cdot\}$ is a Lie bracket on $C^\infty(M)$, which is compatible with the pointwise product on $C^\infty(M)$ via the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \quad f, g, h \in C^\infty(M).$$

The Leibniz rule implies that any Poisson structure $\{\cdot, \cdot\}$ is equivalently described by a bivector field $\pi \in \mathcal{X}^2(M)$, via

$$\{f, g\} = \pi(df, dg),$$

satisfying the additional condition (accounting for the Jacobi identity of $\{\cdot, \cdot\}$) that $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is an extension to $\mathcal{X}^\bullet(M)$ of the Lie bracket of vector fields, known as the *Schouten bracket.* The pair (M, π) is called a *Poisson manifold* (see e.g. [**7**] for more on Poisson geometry). If a star product $\star$ corresponds to a Poisson structure π via (3.4), we say that $\star$ *quantizes* π, or that $\star$ is a *deformation quantization of the Poisson manifold* (M, π).

A Poisson structure π on M defines a bundle map

$$\pi^\sharp : T^*M \to TM, \quad \alpha \mapsto i_\alpha \pi = \pi(\alpha, \cdot). \tag{3.5}$$

We say that π is *nondegenerate* if (3.5) is an isomorphism, in which case π is equivalent to a symplectic structure $\omega \in \Omega^2(M)$, defined by

$$\omega(\pi^\sharp(\alpha), \pi^\sharp(\beta)) = \pi(\beta, \alpha); \tag{3.6}$$

alternatively, the 2-form ω is defined by the condition that the map $TM \to T^*M$, $X \mapsto i_X\omega$, is inverse to (3.5).

EXAMPLE 3.2. *The star product* $\star_{\mathrm{Std}}$ *on* $T^*\mathbb{R}^n$ *quantizes the classical Poisson bracket*

$$\{f, g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j},$$

defined by the (nondegenerate) bivector field $\pi = \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}$. *The same holds for* $\star_{\mathrm{Weyl}}$.

3.3. Existence and classification of star products. A direct computation shows that if $\star$ and $\star'$ are equivalent star products, i.e., define the same element in $\mathrm{Def}(M)$, then they necessarily quantize the same Poisson structure. For a Poisson structure π on M, we denote by

$$\mathrm{Def}(M, \pi) \subset \mathrm{Def}(M)$$

the subset of equivalence classes of star products quantizing π. The central issue in deformation quantization is understanding $\mathrm{Def}(M, \pi)$, for example by finding a concrete parametrization of this space. Concretely, deformation quantization concerns the following fundamental issues:

- Given a Poisson structure π on M, is there a star product quantizing it?
- If there is a star product quantizing π, how many distinct equivalence classes in $\mathrm{Def}(M)$ with this property are there?

The main result on existence and classification of star products on Poisson manifolds follows from Kontsevich's formality theorem [**18**], that we briefly recall.

Let $\mathcal{X}^2(M)[[\hbar]]$ denote the space of formal power series in $\hbar$ with coefficients in bivector fields. A *formal Poisson structure* on M is an element $\pi_\hbar \in \hbar\mathcal{X}^2(M)[[\hbar]]$,

$$\pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \pi_r, \quad \pi_r \in \mathcal{X}^2(M),$$

such that

$$[\pi_\hbar, \pi_\hbar] = 0, \tag{3.7}$$

where $[\cdot,\cdot]$ denotes the $\hbar$-bilinear extension of the Schouten bracket to formal power series. It immediately follows from (3.7) that

$$[\pi_1, \pi_1] = 0,$$

i.e., π_1 is an ordinary Poisson structure on M. So we view $\pi_\hbar$ as a formal deformation of π_1 in the realm of Poisson structures.

A formal Poisson structure $\pi_\hbar$ defines a bracket $\{\cdot,\cdot\}_\hbar$ on $C^\infty(M)[[\hbar]]$ by

$$\{f, g\}_\hbar = \pi_\hbar(df, dg).$$

Two formal Poisson structures $\pi_\hbar$ and $\pi'_\hbar$ are *equivalent* if there is a formal diffeomorphism $T = \exp(\sum_{r=1}^{\infty} \hbar^r X_r) : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$, where each $X_r \in \mathcal{X}^1(M)$ is a vector field, satisfying

$$T\{f, g\}_\hbar = \{Tf, Tg\}'_\hbar.$$

(Here the exponential exp is defined by its formal series, and it gives a well-defined formal power series in $\hbar$ since $\sum_{r=1}^{\infty} \hbar^r X_r$ starts at order $\hbar$.) We define the *moduli space of formal Poisson structures* on M as the set of equivalence classes of formal

Poisson structures, and we denote it by FPois(M). One may readily verify that two equivalent formal Poisson structures necessarily agree to first order in $\hbar$, i.e., deform the same Poisson structure. So, given a Poisson structure π on M, we may consider the subset

$$\mathrm{FPois}(M,\pi) \subset \mathrm{FPois}(M)$$

of equivalence classes of formal Poisson structures deforming π.

We can now state Kontsevich's theorem [**18**].

THEOREM 3.3. *There is a one-to-one correspondence*

$$\mathcal{K}_* : \mathrm{FPois}(M) \xrightarrow{\sim} \mathrm{Def}(M), \quad [\pi_\hbar] = [\hbar\pi_1 + \cdots] \mapsto [\star], \tag{3.8}$$

such that $\frac{1}{\mathrm{i}\hbar}[f,g]_\star\big|_{\hbar=0} = \pi_1(df,dg)$.

For a given star product $\star$ on M, the element in FPois(M) corresponding to $[\star]$ under (3.8) is called its *characteristic class*, or its *Kontsevich class*.

Theorem 3.3 answers the existence and classification questions for star products as follows:

- Any Poisson structure π on M may be seen as a formal Poisson structure $\hbar\pi$. So it defines a class $[\hbar\pi] \in \mathrm{FPois}(M)$, which is quantized by any star product $\star$ such that $[\star] = \mathcal{K}_*([\hbar\pi_1])$.
- For any Poisson structure π on M, the map (3.8) restricts to a bijection

$$\mathrm{FPois}(M,\pi) \xrightarrow{\sim} \mathrm{Def}(M,\pi). \tag{3.9}$$

  This means that the distinct classes of star products quantizing π are in one-to-one correspondence with the distinct classes of formal Poisson structures deforming π.

REMARK 3.4.

(a) *Theorem 3.3 is a consequence of a much more general result, known as Kontsevich's formality theorem* [**18**]*; this theorem asserts that, for any manifold M, there is an L_∞-quasi-isomorphism from the differential graded Lie algebra (DGLA) $\mathcal{X}(M)$ of multivector fields on M to the DGLA $\mathcal{D}(M)$ of multidifferential operators on M, and moreover the first Taylor coefficient of this L_∞-morphism agrees with the natural map $\mathcal{X}(M) \to \mathcal{D}(M)$ (defined by viewing vector fields as differential operators). It is a general fact that any L_∞-quasi-isomorphism between DGLAs induces a one-to-one correspondence between equivalence classes of Maurer-Cartan elements. Theorem 3.3 follows from the observation that the Maurer-Cartan elements in $\mathcal{X}(M)[[\hbar]]$ are formal Poisson structures, whereas the Maurer-Cartan elements in $\mathcal{D}(M)[[\hbar]]$ are star products.*

(b) *We recall that the L_∞-quasi-isomorphism from $\mathcal{X}(M)$ to $\mathcal{D}(M)$, also called a* formality*, is not unique, and the map* (3.8) *may depend upon this choice (see e.g.* [**12**] *for more details and references). Just as in* [**3**]*, for the purposes of these notes, we will consider the specific global formality constructed in* [**10**]*. The specific properties of the global formality that we will need are explicitly listed in* [**3**, Sec. 2.2].

In general, not much is known about the space FPois(M,π), which parametrizes Def(M,π), according to (3.9). An exception is when the Poisson structure π is nondegenerate, i.e., defined by a symplectic structure $\omega \in \Omega^2(M)$. In this case, any formal Poisson structure $\pi_\hbar = \hbar\pi + \sum_{r=2}^{\infty} \hbar^r \pi_r$ is equivalent (similarly to (3.6)) to a formal series $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega_r$, where each $\omega_r \in \Omega^2(M)$ is closed (see e.g. [**15**, Prop. 13]); moreover, two formal Poisson structures $\pi_\hbar$, $\pi'_\hbar$ deforming the same Poisson structure π, and corresponding to $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega_r$ and $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega'_r$, define the same class in FPois(M, π) if and only if, for all $r \geq 1$, ω_r and ω'_r are cohomologous. As a result, FPois(M, π) is in bijection with $H^2_{dR}(M, \mathbb{C})[[\hbar]]$. But in order to keep track of the symplectic form ω, one usually replaces $H^2_{dR}(M, \mathbb{C})[[\hbar]]$ by the affine space $\frac{[\omega]}{\hbar} + H^2_{dR}(M, \mathbb{C})[[\hbar]]$ and considers the identification

$$\frac{1}{\hbar}[\omega] + H^2_{dR}(M, \mathbb{C})[[\hbar]] \cong \mathrm{FPois}(M, \pi). \tag{3.10}$$

By (3.9), the map $\mathcal{K}_*$ induces a bijection

$$\frac{1}{\hbar}[\omega] + H^2_{dR}(M, \mathbb{C})[[\hbar]] \overset{\sim}{\to} \mathrm{Def}(M, \pi), \tag{3.11}$$

which gives an explicit parametrization of star products on the symplectic manifold (M, ω). The map (3.11) is proven in [**3**, Sec. 4] to coincide with the known classification of symplectic star products (see e.g. [**15, 16**] for an exposition with original references), which is intrinsic and prior to Kontsevich's general result. The element $c(\star) \in \frac{1}{\hbar}[\omega] + H^2_{dR}(M, \mathbb{C})[[\hbar]]$ corresponding to a star product $\star$ on (M, ω) under (3.11) is known as its *Fedosov-Deligne characteristic class*. In particular, if $H^2_{dR}(M) = \{0\}$, all star products quantizing a fixed symplectic structure on M are equivalent to one another. For star products satisfying the additional compatibility condition (2.13), a classification is discussed in [**23**].

We now move on to the main issue addressed in these notes: characterizing star products on a manifold M which are *Morita equivalent* in terms of their characteristic classes. We first recall basic facts about Morita equivalence.

4. Morita equivalence reminder

In this section, we will consider k-algebras (always taken to be associative and unital), where k is a commutative, unital, ground ring; we will be mostly interested in the cases $k = \mathbb{C}$ or $\mathbb{C}[[\hbar]]$.

Morita equivalence aims at characterizing a k-algebra in terms of its representation theory, i.e., its category of modules. Let us consider unital k-algebras $\mathcal{A}$, $\mathcal{B}$, and denote their categories of left modules by ${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{B}}\mathfrak{M}$. In order to compare ${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{B}}\mathfrak{M}$, we observe that any $(\mathcal{B}, \mathcal{A})$-bimodule X (which we may also denote by ${}_{\mathcal{B}}X_{\mathcal{A}}$, to stress the left $\mathcal{B}$-action and right $\mathcal{A}$-action) gives rise to a functor ${}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$, defined on objects by tensor product:

$$V \mapsto X \otimes_{\mathcal{A}} V.$$

We call ${}_{\mathcal{B}}X_{\mathcal{A}}$ *invertible* if there is an $(\mathcal{A}, \mathcal{B})$-bimodule ${}_{\mathcal{A}}Y_{\mathcal{B}}$ such that $X \otimes_{\mathcal{A}} Y \cong \mathcal{B}$ as $(\mathcal{B}, \mathcal{B})$-bimodules, and $Y \otimes_{\mathcal{B}} X \cong \mathcal{A}$ as $(\mathcal{A}, \mathcal{A})$-bimodules. In this case, the functor ${}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ defined by ${}_{\mathcal{B}}X_{\mathcal{A}}$ is an equivalence of categories.

We say that two unital k-algebras $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exists an invertible bimodule ${}_{\mathcal{B}}X_{\mathcal{A}}$. Note that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic algebras, through an isomorphism $\psi : \mathcal{B} \to \mathcal{A}$, then they are necessarily Morita equivalent: $\mathcal{A}$ itself may be viewed as an invertible $(\mathcal{B}, \mathcal{A})$-bimodule, with right $\mathcal{A}$-action given by algebra multiplication on the right, and left $\mathcal{B}$-action given by left multiplication via ψ, $(b, a) \mapsto \psi(b)a$. One readily verifies that Morita equivalence is a reflexive and symmetric relation; to see that it is transitive, hence an equivalence relation for unital k-algebras, the key observation is that if Y is a $(\mathcal{C},\mathcal{B})$-bimodule and X is a $(\mathcal{B},\mathcal{A})$-bimodule, then the tensor product

$$Y \otimes_{\mathcal{B}} X$$

is a $(\mathcal{C},\mathcal{A})$-bimodule, which is invertible provided X and Y are.

For any unital k-algebra $\mathcal{A}$, the set of isomorphism classes of invertible $(\mathcal{A},\mathcal{A})$-bimodules has a natural group structure with respect to bimodule tensor product; we denote this group of "self-Morita equivalences" of $\mathcal{A}$ by $\mathrm{Pic}(\mathcal{A})$, and call it the *Picard group* of $\mathcal{A}$.

The main characterization of invertible bimodules is given by Morita's theorem [**21**] (see e.g. [**19**, Sec. 18]):

THEOREM 4.1. *A $(\mathcal{B},\mathcal{A})$-bimodule X is invertible if and only if the following holds: as a right $\mathcal{A}$-module, $X_{\mathcal{A}}$ is finitely generated, projective, and full, and the natural map $\mathcal{B} \to \mathrm{End}(X_{\mathcal{A}})$ is an algebra isomorphism.*

In other words, the theorem asserts that a bimodule ${}_{\mathcal{B}}X_{\mathcal{A}}$ is invertible if and only if the following is satisfied: there exists a projection $P \in M_n(\mathcal{A})$, $P^2 = P$, for some $n \in \mathbb{N}$, so that, as a right $\mathcal{A}$-module, $X_{\mathcal{A}} \cong P\mathcal{A}^n$; additionally, $X_{\mathcal{A}}$ being full means that the ideal in $\mathcal{A}$ generated by the entries of P agrees with $\mathcal{A}$; furthermore, the left $\mathcal{B}$-action on X identifies $\mathcal{B}$ with $\mathrm{End}_{\mathcal{A}}(P\mathcal{A}^n) = PM_n(\mathcal{A})P$.

A simple example of Morita equivalent algebras is $\mathcal{A}$ and $M_n(\mathcal{A})$ for any $n \geq 1$; in this case, an invertible bimodule is given by the free $(M_n(\mathcal{A}),\mathcal{A})$-bimodule $\mathcal{A}^n$. Amongst commutative algebras, Morita equivalence boils down to algebra isomorphism; nevertheless, the Picard group of a commutative algebra is generally larger than its group of algebra automorphisms, as illustrated by the next example.

EXAMPLE 4.2. *Let $\mathcal{A} = C^\infty(M)$, equipped with the pointwise product. By the smooth version of Serre-Swan's theorem, see e.g.* [**22**, Thm. 11.32], *finitely generated projective modules $X_{\mathcal{A}}$ are given by the space of smooth sections of vector bundles $E \to M$,*

$$X_{\mathcal{A}} = \Gamma(E).$$

Writing $E = P\mathcal{A}^n$ for a projection P, we see that $\mathrm{tr}(P) = \mathrm{rank}(E)$, so the module $X_{\mathcal{A}}$ is full whenever E has nonzero rank, in which case $\Gamma(E)$ is an invertible $(\Gamma(\mathrm{End}(E)), C^\infty(M))$-bimodule. We conclude that all the algebras Morita equivalent to $C^\infty(M)$ are (isomorphic to one) of the form $\Gamma(\mathrm{End}(E))$. In particular, for any line bundle $L \to M$, $\Gamma(L)$ defines a self Morita equivalence of $C^\infty(M)$, since $\mathrm{End}(L)$ is the trivial line bundle $M \times \mathbb{C}$, so $\Gamma(\mathrm{End}(L)) \cong C^\infty(M)$. Recall that the set of isomorphism classes of complex line bundles over M forms a group (under tensor product), denoted by $\mathrm{Pic}(M)$, which is isomorphic to the additive group $H^2(M,\mathbb{Z})$.

To obtain a complete description of the Picard group of the algebra $C^\infty(M)$, recall that any automorphism of $C^\infty(M)$ is realized by a diffeomorphism $\varphi : M \to M$ via pullback,

$$f \mapsto \varphi^* f = f \circ \varphi,$$

so the group of algebra automorphisms of $C^\infty(M)$ is identified with $\mathrm{Diff}(M)$. Putting these ingredients together, one verifies that

$$\mathrm{Pic}(C^\infty(M)) = \mathrm{Diff}(M) \ltimes \mathrm{Pic}(M) = \mathrm{Diff}(M) \ltimes H^2(M,\mathbb{Z}),$$

where the semi-direct product is with respect to the action of $\mathrm{Diff}(M)$ *on line bundles (or integral cohomology classes) by pullback.*

5. Morita equivalence of star products

We now address the issue of describing when two star products on a manifold M define Morita equivalent $\mathbb{C}[[\hbar]]$-algebras. The main observation in this section is that Morita equivalence can be described as orbits of an action on $\mathrm{Def}(M)$. Let us start by describing when two star products define isomorphic $\mathbb{C}[[\hbar]]$-algebras.

5.1. Isomorphic star products. Any equivalence T between star products $\star$ and $\star'$, in the sense of Section 3.1, is an algebra isomorphism (by definition, $T = \mathrm{Id} + O(\hbar)$, so it is automatically invertible as a formal power series). But not every isomorphism is an equivalence. In general, a $\mathbb{C}[[\hbar]]$-linear isomorphism between star products $\star$ and $\star'$ on M is of the form

$$T = \sum_{r=0}^{\infty} \hbar^r T_r,$$

where each $T_r : C^\infty(M) \to C^\infty(M)$ is a differential operator, such that (3.3) holds (c.f. [**15**], Prop. 14 and Prop. 29); note that this forces $T_0 : C^\infty(M) \to C^\infty(M)$ to be an isomorphism of commutative algebras (relative to the pointwise product), but not necessarily the identity. In particular, there is a diffeomorphism $\varphi : M \to M$ such that

$$T_0 = \varphi^*.$$

If we consider the natural action of the diffeomorphism group $\mathrm{Diff}(M)$ on star products: $\star \mapsto \star_\varphi$, where

$$f \star_\varphi g = (\varphi^{-1})^*(\varphi^* f \star \varphi^* g), \quad \varphi \in \mathrm{Diff}(M), \tag{5.1}$$

we see that it descends to an action of $\mathrm{Diff}(M)$ on $\mathrm{Def}(M)$,

$$\mathrm{Diff}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (\varphi, [\star]) \mapsto [\star_\varphi], \tag{5.2}$$

in such a way that two star products $\star$, $\star'$ define isomorphic $\mathbb{C}[[\hbar]]$-algebras if and only if their classes in $\mathrm{Def}(M)$ lie on the same $\mathrm{Diff}(M)$-orbit.

Remark 5.1. *Similarly, given a Poisson structure* π *on* M *and denoting by*

$$\mathrm{Diff}_\pi(M) \subseteq \mathrm{Diff}(M)$$

the group of Poisson automorphisms of (M, π)*, we see that the action* (5.2) *restricts to an action of* $\mathrm{Diff}_\pi(M)$ *on* $\mathrm{Def}(M, \pi)$ *whose orbits characterize isomorphic star products quantizing* π*.*

We will see that there is a larger group acting on $\mathrm{Def}(M)$ whose orbits characterize Morita equivalence.

5.2. An action of $\mathrm{Pic}(M)$. By Morita's characterization of invertible bimodules in Theorem 4.1, the first step in describing Morita equivalent star products is understanding, for a given star product $\star$ on M, the right modules over $(C^\infty(M)[[\hbar]], \star)$ that are finitely generated, projective, and full.

One obtains modules over $(C^\infty(M)[[\hbar]], \star)$ by starting with a classical finitely generated projective module over $C^\infty(M)$, defined by a vector bundle $E \to M$, and then performing a deformation-quantization type procedure: one searches for bilinear operators $R_r : \Gamma(E) \times C^\infty(M) \to \Gamma(E)$, $r = 1, 2, \ldots$, so that

$$s \bullet f := sf + \sum_{r=1}^{\infty} \hbar^r R_r(s, f), \quad s \in \Gamma(E),\ f \in C^\infty(M), \tag{5.3}$$

defines a right module structure on $\Gamma(E)[[\hbar]]$ over $(C^\infty(M)[[\hbar]], \star)$; i.e.,

$$(s \bullet f) \bullet g = s \bullet (f \star g).$$

One may show [**4**] that the deformation (5.3) is always unobstructed, for any choice of $\star$; moreover, the resulting module structure on $\Gamma(E)[[\hbar]]$ is unique, up to a natural notion of equivalence. Also, the module $(\Gamma(E)[[\hbar]], \bullet)$ over $(C^\infty(M)[[\hbar]], \star)$ is finitely generated, projective, and full, and any module with these properties arises in this way.

The endomorphism algebra $\mathrm{End}(\Gamma(E)[[\hbar]], \bullet)$ may be identified, as a $\mathbb{C}[[\hbar]]$-module, with $\Gamma(\mathrm{End}(E))[[\hbar]]$. As a result, it induces an associative product $\star'$ on $\Gamma(\mathrm{End}(E))[[\hbar]]$, deforming the (generally noncommutative) algebra $\Gamma(\mathrm{End}(E))$. For a line bundle $L \to M$, since $\Gamma(\mathrm{End}(L)) \cong C^\infty(M)$, it follows that $\star'$ defines a new star product on M. The equivalence class $[\star'] \in \mathrm{Def}(M)$ is well-defined, i.e., it is independent of the specific module deformation $\bullet$ or identification $(C^\infty(M)[[\hbar]], \star') \cong \mathrm{End}(\Gamma(L)[[\hbar]], \bullet)$, and it is completely determined by the isomorphism class of L in $\mathrm{Pic}(M)$. The construction of $\star'$ from $\star$ and L gives rise to a canonical action [**2**]

$$\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (L, [\star]) \mapsto \Phi_L([\star]). \tag{5.4}$$

Additionally, for any Poisson structure π, this action restricts to a well defined action of $\mathrm{Pic}(M)$ on $\mathrm{Def}(M, \pi)$.

5.3. Morita equivalence as orbits.

Let us consider the semi-direct product

$$\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M),$$

which is nothing but the Picard group of $C^\infty(M)$, see Example 4.2. By combining the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M)$ on $\mathrm{Def}(M)$, described in (5.2) and (5.4), one obtains a $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-action on $\mathrm{Def}(M)$, which leads to the following characterization of Morita equivalent star products, see [**2**]:

THEOREM 5.2. *Two star products $\star$ and $\star'$ on M are Morita equivalent if and only if $[\star]$, $[\star']$ lie in the same $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-orbit:*

$$[\star'] = \Phi_L([\star_\varphi])$$

Similarly (see Remark 5.1), two star products on M quantizing the same Poisson structure π are Morita equivalent if and only if the lie in the same orbit of $\mathrm{Diff}_\pi(M) \ltimes \mathrm{Pic}(M)$ on $\mathrm{Def}(M, \pi)$.

Our next step is to transfer the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M) = H^2(M, \mathbb{Z})$ to $\mathrm{FPois}(M)$ via $\mathcal{K}_*$ in (3.8); i.e, we will find explicit actions of $\mathrm{Diff}(M)$ and $H^2(M, \mathbb{Z})$ on $\mathrm{FPois}(M)$ making $\mathcal{K}_*$ equivariant with respect to $\mathrm{Diff}(M) \ltimes H^2(M, \mathbb{Z})$.

6. B-field action on formal Poisson structures

There is a natural way in which Poisson structures may be modified by closed 2-forms. In the context of formal Poisson structures, this leads to a natural action of the abelian group $H^2_{dR}(M,\mathbb{C})[[\hbar]]$ on $\mathrm{FPois}(M)$,

$$H^2_{dR}(M,\mathbb{C})[[\hbar]] \times \mathrm{FPois}(M) \to \mathrm{FPois}(M), \tag{6.1}$$

that we will refer to as the *B-field action*, to be discussed in this section.

6.1. B-field transformations of Poisson structures. A convenient way to describe how closed 2-forms may "act" on Poisson structures is to take a broader perspective on Poisson geometry, following [**9, 26**], see also [**14**]. The starting point is considering, for a manifold M, the direct sum $TM \oplus T^*M$. This bundle is naturally equipped with two additional structures: a symmetric, nondegenerate, fibrewise pairing, given for each $x \in M$ by

$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y), \quad X, Y \in T_xM,\ \alpha,\beta \in T^*_xM, \tag{6.2}$$

as well as a bilinear operation on $\Gamma(TM \oplus T^*M)$, known as the *Courant bracket*, given by

$$[\![(X,\alpha),(Y,\beta)]\!] := ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha), \tag{6.3}$$

for $X, Y \in \mathcal{X}^1(M)$ and $\alpha, \beta \in \Omega^1(M)$. The Courant bracket extends the usual Lie bracket of vector fields, but it is not a Lie bracket itself.

One may use (6.2) and (6.3) to obtain an alternative description of Poisson structures. Specifically, Poisson structures on M are in one-to-one correspondence with subbundles $L \subset TM \oplus T^*M$ satisfying the following conditions:

(a) $L \cap TM = \{0\}$,
(b) $L = L^\perp$ (i.e., L is self orthogonal with respect to the pairing (6.2)), and
(c) $\Gamma(L)$ is involutive with respect to the Courant bracket.

For a Poisson structure π, the bundle $L \subset TM \oplus T^*M$ corresponding to it is (see (3.5))

$$L = \mathrm{graph}(\pi^\sharp) = \{(\pi^\sharp(\alpha),\alpha) \mid \alpha \in T^*M\}.$$

Indeed, (a) means that L is the graph of a bundle map $\rho : T^*M \to TM$, while (b) says that $\rho^* = -\rho$, so that $\rho = \pi^\sharp$ for $\pi \in \mathcal{X}^2(M)$; finally, (c) accounts for the condition $[\pi,\pi] = 0$. In general, subbundles $L \subset TM \oplus T^*M$ satisfying only (b) and (c) are referred to as *Dirac structures* [**9**]; Poisson structures are particular cases also satisfying (a).

We will be interested in the group of bundle automorphisms of $TM \oplus T^*M$ which preserve the pairing (6.2) and the Courant bracket (6.3); we refer to such automorphisms as *Courant symmetries*. Any diffeomorphism $\varphi : M \to M$ naturally lifts to a Courant symmetry, through its natural lifts to TM and T^*M. Another type of Courant symmetry, known as *B-field (or gauge) transformation*, is defined by closed 2-forms [**26**]: any $B \in \Omega^2_{cl}(M)$ acts on $TM \oplus T^*M$ via

$$(X,\alpha) \overset{\tau_B}{\mapsto} (X, \alpha + i_X B). \tag{6.4}$$

The full group of Courant symmetries turns out to be exactly $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$.

For a Poisson structure π on M, the B-field transformation (6.4) takes $L = \mathrm{graph}(\pi^\sharp)$ to the subbundle

$$\tau_B(L) = \{(\pi^\sharp(\alpha), \alpha + i_{\pi^\sharp(\alpha)}B) \mid \alpha \in T^*M\} \subset TM \oplus T^*M,$$

and since τ_B preserves (6.2) and (6.3), $\tau_B(L)$ automatically satisfies (b) and (c) (i.e., it is a Dirac structure). It follows that $\tau_B(L)$ determines a new Poisson structure π^B, via $\tau_B(L) = \mathrm{graph}((\pi^B)^\sharp)$, if and only if $\tau_B(L) \cap TM = \{0\}$, which is equivalent to the condition that

$$\mathrm{Id} + B^\sharp \pi^\sharp : T^*M \to T^*M \ \text{ is invertible,} \tag{6.5}$$

where $B^\sharp : TM \to T^*M$, $B^\sharp(X) = i_X B$. In this case, the Poisson structure π^B is completely characterized by

$$(\pi^B)^\sharp = \pi^\sharp \circ (\mathrm{Id} + B^\sharp \pi^\sharp)^{-1}. \tag{6.6}$$

In conclusion, given a Poisson structure π and a closed 2-form B, if the compatibility condition (6.5) holds, then (6.6) defines a new Poisson structure π^B. A simple case is when π is nondegenerate, hence equivalent to a symplectic form ω; then condition (6.5) says that $\omega + B$ is nondegenerate, and π^B is the Poisson structure associated with it.

6.2. Formal Poisson structures and the B-field action. The whole discussion about B-field transformations carries over to formal Poisson structures – and even simplifies in this context. Given a formal Poisson structure

$$\pi_\hbar = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\mathcal{X}^2(M)[[\hbar]]$$

and any $B \in \Omega^2_{cl}(M)[[\hbar]]$, then $B^\sharp \pi^\sharp_\hbar = O(\hbar)$, where here $\pi^\sharp_\hbar$ and $B^\sharp$ are the associated (formal series of) bundle maps. Hence $(\mathrm{Id} + B^\sharp \pi^\sharp_\hbar)$ is automatically invertible as a formal power series (i.e., (6.5) is automatically satisfied),

$$(\mathrm{Id} + B^\sharp \pi^\sharp_\hbar)^{-1} = \sum_{n=0}^{\infty} (-1)^n (B^\sharp \pi^\sharp_\hbar)^n,$$

and the same formula as (6.6) defines an action of the abelian group $\Omega^2_{cl}(M)[[\hbar]]$ on formal Poisson structures: $\pi_\hbar \mapsto \pi^B_\hbar$, where

$$(\pi^B_\hbar)^\sharp = \pi^\sharp_\hbar \circ (\mathrm{Id} + B^\sharp \pi^\sharp_\hbar)^{-1}.$$

There are two key observations concerning this action: First, the B-field transformations of equivalent formal Poisson structures remain equivalent; second, the B-field transformation by an exact 2-form $B = dA$ does not change the equivalence class of a formal Poisson structure. This leads to the next result [**3**, Prop. 3.10]:

THEOREM 6.1. *The action of $\Omega^2_{cl}(M)[[\hbar]]$ on formal Poisson structures descends to an action*

$$H^2_{dR}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPois}(M) \to \mathrm{FPois}(M),\ [\pi_\hbar] \mapsto [\pi^B_\hbar] \tag{6.7}$$

This action is the identity to first order in $\hbar$:

$$\pi_\hbar = \hbar\pi_1 + O(\hbar) \implies \pi^B_\hbar = \hbar\pi_1 + O(\hbar).$$

So, for any Poisson structure $\pi \in \mathcal{X}^2(M)$, the action (6.7) restricts to

$$H^2_{dR}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPois}(M, \pi) \to \mathrm{FPois}(M, \pi).$$

When π is symplectic, so that we have the identification (3.10), this action is simply

$$[\omega_\hbar] \mapsto [\omega_\hbar] + [B], \tag{6.8}$$

for $[\omega_\hbar] \in \frac{1}{\hbar}[\omega] + H^2_{dR}(M, \mathbb{C})[[\hbar]]$ and $[B] \in H^2_{dR}(M, \mathbb{C})[[\hbar]]$.

7. Morita equivalent star products via Kontsevich's classes

Our final goal is to present the description of Morita equivalent star products in terms of their Kontsevich classes, i.e., by means of the correspondence

$$\mathcal{K}_* : \mathrm{FPois}(M) \xrightarrow{\sim} \mathrm{Def}(M)$$

of Thm. 3.3. By Thm. 5.2, Morita equivalent star products are characterized by lying on the same orbit of $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ on $\mathrm{Def}(M)$. In order to find the corresponding action on $\mathrm{FPois}(M)$, we treat the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ independently.

The group $\mathrm{Diff}(M)$ naturally acts on formal Poisson structures: for $\varphi \in \mathrm{Diff}(M)$,

$$\pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \pi_r \overset{\varphi}{\mapsto} \varphi_* \pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \varphi_* \pi_r,$$

and this action induces an action of $\mathrm{Diff}(M)$ on $\mathrm{FPois}(M)$, with respect to which $\mathcal{K}_*$ is equivariant [**11**]. This accounts for the classification of isomorphic star products in terms of their Kontsevich classes, which is (the easy) part of the classification up to Morita equivalence.

The less trivial part is due to the action (5.4) of $\mathrm{Pic}(M)$. In this respect, the main result asserts that the transformation Φ_L on $\mathrm{Def}(M)$ defined by a line bundle $L \in \mathrm{Pic}(M)$ corresponds to the B-field transformation on $\mathrm{FPois}(M)$ by a curvature form of L, see [**3**, Thm. 3.11]:

THEOREM 7.1. *The action $\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M)$ satisfies*

$$\Phi_L([\star]) = \mathcal{K}_*([\pi_\hbar^B]),$$

where B is a closed 2-form representing the cohomology class $2\pi i c_1(L)$, where $c_1(L)$ is the Chern class of $L \to M$.

A direct consequence is that flat line bundles (which are the torsion elements in $\mathrm{Pic}(M)$) act trivially on $\mathrm{Def}(M)$ under (5.4).

The theorem establishes a direct connection between the B-field action (6.7) on formal Poisson structures and algebraic Morita equivalence: If a closed 2-form $B \in \Omega^2_{cl}(M)$ is 2πi-integral, then B-field related formal Poisson structures quantize under $\mathcal{K}_*$ to Morita equivalent star products.

The following classification results for Morita equivalent star products in terms of their characteristic classes readily follow from Thm. 7.1:

- Two star products $\star$ and $\star'$, with Kontsevich classes $[\pi_\hbar]$ and $[\pi'_\hbar]$, are Morita equivalent if and only if there is a diffeomorphism $\varphi : M \to M$ and a closed 2-form B whose cohomology class is 2πi-integral, such that

$$[\pi'_\hbar] = [(\varphi_* \pi_\hbar)^B]. \tag{7.1}$$

- If $\star$ and $\star'$ quantize the same Poisson structure π, so that $[\pi_\hbar], [\pi'_\hbar] \in \mathrm{FPois}(M, \pi)$, then the result is analogous: $\star$ and $\star'$ are Morita equivalent if and only if (7.1) holds, but now $\varphi : M \to M$ is a Poisson automorphism.
- Assume that $\star$ and $\star'$ quantize the same nondegenerate Poisson structure, defined by a symplectic form ω. In this case, by (6.8), Theorem 7.1 recovers the characterization of Morita equivalent star products on symplectic manifolds obtained in [**5**], in terms of Fedosov-Deligne classes

(3.11): $\star$ and $\star'$ are Morita equivalent if and only if there is a symplectomorphism $\varphi : M \to M$ for which the difference $c(\star) - c(\star'_\varphi)$ is a 2πi-integral class in $H^2_{dR}(M, \mathbb{C})$ (viewed as a subspace of $H^2_{dR}(M, \mathbb{C})[[\hbar]] = H^2_{dR}(M, \mathbb{C}) + \hbar H^2_{dR}(M, \mathbb{C})[[\hbar]]$).

REMARK 7.2. *These notes treat star-product algebras simply as associative unital* $\mathbb{C}[[\hbar]]$*-algebras. But these algebras often carry additional structure: one may consider star products for which complex conjugation is an algebra involution (e.g. the Weyl star product, see* (2.13)*), and use the fact that* $\mathbb{R}[[\hbar]]$ *is an ordered ring to obtain suitable notions of positivity on these algebras (e.g. positive elements, positive linear functionals). One can develop refined notions of Morita equivalence for star products, parallel to strong Morita equivalence of* C^**-algebras, taking these additional properties into account, see e.g.* [**6**]*. An overview of these more elaborate aspects of Morita theory for star products can be found e.g. in* [**29**].

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Noncommutative Calculus and Operads

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1. Introduction

This expository paper is based on lecture courses that the author taught at the Hebrew University of Jerusalem in the year of 2009–2010 and at the Winter School on Noncommutative Geometry at Buenos Aires in July-August of 2010. It gives an overview of works on the topics of noncommutative calculus, operads and index theorems.

Noncommutative calculus is a theory that defines classical algebraic structures arising from the usual calculus on manifolds in terms of the algebra of functions on this manifold, in a way that is valid for any associative algebra, commutative or not. It turns out that noncommutative analogs of the basic spaces arising in calculus are well-known complexes from homological algebra. For example, the role of noncommutative multivector fields is played by the Hochschild cochain complex of the algebra; the the role of noncommutative forms is played by the Hochschild chain complex, and the role of the noncommutative de Rham complex by the periodic cyclic complex of the algebra. These complexes turn out to carry a very rich algebraic structure, similar to the one carried by their classical counterparts. Moreover, when the algebra in question is the algebra of functions, the general structures from noncommutative geometry are equivalent to the classical ones. These statements rely on the Kontsevich formality theorem [**72**] and its analogs and generalizations. We rely on the method of proof developed by Tamarkin in [**104**], [**105**]. The main tool in this method is the theory of operads [**86**].

A consequence of the Kontsevich formality theorem is the classification of all deformation quantizations [**5**] of a given manifold. Another consequence is the algebraic index theorem for deformation quantizations. This is a statement about a trace of a compactly supported difference of projections in the algebra of matrices over a deformed algebra. It turns out that all the data entering into this problem (namely, a deformed algebra, a trace on it, and projections in it) can be classified using formal Poisson structures on the manifold. The answer is an expression very similar to the right hand side of the Atiyah-Singer index theorem. For a deformation of a symplectic structure, all the results mentioned above were obtained by Fedosov

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[**44**]; they imply the Atiyah-Singer index theorem and its various generalizations [**9**].

The algebraic index theorem admits a generalization for deformation quantizations of complex analytic manifolds. In this new setting, a deformation quantization as an algebra is replaced by a deformation quantization as an algebroid stack, a trace by a Hochschild cocycle, and a difference of two projections by a perfect complex of (twisted) modules. The situation becomes much more mysterious than before, because both the classification of the data entering into the problem and the final answer depend on a Drinfeld associator [**36**]. The algebraic index theorem for deformation quantization of complex manifolds in its final form is due to Willwacher ([**118**], [**119**], and to appear).

The author is greatly indebted to the organizers and the members of the audience of his course and talks in Jerusalem for the wonderful stimulating atmosphere. He is especially thankful to Volodya Hinich and David Kazhdan with whom he had multiple discussions on the subject and its relations to the theory of infinity-categories, as well as to Ilan Barnea for a key argument on which Section 5 is based. He is grateful to Oren Ben Bassat, Emmanouil Farjoun, Jake Solomon, Ran Tesler and Amitai Zernik for very interesting and enjoyable discussions. It is my great pleasure to thank Willie Cortiñas and of the other co-organizers of the Winter School in Buenos Aires, as well as the audience of my lectures there.

2. Hochschild and cyclic homology of algebras

Let k denote a commutative algebra over a field of characteristic zero and let A be a flat k-algebra with unit, not necessarily commutative. Let $\overline{A} = A/k\cdot 1$. For $p \geq 0$, let $C_p(A) \overset{def}{=} A \otimes_k \overline{A}^{\otimes_k p}$. Define

$$
\begin{aligned}
b : C_p(A) &\to C_{p-1}(A) \\
a_0 \otimes \ldots \otimes a_p &\mapsto (-1)^p a_p a_0 \otimes \ldots \otimes a_{p-1} + \\
&\quad \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_p \ .
\end{aligned}
\tag{2.1}
$$

Then $b^2 = 0$ and one gets the complex $(C_\bullet, b)$, called *the standard Hochschild complex of A*. The homology of this complex is denoted by $H_\bullet(A, A)$, or by $HH_\bullet(A)$.

PROPOSITION 2.0.1. *The map*

$$
\begin{aligned}
B : C_p(A) &\to C_{p+1}(A) \\
a_0 \otimes \ldots \otimes a_p &\mapsto \sum_{i=0}^{p} (-1)^{pi} 1 \otimes a_i \otimes \ldots \otimes a_p \otimes a_0 \otimes \ldots \otimes a_{i-1}
\end{aligned}
\tag{2.2}
$$

satisfies $B^2 = 0$ and $bB + Bb = 0$ and therefore defines a map of complexes

$$B : C_\bullet(A) \to C_\bullet(A)[-1]$$

DEFINITION 2.0.2. *For $p \in \mathbb{Z}$ let*

$$
\begin{aligned}
CC_p^-(A) &= \prod_{\substack{i \geq p \\ i \equiv p \mod 2}} C_i(A) \\
CC_p^{\mathrm{per}}(A) &= \prod_{i \equiv p \mod 2} C_i(A)
\end{aligned}
$$

$$CC_p(A) = \bigoplus_{\substack{i \le p \\ i \equiv p \mod 2}} C_i(A)$$

Since $i \geq 0$, the third formula has a finite sum in the right hand side. The complex $(CC_\bullet^-(A), B+b)$ (respectively $(CC_\bullet^{\mathrm{per}}(A), B+b)$, respectively $(CC_\bullet(A), B+b)$) is called the *negative cyclic* (respectively *periodic cyclic*, respectively *cyclic*) complex of A. The homology of these complexes is denoted by $HC_\bullet^-(A)$ (respectively by $HC_\bullet^{\mathrm{per}}(A)$, respectively by $HC_\bullet(A)$).

There are inclusions of complexes

$$CC_\bullet^-(A)[-2] \hookrightarrow CC_\bullet^-(A) \hookrightarrow CC_\bullet^{per}(A) \tag{2.3}$$

and the short exact sequences

$$0 \to CC_\bullet^-(A)[-2] \to CC_\bullet^-(A) \to C_\bullet(A) \to 0 \tag{2.4}$$

$$0 \to C_\bullet(A) \to CC_\bullet(A) \xrightarrow{S} CC_\bullet(A)[2] \to 0 \tag{2.5}$$

To the double complex $CC_\bullet(A)$ one associates the spectral sequence

$$E^2_{pq} = H_{p-q}(A, A) \tag{2.6}$$

converging to $HC_{p+q}(A)$.

In what follows we will use the notation of Getzler and Jones ([**54**]). Let u denote a variable of degree -2.

Definition 2.0.3. *For any k-module M we denote by $M[u]$ M-valued polynomials in u, by $M[[u]]$ M-valued power series, and by $M((u))$ M-valued Laurent series in u.*

The negative and periodic cyclic complexes are described by the following formulas:

$$CC_\bullet^-(A) = (C_\bullet(A)[[u]], b + uB) \tag{2.7}$$

$$CC_\bullet^{\mathrm{per}}(A) = (C_\bullet(A)((u)), b + uB) \tag{2.8}$$

$$CC_\bullet(A) = (C_\bullet(A)((u))/uC_\bullet(A)[[u]], b + uB) \tag{2.9}$$

In this language, the map S is just multiplication by u.

Remark 2.0.4. For an algebra A without unit, let $\tilde{A} = A + k \cdot 1$ and put

$$CC_\bullet(A) = \mathrm{Ker}(CC_\bullet(\tilde{A}) \to CC_\bullet(k));$$

similarly for the negative and periodic cyclic complexes. If A is a unital algebra then these complexes are quasi-isomorphic to the ones defined above.

2.1. Homology of differential graded algebras. One can easily generalize all the above constructions to the case when A is a differential graded algebra (DGA) with the differential δ (i.e. A is a graded algebra and δ is a derivation of degree 1 such that $\delta^2 = 0$).

The action of δ extends to an action on Hochschild chains by the Leibniz rule:

$$\delta(a_0 \otimes \ldots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1}(a_0 \otimes \ldots \otimes \delta a_i \otimes ldots \otimes a_p)$$

The maps b and B are modified to include signs:

$$b(a_0 \otimes ldots \otimes a_p) = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k} (|a_i|+1)+1} a_0 \otimes ldots \otimes a_k a_{k+1} \otimes ldots \otimes a_p \tag{2.10}$$

$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes ldots \otimes a_{p-1}$$

$$B(a_0 \otimes ldots \otimes a_p) = \sum_{k=0}^{p} (-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)} 1 \otimes a_{k+1} \otimes ldots \otimes a_p \otimes \tag{2.11}$$

$$\otimes a_0 \otimes ldots \otimes a_k$$

The complex $C_\bullet(A)$ now becomes the total complex of the double complex with the differential $b + \delta$:

$$C_p(A) = \bigoplus_{j-i=p} (A \otimes \overline{A}^{\otimes j})^i$$

The negative and the periodic cyclic complexes are defined as before in terms of the new definition of $C_\bullet(A)$. All the results of this section extend to the differential graded case.

Remark 2.1.1. Note that the total complex consists of direct sums rather than direct products. This choice, as well as the choice of defining the periodic cyclic complex using Laurent series, is made so that a quasi-isomorphism of DG algebras would induce a quasi-isomorphism of corresponding complexes.

2.2. The Hochschild cochain complex. Let A be a graded algebra with unit over a commutative unital ring k of characteristic zero. A Hochschild d-cochain is a linear map $A^{\otimes d} \to A$. Put, for $d \geq 0$,

$$C^d(A) = C^d(A, A) = \mathrm{Hom}_k(\overline{A}^{\otimes d}, A) \tag{2.12}$$

where $\overline{A} = A/k \cdot 1$. Put

$$|D| = (\text{degree of the linear map } D) + d \tag{2.13}$$

Put for cochains D and E from $C^\bullet(A, A)$

$$(D \smile E)(a_1, \ldots, a_{d+e}) = (-1)^{|E|\sum_{i\le d}(|a_i|+1)} D(a_1, \ldots, a_d) \times \tag{2.14}$$

$$\times E(a_{d+1}, \ldots, a_{d+e}); \tag{2.15}$$

$$(D \circ E)(a_1, \ldots, a_{d+e-1}) = \sum_{j\ge 0} (-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)} \times \tag{2.16}$$

$$\times D(a_1, \ldots, a_j, E(a_{j+1}, \ldots, a_{j+e}), \ldots);$$

$$[D,\ E] = D \circ E - (-1)^{(|D|+1)(|E|+1)} E \circ D \tag{2.17}$$

These operations define the graded associative algebra $(C^\bullet(A, A)\ , \smile)$ and the graded Lie algebra $(C^{\bullet+1}(A, A), [\ ,\ ])$ (cf. [**19**]; [**50**]). Let

$$m(a_1, a_2) = (-1)^{|a_1|}\ a_1 a_2; \tag{2.18}$$

this is a 2-cochain of A (not in C^2). Put

$$\delta D = [m, D]; \tag{2.19}$$

$$(\delta D)(a_1, \ldots, a_{d+1}) = (-1)^{|a_1||D|+|D|+1} \times \tag{2.20}$$

$$\times a_1 D(a_2, \ldots, a_{d+1}) + \tag{2.21}$$

$$+\sum_{j=1}^{d}(-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j a_{j+1}, \ldots, a_{d+1})$$

$$+(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)} D(a_1, \ldots, a_d) a_{d+1}$$

One has

$$\delta^2 = 0; \quad \delta(D \smile E) = \delta D \smile E + (-1)^{|D|} D \smile \delta E \tag{2.22}$$

$$\delta[D, E] = [\delta D, E] + (-1)^{|D|+1}\, [D, \delta E] \tag{2.23}$$

($\delta^2 = 0$ follows from $[m, m] = 0$).

Thus $C^\bullet(A, A)$ becomes a complex; we will denote it also by $C^\bullet(A)$. The cohomology of this complex is $H^\bullet(A, A)$ or the Hochschild cohomology. We denote it also by $H^\bullet(A)$. The $\smile$ product induces the Yoneda product on $H^\bullet(A, A) = \mathrm{Ext}^\bullet_{A\otimes A^0}(A, A)$. The operation [,] is the Gerstenhaber bracket [**50**].

If $(A,\ \partial)$ is a differential graded algebra then one can define the differential ∂ acting on $C^\bullet(A)$ by:

$$\partial D \ = \ [\partial, D] \tag{2.24}$$

THEOREM 2.2.1. [**50**] *The cup product and the Gerstenhaber bracket induce a Gerstenhaber algebra structure on $H^\bullet(A)$ (cf. 3.6.2 for the definition of a Gerstenhaber algebra).*

For cochains D and D_i define a new Hochschild cochain by the following formula of Gerstenhaber ([**50**]) and Getzler ([**52**]):

$$D_0\{D_1, \ldots, D_m\}(a_1, \ldots, a_n) = \tag{2.25}$$

$$= \sum (-1)^{\sum_{k\le i_p}(|a_k|+1)(|D_p|+1)} D_0(a_1, \ldots, a_{i_1}, D_1(a_{i_1+1}, \ldots), \ldots,$$

$$D_m(a_{i_m+1}, \ldots), \ldots)$$

PROPOSITION 2.2.2. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q\le i_p}(|E_p|+1)(|F_q|+1)} \times$$

$$\times D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots, \}, \ldots, E_k\{F_{i_k+1}, \ldots, \}, \ldots, \}$$

The above proposition can be restated as follows. For a cochain D let $D^{(k)}$ be the following k-cochain of $C^\bullet(A)$:

$$D^{(k)}(D_1, \ldots, D_k) = D\{D_1, \ldots, D_k\}$$

PROPOSITION 2.2.3. *The map*

$$D \mapsto \sum_{k\ge 0} D^{(k)}$$

is a morphism of differential graded algebras

$$C^\bullet(A) \to C^\bullet(C^\bullet(A))$$

2.3. Products on Hochschild and cyclic complexes. Unless otherwise specified, the reference for this subsection is [**85**].

2.3.1. Product and coproduct; the Künneth exact sequence. For an algebra A define the shuffle product

$$\mathrm{sh} : C_p(A) \otimes C_q(A) \to C_{p+q}(A) \tag{2.26}$$

as follows.

$$(a_0 \otimes \ldots \otimes a_p) \otimes (c_0 \otimes ldots \otimes c_q) = a_0 c_0 \otimes \mathrm{sh}_{pq}(a_1, ldots, a_p,\, c_1, ldots, c_q) \tag{2.27}$$

where

$$\mathrm{sh}_{pq}(x_1, ldots, x_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma) x_{\sigma^{-1}1} \otimes ldots \otimes x_{\sigma^{-1}(p+q)} \tag{2.28}$$

and

$$\mathrm{Sh}(p,q) = \{\sigma \in \Sigma_{p+q} \,|\sigma 1 < \ldots < \sigma p;\ \sigma(p+1) < ldots < \sigma(p+q)\}$$

In the graded case, $\mathrm{sgn}(\sigma)$ gets replaced by the sign computed by the following rule: in all transpositions, the parity of a_i is equal to $|a_i| + 1$ if $i > 0$, and similarly for c_i. A transposition contributes a product of parities.

The shuffle product is not a morphism of complexes unless A is commutative. It defines, however, an exterior product as shown in the following theorem. For two unital algebras A and C, let i_A, i_C be the embeddings $a \mapsto a \otimes 1$, resp. $c \mapsto 1 \otimes c$ of A, resp. C, to $A \otimes C$. We will use the same notation for the embeddings that i_A, i_C induce on all the chain complexes considered by us.

THEOREM 2.3.1. *For two unital algebras A and C the composition*

$$C_p(A) \otimes C_q(C) \xrightarrow{i_A \otimes i_C} C_p(A \otimes C) \otimes C_q(A \otimes C) \xrightarrow{\mathrm{sh}} C_{p+q}(A \otimes C)$$

defines a quasi-isomorphism

$$\mathrm{sh} : C_\bullet(A) \otimes C_\bullet(C) \to C_\bullet(A \otimes C)$$

To extend this theorem to cyclic complexes, define

$$\mathrm{sh}' : C_p(A) \otimes C_q(A) \to C_{p+q+2}(A) \tag{2.29}$$

as follows.

$$(a_0 \otimes \ldots \otimes a_p) \otimes (c_0 \otimes \ldots \otimes c_q) \mapsto 1 \otimes \mathrm{sh}'_{p+1,\, q+1}(a_0, \ldots, a_p,\, c_0, \ldots, c_q) \tag{2.30}$$

where

$$\mathrm{sh}'_{p+1,q+1}(x_0, \ldots, x_{p+q+1}) = \sum_{\sigma \in \mathrm{Sh}'(p+1,q+1)} \mathrm{sgn}(\sigma) x_{\sigma^{-1}0} \otimes \ldots \otimes x_{\sigma^{-1}(p+q+1)} \tag{2.31}$$

and $\mathrm{Sh}'(p+1, q+1)$ is the set of all permutations $\sigma \in \Sigma_{p+q+2}$ such that $\sigma 0 < \ldots < \sigma p$, $\sigma(p+1) < \ldots < \sigma(p+q+1)$, and $\sigma 0 < \sigma(p+1)$.

Now define (2.29) to be the composition

$$C_p(A) \otimes C_q(C) \xrightarrow{i_A \otimes i_C} C_p(A \otimes C) \otimes C_q(A \otimes C) \xrightarrow{\mathrm{sh}'} C_{p+q+2}(A \otimes C)$$

In the graded case, the sign rule is as follows: any a_i has parity $|a_i| + 1$, and similarly for c_i.

THEOREM 2.3.2. *The map $\mathrm{sh} + u\mathrm{sh}'$ defines a $k[[u]]$-linear, (u)-adically continuous quasi-isomorphism*

$$(C_\bullet(A) \otimes C_\bullet(C))[[u]] \to CC^-_\bullet(A \otimes C)$$

as well as

$$(C_\bullet(A) \otimes C_\bullet(C))((u)) \to CC_\bullet^{\mathrm{per}}(A \otimes C)$$
$$(C_\bullet(A) \otimes C_\bullet(C))((u))/u(C_\bullet(A) \otimes C_\bullet(C))[[u]] \to CC_\bullet(A \otimes C)$$

(differentials on the left hand sides are equal to $b \otimes 1 + 1 \otimes b + u(B \otimes 1 + 1 \otimes B)$).

Note that the left hand side of the last formula maps to the tensor product $CC_\bullet(A) \otimes CC_\bullet(C) : \Delta(u^{-p} c \otimes c') = (u^{-1} \otimes 1 + 1 \otimes u^{-1})^p c \otimes c'$. One checks that this map is an embedding whose cokernel is the kernel of the map $u \otimes 1 - 1 \otimes u$, or $S \otimes 1 - 1 \otimes S$ where S is as in (2.5). From this we get

THEOREM 2.3.3. *There is a long exact sequence*

$$\to HC_n(A \otimes C) \xrightarrow{\Delta} \bigoplus_{p+q=n} HC_p(A) \otimes HC_q(C) \xrightarrow{S \otimes 1 - 1 \otimes S}$$
$$\bigoplus_{p+q=n-2} HC_p(A) \otimes HC_q(C) \xrightarrow{\times} HC_{n-1}(A \otimes C) \xrightarrow{\Delta}$$

2.4. Pairings between chains and cochains. Let us start with a motivation for what follows. We will see below that, when the ring of functions on a manifold is replaced by an arbitrary algebra, then Hochschild chains play the role of differential forms (with the differential B replacing the de Rham differential) and Hochschild cochains play the role of multivector fields. We are looking for an analog of pairings that are defined in the classical context, namely the contraction of a form by a multivector field and the Lie derivative. In classical geometry, those pairings satisfy various algebraic relations that we try to reproduce in general. We will show that these relations are true up to homotopy; a much more complicated question whether they are true up to all higher homotopies is postponed until section 8. For a graded algebra A, for $D \in C^d(A, A)$, define

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D| \sum_{i \le d}(|a_i|+1)} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n \tag{2.32}$$

PROPOSITION 2.4.1.

$$[b, i_D] = i_{\delta D}$$
$$i_D i_E = (-1)^{|D||E|} i_{E \smile D}$$

Now, put

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \tag{2.33}$$
$$\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k$$

(The second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D). The signs are given by

$$\epsilon_k = (|D| + 1) \sum_{i=0}^{k} (|a_i| + 1)$$

and

$$\eta_k = |D| + 1 + \sum_{i \le k} (|a_i| + 1) \sum_{i \ge k} (|a_i| + 1)$$

PROPOSITION 2.4.2.

$$[L_D, L_E] = L_{[D,E]}$$

$$[b, L_D] + L_{\delta D} = 0$$

$$[L_D, B] = 0$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j\geq 0;\ k\geq j+d} \epsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes \tag{2.34}$$

$$D(a_{j+1}, \ldots, a_{j+d}) \otimes \ldots \otimes a_k$$

(The sum is taken over all cyclic permutations; a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = |D|(|a_0| + \sum_{i=1}^{n}(|a_i|+1)) + (|D|+1)\sum_{j+1}^{k}(|a_i|+1) + \sum_{i\leq k}(|a_i|+1)\sum_{i\geq k}(|a_i|+1)$$

PROPOSITION 2.4.3. *(*[**96**]*)*

$$[b + uB, i_D + uS_D] - i_{\delta D} - uS_{\delta D} = L_D$$

PROPOSITION 2.4.4. *(*[**26**]*) There exists a linear transformation $T(D,E)$ of the Hochschild chain complex, bilinear in $D,\ E \in C^\bullet(A,A)$, such that*

$$[b + uB, T(D,E)] - T(\delta D, E) - (-1)^{|D|}T(D, \delta E) =$$

$$= [L_D, i_E + uS_E] - (-1)^{|D|+1}(i_{[D,E]} + uS_{[D,E]})$$

2.5. Hochschild and cyclic complexes of A_∞ algebras. They are defined exactly as for DG algebras, the chain differential b being replaced by L_m and the cochain differential δ by $[m, ?]$ where m is the Hochschild cochain from the definition of an A_∞ algebra.

2.6. Rigidity of periodic cyclic homology. The following is the Goodwillie rigidity theorem [**59**]. A proof using operations on Hochschild and cyclic complexes is given in [**90**]. Let A be an associative algebra over a ring k of characteristic zero. Let I be a nilpotent two-sided ideal of A. Denote $A_0 = A/I$.

THEOREM 2.6.1. *(Goodwillie) The natural map $CC^{\mathrm{per}}_\bullet(A) \to CC^{\mathrm{per}}_\bullet(A/I)$ is a quasi-isomorphism.*

2.7. Smooth functions. For a smooth manifold M one can compute the Hochschild and cyclic homology of the algebra $C^\infty(M)$ where the tensor product in the definition of the Hochschild complex is one of the following three:

$$C^\infty(M)^{\otimes n} = C^\infty(M^n); \tag{2.35}$$

$$C^\infty(M)^{\otimes n} = \mathrm{germs}_\Delta\, C^\infty(M^n); \tag{2.36}$$

$$C^\infty(M)^{\otimes n} = \mathrm{jets}_\Delta\, C^\infty(M^n) \tag{2.37}$$

where Δ is the diagonal.

THEOREM 2.7.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes

$$C_\bullet(C^\infty(M)) \to (\Omega^\bullet(M), 0)$$

and a $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$CC_\bullet^-(C^\infty(M)) \to (\Omega^\bullet(M)[[u]], ud)$$

Localizing with respect to u, we also get quasi-isomorphisms

$$CC_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]]/u\Omega^\bullet(M)[[u]], ud)$$

$$CC_\bullet^{\mathrm{per}}(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]], ud)$$

This theorem, for the tensor products (2.36, 2.37), is due essentially to Hochschild, Kostant and Rosenberg (the Hochschild case) and to Connes (the cyclic cases). For the tensor product (2.35), see [**110**].

2.7.1. Holomorphic functions. Let M be a complex manifold with the structure sheaf $\mathcal{O}_M$ and the sheaf of holomorphic forms $\Omega_M^\bullet$. If one uses one of the following definitions of the tensor product, then $C_\bullet(\mathcal{O}_M)$, etc. are complexes of sheaves:

$$\mathcal{O}_M^{\otimes n} = \mathrm{germs}_\Delta\, \mathcal{O}_{M^n}; \tag{2.38}$$

$$\mathcal{O}_M^{\otimes n} = \mathrm{jets}_\Delta\, \mathcal{O}_{M^n} \tag{2.39}$$

where Δ is the diagonal.

THEOREM 2.7.2. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes of sheaves

$$C_\bullet(\mathcal{O}_M) \to (\Omega_M^\bullet, 0)$$

and a $\mathbb{C}[[u]]$*-linear,* (u)*-adically quasi-isomorphism of complexes of sheaves*

$$CC_\bullet^-(\mathcal{O}_M) \to (\Omega_M^\bullet[[u]], ud)$$

Similarly for the complexes $CC_\bullet$ and CC^{per}.

3. Operads

3.1. Definition and basic properties.

DEFINITION 3.1.1. *An operad* $\mathcal{P}$ *in a symmetric monoidal category with direct sums and products* $\mathcal{C}$ *is:*

a) a collection of objects $\mathcal{P}(n)$*,* $n \geq 1$*, with an action of the symmetric group* Σ_n *on* $\mathcal{P}(n)$ *for every n;*

b) morphisms

$$\mathrm{op}_{n_1,\ldots,n_k} : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \ldots \otimes \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \ldots + n_k)$$

such that:

(i)

$$\bigoplus_{\sigma\in\Sigma_k}\mathrm{op}_{n_{\sigma(1)},\ldots,n_{\sigma(k)}}:\bigoplus_{\sigma\in\Sigma_k}\mathcal{P}(k)\otimes\mathcal{P}(n_{\sigma(1)})\otimes\ldots otimescP(n_{\sigma(k)})\to\mathcal{P}(n_1+\ldots+n_k)$$

is invariant under the action of the cross product $\Sigma_k \ltimes (\Sigma_{n_1}\times\ldots\times\Sigma_{n_k})$*;*

(ii) the diagram

$$\begin{array}{ccc}\mathcal{P}(k)\otimes\bigotimes_i\mathcal{P}(l_i)\otimes\bigotimes_{i,j}\mathcal{P}(m_{i,j}) & \longrightarrow & \mathcal{P}(k)\otimes\bigotimes_i\mathcal{P}(\sum_j m_{i,j})\\ \downarrow & & \downarrow\\ \mathcal{P}(\sum_i l_i)\otimes\bigotimes_{i,j}\mathcal{P}(m_{i,j}) & \longrightarrow & \mathcal{P}(\sum_{i,j} m_{i,j})\end{array}$$

is commutative.

Here is an equivalent definition: an operad is an object $\mathcal{P}(I)$ for any nonempty finite set I, functorial with respect to bijections of finite sets, together with a morphism

$$\mathrm{op}_f:\mathcal{P}(f)\to\mathcal{P}(I)$$

for every surjective map $f: I\to J$, where we put

$$\mathcal{P}(f)=\mathcal{P}(J)\otimes\bigotimes_{j\in J}\mathcal{P}(f^{-1}(\{j\});$$

for every pair of surjections $I\xrightarrow{g}J\xrightarrow{f}K$, and any element k of K, set

$$g_k=g|(fg)^{-1}(\{k\}):(fg)^{-1}(\{k\})\to g^{-1}(\{k\}).$$

We require the diagram

$$\begin{array}{ccc}\mathcal{P}(K)\otimes\bigotimes_{k\in K}\mathcal{P}(g_k) & \longrightarrow & \mathcal{P}(fg)\\ \downarrow & & \downarrow\\ \mathcal{P}(g) & \longrightarrow & \mathcal{P}(I)\end{array}\tag{3.1}$$

to be commutative.

It is easy to see that the two definitions are equivalent. Indeed, starting from Definition 3.1.1, put

$$\mathcal{P}(I)=\bigoplus_{\phi:\{1,\ldots,k\}\xrightarrow{\sim}I}\mathcal{P}(k)/\sim$$

where $(\psi,p)\sim(\phi,\phi\psi^{-1}p)$. In the opposite direction, define $\mathcal{P}(k)=\mathcal{P}(\{1,\ldots,k\})$.

An element e of $\mathcal{P}(1)$ is a unit of $\mathcal{P}$ if $\mathrm{op}_1(p,e)=p$ for all $p\in\mathcal{P}(1)$, $\mathrm{op}_n(e,p)=p$ for all $p\in\mathcal{P}(n)$ for the operation $\mathrm{op}_n:\mathcal{P}(1)\otimes\mathcal{P}(n)\to\mathcal{P}(n)$. (This definition works for categories such as spaces, complexes, etc.; in general, instead of an object e, one should talk about a morphism from the object $\mathbf{1}$ to $\mathcal{P}(1)$). An operad is unital if it has a unit. For a unital operad $\mathcal{P}$, and for every map, surjective or not, morphisms

$$\mathrm{op}_f:\mathcal{P}(f)\to\mathcal{P}(\widetilde{I}),\ \widetilde{I}_f=I\coprod(J-f(I)),\tag{3.2}$$

can be defined by mapping $\mathbf{1}$ to $\mathcal{P}$ using the unit, and then constructing the operation $\mathrm{op}_{\tilde{f}}$, $\overline{f}(i)=f(i)$ for $i\in I$, $\overline{f}(j)=j$ for $j\in J$. In particular, taking f to be a map whose image consists of one point, we get morphisms $\circ_i:\mathcal{P}(k)\otimes\mathcal{P}(n)\to\mathcal{P}(n+k-1)$ for $1\le k\le n$.

REMARK 3.1.2. We can define an operad $\mathcal{P}$ as a collection $\mathcal{P}(n)$ with actions of Σ_n and with products op_f as in (3.2) for any map $f : I \to J$, surjective or not, subject to the condition of invariance under Σ_n associative in the following sense. For maps $I \xrightarrow{f} J \xrightarrow{f} K$, define $\tilde{g} : I \to \widetilde{J}_f$ as the composition $I \xrightarrow{g} J \to \widetilde{J}_f$, and $\widetilde{fg} : \widetilde{I}_f \to K$ as fg on I and f on $J - f(I)$. Observe that

$$\mathcal{P}(K) \otimes \bigotimes_{k\in K} \mathcal{P}(f^{-1}(\{k\})) \otimes_{j\in J} \mathcal{P}(g^{-1}(\{j\})) \xrightarrow{\sim} \mathcal{P}(K) \otimes \bigotimes_{k\in K} \mathcal{P}(g_k);$$

$$\mathcal{P}(\widetilde{J}_f) \otimes \bigotimes_{j\in J} \mathcal{P}(g^{-1}(\{j\})) \xrightarrow{\sim} \mathcal{P}(\widetilde{g});$$

$$\mathcal{P}(K) \otimes \bigotimes_{k\in K} \mathcal{P}(\widetilde{f^{-1}(\{k\})}_{g_k}) \xrightarrow{\sim} \mathcal{P}(\widetilde{fg});$$

we get the diagram

$$\begin{array}{ccc} \mathcal{P}(K) \otimes \bigotimes_{k\in K} \mathcal{P}(g_k) & \longrightarrow & \mathcal{P}(\widetilde{fg}) \\ \downarrow & & \downarrow \\ \mathcal{P}(\widetilde{g}) & \longrightarrow & \mathcal{P}(\widetilde{I}_{fg}) \end{array} \tag{3.3}$$

that is required to be commutative. We can take this for the definition of an operad. Any unital operad is an example, but there are others which are not exactly unital.

EXAMPLE 3.1.3. For an object A, put $\mathrm{End}_A(n) = \mathrm{Hom}(A^{\otimes n}, A)$. The action of Σ_n and the operations op are the obvious ones. This is the operad of endomorphisms of A.

A morphism of operads $\mathcal{P} \to \mathcal{Q}$ is a collection of morphisms $\mathcal{P}(n) \to \mathcal{Q}(n)$ that agree with the action of Σ_n and with the operations $\mathrm{op}_{n_1,\ldots,n_k}$. A morphism of unital operads is a morphism that sends the unit of $\mathcal{P}$ to the unit of $\mathcal{Q}$.

3.1.1. Algebras over operads. An algebra over an operad $\mathcal{P}$ is an object A with a morphism $\mathcal{P} \to \mathrm{End}_A$. In other words, an algebra over $\mathcal{P}$ is an object A together with Σ_n-invariant morphisms

$$\mathcal{P}(n) \otimes A^{\otimes n} \to A$$

such that the diagram

$$\begin{array}{ccc} \mathcal{P}(k) \otimes \bigotimes_{i=1}^k \mathcal{P}(n_i) \otimes A^{\otimes \sum_{i=1}^k n_i} & \longrightarrow & \mathcal{P}(\sum_{i=1}^k n_i) \otimes A^{\otimes \sum_{i=1}^k n_i} \\ \downarrow & & \downarrow \\ \mathcal{P}(k) \otimes A^{\otimes k} & \longrightarrow & A \end{array}$$

is commutative. For an algebra over a unital operad $\mathcal{P}$, one assumes in addition that the composition $A \xrightarrow{\sim} \mathbf{1} \otimes A \to \mathcal{P}(1) \otimes A \to A$ is the identity.

A free algebra over $\mathcal{P}$ generated by V is

$$\mathrm{Free}_{\mathcal{P}}(V) = \bigoplus_n \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$$

The action of $\mathcal{P}$ combines the operadic products on $\mathcal{P}$ and the free (tensor) product on $V^{\otimes \bullet}$. The free algebra satisfies the usual universal property: For any $\mathcal{P}$-algebra A, a morphism of objects $V \to A$ extends to a unique morphism of $\mathcal{P}$-algebras $\mathrm{Free}_{\mathcal{P}}(V) \to A$.

3.1.2. Colored operads. A colored operad is a set X (whose elements are called colors), an object $\mathcal{P}(x_1, \ldots, x_n; y)$ for every finite subset $\{x_1, \ldots, x_n\}$ and every element y of X, an action of $\mathrm{Aut}(\{x_1, \ldots, x_n\})$ on $\mathcal{P}(x_1, \ldots, x_n; y)$, and morphisms

$$\mathrm{op} : \mathcal{P}(y_1, \ldots, y_k; z) \otimes \bigotimes_{i=1}^{k} \mathcal{P}(\{x_{ij}\}_{1\leq j\leq n_i}; y_i) \to \mathcal{P}(\{x_{ij}\}_{1\leq i\leq k\ 1\leq j\leq n_i}; z),$$

subject to the axioms of invariance and associativity generalizing the ones in Definition 3.1.1. An algebra over a colored operad $\mathcal{P}$ is a collection of objects A_x, $x \in X$, together with operations

$$\mathcal{P}(x_1, \ldots, x_n; y) \otimes A_{x_1} \otimes \ldots \otimes A_{x_n} \to A_y,$$

subject to axioms of invariance and associativity.

3.1.3. Topological operads. A topological operad is an operad in the category of topological spaces where $\otimes$ stands for the Cartesian product. If $\mathcal{P}$ is a topological operad then $C_{-\bullet}(\mathcal{P})$ is an operad in the category of complexes. (We use the minus sign to keep all our complexes cohomological, i.e. with differential of degree $+1$). Its nth term is the singular complex of the space $\mathcal{P}(n)$.

3.2. DG operads. A DG operad is an operad in the category of complexes. A DG operad for which $\mathcal{P}(n) = 0$ for $n \neq 1$ is the same as an associative DG algebra.

3.3. Cofibrant DG operads and algebras. A free DG operad generated by a collection of complexes $V(n)$ with an action of $\Sigma(n)$ is defined as follows. Let $\mathrm{FreeOp}(V)(n)$ be the direct sum over isomorphism classes of rooted trees T whose external vertices are labeled by indexes $1, \ldots, n$:

$$\mathrm{FreeOp}(V)(n) = \bigoplus_{T} \bigotimes_{\text{Internal vertices } v \text{ of } T} V(\{\text{edges outgoing from } v\})$$

The action of the symmetric group relabels the external vertices; the operadic products graft the root of the tree corresponding to the argument in $\mathrm{FreeOp}(V)(n_i)$ to the vertex labeled by the index i of the tree corresponding to the factor in $\mathrm{FreeOp}(V)(k)$. A free operad has the usual universal property: for a DG operad $\mathcal{P}$, a morphism of collections of Σ_n modules $V(n) \to \mathcal{P}(n)$ extends to a unique morphism of operads $\mathrm{FreeOp}(V) \to \mathcal{P}$.

3.3.1. Semifree operads and algebras. An algebra over a DG operad $\mathcal{P}$ is semifree if:

(i) its underlying graded k-module is a free algebra generated by a graded k-module V over the underlying graded operad of $\mathcal{P}$;

(ii) there is a filtration on $V : 0 = V_0 \subset V_1 \subset \ldots,\ V = \cup_n V_n$, such that the differential sends V_n to the suboperad generated by $V_k,\ k < n$.

One defines a semifree DG operad exactly in the same way, denoting by V a collection of Σ_n-modules.

A DG operad R (resp. an algebra R over a DG operad $\mathcal{P}$) is cofibrant if it is a retract of a semifree DG operad (resp. algebra), i.e. if there is a semifree Q and maps $R \xrightarrow{i} Q \xrightarrow{j} R$ such that $ji = \mathrm{id}_R$.

We say that a morphism of DG operads (resp. of algebras over a DG operad) is a fibration if it is surjective. We say that a morphism is a weak equivalence if it is a quasi-isomorphism. It is easy to see that the above definition of a cofibrant object is equivalent to the usual one: for every morphism $p : P \to Q$ that is a fibration and a weak equivalence, and for every $f : R \to Q$, there is a morphism $\tilde{f} : R \to P$ such that $p\tilde{f} = f$.

3.3.2. Cofibrant resolutions. A cofibrant resolution of a DG operad $\mathcal{P}$ is a cofibrant DG operad $\mathcal{R}$ together with a surjective quasi-isomorphism of DG operads $\mathcal{R} \to \mathcal{P}$. Every DG operad has a cofibrant resolution. For two such resolutions $\mathcal{R}_1$ and $\mathcal{R}_2$, there is a morphism $\mathcal{R}_1 \to \mathcal{R}_2$ over $\mathcal{P}$. Any two such morphisms are homotopic in the following sense. Let $\Omega^\bullet([0,1])$ be the DG algebra $k[t, dt]$ with the differential sending t to dt. Let $\mathrm{ev}_a : \Omega^\bullet([0,1]) \to k$ be the morphism of algebras sending t to a and dt to zero. Two morphisms $f_0, f_1 : \mathcal{R}_1 \to \mathcal{R}_2$ are homotopic if there is a morphism $f : \mathcal{R}_1 \to \mathcal{R}_2 \otimes \Omega^\bullet([0,1])$ such that $\mathrm{id}_{\mathcal{R}_2} \otimes \mathrm{ev}_a = f_a$ for $f = 0, 1$.

3.4. Bar and cobar constructions. The references for this subsection are [**56**] for the case of operads and [**54**] for the case of DG operads.

3.4.1. Cooperads and coalgebras. The definition a cooperad and a coalgebra over it is dual to that of an operad and an algebra over it. In particular, a cooperad is a collection of objects $\mathcal{B}(n)$ with actions of Σ_n, together with morphisms

$$\mathcal{B}(n_1 + \ldots + n_k) \to \mathcal{B}(k) \otimes \mathcal{B}(n_1) \otimes \ldots \otimes \mathcal{B}(n_k),$$

and a coalgebra C over $\mathcal{B}$ is an object C together with morphisms

$$C \to \mathcal{B}(n) \otimes C^{\otimes n},$$

subject to the conditions of Σ_n-invariance and coassociativity. A cofree coalgebra over $\mathcal{B}$ (co)generated by a complex W is defined as

$$\mathrm{Cofree}_{\mathcal{B}}(W) = \prod_{n \geq 1} (\mathcal{B}(n) \otimes W^{\otimes n})^{\Sigma_n};$$

a cofree cooperad (co)generated by a collection of Σ_n-modules $W = \{W(n)\}$ is by definition

$$\mathrm{CofreeCoop}(W)(n) = \prod_T \bigotimes_{\text{Interior vertices } v \text{ of } T} W(\{\text{edges outgoing from } v\})$$

The cooperadic coproducts are induced by cutting a tree in all possible ways into a subtree containing the root and k subtrees $T_1, \ldots, T_k$, such that the external vertices of T_i are exactly the external vertices of T labeled by $n_1 + \ldots + n_{i-1} + 1, \ldots, n_1 + \ldots + n_i$. The coaction of $\mathcal{B}$ on the cofree coalgebra is a combination of the cooperadic coproducts on $\mathcal{B}$ and the cofree coproduct on the tensor coalgebra $W^{\otimes \bullet}$.

3.4.2. The bar construction. Let $\mathcal{P}$ be a DG operad as in Remark 3.1.2. The bar construction of $\mathcal{P}$ is the cofree DG cooperad $\mathrm{CofreeCoop}(\mathcal{P}[-1])$ with the differential defined by $d = d_1 + d_2$ where, for a rooted tree T,

$$d_1(\otimes_{\text{Internal vertices } v \text{ of } T}(p(v))) = \sum \pm \otimes_{v' \neq v} p(v') \otimes d_{\mathcal{P}} p(v),$$

$p(v) \in \mathcal{P}(\{\text{edges outgoing from } v\})[1]$, where $d_{\mathcal{P}}$ is the differential on $\mathcal{P}[1]$;

$$d_2(\otimes_v(p(v))) = \sum_{\text{Internal edges } e \text{ of } T} \pm \mathbf{c}(e)(\otimes_v(p(v))).$$

Here $\mathbf{c}(e)$ is the operator of contracting the edge e that acts as follows. Let v_1 and v_2 be vertices adjacent to e, v_1 closer to the root than v_2. Let T_e be the tree obtained from T by contracting the edge e. Consider the operation

$$\mathcal{P}(\{\text{ edges of } T \text{ outgoing from } v_1\}) \otimes \mathcal{P}(\{\text{ edges of } T \text{ outgoing from } v_2\})$$

$$\stackrel{\mathrm{op}_{f_e}}{\to} \mathcal{P}(\{\text{edges of } T_e \text{ outgoing from } v_1\})$$

corresponding to the map

$$f_e : \{\text{ edges of } T \text{ outgoing from } v_2\} \to \{\text{edges of } T \text{ outgoing from } v_1\}$$

sending all edges to e. The operator $\mathbf{c}(e)$ replaces T by T_e and the tensor factor $p(v_1) \otimes p(v_2)$ by its image under op_{f_e}. The signs both in d_1 and d_2 are computed according to the following rule: start from the root of T and advance to the vertex, resp. to the edge. Passage through every factor $p(v)$ at a vertex v introduces the factor $(-1)^{|p(v|}$ (the degree in $\mathcal{P}[1]$).

It is easy to see that this differential defines a DG cooperad structure on $\mathrm{CofreeCoop}(\mathcal{P}[-1])$. We call this DG cooperad the bar construction of $\mathcal{P}$ and denote it by $\mathrm{Bar}(\mathcal{P})$.

The dual definition starts with a DG cooperad $\mathcal{B}$ and produces the DG operad $\mathrm{Cobar}(\mathcal{B})$.

LEMMA 3.4.1. *Let $V = \{V(n)\}$ be a collection of Σ_n-modules. The embedding of V into* $\mathrm{BarFreeOp}(V)$ *that sends an element of $V(n)$ into itself attached to a corolla with n external vertices is a quasi-isomorphism of complexes.*

Let $\mathcal{P}$ be a DG operad as in Remark 3.1.2. Consider the map $\mathrm{CobarBar}(\mathcal{P}) \to \mathcal{P}$ defined as follows. A free generator which is an element of $\mathrm{CofreeCoop}(\mathcal{P}[1])[-1]$ corresponding to a tree T is sent to zero unless T is a corolla, in which case it is sent to the corresponding element of $\mathcal{P}(n)$.

PROPOSITION 3.4.2. *The above map* $\mathrm{CobarBar}(\mathcal{P}) \to \mathcal{P}$ *is a surjective quasi-isomorphism of DG operads.*

The DG operad $\mathrm{CobarBar}(\mathcal{P})$ is the standard cofibrant resolution of $\mathcal{P}$.

3.5. Koszul operads. The reference for this subsection is [**56**]. We give a very brief sketch of the main definitions and results. Let $V(2)$ be a k-module with an action of Σ_2. A quadratic operad generated by $V(2)$ is a quotient of the free operad $\mathrm{FreeOp}(\{V(2)\})$ by the ideal generated by a subspace R of $(\mathrm{FreeOp}(\{V(2)\}))(3)$.

For a k-module X, let $X^* = \mathrm{Hom}_k(X, k)$. Let $V(2)$ and S be free k-modules of finite rank. The Koszul dual operad to a quadratic operad $\mathcal{P}$ generated by $V(2)$ with relations R is the quadratic operad $\mathcal{P}^\vee$ generated by $V(2)[1]^*$ subject to the orthogonal complement $R^\perp$ to R.

By definition, $(\mathcal{P}^\vee)^\vee = \mathcal{P}$. There is a natural morphism of operads $\mathcal{P}^\vee \to \mathrm{Bar}(\mathcal{P})^*$. The quadratic operad $\mathcal{P}$ is *Koszul* if this map is a quasi-isomorphism.

A quadratic operad $\mathcal{P}$ is Koszul if and only if $\mathcal{P}$ is.

The above constructions may be carried out if $V(2)$ is replaced by a pair $(V(1), V(2))$.

For a Koszul operad $\mathcal{P}$, the DG operad $\mathrm{Cobar}(\mathcal{P}^\vee)$ is a cofibrant resolution of $\mathcal{P}$. We will denote it by $\mathcal{P}_\infty$.

3.6. Operads As, Com, Lie, Gerst, Calc, BV**, and their ∞ analogs.**

3.6.1. As**,** Com**, and** Lie**.** Algebras over them are, respectively, graded associative algebras, graded commutative algebras, and graded Lie algebras.

3.6.2. Gerstenhaber algebras. Let k be the ground ring of characteristic zero. A *Gerstenhaber algebra* is a graded space $\mathcal{A}$ together with

- A graded commutative associative algebra structure on $\mathcal{A}$;
- a graded Lie algebra structure on $\mathcal{A}^{\bullet+1}$ such that

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|)}b[a, c]$$

Example 3.6.1. Let M be a smooth manifold. Then

$$\mathcal{V}_M^\bullet = \wedge^\bullet T_M$$

is a sheaf of Gerstenhaber algebras.

The product is the exterior product, and the bracket is the Schouten bracket. We denote by $\mathcal{V}(M)$ the Gerstenhaber algebra of global sections of this sheaf.

Example 3.6.2. Let $\mathfrak{g}$ be a Lie algebra. Then

$$C_\bullet(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}$$

is a Gerstenhaber algebra.

The product is the exterior product, and the bracket is the unique bracket which turns $C_\bullet(\mathfrak{g})$ into a Gerstenhaber algebra and which is the Lie bracket on $\mathfrak{g} = \wedge^1(\mathfrak{g})$.

3.6.3. Calculi.

Definition 3.6.3. *A precalculus is a pair of a Gerstenhaber algebra $\mathcal{V}^\bullet$ and a graded space $\Omega^\bullet$ together with*

- *a structure of a graded module over the graded commutative algebra $\mathcal{V}^\bullet$ on $\Omega^{-\bullet}$ (the corresponding action is denoted by i_a, $a \in \mathcal{V}^\bullet$);*
- *a structure of a graded module over the graded Lie algebra $\mathcal{V}^{\bullet+1}$ on $\Omega^{-\bullet}$ (the corresponding action is denoted by L_a, $a \in \mathcal{V}^\bullet$) such that*

$$[L_a, i_b] = i_{[a,b]}$$

and

$$L_{ab} = (-1)^{|b|} L_a i_b + i_a L_b$$

Definition 3.6.4. *A calculus is a precalculus together with an operator d of degree 1 on $\Omega^\bullet$ such that $d^2 = 0$ and*

$$[d, i_a] = (-1)^{|a|-1} L_a.$$

Example 3.6.5. For any manifold one defines a calculus $\mathrm{Calc}(M)$ with $\mathcal{V}^\bullet$ being the algebra of multivector fields, $\Omega^\bullet$ the space of differential forms, and d the de Rham differential. The operator i_a is the contraction of a form by a multivector field.

Example 3.6.6. For any associative algebra A one defines a calculus $\mathrm{Calc}_0(A)$ by putting $\mathcal{V}^\bullet = H^\bullet(A, A)$ and $\Omega^\bullet = H_\bullet(A, A)$. The five operations from Definition 3.6.4 are the cup product, the Gerstenhaber bracket, the pairings i_D and L_D, and the differential B, as in 2.4. The fact that it is indeed a calculus follows from Theorem 2.4.4.

A differential graded (dg) calculus is a calculus with extra differentials δ of degree 1 on $\mathcal{V}^\bullet$ and b of degree -1 on $\Omega^\bullet$ which are derivations with respect to all the structures.

DEFINITION 3.6.7. *1) An $\hbar$-calculus is a precalculus over the algebra $k[\hbar]$, $|\hbar| = 0$, together with a $k[\hbar]$-linear operator of degree $+1$ on $\Omega^{-\bullet}$ satisfying*

$$d^2 = 0;\ [d, \iota_a] = (-1)^{|a|-1}\hbar L_a$$

2) A u-calculus is a precalculus over the algebra $k[u]$, $|u| = 2$, together with a $k[u]$-linear operator of degree -1 on $\Omega^{-\bullet}$ satisfying

$$d^2 = 0;\ [d, \iota_a] = (-1)^{|a|-1}uL_a$$

3.6.4. BV algebras.

DEFINITION 3.6.8. *A Batalin-Vilkovisky (BV) algebra is a Gerstenhaber algebra together with an operator $\Delta : \mathcal{A} \to \mathcal{A}$ of degree -1 satisfying*

$$\Delta^2 = 0$$

and

$$\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = (-1)^{|a|-1}[a, b] \tag{3.4}$$

Note that the above axioms imply

$$\Delta([a, b]) - [\Delta(a), b] + (-1)^{|a|-1}[a, \Delta(b)] = 0 \tag{3.5}$$

There are two variations of this definition.

DEFINITION 3.6.9. *1) A $\mathrm{BV}_\hbar$-algebra is a Gerstenhaber algebra over the algebra $k[\hbar]$, $|\hbar| = 0$, with a $k[\hbar]$-linear operator $\Delta : \mathcal{A} \to \mathcal{A}$ of degree -1 satisfying*

$$\Delta^2 = 0,$$

the identity (3.5)*, and*

$$\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = (-1)^{|a|-1}\hbar[a, b] \tag{3.6}$$

2) 1) A BV_u-algebra is a Gerstenhaber algebra over the algebra $k[u]$, $|u| = 2$, with a $k[u]$-linear operator $\Delta : \mathcal{A} \to \mathcal{A}$ of degree $+1$ satisfying

$$\Delta^2 = 0,$$

the identity (3.5)*, and*

$$\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = (-1)^{|a|-1}u[a, b] \tag{3.7}$$

PROPOSITION 3.6.10. *For a DG operad $\mathcal{P}$, denote by $\mathcal{P}^\vee$ its Koszul dual.*

1. $\mathrm{As}^\vee = \mathrm{As}$; $\mathrm{Com}^\vee = \mathrm{Lie}$; $\mathrm{Lie}^\vee = \mathrm{Com}$;
2. *a complex A is an algebra over* $\mathrm{Gerst}^\vee$ *if and only if $A[1]$ is an algebra over* Gerst;
3. *a complex A is an algebra over* $\mathrm{BV}_u^\vee$ *if and only if $A[1]$ is an algebra over* $\mathrm{BV}_\hbar$;
4. *a complex A is an algebra over* $\mathrm{BV}_\hbar^\vee$ *if and only if $A[1]$ is an algebra over* BV_u;
5. *a pair of complexes (A, Ω) is an algebra over* $\mathrm{Calc}_u^\vee$ *if and only if $(A[1], \Omega)$ is an algebra over* $\mathrm{Calc}_\hbar$;

(6) *a pair of complexes* (A, Ω) *is an algebra over* $\mathrm{BV}_u^\vee$ *if and only if* $(A[1], \Omega)$ *is an algebra over* $\mathrm{Calc}_\hbar$.

(7) *All the operads above are Koszul.*

The above result was proved in [**56**] for As, Com and Lie; in [**54**] for Gerst; and in [**49**] for BV.

3.7. The Boardman-Vogt construction. For a topological operad $\mathcal{P}$, Boardman and Vogt constructed in [**7**] another topological operad $W\mathcal{P}$, together with a weak homotopy equivalence of topological operads $W\mathcal{P} \stackrel{\sim}{\to} \mathcal{P}$. (In fact $W\mathcal{P}$ is a cofibrant replacement of $\mathcal{P}$). The space $W\mathcal{P}(n)$ consists of planar rooted trees T with the following additional data:

(1) internal vertices of T of valency $j+1$ are decorated by points of $\mathcal{P}(j)$;
(2) external vertices of T are decorated by numbers from 1 to n, so that the map sending a vertex to its label is a bijection between the set of internal vertices and $\{1, \ldots, n\}$;
(3) internal edges of T are decorated by numbers $0 \leq r \leq 1$. The label r is called the length of the edge.

If the length of an edge of a tree is zero, this tree is equivalent to the tree obtained by contracting the edge, the label of the new vertex defined via operadic composition from the labels of the two vertices incident to e.

3.8. Operads of little discs. Let D be the standard k-disc $\{x \in \mathbb{R}^k | \ |x| \leq 1.\}$. For $1 \leq i \leq n$, denote by D_i a copy of D. Let $\mathrm{LD}_k(n)$ be the space of embeddings

$$\coprod_{i=1}^{n} D_i \to D \tag{3.8}$$

whose restriction to every component is affine Euclidean. The collection $\{\mathrm{LD}_k(n)\}$ is an operad in the category of topological spaces. The action of S_n is induced from the action by permutations of the n copies of D. Operadic composition is as follows. For embeddings

$$f\colon \coprod_{i=1}^{m} D_i \to D$$

and

$$f_i\colon \coprod_{j_i=1}^{n_i} D_{j_i} \to D_i,$$

the embedding

$$\mathrm{op}_{n_1,\ldots,n_m}(f; f_1, \ldots, f_m)\colon \coprod_{i=1}^{m} \coprod_{j=1}^{n_i} D_j \to D \tag{3.9}$$

acts on every component D_{j_i} by the composition $f \circ f_i$.

3.9. Fulton-MacPherson operads. The spaces $\mathrm{FM}_k(n)$ were defined by Fulton and MacPherson in [**48**]. The operadic structure on them was defined in [**54**] by Getzler and Jones.

For $k > 0$, let $\mathbb{R}^+ \ltimes \mathbb{R}^k$ be the group of affine transformations of $\mathbb{R}^k$ generated by positive dilations and translations. Define the configuration spaces to be

$$\mathrm{Conf}_k(n) = \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}^k,\ x_i \neq x_j\} / (\mathbb{R}^+ \ltimes \mathbb{R}^k) \tag{3.10}$$

There are compactifications $\mathrm{FM}_k(n)$ of $\mathrm{Conf}_k(n)$ that form an operad in the category of topological spaces for each $k > 0$. As an operad of sets, $\mathrm{FM}_k(n)$ is the free operad generated by the collection of sets $\mathrm{Conf}_k(n)$ with the action of S_n. In fact there are continuous bijections

$$\mathrm{FreeOp}(\{\mathrm{Conf}_k(n)\}) \to \mathrm{FM}_k(n) \tag{3.11}$$

The spaces $\mathrm{FM}_k(n)$ are manifolds with corners. They can be defined explicitly as follows. Consider the functions $\theta_{ij} : \mathrm{Conf}_k(n) \to S^{k-1}$ and $\rho_{ijk} : \mathrm{Conf}_k(n) \to \mathbb{R}$ by

$$\theta_{ij}(x_1, \ldots, x_n) = \frac{x_i - x_j}{|x_i - x_j|};\ \delta_{ijk}(x_1, \ldots, x_n) = \frac{|x_i - x_j|}{|x_i - x_k|} \tag{3.12}$$

The map

$$\mathrm{Conf}_k(n) \to (S^{k-1})^{\binom{n}{2}} \times [0, +\infty]^{\binom{n}{3}} \tag{3.13}$$

defined by all θ_{ij}, $i < j$, and δ_{ijk}, $i < j < k$, can be shown to be an embedding. The space $\mathrm{FM}_k(n)$ can be defined as the closure of the image of this embedding.

Kontsevich and Soibelman proved in [**74**] that the topological operads FM_k and LD_k are weakly homotopy equivalent. In fact there is a homotopy equivalence of topological operads

$$W\mathrm{LD}_k \xrightarrow{\sim} \mathrm{FM}_k\,. \tag{3.14}$$

constructed by Salvatore in [**99**], Prop. 4.9.

3.10. The operad of framed little discs. This operad constructed analogously to the operad $rmLD_2$. By definition, $\mathrm{FLD}_2(n)$ is the space of affine embeddings 3.8 together with points $a_i \in \partial D_i$, $a \in \partial D$. The operadic compositions consist of those for LD_2 and of rotating the discs D_i so that the marked points on the boundaries come together.

3.11. The colored operad of little discs and cylinders. The colored operad LC has two colors that we denote by $\mathbf{c}$ and $\mathbf{h}$. All spaces $\mathrm{LC}(x_1, \ldots, x_n; y)$ are empty if more than one x_i is equal to $\mathbf{h}$ or if one x_i is equal to $\mathbf{h}$ and $y = \mathbf{c}$. For $n \geq 0$, let

$$\mathrm{LC}(n) \stackrel{\mathrm{def}}{=} \mathrm{LC}(\mathbf{c}, \ldots, \mathbf{c}; \mathbf{c}) = \mathrm{LD}_2(n)$$

and

$$\mathrm{LC}(n, 1) \stackrel{\mathrm{def}}{=} \mathrm{LC}(\mathbf{c}, \ldots, \mathbf{c}, \mathbf{h}; \mathbf{h}).$$

The spaces $\mathrm{LD}(n)$ form a suboperad of LC. For $r > 0$, let C_r be the cylinder $S^1 \times [0, r]$. By definition, $\mathrm{LC}(n, 1)$ is the space of data (r, g) where

$$g \colon \coprod_{i=1}^{k} D_i \to C_r$$

is an embedding such that $g|D_i$ is the composition

$$D_i \xrightarrow{\widetilde{g_i}} \mathbb{R} \times [0, r] \xrightarrow{\mathrm{pr}} S^1 \times [0, 1] \tag{3.15}$$

of the projection with an affine Euclidean map $\widetilde{g}$. The action of S_n on $\mathrm{LC}(n, 1)$ is induced by permutations of the components D_i. Let us define operadic compositions of two types. The first is

$$\mathrm{LC}(m, 1) \times \mathrm{LC}(n_1) \times \ldots \times \mathrm{LC}(n_m) \to \mathrm{LC}(n_1 + \ldots + n_m, 1);$$

it is defined exactly as the operadic composition in (3.9), with D replaced by C_r. The second is

$$\mathrm{LC}(n, 1) \times \mathrm{LC}(m, 1) \to \mathrm{LC}(n + m, 1) \tag{3.16}$$

For $\widetilde{g_i} : D_i \to C_r, 1 \le i \le n$, and $\widetilde{g'_j} : D_j \to C_{r'}, 1 \le j \le m$, as in (3.15), define

$$\widetilde{g''} : (\coprod_{i=1}^{n} D_i) \coprod (\coprod_{j=1}^{m} D_j) \to \mathbb{R} \times [0, r + r'] \tag{3.17}$$

that sends $z \in D_i, 1 \le i \le n$, to the image of $\widetilde{g}(z)$ under the map $\mathbb{R} \times [0, r] \to \mathbb{R} \times [0, r + r']$, $(x, t) \mapsto (x, t)$, and $z \in D_j, 1 \le i \le m$, to the image of $\widetilde{g'}(z)$ under the map $\mathbb{R} \times [0, r'] \to \mathbb{R} \times [0, r+r']$, $(x, t) \mapsto (x, t+r)$. Let $g = \mathrm{pr} \circ \widetilde{g}$ and $g' = \mathrm{pr} \circ \widetilde{g}'$ The composition (3.17) of (g, r) and $g'r')$ is by definition $(\mathrm{pr} \circ \widetilde{g''}, r + r')$.

All other nonempty spaces $\mathrm{LC}(x_1, \ldots, x_n; y)$, in other words spaces $\mathrm{LC}(\mathbf{c}, \ldots \mathbf{c}, \mathbf{h}, \ldots, \mathbf{c}; \mathbf{h})$, together with the actions of symmetric groups and with operadic compositions, are uniquely determined by the above and by the axioms of colored operads.

3.11.1. The colored operad of little discs and framed cylinders. The colored operad LfC is defined exactly as LC above, with the following modifications. First, by definition, $\mathrm{LC}(n, 1)$ is the space of data (r, x_0, x_1, g) where r and g are as above, $x_0 \in S^1 \times \{0\}$, and $x_1 \in \mathbb{R} \times \{r\}$, factorized by the action of the circle by rotations on the factor S^1. The composition

$$\mathrm{LfC}(n, 1) \times \mathrm{LfC}(m, 1) \to \mathrm{LfC}(n + m, 1) \tag{3.18}$$

is defined as follows: given (r, g, x_0, x_1) and (r', g', x'_0, x'_1), their composition is $(r + r', g'', x_0, x'_1 + x_1 - x'_0)$ where $g'' = \mathrm{pr} \circ \widetilde{g''}$ and $\widetilde{g''}$ is exactly as in (3.17), with the only difference that it sends $z \in D_j, 1 \le i \le m$, to the image of $\widetilde{g'}(z)$ under the map $\mathbb{R} \times [0, r'] \to \mathbb{R} \times [0, r + r']$, $(x, t) \mapsto (x + x_1 - x'_0, t + r)$. Note that $\mathrm{LC}(1)$ is contractible but $\mathrm{LfC}(1)$ is homotopy equivalent to S^1.

3.11.2. The Fulton-MacPherson version of LC **and of** LfC**.** Note first that the colored operad LC can be alternatively defined as follows: the spaces $\mathrm{LC}(n)$ are as above; the spaces $\mathrm{LC}(n, 1)$ are defined as subspaces of $\mathrm{LD}_2(n + 1)$ consisting of those embeddings (3.8) that map the center of D_{n+1} to the center of D. The action of the symmetric groups and the operadic compositions are induced from those of LD_2. Similarly, define the two-colored operad FMC as follows. Put $\mathrm{FMC}(n) = \mathrm{FM}(n)$; define $\mathrm{FMC}(n, 1)$ to be the subspace of $\mathrm{FM}(n + 1)$ consisting of data

$$(T, \{c_v\} | v \in \{\text{external vertices of } T\})$$

such that:

(1) T is a rooted tree;
(2) $c_v \in \mathrm{Conf}(\{\text{edges outgoing from } v\})$;
(3) Consider the path from the root of T to the external vertex labeled by $n+1$ (*the trunk* of T). Let e_0 be the edge on this path that goes out of a vertex v. Let $c_v = (x_e)$ where e are all edges outgoing from v. Then $x_{e_0} = 0$.

We leave to the reader to define the operadic compositions and the action of the symmetric groups, as well as the Fulton-MacPherson analog FMfC of the two-colored operad LfC.

PROPOSITION 3.11.1. *The two-colored operads* FMC *and* LC, *resp.* FMfC *and* LfC, *are weakly equivalent.*

4. DG categories

The contents of this section are taken mostly from [**37**], [**69**], and [**111**].

4.1. Definition and basic properties. A differential graded (DG) category A over k is a collection $\mathrm{Ob}(A)$ of elements called objects and of complexes $A(x,y)$ of k-modules for every $x, y \in \mathrm{Ob}(A)$, together with morphisms of complexes

$$A(x,y) \otimes A(y,z) \to A(x,z),\ a \otimes b \mapsto ab, \tag{4.1}$$

and zero-cycles $\mathbf{1}_x \in A(x,x)$, such that (4.1) is associative and $\mathbf{1}_x a = a\mathbf{1}_y = a$ for any $a \in A(x,y)$. For a DG category, its homotopy category is the k-linear category $\mathrm{Ho}(A)$ such that $\mathrm{Ob}(\mathrm{Ho}(A)) = \mathrm{Ob}(A)$ and $\mathrm{Ho}(A)(x,y) = H^0(A(x,y)$, with the units being the classes of $\mathbf{1}_x$ and the composition induced by (4.1).

A DG functor $A \to B$ is a map $\mathrm{Ob}(A) \to \mathrm{Ob}(B)$, $x \mapsto Fx$, and a collection of morphisms of complexes $F_{x,y}\colon A(x,y) \to B(Fx,Fy), x, y \in \mathrm{Ob}(A)$, which commutes with the composition (4.1) and such that $F_{x,x}(\mathbf{1}_x) = \mathbf{1}_{Fx}$ for all x.

The opposite DG category of A is defined by $\mathrm{Ob}(A^{\mathrm{op}}) = \mathrm{Ob}(A)$, $A^{\mathrm{op}}(x,y) = A(y,x)$, the unit elements are the same as in A, and the composition (4.1) is the one from A, composed with the transposition of tensor factors.

For two DG categories A and B, the tensor product $A \otimes B$ is defined as follows: $\mathrm{Ob}(A \otimes B) = \mathrm{Ob}(A) \times \mathrm{Ob}(B)$; we denote the object (x,y) by $x \otimes y$;

$$(A \otimes B)(x \otimes y, x' \otimes y') = A(x,y) \otimes B(x',y');$$

$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'; \mathbf{1}_{x \otimes y} = \mathbf{1}_x \otimes \mathbf{1}_y$.

4.2. Cofibrant DG categories. Cofibrant DG categories are defined exactly following the general principle of 3.3.

4.3. Quasi-equivalences. A quasi-equivalence [**101**] between DG categories A and B is a DG functor $F : A \to B$ such that a) F induces an equivalence of homotopy categories and b) for any $x, y \in \mathrm{Ob}(A)$, $F_{x,y}\colon A(x,y) \to B(Fx,Fy)$ is a quasi-isomorphism.

4.4. Drinfeld localization. For a full DG subcategory C of a DG category A, the localization of A with respect to C is obtained from A as follows. Consider DG categories k_C and $\mathcal{N}_C$; $\mathrm{Ob}(k_C) = \mathrm{Ob}(\mathcal{N}_C) = \mathrm{Ob}(C)$; $k_C(x,y) = \mathcal{N}_C(x,y) = 0$ if $x \neq y$; $k_C(x,x) = k \cdot \mathbf{1}_x$; $\mathcal{N}_C(x,x)$ is equal to the free algebra generated by one element ϵ_x of degree -1 satisfying $d\epsilon_x = \mathbf{1}_x$ for all $x \in Ob(C)$. The localization of A is the free product $A *_{k_C} \mathcal{N}_C$. In other words, it is a DG category $\mathcal{A}$ such that:

(1) $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(x)$;
(2) there is a DG functor $i : A \to \mathcal{A}$ which is the identity on objects;
(3) for every $x \in \mathrm{Ob}(C)$, there is an element ϵ_x of degree -1 in $\mathcal{A}(x, x)$ satisfying $d\epsilon_x = \mathbf{1}_x$;
(4) for any other DG category $\mathcal{A}'$ together with a DG functor $i' : A \to \mathcal{A}'$ and elements ϵ'_x as above, there is unique DG functor $f : \mathcal{A} \to \mathcal{A}'$ such that $i' = f \circ i$ and $\epsilon_x \mapsto \epsilon'_x$.

One has

$$\mathcal{A}(x, y) = \bigoplus_{n\geq 0} \bigoplus_{x_1,\ldots,x_n\in \mathrm{Ob}(C)} A(x, x_1)\epsilon_{x_1} A(x_1, x_2)\epsilon_{x_2} \ldots \epsilon_{x_n} A(x_n, y);$$

it is easy to define the composition and the differential explicitly.

4.5. DG modules over DG categories. A DG module over a DG category A is a collection of complexes of k-modules $M(x)$, $x \in \mathrm{Ob}(A)$, together with morphisms of complexes

$$A(x, y) \otimes M(y) \to A(x),\ a \otimes m \mapsto am, \tag{4.2}$$

which is compatible with the composition (4.1) and such that $\mathbf{1}_x m = m$ for all x and all $m \in M(x)$. A DG bimodule over A is a collection of complexes $M(x, y)$ together with morphisms of complexes

$$A(x, y) \otimes M(y, z) \otimes A(z, w) \to M(x, w),\ a \otimes m \otimes b \mapsto amb, \tag{4.3}$$

that agrees with the composition in A and such that $\mathbf{1}_x m \mathbf{1}_y = m$ for any x, y, m. We put $am = am\mathbf{1}_z$ and $mb = \mathbf{1}_x mb$. A DG bimodule over A is the same as a DG module over $A \otimes A^{\mathrm{op}}$.

4.6. Bar and cobar constructions for DG categories. The bar construction of a DG category A is a DG cocategory $\mathrm{Bar}(A)$ with the same objects where

$$\mathrm{Bar}(A)(x, y) = \bigoplus_{n\geq 0} \bigoplus_{x_1,\ldots,x_n} A(x, x_1)[1] \otimes A(x_1, x_2)[1] \otimes \ldots \otimes A(x_n, x)[1]$$

with the differential

$$d = d_1 + d_2;$$

$$d_1(a_1|\ldots|a_{n+1}) = \sum_{i=1}^{n+1} \pm(a_1|\ldots|da_i|\ldots|a_{n+1});$$

$$d_2(a_1|\ldots|a_{n+1}) = \sum_{i=1}^{n} \pm(a_1|\ldots|a_i a_{i+1}|\ldots|a_{n+1})$$

The signs are $(-1)^{\sum_{j<i}(|a_i|+1)+1}$ for the first sum and $(-1)^{\sum_{j\leq i}(|a_i|+1)}$ for the second. The comultiplication is given by

$$\Delta(a_1|\ldots|a_{n+1}) = \sum_{i=0}^{n+1} (a_1|\ldots|a_i) \otimes (a_{i+1}|\ldots|a_{n+1})$$

Dually, for a DG cocategory B one defines the DG category $\mathrm{Cobar}(B)$. The DG category $\mathrm{CobarBar}(A)$ is a cofibrant resolution of A.

4.6.1. Units and counits. It is convenient for us to work with DG (co)categories without (co)units. For example, this is the case for $\mathrm{Bar}(A)$ and $\mathrm{Cobar}(B)$ (we sum, by definition, over all tensor products with at least one factor). Let A^+ be the (co)category A with the (co)units added, i.e. $A^+(x,y) = A(x,y)$ for $x \neq y$ and $A^+(x,x) = A(x,x) \oplus k\,\mathrm{id}_x$. If A is a DG category then A^+ is an augmented DG category with units, i.e. there is a DG functor $\epsilon : A^+ \to k_{\mathrm{Ob}(A)}$. The latter is the DG category with the same objects as A and with $k_I(x,y) = 0$ for $x \neq y$, $k_I(x,x) = k$. Dually, one defines the DG cocategory $k^{\mathrm{Ob}(B)}$ and the DG functor $\eta : k^{\mathrm{Ob}(B)} \to B^+$ for a DG cocategory B.

4.6.2. Tensor products. For DG (co)categories with (co)units, define $A \otimes B$ as follows: $\mathrm{Ob}(A \otimes B) = \mathrm{Ob}(A) \times \mathrm{Ob}(B)$; $(A \otimes B)((x_1,y_1),(x_2,y_2)) = A(x_1,y_1) \otimes B(x_2,y_2)$; the product is defined as $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1a_2 \otimes b_1b_2$, and the coproduct in the dual way. This tensor product, when applied to two (co)augmented DG (co)categories with (co)units, is again a (co)augmented DG (co)category with (co)units: the (co)augmentation is given by $\epsilon \otimes \epsilon$, resp. $\eta \otimes \eta$.

Definition 4.6.1. *For DG categories A and B without units, put*

$$A \otimes B = \mathrm{Ker}(\epsilon \otimes \epsilon : A^+ \otimes B^+ \to k_{\mathrm{Ob}(A)} \otimes k_{\mathrm{Ob}(B)}).$$

Dually, for For DG cocategories A and B without counits, put

$$A \otimes B = \mathrm{Coker}(\eta \otimes \eta : k^{\mathrm{Ob}(A)} \otimes k^{\mathrm{Ob}(B)} \to A^+ \otimes B^+).$$

One defines a morphism of DG cocategories

$$\mathrm{Bar}(A) \otimes \mathrm{Bar}(B) \to \mathrm{Bar}(A \otimes B) \tag{4.4}$$

by the standard formula for the shuffle product

$$(a_1|\ldots|a_m)(b_1|\ldots|b_n) = \sum \pm(\ldots|a_i|\ldots|b_j|\ldots) \tag{4.5}$$

The sum is taken over all shuffle permutations of the $m+n$ symbols $a_1, \ldots, a_m, b_1, \ldots, b_n$), i.e. over all permutations that preserve the order of the a_i's and the order of the b_j's. The sign is computed as follows: a transposition of a_i and b_j introduces a factor $(-1)^{(|a_i|+1)(b_j|+1)}$. Let us explain the meaning of the factors a_i and b_j in the formula. We assume $a_i \in A(x_{i-1}, x_i)$ and $b_j \in B(y_{j-1}, y_j)$ for $x_i \in \mathrm{Ob}(A)$ and $y_j \in \mathrm{Ob}(B)$, $0 \leq i \leq m$, $0 \leq j \leq m$. Consider a summand $(\ldots|a_i|b_j|b_{j+1}|\ldots|b_k|a_{i+1}|\ldots)$. In this summand, all b_p, $j \leq p \leq k$, are interpreted as $\mathrm{id}_{x_i} \otimes b_p \in (A \otimes B)((x_i, y_{p-1}), (x_i, y_p))$. Similarly, in the summand $(\ldots|b_i|a_j|a_{j+1}|\ldots|a_k|a_{i+1}|\ldots)$, all a_p, $j \leq p \leq k$, are interpreted as $a_p \otimes \mathrm{id}_{y_i} \in (A \otimes B)((x_{p-1}, y_i), (x_p, y_i))$. Dually, one defines the morphism of DG cocategories

$$\mathrm{Cobar}(A \otimes B) \to \mathrm{Cobar}(A) \otimes \mathrm{Cobar}(B) \tag{4.6}$$

4.7. A_∞ categories. An A_∞ category is a natural generalization of both a DG category and an A_∞ algebra. We refer the reader, for example, to [**75**].

4.7.1. DG category $\mathbf{C}^\bullet(A,B)$. For two DG categories A and B, define the DG category $\mathbf{C}^\bullet(A,B)$ as follows. Its objects are A_∞ functors $f : A \to B$. Define the complex of morphisms as

$$\mathbf{C}^\bullet(A,B)(f,g) = C^\bullet(A, {}_fB_g)$$

where ${}_fB_g$ is the complex B viewed as an A_∞ bimodule on which A acts on the left via f and on the right via g. The composition is defined by the cup product as in the formula (2.14).

REMARK 4.7.1. Every A_∞ functor $f : A \to B$ defines an A_∞ (A, B)-bimodule ${}_fB$, namely the complex B on which A acts on the left via f and B on the right in the standard way. If for example f, $g : A \to B$ are morphisms of algebras then $C^\bullet(A,{}_f B_g)$ computes $\mathrm{Ext}^\bullet_{A\otimes B^{\mathrm{op}}}({}_fB,{}_g B)$. What we are going to construct below does not seem to extend literally to all (A_∞) bimodules. This applies also to related constructions of the category of internal homomorphisms, such as in [**68**] and [**112**]. One can overcome this by replacing A by the category of A-modules, since every (A, B)-bimodule defines a functor between the categories of modules.

4.7.2. The bialgebra structure on $\mathrm{Bar}(C^\bullet(A, A))$**.** Let us first recall the product on the bar construction $\mathrm{Bar}(C^\bullet(A, A))$ where $C^\bullet(A, A)$ is the algebra of Hochschild cochains of A with coefficients in A (cf. [**54**], [**51**]). For cochains D_i and E_j, define

$$(D_1|\dots|D_m)\bullet(E_1|\dots|E_n) = \sum \pm(\dots|D_1\{\dots\}|\dots|D_m\{\dots\}|\dots)$$

Here the space denoted by ... inside the braces contains $E_{j+1},\dots,E_k$; outside the braces, it contains $E_{j+1}|\dots|E_k$. The factor $D_i\{E_{j+1},\dots,E_k\}$ is the brace operation as in (2.25). The sum is taken over all possible combinations for which the natural order of E_j's is preserved. The signs are computed as follows: a transposition of D_i and E_j introduces a sign $(-1)^{(|D_i|+1)(|E_j|+1)}$. In other words, the right hand side is the sum over all tensor products of $D_i\{E_{j+1},\dots,E_k\}$, $k \ge j$, and E_p, so that the natural orders of D_i's and of E_j's are preserved. For example,

$$(D)\bullet(E) = (D|E) + (-1)^{(|D|+1)(|E|+1)}(E|D) + D\{E\}$$

PROPOSITION 4.7.2. *The product* $\bullet$ *together with the comultiplication* Δ *makes* $\mathrm{Bar}(C^\bullet(A, A))$ *an associative bialgebra.*

Now let us explain how to modify the product $\bullet$ and to get a DG functor

$$\bullet : \mathrm{Bar}(\mathbf{C}^\bullet(A, B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B, C)) \to \mathrm{Bar}(\mathbf{C}^\bullet(A, C)) \tag{4.7}$$

4.7.3. The brace operations on $\mathbf{C}^\bullet(A, B)$**.** For Hochschild cochains $D \in C^\bullet(B,{}_{f_0} C_{f_1})$ and $E_i \in \mathbf{C}^\bullet(A,{}_{g_{i-1}} B_{g_i}$, $1 \le i \le n$, define the cochain

$$D\{E_1,\dots,E_n\} \in C^\bullet(A,{}_{f_0g_0} C_{f_1g_n})$$

by

$$D\{E_1,\dots,E_n\}(a_1,\dots,a_N) = \sum \pm D(\dots,E_1(\underline{\dots}),\dots,E_n(\underline{\dots}),\dots) \tag{4.8}$$

where the space denoted by $\underline{\dots}$ within $E_k(\underline{\dots})$ stands for $a_{i_k+1},\dots,a_{j_k}$, and the space denoted by ... between $E_k(\underline{\dots})$ and $E_{k+1}(\underline{\dots})$ stands for $g_k(a_{j_k+1},\dots,), g_k(\dots)$, $\dots, g_k(\dots,a_{i_{k+1}})$. The sum is taken over all possible combinations such that $i_k \le j_k \le i_{k+1}$. The signs are as in (2.25).

4.7.4. The $\bullet$ product on $\mathrm{Bar}(\mathbf{C}(A,B))$. For Hochschild cochains $D_i \in C^\bullet(B, {}_{f_{i-1}}C_{f_i})$ and $E_j \in C^\bullet(A, {}_{g_{j-1}}B_{g_j})$, $1 \le i \le m$, $1 \le j \le n$, we have

$$(D_1|\dots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(B,C))(f_0, f_m);$$

$$(D_1|\dots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,B))(g_0, g_m);$$

define

$$(D_1|\dots|D_m) \bullet (E_1|\dots|E_n) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,C))(f_0g_0, f_mg_n)$$

by the formula in the beginning of 4.7.2, with the following modification. The expression $D_i\{E_{j+1},\dots,E_k\}$ is now in $\mathbf{C}(A,C)(f_{i-1}g_{j+1}, f_ig_j)$, as explained above. The space denoted by $\dots$ between $D_i\{E_{j+1},\dots,E_k\}$ and $D_{i+1}\{E_{p+1},\dots,E_q\}$ contains $f_i(E_{k+1}|\dots)|f_i(\dots)|\dots|f_i(\dots,E_p)$. Here, for an A_∞ functor f and for cochains $E_1,\dots,E_k$,

$$f(E_1,\dots,E_k)(a_1,\dots,a_N) = \sum f(E_1(a_1,\dots,a_{i_2-1}),\dots,E_k(a_{i_k+1},\dots,a_n)) \tag{4.9}$$

The sum is taken over all possible combinations $1 \le i_1 \le i_2 \le \dots i_k \le n$.

LEMMA 4.7.3. *1) The product $\bullet$ is associative.*
2) It is a morphism of DG cocategories. In other words, one has

$$\Delta \circ \bullet = (\bullet_{13} \otimes \bullet_{24}) \circ (\Delta \otimes \Delta)$$

as morphisms

$$\mathrm{Bar}(C^\bullet(A,B))(f_0,f_1) \otimes \mathrm{Bar}(C^\bullet(B,C))(g_0,g_1) \to$$

$$\mathrm{Bar}(C^\bullet(A,C))(f_0g_0, fg) \otimes \mathrm{Bar}(C^\bullet(A,C))(fg, f_1g_1)$$

4.7.5. Internal $\underline{\mathrm{Hom}}$ of DG cocategories. Following the exposition of [**68**], we explain the construction of Keller, Lyubashenko, Manzyuk, Kontsevich and Soibelman. For two k-modules V and W, let $\mathrm{Hom}(V,W)$ be the set of homomorphisms from V to W, and let $\underline{\mathrm{Hom}}(V,W)$ be the same set viewed as a k-module. The two satisfy the property

$$\mathrm{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}(U, \underline{\mathrm{Hom}}(V,W)). \tag{4.10}$$

In other words, $\underline{\mathrm{Hom}}(V,W)$ is the internal object of morphisms in the symmetric monoidal category $k-\mathrm{mod}$. The above equation automatically implies the existence of an associative morphism

$$\underline{\mathrm{Hom}}(U,V) \otimes \underline{\mathrm{Hom}}(V,W) \to \underline{\mathrm{Hom}}(U,W) \tag{4.11}$$

If we replace the category of modules by the category of algebras, there is not much chance of constructing anything like the internal object of morphisms. However, if we replace $k-\mathrm{mod}$ by the category of coalgebras, the prospects are much better. For our applications, it is better to consider counital coaugmented coalgebras. In this category, objects $\underline{\mathrm{Hom}}$ do not exist because the equation (4.10) does not agree with coaugmentations. However, as explained in [**68**], the following is true.

PROPOSITION 4.7.4. *The category of coaugmented counital conilpotent cocategories admits internal $\underline{\mathrm{Hom}}$s. For two DG categories A and B, one has*

$$\underline{\mathrm{Hom}}(\mathrm{Bar}(A), \mathrm{Bar}(B)) = \mathrm{Bar}(\mathbf{C}(A,B)) \tag{4.12}$$

4.8. Hochschild and cyclic complexes of DG categories and A_∞ categories. These are direct generalizations of the corresponding constructions for DG algebras. The Hochschild chain complex of a DG category (or, more generally, of an A_∞ category) A is defined as

$$C_p(A) = \bigoplus_{k-j=p} \bigoplus_{i_0,\ldots,i_p \in \mathrm{Ob}(A)} (A(i_0,i_1) \otimes \overline{A}(i_1,i_2) \otimes \ldots \otimes \overline{A}(i_p,i_0))^j;$$

the Hochschild cochain complex, as

$$C^p(A) = \prod_{k+j=p} \prod_{i_0,\ldots,i_p \in \mathrm{Ob}(A)} \mathrm{Hom}(\overline{A}(i_0,i_1) \otimes \ldots \otimes \overline{A}(i_{p-1},i_p), A(i_0,i_p))^j;$$

the formulas for the differentials b, B, and δ are identical to those defined above for DG and A_∞ algebras.

5. Infinity algebras and categories

We develop a version of the definitions of an infinity algebra over an operad, an infinity category, and an infinity n-category. These definitions are closer to the work of Lurie, and of Batanin, than the ones developed in 3. We compare the two. We show that Hochschild cochains of a DG algebra (or DG category) form an infinity two-category. We extend some of this discussion to the case of Hochschild chains.

5.1. Infinity algebras over an operad. Let $\mathcal{P}$ be an operad in sets. Define the category $\mathcal{P}^\#$ as the PROP associated to $\mathcal{P}$. In other words, let $\mathcal{P}^\#$ be the category whose objects are $[n]$, $n = 1, 2, 3, \ldots$, and whose morphisms are defined by

$$\mathcal{P}^\#([n],[m]) = \{\text{Natural maps } X^n \to X^m\} \tag{5.1}$$

where X is any set which is an algebra over $\mathcal{P}$. By this we mean that morphisms from $[n]$ to $[m]$ are all maps that you can construct universally, using the algebra structure, from X^n to X^m where X is any set that is a $\mathcal{P}$-algebra, so that every component x_j in the argument $(x_1, \ldots, x_n)$ is used exactly once.

REMARK 5.1.1. When $\mathcal{P} = \mathrm{As}$, a $\mathcal{P}$-algebra is an associative monoid. We will, however, modify the definition slightly and require it to be a unital monoid. The set of objects will be $\{[0], [1], [2], \ldots\}$. Morphisms in $\mathrm{As}^\#([n],[m])$ can be identified with data

$$(f : \{1,\ldots,n\} \to \{1,\ldots,m\}; <_1, \ldots, <_m)$$

where $<_i$ is a linear order on $f^{-1}(\{i\})$. A natural morphism associated to such data is defined by

$$(x_1,\ldots,x_n) \mapsto (\prod_{f(j)=1} x_j, \ldots, \prod_{f(j)=m} x_j) \tag{5.2}$$

where the products are taken according to the orders $<_i$ and the product over the empty set is 1. This category was introduced in [**42**].

The category $\mathcal{P}^\#$ has a symmetric monoidal structure as follows. On objects, $[n] \otimes [m] = [n+m]$; on morphisms, $f \otimes f' : [n+n'] \to [m+m']$ is the natural morphism obtained by concatenation of f and f'.

The following definition is due to Leinster [**80**].

DEFINITION 5.1.2. *Let $\mathfrak{C}$ be a symmetric monoidal category with weak equivalences. An infinity-algebra over $\mathcal{P}$ in $\mathfrak{C}$ is a functor*

$$A : \mathcal{P}^{\#} \to \mathfrak{C};\ [n] \mapsto A(n)$$

together with a natural transformation

$$\Delta(n,m) :\ A(n+m) \to A(n) \otimes A(m) \tag{5.3}$$

which is a weak equivalence for every pair $(n,\, m)$ and is coassociative, i.e.

$$\mathrm{id}_{A(n)} \otimes \Delta(m,k) = \Delta(n,m) \otimes \mathrm{id}_{A(k)} : A(n+m+k) \to A(n) \otimes A(m) \otimes A(k)$$

LEMMA 5.1.3. *For an infinity algebra A in the category of complexes, there exists a $k[\mathcal{P}]_\infty$-algebra structure on $A(1)$ such that the composition*

$$\mathcal{P}(n) \otimes A(n) \stackrel{\mathrm{id}_{\mathcal{P}} \otimes \Delta}{\longrightarrow} \mathcal{P}(n) \otimes A(1)^{\otimes n} \to A(1)$$

is homotopic to

$$\mathcal{P}(n) \otimes A(n) \to \mathcal{P}^{\#}(n,1) \otimes A(n) \to A(1).$$

This structure can be chosen canonically up to homotopy.

PROOF. One can define the DG coalgebra

$$\prod_n (\mathrm{Bar}(\mathcal{P})(n) \otimes A(n))^{\Sigma_n}$$

over $\mathrm{Bar}(\mathcal{P})$ together with a coderivation d of degree one and square zero, using the infinity algebra structure on A. Then one transfers the DG coalgebra structure to the quasi-isomorphic complex

$$\prod_n (\mathrm{Bar}(\mathcal{P})(n) \otimes A^{\otimes n})^{\Sigma_n}$$

which is the cofree coalgebra over $\mathrm{Bar}(\mathcal{P})$ generated by A. The resulting coderivation gives a $\mathcal{P}_\infty$-algebra structure on A. □

REMARK 5.1.4. In [**23**], Costello uses a different definition of an infinity algebra over a PROP in simplicial sets. For such a PROP $\mathbf{P}$, an infinity $\mathbf{P}$-algebra A is defined as a functor $\mathbf{P} \to \mathfrak{C}$ together with an associative natural transformation $A(n) \otimes A(m) \to A(n+m)$ which is a weak equivalence for every m and n. But, when $\mathbf{P} = \mathcal{P}^{\#}$ for an operad $\mathcal{P}$, what we get is a strict algebra over $\mathcal{P}$.

REMARK 5.1.5. [**80**] When $\mathfrak{C} = \mathrm{Top}$, then the definition of an infinity associative algebra leads to the definition of a Segal space X with $X_0 = \mathrm{pt}$. Indeed, put $X_n = A(n)$. Define $d_i : A(n) \to A(n-1)$ as follows. For $1 \leq i \leq n-1$, d_i is induced by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i x_{i+1}, \ldots, x_n)$ in $\mathrm{As}^{\#}([n], [n-1])$. For $i = 0$, resp. $i = n$, define d_i to be the composition $A(n) \to A(1) \times A(n-1) \to A(n-1)$, resp. $A(n) \to A(n-1) \times A(1) \to A(n-1)$. Degeneracy operators s_i are induced by maps $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n)$.

5.1.1. Multiple infinity algebras. A morphism $A_1 \to A_2$ of infinity $\mathcal{P}$-algebras is a morphism of functors which is compatible with the underlying structure. By definition, a morphism is a weak equivalence if every map $A_1(n) \to A_2(n)$ is a weak equivalence.

Infinity $\mathcal{P}$-algebras form a symmetric monoidal category: for two such algebras A_1 and A_2, put $(A_1 \otimes A_2)(n) = A_1(n) \otimes A_2(n)$; the action of morphisms from $\mathcal{P}^{\#}$ and the comultiplication Δ are defined as tensor products of those for A_1 and A_2.

DEFINITION 5.1.6. *An infinity $(\mathcal{P}, \mathcal{Q})$-algebra is an infinity $\mathcal{P}$-algebra in the symmetric monoidal category of infinity $\mathcal{Q}$-algebras.*

In other words, an infinity $(\mathcal{P}, \mathcal{Q})$-algebra is a collection of objects $A(m,n)$, morphisms $\mathcal{P}^{\#}(m_1, m_2) \otimes A(m_1, n) \to A(m_2, n)$, and weak equivalences $\mathcal{Q}^{\#}(n_1, n_2) \otimes A(m, n_1) \to A(m, n_2)$, $A(m_1+m_2, n) \to A(m_1, n) \otimes A(m_2, n)$, and $A(m, n_1+n_2) \to A(m, n_1) \otimes A(m, n_2)$ subject to various compatibilities.

EXAMPLE 5.1.7. Let $\mathcal{P} \otimes \mathcal{Q}$ be the tensor product as in [**39**]; it is defined as the free product of $\mathcal{P}$ and $\mathcal{Q}$ factorized by relations

$$\alpha(\beta, \ldots, \beta) = \beta(\alpha, \ldots, \alpha) \in (\mathcal{P} \otimes \mathcal{Q})(mn)$$

for all $\alpha \in \mathcal{P}(m)$ and $\beta \in \mathcal{Q}(n)$; here $\alpha(\beta, \ldots, \beta)$ denotes $\mathrm{op}_{n,\ldots,n}(\alpha \otimes (\beta \otimes \ldots \otimes \beta))$ and $\beta(\alpha, \ldots, \alpha)$ denotes $\mathrm{op}_{m,\ldots,m}(\beta \otimes (\alpha \otimes \ldots \otimes \alpha))$. For a $\mathcal{P} \otimes \mathcal{Q}$-algebra A one can define an infinity $(\mathcal{P}, \mathcal{Q})$-algebra with $A(m,n) = A^{\otimes mn}$.

5.2. Infinity categories.

DEFINITION 5.2.1. *For a set I, let $\mathrm{As}_I^{\#}$ be the following category. Its objects are directed graphs with the set of vertices I and with a finite number of edges. For two such graphs Γ and Γ', $\mathrm{As}_I^{\#}(\Gamma, \Gamma')$ is the set of all natural maps*

$$\prod_{\mathrm{edges}(\Gamma)} X(\mathrm{source}(e), \mathrm{target}(e)) \to \prod_{\mathrm{edges}(\Gamma')} X(\mathrm{source}(e), \mathrm{target}(e))$$

for any category X with $\mathrm{Ob}(X) = I$; we require any argument $x_e \in X(\mathrm{source}(e), \mathrm{target}(e))$ to enter exactly once.

Note that $\mathrm{As}_I^{\#}$ is a symmetric monoidal category if we put $\Gamma \otimes \Gamma' = \Gamma \coprod \Gamma'$ (disjoint union of edges with the same set of vertices). If I is a one-element set then $\mathrm{As}_I^{\#}$ is the category $\mathrm{As}^{\#}$ as in 5.

A map of sets $F : I_1 \to I_2$ induces a monoidal functor $F_* : \mathrm{As}_{I_1}^{\#} \to \mathrm{As}_{I_2}^{\#}$.

DEFINITION 5.2.2. *An infinity category A in a symmetric monoidal category $\mathfrak{C}$ with weak equivalences is a set I and a functor $A : \mathrm{As}_I^{\#} \to \mathfrak{C}$ together with a coassociative natural transformation*

$$\Delta(\Gamma, \Gamma') : A(\Gamma \coprod \Gamma') \to A(\Gamma) \otimes A(\Gamma')$$

which is a weak equivalence for all Γ, Γ' in $\mathrm{Ob}(\mathrm{As}_I^{\#})$.

5.3. Infinity 2-categories. Let $\mathfrak{C}$ be the category of complexes, of simplicial sets, or of topological spaces. For an infinity category A in $\mathfrak{C}$, define the homotopy category $\mathrm{Ho}(A)$ by

$$\mathrm{Ob}\mathrm{Ho}(A) = I; \ \mathrm{Ho}(A)(i,j) = H^0(A(i \to j))$$

in the case of complexes, or π_0 in the other cases. (By $i \to j$ we denote the graph with two vertices marked by i and j and one arrow from i to j).

Definition 5.3.1. *A morphism of infinity categories* $(I_1, A_1) \to (I_2, A_2)$ *is:*
a) a map of sets $F : I_1 \to I_2$;
b) a morphism of functors $A_1 \to A_2 \circ F_*$ *which is compatible with* Δ.
A morphism is by definition a weak equivalence if it induces an equivalence of homotopy categories and every morphism $A_1(\Gamma) \to A_2(F_*(\Gamma))$ *is a weak equivalence.*

The category of infinity categories is symmetric monoidal if one puts $(I_1, A_1) \otimes (I_2, A_2) = (I_1 \times I_2, A)$ where

$$A(\Gamma) = A_1(\Gamma_1) \otimes A_2(\Gamma_2);$$

here Γ_1 has one edge $i_1 \to j_1$ for every edge $(i_1, i_2) \to (j_1, j_2)$ and Γ_2 has one edge $i_2 \to j_2$ for every edge $(i_1, i_2) \to (j_1, j_2)$.

Definition 5.3.2. *An infinity two-category is an infinity category in the symmetric monoidal category of infinity categories (the monoidal structure and weak equivalences on the latter are defined above).*

5.4. Hochschild cochains as an infinity two-category. It is well known that categories form a two-category where one-morphisms are functors and two-morphisms are natural transformations. Associative algebras also form a two-category: one-morphisms between A and B are (A, B)-bimodules; two-morphisms between (A, B)-bimodules M and N are morphisms of bimodules. In other words, to any algebras A and B we can associate a category $\mathcal{C}(A, B) = (A, B) - \text{bimod}$; for any three algebras there is a functor

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{5.4}$$

that satisfies the associativity property; it sends (M, N) to $M \otimes_B N$. For any A, there is the unit object id_A of $\mathcal{C}(A)$ with respect to the above product. In fact $\mathrm{id}_A = A$ viewed as a bimodule. Note that

$$\mathrm{End}_{\mathcal{C}(A,A)}(\mathrm{id}(A)) = \mathrm{Center}(A).$$

Note also that the two-category of algebras maps to the two-category of categories: an algebra A maps to the category $A - \text{mod}$, and a bimodule M to the functor $M \otimes -$.

Our aim is to construct an infinity version of the above, namely an infinity 2-category whose objects are DG categories.

5.4.1. The construction of the infinity 2-category of Hochschild cochains. Let I be any set of DG categories. We first define the infinity-category $\mathcal{C}$ in the category of DG categories with the set of objects I. To do that, for any directed graph Γ with set of vertices I and with finitely many edges, put

$$\mathcal{C}(\Gamma) = \mathrm{Cobar}(\bigotimes_{\mathrm{edges}(\Gamma)} \mathrm{Bar}(\mathbf{C}(\mathrm{source}(e), \mathrm{target}(e)))) \tag{5.5}$$

(recall that DG categories $\mathbf{C}(A, B)$ were defined in 4.7.4). For any $f : \Gamma \to \Gamma'$ in $\mathrm{As}^{\#}(\Gamma, \Gamma')$, the corresponding map is induced by the $\bullet$ product and by insertion of 1. The coproduct $\Delta : \mathcal{C}(\Gamma \coprod \Gamma') \to \mathcal{C}(\Gamma) \otimes \mathcal{C}(\Gamma')$ is a partial case of the coproduct (4.6).

5.4.2. The module structure. Similarly to the above, one can define the notion of an infinity algebra and an infinity module in a monoidal category $\mathcal{C}$ with weak equivalences. Such an object is an infinity algebra $\{A(n)\}$ and a collection of objects $\{M(n-1,1)\}$ subject to various axioms that we leave to the reader. (Alternatively, one can replace the operad $\mathcal{P}$ in Definition 5.1.2 by the colored operad As^+). Similarly one can define an infinity functor from an infinity category to $\mathcal{C}$. The latter is a collection of objects $M(\Gamma, v)$ where Γ is a graph as above and v is a vertex of Γ.

Recall that we have constructed in 5.4.1 an infinity category $\mathcal{C}$ in the category DGCat of DG categories such that the its value at the graph $A \to B$ with two vertices and one edge is equal to

$$\mathcal{C}(A \to B) = \mathrm{CobarBar}(\mathbf{C}(A,B)).$$

One can extend this definition by constructing an infinity functor M from $\mathcal{C}$ to DGCat such that $M(A) = \mathrm{CobarBar}(A)$. To do this, just observe that there is a morphism of DG cocategories

$$\mathrm{Bar}(\mathbf{C}(A,B)) \times \mathrm{Bar}(A) \to \mathrm{Bar}(B) \tag{5.6}$$

that agrees with the product from Lemma 4.7.3.

5.4.3. The A_∞ structure on chains of cochains. As a consequence of the above, we get

PROPOSITION 5.4.1. *1) The complex $C_{-\bullet}(C^\bullet(A,A),(C^\bullet(A,A))$ carries a natural A_∞ algebra structure such that*

- *All m_n are $k[[u]]$-linear, (u)-adically continuous*
- *$m_1 = b + \delta + uB$ For $x,\ y \in C_\bullet(A)$:*
- *$(-1)^{|x|}m_2(x,y) = (\mathrm{sh} + u\,\mathrm{sh}')(x,y)$*
  For $D,\ E \in C^\bullet(A,A)$:
- *$(-1)^{|D|}m_2(D,E) = D \smile E$*
- *$m_2(1 \otimes D,\ 1 \otimes E) + (-1)^{|D||E|}m_2(1 \otimes E,\ 1 \otimes D) = (-1)^{|D|}1 \otimes [D,\ E]$*
- *$m_2(D,\ 1 \otimes E) + (-1)^{(|D|+1)|E|}m_2(1 \otimes E,\ D) = (-1)^{|D|+1}[D,\ E]$*

(we use the shuffle products as defined in 2.3.1).

2) The complex $C_{-\bullet}(A,A)$ carries a natural structure of an A_∞ module over the A_∞ algebra from 1), such that

- *All μ_n are $k[[u]]$-linear, (u)-adically continuous*
- *$\mu_1 = b + uB$ on $C_\bullet(A)[[u]]$*
  For $a \in C_\bullet(A)[[u]]$:
- *$\mu_2(a, D) = (-1)^{|a||D|+|a|}(i_D + uS_D)a$*
- *$\mu_2(a, 1 \otimes D) = (-1)^{|a||D|}L_D a$*
  For $a,\ x \in C_\bullet(A)[[u]]$: $(-1)^{|a|}\mu_2(a,x) = (\mathrm{sh} + u\,\mathrm{sh}')(a,x)$

3) The above structures extend to negative cyclic complexes $\mathrm{CC}_\bullet^-$.

PROOF. In fact, the above is true if we replace $C_{-\bullet}$ or $\mathrm{CC}_\bullet^-$ by any functor which is multiplicative, i.e admits an associative Künneth map. ∎

REMARK 5.4.2. An A_∞ structure as above was constructed in [**114**]. It was used in [**33**] to construct a Gauss-Manin connection on the periodic cyclic complex.

5.5. Hochschild chains.

5.5.1. A 2-category with a trace functor. The two-category of algebras and bimodules has an additional structure: a functor $\mathrm{Tr}_A : \mathcal{C}(A, A) \to k - \mathrm{mod}$ such that the two functors

$$\mathcal{C}(A, B) \times \mathcal{C}(B, A) \to \mathcal{C}(A, A) \xrightarrow{\mathrm{Tr}_A} k - \mathrm{mod} \tag{5.7}$$

and

$$\mathcal{C}(A, B) \times \mathcal{C}(B, A) \to \mathcal{C}(B, A) \times \mathcal{C}(A, B) \to \mathcal{C}(B, B) \xrightarrow{\mathrm{Tr}_B} k - \mathrm{mod} \tag{5.8}$$

coincide. Here the first functor from the left in (5.7) and the second from the left in (5.8) are the products as in (5.4); the first functor from the left in (5.8) is the permutation of factors. We call a two-category with a functor as above *a two-category with a trace functor.*

For the two-category of algebras, the trace functor is defined as

$$\mathrm{Tr}_A(M) = M/[A, M] = M \otimes_{A\otimes A^{\mathrm{op}}} A = H_0(A, M) \tag{5.9}$$

5.5.2. A dimodule over a 2-category. When we consider only those bimodules that come from morphisms of algebras, we get another algebraic structure on the two-category of algebras.

For an (A, B)-bimodule M, put

$$M^\vee = \mathrm{Hom}(M, B) \tag{5.10}$$

which is a (B, A)-bimodule. We have a morphism of (B, B)-bimodules

$$M^\vee \otimes_A M \to B \tag{5.11}$$

For bimodules of the type that we will consider below, there is also a morphism of (A, A)-bimodules

$$A \to M \otimes_B M^\vee \tag{5.12}$$

such that the compositions

$$M = A \otimes_A M \to (M \otimes_B M^\vee) \otimes_A M \xrightarrow{\sim} M \otimes_B (M^\vee \otimes_A M) \to M \tag{5.13}$$

$$M^\vee = M^\vee \otimes_A \mathbf{1} \to M^\vee \otimes_A (M \otimes_B M^\vee) \xrightarrow{\sim} (M^\vee \otimes_A M) \otimes_B M^\vee \to M^\vee \tag{5.14}$$

are the identity morphisms. There is the second way to define a dual bimodule; namely, for an (A, B)-bimodule M, define a (B, A)-bimodule

$$M^\dagger = \mathrm{Hom}_A(M, A). \tag{5.15}$$

There are bimodule morphisms $M \to M^{\vee\dagger}$ and $M \to M^{\dagger\vee}$. The first one is an isomorphism for $M =_f B$ as above, the second for $M = B_f \xrightarrow{\sim} ({}_fB)^\vee$. Put

$$\langle M, N\rangle = M \otimes_{B\otimes A^{\mathrm{op}}} N^\vee = (M \otimes_B N^\vee) \otimes_{A\otimes A^{\mathrm{op}}} A \tag{5.16}$$

Let us describe the pairing $\langle M, N\rangle$, and the algebraic structure it is an example of, in the special case when our bimodules are of the form ${}_fB$ where f is a homomorphism of algebras. Denote, as above, by ${}_fB_g$ the algebra B viewed as an A-bimodule on which A acts on the left via f and on the right via g. Here f and g are two homomorphisms $A \to B$. We have

$$\langle {}_gB, {}_fB\rangle = \mathrm{Tr}_B({}_fB_g) = B/\langle f(a)b - bg(a) | a \in A\rangle. \tag{5.17}$$

Denote

$$T(A, B)(f, g) = \langle {}_gB, {}_fB\rangle \tag{5.18}$$

Note also that

$$\mathcal{C}(A,B)(f,g) = \mathrm{Hom}_{A-B}({}_fB,{}_gB) = \{b \in B | \forall a \in A : g(a)b = bf(a)\}. \tag{5.19}$$

The collection $T(A,B)$ of k-modules $T(A,B)(f,g)$ carries the following structure.

1. For every A and B, the collection $T(A,B)$ is a bimodule over the category $\mathcal{C}(A,B)$.

2. For every three algebras A, B, C, there are pairings

$$T(A,B)(g_0,g_1) \times \mathcal{C}(B,C)(f_0,f_1) \to T(A,C)(f_0g_0,f_1g_1) \tag{5.20}$$

and

$$T(A,C)(f_0g_0,f_1g_1) \times \mathcal{C}(A,B)(g_1,g_0) \to T(B,C)(f_0,f_1) \tag{5.21}$$

such that the following three compatibility conditions hold:

(1) the functors

$$T(A,B)(h_0,h_1) \times \mathcal{C}(B,C)(g_0,g_1) \times \mathcal{C}(C,D)(f_0,f_1) \to$$
$$T(A,B)(h_0,h_1) \times \mathcal{C}(B,D)(g_0h_0,g_1h_1) \to T(A,D)(f_0g_0h_0,f_1g_1h_1)$$

and

$$T(A,B)(h_0,h_1) \times \mathcal{C}(B,C)(g_0,g_1) \times \mathcal{C}(C,D)(f_0,f_1) \to$$
$$T(A,C)(g_0h_0,g_1h_1) \times \mathcal{C}(C,D)(h_0,h_1) \to T(A,D)(f_0g_0h_0,f_1g_1h_1)$$

are equal;

(2) the functors

$$T(A,D)(f_0g_0h_0,f_1g_1h_1) \times \mathcal{C}(A,B)(h_1,h_0) \times \mathcal{C}(B,C)(g_1,g_0) \to$$
$$T(A,D)(f_0g_0h_0,f_1g_1h_1) \times \mathcal{C}(A,C)(g_1h_0,g_1h_1) \to T(C,D)(f_0,f_1)$$

and

$$T(A,D)(f_0g_0h_0,f_1g_1h_1) \times \mathcal{C}(A,B)(h_1,h_0) \times \mathcal{C}(B,C)(g_1,g_0) \to$$
$$T(B,D)(f_0g_0,f_1g_1) \times \mathcal{C}(B,C)(g_1,g_0) \to T(C,D)(f_0,f_1)$$

are equal;

(3)

$$T(A,C)(g_0h_0,g_1h_1) \times \mathcal{C}(A,B)(h_1,h_0) \times \mathcal{C}(C,D)(f_0,f_1) \to$$
$$T(B,C)(g_0,g_1) \times \mathcal{C}(C,D)(f_0,f_1) \to T(B,D)(f_0g_0,f_1g_1)$$

and

$$T(A,C)(g_0h_0,g_1h_1) \times \mathcal{C}(A,B)(h_1,h_0) \times \mathcal{C}(C,D)(f_0,f_1) \to$$
$$T(A,D)(f_0g_0h_0,f_1g_1h_1) \times \mathcal{C}(A,B)(h_1,h_0) \to T(B,D)(f_0g_0,f_1g_1)$$

are equal.

3. The pairings (5.20), (5.21) are compatible with the $\mathcal{C}(A,B)$-bimodule structures on $T(A,B)$.

We call a 2-category and a collection of $T(A,B)(f,g)$ subject to the conditions above *a 2-category with a dimodule* (for want of a better term).

When $\mathcal{C}$ is the 2-category of algebras and bimodules, and $T(A,B)(f,g)$ are as in (5.18), then the action (5.20) is defined as

$$b \otimes c \mapsto f_1(b)c = cf_0(b) \tag{5.22}$$

for $b \in {}_{g_1}B_{g_0}$ and $c \in \mathcal{C}(B,C)(f_0,f_1)$; the action (5.21) is defined as

$$c \otimes b \mapsto f_1(b)c \sim cf_0(b) \in T(B,C)(g_1,g_0) \tag{5.23}$$

for $b \in \mathcal{C}(A,B)(g_1,g_0) = \{b \in B | \forall A : g_1(a)b = bg_0(a)$ and $c \in_{f_1 g_1} C_{f_0 g_0}$.

The definition of a dimodule is rather peculiar. If we replace categories $\mathcal{C}(A,B)$ by sets, and therefore consider a category $\mathcal{C}$ instead of a two-category, we get the definition of a $(\mathcal{C}^{\mathrm{op}}, \mathcal{C})$-bimodule. In the case of 2-categories that we are working with, the notion of a dimodule is more subtle. If we put

$$T^{\mathrm{dual}}(A,B)(f,g) = \mathrm{Hom}_k(T(A,B)(g,f),k)$$

then a dimodule defines two compatible actions

$$T(A,B) \times \mathcal{C}(B,C) \to T(A,C)$$

$$\mathcal{C}(A,B) \times T^{\mathrm{dual}}(B,C) \to T^{\mathrm{dual}}(A,C)$$

For any dimodule T over a 2-category $\mathcal{C}$, the action (5.21) of the morphism id_g, $f \in \mathrm{Ob}\, C(A,B)$, defines the morphism the action (5.20) of the morphism id_f, $f \in \mathrm{Ob}\, C(B,C)$, defines the morphism

$$f_* : T(A,B)(g_0,g_1) \to T(A,C)(fg_0, fg_1); \tag{5.24}$$

the action (5.21) of the morphism id_g, $g \in \mathrm{Ob}\, C(A,B)$, defines the morphism

$$g^* : T(A,C)(f_0 g, f_1 g) \to T(B,C)(f_0, f_1). \tag{5.25}$$

Our dimodule T has the following extra property (which does not seem to follow from the axioms).

Lemma 5.5.1. *Let $f_0, f_1 : B \to C$ and $g_0, g_1 : A \to B$ be one-morphisms in $\mathcal{C}$ such that $f_0 g_0 = f_1 g_1$. Then the diagram*

$$\begin{array}{ccc}
T(A,B)(g_0,g_1) & \xrightarrow{f_{1*}} & T(A,C)(f_1g_0, f_1g_1) \\
{\scriptstyle f_{0*}}\downarrow & & \downarrow{\scriptstyle =} \\
T(A,C)(f_0g_0, f_0g_1) & \longrightarrow & T(A,C)(f_1g_0, f_0g_0) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle g_0{}^*} \\
T(A,C)(f_1g_1, f_0g_1) & \xrightarrow{g_1{}^*} & T(B,C)(f_1, f_0)
\end{array}$$

is commutative.

Proof. In fact, for $b \in_{g_1} B_{g_0}$, $g_1{}^* f_{0*}(b) = f_0(b) \in {}_{f_0}C_{f_1}$; $g_0{}^* f_{1*}(b) = f_1(b) \in {}_{f_0}C_{f_1}$; the two are equal in $H_0(B, {}_{f_0}C_{f_1})$ (their difference is equal to the Hochschild chain differential of $1 \otimes b$; here is the origin of the cyclic differential B, see below). ☐

5.5.3. The higher structure on Hochschild chains: the first step. We expect that, when we replace $\mathcal{C}(A,B)(f_0,f_1)$ by $C^\bullet(A, {}_{f_1}B_{f_0})$ and $T(A,B)(f_0,f_1)$ by $C_\bullet(A, {}_{f_1}B_{f_0})$, the result will carry a structure of an infinity dimodule with property (5.5.1). Observe first that the morphisms (5.24), (5.25) can be written down easily:

$$f_*(b_0 \otimes a_1 \ldots \otimes \otimes a_n) = f(b_0) \otimes a_1 \ldots \otimes a_n; \tag{5.26}$$

$$g^*(c_0 \otimes a_1 \otimes \ldots \otimes a_n) = c_0 \otimes g(a_1) \otimes \ldots \otimes g(a_n). \tag{5.27}$$

5.5.4. The origin of the differential B. Consider the statement of Lemma 5.5.1 in the partial case $A = B = C$, $f_1 = g_0 = f$, $g_1 = f_0 = \mathrm{id}$. We see that the two maps

$$\mathrm{id},\ f : C_\bullet(A, {}_fA) \to C_\bullet(A, {}_fA)$$

should be homotopic. Here

$$f(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = f(a_0) \otimes f(a_1) \otimes \ldots \otimes f(a_n).$$

In particular, $C_\bullet(A, A)$ should carry an endomorphism of degree plus one. Such a homotopy can be easily written down as

$$B(f)(a_0 \otimes a_1 \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes f(a_i) \otimes \ldots \otimes f(a_n) \otimes a_0 \ldots \otimes a_{i-1} \tag{5.28}$$

6. Deligne conjecture

6.1. Deligne conjecture for Hochschild cochains. In the early 90s, Deligne conjectured that Hochschild cochains form a homotopy algebra over the operad of chain complexes of the little discs operad. This conjecture was proved by McClure and Smith in [**88**]. Subsequent proofs are contained in [**4**], [**6**], [**63**], [**67**], [**74**], [**?**], [**104**].

THEOREM 6.1.1. *For any A_∞ category A there is an action of a cofibrant resolution of the DG operad $C_{-\bullet}(\mathrm{LD}_2)$ on the Hochschild complex $C^\bullet(A, A)$ such that at the level of cohomology:*

(1) *the generator of $H_0(\mathrm{LD}_2(2))$ acts by the cup product on $H^\bullet(A, A)$;*
(2) *the generator of $H_1(\mathrm{LD}_2(2))$ acts by the Gerstenhaber bracket on $H^\bullet(A, A)$.*
(3) *This structure is natural with respect to isomorphisms.*

6.2. Deligne conjecture for Hochschild chains. An extension of the Deligne conjecture to chains maintains that the pair of complexes of Hochschild cochains and chains is a homotopy algebra over the two-colored operad of little discs and cylinders.

THEOREM 6.2.1. *For any A_∞ category A there is an action of a cofibrant resolution of the DG operad $C_{-\bullet}(\mathrm{LD}_2)$ on the pair of Hochschild complexes $(C^\bullet(A, A), C_{-\bullet}$ (A, A) such that at the level of cohomology:*

(1) *the generator of $H_0(\mathrm{LC}(1, 1))$ acts by the pairing $H^\bullet(A, A) \otimes H_{-\bullet}(A, A) \to H_{-\bullet}(A, A)$;*
(2) *the generator of $H_1(\mathrm{LC}(1, 1))$ acts by the pairing $H^\bullet(A, A) \otimes H_{-\bullet}(A, A) \to H_{-\bullet+1}(A, A)$.*
(3) *This structure is natural with respect to isomorphisms.*

7. Formality of the operad of little two-discs

7.1. Associators. We follow the exposition in [**3**], [**105**], and [**97**].

7.1.1. The operad in categories PaB. Define the category $\mathbf{PaB}(n)$ as follows. Its object is a parenthesized permutation, i.e. a pair (σ, π) of a permutation $\sigma \in S_n$ and a parenthesization π of length n. A parenthesization is by definition an element of the free non-associative monoid with one generator $\bullet$. Example ($n = 6$):

$$\pi = ((\bullet\bullet)((\bullet\bullet)(\bullet\bullet))) \tag{7.1}$$

A morphism from (σ_1, π_1) to (σ_2, π_2) is an element of the braid group B_n whose projection to S_n is equal to $\sigma_2^{-1}\sigma_1$. The composition of morphisms is given by the multiplication of braids.

To describe the operadic structure, it is more convenient to use a slightly different definition of $\mathbf{PaB}(n)$. A parenthesization of a finite ordered set A is a parenthesization of length $n = |A|$ where the jth symbol $\bullet$ is replaced by a_j for all j.

For two total orders $<_1$ and $<_2$ on a finite set A, a pure braid between $(A, <_1)$ and $(A, <_2)$ is a braid whose lower ends are decorated by elements of A in the order $<_1$, whose upper ends are decorated by elements of A in the order $<_2$, and whose strands all go from a to the same element a. For a finite set A, the category $\mathbf{PaB}(A)$ is defined as follows:

(1) Objects of $\mathbf{PaB}(A)$ are pairs $(<, \pi)$ where $<$ is a total order on A and a parenthesization of A;
(2) a morphism from $(<_1, \pi_1)$ to $(<_2, \pi_2)$ is a pure braid from $(A, <_1)$ to $(A, <_2)$;
(3) the composition is the multiplication of braids.

Now let us define the operadic composition. Let A and B be totally ordered finite sets. Consider the surjection $A \coprod B \to A \coprod \{c\}$ that is the identity on A and that sends all elements of B to c. The operadic composition

$$\mathbf{PaB}(B) \times \mathbf{PaB}(A \coprod \{c\}) \to \mathbf{PaB}(A \coprod B) \tag{7.2}$$

corresponding to this surjection acts as follows: Let $<_1$ be a total order on B, π_1 a parenthesization of B, $<_2$ a total order on $A \coprod \{c\}$, and π_2 a parenthesization of $A \coprod \{c\}$. Then the value of the functor (7.2) on $((<_1, \pi_1), (<_2, \pi_2)$ is $(<, \pi)$ where

(1) $<$ is the total order for which $a < a'$ iff $a <_2 a'$; $b < b'$ iff $b <_1 b'$; $a < b$ iff $a <_2 c$;
(2) π is obtained from the parenthesization Π_2 by replacing the symbol c with the set B, parenthesized by π_1.

Note that the operad of sets $\mathrm{Ob}\,\mathbf{PaB}$ is the free operad generated by one binary operation. At the level of morphisms, let γ be a pure braid between $(B, <_1)$ and $(B, <_1')$; let γ' be a pure braid between $(A \coprod \{c\}, <_2)$ and $(A \coprod \{c\}, <_2')$. The functor (7.2) sends (γ, γ') to γ'' defined as γ' in which the strand from c to c is replaced by the pure braid γ.

7.1.2. The operad in Lie algebras $\mathfrak{t}$. For a finite set A, let $\mathfrak{t}(A)$ be the Lie algebra with generators t_{ij}, i, $j \in A$, subject to relations

$$[t_{ij}, t_{kl}] = 0 \tag{7.3}$$

if i, j, k, l are all different;

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \tag{7.4}$$

if i, j, k are all different. We put $\mathfrak{t}(n) = \mathfrak{t}(\{1, \ldots, n\})$. These Lie algebras form an operad in the category of Lie algebras where the monoidal structure is the direct sum. The operadic compositions are uniquely defined by the compositions $\circ_j : \mathfrak{t}(m) \oplus \mathfrak{t}(n) \to \mathfrak{t}(n + m - 1)$ acting as follows. Let A and B be finite sets. Consider the surjection $A \coprod B \to A \coprod \{c\}$ that is the identity on A and that sends all elements of B to c. The operadic composition $\mathfrak{t}(B) \oplus \mathfrak{t}(A \coprod \{c\}) \to \mathfrak{t}(A \coprod B)$ corresponding to this surjection acts as follows:

$$(t_{bb'}, t_{aa'}) \mapsto t_{bb'} + t_{aa'}; (t_{bb'}, t_{ac}) \mapsto t_{bb'} + \sum_{b'' \in B} t_{ab''} \tag{7.5}$$

for a, $a' \in A, b$, $b' \in B$. The action of the symmetric group on $U(\mathfrak{t}(n))$ is by permutation of pairs of indices (ij).

The operad $\mathfrak{t}$ gives rise to the operads $U(\mathfrak{t})$ and $\widehat{U(\mathfrak{t})}$ in the category of algebras and to the operad $\widehat{U(\mathfrak{t})}^{\mathrm{group}}$ in the category of groups. Here $U(\mathfrak{t})$ is the universal enveloping algebra of $\mathfrak{t}$, $\widehat{U(\mathfrak{t})}$ its completion with respect to the augmentation ideal, and $\widehat{U(\mathfrak{t})}^{\mathrm{group}}$ the set of grouplike elements of this completion (with respect to the coproduct for which all t_{ij} are primitive). Since every group is a category with one object, we can consider $\widehat{U(\mathfrak{t})}^{\mathrm{group}}$ as an operad in categories.

7.1.3. Definition of an associator. Let σ be the morphism in $\mathbf{PaB}(2)$ between (12) and (21) corresponding to the generator of the pure braidgroup $\mathrm{PB}_2 \xrightarrow{\sim} \mathbb{Z}$. Let a be the morphism in $\mathbf{PaB}(3)$ between (12)3) and (1(23)) corresponding to the trivial pure braid e.

DEFINITION 7.1.1. *An associator is a group element* $\Phi \in \widehat{U(\mathfrak{t}(3))}^{\mathrm{group}}$ *such that there is a morphism of operads in categories*

$$\mathbf{PaB} \to \widehat{U(\mathfrak{t})}^{\mathrm{group}}$$

that sends σ *to* $\exp(\frac{t_{12}}{2})$ *and* a *to* Φ.

The following theorem is essentially proven in [**36**]. It is formulated in the language of operads in [**105**] which is based on [**3**].

THEOREM 7.1.2. *There exists an associator* Φ.

REMARK 7.1.3. The above theorem is plausible because the relations (7.3), (7.4) are infinitesimal analogs of the defining relations in pure braid groups. Naïvely, $\mathfrak{t}(n)$ is the Lie algebra of PB_n. If the latter were nilpotent, the theorem would follow from rational homotopy theory. However, pure braid groups are far from being nilpotent, so the existence of an associator is not easy to prove.

7.1.4. Parenthesized braids and little discs. Consider the embedding

$$\mathrm{FM}_1 \to \mathrm{FM}_2 \tag{7.6}$$

induced by the embedding $\mathbb{R} \to \mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$. Note that the zero strata of FM_1 form an operad in sets that is isomorphic to the operad $\mathrm{Ob}\,\mathbf{PaB}$. We denote this suboperad of FM_1 (and FM_2) by $\mathbf{PaP}$. Denote by $\pi_1(\mathrm{FM}_2(n), \mathbf{PaP}(n))$ the full subcategory of the fundamental groupoid of FM_2 with the set of objects $\mathbf{PaP}(n)$. The collection of categories $\pi_1(\mathrm{FM}_2(n), \mathbf{PaP}(n))$ is an operad that we denote by $\pi_1(\mathrm{FM}_2, \mathbf{PaP})$.

LEMMA 7.1.4. *There is an isomorphism of operads in categories*

$$\pi_1(\mathrm{FM}_2, \mathbf{PaP}) \xrightarrow{\sim} \mathbf{PaB}$$

7.2. Formality of the operad of chains of little two-discs.

THEOREM 7.2.1. [**105**] *There is a chain of weak equivalences between DG operads* $C_{-\bullet}(\mathrm{LD}_2)$ *and* $H_{-\bullet}(\mathrm{LD}_2)$.

PROOF. There is a chain of equivalences of topological operads:

$$\text{Nerve }\pi_1(\mathrm{FM}_2, \mathbf{PaP}) \overset{\sim}{\leftarrow} \text{Nerve }\pi_1(\mathrm{FM}_2) \overset{\sim}{\leftarrow} \mathrm{FM}_2$$

The morphism of nerves on the left is induced by an equivalence of categories and therefore an equivalence. The map on the right is the classifying map which is an equivalence because all $\mathrm{FM}_2(n)$ are $K(\pi, 1)$. By Lemma 7.1.4, there is a chain of equivalences between Nerve **PaB** and FM_2. Applying the functor $C_{-\bullet}$ to this chain of equivalences, we see that it is enough to construct a chain of weak equivalences between $C_{-\bullet}(\text{Nerve }\mathbf{PaB})$ and $H_{-\bullet}(\mathrm{LD}_2)$, which is the same as $H_{-\bullet}(\mathrm{FM}_2)$. An associator Φ provides an equivalence

$$C_{-\bullet}(\text{Nerve }\mathbf{PaB}) \overset{\sim}{\to} C_{-\bullet}(\text{Nerve }\widehat{U(\mathfrak{t})}^{\text{group}}) \tag{7.7}$$

The right hand side of the above (if we replace singular chains of the geometric realization by simplicial chains) is the completed version of the chain complex of the group $\widehat{U(\mathfrak{t})}^{\text{group}}$. It is not difficult to define the chain of equivalence below, where Cobar stands for the cobar construction of the augmentation ideal or, what is the same, the standard complex for computing $\mathrm{Tor}_{-\bullet}^{U(\mathfrak{t})}(k, k)$.

$$C_{-\bullet}(\text{Nerve }\widehat{U(\mathfrak{t})}^{\text{group}}) \overset{\sim}{\to} \widehat{\mathrm{Cobar}}_{-\bullet}(U(\mathfrak{t})^+) \overset{\sim}{\leftarrow} \mathrm{Cobar}_{-\bullet}(U(\mathfrak{t})^+) \overset{\sim}{\leftarrow} C_{-\bullet}^{\mathrm{Lie}}(\mathfrak{t})$$

Finally, the right hand side is quasi-isomorphic to Gerst $\overset{\sim}{\to} H_{-\bullet}(\mathrm{LD}_2)$. □

7.3. Formality of the colored operad of little discs and cylinders.

THEOREM 7.3.1. *There are chains of weak equivalences between two-colored DG operads* $C_{-\bullet}(\mathrm{LC})$ *and* $H_{-\bullet}(\mathrm{LC})$, *and between* $C_{-\bullet}(\mathrm{LfC})$ *and* $H_{-\bullet}(\mathrm{LfC})$.

PROOF. The proof for the case of LC is virtually identical to the proof of Theorem 7.2.1. The proof for LfC requires a modification regarding the action of S^1. We omit it here. □

7.3.1. Gamma function of an associator. Note that

$$\mathfrak{t}(3) \overset{\sim}{\to} \mathrm{FreeLie}(t_{12}, t_{23}) \oplus k \cdot (t_{12} + t_{13} + t_{23})$$

It is easy to see [**36**], [**3**] that one can choose $\Phi = \Phi(t_{12}, t_{23})$. Since Φ is grouplike, $\log \Phi$ is a Lie series in two variables. Put

$$\log \Phi(x, y) = -\sum_{k=1}^{\infty} \zeta_\Phi(k+1)\, \mathrm{ad}_x^k(y) + O(y^2) \tag{7.8}$$

and

$$\Gamma_\Phi(u) = \exp(\sum_{n=2}^{\infty}(-1)^n \zeta_\Phi(n) u^n/n) \tag{7.9}$$

It is known that

$$\exp(\sum_{n=1}^{\infty} \zeta_\Phi(2n) u^{2n}) = -\frac{1}{2}(\frac{u}{e^u - 1} - 1 + \frac{u}{2}) \tag{7.10}$$

8. Noncommutative differential calculus

We deduce from 6 and 7 that the Hochschild cochain complex is an infinity Gerstenhaber algebra and, more generally, the pair of the cochain and the chain complexes is an infinity calculus. This admits the interpretation below, due to the fact that infinity algebras can be rectified (cf. 3).

8.1. The Gerst_∞ **structure on Hochschild cochains.** Below is the theorem from [**104**].

THEOREM 8.1.1. *For every associative algebra* A *and every associator* Φ, *there exists a* Gerst_∞ *algebra structure on* $C^\bullet(A, A)$, *natural with respect to isomorphisms of algebras, such that*

(1) *The induced Gerstenhaber algebra structure on* $H^\bullet(A, A)$ *is the standard one, defined by the cup product and the Gerstenhaber bracket as in 2.2.*
(2) *The underlying* L_∞ *structure on* $C^{\bullet+1}(A, A)$ *is given by the Gerstenhaber bracket.*

8.2. The Calc_∞ **structure on Hochschild chains.**

THEOREM 8.2.1. [**106**], [**35**] *For every associative algebra* A *and every associator* Φ, *there exists a* Calc_∞ *algebra structure on* $(C^\bullet(A, A), C_\bullet(A, A))$, *such that*

(1) *The induced calculus structure on* $(H^\bullet(A, A), H_\bullet(A, A))$ *is defined by the Gerstenhaber bracket, the cup product, the actions* ι_D *and* L_D *from 2.4, and the cyclic differential* B, *as in Example 3.6.6.*
(2) *The induced structure of an* L_∞ *module over* $C^{\bullet+1}(A, A)$ *on* $C_\bullet(A)[[u]]$ *is defined by the differential* $b + uB$ *and the DG Lie algebra action* L_D *from 2.4.*

8.3. Enveloping algebra of a Gerstenhaber algebra. The following construction is motivated by Example 3.6.5. For a Gerstenhaber algebra $\mathcal{V}^\bullet$, let $Y(\mathcal{V}^\bullet)$ be the associative algebra generated by two sets of generators i_a, L_a, $a \in \mathcal{V}^\bullet$, both i and L linear in a,

$$|i_a| = |a|; \ |L_a| = |a| - 1$$

subject to relations

$$i_a i_b = i_{ab}; \ \ [L_a, L_b] = L_{[a,b]};$$

$$[L_a, i_b] = i_{[a,b]}; \ L_{ab} = (-1)^{|b|} L_a i_b + i_a L_b$$

The algebra $Y(\mathcal{V}^\bullet)$ is equipped with the differential d of degree one which is defined as a derivation sending i_a to $(-1)^{|a|-1} L_a$ and L_a to zero.

For a smooth manifold M one has a homomorphism

$$Y(\mathcal{V}^\bullet(M)) \to \mathrm{D}(\Omega^\bullet(M))$$

The right hand side is the algebra of differential operators on differential forms on M, and the above homomorphism sends the generators i_a, L_a to corresponding differential operators on forms (cf. Example 3.6.5). It is easy to see that the above map is in fact an isomorphism.

8.3.1. Differential operators on forms in noncommutative calculus. Using a standard rectification argument one can restate Theorem 8.2.1 as follows:

THEOREM 8.3.1. *For every associative algebra A and every associator Φ, there exists a DG calculus $(\mathcal{V}^\bullet(A), \Omega^\bullet(A))$, natural with respect to isomorphisms of algebras, such that:*

1) there is a quasi-isomorphism of DGLA

$$\mathcal{V}^{\bullet+1}(A) \to C^{\bullet+1}(A, A)$$

and a compatible quasi-isomorphism of DG modules

$$(\Omega^\bullet(A)[[u]], \delta + ud) \to (C_\bullet(A, A)[[u]], b + uB)$$

where the right hand sides are equipped with the standard structures given by the Gerstenhaber bracket and the operation L_D; both maps are natural with respect to isomorphisms of algebras;

2) The statement 1) of Theorem 8.2.1 holds.

PROPOSITION 8.3.2. *There is an A_∞ quasi-isomorphism of A_∞ algebras, natural with respect to isomorphisms of algebras:*

$$Y(\mathcal{V}^\bullet(A)) \to C_{-\bullet}(C^\bullet(A, A), C^\bullet(A, A))$$

that extends to an A_∞ quasi-isomorphism

$$(Y(\mathcal{V}^\bullet(A))[[u]], \delta + ud) \to \mathrm{CC}^-_{-\bullet}(C^\bullet(A, A), C^\bullet(A, A))$$

(the A_∞ structures on the right hand side were defined in 5.4.1).

The proof is given in [**107**].

9. Formality theorems

For an associative algebra A and an associator Φ, let

$$(C^\bullet(A, A), C_\bullet(A, A))_\Phi$$

denote the Calc_∞ algebra given by Theorem 8.2.1. Let X be a smooth manifold (real, complex analytic, or algebraic over a field of characteristic zero).

THEOREM 9.0.3. *There is a Calc_∞ quasi-isomorphism between the sheaves of Calc_∞ algebras $(C^\bullet(\mathcal{O}_X, \mathcal{O}_X), C_\bullet(\mathcal{O}_X, \mathcal{O}_X))_\Phi$ and Calc_X such that:*

(1) *the induced isomorphism*

$$\mathbf{H}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \to H^\bullet(X, \wedge^\bullet T_X)$$

is given by

$$c \mapsto \iota(\sqrt{\widehat{A}}_\Phi(T_X)) I_{\mathrm{HKR}}(c);$$

(2) *the induced isomorphism*

$$\mathbf{H}_\bullet(\mathcal{O}_X, \mathcal{O}_X) \to H^{-\bullet}(X, \Omega)$$

is given by

$$c \mapsto \sqrt{\widehat{A}}_\Phi(T_X) \wedge I_{\mathrm{HKR}}(c)$$

where the left hand side stands for the hypercohomology of X in the sheaf of Hochschild complexes, and $\sqrt{\widehat{A}}_\Phi(T_X)$ is the characteristic class of the tangent bundle T_X corresponding to the symmetric power series $\Gamma_\Phi(x_1) \ldots \Gamma_\Phi(x_n)$. Here Γ_Φ denotes the gamma function of the associator Φ.

The proof can be obtained from [**118**], [**119**].

10. Deformation quantization

Let M be a smooth manifold. By a deformation quantization of M we mean a formal product

$$f * g = fg + \sum_{k=1}^{\infty} (i\hbar)^k P_k(f,g) \tag{10.1}$$

where P_k are bidifferential expressions, $*$ is associative, and $1 * f = f * 1 = f$ for all f. Given such a product (which is called a star product), we define

$$\mathbb{A}^\hbar(M) = (C^\infty(M)[[\hbar]], *) \tag{10.2}$$

This is an associative algebra over $\mathbb{C}[[\hbar]]$. By $\mathbb{A}^\hbar_c(M)$ we denote the ideal $C^\infty_c(M)[[\hbar]]$ of this algebra. An isomorphism of two deformations is by definition a power series $T(f) = f + (i\hbar)^k \sum_{k=1}^{\infty} T_k(f)$ where all T_k are differential operators and which is an isomorphism of algebras.

Given a star product on M, for $f,\ g \in C^\infty(M)$ let

$$\{f,g\} = P_1(f,g) - P_1(g,f) = \frac{1}{t}[f,g]|\hbar = 0. \tag{10.3}$$

This is a Poisson bracket corresponding to some Poisson structure on M. If this Poisson structure is defined by a symplectic form ω, we say that $\mathbb{A}^\hbar(M)$ is a deformation of the symplectic manifold $(M,\ \omega)$.

Recall the following classification result from [**27**], [**29**], [**44**], [**91**].

THEOREM 10.0.4. *Isomorphism classes of deformation quantizations of a symplectic manifold (M,ω) are in a one-to-one correspondence with the set*

$$\frac{1}{i\hbar}[\omega] + H^2(M, \mathbb{C}[[\hbar]])$$

where $[\omega]$ is the cohomology class of the symplectic structure ω.

In defining the Hochschild and cyclic complexes, we use $k = \mathbb{C}[[\hbar]]$ as the ring of scalars, and put

$$\mathbb{A}^\hbar(M)^{\otimes n} = \mathrm{jets}_\Delta\, C^\infty(M^n)[[\hbar]] \tag{10.4}$$

Sometimes we are interested in the homology defined using $\mathbb{C}$ as the ring of scalars. Then we use the standard definitions where the tensor products over $\mathbb{C}$ are defined by

$$\mathbb{A}^\hbar(M)^{\otimes_{\mathbb{C}} n} = \mathrm{jets}_\Delta\, C^\infty(M^n)[[\hbar_1, \ldots, \hbar_n]] \tag{10.5}$$

Let $\mathbb{A}^\hbar(M)$ be a deformation of a symplectic manifold (M,ω).

THEOREM 10.0.5. *There exists a quasi-isomorphism*

$$C_\bullet(\mathbb{A}^\hbar(M), \mathbb{A}^\hbar(M))[\hbar^{-1}] \to (\Omega^{2n-\bullet}(M)((\hbar)), i\hbar d)$$

which extends to a $\mathbb{C}[[\hbar,u]]$-linear, $(\hbar,u)$-adically continuous quasi-isomorphism

$$\mathrm{CC}^-_\bullet(\mathbb{A}^\hbar(M))[\hbar^{-1}] \to (\Omega^{2n-\bullet}(M)[[u]]((\hbar)), i\hbar d)$$

An analogous theorem holds for $\mathbb{A}^\hbar_c(M)$ if we replace $\Omega^\bullet$ by $\Omega^\bullet_c$.

10.0.2. The canonical trace. Combining the first map from Theorem 10.0.5 in the compactly supported case with integrating over M and dividing by $\frac{1}{n!}$, one gets the canonical trace of Fedosov

$$Tr : \mathbb{A}_c^\hbar(M) \to \mathbb{C}((\hbar))$$

It follows from Theorem 10.0.5 that, for M connected, this trace is unique up to multiplication by an element of $\mathbb{C}((i\hbar))$.

11. Applications of formality theorems to deformation quantization

11.1. Kontsevich formality theorem and classification of deformation quantizations. From Theorem 9.0.3 we recover the formality theorem of Kontsevich [**71**], [**72**]:

THEOREM 11.1.1. *For a C^∞ manifold X there exists an L_∞ quasi-isomorphism of DGLA*

$$\Gamma(X, \wedge^{\bullet+1}(T_X)) \to C^{\bullet+1}(C^\infty(X), C^\infty(X))$$

For a complex manifold X, or for a smooth algebraic variety X over a field of characteristic zero, there exists an L_∞ quasi-isomorphism of sheaves of DGLA

$$\wedge^{\bullet+1}(T_X) \to C^{\bullet+1}(\mathcal{O}_X, \mathcal{O}_X)$$

DEFINITION 11.1.2. *A formal Poisson structure on a C^∞ manifold X is a power series $\pi = \sum_{n=0}^\infty (i\hbar)^{n+1}\pi_n$ where π_n are bivector fields and $[\pi, \pi]_{\mathrm{Sch}} = 0$ (here $[\ ]_{\mathrm{Sch}}$ denotes the Schouten bracket, extended bilinearly to power series in $\hbar$ with values in multivector fields). An equivalence between two formal Poisson structures π and π' is a series $X = \sum_{n=1}^\infty (i\hbar)^{n+1} X_n$ such that $\pi' = \exp(L_X)\pi$.*

From Theorem 11.1.1 one deduces [**71**], [**72**]

THEOREM 11.1.3. *There is a bijection between isomorphism classes of deformation quantizations of a C^∞ manifold X and equivalence classes of formal Poisson structures on X.*

This theorem admits an analog for complex analytic manifolds and for smooth algebraic varieties in characteristic zero. The correct generalization of a deformation quantization is a formal deformation of the structure sheaf $\mathcal{O}_X$ as an *algebroid stack* (cf. [**66**], [**73**] for definitions).

THEOREM 11.1.4. [**11**], [**12**] *For any associator Φ, there is a bijection between isomorphism classes of deformation quantizations of a complex manifold X and equivalence classes of Maurer-Cartan elements of the DGLA*

$$(\hbar\Omega^{0,\bullet}(X, \wedge^{\bullet+1}T_X)[[\hbar]], \overline{\partial})$$

11.1.1. Hochschild cohomology of deformed algebras. Let π be a formal Poisson structure on a smooth manifold X. Denote by $\mathbb{A}^\pi$ the deformation quantization algebra given by Theorem 11.1.3. The Hochschild cochain complex $C^\bullet(\mathbb{A}^\pi, \mathbb{A}^\pi)$ is by definition the complex of multidifferential, $\mathbb{C}[[\hbar]]$-linear cochains. One deduces from Theorem 11.1.1

THEOREM 11.1.5. [**71**], [**72**] *There is an L_∞ quasi-isomorphism of DGLA*

$$(\Gamma(X, \wedge^{\bullet+1}(T_X))[[\hbar]], [\pi, -]_{\mathrm{Sch}}) \xrightarrow{\sim} C^\bullet(\mathbb{A}^\pi, \mathbb{A}^\pi)$$

11.2. Formality theorem for chains and the Hochschild and cyclic homology of deformed algebras. Note that, by Theorems 11.1.1 and 11.1.5, the Hochschild and negative cyclic complexes of $C^\infty(X)$, resp. of $\mathbb{A}^\pi$, are L_∞-modules over $\Gamma^{\bullet+1}(T_X)$, resp. over $(\Gamma(X,\wedge^{\bullet+1}(T_X))[[\hbar]],[\pi,-]_{\mathrm{Sch}})$.

THEOREM 11.2.1. [**31**], [**32**], [**98**].

(1) *There is a* $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous* L_∞ *quasi-isomorphism of DG modules over the DGLA* $\Gamma(X,\wedge^{\bullet+1}(T_X))$

$$\mathrm{CC}^-_{-\bullet}(C^\infty(X)) \xrightarrow{\sim} (\Omega^{-\bullet}[[u]], ud_{\mathrm{DR}})$$

whose reduction modulo u is an L_∞ *quasi-isomorphism*

$$C_{-\bullet}(C^\infty(X), C^\infty(X)) \xrightarrow{\sim} \Omega^{-\bullet}(X)$$

(2) *There is a* $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous* L_∞ *quasi-isomorphism of DG modules over the DGLA* $(\Gamma(X,\wedge^{\bullet+1}(T_X))[[\hbar]],[\pi,-]_{\mathrm{Sch}})$

$$\mathrm{CC}^-_{-\bullet}(\mathbb{A}_\pi) \xrightarrow{\sim} (\Omega^{-\bullet}(X)[[\hbar,u]], L_\pi + ud_{\mathrm{DR}})$$

whose reduction modulo u is an L_∞ *quasi-isomorphism*

$$C_{-\bullet}(\mathbb{A}^\pi,\mathbb{A}^\pi) \xrightarrow{\sim} (\Omega^{-\bullet}[[\hbar]], L_\pi)$$

11.2.1. The complex analytic case. Let π be a Maurer-Cartan element of the DGLA $(\hbar\Omega^{0,\bullet}(X,\wedge^{\bullet+1}T_X)[[\hbar]],\overline{\partial})$. Let $\mathbb{A}^\pi_\Phi$ be the algebroid stack deformation corresponding to π by Theorem 11.1.4. A Hochschild cochain complex $C^\bullet(\mathcal{A})$ of any algebroid stack $\mathcal{A}$ was defined in [**11**]; the complexes $C_-\bullet(\mathcal{A})$, $\mathrm{CC}^-_{-\bullet}(\mathcal{A})$, and $\mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathcal{A})$ were defined in [**14**]. As in the usual case, $C^{\bullet+1}(\mathcal{A})$ is a DGLA and the chain complexes are *DG* modules over it.

THEOREM 11.2.2. (1) *There is* L_∞ *quasi-isomorphism*

$$\Omega^{0,\bullet}(X,\wedge^{\bullet+1}(T_X))[[\hbar]],[\pi,-]_{\mathrm{Sch}}) \xrightarrow{\sim} C^{\bullet+1}(\mathbb{A}^\pi_\Phi)$$

(2) *There is a* $\mathbb{C}[[u]]$*-linear* (u)*-adically continuous quasi-isomorphism of* L_∞ *modules over the left hand side of the above formula*

$$\mathrm{CC}^-_{-\bullet}(\mathbb{A}^\pi_\Phi) \xrightarrow{\sim} (\Omega^{0,\bullet}(X,\Omega^\bullet_X)[[\hbar,u]],\overline{\partial}+L_\pi+u\partial)$$

11.3. Algebraic index theorem for deformations of symplectic structures. Let M be a smooth symplectic manifold. Let $\mathbb{A}^\hbar_M$ be a deformation quantization of a smooth symplectic manifold M. Recall that there exists canonical up to homotopy equivalence quasi-isomorphism

$$\mu^\hbar:\ CC^-_\bullet(\mathbb{A}^\hbar(M))[\hbar^{-1}] \to (\Omega^{2n-\bullet}(M)[[u]]((\hbar))], i\hbar d) \tag{11.1}$$

Localizing in u, we obtain a quasi-isomorphism

$$\mu^\hbar:\ CC^{\mathrm{per}}_\bullet(\mathbb{A}^\hbar(M))[\hbar^{-1}] \to (\Omega^{2n-\bullet}(M)((u))((\hbar)), i\hbar d) \tag{11.2}$$

(recall the notation from Definition 2.0.3).

DEFINITION 11.3.1. *The above morphisms are called the trace density morphisms.*

The index theorem compares the trace density morphism to the principal symbol morphism. To define the latter, consider the cyclic complex of the deformed algebra where the scalar ring is $\mathbb{C}$ instead of $\mathbb{C}[[\hbar]]$. Consider the composition

$$CC^-_\bullet(\mathbb{A}^\hbar M)) \to CC^-_\bullet(C^\infty(M)) \to$$

$$\to (\Omega^\bullet(M)[[u]], ud) \to (\Omega^\bullet(M)[[u]][[\hbar]], ud)$$

where the first morphism is reduction modulo $\hbar$, the second one is μ from Theorem 2.7.1, and the third one is induced by the embedding $\mathbb{C} \to \mathbb{C}[[\hbar]]$. We will denote this composition, followed by localization in $\hbar$, by

$$\mu : CC^{\mathrm{per}}_\bullet(\mathbb{A}^\hbar(M)) \to (\Omega^\bullet(M)((u))((\hbar)), ud) \tag{11.3}$$

To compare μ and $\mu^\hbar$, let us identify the right hand sides by the isomorphism

$$(\Omega^{2n-\bullet}(M)((u))((\hbar)), i\hbar d) \to (\Omega^\bullet(M)((u))((\hbar)), ud)$$

which is equal to $(\frac{t}{u})^{n-k}$ on $\Omega^k(M)((u))((\hbar))$. After this identification, we obtain two morphisms

$$\mu,\ \mu^t : CC^{\mathrm{per}}_\bullet(\mathbb{A}^t(M)) \to (\Omega^\bullet(M)((u))((\hbar)), ud)$$

where the left hand side is defined as the periodic cyclic complex with respect to the ground ring $\mathbb{C}$.

THEOREM 11.3.2. *At the level of cohomology,*

$$\mu^\hbar = \sum_{p=0}^{\infty} u^p(\widehat{A}(M)e^\theta)_{2p} \cdot \mu$$

where $\widehat{A}(M)$ is the $\widehat{A}$ class of the tangent bundle of M viewed as a complex bundle (with an almost complex structure compatible with the symplectic form), and $\theta \in \frac{1}{i\hbar}[\omega] + H^2(M, \mathbb{C}[[\hbar]])$ is the characteristic class of the deformation (cf. Theorem 10.0.4).

Note that the canonical trace $\mathrm{Tr}_{\mathrm{can}}$ is the composition of $\mu^\hbar$ with the integration $\Omega^{2n}((\hbar)) \to \mathbb{C}((\hbar))$. Let P and Q be $N \times N$ matrices over $\mathbb{A}^t(M)$ such that $P^2 = P$, $Q^2 = Q$, and $P - Q$ is compactly supported. Let P_0, Q_0 be reductions of P, Q modulo $\hbar$. They are idempotent matrix-valued functions; their images $P_0\mathbb{C}^N$, $Q_0\mathbb{C}^N$ are vector bundles on M. Applying $\mu^\hbar$ to the the difference of Chern characters of P and Q, we obtain the following index theorem of Fedosov [**44**] (cf. also [**91**]).

THEOREM 11.3.3.

$$\mathrm{Tr}_{\mathrm{can}}(P - Q) = \int_M (\mathrm{ch}(P_0\mathbb{C}^N) - \mathrm{ch}(Q_0\mathbb{C}^N))\widehat{A}(M)e^\theta$$

11.4. Algebraic index theorem. The algebraic index theorem compares two morphisms from the periodic cyclic homology of a deformed algebra to the de Rham cohomology of the underlying manifold.

11.4.1. The trace density map.

DEFINITION 11.4.1. *For a C^∞ manifold X, a formal Poisson structure π on X, and for the deformation quantization algebra $\mathbb{A}_\pi$, define the trace density map*

$$\mathrm{TR}\colon \mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}_\pi) \xrightarrow{\sim} (\Omega^{-\bullet}(X)[[\hbar]]((u)), ud_{\mathrm{DR}})$$

to be the composition

$$\mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}^\pi) \xrightarrow{\sim} (\Omega^{-\bullet}(X)[[\hbar]]((u)), L_\pi + ud_{\mathrm{DR}}) \xrightarrow{\sim} (\Omega^{-\bullet}(X)[[\hbar]]((u)), ud_{\mathrm{DR}})$$

where the map on the right is (the first component of) the first quasi-isomorphism (2), Theorem 11.2.1, localized with respect to u, and the map on the right is the isomorphism $\exp(\frac{\iota_\pi}{u})$.

11.4.2. The principal symbol map. Denote by $\mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}^\pi)_{\mathbb{C}}$ the periodic cyclic chain complex of $\mathbb{A}^\pi$ where the ring of scalars is defined as $\mathbb{C}$, not $\mathbb{C}[[\hbar]]$.

DEFINITION 11.4.2. *Define the principal symbol map*

$$\sigma\colon \mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}^\pi)_{\mathbb{C}} \xrightarrow{\sim} (\Omega^{-\bullet}(X)((u)), ud_{\mathrm{DR}})$$

to be the composition

$$\mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}^\pi)_{\mathbb{C}} \xrightarrow{\sim} \mathrm{CC}^{\mathrm{per}}_{-\bullet}(C^\infty(X)) \xrightarrow{\sim} (\Omega^{-\bullet}(X)((u)), ud_{\mathrm{DR}})$$

where the map on the left is induced by the corresponding morphism of algebras (reduction modulo $\hbar$, a quasi-isomorphism by the Goodwillie rigidity theorem), and the map on the right is the HKR quasi-isomorphism.

THEOREM 11.4.3. *For $a \in \mathrm{HC}^{\mathrm{per}}_{-\bullet}(\mathbb{A}^\pi)_{\mathbb{C}}$,*

$$\mathrm{TR}(a) = \iota(\sigma(a)) \wedge \sqrt{\widehat{A}}(T_X)_u$$

where $\iota : \Omega^{-\bullet}(X)((u)) \to \Omega^{-\bullet}(X)[[\hbar]]((u))$ is the inclusion and

$$\sqrt{\widehat{A}}(T_X)_u = (\sqrt{\widehat{A}}(T_X))_{2p} u^{\pm p}$$

11.4.3. The complex analytic case. One defines, exactly as in 11.4.1 and in 11.4.2, the quasi-isomorphisms

$$\mathrm{TR}_\Phi\colon \mathrm{CC}^{-}_{-\bullet}(\mathbb{A}^\pi_\Phi) \xrightarrow{\sim} (\Omega^{0,\bullet}(X, \Omega^\bullet_X)[[\hbar, u]], \overline{\partial} + u\partial)$$

and

$$\sigma_\Phi\colon \mathrm{CC}^{-}_{-\bullet}(\mathbb{A}^\pi_\Phi)_{\mathbb{C}} \xrightarrow{\sim} (\Omega^{0,\bullet}(X, \Omega^\bullet_X)((u)), \overline{\partial} + u\partial)$$

THEOREM 11.4.4. *For $a \in \mathrm{HC}^{-}_{-\bullet}(\mathbb{A}^\pi_\Phi)_{\mathbb{C}}$,*

$$\mathrm{TR}_\Phi(a) = i(\sigma_\Phi(a)) \wedge (\sqrt{\widehat{A}}_\Phi(T_X))_u$$

11.4.4. Algebraic index theorem for traces.

THEOREM 11.4.5. *Let $\mathbb{A}^\pi$ be the deformation quantization of a C^∞ manifold M corresponding to a formal Poisson structure π. Let $\mathrm{Tr}\colon \mathbb{A}^\pi_c \to \mathbb{C}[[\hbar]]$ be a trace on the subalgebra of compactly supported functions. There exists a Poisson trace $\tau : C^\infty(M)[[\hbar]] \to \mathbb{C}[[\hbar]]$ with respect to π such that, for any two idempotents P and Q in $\mathrm{Matr}_N(\mathbb{A}^\pi)$ such that $P - Q$ is compactly supported,*

$$\mathrm{Tr}(P - Q) = \langle \tau, \exp(\iota_\pi)(\mathrm{ch}(P_0 - Q_0)\widehat{A}^{\frac{1}{2}}(M)) \rangle$$

where $P_0 = P(\operatorname{mod}\hbar)$, $Q_0 = Q(\operatorname{mod}\hbar)$, *and* Tr *is extended to the trace on the matrix algebra by* $\operatorname{Tr}(a) = \sum \operatorname{Tr}(a_{ii})$.

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Some Elementary Operadic Homotopy Equivalences

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Abstract. In this work we present an elementary construction of an operad morphism that is also a homotopy equivalence between the operad given by the Fulton-MacPherson compactification of configuration spaces and the little n-disks operad. In particular, the construction gives an operadic homotopy equivalence between the associahedra and the little intervals. It can also be extended to the case of the Kontsevich compactification and the Voronov Swiss-cheese operad.

The little cubes operad $\mathcal{C}_n$ was introduced by Boardman and Vogt [**2**] and extensively used by many authors, including Peter May in his famous proof of the Recognition Principle of n-fold loop spaces [**11**]. On the other hand, the real version of the Fulton-MacPherson compactification of configuration spaces of points was defined by Axelrod and Singer (see [**1**] where the manifold with corners structure is presented in detail). In the case of Euclidean n-space, the Axelrod-Singer compactification results in an operad $\mathcal{F}_n$. This operad has also been studied by many authors. Here we will just mention Markl's characterization of $\mathcal{F}_n$ as an operadic completion [**9**] and Salvatore's proof of its cofibrancy [**13**]. It is also well known that $\mathcal{F}_1$ gives Stasheff's associahedra [**15**]. As a consequence of the cofibrancy proven by Salvatore, the operads $\mathcal{F}_n$ and $\mathcal{C}_n$ are related by the existence of an operad morphism $\nu : \mathcal{F}_n \to \mathcal{C}_n$ that is also a homotopy equivalence, i.e. an operadic homotopy equivalence. For more details and an extensive historical review, we refer the reader to [**10**].

In this work we construct an operadic homotopy equivalence between $\mathcal{F}_n$ and $\mathcal{D}_n$ explicitly by using elementary techniques, where $\mathcal{D}_n$ is the little disks analogue of the little cubes operad. The constructions also applies to the Swiss-cheese operad and the Kontesevich compactification. The Swiss-cheese operad was originally defined by Voronov in [**16**] and a slightly different definition was given by Kontsevich in [**7**]. The difference between the two versions of the Swiss-cheese operad from the point of view of algebras over Koszul operads is explored in detail in [**8**], where the second version is called the unital Swiss-cheese operad. In this paper we will

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restrict attention to the unital Swiss-cheese operad and show that it is operadically homotopy equivalent to the Kontsevich compactification.

In Section 1 we review the little disks and the (unital) Swiss-cheese operad. The manifold with corners structure, given by Axelrod and Singer, on the real version of the Fulton-MacPherson compactification of the configurations spaces is reviewed in Section 2 for the case of points in Euclidean space. The Kontsevich compactification is also defined in Section 2. The construction of the explicit operadic homotopy equivalence is given in Section 3. It is assumed that the reader has some familiarity with the Fulton-MacPherson or the Axelrod-Singer compactification.

1. Little disks and Swiss-cheese

1.1. Little disks. Let D^2 denote the standard unit disk in the complex plane $\mathbb{C}$. By a configuration of n disks in D^2 we mean a map

$$d : \coprod_{1\leqslant s\leqslant n} D^2_s \to D^2$$

from the disjoint union of n numbered standard disks $D^2_1, \dots, D^2_n$ to D^2 such that d, when restricted to each disk, is a composition of translations and dilations. The image of each restriction is called a little disk. The interiors of the little disks are required to be disjoint. The space of all configurations of n disks is denoted by $\mathcal{D}_2(n)$ and is topologized as a subspace of $(\mathbb{R}^2 \times \mathbb{R}^+)^n$ containing the coordinates of the center and radius of each little disk. The symmetric group acts on $\mathcal{D}_2(n)$ by renumbering the disks. For $n = 0$, we define $\mathcal{D}_2(0) = \emptyset$. The Σ-module $\mathcal{D}_2 = \{\mathcal{D}_2(n)\}_{n\geqslant 0}$ admits a well known structure of operad given by gluing configurations of disks into little disks, see [**10**].

1.2. Swiss-cheese. For $m, n \geqslant 0$ such that $m+n > 0$, let us define $\mathcal{SC}(n, m; \mathfrak{o})$ as the space of those configurations $d \in \mathcal{D}_2(2n + m)$ such that its image in D^2 is invariant under complex conjugation and exactly m little disks are left fixed by conjugation. A little disk that is fixed by conjugation must be centered at the real line; in this case it is called *open*. Otherwise, it is called *closed*. The little disks in $\mathcal{SC}(n, m; \mathfrak{o})$ are labeled according the following rules:

i) Open disks have labels in $\{1, \dots, m\}$ and closed disks have labels in $\{1, \dots, 2n\}$.
ii) Closed disks in the upper half-plane have labels in $\{1, \dots, n\}$. If conjugation interchanges the images of two closed disks, their labels must be congruent modulo n.

There is an action of $S_n \times S_m$ on $\mathcal{SC}(n, m; \mathfrak{o})$ extending the action of $S_n \times \{e\}$ on pairs of closed disks having modulo n congruent labels and the action of $\{e\} \times S_m$ on open disks. Figure 1 illustrates a point in the space $\mathcal{SC}(n, m; \mathfrak{o})$.

Definition 1.2.1 (Swiss cheese operad). The 2-colored operad $\mathcal{SC}$ is defined as follows. For $m, n \geqslant 0$ with $m + n > 0$, $\mathcal{SC}(n, m; \mathfrak{o})$ is the configuration space defined above and $\mathcal{SC}(0, 0; \mathfrak{o}) = \emptyset$. For $n \geqslant 0$, $\mathcal{SC}(n, 0; \mathfrak{c})$ is defined as $\mathcal{D}_2(n)$ and $\mathcal{SC}(n, m; \mathfrak{c}) = \emptyset$ for $m \geqslant 1$. The operad structure in $\mathcal{SC}$ is given by:

$$\circ^{\mathfrak{c}}_i : \mathcal{SC}(n, m; x) \times \mathcal{SC}(n', 0; \mathfrak{c}) \to \mathcal{SC}(n + n' - 1, 0; x), \text{ for } 1 \leqslant i \leqslant n$$
$$\circ^{\mathfrak{o}}_i : \mathcal{SC}(n, m; x) \times \mathcal{SC}(n', m'; \mathfrak{o}) \to \mathcal{SC}(n + n', m + m' - 1; x), \text{ for } 1 \leqslant i \leqslant m$$

When $x = \mathfrak{c}$ and $m = 0$, $\circ^{\mathfrak{c}}_i$ is the usual gluing of little disks in $\mathcal{D}_2$. If $x = \mathfrak{o}$, then $\circ^{\mathfrak{c}}_i$ is defined by gluing each configuration of $\mathcal{SC}(n', 0; \mathfrak{c})$ in the little disk labeled by

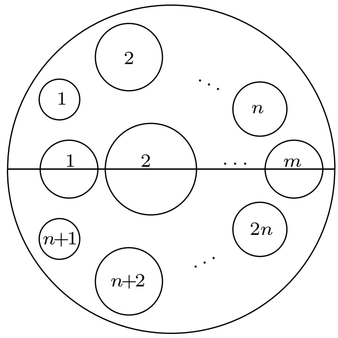

FIGURE 1. A configuration in $\mathcal{SC}(n, m; \mathfrak{o})$

i and then taking the complex conjugate of the same configuration and gluing the resulting configuration in the little disk labeled by $i + n$. Since $\mathcal{SC}(n, m; \mathfrak{c}) = \emptyset$ for $m \geqslant 1$, $\circ_i^{\mathfrak{o}}$ is only defined for $x = \mathfrak{o}$ and is given by the usual operation of $\mathcal{D}_2$.

2. Compactified Configurations Spaces

Let p, q be non-negative integers satisfying the inequality $2p+q \geqslant 2$. We denote by $\mathrm{Conf}(p, q)$ the configuration space of marked points on the upper closed half-plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geqslant 0\}$ with p points in the interior and q points on the boundary (real line)

$$\{(z_1, \dots, z_p, x_1, \dots, x_q) \in H^{p+q} \mid z_{i_1} \neq z_{i_2},\, x_{j_1} \neq x_{j_2}\ \mathrm{Im}(z_i) > 0,\, \mathrm{Im}(x_j) = 0\}.$$

The above configuration space $\mathrm{Conf}(p, q)$ is the Cartesian product of an open subset of H^p and an open subset of $\mathbb{R}^q$ and, consequently, is a $(2p+q)$-dimensional smooth manifold. Let $C(p, q)$ be the quotient of $\mathrm{Conf}(p, q)$ by the action of the group of orientation preserving affine transformations that leaves the real line fixed: $C(p, q) = \mathrm{Conf}(p, q)\big/(z \mapsto az + b)$ where $a, b \in \mathbb{R},\ a > 0$. The condition $2p + q \geqslant 2$ ensures that the action is free and thus $C(p, q)$ is a $(2p + q - 2)$-dimensional smooth manifold. In the case of points in the complex plane we have: $\mathrm{Conf}(n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, \forall i \neq j\}$ and $C(n) = \mathrm{Conf}(n)\big/(z \mapsto az + b)$ where $a \in \mathbb{R},\ a > 0$ and $b \in \mathbb{C}$. The manifold $C(n)$ is $(2n - 3)$-dimensional and its real Fulton-MacPherson compactification is denoted by $\overline{C(n)}$ (see [**1**]).

Let ϕ be the embedding $\phi : C(p, q) \longrightarrow C(2p + q)$ defined by

$$\phi(z_1, \dots, z_p, x_1, \dots, x_q) = (z_1, \bar{z}_1, \dots, z_p, \bar{z}_p, x_1, \dots, x_q) \tag{1}$$

where $\bar{z}$ denotes complex conjugation. The Fulton-MacPherson compactification of $C(p, q)$ is defined as the closure in $\overline{C(2p + q)}$ of the image of ϕ and is denoted by $\overline{C(p, q)}$. For a detailed combinatorial and geometrical study of $\overline{C(p, q)}$, we refer the reader to [**3**].

Both compactifications $\overline{C(n)}$ and $\overline{C(p, q)}$ have the structure of manifolds with corners whose boundary strata are labeled by trees (for details, see: [**5, 14, 7, 12**]). This labelling by trees defines a 2-colored operad structure, denoted by $\mathcal{H}_2$. The

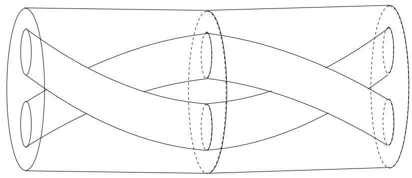

FIGURE 2. The manifold $\overline{C(3)}$ is obtained from the 3-manifold shown in this picture after identifying the two sides of the cylinder through the identity map.

set of colors is $\{\mathfrak{o}, \mathfrak{c}\}$ and

$$\mathcal{H}_2(p,q;x) := \begin{cases} \overline{C(p,q)}, & \text{if } x = \mathfrak{o} \text{ and } 2p+q \geqslant 2, \\ \overline{C(p)}, & \text{if } x = \mathfrak{c},\ q = 0 \text{ and } p \geqslant 2, \\ \emptyset, & \text{if } x = \mathfrak{c} \text{ and } q \geqslant 1. \end{cases} \tag{2}$$

In addition, we define $\mathcal{H}_2(1,0;\mathfrak{c})$ and $\mathcal{H}_2(0,1;\mathfrak{o})$ as one point spaces.

On the other hand, the sequence of manifolds $\{\overline{C(n)}\}_{n\geqslant 1}$ gives the well known operad $\mathcal{F}_2$, where $\overline{C(1)}$ is defined as the one-point space. The manifold $\overline{C(2)}$ is the circle S^1, while $\overline{C(3)}$ is the 3-manifold shown in Figure 2.

Since the three boundary components of $\overline{C(3)}$ are equivalent to tori, the boundary defines three ways of embedding $S^1 \times S^1$ into $\overline{C(3)}$ which is part of the operad structure of $\mathcal{F}_2$. The manifold $\overline{C(3)}$ is called the Jacobi manifold because its fundamental class provides a parametrization for the Jacobiator J, the homotopy operator for the Jacobi identity in a L_∞-algebra. More about $\overline{C(n)}$ in relation to L_∞-algebras and the Deligne-Knudsen-Mumford compactification can be found in [**6, 4**].

2.1. Coordinates on $\overline{C(n)}$. Before we proceed, let us review some properties of $\overline{C(n)}$. The codimension k boundary stratum of $\overline{C(n)}$ will be denoted by $\partial_k\overline{C(n)}$. It consists of a disjoint union of open submanifolds. More explicitly, we have

$$\partial_k\overline{C(n)} = \bigsqcup_{|T|=k} C(n)(T) \tag{3}$$

where the disjoint union is taken for all labelled trees T and $|T|$ denotes the number of internal edges of T. Each stratum $C(n)(T)$ is open in $\partial_k\overline{C(n)}$ and the strata satisfy the following properties:

1) If T is a corolla δ_k, then $C(n)(\delta_k)$ is homeomorphic to $C(k)$;
2) If $T = S_1 \circ_i S_2$, then $C(n)(S_1 \circ_i S_2)$ is homeomorphic to $C(n)(S_1) \times C(n)(S_2)$.

It is also worth mentioning that the closure of each stratum is given by

$$\overline{C(n)(T)} = \bigsqcup_{T'\to T} C(n)(T')$$

where $T' \to T$ means that T can be obtained from T' by contracting a finite number of internal edges. Hence, the closure of $\partial_k\overline{C(n)}$ is $\bigsqcup_{|T|\geqslant k} C(n)(T)$.

After modding out by translations and dilations, a configuration $\vec{z} \in C(p)$ may be seen as a sequence of pairwise distinct points $(z_1,\ldots,z_p) \in \mathbb{C}^{\times p}$ that is in normal form, i.e., such that $\sum_{i\in[p]} z_i = 0$ and $\sum_{i\in[p]} |z_i|^2 = 1$. In order to show that $\overline{C(n)}$

is a manifold with corners, Axelrod and Singer define ([**1**], formula 5.71), for each tree S with n leaves and k internal edges, a map

$$\mathcal{M}_S : C(n)(S) \times (\mathbb{R}_{\geqslant 0})^k \to \mathbb{C}^n. \tag{4}$$

In [**1**], the points are in a Riemannian manifold and the coordinate system is defined through the exponential. In our case, the manifold is $\mathbb{C}$ and the exponential map is hidden in the affine structure of the complex plane. The family of maps $\mathcal{M}_S$ are characterized by the following properties:

i) For a corolla δ_n, it is defined as the identity $\mathcal{M}_{\delta_n} = \mathrm{Id} : C(n) \to C(n) \subseteq \mathbb{C}^n$;
ii) if $\mathcal{M}_S$ and $\mathcal{M}_T$ are already defined, where S is a tree with n_1 leaves and k internal edges and T is a tree with n_2 leaves and l internal edges, then $\mathcal{M}_{S\circ_i T}$ is defined as follows. First identify $C(n)(S \circ_i T) \times (\mathbb{R}_{\geqslant 0})^{(k+l+1)} = C(n)(S) \times (\mathbb{R}_{\geqslant 0})^k \times C(n)(T) \times (\mathbb{R}_{\geqslant 0})^l \times R_{\geqslant 0}$, and then define

$$C(n)(S \circ_i T) \times (\mathbb{R}_{\geqslant 0})^{(k+l+1)} \xrightarrow{\mathcal{M}_S \times \mathcal{M}_T \times \mathrm{Id}_{\mathbb{R}}} C(n_1) \times C(n_2) \times R_{\geqslant 0} \xrightarrow{\gamma_i} \mathbb{C}^n$$

where $n = n_1 + n_2 - 1$ and $\gamma_i : C(n_1) \times C(n_2) \times R_{\geqslant 0} \to \mathbb{C}^n$ is given by

$$\gamma_i(\vec{x}, \vec{y}, t) = (x_1, \ldots, x_{i-1}, x_i + t(y_1, \ldots, y_{n_2}), x_{i+1}, \ldots, x_{n_1}), \tag{5}$$

where $\vec{x} = (x_1, \ldots, x_{n_1}) \in C(n_1)$ and $\vec{y} = (y_1, \ldots, y_{n_2}) \in C(n_2)$;
iii) the maps $\mathcal{M}_S$ are Σ_n-equivariant in the following sense

$$\mathcal{M}_{(S\sigma)} = (\mathcal{M}_S)\sigma, \quad \forall \sigma \in \Sigma_n,$$

where the σ-action on the left hand side is the right Σ_n-action on trees, while the action on the right hand side is the right Σ_n-action on $\mathbb{C}^n$.

REMARK 2.1.1. The reader should compare the above γ_i maps with Markl's pseudo-operad structure on $\mathrm{Conf}(n)$ [**9**]. The local charts on $\overline{C(n)}$ are given by the following proposition proven in [**1**].

PROPOSITION 2.1.2 (Axelrod-Singer). *For any n-tree S with k internal edges and any point $p \in C(n)(S)$, there is an open neighborhood U of p in $C(n)(S)$ and an open neighborhood W of 0 in $(\mathbb{R}_{\geqslant 0})^k$ such that $\mathcal{M}_S$ maps $U \times (W \setminus \partial W)$ into* $\mathrm{Conf}(n)$ *and is a diffeomorphism onto its image.*

Modding out by translations and dilations if necessary, we can assume that the local $\mathcal{M}_S$ maps assume values in $C(n)$. Axelrod and Singer showed that the local $\mathcal{M}_S$ maps can be continuously extended to maps of the form $\mathcal{M}_S : U \times W \to \overline{C(n)}$ and that this set of local $\mathcal{M}_S$ maps define a coordinate system on $\overline{C(n)}$ giving it a structure of manifold with corners (see also: [**12**]).

3. Operadic Homotopy Equivalence

The explicit homotopy equivalence will use the coordinate system defined by the local $\mathcal{M}_S$ maps. The basic idea is to define the map from $\overline{C(n)} \to \mathcal{D}_2(n)$ in the obvious way on the interior of $\overline{C(n)}$ and extend it to the boundary as an operad morphism. The continuity problem can be solved through a collar neighborhood around the boundary.

3.1. Collar Neighborhood. Let U be a collar neighborhood of $\partial\overline{C(n)}$ in $\overline{C(n)}$, with a homeomorphism

$$h : \partial\overline{C(n)} \times [0,1) \to U \subseteq \overline{C(n)}, \tag{6}$$

such that for any $p \in \partial\overline{C(n)}$ there is a neighborhood W of p such that $h(W \times [0,1))$ is a coordinate neighborhood of p in $\overline{C(n)}$. For any such $p \in \partial\overline{C(n)}$, the subset $h(p \times [0,1))$ is called the fiber of p in the collar U and $h(p \times (0,1))$ is the open fiber of p in the collar U. In view of the description of the coordinate system in $\overline{C(n)}$ given by the local $\mathcal{M}_S$ maps, all the configurations in a fiber are obtained from the infinitesimal components of p by applying the compositions of the form (5) a finite number of times for different values of i.

The projection $\pi : \mathcal{D}_2(n) \to C(n)$ taking each configuration of little disks into the configuration of their centers modded out by translations and dilations will be called the *center projection*.

Lemma 3.1.1. *For any $\vec{x} \in C(n)$, the inverse image $\pi^{-1}(\vec{x})$ is convex in $\mathcal{D}_2(n)$.*

Proof. It is enough to show that if d_1 and d_2 are two configurations of little disks in $\mathcal{D}_2(n)$ such that the centers of d_1 and d_2 define two configurations of points in $C(n)$ that are the same modulo translation and dilation then

$$\delta d_1 + (1-\delta)d_2 \tag{7}$$

gives a well defined configuration of little disks in $\mathcal{D}_2(n)$ for all $\delta \in [0,1]$. Indeed, note that the configurations can be presented in terms of centers and radii as follows:

$$d_1 = ((a_1,\alpha_1),\dots,(a_n,\alpha_n)) \quad \text{and} \quad d_2 = ((b_1,\beta_1),\dots,(b_n,\beta_n)).$$

The disjointness between the interiors of two disks is given by

$$\|a_i - a_j\| \geqslant \alpha_i + \alpha_j \quad \text{and} \quad \|b_i - b_j\| \geqslant \beta_i + \beta_j. \tag{8}$$

We denote by $\vec{a}$ and $\vec{b}$ the configurations of the centers in d_1 and d_2. Since $\vec{a} = \lambda\vec{b} + d$ for some $\lambda > 0$ and $d \in \mathbb{C}$, a straightforward computation shows that

$$\|(\delta a_i + (1-\delta)b_i) - (\delta a_j + (1-\delta)b_j)\| \geqslant (\delta\alpha_i + (1-\delta)\beta_i) + (\delta\alpha_j + (1-\delta)\beta_j).$$

Hence $\delta d_1 + (1-\delta)d_2$ is a well defined configuration of little disks in $\mathcal{D}_2(n)$. □

Corollary 3.1.2. *For all $p \in \partial\overline{C(n)}$ and $d_1, d_2 \in \pi^{-1}[h(p \times [0,1))]$ any convex combination*

$$\delta d_1 + (1-\delta)d_2, \qquad \delta \in [0,1]$$

gives a well defined configuration in $\mathcal{D}_2(n)$.

Proof. In the previous lemma we have seen that if the centers of little discs are related by translations and dilations, then the convex combination of the two configurations of little disks is well defined in $\mathcal{D}_2$. From the definition of the local $\mathcal{M}_S$ maps in the previous section, if the centers of d_1 and d_2 are in the same fiber of the tubular neighborhood, it follows that one is obtained from the other by a sequence of translations and dilations. The result then follows from the previous lemma. □

THEOREM 3.1.3. *There is an operad morphism $\nu : \mathcal{F}_2 \to \mathcal{D}_2$ such that the diagram*

$$\begin{array}{ccc} \mathcal{F}_2(n) & \xrightarrow{\nu(n)} & \\ \uparrow \iota & & \searrow \\ C(n) & \longrightarrow & \mathcal{D}_2(n) \end{array} \tag{9}$$

is homotopy commutative for each $n \geqslant 1$, where ι is the canonical inclusion $C(n) \hookrightarrow \overline{C(n)}$ and $C(n) \to \mathcal{D}_2(n)$ is a right inverse to the center projection $\pi : \mathcal{D}_2(n) \to C(n)$.

PROOF. The open submanifold $\overline{C(n)} \setminus \partial\overline{C(n)}$ of $\overline{C(n)}$ is homotopy equivalent to $C(n)$ which in turn is homeomorphic to the configuration space of n points in the plane modded out by translations and dilations. After modding out by translations and dilations, the configurations $(x_i)_{i\in[n]}$ can be thought of as configurations in normal form, i.e., such that $\sum_{i\in[n]} x_i = 0$ and $\sum_{i\in[n]} |x_i|^2 = 1$. By assigning to each point x_i a disk centered at it with radius $r = \min\{|x_i - x_j|, 1 - |x_i|\}_{1\leqslant i<j\leqslant n}$, we get a continuous map $\nu(n)_1 : C(n) \to \mathcal{D}_2(n)$ which is clearly a homotopy equivalence.

We will show that the $\nu(n)_1$ can be extended to an operad morphism on $\overline{C(n)}$. If $n = 2$ we are done, because $\overline{C(2)}$ is just the circle S^1, hence $\overline{C(2)} = C(2)$. Now, assuming that those maps are already extended for all $\overline{C(k)}$ with $k < n$, let us show how to extend them to $\overline{C(n)}$. Since the boundary of $\overline{C(n)}$ has only strata that are products of $\overline{C(k)}$ with $k < n$, we define $\nu(n)_2 : \partial\overline{C(n)} \to \mathcal{D}_2(n)$ as an operad morphism.

Now take a collar neighborhood U around the boundary in $\overline{C(n)}$ given by the coordinate system of the previous section and extend $\nu(n)_2$ to the collar neighborhood so that it is constant along each fiber. Since $\overline{C(n)}$ is compact, there is a continuous function $u : \overline{C(n)} \to [0,1]$ that is 1 on $\partial\overline{C(n)}$ and vanishes outside the collar neighborhood. We define $\nu(n) = (1-u)\nu(n)_1 + u\nu(n)_2$. For each p in the collar U, we have that $\nu(n)_1(p)$ and $\nu(n)_2(p)$ belong to $\pi^{-1}[h(p \times [0,1))]$, hence the map $\nu(n)$ is well defined by Corollary 3.1.2. So we have an operad morphism $\nu : \mathcal{F}_2 \to \mathcal{D}_2$. To see that the diagram (9) is homotopy commutative, we observe that it is strict commutative on $C(n) \setminus \overline{U}$ which in turn is a deformation retract of $\overline{C(n)} = \mathcal{F}_2(n)$. □

REMARK 3.1.4. Notice that $\nu(n) : \mathcal{F}_2(n) \to \mathcal{D}_2(n)$ is a homotopy equivalence for each $n \leqslant 1$. So ν is an operadic homotopy equivalence. With the same argument, one can construct operad morphisms $\nu_k : \mathcal{F}_k \to \mathcal{D}_k$ lying in analogous homotopy commutative diagrams. Hence each ν_k is an operadic homotopy equivalence. If $k = 1$, it is well known that $\mathcal{F}_1$ is the operad given by Stasheff's associahedra and is operadically homotopy equivalent to $\mathcal{D}_1$.

Analogous results hold in the case of the Kontsevich compactification and Swiss-cheese operad.

COROLLARY 3.1.5. *There is a morphism of 2-colored operads $\mu : \mathcal{H}_2 \to \mathcal{SC}$ which coincides with the morphism ν of Theorem 3.1.3 in color $\mathfrak{c}$ and is such that* *the diagram*

$$\begin{array}{ccc} \mathcal{H}_2(p,q;o) & & \\ \uparrow \iota & \searrow^{\mu(p,q;o)} & \\ C(p,q) & \longrightarrow & \mathcal{SC}(p,q;o) \end{array} \tag{10}$$

is homotopy commutative for $p, q \geqslant 0$ *and* $p + q \geqslant 1$*, where* ι *is the canonical inclusion* $C(p,q) \hookrightarrow \overline{C(p,q)}$ *and* $C(p,q) \to \mathcal{SC}(p,q;o)$ *is a right inverse to the center projection* $\pi : \mathcal{SC}(p,q;o) \to C(p,q)$*.*

Proof. The manifold $\overline{C(p,q)}$ is embedded in $\overline{C(2p+q)}$ in the same way that $\mathcal{SC}(p,q;o)$ is embedded in $\mathcal{D}_2(2p+q)$. Hence the operadic homotopy equivalence $\overline{C(2p+q)} \to \mathcal{D}_2(2p+q)$ naturally restricts to a homotopy equivalence between $\overline{C(p,q)}$ and $SC(p,q;o)$. □

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Actions of Higher Categories on C*-Algebras

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Abstract. We examine crossed products for twisted group actions and are led by this to introduce notions from higher category theory into the study of operator algebras. These lectures are based on joint work with Alcides Buss and Chenchang Zhu.

1. Crossed products and their universal property

A C^*-*dynamical system* consists of a C^*-algebra A, a (locally compact) group G, and a (strongly continuous) action of G on A by *-automorphisms. Here we will only consider the case where G is discrete for simplicity.

Already the most classical case is interesting: $A = C(X)$ for some compact space X, $G = \mathbb{Z}$. An action α corresponds to a single homeomorphism $\Phi\colon X \to X$ by $(\alpha_n f)(x) := f(\Phi^{-n}x)$ for all $f \in C(X)$, $x \in X$, $n \in \mathbb{Z}$. This is a (discrete) *dynamical system.*

In this section, we briefly explain how to associate a crossed product C^*-algebra to a C^*-dynamical system. The idea is to get interesting invariants of dynamical systems by studying this single C^*-algebra. First, we recall some facts about multipliers and introduce the notion of a morphism of C^*-algebras. This slightly non-standard category is crucial to characterise crossed products by a universal property. It is also used frequently to study locally compact quantum groups.

1.1. Multipliers.

Let A be a C^*-algebra.

Definition 1.1. A *multiplier* of A is a map $m\colon A \to A$ for which there exists an adjoint map $m^*\colon A \to A$ such that $a^* \cdot m(b) = (m^*(a))^* \cdot b$ for all $a, b \in A$.

Multipliers are linear and right A-module homomorphisms for the obvious right A-module structure on A. The norm of a multiplier is the usual operator norm,

$$\|m\| := \sup\{\|m(a)\| \mid a \in A,\ \|a\| \leq 1\}.$$

If m is a multiplier, then m^* is uniquely determined and a multiplier as well, with $(m^*)^* = m$. If m_1 and m_2 are two multipliers of A, then so are linear combinations of them and $m_1 \cdot m_2 := m_1 \circ m_2$, with adjoints $(c_1 m_1 + c_2 m_2)^* = \overline{c_1} m_1^* + \overline{c_2} m_2^*$ for $c_1, c_2 \in \mathbb{C}$ and $(m_1 \circ m_2)^* = m_2^* \circ m_1^*$. The identity map is a multiplier, it is its own adjoint.

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With the norm and algebraic operations described above, the multipliers form a unital C*-algebra, denoted $\mathcal{M}(A)$.

We will denote $m(a)$ for $m \in \mathcal{M}(A)$ and $a \in A$ by $m \cdot a$ or simply ma. We also define $am = a \cdot m := (m^* \cdot a^*)^*$.

Lemma 1.2. *The unitary multipliers in $\mathcal{M}(A)$ are precisely those isometric right A-module isomorphisms $u\colon A \to A$ for which the adjoint of u is u^{-1}.*

Every $a \in A$ defines a multiplier m_a by $m_a b := a \cdot b$, with $m_a^* = m_{a^*}$. This embeds A as a closed *-ideal in $\mathcal{M}(A)$.

Exercise 1.3. *If A is unital, then $A \cong \mathcal{M}(A)$ via the embedding just described.*

More generally, let B be a C*-algebra containing A as an ideal. Then each $b \in B$ defines a multiplier m_b of A by $m_b a := b \cdot a$. This defines a *-homomorphism $B \to \mathcal{M}(A)$. It is injective if and only if A is an *essential* ideal in B, that is, $b \cdot a = 0$ for all $a \in A$ implies $b = 0$. Thus $\mathcal{M}(A)$ is the largest C*-algebra containing A as an essential ideal.

Definition 1.4. The *strict topology* on $\mathcal{M}(A)$ is defined by requiring that a net of multipliers $(m_i)_{i\in I}$ converges if and only if the nets $(m_i \cdot a)$ and $(m_i^* \cdot a)$ are norm convergent for all $a \in A$.

The subspace A is dense in $\mathcal{M}(A)$ in the strict topology: if (u_i) is an approximate identity in A, then $(m \cdot u_i)$ converges strictly to m for any $m \in \mathcal{M}(A)$. The multiplier algebra is complete in the strict topology, that is, any strict Cauchy net converges strictly to some limit in $\mathcal{M}(A)$. Thus $\mathcal{M}(A)$ is the completion of A in the strict topology (restricted to A).

Example 1.5. For the C*-algebra $\mathbb{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$, the multiplier algebra is $\mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H})$, the C*-algebra of all bounded operators on $\mathcal{H}$. We get an injective *-homomorphism $\mathbb{B}(\mathcal{H}) \to \mathcal{M}(\mathbb{K}(\mathcal{H}))$ from the general theory. Surjectivity follows by examining the action of a multiplier on rank-one operators.

Example 1.6. For the C*-algebra $C_0(X)$ of continuous functions vanishing at infinity on a locally compact space X, we get $\mathcal{M}(C_0(X)) \cong C_b(X)$, the C*-algebra of all continuous bounded functions on X. Once again, the general theory already provides an injective *-homomorphism $C_b(X) \to \mathcal{M}(C_0(X))$.

Recall that the spectrum of $C_b(X)$ is the Stone–Čech compactification of X. For this reason, the multiplier algebra may also be viewed as a non-commutative generalisation of the Stone–Čech compactification for locally compact spaces.

Example 1.7. More generally, consider the C*-algebra $C_0(X, A)$ of continuous functions $X \to A$ that vanish at infinity, for a C*-algebra A. Then $\mathcal{M}(C_0(X, A))$ is the C*-algebra of all *strictly* continuous bounded functions $X \to \mathcal{M}(A)$.

1.2. Morphisms of C*-algebras.

Definition 1.8. Let A and B be C*-algebras. A *-homomorphism $f\colon A \to \mathcal{M}(B)$ is called *essential* or *non-degenerate* if the linear span of $f(A) \cdot B$ is dense in B.

If A is unital, f is essential if and only if f is unital.

PROPOSITION 1.9. *A *-homomorphism $f\colon A \to \mathcal{M}(B)$ is essential if and only if it extends to a strictly continuous, unital *-homomorphism $\bar{f}\colon \mathcal{M}(A) \to \mathcal{M}(B)$. This extension is defined by $\bar{f}(m) \cdot f(a) \cdot b = f(m \cdot a) \cdot b$ for all $m \in \mathcal{M}(A)$, $a \in A$, $b \in B$.*

In the following, we will write f for $\bar{f}$, not distinguishing in our notation between an essential *-homomorphism and its unique strictly continuous extension to the multiplier algebra.

Clearly, the composition of two strictly continuous, unital *-homomorphisms is again a strictly continuous, unital *-homomorphism. This also defines a composition for essential *-homomorphisms, using Proposition 1.9. It is easy to see that this composition is associative. Since identity maps are essential *-homomorphisms, we get a category whose objects are the C*-algebras and whose morphisms are the essential *-homomorphisms. This will be our preferred category of C*-algebras, so that we briefly call essential *-homomorphisms $A \to \mathcal{M}(B)$ *morphisms* from A to B.

In the following, when I write something like "a morphism $f\colon A \to \mathcal{M}(B)$," I mean that f is a morphism from A to B. I will never use morphisms from A to $\mathcal{M}(B)$. There are no such morphisms unless A is unital, in which case the morphisms from A to B are the same as morphisms from A to $\mathcal{M}(B)$, namely, unital *-homomorphisms $A \to \mathcal{M}(B)$.

PROPOSITION 1.10. *The invertible morphisms between two* C**-algebras A and B are exactly the *-isomorphisms $f\colon A \to B$.*

PROOF. It is clear that *-isomorphisms remain invertible when we view them as morphisms. The point is that any isomorphism in the category of C*-algebras described above is of this form. It suffices to prove that an invertible morphism must map A to B, not just to $\mathcal{M}(B)$ because then its inverse will also map B to A. If $f\colon A \to \mathcal{M}(B)$ is invertible with inverse $g\colon B \to \mathcal{M}(A)$, then

$$f(A) = f(g(B) \cdot A) = B \cdot f(A) \subseteq B$$

because g is essential and $f \circ g = \mathrm{Id}_B$. □

1.3. Crossed products. Let G be a (discrete) group and let A be a C*-algebra equipped with an action of G by automorphisms, that is, a group homomorphism α from G to the automorphism group $\mathrm{Aut}(A)$.

How should we represent this dynamics on a Hilbert space? Let us consider a classical example.

EXAMPLE 1.11. Let $A = \mathrm{C}_0(X)$, $G = \mathbb{Z}$, and let $\Phi\colon X \to X$ be the homeomorphism that induces the action α of $\mathbb{Z}$ on A. Let μ be a Φ-invariant measure on X, that is, $\mu(\Phi(A)) = \mu(A)$ for all measurable subsets A of X. Let $\mathcal{H}$ be the Hilbert space $L^2(X, \mu)$. We let A act on $\mathcal{H}$ by pointwise multiplication, that is, by the representation $\pi\colon A \to \mathbb{B}(\mathcal{H})$ defined by $(\pi(a)h)(x) := a(x) \cdot h(x)$ for all $a \in A$, $h \in \mathcal{H}$, $x \in X$. We let G act on $\mathcal{H}$ by the induced action, $\rho\colon G \to \mathcal{U}(\mathcal{H})$ defined by $(\rho(n)h)(x) := h(\Phi^{-n}x)$ for all $n \in G$, $h \in \mathcal{H}$, $x \in X$.

This should be a nice representation of the dynamical system (A, G, α). In what sense are the representations π and ρ in this example compatible with each other? — They satisfy the covariance condition in the next definition, so that (π, ρ) is a covariant representation of (A, G, α).

Definition 1.12. A *covariant representation* of (A, G, α) on a C*-algebra D is a pair (π, ρ) consisting of a morphism $\pi\colon A \to \mathcal{M}(D)$ and a homomorphism ρ from G to the group of unitary multipliers in D, satisfying the covariance condition $\rho(g)\pi(a)\rho(g)^{-1} = \pi(\alpha_g(a))$ for all $g \in G$, $a \in A$.

Definition 1.13. A *crossed product* for (A, G, α) is a representing object for covariant representations, that is, a C*-algebra B with a covariant representation (π_0, ρ_0), such that any covariant representation (π, ρ) on any D is of the form $(f \circ \pi_0, f \circ \rho_0)$ for a unique morphism $f\colon B \to D$.

By general category theory, such a crossed product is determined uniquely if it exists. We may construct a crossed product as follows. We let $\mathbb{C}[G, A]$ be the vector space of finitely supported maps $G \to A$, that is, finite formal linear combinations $\sum_{g\in G} a_g\lambda_g$. We define a *-algebra structure on $\mathbb{C}[G, A]$ by

$$\sum_{g\in G} a_g\lambda_g \cdot \sum_{g\in G} b_g\lambda_g := \sum_{g\in G}\sum_{h\in G} a_h\alpha_h(b_{h^{-1}g})\lambda_g,$$

$$\left(\sum_{g\in G} a_g\lambda_g\right)^* := \sum_{g\in G} \alpha_g(a_{g^{-1}})^*\lambda_g.$$

Any C*-seminorm on $\mathbb{C}[G, A]$ is dominated by the norm $\sum_{g\in G}\|a_g\|$. Hence the supremum of all C*-seminorms on $\mathbb{C}[G, A]$ is a C*-seminorm (even a C*-norm). The completion of $\mathbb{C}[G, A]$ together with the obvious covariant representation $a \mapsto a\lambda_1$, $g \mapsto \lambda_g$, is a crossed product in the sense of the above definition.

Example 1.14. For $G = \mathbb{Z}$, the action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(G)$ is determined by a single automorphism $\alpha(1)$ because $\alpha(n) = \alpha^n$ for all $n \in \mathbb{Z}$. Thus our construction above contains a crossed product for pairs (A, α) with $\alpha \in \mathrm{Aut}(A)$ a single automorphism. A covariant representation in this case is equivalent to a morphism $f\colon A \to D$ together with a unitary multiplier u of D such that $uf(a)u^* = f(\alpha(a))$.

2. How trivial are inner automorphisms?

As we shall see, we may consider inner automorphisms to be trivial in connection with crossed products by a single automorphism, but not for more general crossed products. Roughly speaking, inner automorphisms are non-trivial but more trivial than general automorphisms. To make sense of this, we introduce 2-categories: in this setting, inner automorphisms are *equivalent* to but not *equal* to the trivial automorphism.

2.1. Isomorphism of crossed products for automorphisms. Let A be a C*-algebra. For an automorphism $\alpha \in \mathrm{Aut}(A)$, we define a crossed product $C^*(A, \alpha) = A \rtimes_\alpha \mathbb{Z}$ by a universal property as in Section 1.3. Recall that this is a completion of $\mathbb{C}[\mathbb{Z}, A]$. Although this depends on the automorphism α, it turns out that many automorphisms induce isomorphic crossed products in a canonical way.

Definition 2.1. Let u be a unitary multiplier of A. Then we define $\mathrm{Ad}_u \in \mathrm{Aut}(A)$ by $\mathrm{Ad}_u(a) := uau^*$. Automorphisms of this form are called *inner automorphisms*. The inner automorphisms form a normal subgroup in $\mathrm{Aut}(A)$. The quotient $\mathrm{Aut}(A)$ by this subgroup is called the *outer automorphism group* $\mathrm{Out}(A)$.

PROPOSITION 2.2. *Let* $\alpha \in \mathrm{Aut}(A)$ *and* $u \in \mathcal{UM}(A)$*. Then*

$$\mathrm{C}^*(A,\alpha) \cong \mathrm{C}^*(A, \mathrm{Ad}_u \circ \alpha).$$

Thus $\mathrm{C}^*(A,\alpha)$ *depends, up to isomorphism, only on the class of* α *in* $\mathrm{Out}(A)$*.*

PROOF. Abbreviate $\beta := \mathrm{Ad}_u \circ \alpha$. Let (π, V) be a covariant representation of (A,α), that is, $\pi\colon A \to \mathcal{M}(D)$ is a morphism and $V \in \mathcal{UM}(D)$, such that $V\pi(a)V^* = \pi(\alpha(a))$. Then $(\pi, \pi(u)\cdot V)$ is a covariant representation of (A,β) because

$$\pi(u)V\pi(a)(\pi(u)V)^* = \pi(u\alpha(a)u^*) = \pi(\beta(a)).$$

This is a natural bijection between covariant representations. Hence the universal objects $\mathrm{C}^*(A,\alpha)$ and $\mathrm{C}^*(A,\beta)$ must be isomorphic. □

EXERCISE 2.3. *Describe the isomorphism between* $\mathrm{C}^*(A,\alpha)$ *and* $\mathrm{C}^*(B,\beta)$ *explicitly as an isomorphism between the dense* **-subalgebras* $\mathbb{C}[\mathbb{Z}, A]$*.*

2.2. A counterexample. The equivalence result for crossed products by a single automorphism does *not* generalise to actions of other groups. For instance, it fails for actions of $\mathbb{Z}^2$ on the C^*-algebra of compact operators $\mathbb{K}$ on the separable Hilbert space $\mathcal{H} := \ell^2\mathbb{Z}$. Recall that any automorphism of $\mathbb{K}$ is of the form $T \mapsto uTu^*$ for a unitary operator $u\colon \mathcal{H} \to \mathcal{H}$. Since $\mathcal{M}(\mathbb{K}) = \mathbb{B}(\mathcal{H})$, this says that all automorphisms of $\mathbb{K}$ are inner. Hence crossed products for $\mathbb{Z}$-actions on $\mathbb{K}$ are all isomorphic (to the tensor product $\mathbb{K} \otimes \mathrm{C}^*(\mathbb{Z})$). A representation of $\mathbb{Z}^2$ on $\mathbb{K}$ by automorphisms is equivalent to a pair (α,β) of *commuting* automorphisms of $\mathbb{K}$. Let (U,V) be unitaries on $\mathcal{H}$ with $\alpha = \mathrm{Ad}_U$, $\beta = \mathrm{Ad}_V$. Then $U^*\lambda_{(1,0)}$ and $V^*\lambda_{(0,1)}$ commute with $\mathbb{K}$ in $\mathcal{M}(\mathbb{K} \rtimes \mathbb{Z}^2)$. Since their products with elements of $\mathbb{K}$ generate the crossed product, it follows that $\mathbb{K} \rtimes \mathbb{Z}^2$ is isomorphic to a C^*-tensor product of $\mathbb{K}$ with $\mathrm{C}^*(U^*\lambda_{(1,0)}, V^*\lambda_{(0,1)})$. We compute

$$U^*\lambda_{(1,0)} \cdot V^*\lambda_{(0,1)} = U^*\alpha(V^*)\lambda_{(1,1)} = U^*UV^*U^*\lambda_{(1,1)} = V^*U^*\lambda_{(1,1)},$$
$$V^*\lambda_{(0,1)} \cdot U^*\lambda_{(1,0)} = V^*\beta(U^*)\lambda_{(1,1)} = V^*VU^*V^*\lambda_{(1,1)} = U^*V^*\lambda_{(1,1)}.$$

Here we use that the covariance condition $\lambda_g a = \alpha_g(a)\lambda_g$ for $g \in G$, $a \in A$ continues to hold in $\mathcal{M}(A \rtimes_\alpha G)$ if $a \in \mathcal{M}(A)$, provided we use the unique extension of α_g to an automorphism of $\mathcal{M}(A)$.

Now $\mathrm{Ad}_{UV} = \mathrm{Ad}_{VU}$ because α and β commute. But this only implies $UV = cVU$ for some $c \in \mathbb{C}$ with $|c| = 1$. Thus $\mathrm{C}^*(U^*\lambda_{(1,0)}, V^*\lambda_{(0,1)})$ is a rotation algebra with parameter $\vartheta := \log(c)/2\pi\mathrm{i}$, and

$$\mathbb{K} \rtimes \mathbb{Z}^2 \cong \mathbb{K} \otimes A_\vartheta.$$

This depends on the parameter c. Since $\mathrm{Out}(\mathbb{K})$ is trivial, the composite homomorphism $\mathbb{Z}^2 \to \mathrm{Aut}(\mathbb{K}) \to \mathrm{Out}(\mathbb{K})$ is not enough to recover the crossed product.

2.3. Cocycle equivalence. The above counterexample shows that we must be more careful in order to understand in what sense crossed products are not affected by inner automorphisms. Let us carry over the proof method of the isomorphism $A \rtimes_\alpha \mathbb{Z} \cong A \rtimes_\beta \mathbb{Z}$ if $\beta = \mathrm{Ad}_u \circ \alpha$. That is, let G be a group, let A be a C^*-algebra and let α and β be actions of G on A by automorphisms. Let D be an auxiliary C^*-algebra. We want to construct a bijection between covariant representations of (A,α,G) and (A,β,G) on D of the form $(\pi,\rho) \mapsto (\pi,\rho')$ with $\rho'_g = \pi(U_g)\rho_g$ for unitary multipliers $U_g \in \mathcal{M}(A)$ for all $g \in G$.

The pair (π, ρ') as defined above is a covariant representation of (A, β, G) if and only if the following holds:

- $\pi(U_g)\rho_g\pi(U_h)\rho_h = \pi(U_{gh})\rho_{gh}$ for all $g, h \in G$;
- $\pi(U_g)\rho_g\pi(a)\rho_g^*\pi(U_g)^* = \pi(\beta_g(a))$ for all $g \in G$, $a \in A$.

Using the covariance condition, we may simplify this to $\pi(U_g\alpha_g(U_h)) = \pi(U_{gh})$ and $\pi(U_g\alpha_g(a)U_g^*) = \pi(\beta_g(a))$. If we want the same U_g to work for all covariant pairs (π, ρ), then we may as well assume that π is faithful, so that we arrive at the conditions

$$U_g\alpha_g(U_h) = U_{gh} \quad \text{and} \quad \mathrm{Ad}_{U_g} \circ \alpha_g = \beta_g \qquad \text{for all } g, h \in G.$$

We take note of this in a definition:

Definition 2.4. The actions α and β are *cocycle equivalent* if there is a map $U\colon G \to \mathcal{UM}(A)$ with $\mathrm{Ad}_{U_g} \circ \alpha_g = \beta_g$ for all $g \in G$ and $U_g\alpha_g(U_h) = U_{gh}$ for all $g, h \in G$. (This involves the unique strictly continuous extension of α_g to $\mathcal{M}(A)$.)

The same argument as for $G = \mathbb{Z}$ shows:

Theorem 2.5. *A cocycle equivalence between two group actions induces an isomorphism between the crossed product* C^**-algebras.*

Unitaries U_g with $\mathrm{Ad}_{U_g} \circ \alpha_g = \beta_g$ for all $g \in G$ exist if and only if α and β become equal as maps to $\mathrm{Out}(A)$. Furthermore, if U_g exists at all, it is unique up to multiplication by a central unitary. If the U_g are chosen to verify $\mathrm{Ad}_{U_g} \circ \alpha_g = \beta_g$ for all $g \in G$, then the unitaries $U_g\alpha_g(U_h)U_{gh}^*$ for $g, h \in G$ are necessarily central, but they are not necessarily 1.

2.4. Interpretation. We have seen the following: automorphisms that differ by an inner automorphism, may often be considered equivalent; but we must be careful when several automorphisms interact. The notion of cocycle equivalence makes precise what additional information is needed to get an isomorphism between the crossed products for two actions that differ by inner automorphisms.

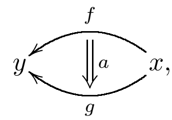
A better understanding of this phenomenon is crucial in order to treat other problems of a similar nature. For instance, suppose that we are only given a group homomorphism $G \to \mathrm{Out}(A)$. What additional information is needed to define a crossed product in such a situation? We certainly need something because different group actions in the usual sense that give the same map $G \to \mathrm{Out}(A)$ may have non-isomorphic crossed products.

When we pass from $\mathrm{Aut}(A)$ to $\mathrm{Out}(A)$, then we form a quotient group. Noncommutative geometry suggests to replace quotient spaces by groupoids (see also Section 4.1). Following this general paradigm, we should replace $\mathrm{Out}(A)$ by a groupoid. The object space of this groupoid is $\mathrm{Aut}(A)$. The set of arrows between automorphisms $f, g \in \mathrm{Aut}(A)$ is the set of all unitary multipliers $u \in \mathcal{M}(A)$ with $\mathrm{Ad}_u \circ f = g$. The composition in this groupoid is the multiplication of unitaries. The identity morphism on an automorphism f is the unitary 1, and the inverse of u is $u^* = u^{-1}$.

The groupoid just described treats $\mathrm{Aut}(A)$ merely as a set. In order to understand group actions by automorphisms, we must incorporate further structure into this groupoid that reflects the multiplication in $\mathrm{Aut}(A)$ and its interaction with unitaries. This leads us to the structure of a 2-*category*. Our first task is to define 2-categories. We will only define *strict* 2-categories, following [4], and then give several examples. The most relevant example for us is the 2-category of C*-algebras with morphisms as arrows and unitaries as 2-arrows. This setup allows us to interpret the cocycle relation appearing above, and to derive similar notions.

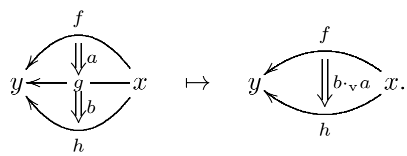
2.5. Strict 2-categories. The quick definition of a strict 2-category describes it as a category enriched over categories. That is, for two objects x and y of our first order category, we have a *category* of morphisms from x to y, and the composition of morphisms lifts to a bifunctor between these morphism categories. This definition is similar to the definition of a topological category: the latter is nothing but a category enriched over topological spaces. We now write down more explicitly what a category enriched over categories is (see also [1]).

Having categories of morphisms boils down to having *arrows* between objects $x \to y$, also called 1-arrows or 1-morphisms, and arrows between arrows



which are called 2-arrows, 2-morphisms, or *bigons* because of their shape. We prefer to call them bigons because there are other ways to describe 2-categories that use triangles or even more complicated shapes as 2-morphisms (see [1]).

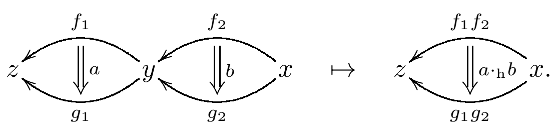
The category structure on the space of arrows $x \to y$ provides a *vertical composition* of bigons



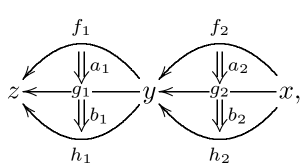
The composition functor between the arrow categories provides both a composition of arrows

$$z \xleftarrow{f} y \xleftarrow{g} x \quad \mapsto \quad z \xleftarrow{fg} x$$

and a *horizontal composition* of bigons



These three compositions of arrows and bigons are associative and unital in an appropriate sense. Furthermore, the horizontal and vertical products commute: given a diagram



composing first vertically and then horizontally or the other way around produces the same bigon $f_1 f_2 \Rightarrow h_1 h_2$.

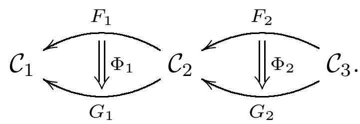
Let me emphasise again that the three composition operations and the associativity, unitality, and interchange conditions above only make explicit what is meant by a category enriched over categories.

In any strict 2-category, the objects and arrows form an ordinary category. So do the arrows and bigons with vertical composition of bigons as composition.

Example 2.6. Categories form a strict 2-category with small categories as objects, functors between categories as arrows, and natural transformations between functors as bigons. The composition of arrows is the composition of functors and the vertical composition of bigons is the composition of natural transformations. The horizontal composition of bigons yields the canonical natural transformation

$$\Phi_{1,G_2(A)} \circ F_1(\Phi_{2,A}) = G_1(\Phi_{2,A}) \circ \Phi_{1,F_2(A)} \colon F_1\bigl(F_2(A)\bigr) \to G_1\bigl(G_2(A)\bigr)$$

for the diagram



This 2-categorical structure was, in fact, the reason to introduce categories in the first place: a good understanding of categories is needed to understand natural transformations, not to understand functors.

2.5.1. *The strict* 2*-category* $\mathfrak{C}^*(2)$ *of* C**-algebras.* Now we turn C*-algebras into a strict 2-category. We take C*-algebras as objects, of course. We take "morphisms" (essential *-homomorphisms $A \to \mathcal{M}(B)$) as arrows (see Sections 1.1 and 1.2).

Let f and g be two morphisms from A to B; view them as strictly continuous, unital *-homomorphisms from $\mathcal{M}(A)$ to $\mathcal{M}(B)$. An element $b \in \mathcal{M}(B)$ is called an *intertwiner* from f to g if $b \cdot f(a) = g(a) \cdot b$ for all $a \in \mathcal{M}(A)$ or, equivalently, for all $a \in A$. If b is unitary, this is equivalent to $g = \mathrm{Ad}_b \circ f$, where Ad_b is the inner automorphism generated by b. The set of bigons from f to g in $\mathfrak{C}^*(2)$ is the set of unitary intertwiners from f to g. (We also get a strict 2-category if we use arbitrary intertwiners here; the restriction to unitaries is convenient because it ensures that all 2-arrows are invertible.)

The vertical composition of intertwiners is the product in $\mathcal{M}(B)$. The horizontal composition of two bigons $c\colon f_1 \Rightarrow g_1$ and $b\colon f_2 \Rightarrow g_2$ for composable pairs of arrows $f_1, g_1\colon B \rightrightarrows C$ and $f_2, g_2\colon A \rightrightarrows B$ is

$$c \cdot_{\mathrm{h}} b := c \cdot f_1(b) = g_1(b) \cdot c. \tag{2.7}$$

Notice that $c \cdot_{\mathrm{h}} b$ is the unique intertwiner between $f_1 \circ f_2$ and $g_1 \circ g_2$ that we get from c and b by a simple explicit formula. It is auspicious that the two possible ways of defining such an intertwiner yield the same result.

Exercise 2.8. *Verify that $\mathfrak{C}^*(2)$ as defined above is a strict* 2*-category, that is, verify that the composition products are associative and unital in the appropriate sense, and verify the interchange law. The latter reduces to* (2.7)*, by the way.*

Definition 2.9. An arrow f in a 2-category is called an *equivalence* if there is an arrow g (called a quasi-inverse of f) and invertible bigons between $f \circ g$ and an identity map, and between $g \circ f$ and an identity map.

Exercise 2.10. *Verify that any equivalence in $\mathfrak{C}^*(2)$ is a *-isomorphism.*

How does our 2-category help to understand the cocycle relation?

Consider two group actions α and β of G on a C*-algebra A. They are equal if $\alpha_g = \beta_g$ for all $g \in G$. We have seen in examples that it is not enough to replace equality by equivalence. We must, as additional data, specify the unitaries that implement the equivalence explicitly. Thus an equivalence between the actions α and β specifies bigons $u_g\colon \alpha_g \to \beta_g$ for all $g \in G$. These bigons are nothing but unitary multipliers u_g of A with $u_g\alpha_g(a)u_g^* = \beta_g(a)$ for all $a \in A$. Roughly speaking, we must specify a reason for α_g and β_g to be equivalent.

Bigons $u_g\colon \alpha_g \to \beta_g$ and $u_h\colon \alpha_h \to \beta_h$ yield a bigon $u_g \cdot_{\mathrm{h}} u_h\colon \alpha_g \circ \alpha_h \to \beta_g \circ \beta_h$. Since we are dealing with group actions, $\alpha_g \circ \alpha_h = \alpha_{gh}$ and $\beta_{gh} = \beta_g \circ \beta_h$. Thus $u_g \cdot_{\mathrm{h}} u_h = u_g\alpha_g(u_h) = \beta_g(u_h)u_g$ and u_{gh} are two reasons for α_{gh} and β_{gh} to be equivalent. The cocycle relation says $u_g \cdot_{\mathrm{h}} u_h = u_{gh}$. We have understood the combination $u_g\alpha_g(u_h)$ as an elementary operation with unitary intertwiners: it is their horizontal product.

2.6. Group actions up to inner automorphisms. We may also view a group action on a C*-algebra as a functor from the group (viewed as a category with one object) to the category of C*-algebras. Now we treat a group as a 2-category with one object and only identity bigons. We want to study functors from this 2-category to the 2-category of C*-algebras just introduced.

At this point we have a choice. The most obvious notion of functor is that of a strict functor. This consists of maps between objects, arrows, and bigons that preserve all the extra structure. If we do this, we get nothing new, so that we do not discuss this further. But in the setting of 2-categories, it is customary to allow functors that are only functorial in a weaker sense, where all equalities of arrows are replaced by equivalences. These equivalences are given by bigons that are part of the data of the functor. And there are certain coherence conditions, which appear automatically, like the cocycle relation in the definition of cocycle equivalence for group actions.

Let us build up these weak functors. To begin with, we need the same data as for a usual group action: a C*-algebra A and arrows (that is, morphisms) $\alpha_g\colon A \to A$ for all $g \in G$. Further conditions that we will impose later imply that the α_g are *-isomorphisms, not just morphisms.

For a group action in the usual sense, we would require the equalities $\alpha_g\alpha_h = \alpha_{gh}$ for all $g, h \in G$, and $\alpha_1 = \mathrm{Id}_A$. Now we replace these equations by additional data: bigons $\omega_{g,h}\colon \alpha_g\alpha_h \Rightarrow \alpha_{gh}$ for all $g, h \in G$ and $u\colon \mathrm{Id}_A \Rightarrow \alpha_1$. More concretely, these are unitary multipliers of A such that

$$\omega_{g,h}\alpha_g\big(\alpha_h(a)\big)\omega_{g,h}^* = \alpha_{gh}(a) \qquad \text{for all } g, h \in G,\ a \in A, \tag{2.11}$$

$$uau^* = \alpha_1(a) \qquad \text{for all } a \in A. \tag{2.12}$$

In the following, we will use the inverse bigons $\omega_{g,h}^*$ because the resulting formulas are more familiar: they lead to Busby–Smith twisted group actions.

Given a group action α_g, we get many more complicated equalities from the basic ones above, for instance, $\alpha_1\alpha_g = \alpha_g$ for all $g \in G$. In fact, there are two ways to prove $\alpha_1\alpha_g = \alpha_g$, namely, $\alpha_1\alpha_g = \mathrm{Id}_A\alpha_g = \alpha_g$ or $\alpha_1\alpha_g = \alpha_{1\cdot g} = \alpha_g$. If we replace equalities by bigons, then these two ways to prove an equation yield two unitary intertwiners between the same arrows. In our example, we get the unitary intertwiners $u^* \cdot_{\mathrm{h}} 1_{\alpha_g}$ and $\omega_{1,g}$ from $\alpha_1\alpha_g$ to α_g, respectively, where 1_{α_g} denotes the identity bigon on the arrow α_g, that is, the identity unitary.

Now we can formulate a meta-coherence law: *whenever an equation of arrows for group actions may be proved in two different ways, the bigons that we get by lifting these computations must be equal.* For instance, we require the identity $u^* \cdot_{\mathrm{h}} 1_{\alpha_g} = \omega_{1,g}$ for all $g \in G$, that is, $u = \omega_{1,g}^*$ as unitary multipliers of A.

Similarly, the two obvious ways of proving $g \cdot 1 = g$ lead to an identity $1_{\alpha_g} \cdot_{\mathrm{h}} u^* = \omega_{g,1}$ for all $g \in G$, that is, $\alpha_g(u) = \omega_{g,1}^*$ as unitary multipliers of A. Notice that the recipe for horizontal products brings in α_g. We may prove $\alpha_g \alpha_h \alpha_k = \alpha_{ghk}$ in two ways, via $\alpha_{gh}\alpha_k$ or $\alpha_g \alpha_{hk}$. This leads to a coherence condition

$$\omega_{gh,k} \cdot_{\mathrm{v}} (\omega_{g,h} \cdot_{\mathrm{h}} 1_{\alpha_k}) = \omega_{g,hk} \cdot_{\mathrm{v}} (1_{\alpha_g} \cdot_{\mathrm{h}} \omega_{h,k})$$

or, explicitly,

$$\omega_{g,h}^* \cdot \omega_{gh,k}^* = \alpha_g(\omega_{h,k}^*) \cdot \omega_{g,hk}^*. \tag{2.13}$$

Now it turns out that all other coherence conditions that are contained in our meta-coherence law follow from the ones we already have. We do not prove this fact here. Thus a functor from G to $\mathfrak{C}^*(2)$ is defined as a C*-algebra A with morphisms α_g for all $g \in G$ and unitaries $\omega_{g,h}^*$ for all $g, h \in G$ and u satisfying the three coherence conditions just listed.

EXERCISE 2.14. *Since $u = \omega_{1,1}^*$, the unitary u is redundant. Show that the relations $u = \omega_{1,g}^*$ and $\alpha_g(u) = \omega_{g,1}^*$ follow from* (2.13). *Thus a functor from G to $\mathfrak{C}^*(2)$ is equivalent to morphisms α_g for all $g \in G$ and unitaries $\omega_{g,h}^*$ for all $g, h \in G$ satisfying* (2.11) *and* (2.13).

This should not be surprising because the equation $\alpha_1 = 1$ for group *actions is redundant: it follows from $\alpha_1\alpha_1 = \alpha_1$ because α_1 is invertible.* (*Semigroup actions would be a different matter.*)

The above notion of a functor is exactly the notion of a *Busby–Smith twisted group action* as defined in [5]. The cocycle relation (2.13) becomes completely natural from the higher category point of view.

The notion we have just defined is a group action that only satisfies the usual multiplicativity condition up to inner automorphisms. To get a well-behaved theory, we also specify the unitaries that generate these inner automorphisms explicitly, and we require these unitaries to satisfy some coherence conditions.

2.7. Transformations between group actions. What would be the appropriate notion of cocycle equivalence for Busby–Smith twisted group actions? To answer this question, we study natural isomorphisms of functors between 2-categories. Let $(A, \alpha_g, \omega_{g,h})$ and $(B, \beta_g, \psi_{g,h})$ be two functors from the same group G to $\mathfrak{C}^*(2)$. A natural transformation between them contains an arrow $f \colon A \to B$. If we were dealing with functors between ordinary categories, this arrow would be required to satisfy $\beta_g \circ f = f \circ \alpha_g$ for all $g \in G$. In the world of 2-categories, we weaken this equality of arrows to an equivalence.

As before, we specify explicitly the bigons $W_g \colon \beta_g f \Rightarrow f\alpha_g$ that implement this equivalence. That is, W_g is a unitary multiplier of B and satisfies

$$W_g \beta_g\bigl(f(a)\bigr) W_g^* = f\bigl(\alpha_g(a)\bigr) \qquad \text{for all } g \in G,\ a \in A.$$

It remains to determine the coherence conditions. The two ways of simplifying $\beta_g\beta_h f$ to $f\alpha_{gh}$ via $\beta_g\beta_h f \Rightarrow \beta_g f \alpha_h \Rightarrow f\alpha_g\alpha_h \Rightarrow f\alpha_{gh}$ and via $\beta_g\beta_h f \Rightarrow \beta_{gh} f \Rightarrow f\alpha_{gh}$ lead to a coherence law. It turns out that this single coherence condition implies all other coherence conditions.

EXERCISE 2.15. *Formulate this coherence law explicitly.*

Specialising to the case $f = \mathrm{Id}_A$, we get a notion of cocycle equivalence for Busby–Smith twisted actions. Of course, this yields the notion already used in the literature.

By the way, if we are given only A, $(\alpha_g)_{g\in G}$, $(\omega_{g,h})_{g,h\in G}$, and $(W_g)_{g\in G}$, then there is a unique way to define β_g and $\psi_{g,h}$ so that the W_g form a natural isomorphism between $(A, \alpha_g, \omega_{g,h})$ and $(B, \beta_g, \psi_{g,h})$. That is, we may conjugate a Busby–Smith twisted action by an arbitrary cochain (W_g) and still get a Busby–Smith twisted action.

Summing up, the mathematical structure in the 2-category of C*-algebras $\mathfrak{C}^*(2)$ explains the notions of Busby–Smith twisted action and cocycle equivalence for such actions.

3. Group actions by correspondences

We may also study another 2-category of C*-algebras that is related to Morita–Rieffel equivalence. Many important C*-algebras are constructed from groupoids. We will consider groupoids later. For the time being, one observation is important: equivalent groupoids yield Morita–Rieffel equivalent C*-algebras. Therefore, it would be nice to have a category in which Morita–Rieffel equivalent C*-algebras become isomorphic. We will even construct a 2-category in which the Morita–Rieffel equivalences are exactly the equivalences (see Definition 2.9).

3.1. Hilbert modules. Let B be a C*-algebra. A *Hilbert B-module* is a right B-module $\mathcal{H}$ with a B-valued inner product

$$\mathcal{H} \times \mathcal{H} \to B, \qquad (\xi, \eta) \mapsto \langle \xi, \eta \rangle,$$

with the following properties:

- the inner product is conjugate-linear in the first and linear in the second variable;
- $\langle \xi_1 \cdot b_1, \xi_2 \cdot b_2 \rangle = b_1^* \cdot \langle \xi_1, \xi_2 \rangle \cdot b_2$ for all $\xi_1, \xi_2 \in \mathcal{H}$, $b_1, b_2 \in B$;
- $\langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle^*$ for all $\xi_1, \xi_2 \in \mathcal{H}$;
- $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$;
- $\mathcal{H}$ is complete for the norm defined by $\|\xi\|^2 := \langle \xi, \xi \rangle$ for all $\xi \in \mathcal{H}$. (This is indeed a norm because of a Hilbert module generalisation of the Cauchy–Schwarz inequality).

EXAMPLE 3.1. A Hilbert $\mathbb{C}$-module is exactly the same as a Hilbert space. We get the above definition from the definition of a Hilbert space by replacing the algebra of scalars by B everywhere.

EXAMPLE 3.2. Let B be a C*-algebra. Then B is a Hilbert B-module with respect to the obvious right module structure and the inner product $\langle b_1, b_2 \rangle := b_1^* b_2$. More generally, the same module structure and inner product work if we replace B by a right ideal in B.

DEFINITION 3.3. A map $f\colon \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert B-modules is an *adjointable* operator if there is an adjoint map $f^*\colon \mathcal{H}_2 \to \mathcal{H}_1$ such that $\langle f^*\xi, \eta \rangle = \langle \xi, f\eta \rangle$ for all $\xi \in \mathcal{H}_2$, $\eta \in \mathcal{H}_1$.

The adjoint of f is unique if it exists and is again adjointable with $(f^*)^* = f$. An adjointable operator is necessarily bounded, linear, and a B-module homomorphism. The adjointable operators on a Hilbert B-module $\mathcal{H}$ form a C^*-algebra in a canonical way, which we denote by $\mathbb{B}(\mathcal{H})$.

EXERCISE 3.4. *If we view a* C^**-algebra as a Hilbert module as above, then* $\mathbb{B}(B) = \mathcal{M}(B)$.

DEFINITION 3.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert B-modules. If $\xi \in \mathcal{H}_1$, $\eta \in \mathcal{H}_2$, then we define a map $|\xi\rangle\langle\eta| : \mathcal{H}_2 \to \mathcal{H}_1$ by $|\xi\rangle\langle\eta|\,\zeta := \xi \cdot \langle\eta,\zeta\rangle$. The closed linear span of these operators is denoted by $\mathbb{K}(\mathcal{H}_2, \mathcal{H}_1)$ or just $\mathbb{K}(\mathcal{H})$ if $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$. Its elements are called *compact* operators (although they are not compact in the sense of Banach space theory).

Since $T \circ |\xi\rangle\langle\eta| = |T(\xi)\rangle\langle\eta|$ and $|\xi\rangle\langle\eta|^* = |\eta\rangle\langle\xi|$, the compact operators $\mathbb{K}(\mathcal{H})$ form a *-ideal in $\mathbb{B}(\mathcal{H})$.

PROPOSITION 3.6. $\mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H})$.

3.2. Correspondences. Let A and B be C^*-algebras.

DEFINITION 3.7. A *correspondence* from A to B is a Hilbert B-module $\mathcal{H}$ together with an essential *-homomorphism (morphism) $A \to \mathbb{B}(\mathcal{H}) = \mathcal{M}(\mathbb{K}(\mathcal{H}))$.

LEMMA 3.8. *A* **-homomorphism* $A \to \mathbb{B}(\mathcal{H})$ *is essential if and only if* $A \cdot \mathcal{H}$ *is dense in* $\mathcal{H}$.

EXAMPLE 3.9. Let $f\colon A \to \mathcal{M}(B)$ be a morphism. We may view f as a correspondence by interpreting $\mathcal{M}(B) \cong \mathbb{B}(B)$ for B viewed as a Hilbert module over itself.

We may compose correspondences by a tensor product construction. Let $\mathcal{H}_1$ be a correspondence from A to B and let $\mathcal{H}_2$ be a correspondence from B to C. The product is obtained from $\mathcal{H}_1 \otimes \mathcal{H}_2$ by completing with respect to the C-valued inner product

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle := \langle \xi_2, \langle \xi_1, \eta_1\rangle_B \cdot \eta_2\rangle_C$$

for all $\xi_1, \eta_1 \in \mathcal{H}_1$, $\xi_2, \eta_2 \in \mathcal{H}_2$. We denote this completion also by $\mathcal{H}_1 \otimes_B \mathcal{H}_2$.

EXERCISE 3.10. *Let* $\mathcal{H}_1$ *be the correspondence associated to a morphism* $f\colon A \to \mathcal{M}(B)$ *as in Example* 3.9*. Then the composition is isomorphic to* $\mathcal{H}_2$ *as a Hilbert* C*-module with the left* A*-module structure* $a \cdot \xi := f(a) \cdot \xi$ *for all* $a \in A$, $\xi \in \mathcal{H}_2$.

In particular, if f *is an identity morphism, then it acts as a left identity for the composition of correspondences, up to isomorphism. Check that the identity also acts as a right identity.*

If $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *come from morphisms* $f\colon A \to \mathcal{M}(B)$ *and* $g\colon B \to \mathcal{M}(C)$*, then* $\mathcal{H}_1 \otimes_B \mathcal{H}_2$ *comes from the morphism* $g \circ f\colon A \to \mathcal{M}(C)$.

We can also turn C^*-algebras into a 2-category using correspondences as arrows. The bigons are isomorphisms of correspondences, that is, Hilbert module isomorphisms intertwining the given left module structures. But this leads to a technical nuisance: the composition of correspondences is only associative up to isomorphism, and units also work only up to isomorphism. 2-categories with this technical problem are also called bicategories or *weak* 2-categories. They can be treated by specifying the 2-arrows that make associativity and units work and requiring suitable coherence laws for them. We will not discuss this here.

Definition 3.11. A Hilbert B-module $\mathcal{H}$ is called *full* if the inner products $\langle\xi,\eta\rangle$ for $\xi,\eta\in\mathcal{H}$ span a dense subspace of B.

Definition 3.12. A *Morita–Rieffel equivalence* between two C*-algebras A and B is a full correspondence $\mathcal{H}$ from A to B where the left action of A is given by an isomorphism $A\cong\mathbb{K}(\mathcal{H})$.

Let $\mathcal{H}$ be a Morita–Rieffel equivalence from A to B. Then $\mathcal{H}$ is a full Hilbert B-module. We turn $\mathcal{H}$ into a left A-module using the isomorphism $A\cong\mathbb{K}(\mathcal{H})$, and we use this isomorphism to view the map $(\xi,\eta)\mapsto|\xi\rangle\langle\eta|$ as an A-valued inner product on $\mathcal{H}$. With this structure, $\mathcal{H}$ becomes a full left Hilbert A-module. Furthermore, the two inner products are related by the condition

$$\langle\xi,\eta\rangle_A\cdot\zeta=\xi\cdot\langle\eta,\zeta\rangle_B\qquad\text{for all } \xi,\eta,\zeta\in\mathcal{H}.$$

This more symmetric definition of a Morita–Rieffel equivalence is Rieffel's original definition.

Lemma 3.13. *A correspondence $\mathcal{H}$ from A to B is a Morita–Rieffel equivalence if and only it is an equivalence in the correspondence 2-category, that is, there is a correspondence $\mathcal{H}^*$ from B to A such that $\mathcal{H}\otimes_B\mathcal{H}^*\cong A$ and $\mathcal{H}^*\otimes_A\mathcal{H}\cong B$.*

3.3. Crossed products for group actions by correspondences. Our new correspondence 2-category also yields a more general notion of group action where automorphisms α_g of A are replaced by correspondences from A to A. Since these correspondences must be equivalences, they are actually Morita–Rieffel equivalences. Here we briefly want to observe that the construction of crossed product C*-algebras extends very naturally to such more general actions. In fact, it could be said that the construction becomes more natural.

A group action by correspondences of a group G on a C*-algebra A consists of correspondences α_g for $g\in G$, an isomorphism $u\colon A\cong\alpha_1$, and isomorphisms $\omega_{g,h}\colon\alpha_g\otimes_A\alpha_h\cong\alpha_{gh}$. These are subject to coherence conditions as in Section 2.6.

There is no need to require analogues of (2.11) and (2.12): these two equations are already expressed by the requirement that $\omega_{g,h}$ is an isomorphism from $\alpha_g\otimes_A\alpha_h$ to α_{gh} and u is an isomorphism from the unit correspondence A to α_1.

The coherence laws regarding u become $u\otimes_A\mathrm{Id}_{\alpha_g}=\omega_{1,g}^*$ and $\mathrm{Id}_{\alpha_g}\otimes_A u=\omega_{g,1}^*$ for all $g\in G$. Equation 2.13 becomes

$$(\omega_{g,h}^*\otimes_A\mathrm{Id}_{\alpha_k})\cdot\omega_{gh,k}^*=(\mathrm{Id}_{\alpha_g}\otimes_A\omega_{h,k}^*)\cdot\omega_{g,hk}^*$$

for all $g,h,k\in G$; both sides are unitaries from $\alpha_g\otimes_A\alpha_h\otimes_A\alpha_k$ to α_{ghk}. As in Exercise 2.14, these coherence laws show that u is redundant provided all α_g are equivalences.

Here is what covariant representations are:

Definition 3.14. A *covariant* representation of a group action by correspondence is a transformation in the correspondence 2-category to the trivial action on $\mathbb{C}$.

More explicitly, a transformation from a group action $(A,\alpha_g,\omega_{g,h})$ to $\mathbb{C}$ involves a correspondence $\mathcal{H}$ from A to $\mathbb{C}$ and isomorphisms $V_g\colon\mathcal{H}\to\alpha_g\otimes_A\mathcal{H}$ for all $g\in G$

such that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathcal{H} & \xrightarrow{V_g} & \alpha_g \otimes_A \mathcal{H} & \xrightarrow{\mathrm{Id}_{\alpha_g} \otimes_A V_h} & \alpha_g \otimes_A \alpha_h \otimes_A \mathcal{H} \\
 & \searrow^{V_{gh}} & & & \downarrow{\scriptstyle \omega_{g,h} \otimes \mathrm{Id}_{\mathcal{H}}} \\
 & & & & \alpha_{gh} \otimes_A \mathcal{H}.
\end{array}
\tag{3.15}
$$

A correspondence from A to $\mathbb{C}$ is just a non-degenerate representation π of A on a Hilbert space. If the action α_g is by automorphisms in the usual sense, then $\alpha_g \otimes_A \mathcal{H}$ is the representation $\pi \circ \alpha_g \colon A \to \mathbb{B}(\mathcal{H})$ on the same Hilbert space. Thus the isomorphisms of correspondences V_g are simply unitary intertwiners on $\mathcal{H}$ from π to $\pi \circ \alpha_g$, that is, we get the condition

$$\pi(\alpha_g(a)) = V_g^* \pi(a) V_g.$$

This differs from the usual definition of a covariant representation only in that we have replaced V_g by V_g^*. The commutative diagram (3.15) becomes $\pi(\omega_{g,h}) \cdot V_h \cdot V_g = V_{gh}$, that is, $g \mapsto V_g^*$ is a representation of G up to a correction by $\pi(\omega_{g,h}^*)$. Thus Definition 3.14 reduces to the usual definition of a covariant representation for group actions by automorphisms. Needless to say, we get the expected notion of a covariant representation for Busby–Smith twisted actions.

Given a general action by correspondences, we may also define covariant representations on a C*-algebra D as transformations in the correspondence 2-category to D with trivial G-action. Using this notion, we may then define the crossed product by the following universal property: its morphisms to D are in natural bijection with covariant representations by multipliers of D.

A concrete construction is also not very difficult. The unitary $V_g^* \colon \alpha_g \otimes_A \mathcal{H} \to \mathcal{H}$ induces a map $\phi_g \colon \alpha_g \to \mathbb{B}(\mathcal{H})$, with $\phi_g(x)$ mapping $\xi \in \mathcal{H}$ to $V_g^*(x \otimes \xi) \in \mathcal{H}$. Let

$$A[G] := \bigoplus_{g \in G} \alpha_g,$$

then a covariant representation induces a map $\bigoplus \phi_g \colon A[G] \to \mathbb{B}(\mathcal{H})$. There is a canonical *-algebra structure on $A[G]$ for which all these maps are *-homomorphisms. The enveloping C*-algebra of this *-algebra has the correct universal property.

Once again, the notion of a group action by correspondences is not new: these generalised group actions are equivalent to *Fell bundles*, and the above notion of covariant representation is the traditional notion of representation of a Fell bundle. The crossed product described above is the sectional C*-algebra of a Fell bundle.

We may also let semigroups act on C*-algebras by correspondences. The resulting notion of a semigroup action by correspondences is essentially equivalent to the notion of a *product system*. In our setting, however, the multiplication becomes a map $E_s \times E_t \to E_{ts}$, not $E_s \times E_t \to E_{st}$. That is, we replace any semigroup by its opposite semigroup. This ensures that a semigroup homomorphism from a semigroup to the endomorphism semigroup of a C*-algebra induces an action by correspondences of the same semigroup.

EXAMPLE 3.16. Let a group G act on a C*-algebra A by automorphisms $(\alpha_g)_{g \in G}$ in the usual sense and let B be Morita equivalent to A by some equivalence A,B-bimodule $\mathcal{H}$. Then $\beta_g := \mathcal{H}^* \otimes_A \alpha_g \otimes_A \mathcal{H}$ is a self-correspondence on B, and these correspondences together with the canonical isomorphisms $\beta_g \otimes_B \beta_h \to \beta_{hg}$ and $B \to \beta_1$ define an action of G on B by correspondences.

Conversely, it is shown in [4] that any group action by correspondences is equivalent to one of this form. Roughly speaking, the notion of a group action by correspondences captures what is Morita-invariant about group actions.

4. Higher groupoids as symmetries of non-commutative spaces

The results above show that the 2-category structure on C*-algebras is useful to study Morita–Rieffel equivalence, equivalence of crossed products, and generalisations of crossed products to twisted group actions or group actions by correspondences. But so far, we have only considered actions of groups in the usual sense. We may, of course, generalise all this to actions of 2-groupoids. We believe that this generalisation is very natural because we consider 2-groupoids to be the most natural symmetry objects in non-commutative geometry, following [3].

Non-commutative spaces are often quotient spaces encoded by groupoids. We should expect their symmetries to be quotient groups. And these quotient groups are described by 2-categories. We first explain this point of view for groupoids.

4.1. Groupoids. A groupoid consists of two sets, the objects and the arrows, together with some additional algebraic structure: each arrow has a source and a range object, and there is an associative composition defined for arrows with compatible range and source; there are unit arrows for all objects, and each arrow is invertible. Thus groupoids may either be viewed as generalised groups or as generalised spaces, emphasising the arrows or the objects.

A useful picture for our purposes is that a groupoid encodes a parametrisation of a space. The objects of the groupoid are the parameters for points in our space. The same point may be described by different parameters. The arrows are reasons for two parameters to give the same point. There may be several reasons for two parameters x and y to give the same point. There may even be interesting reasons for x and x to give the same point in the quotient space. This leads to the isotropy of the groupoid (arrows from x to x). Of course, if we have reasons why x and y, and y and z yield the same point, then there will be a reason for x and z to yield the same point. This leads to the composition of arrows, and identities and inverses are also contained in this interpretation: the identity on x is the obvious reason why x and x give the same point. And if x and y give the same point, so do y and x, and a reason for the former yields a reason for the latter. In addition, to get a groupoid we require algebraic assumptions for units and inverses and associativity. These ensure that our reasoning is tame enough to work with it.

Example 4.1. Suppose we want to parametrise the space of all subspaces of $\mathbb{R}^n$. We may parametrise such a subspace by specifying a set of vectors that span it. For dimension reasons, each subspace of $\mathbb{R}^n$ may be spanned by n vectors. Thus we take $(\mathbb{R}^n)^n \cong \mathbb{M}_n(\mathbb{R})$ as our parameter space. A matrix A parametrises the subspace spanned by its columns or, equivalently, the range of the linear map associated to A.

Of course, a subspace of $\mathbb{R}^n$ may correspond to many different matrices. If A and B have the same range, then there is an invertible matrix T with $A = BT$. Thus the arrow space of our groupoid is $\mathbb{M}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$, where (A, T) is the reason why A and AT describe the same subspace. The composition of arrows is essentially the multiplication in $\mathrm{GL}_n(\mathbb{R})$. This groupoid has rather large isotropy.

Alternatively, we could first fix the dimension k of a subspace and then pick k linearly independent vectors that span a subspace of dimension k. This leads to a different, non-equivalent groupoid whose objects are (ordered) families of linearly independent vectors in $\mathbb{R}^n$. Two such families of different cardinality certainly give different subspaces, so that there are no arrows between them. Two families of the same cardinality k give the same subspace if and only if they are related by a matrix in $\mathrm{GL}_k(\mathbb{R})$. The resulting groupoid has trivial isotropy.

This view on groupoids also leads to a definition of higher groupoids: here the arrow space is parametrised by a space of arrows, with 2-arrows giving reasons for arrows to be equivalent. Since the arrows themselves are reasons for points to be equivalent, we expect to find the three composition operations in a 2-category. In addition, we must require some algebraic conditions like associativity of the various compositions.

4.2. The symmetries of rotation algebras. Now we argue that 2-groupoids naturally appear as symmetries of non-commutative spaces.

Let $\vartheta \in [0,1)$ and let $\lambda := \exp(2\pi i\vartheta)$. Recall that the rotation algebra A_ϑ is the universal C*-algebra generated by two unitaries U and V that satisfy the commutation relation $VU = \lambda UV$. This C*-algebra carries a natural *gauge action* of the 2-torus $\mathbb{T}^2$ by $\alpha_{(z,w)}(U^mV^n) := z^m w^n U^m V^n$ for all $m, n \in \mathbb{Z}$. This action is effective, that is, $\alpha_{z,w} = \mathrm{Id}$ only for $z = w = 1$. However, there are many parameters z, w for which $\alpha_{z,w}$ is inner because

$$\mathrm{Ad}_{U^aV^b}(U^mV^n) = U^aV^bU^mV^nV^{-b}U^{-a} = \lambda^{bm-an}U^mV^n.$$

Thus $\mathrm{Ad}_{U^aV^b} = \alpha_{\lambda^b,\lambda^{-a}}$.

To take this into account, we turn $\mathbb{T}^2$ into a 2-groupoid by adding 2-arrows $(a,b)\colon (z,w) \Rightarrow (\lambda^b z, \lambda^{-a} w)$ for all $a,b \in \mathbb{Z}$, $z,w \in \mathbb{T}$. This 2-arrow is the reason why $\alpha_{z,w}$ and $\alpha_{\lambda^b z,\lambda^{-a} w}$ are equivalent automorphisms, that is, differ by inner automorphisms. It is easy to define horizontal and vertical products for these bigons. The map that sends $(z,w) \mapsto \alpha_{z,w}$ and $(a,b)\colon (z,w) \Rightarrow (\lambda^b z, \lambda^{-a} w)$ to the unitary U^aV^b is an action of the 2-groupoid just described on the rotation algebra A_ϑ. This 2-groupoid is the non-commutative substitute for the quotient group $(\mathbb{T}/\lambda^{\mathbb{Z}})^2$, which is either non-Hausdorff (for irrational ϑ) or has large isotropy (for rational ϑ).

4.3. Other notions of symmetry? The above example shows in which way a 2-groupoid may act on a C*-algebra and thus encode its symmetries. There are also other interesting ways to describe symmetries of C*-algebras.

Locally compact quantum groups provide an established notion of this kind. The idea of a locally compact quantum group is to equip the C*-algebra $\mathrm{C}_0(G)$ for a locally compact group G with additional structure that reflects the group structure on G and then to allow arbitrary C*-algebras with the same kind of extra structure. The multiplication on G clearly induces a morphism from $\mathrm{C}_0(G)$ to $\mathrm{C}_0(G) \otimes \mathrm{C}_0(G) \cong \mathrm{C}_0(G \times G)$, and this is enough to uniquely determine the group structure on G. Correspondingly, a locally compact quantum group is a pair (A, Δ), where A is a C*-algebra and Δ is a morphism from A to $A \otimes A$, subject to several conditions. To state these conditions, we require the existence of further structure like Haar weights or multiplicative unitaries. But the isomorphism type of the locally compact quantum group only depends on the pair (A, Δ).

Unfortunately, locally compact quantum groups cannot be used to encode the group structure on the irrational rotation algebras. Recall that the irrational rotation algebra encodes the non-commutative space $\mathbb{T}/\lambda^{\mathbb{Z}}$, which is a group. But Piotr Sołtan [6] has shown that an irrational rotation algebra carries no structure of locally compact quantum group whatsoever. The following exercise gives an idea why there exist locally compact spaces with no group structure, and hence C*-algebras with no quantum group structure.

Exercise 4.2. *Show that there is no group structure on the locally compact space $G = [0, 1)$. Use that small open neighbourhoods of the points $0 \in G$ and $1/2 \in G$ are not homeomorphic.*

It would be highly desirable to encode the group structure on the quotient space $\mathbb{T}/\lambda^{\mathbb{Z}}$ in a non-commutative object like the irrational rotation algebra, but this seems to be impossible. This problem is a symptom of a more fundamental problem: the construction of groupoid C*-algebras is not functorial.

To see this, we do not even have to understand the definition of groupoid C*-algebras (which is not discussed above). It suffices to study two special classes of groupoids.

First, we may consider groups as groupoids with only one object. The universal property of the group C*-algebra shows that any continuous group homomorphism $f\colon G \to H$ between two locally compact groups induces a morphism from $\mathrm{C}^*(G)$ to $\mathrm{C}^*(H)$. Thus (full) group C*-algebras are *covariantly* functorial for group homomorphisms.

Secondly, we may consider spaces as groupoids with only identity arrows. Continuous functors in this case amount to continuous maps. And a continuous map $f\colon X \to Y$ induces a morphism from $\mathrm{C}_0(Y)$ to $\mathrm{C}_0(X)$. Since the groupoid C*-algebra for a space X viewed as a groupoid is just $\mathrm{C}_0(X)$, this shows that groupoid C*-algebras are *contravariantly* functorial for spaces viewed as groupoids.

Taken together, groupoid C*-algebras are sometimes covariantly functorial, sometimes contravariantly functorial. But these things cannot be combined. When taken on the category of all groupoids, the groupoid C*-algebras are neither a covariant nor a contravariant functor. Isomorphisms of locally compact groupoids induce isomorphisms of groupoid C*-algebras, but general continuous functors induce nothing.

The multiplication on $\mathbb{T}/\lambda^{\mathbb{Z}}$ may be encoded by a functor between appropriate groupoids describing $\mathbb{T}/\lambda^{\mathbb{Z}} \times \mathbb{T}/\lambda^{\mathbb{Z}}$ and $\mathbb{T}/\lambda^{\mathbb{Z}}$, but this functor induces nothing on the level of C*-algebras.

A way out is to use a different category of groupoids where the arrows are not functors (see [2]). But in this category, $\mathbb{T}/\lambda^{\mathbb{Z}}$ no longer carries a group structure.

The above problem is improved by passing to Kasparov theory. A basic ingredient in index theory is wrong-way maps

$$f_!\colon \mathrm{K}^*(X) \to \mathrm{K}^*(Y)$$

associated to certain maps $f\colon X \to Y$ (say, take f to be smooth and K-oriented). This construction yields a covariant functor from the category of smooth manifolds with smooth K-oriented maps as morphisms to Kasparov theory. It is possible to extend this to proper Lie groupoids, using an appropriate notion of smooth K-oriented map. On this category of groupoids, taking groupoid C*-algebras is a covariant functor to Kasparov theory. But this homotopy-invariant construction is not fine enough for some applications. Here algebraic K-theory would be a more suitable invariant. But in what sense and generality does wrong-way functoriality work in algebraic K-theory, as opposed to topological K-theory?

Working with 2-groupoids to some extent solves these problems: at least we may describe in what sense the group $\mathbb{T}/\lambda^{\mathbb{Z}}$ acts on itself by left translations; this is exactly the action of the 2-group described above on the rotation algebra. But this passage to 2-groupoids has not completely resolved the problem. When we try to formulate an analogue of the Baum–Connes conjecture for 2-groupoids such as the one discussed above, then the problem reappears.

Is there a good analogue of the Baum–Connes conjecture for 2-groupoids?

These questions lead us way beyond the scope of these introductory lectures.

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Examples and Applications of Noncommutative Geometry and K-Theory

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Abstract. These are informal notes from my course at the 3^{era} Escuela de Invierno Luis Santaló-CIMPA Research School on Topics in Noncommutative Geometry. The notes follow the style of the lectures, so I've attempted to give the reader the flavor of the subject without burdening him or her with a lot of technicalities. So I apologize to the experts for oversimplifying some things and for not giving complete references. However, in some cases I have included topics which were skipped in the original lectures due to lack of time.

The course basically is divided into two (related) parts. Sections 1–3 deal with Kasparov's KK-theory and some of its applications. Sections 4–5 deal with one of the most fundamental examples in noncommutative geometry, the noncommutative 2-torus.

Contents

1. Introduction to Kasparov's KK-theory

1.1. Why KK? KK-theory is a bivariant version of topological K-theory, defined for C^*-algebras, with or without a group action. It can be defined for either real or complex algebras, but in these notes we will stick to complex algebras for simplicity. Thus if A and B are complex C^*-algebras, subject to a minor

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technical requirement (that B be σ-unital, which is certainly the case if it is either unital or separable), an abelian group $KK(A, B)$ is defined, with the property that $KK(\mathbb{C}, B) = K(B) = K_0(B)$ if the first algebra A is just the scalars. (For the basic properties of K_0, I refer you to the courses by Reich and Karoubi.) Dually, $KK(A, \mathbb{C})$ is *contravariant* in A and behaves like a "dual" to $K(A)$. Furthermore, there is an associative bilinear product

$$\otimes_B\colon\ KK(A, B) \times KK(B, C) \to KK(A, C),$$

about which we will say a lot more in Section 1.3. The theory was defined by Gennadi Kasparov in a remarkable series of papers: [**39, 40, 41**]. However, the definition at first seems highly technical and unmotivated, so it's worth first seeing where the theory comes from and why one might be interested in it. For purposes of this introduction, we will only be concerned with the case where A and B are commutative. Thus $A = C_0(X)$ and $B = C_0(Y)$, where X and Y are locally compact Hausdorff spaces. We will abbreviate $KK(C_0(X), C_0(Y))$ to $KK(X, Y)$. It is worth pointing out that the study of $KK(X, Y)$ (without considering $KK(A, B)$ more generally) is already highly nontrivial, and encompasses most of the features of the general theory. Note that we expect to have $KK(\mathbb{C}, C_0(Y)) = KK(\mathrm{pt}, Y) = K(Y)$, the K-theory of Y with compact support. Recall that this is the Grothendieck group of homotopy classes of complexes of vector bundles over Y that are exact off a compact set. (See [**72**, §3] for this point of view.) It's actually enough to take complexes of length 2, so an element of $K(Y)$ is represented by a pair of vector bundles V and V' over Y, together with a morphism of vector bundles $V \xrightarrow{\varphi} V'$ that is an isomorphism off a compact set. (Note that when Y is compact, the condition on φ is vacuous, and hence φ can be homotoped to 0, so we can dispense with it entirely in this case. Thus for Y compact, we just get usual K-theory, the Grothendieck group of isomorphism classes of vector bundles.) Alternatively, $K(Y)$ can be identified with the reduced K-theory $\widetilde{K}(Y_+)$ of the one-point compactification Y_+ of Y.

A good place to start in trying to understand KK is Atiyah's paper [**4**] on the Bott periodicity theorem. Bott periodicity, or more generally, the Thom isomorphism theorem for a complex vector bundle, asserts that if $p\colon E \to X$ is a complex vector bundle (more generally, one could take an even-dimensional real vector bundle with a spin^c structure), then there is a natural isomorphism $\beta_E\colon K(X) \to K(E)$, called the Thom isomorphism in K-theory. In the special case where $X = \mathrm{pt}$, E is just $\mathbb{C}^n$ for some n, and we are asserting that there is a natural isomorphism $\mathbb{Z} = K(\mathrm{pt}) \to K(\mathbb{C}^n) = K(\mathbb{R}^{2n}) = \widetilde{K}(S^{2n})$, the Bott periodicity map. The map β_E can be described by the formula $\beta_E(a) = p^*(a) \cdot \tau_E$. Here $p^*(a)$ is the pull-back of $a \in K(X)$ to E. Since a had compact support, $p^*(a)$ has compact support in the base direction of E, but is constant on fibers of p, so it certainly does not have compact support in the fiber direction. However, we can multiply it by the *Thom class* τ_E, which does have compact support along the fibers, and the product will have compact support in both directions, and will thus give a class in $K(E)$ (remember that since E is necessarily noncompact, assuming $n > 0$, we need to use K-theory with compact support). The Thom class τ_E, in turn, can be described [**72**, §3] as an explicit complex $\bigwedge^\bullet p^*E$ over E. The vector bundles in this complex are the exterior powers of E pulled back from X to E, and the map at a point $e \in E_x$ from $\bigwedge^j E_x$ to $\bigwedge^{j+1} E_x$ is simply exterior product with

e. This complex has compact support in the fiber directions since it is exact off the zero-section of E. (If $e \neq 0$, then the kernel of $e \wedge —$ is spanned by products $e \wedge \omega$.)

So far this is all simple vector bundle theory and KK is not needed. But it comes in at the next step. How do we prove that β_E is an isomorphism? The simplest way would be to construct an inverse map $\alpha_E : K(E) \to K(X)$. But there is no easy way to describe such a map using topology alone. As Atiyah recognized, the easiest way to construct α_E uses elliptic operators, in fact the family of Dolbeault operators along the fibers of E. Thus whether we like it or not, some analysis comes in at this stage. In more modern language, what we really want is the class α_E in $KK(E, X)$ corresponding to this family of operators, and the verification of the Thom isomorphism theorem is a Kasparov product calculation, the fact that α_E is a KK inverse to the class $\beta_E \in KK(X, E)$ described (in slightly different terms) before. Atiyah also noticed [**4**] that it's really just enough (because of certain identities about products) to prove that α_E is a one-way inverse to β_E, or in other words, in the language of Kasparov theory, that $\beta_E \otimes_E \alpha_E = 1_X$. This comes down to an index calculation, which because of naturality comes down to the single calculation $\beta \otimes_{\mathbb{C}} \alpha = 1 \in KK(\mathrm{pt}, \mathrm{pt})$ when X is a point and $E = \mathbb{C}$, which amounts to the Riemann-Roch theorem for $\mathbb{CP}^1$.

What then is $KK(X, Y)$ when X and Y are locally compact spaces? An element of $KK(X, Y)$ defines a map of K-groups $K(X) \to K(Y)$, but is more than this; it is in effect a *natural* family of maps of K-groups $K(X \times Z) \to K(Y \times Z)$ for arbitrary Z. Naturality of course means that one gets a natural transformation of functors, from $Z \mapsto K(X \times Z)$ to $Z \mapsto K(Y \times Z)$. (Nigel Higson has pointed out that one can use this in reverse to *define* $KK(X, Y)$ as a *natural* family of maps of K-groups $K(X \times Z) \to K(Y \times Z)$ for arbitrary Z. The reason why this works will be explained in Section 3.5.) In particular, since $KK(X \times \mathbb{R}^j)$ can be identified with $K^{-j}(X)$, an element of $KK(X, Y)$ defines a graded map of K-groups $K^j(X) \to K^j(Y)$, at least for $j \leq 0$ (but then for arbitrary j because of Bott periodicity). The example of Atiyah's class $\alpha_E \in KK(E, X)$, based on a family of elliptic operators over E parametrized by X, shows that one gets an element of the bivariant K-group $KK(X, Y)$ from a family of elliptic operators over X parametrized by Y. The element that one gets should be invariant under homotopies of such operators. Hence Kasparov's definition of $KK(A, B)$ is based on a notion of homotopy classes of generalized elliptic operators for the first algebra A, "parametrized" by the second algebra B (and thus commuting with a B-module structure).

1.2. Kasparov's original definition. As indicated above in Section 1.1, an element of $KK(A, B)$ is roughly speaking supposed to be a homotopy class of families of elliptic pseudodifferential operators[1] over A parametrized by B. For

[1]For those familiar with differential operators but not pseudodifferential operators, the latter are a larger class in which one has a good "functional calculus." Even though the elliptic operators of greatest interest are in fact differential operators, one needs this larger class because the inverse or the square root of a differential operator (when this makes sense) is a pseudodifferential operator, but not a differential operator. Differential operators D are local; for a C^∞ function f, $Df(x)$ only depends on f and a finite number of its derivatives at x. Pseudodifferential operators are instead *pseudolocal*; they are *approximately local* up to terms of lower order.

technical reasons, it's convenient to work with self-adjoint bounded operators[2], but it's well-known that the most interesting elliptic operators send sections of one vector bundle to sections of another. The way to get around this is to take our operators to be self-adjoint, but odd with respect to a grading, i.e., of the form

$$T = T^* = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}. \tag{1.1}$$

The operator F here really does act between different spaces, but T, built from F and F^*, is self-adjoint, making it easier to work with. Then we need various conditions on T that correspond to the terms "elliptic," "pseudodifferential," and "parametrized by B." So this boils down to the following. A class in $KK(A, B)$ is represented by a *Kasparov A-B-bimodule*, that is, a $\mathbb{Z}/2$-graded (right) Hilbert B-module $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, together with a B-linear operator $T \in \mathcal{L}(\mathcal{H})$ of the form (1.1), and a (grading-preserving) $*$-representation ϕ of A on $\mathcal{H}$, subject to the conditions that $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ and $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ for all $a \in A$. These conditions require a few comments. The condition that $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ is "ellipticity" and the condition that $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ is "pseudolocality." If $B = \mathbb{C}$, a Hilbert B-module is just a Hilbert space, $\mathcal{L}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, and $\mathcal{K}$ is the algebra of compact operators on $\mathcal{H}$. If $B = C_0(Y)$, a Hilbert B-module is equivalent to a continuous field of Hilbert spaces over Y. In this case, $\mathcal{K}(\mathcal{H})$ is the associated algebra of norm-continuous fields of compact operators, while $\mathcal{L}(\mathcal{H})$ consists of continuous fields (continuity taken in the strong-$*$ operator topology) of bounded Hilbert space operators. If X is another locally compact space, then it is easy to see that Kasparov's conditions are an abstraction of a continuous family of elliptic pseudolocal Hilbert space operators over X, parametrized by Y. Finally, if B is arbitrary, a Hilbert B-module means a right B-module equipped with a B-valued inner product $\langle -, - \rangle_B$, right B-linear in the second variable, satisfying $\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*$ and $\langle \xi, \xi \rangle_B \geq 0$ (in the sense of self-adjoint elements of B), with equality only if $\xi = 0$. Such an inner product gives rise to a norm on $\mathcal{H}$: $\|\xi\| = \|\langle \xi, \xi \rangle_B\|_B^{1/2}$, and we require $\mathcal{H}$ to be complete with respect to this norm. Given a Hilbert B-module $\mathcal{H}$, there are two special C^*-algebras associated to it. The first, called $\mathcal{L}(\mathcal{H})$, consists of bounded B-linear operators a on $\mathcal{H}$, admitting an adjoint a^* with the usual property that $\langle a\xi, \eta \rangle_B = \langle \xi, a^*\eta \rangle_B$ for all ξ, $\eta \in \mathcal{H}$. Unlike the case where $B = \mathbb{C}$, existence of an adjoint is not automatic, so it must be explicitly assumed. Then inside $\mathcal{L}(\mathcal{H})$ is the ideal of *B-compact operators*. This is the closed linear span of the "rank-one operators" $T_{\xi,\eta}$ defined by $T_{\xi,\eta}(\nu) = \xi\langle \eta, \nu \rangle_B$. Note that

$$\begin{aligned}
\langle T_{\xi,\eta}(\nu), \omega \rangle_B &= \langle \xi\langle \eta, \nu \rangle_B, \omega \rangle_B = \langle \omega, \xi\langle \eta, \nu \rangle_B \rangle_B^* \\
&= \left(\langle \omega, \xi \rangle_B \langle \eta, \nu \rangle_B\right)^* = \langle \eta, \nu \rangle_B^* \langle \omega, \xi \rangle_B^* \\
&= \langle \nu, \eta \rangle_B \langle \xi, \omega \rangle_B = \langle \nu, \eta\langle \xi, \omega \rangle_B \rangle_B \\
&= \langle \nu, T_{\eta,\xi}(\omega) \rangle_B,
\end{aligned}$$

so that $T_{\xi,\eta}^* = T_{\eta,\xi}$. It is also obvious that if $a \in \mathcal{L}(\mathcal{H})$, then $aT_{\xi,\eta} = T_{a\xi,\eta}$, while $T_{\xi,\eta}a = T_{\eta,\xi}^*\big(a^*\big)^* = \big(a^*T_{\eta,\xi}\big)^* = T_{a^*\eta,\xi}^* = T_{\xi,a^*\eta}$, so these rank-one operators generate an ideal in $\mathcal{L}(\mathcal{H})$, which is just the usual ideal of compact operators in

[2]Often we want to apply the theory to self-adjoint differential operators D, which are never bounded on L^2 spaces. The trick is to replace D by the pseudodifferential operator $D(1+D^2)^{-\frac{1}{2}}$, which has the same index theory as D and is bounded.

case $B = \mathbb{C}$. For more on Hilbert C^*-modules and the C^*-algebras acting on them, see [**46**] or [**59**, Ch. 2].

The simplest kind of Kasparov bimodule is associated to a homomorphism $\phi\colon A \to B$. In this case, we simply take $\mathcal{H} = \mathcal{H}_0 = B$, viewed as a right B-module, with the B-valued inner product $\langle b_1, b_2\rangle_B = b_1^* b_2$, and take $\mathcal{H}_1 = 0$ and $T = 0$. In this case, $\mathcal{L}(\mathcal{H}) = M(B)$ (the multiplier algebra of B, the largest C^*-algebra containing B as an essential ideal), and $\mathcal{K}(\mathcal{H}) = B$. So ϕ maps A into $\mathcal{K}(\mathcal{H})$, and even though $T = 0$, the condition that $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ is satisfied for any $a \in A$.

One special case which is especially important is the case where $A = B$ and ϕ is the identity map. The above construction then yields a distinguished element $1_A \in KK(A, A)$, which will play an important role later.

In applications to index theory, Kasparov A-B-bimodules typically arise from elliptic (or hypoelliptic) pseudodifferential operators. However, there are other ways to generate Kasparov bimodules, which we will discuss in Section 1.4 below.

So far we have explained what the *cycles* are for KK-theory, but not the equivalence relation that determines when two such cycles give the same KK-element. First of all, there is a natural associative addition on Kasparov bimodules, obtained by taking the direct sum of Hilbert B-modules and the block direct sum of homomorphisms and operators. Then we divide out by the equivalence relation generated by addition of *degenerate* Kasparov bimodules (those for which for all $a \in A$, $\phi(a)(T^2 - 1) = 0$ and $[\phi(a), T] = 0$) and by *homotopy.* (A homotopy of Kasparov A-B-bimodules is just a Kasparov A-$C([0, 1], B)$-bimodule.) Then it turns out that the resulting semigroup $KK(A, B)$ is actually an abelian group, with inversion given by reversing the grading, i.e., reversing the roles of $\mathcal{H}_0$ and $\mathcal{H}_1$, and interchanging F and F^*. Actually, it was not really necessary to divide out by degenerate bimodules, since if $(\mathcal{H}, \phi, T)$ is degenerate, then $(C_0((0, 1], \mathcal{H})$ (along with the action of A and the operator which are given by ϕ and T at each point of $(0, 1]$) is a homotopy from $(\mathcal{H}, \phi, T)$ to the 0-module.

An interesting exercise is to consider what happens when $A = \mathbb{C}$ and B is a unital C^*-algebra. Then if $\mathcal{H}_0$ and $\mathcal{H}_1$ are finitely generated projective (right) B-modules and we take $T = 0$ and ϕ to be the usual action of $\mathbb{C}$ by scalar multiplication, we get a Kasparov $\mathbb{C}$-B-bimodule corresponding to the element $[\mathcal{H}_0] - [\mathcal{H}_1]$ of $K_0(B)$. With some work one can show that this gives an isomorphism between the Grothendieck group $K_0(B)$ of usual K-theory and $KK(\mathbb{C}, B)$. By considering what happens when one adjoins a unit, one can then show that there is still a natural isomorphism between $K_0(B)$ and $KK(\mathbb{C}, B)$, even if B is nonunital.

Another important special case is when A and B are Morita equivalent in the sense of Rieffel [**60, 63**] — see [**59**] for a very good textbook treatment. That means we have an A-B-bimodule X with the following special properties:

(1) X is a right Hilbert B-module and a left Hilbert A-module.
(2) The left action of A is by bounded adjointable operators for the B-valued inner product, and the right action of B is by bounded adjointable operators for the A-valued inner product.
(3) The A- and B-valued inner products on X are compatible in the sense that if $\xi, \eta, \nu \in X$, then ${}_A\langle \xi, \eta\rangle \nu = \xi \langle \eta, \nu\rangle_B$.
(4) The inner products are "full," in the sense that the image of ${}_A\langle —, —\rangle$ is dense in A, and the image of $\langle —, —\rangle_B$ is dense in B.

Under these circumstances, X defines classes in $[X] \in KK(A,B)$ and $[\widetilde{X}] \in KK(B,A)$ which are inverses to each other (with respect to the product discussed below in Section 1.3). Thus as far as KK-theory is concerned, A and B are essentially equivalent. The construction of $[X]$ and of $[\widetilde{X}]$ is fairly straightforward; for example, to construct $[X]$, take $\mathcal{H}_0 = X$ (viewed as a right Hilbert B-module), $\mathcal{H}_1 = 0$, and $T = 0$, and let $\phi\colon A \to \mathcal{L}(\mathcal{H})$ be the left action of A (which factors through $\mathcal{L}(\mathcal{H})$ by axiom (2)). By axiom (4) (which is really the key property), any element of A can be approximated by linear combinations of inner products ${}_A\langle \xi, \eta\rangle$. For such an inner product, we have

$$\phi({}_A\langle \xi, \eta\rangle)\nu = \xi\langle \eta, \nu\rangle_B = T_{\xi,\eta}(\nu),$$

so the action of A on $\mathcal{H}$ is by operators in $\mathcal{K}(\mathcal{H})$, which is what is needed for the conditions for a Kasparov bimodule.

The prototype example of a Morita equivalence has $A = \mathbb{C}$, $B = \mathcal{K}(\mathcal{H})$ (we usually drop the $\mathcal{H}$ and just write $\mathcal{K}$ if the Hilbert space is infinite-dimensional and separable), and $X = \mathcal{H}$, with the B-valued inner product taking a pair of vectors in $\mathcal{H}$ to the corresponding rank-one operator. Thus from the point of KK-theory, $\mathbb{C}$ and $\mathcal{K}$ are essentially indistinguishable, and so are B and $B \otimes \mathcal{K}$ for any B. There is a converse [**11**]; separable C^*-algebras A and B are Morita equivalent if and only if $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic. (This condition, called *stable isomorphism*, is obviously satisfied by B and $B \otimes \mathcal{K}$, since $(B \otimes \mathcal{K}) \otimes \mathcal{K} \cong B \otimes (\mathcal{K} \otimes \mathcal{K}) \cong B \otimes \mathcal{K}$.) However, a Morita equivalence between A and B leads directly to a KK-equivalence, but not directly to an isomorphism $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ (which requires some arbitrary choices).

The most readable references for the material of this section are the book by Blackadar [**7**], Chapter VIII, and the "primer" of Higson [**32**].

1.3. Connections and the product. The hardest aspect of Kasparov's approach to KK is to prove that there is a well-defined, functorial, bilinear, and associative product $\otimes_B\colon KK(A,B) \times KK(B,C) \to KK(A,C)$. There is also an *external* product $\boxtimes\colon KK(A,B) \times KK(C,D) \to KK(A \otimes C, B \otimes D)$, where $\otimes$ denotes the completed tensor product. (For our purposes, the *minimal* or *spatial* C^*-tensor product will suffice. This is defined as follows. Suppose A and B are C^*-algebras represented on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $A \otimes B$ is the completion of the algebraic tensor product for the operator norm on the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. It turns out that up to $*$-isomorphism, this is independent of the choice of representations of A and of B. There is a big class of C^*-algebras, called the *nuclear* C^*-algebras, with the property that if one of the algebras A and B is nuclear, all C^*-tensor products of A with B coincide, in which case the spatial tensor product is the only C^*-tensor product. All commutative and type I C^*-algebras are nuclear.) The external product is actually built from the usual product using an operation called *dilation* (external product with 1). We can dilate a class $a \in KK(A,B)$ to a class $a \boxtimes 1_C \in KK(A\otimes C, B\otimes C)$, by taking a representative $(\mathcal{H}, \phi, T)$ for a to the bimodule $(\mathcal{H}\otimes C, \phi\otimes 1_C, T\otimes 1)$. Similarly, we can dilate a class $b \in KK(C,D)$ (on the other side) to a class $1_B \boxtimes b \in KK(B\otimes C, B\otimes D)$. Then

$$a \boxtimes b = (a \boxtimes 1_C) \otimes_{B\otimes C} (1_B \boxtimes b) \in KK(A \otimes C, B \otimes D),$$

and one can check that this is the same as what one gets by computing in the other order as $(1_A \boxtimes b) \otimes_{A\otimes D} (a \boxtimes 1_D)$.

The *Kasparov products*, as they are called, encompass the usual cup and cap products relating K-theory and K-homology. For example, the cup product in ordinary topological K-theory for a compact space X, $\cup\colon\ K(X)\times K(X)\to K(X)$, is a composite of two products. Given $a\in K(X)=KK(\mathbb{C},C(X))$ and $b\in K(X)=KK(\mathbb{C},C(X))$, we first form the external product $a\boxtimes b\in KK(\mathbb{C},C(X)\otimes C(X))=KK(\mathbb{C},C(X\times X))$. Then we have

$$a\cup b=(a\boxtimes b)\otimes_{C(X\times X)}\Delta,$$

where $\Delta\in KK(C(X\times X),C(X))$ is the class of the homomorphism defined by restriction of functions on $X\times X$ to the diagonal copy of X.

Similarly, we can obtain the cap product $\cap\colon\ K(X)\times K_0(X)\to K_0(X)$ as follows. The K-homology group $K_0(X)$ is "dual" to $K(X)$, and is given by the Kasparov group $KK(C(X),\mathbb{C})$. If $a\in K(X)=KK(\mathbb{C},C(X))$ and $b\in K_0(X)=KK(C(X),\mathbb{C})$, we view a as a class $\overline{a}\in KK(C(X),C(X))$ (by letting $C(X)$ act on the Kasparov module representing a on both the left and the right, which we can do since $C(X)$ is commutative)[3], and then form the Kasparov product $\overline{a}\otimes_{C(X)}b\in KK(C(X),\mathbb{C})=K_0(X)$.

In any event, it still remains to construct the product $KK(A,B)\times KK(B,C)\to KK(A,C)$. Suppose we have classes represented by $(\mathcal{E}_1,\phi_1,T_1)$ and $(\mathcal{E}_2,\phi_2,T_2)$, where $\mathcal{E}_1$ is a right Hilbert B-module, $\mathcal{E}_2$ is a right Hilbert C-module, $\phi_1\colon\ A\to\mathcal{L}(\mathcal{E}_1)$, $\phi_2\colon\ B\to\mathcal{L}(\mathcal{E}_2)$, T_1 essentially commutes with the image of ϕ_1, and T_2 essentially commutes with the image of ϕ_2. It is clear that we want to construct the product using $\mathcal{H}=\mathcal{E}_1\otimes_{B,\phi_2}\mathcal{E}_2$ and $\phi=\phi_1\otimes 1\colon\ A\to\mathcal{L}(\mathcal{H})$. The main difficulty is getting the correct operator T. In fact there is no canonical choice; the choice is only unique up to homotopy. The most convenient method of doing the construction seems to be using the notion of a *connection* due to Connes and Skandalis [**14**], nicely explained in [**7**, §18] or [**75**]. However, one should be aware that the existence of connections is not at all "abstract nonsense," but depends on a fairly deep result, the "Kasparov Technical Theorem" [**31**] (or one of its variants).

To motivate this, let's just consider a simple example that comes up in index theory, the construction of an "elliptic operator with coefficients in a vector bundle." Let T be an elliptic operator on a compact manifold M, which we take to be a bounded operator of the form (1.1) (acting on a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}$), and let E be a complex vector bundle over M. Often we want to form T_E, the same operator with coefficients in the vector bundle E. This is actually a special case of the Kasparov product, or of the cap product $[E]\cap[T]$. The sections $\Gamma(M,E)$ are a finitely generated projective $C(M)$-module $\mathcal{E}$; since $C(M)$ is commutative, we can regard this as a $C(M)$-$C(M)$-bimodule, with the same action on the left and on the right. Then $\mathcal{E}$ (concentrated entirely in degree 0, together with the 0-operator), defines a KK-class $[[E]]\in KK(M,M)$, while T defines a class $[T]$ in $KK(M,\mathrm{pt})$. Note that forgetting the left $C(M)$-action on $\mathcal{E}$ is the same as composing with inclusion of the scalars $\mathbb{C}\hookrightarrow C(M)$ to get from $[[E]]$ a class $[E]\in KK(\mathrm{pt},M)=K(M)$, which is the usual K-theory class of E. The class of the operator T_E will be the Kasparov product $[[E]]\otimes_M[T]\in KK(M,\mathrm{pt})$. Defining the operator, however, requires a choice of connection on the bundle E. One way to get this is to embed E as a direct summand in a trivial bundle $M\times\mathbb{C}^n$. Then orthogonal projection onto E is given by a self-adjoint projection $p\in C(M,M_n(\mathbb{C}))$. We can certainly form $T\otimes 1$

[3]Alternatively, one can also check that $\overline{a}=(a\boxtimes 1)\otimes_{C(X\times X)}\Delta$.

acting on $\mathcal{H} \otimes \mathbb{C}^n$, on which we have an obvious action of $C(M) \otimes M_n(\mathbb{C})$, but there is no reason why $T \otimes 1$ and p should commute, so there is no "natural" cut-down of T to E. Thus we simply take the compression $T' = p(T \otimes 1)p$ acting on $\mathcal{H}' = p\mathcal{H}$ with the obvious action ϕ' of $C(M)$. Since T commutes with the action of $C(M)$ up to compact operators, the commutator $[p, T \otimes 1]$ is also compact, so T' satisfies the requirements that $(T')^2 - 1 \in \mathcal{K}(\mathcal{H}')$ and $[\phi'(f), T'] \in \mathcal{K}(\mathcal{H}')$. Its Kasparov class is well-defined, even though there is great freedom in choosing the operator (corresponding to the freedom to embed E in a trivial bundle in many different ways). The reason is that when n is large enough, all vector bundle embeddings of E into $M \times \mathbb{C}^n$ are isotopic, and thus the operators obtained by the above construction will be homotopic in a way preserving the Kasparov requirements.

1.4. Cuntz's approach. Joachim Cuntz noticed in [**19**] that all Kasparov bimodules can be taken to come from the basic notion of a *quasihomomorphism* between C^*-algebras A and B. A quasihomomorphism $A \rightrightarrows D \trianglerighteq B$ is roughly speaking a formal difference of two homomorphisms $f_\pm \colon A \to D$, neither of which maps into B itself, but which agree modulo an ideal isomorphic to B. Thus $a \mapsto f_+(a) - f_-(a)$ is a linear map $A \to B$. Suppose for simplicity (one can always reduce to this case) that $D/B \cong A$, so that $f_\pm$ are two splittings for an extension

$$0 \to B \to D \to A \to 0.$$

Then for any *split-exact* functor F from C^*-algebras to abelian groups (meaning it sends split extensions to short exact sequences — an example would be $F(A) = K(A \otimes C)$ for some coefficient algebra C), we get an exact sequence with two splittings

$$0 \longrightarrow F(B) \longrightarrow F(D) \underset{(f_-)_*}{\overset{(f_+)_*}{\rightleftarrows}} F(A) \longrightarrow 0.$$

Thus $(f_+)_* - (f_-)_*$ gives a well-defined homomorphism $F(A) \to F(B)$, which we might well imagine should come from a class in $KK(A, B)$. (Think about Section 1.1, where we mentioned Higson's idea of *defining* $KK(X, Y)$ in terms of natural transformations of functors, from $Z \mapsto K(X \times Z)$ to $Z \mapsto K(Y \times Z)$. We will certainly get such a natural transformation from a quasihomomorphism $C_0(X) \rightrightarrows D \trianglerighteq C_0(Y) \otimes \mathcal{K}$, since $C_0(Y) \otimes \mathcal{K}$ and Y have the same K-theory.) And indeed, given a quasihomomorphism as above, we get a Kasparov A-B-bimodule, with $B \oplus B$ as the Hilbert B-module (with the obvious grading), with $\phi \colon A \to \mathcal{L}(B \oplus B)$ defined by

$$\begin{pmatrix} f_+ & 0 \\ 0 & f_- \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The "almost commutation" relation is

$$\left[\begin{pmatrix} f_+(a) & 0 \\ 0 & f_-(a) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & f_+(a) - f_-(a) \\ f_-(a) - f_+(a) & 0 \end{pmatrix} \in \mathcal{K}(B \oplus B),$$

since $\mathcal{K}(B \oplus B) = M_2(B)$. In the other direction, given a Kasparov A-B-bimodule, one can add on a degenerate bimodule and do a homotopy to reduce it to something roughly of this form, showing that all of $KK(A, B)$ comes from quasihomomorphisms (see [**7**, §17.6]).

The quasihomomorphism approach to KK makes it possible to define $KK(A, B)$ in a seemingly simpler way [**20**]. To do this, Cuntz observed that a quasihomomorphism $A \rightrightarrows D \trianglerighteq B$ factors through a *universal* algebra qA constructed as follows. Start with the *free product* C^*-algebra $QA = A * A$, the completion of linear combinations of words in two copies of A. QA is characterized by the universal property that its representations (on Hilbert spaces) are generated by two representations of A (that may not commute). There is an obvious surjective homomorphism $QA \twoheadrightarrow A$ obtained by identifying the two copies of A. The kernel of $QA \twoheadrightarrow A$ is called qA, and if

$$0 \longrightarrow B \longrightarrow D \underset{f_-}{\overset{f_+}{\leftleftarrows\!\!\!\longrightarrow}} A \longrightarrow 0$$

is a quasihomomorphism, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & qA & \longrightarrow & QA & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

with the first copy of A in QA mapping to D via f_+, and the second copy of A in QA mapping to D via f_-. Thus homotopy classes of (strict) quasihomomorphisms from A to B can be identified with homotopy classes of $*$-homomorphisms from qA to B, and $KK(A, B)$ turns out to be simply the set of homotopy classes of $*$-homomorphisms from qA to $B \otimes \mathcal{K}$.

1.5. Higson's approach. There is still another very elegant approach to KK-theory due to Nigel Higson [**30**]. Namely, one can construct an additive category **KK** whose objects are the separable C^*-algebras, and where the morphisms from A to B are given by $KK(A, B)$. Associativity and bilinearity of the Kasparov product, along with properties of the special elements $1_A \in KK(A, A)$, ensure that this is indeed an additive category. What Higson did is to give an alternative construction of this category. Namely, start with the *homotopy category* of separable C^*-algebras, where the morphisms from A to B are the homotopy classes of $*$-homomorphisms $A \to B$. Then **KK** is the smallest additive category with the same objects, these morphisms, plus enough additional morphisms so that two basic properties are satisfied:

(1) Matrix stability. If A is an object in **KK** (that is, a separable C^*-algebra) and if e is a rank-one projection in $\mathcal{K} = \mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, then the homomorphism $a \mapsto a \otimes e$, viewed as an element of $\mathrm{Hom}(A, A \otimes \mathcal{K})$, is an equivalence in **KK**, i.e., has an inverse in $KK(A \otimes \mathcal{K}, A)$.

(2) Split exactness. If $0 \longrightarrow A \longrightarrow B \overset{s}{\leftrightarrows} C \longrightarrow 0$ is a split short exact sequence of separable C^*-algebras, then for any separable C^*-algebra D,

$$0 \longrightarrow KK(D, A) \longrightarrow KK(D, B) \overset{s_*}{\leftrightarrows} KK(D, C) \longrightarrow 0$$

and

$$0 \longrightarrow KK(C, D) \overset{s^*}{\leftrightarrows} KK(B, D) \longrightarrow KK(A, D) \longrightarrow 0$$

are split exact.

Incidentally, if one just starts with the homotopy category and requires (1), matrix stability, that is already enough to guarantee that the resulting category has Hom-sets which are commutative monoids and that composition is bilinear [**67**, Theorem 3.1]. So it's not asking much additional to require that one have an additive category.

The proof of Higson's theorem very much depends on the Cuntz construction in Section 1.4 above. Basically, there are two main steps. The first is to show that KK is split exact in both variables. (That KK is homotopy invariant is obvious from the definition, and that it is matrix stable follows from our earlier comment that Morita equivalences give invertible KK elements.) For the other step, let $\mathbf{H}$ be the category obtained from the homotopy category of separable C^*-algebras by Higson's construction, and let $H(A, B)$ be the group of morphisms from A to B in $\mathbf{H}$. The first step gives a canonical map $H(A, B) \to KK(A, B)$, and we need to show this is surjective. Given Cuntz's theorem that any class in $KK(A, B)$ arises from a $*$-homomorphism $qA \to B \otimes \mathcal{K}$, and the fact that matrix stability means B and $B \otimes \mathcal{K}$ are equivalent in $\mathbf{H}$, it suffices to show that A and qA are equivalent in $\mathbf{H}$. For this we use the commutative diagram with split exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & qA & \longrightarrow & QA & \overset{s}{\underset{\longrightarrow}{\longleftarrow}} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus A & \overset{s}{\underset{\longrightarrow}{\longleftarrow}} & A & \longrightarrow & 0, \end{array}$$

where the downward arrow ψ sends the two copies of A in QA to the two copies of A in $A \oplus A$. The downward arrows give canonical elements $[\phi] \in H(qA, A)$ and $[\psi] \in H(QA, A \oplus A)$. But $[\psi]$ has an inverse in $H(A \oplus A, QA)$ represented by the map

$$\eta \colon (a_1, a_2) \mapsto \begin{pmatrix} i_1(a_1) & 0 \\ 0 & i_2(a_2) \end{pmatrix} \in M_2(QA),$$

where i_1 and i_2 are the two canonical inclusions of A into QA. Indeed,

$$\psi \circ \eta \colon (a_1, a_2) \mapsto \begin{pmatrix} (a_1, 0), & (0, 0) \\ (0, 0) & (0, a_2) \end{pmatrix} = \left(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix} \right)$$

is homotopic to

$$(a_1, a_2) \mapsto \begin{pmatrix} (a_1, a_2), & (0, 0) \\ (0, 0) & (0, 0) \end{pmatrix} = \left(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

via "rotation" in the second coordinate (conjugation by $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ as θ goes from 0 to $\pi/2$), and similarly $\eta \circ \psi$ is homotopic to the stabilization map. So by property (1), $[\eta]$ and $[\psi]$ are inverses in $\mathbf{H}$. Then by split exactness, property (2), $[\phi]$ is also invertible in $\mathbf{H}$. Thus A and qA are equivalent in $\mathbf{H}$. Finally, we need to show that the natural map $H(A, B) \to KK(A, B)$ is also injective. But this also follows from the same calculation, for if a KK element is trivial, that means its underlying quasihomomorphism is homotopic to 0, and thus is trivial in the homotopy category as a map $qA \to B \otimes \mathcal{K}$.

2. K-theory and KK-theory of crossed products

2.1. Equivariant Kasparov theory. Many of the interesting applications of KK-theory involve actions of groups in some way. For this, Kasparov also invented an equivariant version of the theory. In what follows, G will always be a second-countable locally compact group. A G-C^*-algebra will mean a C^*-algebra A, along with an action of G on A by $*$-automorphisms, continuous in the sense that the map $G \times A \to A$ is jointly continuous. (Another way to say this is that if we give $\operatorname{Aut} A$ the topology of pointwise convergence, then $G \to \operatorname{Aut} A$ is a continuous group homomorphism.) If G is compact, making the theory equivariant is rather straightforward. We just require all algebras and Hilbert modules to be equipped with G-actions, we require $\phi\colon A \to \mathcal{L}(\mathcal{H})$ to be G-equivariant, and we require the operator $T \in \mathcal{L}(\mathcal{H})$ to be G-invariant. This produces groups $KK^G(A, B)$ for (separable, say) G-C^*-algebras A and B, and the same argument as before shows that $KK^G(\mathbb{C}, B) \cong K_0^G(B)$, equivariant K-theory. (In the commutative case, this is described in [**72**]. A general description may be found in [**7**, §11].) In particular, $KK^G(\mathbb{C}, \mathbb{C}) \cong R(G)$, the *representation ring* of G, in other words, the Grothendieck group of the category of finite-dimensional representations of G, with product coming from the tensor product of representations. The rings $R(G)$ are commutative, Noetherian if G is a compact Lie group, and often easily computable; for example, if G is compact and abelian, $R(G) \cong \mathbb{Z}[\widehat{G}]$, the group ring of the Pontrjagin dual. If G is a compact connected Lie group with maximal torus T and Weyl group $W = N_G(T)/T$, then $R(G) \cong R(T)^W \cong \mathbb{Z}[\widehat{T}]^W$. The properties of the Kasparov product all go through without change, since it is easy to "average" things with respect to a compact group action. Then Kasparov product with $KK^G(\mathbb{C}, \mathbb{C})$ makes all KK^G-groups into modules over the ground ring $R(G)$, so that homological algebra of the ring $R(G)$ comes into play in understanding the equivariant KK-category $\mathbf{KK}^G$.

When G is noncompact, the definition and properties of KK^G are considerably more subtle, and were worked out in [**41**]. A shorter exposition may be found in [**42**]. The problem is that in this case, topological vector spaces with a continuous G-action are very rarely completely decomposable, and there are rarely enough G-equivariant operators to give anything useful. Kasparov's solution was to work with G-continuous rather than G-equivariant Hilbert modules and operators; rather remarkably, these still give a useful theory with all the same formal properties as before. The KK^G-groups are again modules over the commutative ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$, though this ring no longer has such a simple interpretation as before, and in fact, is not known for most connected semisimple Lie groups.

A few functorial properties of the KK^G-groups will be needed below, so we just mention a few of them. First of all, if H is a closed subgroup of G, then any G-C^*-algebra is by restriction also an H-C^*-algebra, and we have restriction maps $KK^G(A, B) \to KK^H(A, B)$. To go the other way, we can "induce" an H-C^*-algebra A to get a G-C^*-algebra $\operatorname{Ind}_H^G(A)$, defined by

$$\operatorname{Ind}_H^G(A) = \{f \in C(G, A) \mid f(gh) = h \cdot f(g) \quad \forall g \in G, h \in H, \\ \|f(g)\| \to 0 \text{ as } g \to \infty \mod H\}.$$

The induced action of G on $\operatorname{Ind}_H^G(A)$ is just left translation. For example, if $A = C_0(X)$ with X a locally compact H-space, $\operatorname{Ind}_H^G(A)$ is just $C_0(G \times_H X)$.

If A and B are H-C^*-algebras, we then have an induction homomorphism

$$KK^H(A, B) \to KK^G(\operatorname{Ind}_H^G(A), \operatorname{Ind}_H^G(B)).$$

The last basic operation on the KK^G-groups depends on *crossed products*, so we consider these next.

2.2. Basic properties of crossed products. Suppose A is a G-C^*-algebra. Then one can define two new C^*-algebras, called the full and reduced *crossed products* of A by G, which capture the essence of the group action. These are easiest to define when G is discrete and A is unital. Then the full crossed product $A \rtimes_\alpha G$ (we often omit the α if there is no possibility of confusion) is the universal C^*-algebra generated by a copy of A and unitaries u_g, $g \in G$, subject to the commutation condition $u_g a u_g^* = \alpha_g(a)$, where α denotes the action of G on A. The reduced crossed product $A \rtimes_{\alpha,r} G$ is the image of $A \rtimes_\alpha G$ in its "regular representation" π on $L^2(G, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space on which A acts faithfully, say by a representation ρ. Here A acts by $(\pi(a)f)(g) = \rho(\alpha_{g^{-1}}(a))f(g)$ and G acts by left translation. The compatibility condition is satisfied since

$$\begin{aligned}\pi(u_g)\pi(a)\pi(u_g^*)f(g') &= (\pi(a)\pi(u_g^*)f)(g^{-1}g')\\ &= \rho(\alpha_{g'^{-1}g}(a))(\pi(u_g^*)f)(g^{-1}g')\\ &= \rho(\alpha_{g'^{-1}g}(a))(f(g'))\\ &= \rho(\alpha_{g'^{-1}}(\alpha_g(a))(f(g')) = \pi(\alpha_g(a))f(g').\end{aligned}$$

In the general case (where A is not necessarily unital and G is not necessarily discrete), the full crossed product is still defined as the universal C^*-algebra for *covariant pairs* of a $*$-representation ρ of A and a unitary representation π of G, satisfying the compatibility condition $\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$. It may be constructed by defining a convolution multiplication on $C_c(G, A)$ and then completing in the greatest C^*-algebra norm. The reduced crossed product $A \rtimes_{\alpha,r} G$ is again the image of $A \rtimes_\alpha G$ in its "regular representation" on $L^2(G, \mathcal{H})$. For details of the construction, see [**53**, §7.6] and [**77**, Ch. 2].

If $A = \mathbb{C}$, the crossed product $A \rtimes G$ is simply the universal C^*-algebra for unitary representations of G, or the group C^*-algebra $C^*(G)$, and $A \rtimes_r G$ is $C_r^*(G)$, the image of $C^*(G)$ in the left regular representation on $L^2(G)$. The natural map $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism if and only if G is amenable.[4] When the action α is trivial (factors through the trivial group $\{1\}$), then $A \rtimes G$ is the maximal tensor product $A \otimes_{\max} C^*(G)$ while $A \rtimes_r G$ is the minimal tensor product $A \otimes C_r^*(G)$. Again, the natural map from $A \otimes_{\max} C^*(G)$ to $A \otimes C_r^*(G)$ is an isomorphism if and only if G is amenable.

When A and the action α are arbitrary, the natural map $A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ is an isomorphism if G is amenable, but also more generally if the action α is amenable in a certain sense. For example, if X is a locally compact G-space, the action is automatically amenable if it is proper, whether or not G is amenable. A good short survey of amenability for group actions may be found in [**1**].

When X is a locally compact G-space, the crossed product $C_0(X) \rtimes G$ often serves as a good substitute for the "quotient space" X/G in cases where the latter is badly behaved. Indeed, if G acts freely and properly on X, then $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. If G acts locally freely and properly on X, then

[4]This is a reformulation of a famous theorem of Hulanicki [**36**].

$C_0(X) \rtimes G$ is Morita equivalent to an "orbifold algebra" that encompasses not only the topology of X/G but also the finite isotropy groups. But if the G-action is not proper, X/G may be highly non-Hausdorff, while $C_0(X) \rtimes G$ may be a perfectly well-behaved noncommutative algebra. A key case later on will the one where $X = \mathbb{T}$ is the circle group, $G = \mathbb{Z}$, and the generator of G acts by multiplication by $e^{2\pi i\theta}$. When θ is irrational, every orbit is dense, so X/G is an indiscrete space (the only open sets are $\emptyset$ and the whole space), and $C(\mathbb{T}) \rtimes \mathbb{Z}$ is what's usually denoted A_θ, an *irrational rotation algebra* or *noncommutative* 2*-torus*.

The understanding of crossed products is often aided by various "imprimitivity theorems" generalizing Mackey's famous characterization of induced representations. For example, an "imprimitivity theorem" due to Green shows that $\operatorname{Ind}_H^G(A) \rtimes G$ and $A \rtimes H$ are Morita equivalent, if the induced actions are defined as in Section 2.1.

Now we can explain the relationships between equivariant KK-theory and crossed products. One connection is that if G is discrete and A is a G-C^*-algebra, there is a natural isomorphism $KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$. Dually, if G is compact, there is a natural *Green-Julg isomorphism* [**7**, §11.7] $KK^G(\mathbb{C}, A) \cong KK(\mathbb{C}, A \rtimes G)$. Still another connection is that there is (for arbitrary G) a functorial homomorphism

$$j\colon\ KK^G(A, B) \to KK(A \rtimes G, B \rtimes G)$$

sending (when $B = A$) 1_A to $1_{A\rtimes G}$. (In fact, j can be viewed as a functor from the equivariant Kasparov category $\mathbf{KK}^G$ to the non-equivariant Kasparov category $\mathbf{KK}$. Later we will study how close it is to being faithful.) There is also a variant of j using reduced crossed products, denoted j_r [**41**, §3.11]. If $B = \mathbb{C}$ and G is discrete, then j can be identified with the map $KK(A\rtimes G, \mathbb{C}) \to KK(A\rtimes G, C^*(G))$ induced by the inclusion of scalars $\mathbb{C} \hookrightarrow C^*(G)$. (The fact that G is discrete means that $C^*(G)$ is unital.) The map j is split injective in this case since it is split by the map induced by $C^*(G) \to \mathbb{C}$, corresponding to the trivial representation of G. Similarly, if G is compact, then via Green-Julg, j can be identified with the map $KK(\mathbb{C}, A \rtimes G) \to KK(C^*(G), A \rtimes G)$ induced by the map $C^*(G) \to \mathbb{C}$ corresponding to the trivial representation of G. The map j is again a split injection since $C^*(G)$ splits as the direct sum of $\mathbb{C}$ and summands associated to other representations.

2.3. The dual action and Takai duality. When the group G is not just locally compact but also abelian, then it has a Pontrjagin dual group $\widehat{G}$. In this case, given any G-C^*-algebra algebra A, say with α denoting the action of G on A, there is a *dual action* $\widehat{\alpha}$ of $\widehat{G}$ on the crossed product $A \rtimes G$. When A is unital and G is discrete, so that $A \rtimes G$ is generated by a copy of A and unitaries u_g, $g \in G$, the dual action is given simply by

$$\widehat{\alpha}_\gamma(au_g) = au_g\langle g, \gamma\rangle.$$

The same formula still applies in general, except that the elements a and u_g don't quite live in the crossed product but in the multiplier algebra. (However, there is a still a sense in which they generate the crossed product.) The key fact about the dual action is the *Takai duality theorem*: $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$, and the double dual action $\hat{\hat{\alpha}}$ of $\tilde{\widehat{G}} \cong G$ on this algebra can be identified with $\alpha \otimes \operatorname{Ad}\lambda$, where λ is the left regular representation of G on $L^2(G)$. Good expositions may be found in [**53**, §7.9] and in [**77**, Ch. 7].

2.4. Connes' "Thom isomorphism". Recall that the Thom isomorphism theorem in K-theory (see Section 1.1) asserts that if E is a complex vector bundle over X, there is an isomorphism of K-groups $K(X) \to K(E)$, implemented by a KK-class in $KK(X, E)$. Now if $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$ — there is no difference since we are just considering the additive group structure) acts on X by a trivial action α, then $C_0(X) \rtimes_\alpha \mathbb{C}^n \cong C_0(X) \otimes C^*(\mathbb{C}^n) \cong C_0(X) \otimes C_0(\widehat{\mathbb{C}}^n) \cong C_0(E)$, where E is a trivial rank-n complex vector bundle over X. (We have used Pontrjagin duality and the fact that abelian groups are amenable.) It follows that $K(C_0(X)) \cong K(C_0(X) \rtimes_\alpha \mathbb{C}^n)$. Since any action α of $\mathbb{C}^n$ is homotopic to the trivial action and "K-theory is supposed to be homotopy invariant," that suggests that perhaps $KK(A) \cong KK(A \rtimes_\alpha \mathbb{C}^n)$ for any C^*-algebra A and for any action α of $\mathbb{C}^n$. This is indeed true and the isomorphism is implemented by classes (which are inverse to one another) in $KK(A, A \rtimes_\alpha \mathbb{C}^n)$ and $KK(A \rtimes_\alpha \mathbb{C}^n, A)$. It is clearly enough to prove this in the case $n = 1$, since we can always break a crossed product by $\mathbb{C}^n$ up as an n-fold iterated crossed product.

That A and $A \rtimes_\alpha \mathbb{C}$ are always KK-equivalent or that they at least have the same K-theory, or (this is equivalent since one can always suspend on both sides) that $A \otimes C_0(\mathbb{R})$ and $A \rtimes_\alpha \mathbb{R}$ are always KK-equivalent or that they at least have the same K-theory for any action of $\mathbb{R}$, is called *Connes' "Thom isomorphism"* (with the name "Thom" in quotes since the only connection with the classical Thom isomorphism is the one we have already explained). Connes' original proof is relatively elementary, but only gives an isomorphism of K-groups, not a KK-equivalence, and can be found in [**13**] or in [**22**, §10.2].

To illustrate Connes' idea, let's suppose A is unital and we have a class in $K_0(A)$ represented by a projection $p \in A$. (One can always reduce to this special case.) If α were to fix p, then $1 \mapsto p$ gives an equivariant map from $\mathbb{C}$ to A and thus would induce a map of crossed products $\mathbb{C} \rtimes \mathbb{R} \cong C_0(\widehat{\mathbb{R}}) \to A \rtimes_\alpha \mathbb{R}$ or $\mathbb{C} \rtimes \mathbb{C} \cong C_0(\widehat{\mathbb{C}}) \to A \rtimes_\alpha \mathbb{C}$ giving a map on K-theory $\beta\colon \mathbb{Z} \to K_0(A \rtimes \mathbb{C})$. The image of $[p]$ under the isomorphism $K_0(A) \to K_0(A \rtimes \mathbb{C})$ will be $\beta(1)$. So the idea is to show that one can modify the action to one fixing p (using a cocycle conjugacy) without changing the isomorphism class of the crossed product.

There are now quite a number of proofs of Connes' theorem available, each using somewhat different techniques. We just mention a few of them. A proof using K-theory of Wiener-Hopf extensions is given in [**62**]. There are also fancier proofs using KK-theory. If α is a given action of $\mathbb{R}$ on A and if β is the trivial action, one can try to construct $KK^{\mathbb{R}}$ elements $c \in KK^{\mathbb{R}}((A, \alpha), (A, \beta))$ and $d \in KK^{\mathbb{R}}((A, \beta), (A, \alpha))$ which are inverses of each other in $\mathbf{KK}^{\mathbb{R}}$. Then the morphism j of Section 2.1 sends these to KK-equivalences $j(c)$ and $j(d)$ between $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_\beta \mathbb{R} \cong A \otimes C_0(\mathbb{R})$.

Another rather elegant approach, using KK-theory but not the equivariant groups, may be found in [**28**]. Fack and Skandalis use the group $KK^1(A, B)$, which we have avoided so far in order to simplify the theory, but it can be defined with triples $(\mathcal{H}, \phi, T)$ like those used for $KK(A, B)$, but with two modifications:

1. $\mathcal{H}$ is no longer graded, and there is no grading condition on ϕ.
2. T is self-adjoint but with no grading condition, and $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ and $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ for all $a \in A$.

It turns out that $KK^1(A,B) \cong KK(A \otimes C_0(\mathbb{R}), B)$, and that the Kasparov product can be extended to a graded commutative product on the direct sum of $KK = KK^0$ and KK^1. The product of two classes in KK^1 can by Bott periodicity be taken to land in KK^0.

We can now explain the proof of Fack and Skandalis as follows. They show that for each separable C^*-algebra A with an action α of $\mathbb{R}$, there is a special element $t_\alpha \in KK^1(A, A \rtimes_\alpha \mathbb{R})$ (constructed using a singular integral operator). Note by the way that doing the construction with the dual action and applying Takai duality gives $t_{\widehat{\alpha}} \in KK^1(A \rtimes_\alpha \mathbb{R}, A)$, since $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong A \otimes \mathcal{K}$, which is Morita equivalent to A. These elements have the following properties:

(1) (Normalization) If $A = \mathbb{C}$ (so that necessarily $\alpha = 1$ is trivial), then $t_1 \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is the usual generator of this group (which is isomorphic to $\mathbb{Z}$).
(2) (Naturality) The elements are natural with respect to equivariant homomorphisms $\rho\colon (A,\alpha) \to (C,\gamma)$, in that if $\bar{\rho}$ denotes the induced map on crossed products, then $\bar{\rho}_*(t_\alpha) = \rho^*(t_\gamma) \in KK(A, C \rtimes_\gamma \mathbb{R})$, and similarly, $\bar{\rho}^*(t_{\widehat{\gamma}}) = \rho_*(t_{\widehat{\alpha}}) \in KK(A \rtimes_\alpha \mathbb{R}, C)$.
(3) (Compatibility with external products) Given $x \in KK(A,B)$ and $y \in KK(C,D)$,
$$(t_{\widehat{\alpha}} \otimes_A x) \boxtimes y = t_{\widehat{\alpha \otimes 1_C}} \otimes_{A \otimes C} (x \boxtimes y).$$
Similarly, given $x \in KK(B,A)$ and $y \in KK(D,C)$,
$$y \boxtimes (x \otimes_A t_\alpha) = (y \boxtimes x) \otimes_{C \otimes A} t_{1_C \otimes \alpha}. \qquad \square$$

Theorem 2.1 (Fack-Skandalis [**28**]). *These properties completely determine t_α, and t_α is a KK-equivalence (of degree 1) between A and $A \rtimes_\alpha \mathbb{R}$.*

Proof. Suppose we have elements t_α satisfying the properties above. Let us first show that $t_\alpha \otimes_{A \rtimes_\alpha \mathbb{R}} t_{\widehat{\alpha}} = 1_A$. For $s \in \mathbb{R}$, let α^s be the rescaled action $\alpha^s_t = \alpha_{st}$. Then define an action β of $\mathbb{R}$ on $B = C([0,1], A)$ by $(\beta_t f)(s) = \alpha^s_t(f(s))$. Let $g_s\colon B \to A$ be evaluation at s, which is clearly an equivariant map $(B,\beta) \twoheadrightarrow (A, \alpha^s)$. We also get maps $\widehat{g}_s\colon B \rtimes_\beta \mathbb{R} \to A \rtimes_{\alpha^s} \mathbb{R}$, and the double dual map $\hat{\hat{g}}_s$ can be identified with $g_s \otimes 1\colon B \otimes \mathcal{K} \to A \otimes \mathcal{K}$. By Axiom (2), $(\bar{g}_s)_*(t_\beta) = g_s^*(t_{\alpha^s})$ and $(g_s)_*(t_{\widehat{\beta}}) = \bar{g}_s^*(t_{\widehat{\alpha}^s})$. Let $\sigma_s = t_{\alpha^s} \otimes_{A \rtimes_{\alpha^s} \mathbb{R}} t_{\widehat{\alpha}^s} \in KK(A,A)$. By associativity of Kasparov products,
$$\begin{aligned}(g_s)_*\big(t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} t_{\widehat{\beta}}\big) &= t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} \big(t_{\widehat{\beta}} \otimes_B [g_s]\big) \\ &= t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} \big([\bar{g}_s] \otimes_{A \rtimes_{\alpha^s} \mathbb{R}} t_{\widehat{\alpha}^s}\big) \\ &= \big(t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} [\bar{g}_s]\big) \otimes_{A \rtimes_{\alpha^s} \mathbb{R}} t_{\widehat{\alpha}^s} \\ &= \big([g_s] \otimes_A t_{\alpha^s}\big) \otimes_{A \rtimes_{\alpha^s} \mathbb{R}} t_{\widehat{\alpha}^s} \\ &= [g_s] \otimes_A \sigma_s.\end{aligned}$$
Since g_s is a homotopy of maps $B \to A$ and KK is homotopy-invariant, $[g_s] = [g_0]$. But g_0 is a homotopy equivalence with homotopy inverse $f\colon a \mapsto a \otimes 1$, so we see that
$$\sigma_s = [f] \otimes_B \big(t_\beta \otimes_{B \rtimes_\beta \mathbb{R}} t_{\widehat{\beta}}\big) \otimes_B [g_0]$$
is independent of s. In particular, $\sigma_1 = t_\alpha \otimes_{A \rtimes_\alpha \mathbb{R}} t_{\widehat{\alpha}}$ agrees with σ_0, which can be computed to be 1_A by Axioms (1) and (3) since the action of $\mathbb{R}$ is trivial in this

case. So $t_\alpha \otimes_{A\rtimes_\alpha\mathbb{R}} t_{\widehat{\alpha}} = 1_A$. Replacing α by $\widehat{\alpha}$ and using Takai duality, this also implies that $t_{\widehat{\alpha}} \otimes_A t_\alpha = 1_{A\rtimes_\alpha\mathbb{R}}$. So t_α and $t_{\widehat{\alpha}}$ give KK-equivalences.

The uniqueness falls out at the same time, since we see from the above that $[g_s] \otimes_A t_{\alpha^s} = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar{g}_s] \in KK(B, A \rtimes_{\alpha^s} \mathbb{R})$, and that all the KK-elements involved are KK-equivalences. Furthermore, we know by Axioms (1) and (3) that $t_{\alpha^0} = 1_A \boxtimes t_1$, where t_1 is the special element of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ mentioned in Axiom (1). This determines t_β (from the identity $[g_0] \otimes_A t_{\alpha^0} = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar{g}_0]$), and then t_α is determined from the identity $[g_0] \otimes_A t_\alpha = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar{g}_1]$. □

2.5. The Pimsner-Voiculescu Theorem. Connes' Theorem from Section 2.4 computes K-theory or KK-theory for crossed products by $\mathbb{R}$. This can be used to compute K-theory or KK-theory for crossed products by $\mathbb{Z}$, using the fact from Section 2.2 that if A is a C^*-algebra equipped with an action α of $\mathbb{Z}$ (or equivalently, a single $*$-automorphism θ, the image of $1 \in \mathbb{Z}$ under the action), then $A \rtimes_\alpha \mathbb{Z}$ is Morita equivalent to $\left(\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)\right) \rtimes \mathbb{R}$. The algebra $T_\theta = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)$ is often called the *mapping torus* of (A, θ); it can be identified with the algebra of continuous functions $f\colon [0,1] \to A$ with $f(1) = \theta(f(0))$. It comes with an obvious short exact sequence

$$0 \to C_0((0,1), A) \to T_\theta \to A \to 0,$$

for which the associated exact sequence in K-theory has the form

$$\cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \to K_0(T_\theta) \to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \to \cdots .$$

Since

$$K_0(A \rtimes_\alpha \mathbb{Z}) \cong K_0(T_\theta \rtimes_{\operatorname{Ind}\alpha} \mathbb{R}) \cong K_1(T_\theta),$$

and similarly for K_0, we obtain the *Pimsner-Voiculescu exact sequence*

$$\begin{aligned} &\cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \to K_1(A \rtimes_\alpha \mathbb{Z}) \to \\ &\to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \to K_0(A \rtimes_\alpha \mathbb{Z}) \to \cdots . \end{aligned} \tag{2.1}$$

Here one can check that the maps $K_j(A) \to K_j(A \rtimes_\alpha \mathbb{Z})$ are induced by the inclusion of A into the crossed product. For another proof, closer to the original argument of Pimsner and Voiculescu, see [**22**, Ch. 5].

2.6. The Baum-Connes Conjecture. The theorems of Connes and Pimsner-Voiculescu on K-theory of crossed products by $\mathbb{R}$ and $\mathbb{Z}$ suggest the question of whether there are similar results for other groups G. In particular, one would like to know if the K-theory of $C_r^*(G)$, or better still, the K-theory of reduced crossed products $A \rtimes G$, can be computed in a "topological" way. The answer in many cases seems to be "yes," and the conjectured answer is what is usually called the *Baum-Connes Conjecture*, with or without coefficients. The special case of the Baum-Connes Conjecture (without coefficients) for connected Lie groups is also known as the *Connes-Kasparov Conjecture*, and is now a known theorem [**76, 45**].

The Baum-Connes conjecture also has other origins, such as the Novikov Conjecture on higher signatures and conjectures about algebraic K-theory of group rings, which will be touched on in Reich's lectures. These other motivations for the conjecture mostly concern the case where G is discrete, which is actually the most interesting case of the conjecture, though there are good reasons for not restricting only to this case. (For example, as we already saw in the case of $\mathbb{Z}$, information about discrete groups can often be obtained by embedding them in a Lie group.)

Here is the formal statement of the conjecture.

Conjecture 2.2 (Baum-Connes). *Let G be a locally compact group, second-countable for convenience. Let $\underline{E}G$ be the universal proper G-space.* (*This is a contractible space on which G acts properly, characterized* [**6**] *up to G-homotopy equivalence by two properties: that every compact subgroup of G has a fixed point in $\underline{E}G$, and that the two projections $\underline{E}G \times \underline{E}G \to \underline{E}G$ are G-homotopic. Here the product space is given the diagonal G-action. If G has no compact subgroups, then $\underline{E}G$ is the usual universal free G-space EG.*) *There is an* assembly map

$$\varinjlim_{\substack{X \subseteq \underline{E}G \\ X/G \text{ compact}}} K_*^G(X) \to K_*(C_r^*(G))$$

defined by taking G-indices of G-invariant elliptic operators, and this map is an isomorphism.

Conjecture 2.3 (Baum-Connes with coefficients). *With notation as in Conjecture* 2.2, *if A is any separable G-C^*-algebra, the assembly map*

$$\varinjlim_{\substack{X \subseteq \underline{E}G \\ X/G \text{ compact}}} KK_*^G(C_0(X), A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Let's see what the conjecture amounts to in some special cases. If G is compact, $\underline{E}G$ can be taken to be a single point. The conjecture then asserts that the *assembly map* $KK_*^G(\text{pt}, \text{pt}) \to K_*(C^*(G))$ is an isomorphism. For G compact, $C^*(G)$ is by the Peter-Weyl Theorem the completed direct sum of matrix algebras $\bigoplus_V \text{End}(V)$, where V runs over a set of representatives for the irreducible representations of G. Thus $K_1(C^*(G))$ (remember this is topological K_1) vanishes and $K_0(C^*(G)) \cong R(G)$. The assembly map in this case is the Green-Julg isomorphism of Section 2.2. In fact, the same holds with coefficients; the assembly map $KK_*^G(\mathbb{C}, A) = K_*^G(A) \to K_*(A \rtimes G)$ is the Green-Julg isomorphism, and Conjecture 2.3 is true.

Next, suppose $G = \mathbb{R}$. Since G has no compact subgroups and is contractible, we can take $\underline{E}G = \mathbb{R}$ with $\mathbb{R}$ acting on itself by translations. If A is an $\mathbb{R}$-C^*-algebra, the assembly map is a map $KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to K_*(A \rtimes \mathbb{R})$. This map turns out to be Kasparov's morphism

$$j\colon\ KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to KK_*(C_0(\mathbb{R}) \rtimes \mathbb{R}, A \rtimes \mathbb{R}) = KK_*(\mathcal{K}, A \rtimes \mathbb{R}) \cong K_*(A \rtimes \mathbb{R}),$$

which is the isomorphism of Connes' Theorem (Section 2.4). (The isomorphism $C_0(\mathbb{R}) \rtimes \mathbb{R} \cong \mathcal{K}$ is a special case of the Imprimitivity Theorem giving a Morita equivalence between $\left(\text{Ind}_{\{1\}}^G A\right) \rtimes G$ and A, or, if you prefer, of Takai duality from Section 2.3.) So again the conjecture is true.

Another good test case is $G = \mathbb{Z}$. Then $\underline{E}G = EG = \mathbb{R}$, with $\mathbb{Z}$ acting by translations and quotient space $\mathbb{T}$. The left-hand side of the conjecture is thus $KK^{\mathbb{Z}}(C_0(\mathbb{R}), A)$, while the right-hand side is $K(A \rtimes \mathbb{Z})$, which is computed by the Pimsner-Voiculescu sequence.

More generally, suppose G is discrete and torsion-free. Then $\underline{E}G = EG$, and the quotient space $\underline{E}G/G$ is the usual classifying space BG. The assembly map (for the conjecture without coefficients) maps $K_*^{\rm cpt}(BG) \to K_*(C_r^*(G))$. (The left-hand side is K-homology with compact supports.) This map can be viewed as an index map, since classes in the K-homology group on the left are represented by generalized Dirac operators D over Spin^c manifolds M with a G-covering, and the assembly map takes such an operator to its "Mishchenko-Fomenko index" with values in the K-theory of the (reduced) group C^*-algebra. The connection between this assembly map and the usual sort of assembly map studied by topologists is discussed in [**68**]. In particular, Conjecture 2.2 implies a strong form of the Novikov Conjecture for G.

2.7. The approach of Meyer and Nest. An interesting alternative approach to the Baum-Connes Conjecture has been proposed by Meyer and Nest [**51, 52**]. This approach is also briefly sketched (in somewhat simplified form) in [**22**, §5.3] and in [**69**, Ch. 5]. Meyer and Nest begin by observing that the equivariant KK-category, $\mathbf{KK}^G$, naturally has the structure of a triangulated category. It has a distinguished class $\mathcal{E}$ of *weak equivalences*, morphisms $f \in KK^G(A, B)$ which restrict to equivalences in $KK^H(A, B)$ for every compact subgroup H of G. (Note that if G has no nontrivial compact subgroups, for example if G is discrete and torsion-free, then this condition just says that f is a KK-equivalence after forgetting the G-equivariant structure.) The Baum-Connes Conjecture with coefficients, Conjecture 2.3, basically amounts to the assertion that if $f \in KK^G(A, B)$ is in $\mathcal{E}$, then $j_r(f) \in KK(A \rtimes_r G, B \rtimes_r G)$ is a KK-equivalence.[5] In particular, suppose G has no nontrivial compact subgroups and satisfies Conjecture 2.3. Then if A is a G-C^*-algebra which, forgetting the G-action, is contractible, then the unique morphism in $KK^G(0, A)$ is a weak equivalence, and so (applying j_r), the unique morphism in $KK(0, A \rtimes_r G)$ is a KK-equivalence. Thus $A \rtimes_r G$ is *K-contractible*, i.e., all of its topological K-groups must vanish. When $G = \mathbb{R}$, this follows from Connes' Theorem, and when $G = \mathbb{Z}$, this follows from the Pimsner-Voiculescu exact sequence, (2.1).

Now that we have several different formulations of the Baum-Connes Conjecture, it is natural to ask how widely the conjecture is valid. Here are some of the things that are known:

(1) There is no known counterexample to Conjecture 2.2 (Baum-Connes for groups, without coefficients). Counterexamples are now known [**29**] to Conjecture 2.3 with G discrete and A even commutative, and to a generalization of Conjecture 2.2 for groupoids.
(2) Conjecture 2.3 is true if G is amenable, or more generally, if it is *a-T-menable*, that is, if it has an affine, isometric and metrically proper action on a Hilbert space [**33**]. Such groups include all Lie groups whose noncompact semisimple factors are all locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$ for some n.
(3) Conjecture 2.2 is true for connected reductive Lie groups, connected reductive p-adic groups, for hyperbolic discrete groups, and for cocompact lattice subgroups of $Sp(n, 1)$ or $SL(3, \mathbb{C})$ [**45**].

[5] The reason for using j_r in place of j for can be seen from the case of G nonamenable with property T. In this case, $C^*(G)$ has a projection corresponding to the trivial representation of G which is "isolated," and thus maps to 0 in $C_r^*(G)$. So these two algebras do not have the same K-theory. It turns out, at least in many examples, that $K_0(C_r^*(G))$ can be described in purely topological terms, but $K_0(C^*(G))$ cannot.

There is now a vast literature on this subject, but our intention here is not to be exhaustive, but just to give the reader some flavor of what's going on.

3. The universal coefficient theorem for KK and some of its applications

3.1. Introduction to the UCT. Now that we have discussed KK and KK^G, a natural question arises: *how computable are they?* In particular, is $KK(A, B)$ determined by $K_*(A)$ and by $K_*(B)$? Is $KK^G(A, B)$ determined by $K^G_*(A)$ and by $K^G_*(B)$?

A first step was taken by Kasparov [**40**]: he pointed out that $KK(X, Y)$ is given by an explicit topological formula when the one-point compactifications X_+ and Y_+ are finite CW complexes: $KK(X, Y) \cong \widetilde{K}(Y_+ \wedge D(X_+))$, where $D(X_+)$ denotes the Spanier-Whitehead dual of X_+.[6]

Let's make a definition — we say the pair of C^*-algebras (A, B) *satisfies the Universal Coefficient Theorem for KK* (or *UCT* for short) if there is an exact sequence

$$0 \to \bigoplus_{* \in \mathbb{Z}/2} \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \to KK(A, B) \xrightarrow{\varphi} \bigoplus_{* \in \mathbb{Z}/2} \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0. \tag{3.1}$$

Here φ sends a KK-class to the induced map on K-groups.

We need one more definition. Let $\mathcal{B}$ be the *bootstrap category*, the smallest full subcategory of the separable C^*-algebras (with the $*$-homomorphisms as morphisms) containing all separable type I algebras, and closed under extensions, countable C^*-inductive limits, and KK-equivalences. Note that KK-equivalences include Morita equivalences, and type I algebras include commutative algebras. Recall from Section 1.2 that stably isomorphic separable C^*-algebras are Morita equivalent, hence KK-equivalent. Furthermore, separable type I C^*-algebras are inductive limits of finite iterated extensions of stably commutative C^*-algebras [**53**, Ch. 6]. Thus we could just as well replace the words "type I" by "commutative" in the definition of $\mathcal{B}$. Furthermore, any compact metric space is a countable (projective) limit of finite CW complexes. Dualizing, this means that any unital separable commutative C^*-algebra is a countable inductive limit (i.e., categorical colimit) of algebras of the form $C(X)$, X a finite CW complex, and any separable commutative C^*-algebra is a countable inductive limit (i.e., colimit) of algebras of the form $C_0(X)$, X_+ a finite CW complex. We will use this fact shortly.

THEOREM 3.1 (Rosenberg-Schochet [**71**]). *The UCT holds for all pairs (A, B) with A an object in $\mathcal{B}$ and B separable.*

Unsolved problem: Is every separable nuclear C^*-algebra in $\mathcal{B}$? Skandalis [**74**] showed that there are *non-nuclear algebras* not in $\mathcal{B}$, for which the UCT fails.

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[6] Spanier-Whitehead duality basically interchanges homology and cohomology. In other words, the (reduced) homology of $D(X_+)$ is the (reduced) cohomology of X_+, and vice versa. $\wedge$ denotes the smash product, the product in the category of spaces with distinguished basepoint.

3.2. The proof of Rosenberg and Schochet. First suppose $K_*(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\mathrm{Hom}_{\mathbb{Z}}(\text{—}, K_*(B))$ is an exact functor, so $A \mapsto \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ gives a cohomology theory on C^*-algebras. In particular, φ is a natural transformation of homology theories for locally compact spaces

$$\big(X \mapsto KK_*(C_0(X), B)\big) \rightsquigarrow \big(X \mapsto \mathrm{Hom}_{\mathbb{Z}}(K^*(X), K_*(B))\big).$$

Since φ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever X_+ is a sphere, and thus (by the analogue of the Eilenberg-Steenrod uniqueness theorem for generalized homology theories) whenever X_+ is a finite CW complex.

We extend to arbitrary locally compact X by taking limits, and then to the rest of $\mathcal{B}$, using the observations we made before the proof of the theorem. In order to know we can pass to countable inductive limits, we need one additional fact about KK, namely that it is "countably additive" (sends countable C^*-algebra direct sums in the first variable to products of abelian groups). This fact is not hard to check from Kasparov's original definition. And the corresponding property for $A \mapsto \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ is clear from the fact that topological K-theory sends C^*-algebra direct sums (categorical coproducts) to direct sums of abelian groups, while $\mathrm{Hom}_{\mathbb{Z}}(\text{—}, K_*(B))$ sends coproducts to products. So the theorem holds when $K_*(B)$ is injective.

The rest of the proof uses an idea due to Atiyah [**3**], of *geometric resolutions.* The idea is that given arbitrary B, we can change it up to KK-equivalence so that it fits into a short exact sequence

$$0 \to C \to B \to D \to 0$$

for which the induced K-theory sequence is short exact:

$$K_*(B) \rightarrowtail K_*(D) \twoheadrightarrow K_{*-1}(C)$$

and $K_*(D)$, $K_*(C)$ are $\mathbb{Z}$-injective. Then we use the theorem for $KK_*(A, D)$ and $KK_*(A, C)$, along with the long exact sequence in KK in the second variable, to get the UCT for (A, B). $\square$

3.3. The equivariant case. If one asks about the UCT in the equivariant case, then the homological algebra of the ground ring $R(G)$ becomes relevant. This is not always well behaved, so, as noticed by Hodgkin [**35**], one needs restrictions on G to get anywhere. But for G a connected compact Lie group with $\pi_1(G)$ torsion-free, $R(G)$ has finite global dimension, and the spectral sequence one ends up with does converge to the right limit.

THEOREM 3.2 (Rosenberg-Schochet [**70**]). *If G is a connected compact Lie group with $\pi_1(G)$ torsion-free, and if A, B are separable G-C^*-algebras with A in a suitable bootstrap category containing all commutative G-C^*-algebras, then there is a convergent spectral sequence*

$$\mathrm{Ext}^p_{R(G)}(K^G_*(A), K^G_{q+*}(A)) \Rightarrow KK^G_*(A, B).$$

The proof is more complicated than in the non-equivariant case, but in the same spirit.

Also along the same lines, there is a UCT for KK of real C^*-algebras, due to Boersema [**9**]. The homological algebra involved in this case is appreciably more complicated than in the complex C^*-algebra case, and is based on ideas of Bousfield [**10**] on the classification of K-local spectra. The trick is this: the coefficient ring for real K-theory is "bad" (it doesn't have finite homological dimension), so one works instead with "united K-theory," based on looking at real, complex, and "self-conjugate" K-theory all at once. Rather remarkably, this gives a category with good homological properties. So one first proves a UCT for "united" KK-theory, then uses this to obtain a calculation of KK for real C^*-algebras in terms of the united K-groups.

3.4. The categorical approach. The UCT implies a lot of interesting facts about the bootstrap category $\mathcal{B}$. Here are a few examples.

THEOREM 3.3 (Rosenberg-Schochet [**71**]). *Let A, B be C^*-algebras in $\mathcal{B}$. Then A and B are KK-equivalent if and only if they have isomorphic topological K-groups.*

PROOF. $\Rightarrow$ is trivial. So suppose $K_*(A) \cong K_*(B)$. Choose an isomorphism

$$\psi\colon\ K_*(A) \to K_*(B).$$

Since the map φ in the UCT (3.1) is surjective, ψ is realized by a class $x \in KK(A, B)$ (not necessarily unique, but just pick one).

Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(K_{*+1}(B), K_*(A)) & \longrightarrow & KK_*(B, A) & \xrightarrow{\varphi} & \operatorname{Hom}(K_*(B), K_*(A)) & \longrightarrow & 0 \\
\Big\| & & \cong \Big\downarrow \psi^* & & \Big\downarrow x\otimes_B _ & & \cong \Big\downarrow \psi^* & & \Big\| \\
0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(K_{*+1}(A), K_*(A)) & \longrightarrow & KK_*(A, A) & \xrightarrow{\varphi} & \operatorname{Hom}(K_*(A), K_*(A)) & \longrightarrow & 0
\end{array}$$

By the 5-Lemma, Kasparov product with x is an isomorphism $KK_*(B, A) \to KK_*(A, A)$. In particular, there exists $y \in KK(B, A)$ with $x \otimes_B y = 1_A$. Similarly, there exists $z \in KK(B, A)$ with $z \otimes_A x = 1_B$. Then by associativity

$$z = z \otimes_A (x \otimes_B y) = (z \otimes_A x) \otimes_B y = y$$

and we have a KK-inverse to x. □

COROLLARY 3.4. *We can also describe $\mathcal{B}$ as the smallest full subcategory of the separable C^*-algebras closed under KK-equivalence and containing the separable commutative C^*-algebras. A separable C^*-algebra A has the property that (A, B) satisfies the UCT for all separable C^*-algebras B if and only if it lies in $\mathcal{B}$.*

PROOF. Let $\mathcal{B}'$ be the smallest full subcategory of the separable C^*-algebras closed under KK-equivalence and containing the separable commutative C^*-algebras. By definition of $\mathcal{B}$, $\mathcal{B}'$ is a subcategory of $\mathcal{B}$. But if A is in $\mathcal{B}$, its K-groups are countable. For any countable groups G_0 and G_1, it is easy to construct a second-countable locally compact space with these K-groups. So there is a separable commutative C^*-algebra $C_0(Y)$ with $K_*(C_0(Y)) \cong K_*(A)$ (just as abelian groups). By Theorem 3.3, there is a KK-equivalence between A and $C_0(Y)$, so A lies in $\mathcal{B}'$.

As far as the last statement is concerned, one direction is the UCT itself. For the other direction, suppose that (A, B) satisfies the UCT for all separable C^*-algebras B. In particular, it holds for a commutative B with the same K-groups as A, and by the argument above, A is KK-equivalent to B, hence lies in $\mathcal{B}$. □

Recall that $KK(A,A) = \mathrm{End}_{\mathbf{KK}}(A)$ is a ring under Kasparov product. We can now compute the ring structure.

THEOREM 3.5 (Rosenberg-Schochet). *Suppose A is in $\mathcal{B}$. In the UCT sequence*

$$0 \to \bigoplus_{i\in\mathbb{Z}/2} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i+1}(A), K_i(A)) \to KK(A,A) \xrightarrow{\varphi} \bigoplus_{i\in\mathbb{Z}/2} \mathrm{End}(K_i(A)) \to 0,$$

φ is a split surjective homomorphism of rings, and $J = \ker\varphi$ (the Ext *term) is an ideal with $J^2 = 0$.*

PROOF. Choose A_0 and A_1 commutative with $K_0(A_0) \cong K_0(A)$, $K_1(A_0) = 0$, $K_0(A_1) = 0$, $K_1(A_1) \cong K_1(A)$. Then by Theorem 3.3, $A_0 \oplus A_1$ is KK-equivalent to A, and without loss of generality, we may assume we have an actual splitting $A = A_0 \oplus A_1$. By the UCT, $KK(A_0,A_0) \cong \mathrm{End}\, K_0(A)$ and $KK(A_1,A_1) \cong \mathrm{End}\, K_1(A)$.

So $KK(A_0,A_0) \oplus KK(A_1,A_1)$ is a subring of $KK(A,A)$ mapping isomorphically under φ. This shows φ is split surjective. We also have $J = KK(A_0,A_1) \oplus KK(A_1,A_0)$. If, say, x lies in the first summand and y in the second, then $x \otimes_{A_1} y$ induces the 0-map on $K_0(A)$ and so is 0 in $KK(A_0,A_0) \cong \mathrm{End}(K_0(A))$. Similarly, $y \otimes_{A_0} x$ induces the 0-map on $K_1(A)$ and so is 0 in $KK(A_1,A_1) \cong \mathrm{End}(K_1(A))$. □

3.5. The homotopy-theoretic approach. There is a homotopy-theoretic approach to the UCT that topologists might find attractive; it seems to have been discovered independently by several people (e.g., [**12, 38**] — see also the review of [**12**] in MathSciNet). Let A and B be C^*-algebras and let $\mathbb{K}(A)$ and $\mathbb{K}(B)$ be their topological K-theory spectra. These are module spectra over $\mathbb{K} = \mathbb{K}(\mathbb{C})$, the usual spectrum of complex K-theory. Then we can define

$$KK^{\mathrm{top}}(A,B) = \pi_0\big(\mathrm{Hom}_{\mathbb{K}}\big(\mathbb{K}(A), \mathbb{K}(B)\big)\big).$$

This again uses ideas of Bousfield [**10**].

THEOREM 3.6. *There is a natural map $KK(A,B) \to KK^{top}(A,B)$, and it's an isomorphism if and only if the UCT holds for the pair (A,B).*

Observe that $KK^{\mathrm{top}}(A,B)$ even makes sense for Banach algebras, and always comes with a UCT.

We promised in Section 1 to show that defining $KK(X,Y)$ to be the set of natural transformations

$$(Z \mapsto K(X\times Z)) \rightsquigarrow (Z \mapsto K(Y \times Z))$$

indeed agrees with Kasparov's $KK(C_0(X), C_0(Y))$. Indeed, $Z \mapsto K(X\times Z)$ is basically the cohomology theory defined by $\mathbb{K}(X)$, and $Z \mapsto K(Y\times Z)$ is similarly the cohomology theory defined by $\mathbb{K}(Y)$. So the natural transformations (commuting with Bott periodicity) are basically a model for $KK^{\mathrm{top}}(C_0(X), C_0(Y))$.

3.6. Topological applications. The UCT can be used to prove facts about topological K-theory which on their face have nothing to do with C^*-algebras or KK. For example, we have the following purely topological fact:

THEOREM 3.7. *Let X and Y be locally compact spaces such that $K^*(X) \cong K^*(Y)$ just as abelian groups. Then the associated K-theory spectra $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are homotopy equivalent.*

PROOF. We have seen (Theorem 3.3) that the hypothesis implies $C_0(X)$ and $C_0(Y)$ are KK-equivalent, which gives the desired conclusion. □

Note that this theorem is quite special to complex K-theory; it fails even for ordinary cohomology (since one needs to consider the action of the Steenrod algebra).

Similarly, the UCT implies facts about cohomology operations in complex K-theory and K-theory mod p. For example, one has:

THEOREM 3.8 (Rosenberg-Schochet [**71**]). *The $\mathbb{Z}/2$-graded ring of homology operations for $K(—;\mathbb{Z}/n)$ on the category of separable C^*-algebras is the exterior algebra over $\mathbb{Z}/n$ on a single generator, the Bockstein β.*

THEOREM 3.9 (Araki-Toda [**2**], new proof by Rosenberg-Schochet in [**71**]). *There are exactly n admissible multiplications on K-theory mod n. When n is odd, exactly one is commutative. When $n = 2$, neither is commutative.*

3.7. Applications to C^*-algebras. Probably the most interesting applications of the UCT for KK are to the classification problem for nuclear C^*-algebras. The *Elliott program* (to quote M. Rørdam from his review of the Kirchberg-Phillips paper [**43**]) is to classify "all separable, nuclear C^*-algebras in terms of an invariant that has K-theory as an important ingredient." Kirchberg and Phillips have shown how to do this for *Kirchberg algebras*, that is simple, purely infinite, separable and nuclear C^*-algebras. The UCT for KK is a key ingredient.

THEOREM 3.10 (Kirchberg-Phillips [**43, 54**]). *Two stable Kirchberg algebras A and B are isomorphic if and only if they are KK-equivalent; and moreover every invertible element in $KK(A, B)$ lifts to an isomorphism $A \to B$. Similarly in the unital case if one keeps track of $[1_A] \in K_0(A)$.*

We will not attempt to explain the proof of Kirchberg-Phillips, but it's based on the idea that a KK-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism. More recent results of a somewhat similar nature may be found in [**24, 23, 47**].

Given the Kirchberg-Phillips result, one is still left with the question of determining when two Kirchberg algebras are KK-equivalent. But those of "Cuntz type" (like $\mathcal{O}_n$)[7] lie in $\mathcal{B}$, and Kirchberg and Phillips show that for all abelian groups G_0 and G_1 and $g \in G_0$, there is a nonunital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups, and there is a unital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups and with $[1_A] = g$. So by the UCT, these algebras are classified by their K-groups.

The original work on the Elliott program dealt with the opposite extreme: stably finite algebras. Here again, KK can play a useful role. Here is a typical result from the vast literature:

THEOREM 3.11 (Elliott [**25**]). *If A and B are C^*-algebras of real rank 0 which are inductive limits of certain "basic building blocks", then any $x \in KK(A, B)$ preserving the "graded dimension range" can be lifted to a $*$-homomorphism $A \to B$. If x is a KK-equivalence, it can be lifted to an isomorphism.*

The algebras considered in this theorem are automatically in the bootstrap category $\mathcal{B}$. This theorem applies, for example, to the irrational rotation algebras

[7]This is the fundamental example of a Kirchberg algebra, invented by Cuntz [**18**]. It is the universal C^*-algebra generated by n isometries whose range projections are orthogonal and add to 1. Cuntz proved that it is simple, and showed that $\mathcal{O}_n \otimes \mathcal{K}$ is a crossed product of a UHF algebra (an inductive limit of matrix algebras) by an action of $\mathbb{Z}$. But crossed products by $\mathbb{Z}$ preserve the category $\mathcal{B}$, because of the arguments in Sections 2.4 and 2.5. Thus $\mathcal{O}_n$ lies in $\mathcal{B}$.

A_θ, because of an amazing result by Elliott and Evans [**26**] that shows that these algebras are indeed inductive limits of the required type.

4. A fundamental example in noncommutative geometry: topology and geometry of the irrational rotation algebra

4.1. Basic facts about A_θ. We previously mentioned the algebra A_θ, defined to be the crossed product $C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$, where $\mathbb{T}$ is the circle group (thought of as the unit circle in $\mathbb{C}$) and where α_θ sends the generator $1 \in \mathbb{Z}$ to multiplication by $e^{2\pi i\theta}$, i.e., rotation of the circle by an angle of $2\pi\theta$. This makes sense for any $\theta \in \mathbb{R}$, but of course only the class of θ mod $\mathbb{Z}$ matters, so we might as well take $\theta \in [0, 1)$. This algebra has two standard names: a *rotation algebra* (with parameter θ), or *irrational rotation algebra* in the most important case of $\theta \notin \mathbb{Q}$, or a *noncommutative* (2-)*torus*, because of the fact that when $\theta = 0$, we get back simply $C(\mathbb{T}^2)$, the continuous functions on the usual 2-torus. It is no exaggeration to say that *these C^*-algebras are the most important examples in* (C^**-algebraic*) *noncommutative geometry.*

In this section we'll try to lay out the basic facts about these algebras, without attempting to prove everything or to explain the history of every result. The standard references for a lot of this material are the fundamental papers of Rieffel [**61, 64**]. A more extensive survey on this material can be found in [**66**].

We can describe the algebra A_θ quite concretely, using the definition of the crossed product in Section 2.2. The algebra has two unitary generators U and V, one of them generating $C(\mathbb{T})$ and the other corresponding to the generator of $\mathbb{Z}$. They satisfy the commutation relation

$$UV = e^{2\pi i\theta} VU. \tag{4.1}$$

The algebra A_θ is the completion of the noncommutative polynomials in U and V. But because of the commutation relation, we can move all U's to the left and all V's to the right in any noncommutative monomial in U and V, at the expense of a scalar factor of modulus 1. Thus A_θ is the completion of the polynomials $\sum_{m,n} c_{m,n} U^m V^n$ (with only finitely many non-zero coefficients). In fact, every element of A_θ is represented by a *formal* such infinite sum, but it is not so easy to describe the C^*-algebra norm in terms of the sequence of *Fourier coefficients* $\{c_{m,n}\}$. The one thing we can say, since $\|U\| = \|V\| = 1$, is that the C^*-norm is bounded by the L^1-norm, so that if the coefficients converge absolutely, then the corresponding infinite sum does represent an element of A_θ. (But the converse is false. This is classical when $\theta = 0$, and amounts to the fact that there are continuous functions whose Fourier series do not converge absolutely.)

The algebra A_θ has a canonical *trace* τ, i.e., a bounded linear functional with $\tau(ab) = \tau(ba)$ for all $a, b \in A_\theta$. We normalize by taking $\tau(1) = 1$. Usually we add the condition that τ should send self-adjoint elements to real values, though when θ is irrational, this is automatic. When $\theta = 0$, τ is just integration with respect to Haar measure on $\mathbb{T}^2$ (normalized to be a probability measure).

There is a basic dichotomy between two cases. If θ is irrational, then no different powers of $e^{2\pi i\theta}$ coincide. It is not too hard to show from this that A_θ is simple and that there is a *unique* trace in this case, defined by the condition that $\tau(U^m V^n) = 0$ if $m \neq 0$ or $n \neq 0$. (Recall we do require $\tau(1) = 1$.) So τ simply picks out the $(0, 0)$ coefficient $c_{0,0}$ from $\sum_{m,n} c_{m,n} U^m V^n$. On the other hand, if $\theta = \frac{p}{q} \in \mathbb{Q}$, then A_θ has a big center, and in fact A_θ is the algebra of sections of a bundle of matrix algebras over T^2. In fact one can show in this case that $A_\theta \cong \mathrm{End}_{T^2}(V)$, the bundle endomorphisms of any complex line bundle V over T^2 with first Chern class $\equiv p \pmod q$ (times the usual generator of $H^2(T^2, \mathbb{Z})$). The algebra has many traces in this case, but it's still convenient to let τ be the one with $\tau(U^m V^n) = 0$ if $m \neq 0$ or $n \neq 0$. (This along with the condition that $\tau(1) = 1$ then determines τ uniquely.)

The K-theory of A_θ can be computed from the Pimsner-Voiculescu sequence of Section 2.5. In fact, the main motivation of Pimsner and Voiculescu for developing this sequence was to compute $K_*(A_\theta)$. Since α_θ is isotopic to the trivial action, regardless of the value of θ, the map $1 - \alpha(1)_*$ in (2.1) is always 0. Hence, just as abelian groups, one always has $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z}^2$. But one wants more than this; one wants a description of the generators. Tracing through the various maps involved shows that one summand in K_0 is generated by the rank-one free module (or the projection 1), and that the two summands in K_1 are generated by U and V, respectively. But the interesting feature is the *order structure* on K_0, which comes from the inclusions of projective modules.[8] Note that the trace gives a homomorphism from $K_0(A_\theta)$ to $\mathbb{R}$, sending a projective module to the trace of a self-adjoint projection (in some matrix algebra) representing it. (It's a fact that every idempotent in a C^*-algebra is similar to a self-adjoint one; see for example [**7**, §4.6]. Since the trace takes real values on self-adjoint elements, the dimension of a projection is real-valued.)

THEOREM 4.1. *If $\theta \notin \mathbb{Q}$, the trace τ induces an isomorphism of $K_0(A_\theta)$ with $\mathbb{Z} + \theta\mathbb{Z}$* as ordered groups. *If $\theta \in \mathbb{Q}$, then τ still sends $K_0(A_\theta)$ to $\mathbb{Z} + \theta\mathbb{Z}$ (which is equal to $\theta\mathbb{Z}$ in this case), but is no longer an isomorphism.*

The original proof of this theorem was nonconstructive, i.e., it did not exhibit a projective module of dimension θ that should be the missing generator of K_0. We will talk about this issue later in Section 4.3.

It follows from Theorem 4.1 that the irrational rotation algebras must split into uncountably many Morita equivalence classes, since it is easy to see that Morita equivalence preserves the ordering on K_0, and since there are uncountably many order isomorphism classes of subgroups of $\mathbb{R}$ of the form $\mathbb{Z} + \theta\mathbb{Z}$. In fact, any order isomorphism $\mathbb{Z} + \theta\mathbb{Z} \to \mathbb{Z} + \theta'\mathbb{Z}$ must be given by multiplication by some $t \neq 0$ in $\mathbb{R}$, with the property that $t \in \mathbb{Z} + \theta'\mathbb{Z}$ and $t\theta \in \mathbb{Z} + \theta'\mathbb{Z}$. If we write $t = c\theta' + d$ and $t\theta = a\theta' + b$, $a, b, c, d \in \mathbb{Z}$, then

$$\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta'$$

for the usual action of 2×2 matrices by linear fractional transformations. Since the Morita equivalence must be invertible, we also have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}).$$

So Morita equivalences of irrational rotation algebras correspond to the action of $GL(2, \mathbb{Z})$ by linear fractional transformations. The converse is also true.

[8] For any unital ring A and finitely generated projective modules P_1 and P_2, we say $[P_1] \leq [P_2]$ in $K_0(A)$ if P_1 is isomorphic to a submodule of P_2.

THEOREM 4.2 (Rieffel). *Any unital C^*-algebra Morita equivalent to an irrational rotation algebra A_θ is a matrix algebra over $A_{\theta'}$ with θ' in the orbit of θ for the action of $GL(2,\mathbb{Z})$ on $\mathbb{RP}^1$ by linear fractional transformations. Every matrix in $GL(2,\mathbb{Z})$ gives rise to such a Morita equivalence.*

This is not true by "general nonsense" but requires an explicit construction, which arises from the following theorem of Rieffel:

THEOREM 4.3 (Rieffel [**63**]). *If G is a locally compact group with closed subgroups H and K, then $H \ltimes (G/K)$ and $(H\backslash G) \rtimes K$ are Morita equivalent.*

If we apply this with $G = \mathbb{R}$, $H = 2\pi\mathbb{Z}$, and $K = 2\pi\theta\mathbb{Z}$, then $H\backslash G$ is the usual model of $\mathbb{T}$ and $(H\backslash G) \rtimes K$ is A_θ, while $H \ltimes (G/K)$ is $A_{1/\theta}$. The Morita equivalence bimodule between these two algebras is a completion of $\mathcal{S}(\mathbb{R})$, with the two generators of each algebra acting by translation and by multiplication by an exponential, respectively. The reason why the two actions commute is that translation by $\mathbb{Z}$ commutes with multiplication by $e^{2\pi i s}$, while translation by $\frac{1}{\theta}\mathbb{Z}$ commutes with multiplication by $e^{2\pi i\theta s}$.

Since $GL(2,\mathbb{Z})$ is generated by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which act by $\theta \mapsto -\theta$, $\theta \mapsto 1+\theta$, and by $\theta \mapsto \frac{1}{\theta}$, respectively, and since A_θ, $A_{1+\theta}$, and $A_{-\theta}$ are all isomorphic (since if U and V satisfy (4.1), U and V^{-1} satisfy the same relation with θ replaced by $-\theta$), Theorem 4.2 follows.

4.2. Basic facts about A_θ^∞. One of the interesting things about A_θ is that it behaves in many ways like a smooth manifold. That means that we should have an analogue of the C^∞ functions inside the algebra A_θ of "continuous" functions. To find this, note that A_θ carries an action of the compact Lie group $\mathbb{T}^2$ via $(z,w)\cdot U = zU$, $(z,w)\cdot V = wV$, for $z, w \in \mathbb{T}$ (viewed as complex numbers of modulus 1). This is analogous to the action of $\mathbb{T}^2$ on itself by translations. The *smooth subalgebra* A_θ^∞ is defined to be the set of C^∞ vectors for this action, i.e., the elements a for which $(z,w) \mapsto (z,w)\cdot a$ is C^∞ as a map $\mathbb{T}^2 \to A_\theta$. Alternatively, we can describe A_θ^∞ as the intersection of the domains of all polynomials in δ_1 and δ_2, the commuting (unbounded) derivations obtained by differentiating the action. Since it is obvious that $\delta_1(U) = 2\pi i U$ and $\delta_2(U) = 0$, while $\delta_2(V) = 2\pi i V$ and $\delta_1(V) = 0$, one readily sees (as in the smooth case) that A_θ^∞ is a subalgebra and that it can be described as

$$A_\theta^\infty = \left\{ \sum_{m,n} c_{m,n} U^m V^n \mid c_{m,n} \text{ is rapidly decreasing} \right\},$$

where "rapidly decreasing" means decreasing faster than the reciprocal of any positive polynomial in m and n. Thus A_θ^∞ is isomorphic as a topological vector space (not as an algebra) to $\mathcal{S}(\mathbb{Z}^2)$ and then by Fourier transform to $C^\infty(\mathbb{T}^2)$.

PROPOSITION 4.4. *The inclusion of A_θ^∞ into A_θ is "isospectral" (i.e., an element of the subalgebra is invertible in the subalgebra if and only if it has an inverse in the larger algebra), and thus the inclusion $A_\theta^\infty \hookrightarrow A_\theta$ induces an isomorphism on K-theory.*

PROOF. Isospectral inclusions preserve K_0 and (topological) K_1, by the "Karoubi density theorem," so it is enough to prove the first statement. But this follows from the characterization of A_θ^∞ in terms of derivations, and the familiar identity $\delta_j(a^{-1}) = -a^{-1}\delta_j(a)a^{-1}$, iterated many times. □

From this Proposition, as well as the fact that there is no essential difference between smooth and purely topological manifold topology in dimension 2, one might be tempted to guess that A_θ and A_θ^∞ behave similarly in all important respects. But a deep fact is that *this is false*; $\operatorname{Aut}(A_\theta)$ and $\operatorname{Aut}(A_\theta^\infty)$ are quite different from one another.

THEOREM 4.5. *If θ is irrational, every automorphism of A_θ^∞ is "orientation-preserving," i.e., the determinant of the induced map on $K_1(A_\theta^\infty) \cong K_1(A_\theta) \cong \mathbb{Z}^2$ is $+1$. On the other hand, A_θ has orientation-reversing automorphisms.*

Comment: The first part of this is due to [**21**]. The second part is due to Elliott and Evans [**26, 25**].

4.3. Geometry of vector bundles. In classical topology, vector bundles play an important role in studying compact manifolds M. Recall *Swan's Theorem* ([**7**, §1.7] or [**22**, §1.3.3]): there is an equivalence of categories between topological (respectively, smooth) vector bundles over M and finitely generated projective modules over $C(M)$ (resp., $C^\infty(M)$), that comes from sending a vector bundle to its module of continuous (or smooth) sections. Thus, in noncommutative geometry, finitely generated projective modules play the same role as vector bundles. Because of Proposition 4.4, when it comes to irrational rotation algebras, the "vector bundle" theory is essentially the same in both the continuous and C^∞ cases, in that every finitely generated projective module over A_θ is isomorphic to one extended from a finitely generated projective module over A_θ^∞, which is unique up to isomorphism.

In general, K-theory gives the *stable* classification of vector bundles. The *unstable* classification is always more delicate, but, for A_θ, this too is known. It turns out that the case of irrational θ is in a sense easier than the "classical" case of $\theta = 0$, since the "dimension" function given by the trace is a complete invariant when θ is irrational, whereas when $\theta = 0$, complex vector bundles over $\mathbb{T}^2$ are classified by the *pair* consisting of the dimension and the first Chern class c_1.

THEOREM 4.6 (Rieffel [**64**]). *For A_θ with θ irrational, complete cancellation holds for finitely generated projective modules, i.e., if $P \oplus Q \cong P' \oplus Q$ as A_θ-modules, for some finitely generated projective A_θ-modules P, P', Q, then P and P' are isomorphic. The isomorphism classes of projective submodules of a free A_θ-module of rank n are distinguished by the trace, and are given exactly by elements of $K_0(A_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ between 0 and n (inclusive).*

Once one knows the classification of the "vector bundles," in both the smooth and continuous categories, a natural next step is to study "geometry" on them. In his fundamental paper [**15**], Alain Connes explained how the theory of connections and curvature in differential geometry can be carried over to the noncommutative case, at least when one has an algebra A like A_θ with an action of a Lie group G for which the "smooth subalgebra" A^∞ is the set of C^∞-vectors for the G-action on A. (This of course applies here with $G = \mathbb{T}^2$ acting as we described above.) Then if V is a finitely generated (right) A^∞-module, a connection on V is a map $\nabla\colon V \to V \otimes \mathfrak{g}^*$ ($\mathfrak{g}$ the Lie algebra of G) satisfying the usual Leibniz rule

$$\nabla_X(v \cdot a) = \nabla_X(v) \cdot a + v \cdot (X \cdot a), \quad v \in V,\ a \in A^\infty,\ X \in \mathfrak{g}.$$

Usually one requires a connection to be compatible with an inner product also. Connections always exist and have a curvature 2-form $\Theta \in \text{End}_A(V) \otimes \bigwedge^2 \mathfrak{g}^*$ defined as usual by

$$\Theta(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

THEOREM 4.7 (Connes [**15**]). *Every finitely generated projective module over* A_θ^∞ *admits a connection of constant curvature (i.e., with the curvature in* $i \bigwedge^2 \mathfrak{g}^*$*). The curvature can be taken to be* 0 *if and only if the module is free. More precisely, on the projective module with "dimension"* $p + q\theta > 0$*,* $p, q \in \mathbb{Z}$*, the constant curvature connections have curvature*

$$\Theta(\delta_1, \delta_2) = \frac{2\pi i q}{p + q\theta}.$$

Connes and Rieffel defined the notion of *Yang-Mills energy* of a connection, precisely analogous to the classical case for smooth vector bundles over manifolds. This is defined by

$$\text{YM}(\nabla) = -\tau_{\text{End}(V)}\big(\{\Theta_\nabla, \Theta_\nabla\}\big),$$

where $\{—, —\}$ is the natural bilinear form on 2-forms.

THEOREM 4.8 (Connes and Rieffel [**17, 65**]). *If* V *is a finitely generated projective module over* A_θ^∞*, a connection* ∇ *on* V *gives a minimum for* YM *if and only if it has constant curvature, and gives a critical point for* YM *if and only if it is a direct sum of constant curvature connections (i.e.,* V *has a decomposition* $V_1 \oplus \cdots \oplus V_n$ *with respect to which* ∇ *has a similar decomposition into connections of constant curvature).*

As we mentioned earlier, the original calculation of $K_0(A_\theta)$ was nonconstructive, and the problem remained of explicitly exhibiting representatives for the finitely generated projective modules. One answer is already implicit in what we have explained: if P is a finitely generated projective A-module, then it gives rise to a Morita equivalence between A and $\text{End}_A(P)$, so constructing all possible P's is equivalent to finding all Morita equivalence bimodules for A. In the case of A_θ, they are all similar to the bimodule we mentioned before between A_θ and $A_{1/\theta}$. But one could ask for another answer to the problem, namely to give explicit representatives for all the equivalence classes of projections in A_θ (or in matrix algebras over it). Here two good solutions have been proposed, one by Rieffel [**61**] and one by Boca [**8**]. Rieffel constructed explicit projections in A_θ of the form $Uf + g + \overline{f}U^*$, where f and g are functions of V. Boca instead constructed projections in terms of theta-functions which can be described as follows: if X is an A-B Morita equivalence bimodule as above, with $A = A_\theta$, and if one can find an element $\psi \in X$ with $\langle \psi, \psi \rangle_B = 1_B$, then ${}_A\langle \psi, \psi \rangle$ will be a projection in A. Boca's projections come from choosing ψ closely related to a Gaussian function in $\mathcal{S}(\mathbb{R})$.

4.4. Miscellaneous other facts about A_θ. Here we just mention a few other things about the algebras A_θ. The work of Elliott and Evans [**26, 25**], which we mentioned before, has more detailed implications for automorphisms and endomorphisms of A_θ. Assuming θ is irrational, given any $A \in GL(2, \mathbb{Z})$, there is an automorphism of A_θ inducing the map A on $K_1(A_\theta) \cong \mathbb{Z}^2$, and given any $B \in \text{End}(\mathbb{Z}^2)$ (including 0!), there is a unital endomorphism of A_θ inducing the identity on $K_0(A_\theta)$ and the map B on $K_1(A_\theta)$. Furthermore, the connected component of the identity in $\text{Aut}(A_\theta)$ is topologically simple, and $\text{Aut}(A_\theta)$ is just an extension of this connected group by $GL(2, \mathbb{Z})$ [**27**]. All of this seems quite strange from the perspective of ordinary manifold topology, since a self-map $T^2 \to T^2$ inducing the identity on $K^0(T^2)$ is of degree 1, and thus cannot induce the 0-map on $K^{-1}(T^2) \cong H^1(T^2)$.

However, the endomorphisms constructed by Elliott's procedure are unlikely to be smooth. Kodaka [**44**] did construct some special smooth proper unital endomorphisms of irrational rotation algebras, but only when θ lies in a real quadratic number field.

And one more structural fact about the algebras A_θ: they have *real rank zero*, that is, finite linear combinations of projections are dense in the set of self-adjoint elements.

5. Applications of the irrational rotation algebra in number theory and physics

5.1. Applications to number theory. In this section we will discuss two ways in which noncommutative tori arise in number theory: as "limit points" of the moduli space of elliptic curves, and as noncommutative elliptic curves themselves. These two points of view are interconnected, as is explained in [**49**, §4], which also provides a good survey of this area, so this division is just for the purpose of giving the reader a quick guide to the subject.

5.1.1. *Noncommutative tori and the moduli space of elliptic curves.* In complex analysis or complex algebraic geometry, an *elliptic curve* is a complex manifold of the form $\mathbb{C}/\Lambda$, where Λ is a *lattice*, that is, a discrete cocompact subgroup, in the vector group $\mathbb{C}$. Note that Λ is then necessarily free abelian on two generators, linearly independent over $\mathbb{R}$. An elliptic curve is the same thing as a Riemann surface of genus 1. The theory of elliptic functions shows that an elliptic curve has an embedding into $\mathbb{CP}^2$ as a complex projective variety of dimension 1 and degree 3. (In other words, it is the solution set of a homogeneous cubic polynomial equation in three homogeneous coordinates.) If there is a complex number $\lambda \neq 0$ with $\lambda\Lambda = \Lambda'$, then multiplication by λ gives a holomorphic isomorphism from $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$, so we can identify these two elliptic curves as being identical (in the holomorphic category). It is therefore no loss of generality to take Λ to be of the form $\mathbb{Z} + \tau\mathbb{Z}$, with $\operatorname{Im}\tau > 0$, i.e., with $\tau \in \mathfrak{h}$, the upper half-plane. Furthermore, the isomorphism class of $\mathbb{C}/\Lambda$ only depends on the orbit of τ under the action of the modular group $\Gamma = SL(2, \mathbb{Z})$ on $\mathfrak{h}$ by linear fractional transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, since Γ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which sends $\tau \mapsto \tau + 1$ and keeps Λ invariant, and by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which sends $\tau \mapsto -\frac{1}{\tau}$, and replaces Λ by $\frac{1}{\tau} \cdot \Lambda$. In the other direction, if there is a holomorphic isomorphism from $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$, then lifting to the universal covers gives a holomorphic map $\mathbb{C} \to \mathbb{C}$ with linear growth, which by an application of Liouville's Theorem has to be given by a linear polynomial. One then quickly deduces that there is a a complex number $\lambda \neq 0$ with $\lambda\Lambda = \Lambda'$, and so the *moduli space* of elliptic curves is precisely the quotient $\mathfrak{h}/\Gamma$. This is itself a complex curve (1-dimensional complex manifold), but is noncompact.

Many problems in algebraic geometry and number theory have to do with understanding limits of elliptic curves as one goes to infinity in the moduli space $\mathfrak{h}/\Gamma$. In other words, one wants some sort of compactification of the moduli space.

If we think of $\mathfrak{h}$ as being an open disk in $\mathbb{CP}^1$, a natural way to compactify would be to adjoin $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ to $\mathfrak{h}$, then take the quotient. The problem, of course, is that the action of Γ on $\mathbb{RP}^1$ is not proper, so the quotient is not Hausdorff. In fact, the orbit of any irrational point $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$ is dense in $\mathbb{RP}^1$.

This is where irrational rotation algebras naturally fit in. In fact, we have already seen that if θ is irrational and $B \in \Gamma$, then $A_{B\cdot\theta}$ and A_θ are Morita equivalent. Thus the orbit of $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$ under Γ [9] naturally parametrizes a Morita equivalence class of noncommutative tori. This observation suggests that noncommutative tori should be viewed as "limits" of degenerating elliptic curves.

5.1.2. *Noncommutative tori as noncommutative elliptic curves.* The other main connection between noncommutative tori and algebraic geometry and arithmetic comes from viewing them as noncommutative elliptic curves, by fixing a complex structure. Of course, there is a big difference from the classical case. In addition to a modular parameter $\tau \in \mathfrak{h}$ defining the complex structure, we also have the noncommutativity parameter θ, which has no classical analogue. Just as there are special elliptic curves (coming from imaginary quadratic number fields) with *complex multiplication*, in other words, "extra" automorphisms with interesting number-theoretic properties, Manin has proposed a program of studying noncommutative elliptic curves with *real multiplication* [**48**], coming from real quadratic number fields.

The study of noncommutative tori as noncommutative elliptic curves was done largely by Polishchuk in a series of papers, notably [**56**], [**57**], and [**58**]. Fix some irrational value of θ. If we think of δ_1 and δ_2 from Section 4.1 as corresponding to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in the complex plane, then the $\overline{\partial}$ operator of complex analysis, $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, is represented by $\frac{1}{2}(\delta_1 + i\delta_2) = \frac{i}{2}(-i\delta_1 + \delta_2)$. More generally, the $\overline{\partial}$ operator for a more general "complex structure" on A_θ can be represented (up to a largely irrelevant complex scalar factor) by the operator $\delta_\tau = \tau\delta_1 + \delta_2$, where τ is in the *lower* half-plane.[10] Then a *holomorphic vector bundle* over A_θ for this choice of complex structure is a finitely generated projective (right) A_θ^∞-module P, equipped with a holomorphic connection, that is, an operator $\overline{\nabla}\colon P \to P$ satisfying

$$\overline{\nabla}(sa) = \overline{\nabla}(s)a + s\delta_\tau(a). \tag{5.1}$$

Polishchuk then proved in [**56**] that the category $\mathcal{C}$ of holomorphic vector bundles $(P, \overline{\nabla})$ on (A_θ, τ) is abelian, and is generated by the standard holomorphic bundles (slight generalizations of the projective modules used above in Section 4.1 to construct the Morita equivalence between A_θ and $A_{1/\theta}$, equipped with standard holomorphic connections). More remarkably, the abelian category $\mathcal{C}$ can be recovered from classical algebraic geometry, in the sense that $\mathcal{C}$ is equivalent to the heart $\mathcal{C}_\theta$ of a (nonstandard) t-structure defined by the parameter θ on the derived category of coherent sheaves on the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The definition of $\mathcal{C}_\theta$ uses something very much like the definition of *stable vector bundle* in geometric invariant theory. In other words, one looks at the degree and rank of holomorphic vector bundles over E_τ, and considers a "slope condition": $\mathcal{C}_\theta$ is built out of bundles

[9] More precisely, we should divide by the action of $GL(2,\mathbb{Z})$, not $SL(2,\mathbb{Z})$, but the usual moduli space $\mathfrak{h}/\Gamma$ can also be written as $(\mathfrak{h} \cup -\mathfrak{h})/GL(2,\mathbb{Z})$, so that $\mathbb{CP}^1/GL(2,\mathbb{Z})$ is a natural compactification.

[10] It might have been easier to use $\delta_1 + \tau\delta_2$ with $\tau \in \mathfrak{h}$, but I'm trying to stick to Polishchuk's notational conventions. The parameter in $\mathfrak{h}$ corresponding to his $\tau \in -\mathfrak{h}$ is $\frac{1}{\tau}$.

P for which $\deg(P) \geq \theta \operatorname{rk}(P)$. Similarly, in [**58**], Polishchuk defines a category of quasicoherent sheaves on (A_θ, τ), and proves that it is abelian. Again he shows that this category is equivalent to a subcategory, specified by the parameter θ, of the derived category of quasicoherent sheaves on E_τ.

The next major step toward Manin's real multiplication program may be found in [**57**]. Here Polishchuk considers the case of A_θ with θ a quadratic irrational number. (Thus $\theta \notin \mathbb{Q}$, but $\mathbb{Q}(\theta)$ is a real quadratic number field.) Then we may assume there is a matrix $g \in SL(2, \mathbb{Z})$ with $g\theta = \theta$, and g can be used to define a nontrivial A_θ^∞-A_θ^∞ bimodule E_g, which can be given a standard holomorphic structure (5.1). Via tensor product over A_θ^∞, we can define the tensor powers $E^{\otimes n}$, and thus an associative graded algebra

$$B_g(\theta, \tau) = \bigoplus_{n=0}^{\infty} H^0(E_g^{\otimes n}), \tag{5.2}$$

where $H^0(E_g^{\otimes n})$ is the space of "holomorphic sections" (i.e., the kernel of $\overline{\nabla}$) in $E_g^{\otimes n}$. This turns out to be finite-dimensional for each n. The algebra structure comes from the fact that the tensor product of two holomorphic sections is again killed by $\overline{\nabla}$. (Note that $H^0(E^{\otimes 0}) = \ker \delta_\tau$ on A_θ^∞, which is just the scalars.) The algebra $B_g(\theta, \tau)$ can be viewed as the coordinate ring of a noncommutative projective variety, and Polishchuk computes its Hilbert function.

A further step toward Manin's theory of real multiplication was taken by Plazas in [**55**]. Plazas actually gives explicit generators and relations for $B_g(\theta, \tau)$ in terms of theta functions and theta constants, making the connection with number theory that Manin had anticipated. In fact, Plazas proves that if the elliptic curve E_τ is algebraic over a number field k, then the algebra $B_g(\theta, \tau)$ admits a rational presentation over a finite algebraic extension of k. He also obtains some other arithmetic results too technical to explain here.

5.2. Applications to physics. That the irrational rotation algebra has shown up frequently in the physics literature is probably not surprising, given that one of the most basic principles of quantum physics is the Heisenberg commutation relation, which in the *Weyl form*[11] becomes the fundamental relation (4.1), with θ playing the role of Planck's constant. In fact, noncommutative tori have appeared in just about all areas of quantum physics, including quantum statistical mechanics, condensed matter physics, and quantum field theory. However, here I will just mention a few of the ways they have appeared in relation to string theory.

String theory is a fundamental particle theory in which point particles are replaced by strings (that is, compact 1-manifolds, possibly with boundary) propagating in space and time. The *fundamental string* is thus a field given by a map from the *string worldsheet* Σ (a 2-manifold, given by the string propagating in time) into a spacetime manifold X. To get a consistent theory of fundamental particles, one usually requires the theory to be supersymmetric[12] (so in particular, it involves

[11]It was Hermann Weyl who had the idea of exponentiating position and momentum to unitary operators, to avoid the necessity of working with unbounded self-adjoint operators.

[12]Supersymmetry is a conjectured symmetry of physics that places bosons, particles like photons that can "accumulate" in a single state, and fermions, particles like electrons that satisfy the Pauli exclusion principle, on an equal footing. Such symmetry has not been observed yet experimentally, but if it holds, it would impose significant constraints on elementary particle theories.

both fermions and bosons); then for reasons of anomaly cancellation, X has to be 10-dimensional. Often it is taken to be a product of 4-dimensional Minkowski space $\mathbb{R}^4$ with a Calabi-Yau 3-fold Y^6, that is, a compact complex Kähler manifold of complex dimension 3 with an everywhere nonvanishing holomorphic 3-form. Physicists often say that the theory is *compactified* on the compact manifold Y.

The basic idea of many of the applications of noncommutative tori to physics is that under some circumstances, it seems that string theory, or other similar theories (which include gauge theories, M-theory, and F-theory), should be compactified on a *noncommutative* compact manifold, of which A_θ is the simplest example. (Indeed, a product of A_θ's with holomorphic structures, which we've argued above should be considered to be noncommutative elliptic curves, can be considered to be a special case of a noncommutative abelian variety, the simplest case of a noncommutative Calabi-Yau.) This point of view is espoused in particular in the two classic papers [**16**] and [**73**].

The first of these deals with *matrix theory*, which is a supersymmetric field theory believed to be closely related to string theory. More precisely, Connes, Douglas, and Schwarz consider two versions of matrix theory, the IKKT model (due to Ishibashi, Kawai, Kitazawa, and Tsuchiya [**37**]) and the BFSS model (due to Banks, Fischler, Shenker, and Susskind [**5**]). Matrix theory is a finite-dimensional quantum theory (that is, the fields (X, Ψ) lie in a finite-dimensional space, which for the IKKT model is $\mathbb{C}^{10|16} \times M_N(\mathbb{C})$); for example, the IKKT model is obtained from ten-dimensional super-Yang-Mills gauge theory by "reduction to a point," or in other words, restricting the action functional to constant fields. But this model leads to the action functional of superstring theory in the limit when the size N of the matrices goes to infinity. The BFSS model roughly speaking corresponds to a Wick rotation[13] of the IKKT model. In any event, Connes, Douglas, and Schwarz consider fields which satisfy a periodicity condition $X_j + R_j = U_j X_j U_j^{-1}$ for $j = 0, 1$, $U_0 X_1 U_0^{-1} = X_1$, $U_1 X_0 U_1^{-1} = X_0$, with the Ψ^α's and X_j's for $j > 1$ commuting with U_0 and U_1, and show that this leads to a theory living on A_θ for some value of θ. From this point of view the significance of the noncommutativity parameter θ is unclear, but later the authors identify this with a physical parameter that transforms under an $SL(2, \mathbb{Z})$ symmetry group acting by linear fractional transformations. (The $SL(2, \mathbb{Z})$ symmetry is what is called S-duality, S for "strong-weak," in super-Yang-Mills gauge theory.)

The Seiberg-Witten paper [**73**] takes a somewhat more intuitive point of view, and shows how string theory in flat space, in the presence of a constant but nonzero B-field, leads to noncommutative tori. Thus we should digress and explain what the B-field is; it is not quite the classical magnetic field of electromagnetism (usually denoted by the same letter), but it serves a quite similar role. The B-field is a differential 2-form on the spacetime manifold X; pulling back to the string worldsheet Σ gives us a 2-form on Σ which can be integrated to give another term (the Wess-Zumino term) in the string action functional. Seiberg and Witten argue that turning on the B-field leads to an effective action which can be understood in terms of spacetime becoming noncommutative.

Still another occurrence of noncommutative tori in string theory can be found in [**50**]. (See also my book [**69**] for a more detailed exposition.) This also has to

[13] This is a trick often used by physicists, in which time t is replaced by it, so as to interchange Lorentzian and Riemannian geometry.

do with the B-field, though in a more subtle way. It turns out that in general the B-field need not be globally well defined, and is only a locally defined 2-form. (One can make rigorous sense of this using the notion of *gerbes*, which are nicely described in [**34**].) However, $H = dB$ is indeed a globally defined integral 3-form, called the H-flux, which need not be exact. In fact, the form representing H can be enhanced to specify a class $H \in H^3(X, \mathbb{Z})$ (the Dixmier-Douady class of the gerbe), which is allowed to have a torsion component, and this is part of the topological data defining a string theory.

The starting point for the appearance of noncommutative geometry is the analysis of what is called T-duality (T for "target space" or "torus"), which is an equivalence between one string theory on a spacetime manifold X and a dual string theory on another spacetime manifold $X^\#$. One expects that when $X \xrightarrow{p} Z$ is a principal torus bundle (with fibers T^n) over a manifold Z, $X^\# \xrightarrow{p^\#} Z$ is another torus bundle over the same base Z, but with fibers that geometrically are the dual tori to the fibers T^n of p. (When $\Lambda \subset \mathbb{R}^n$ is a lattice and $\Lambda^\#$ is the dual lattice in the dual space $(\mathbb{R}^n)^*$, we call $(\mathbb{R}^n)^*/\Lambda^\#$ the dual torus to $\mathbb{R}^n/\Lambda$.) When $n = 1$, one can make this quite explicit, and the pair $([p], H) \in H^2(Z, \mathbb{Z}) \times H^3(X, \mathbb{Z})$, where $[p]$ is the equivalence class of the circle bundle $X \xrightarrow{p} Z$, determines the dual pair $([p^\#], H^\#) \in H^2(Z, \mathbb{Z}) \times H^3(X^\#, \mathbb{Z})$. However, when $n > 1$, it can happen that there is no T-dual in this sense. Varghese and I (in [**50**]) showed that one can often explain these "missing T-duals" in terms of noncommutative geometry. For example, when $n = 2$, the condition for existence of a (classical) T-dual is that the edge homomorphism $H^3(X, \mathbb{Z}) \to E_\infty^{1,2} \subseteq H^1(Z, H^2(T^2, \mathbb{Z})) \cong H^1(Z, \mathbb{Z})$ in the Serre spectral sequence of the bundle p should send the H-flux H to 0. When this is not the case, there is no classical T-dual, but there is a *noncommutative* T-dual, which is a bundle of noncommutative tori over Z. For example, if $X = T^3$, H is the usual generator of $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$, $Z = S^1$, and p is the trivial T^2-bundle $T^3 \to S^1$, then the noncommutative T-dual is the group C^*-algebra A of the discrete Heisenberg group $\Gamma = \langle U, V, W \mid W = [U, V], [U, W] = [V, W] = 1\rangle$. This C^*-algebra is the algebra of sections of a bundle of algebras over S^1, with the fiber over the point $e^{2\pi i\theta}$ being the noncommutative torus A_θ. (This fiber corresponds to unitary representations of Γ sending the central element W to $e^{2\pi i\theta}$.) Confirmation that this is the "right" T-dual comes for example from the calculation of the K-theory. $K_*(A)$ agrees with the H-twisted K-theory of X. However, X is the only principal T^2-bundle over Z, and without twisting, its K-theory is too big. We refer the reader to [**69**] for a more complete explanation.

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Rational Equivariant K-Homology of Low Dimensional Groups

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Abstract. We consider groups G which have a cocompact, 3-manifold model for the classifying space $\underline{E}G$. We provide an algorithm for computing the rationalized equivariant K-homology of $\underline{E}G$. Under the additional hypothesis that the quotient 3-orbifold $\underline{E}G/G$ is geometrizable, the rationalized K-homology groups coincide with the groups $K_*(C^*_{red}G) \otimes \mathbb{Q}$. We illustrate our algorithm on some concrete examples.

1. Introduction

We consider groups G which have a cocompact, 3-manifold model for the classifying space $\underline{E}G$. For such groups, we are interested in computing the equivariant K-homology of $\underline{E}G$. We develop an algorithm to compute the *rational* equivariant K-homology groups. If in addition we assume that the quotient 3-orbifold $\underline{E}G/G$ is geometrizable, then G satisfies the Baum-Connes conjecture, and the rational equivariant K-homology groups coincide with the groups $K_*(C^*_{red}G)$. These are the rationalized (topological) K-theory groups of the reduced C^*-algebra of G.

Some general recipes exist for computing the rational K-theory of an arbitrary group (see Lück and Oliver [**LuO**], as well as Lück [**Lu1**], [**Lu2**]). These general recipes pass via the Chern character. They typically involve identifying certain conjugacy classes of cyclic subgroups, their centralizers, and certain (group) homology computations. Similar formulas (with similar ingredients) appear in p-adic K-theory, after tensoring with $\mathbb{Q}_p$ (see for instance Adem [**Ad**]).

In contrast, our methods rely instead on the low-dimensionality of the model for the classifying space $\underline{E}G$. Given a description of the model space $\underline{E}G$, our procedure is entirely algorithmic, and returns the ranks of the K-homology groups.

Let us briefly outline the contents of this paper. In Section 2, we provide some background material. Section 3 is devoted to explaining our algorithm, and the requisite proofs showing that the algorithm gives the desired K-groups. In Section 4 we implement our algorithm on several concrete classes of examples. Section 5 has some concluding remarks.

2. Background material

2.1. C^*-algebra. Given any discrete group G, one can form the associated reduced C^*-algebra. This Banach algebra is obtained by looking at the action $g \mapsto \lambda_g$ of G on the Hilbert space $l^2(G)$ of square summable complex-valued functions on G, given by the left regular representation:

$$\lambda_g \cdot f(h) = f\big(g^{-1}h\big) \qquad g, h \in G, \qquad f \in l^2(G).$$

The algebra $C_r^*(G)$ is defined to be the operator norm closure of the linear span of the operators λ_g inside the space $B\big(l^2(G)\big)$ of bounded linear operators on $l^2(G)$. The Banach algebra $C_r^*(G)$ encodes various analytic properties of the group G.

2.2. Topological K-theory. For a C^*-algebra A, the corresponding (topological) K-theory groups can be defined in the following manner. The group $K_0(A)$ is defined to be the Grothendieck completion of the semi-group of finitely generated projective A-modules (with group operation given by direct sum). Since the algebra A comes equipped with a topology, one has an induced topology on the space $GL_n(A)$ of invertible $(n \times n)$-matrices with entries in A, and as such one can consider the group $\pi_0\big(GL_n(A)\big)$ of connected components of $GL_n(A)$ (note that this is indeed a group, not just a set). The group $K_1(A)$ is defined to be $\lim \pi_0\big(GL_n(A)\big)$, where the limit is taken with respect to the sequence of natural inclusions of $GL_n(A) \hookrightarrow GL_{n+1}(A)$. The higher K-theory groups $K_q(A)$ are similarly defined to be $\lim \pi_{q-1}\big(GL_n(A)\big)$, for $q \geq 2$. Alternatively, one can identify the functors $K_q(A)$ for all $q \in \mathbb{Z}$ via Bott 2-periodicity in q, i.e. $K_q(A) \cong K_{q+2}(A)$ for all q.

2.3. Baum-Connes conjecture. Let us now recall the statement of the Baum-Connes conjecture (see [**BCH**], [**DL**]). Given a discrete group G, there exists a specific generalized equivariant homology theory having the property that, if one evaluates it on a point $*$ with trivial G-action, the resulting homology groups satisfy $H_n^G(*) \cong K_n(C_r^*(G))$. Now for any G-CW-complex X, one has an obvious equivariant map $X \to *$. It follows from the basic properties of equivariant homology theories that there is an induced *assembly map*:

$$H_n^G(X) \to H_n^G(*) \cong K_n(C_r^*(G)).$$

Associated to a discrete group G, we have a classifying space for proper actions $\underline{E}G$. The G-CW-complex $\underline{E}G$ is well-defined up to G-equivariant homotopy equivalence, and is characterized by the following two properties:

- if $H \leq G$ is any infinite subgroup of G, then $\underline{E}G^H = \emptyset$, and
- if $H \leq G$ is any finite subgroup of G, then $\underline{E}G^H$ is contractible.

The Baum-Connes conjecture states that the assembly map

$$H_n^G(\underline{E}G) \to H_n^G(*) \cong K_n(C_r^*(G))$$

corresponding to $\underline{E}G$ is an isomorphism. For a thorough discussion of this topic, we refer the reader to the book by Mislin and Valette [**MV**] or the survey article by Lück and Reich [**LuR**].

2.4. 3-orbifold groups. We are studying groups G having a cocompact 3-manifold model for $\underline{E}G$. Let X denote this specific model for the classifying space, and for this section, we will further assume that the quotient 3-orbifold X/G is *geometrizable.*

The validity of the Baum-Connes conjecture for fundamental groups of orientable 3-manifolds has been established by Matthey, Oyono-Oyono, and Pitsch [**MOP**, Thm. 1.1] (see also [**MV**, Thm. 5.18] or [**LuR**, Thm. 5.2]). The same argument works in the context of geometrizable 3-orbifolds. We provide some details for the convenience of the reader.

LEMMA 1. *The Baum-Connes conjecture holds for the orbifold fundamental group of geometrizable* 3*-orbifolds.*

PROOF. In fact, the stronger *Baum-Connes property with coefficients* holds for this class of groups. This property states that a certain assembly map, associated to a G-action on a separable C^*-algebra A, is an isomorphism (and recovers the classical Baum-Connes conjecture when $A = \mathbb{C}$). The coefficients version has better inheritance properties, and in particular, is known to be inherited under graph of groups constructions (amalgamations and HNN-extensions), see Oyono-Oyono [**O-O**, Thm. 1.1].

The orbifold fundamental group of a geometrizable 3-orbifold can be expressed as an iterated graph of groups, with all initial vertex groups being orbifold fundamental groups of geometric 3-orbifolds. Geometric 3-orbifolds are cofinite volume quotients of one of the eight 3-dimensional geometries. Combined with Oyono-Oyono's result, the Lemma reduces to establishing the property for the orbifold fundamental group of finite volume geometric 3-orbifolds.

The fundamental work of Higson and Kasparov [**HK**] established the Baum-Connes property with coefficients for all groups satisfying the Haagerup property. We refer the reader to the monograph [**CCJJV**] for a detailed exposition of the Haagerup property. We will merely require the fact that groups acting with cofinite volume on all eight 3-dimensional geometries ($\mathbb{E}^3$, S^3, $S^2 \times \mathbb{E}^1$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{PSL}_2(\mathbb{R})$, Nil, and Sol) always have the Haagerup property, which will conclude the proof of the Lemma.

For the five geometries $\mathbb{E}^3$, S^3, $S^2 \times \mathbb{E}^1$, Nil, and Sol, any group acting on these will be amenable, and hence satisfy the Haagerup property. Lattices inside groups locally isomorphic to $SO(n, 1)$ are Haagerup (see [**CCJJV**, Thm. 4.0.1]), and hence groups acting on the two geometries $\mathbb{H}^3$ and $\widetilde{PSL}_2(\mathbb{R})$ are Haagerup. Finally, the Haagerup property is inherited by amenable extensions of Haagerup groups (see [**CCJJV**, Example 6.1.6]). This implies that groups acting on $\mathbb{H}^2 \times \mathbb{E}^1$ are Haagerup, for any such group is a finite extension of a group which splits as a product of $\mathbb{Z}$ with a lattice in $SO(2, 1)$. This concludes the proof of the Lemma. □

Remark: If one assumes that the G-action is smooth and orientation preserving, then Thurston's geometrization conjecture (now a theorem) predicts that X/G is a geometrizable 3-orbifold. The proof of the orbifold version of the conjecture was originally outlined by Thurston, and was independently established by Boileau, Leeb, and Porti [**BLP**] and Cooper, Hodgson, and Kerckhoff [**CHK**] (both loosely following Thurston's approach). The manifold version of the conjecture (i.e. trivial isotropy groups) is of course due to the recent work of Perelman.

Remark: If the quotient space X/G is not known to be geometrizable (for instance, if the G-action is not smooth, or does not preserve the orientation), then the argument in Lemma 1 does not apply. Nevertheless, our algorithm can still be used to compute the rational equivariant K-homology of $\underline{E}G$. It is however no longer clear that this coincides with $K(C_r^*(G)) \otimes \mathbb{Q}$.

2.5. Polyhedral CW-structures. Let us briefly comment on the G-CW-structure of X. As the quotient space X/G is a connected 3-orbifold, we can assume without loss of generality that the CW-structure contains a single orbit of 3-cell. Taking a representative 3-cell σ for the unique 3-cell orbit, we observe that the closure of σ must *contain* representatives of each lower dimensional orbit of cells. Indeed, if some lower dimensional cell had no orbit representatives contained in $\bar{\sigma}$, then there would be points in that lower dimensional cell with no neighborhood homeomorphic to $\mathbb{R}^3$. Pulling back the 2-skeleton of the CW-structure via the attaching map of the 3-cell σ, we obtain (i) a decomposition of the 2-sphere into the pre-images of the individual cells, and (ii) an equivalence relation on the 2-sphere, identifying together points which have the same image under the attaching map. We note that the quotient space X/G can be reconstructed from this data. If in addition we know the isotropy subgroups of points, then X itself can be reconstructed from X/G. We will assume that we are given the G-action on X, in the form of a partition and equivalence relation on the 2-sphere as above, along with the isotropy data.

In some cases, one can find a G-CW-structure which is particularly simple: the 2-sphere coincides with the boundary of a polyhedron, the partition of the 2-sphere is into the faces of the polyhedron, and the equivalence relation linearly identifies together faces of the polyhedron. More precisely, we make the:

Definition 2. A *polyhedral* CW-structure is a CW-structure where each cell is identified with the interior of a polyhedron $P_i \cong \mathbb{D}^k$, and the attaching map from the boundary $\partial\mathbb{D}^k \cong \partial P_i$ of a k-cell to the $(k-1)$-skeleton, when restricted to each s-dimensional face of ∂P_i, is a combinatorial homeomorphism onto an s-cell in the $(k-1)$-skeleton.

In the case where there is a polyhedral G-CW-structure on X with a single 3-cell orbit, then our algorithms are particularly easy to implement. All the concrete examples we will see in Section 4 come equipped with a polyhedral G-CW-structure.

Remark: It seems plausible that, if a G-CW-structure exists for a (topological) G-action on a 3-manifold X, then a polyhedral G-CW-complex structure should also exist. It also seems likely that, if a polyhedral G-CW-structure exists, then the G-action on the 3-manifold X should be smoothable.

For some concrete examples of polyhedral G-CW-structures, consider the case where X is either hyperbolic space $\mathbb{H}^3$ or Euclidean space $\mathbb{R}^3$, and the G-action is via isometries. Then the desired G-equivariant polyhedral CW-complex structure can be obtained by picking a suitable point $p \in X$, and considering the Voronoi diagram with respect to the collection of points in the orbit $G \cdot p$. Another example, where X is the 3-dimensional Nil-geometry, is discussed in Section 4.2.

3. The algorithm

In this section, we describe the algorithm used to perform our computations. Throughout this section, let G be a group with a smooth action on a 3-manifold, providing a model for $\underline{E}G$. We will assume that $\underline{E}G$ supports a polyhedral G-CW-structure, and that P is a fundamental domain for the G-action on X, as described in Section 2.5. So P is the polyhedron corresponding to the single 3-cell orbit, and the orbit space $\underline{B}G$ is obtained from P by identifying various boundary faces together. We emphasize that the polyhedral G-CW-structure assumption serves only to facilitate the exposition: the algorithm works equally well with an arbitrary G-CW-structure.

3.1. Spectral sequence analysis. As explained in the previous section, the Baum-Connes conjecture provides an isomorphism:

$$H_n^G(\underline{E}G) \to H_n^G(*) \cong K_n(C_r^*(G)).$$

We are interested in computing the equivariant homology group arising on the left hand side of the assembly map. Since our group G is 3-dimensional, we will let X denote the 3-dimensional manifold model for $\underline{E}G$. To compute the equivariant homology of X, one can use an Atiyah-Hirzebruch spectral sequence. Specifically, there exists a spectral sequence (see [**DL**], or [**Q**, Section 8]), converging to the group $H_n^G(X)$, with E^2-terms obtained by taking the homology of the following chain complex:

$$(1) \qquad \cdots \to \bigoplus_{\sigma\in(X/G)^{(p+1)}} K_q\big(C_r^*(G_\sigma)\big) \to \bigoplus_{\sigma\in(X/G)^{(p)}} K_q\big(C_r^*(G_\sigma)\big) \to \cdots$$

In the above chain complex, $(X/G)^{(i)}$ consists of i-dimensional cells in the quotient X/G, or equivalently, G-orbits of i-dimensional cells in X. The groups G_σ denote the stabilizer of a cell in the orbit σ. Since our space X is 3-dimensional, we see that our chain complex can only have non-zero terms in the range $0 \leq p \leq 3$ (the morphisms in the chain complex will be described later, see Section 3.3). Moreover, since X is a model for $\underline{E}G$, all the cell stabilizers G_σ must be finite subgroups of G. For F a finite group, the groups $K_q\big(C_r^*(F)\big)$ are easy to compute:

$$K_q\big(C_r^*(F)\big) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \mathbb{Z}^{c(F)} & \text{if } q \text{ is even.} \end{cases}$$

Here, $c(F)$ denotes the number of conjugacy classes of elements in F. In fact, for q even, $K_q\big(C_r^*(F)\big)$ can be identified with the complex representation ring of F. This immediately tells us that $E^2_{pq} = 0$ for q odd. We will denote by $\mathcal{C}$ the chain complex in equation (1) corresponding to the case where q is even. By the discussion above, we know that $H_p(\mathcal{C}) = 0$ except possibly in the range $0 \leq p \leq 3$. We summarize this discussion in the:

Fact 1: The only potentially non-vanishing terms on the E^2-page (and hence any E^k-page, $k \geq 2$) occur when $0 \leq p \leq 3$ *and* q is even.

Next we note that the differentials on the E^k-page of the spectral sequence have bidegree $(-k, k+1)$, i.e. are of the form $d^k_{p,q} : E^k_{p,q} \to E^k_{p-k,q+k-1}$. When $k = 2$, alternating rows on the E^2-page are zero (see **Fact 1**), which implies that $E^3_{p,q} = E^2_{p,q}$. When $k = 3$, the differentials $d^3_{p,q}$ shift horizontally by three units, and up by two units. So the only potentially non-zero differentials on the E^3-page are (up to vertical translation by the 2-periodicity in q) those of the form

$$d^3_{3,0} : E^2_{3,0} \cong E^3_{3,0} \to E^3_{0,2} \cong E^2_{0,0}.$$

Once we have $k \geq 4$, the differentials $d^k_{p,q}$ shift horizontally by $k \geq 4$ units. But **Fact 1** tells us that the only non-zero terms occur in the vertical strip $0 \leq p \leq 3$, which forces $E^4_{p,q} \cong E^5_{p,q} \cong \cdots$ for all p, q. In other words, the spectral sequence collapses at the E^4-stage. Since the E^2-terms are given by the homology of $\mathcal{C}$, this establishes:

LEMMA 3. *The groups $K_q\big(C^*_r(G)\big)$ can be computed from the E^4-page of the spectral sequence, and coincide with*

$$K_q\big(C^*_r(G)\big) = \begin{cases} H_1(\mathcal{C}) \oplus \ker(d^3_{3,0}) & \text{if } q \text{ is odd,} \\ \operatorname{coker}(d^3_{3,0}) \oplus H_2(\mathcal{C}) & \text{if } q \text{ is even,} \end{cases}$$

where $d^3_{3,0} : H_3(\mathcal{C}) \to H_0(\mathcal{C})$ is the differential appearing on the E^3-page of the spectral sequence.

Since we are only interested in the rationalized equivariant K-homology, we can actually ignore the presence of any differentials: after tensoring with $\mathbb{Q}$ the Atiyah-Hirzebruch spectral sequence collapses at the E^2-page [**Lu1**, Remark 3.9]. Thus for q even,

$$K_q\big(C^*_r(G)\big) \otimes \mathbb{Q} \cong \big(E^2_{0,q} \otimes \mathbb{Q}\big) \oplus \big(E^2_{2,q-2} \otimes \mathbb{Q}\big) \cong (H_0(\mathcal{C}) \otimes \mathbb{Q}) \oplus (H_2(\mathcal{C}) \otimes \mathbb{Q}),$$

and for q odd,

$$K_q\big(C^*_r(G)\big) \otimes \mathbb{Q} \cong \big(E^2_{1,q-1} \otimes \mathbb{Q}\big) \oplus \big(E^2_{3,q-3} \otimes \mathbb{Q}\big) \cong (H_1(\mathcal{C}) \otimes \mathbb{Q}) \oplus (H_3(\mathcal{C}) \otimes \mathbb{Q}).$$

LEMMA 4. *The ranks of the groups $K_q\big(C^*_r(G)\big) \otimes \mathbb{Q}$ are given by*

$$\operatorname{rank}\Big(K_q\big(C^*_r(G)\big) \otimes \mathbb{Q}\Big) = \begin{cases} \operatorname{rank}\big(H_1(\mathcal{C}) \otimes \mathbb{Q}\big) + \operatorname{rank}\big(H_3(\mathcal{C}) \otimes \mathbb{Q}\big) & \text{if } q \text{ is odd,} \\ \operatorname{rank}\big(H_0(\mathcal{C}) \otimes \mathbb{Q}\big) + \operatorname{rank}\big(H_2(\mathcal{C}) \otimes \mathbb{Q}\big) & \text{if } q \text{ is even.} \end{cases}$$

Remark: Alternatively this result follows directly from the equivariant Chern character being a rational isomorphism [**MV**, Thm. 6.1].

In the next four sections, we explain how to algorithmically compute the ranks of the four groups appearing in Lemma 4.

3.2. 1-skeleton of X/G and the group $H_0(\mathcal{C})$. For the group $H_0(\mathcal{C})$, we make use of the result from [**MV**, Theorem 3.19]. For the convenience of the reader, we restate the theorem:

THEOREM 5. *For G an arbitrary group, we have*

$$H_0(\mathcal{C}) \otimes \mathbb{Q} \cong \mathbb{Q}^{cf(G)},$$

where $cf(G)$ denotes the number of conjugacy classes of elements of finite order in the group G.

This reduces the computation of the rank of $H_0(\mathcal{C})\otimes\mathbb{Q}$ to finding some algorithm for computing the number $cf(G)$. We now explain how one can compute the integer $cf(G)$ in terms of the 1-skeleton of the space X/G.

For each cell σ in $\underline{B}G$, we fix a reference cell $\tilde{\sigma} \in \underline{E}G$, having the property that $\tilde{\sigma}$ maps to σ under the quotient map $p : \underline{E}G \to \underline{B}G$. Associated to each cell σ in $\underline{B}G$, we have a finite subgroup $G_{\tilde{\sigma}} \leq G$, which is just the stabilizer of the fixed pre-image $\tilde{\sigma} \in \underline{E}G$. Since the stabilizers of two distinct lifts $\tilde{\sigma}$, $\tilde{\sigma}'$ of the cell σ are conjugate subgroups inside G, we note that the *conjugacy class* of the finite subgroup $G_{\tilde{\sigma}}$ is independent of the choice of lift $\tilde{\sigma}$, and depends solely on the cell $\sigma \in \underline{B}G$. Now given a cell σ in $\underline{B}G$ with a boundary cell τ, we have associated lifts $\tilde{\sigma}$, $\tilde{\tau}$. Of course, the lift $\tilde{\tau}$ might not lie in the boundary of $\tilde{\sigma}$, but there exists some other lift $\tilde{\tau}'$ of τ which *does* lie in the boundary of $\tilde{\sigma}$. Clearly, we have an inclusion $G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}'}$. Fix an element $g_{\sigma,\tau} \in G$ with the property that $g_{\sigma,\tau}$ maps the lift $\tilde{\tau}'$ to the lift $\tilde{\tau}$. This gives us a map $\phi^{\tau}_{\sigma} : G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}}$, obtained by composing the inclusion $G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}'}$ with the isomorphism $G_{\tilde{\tau}'} \to G_{\tilde{\tau}}$ given by conjugation by $g_{\sigma,\tau}$. Now the map ϕ^{τ}_{σ} isn't well-defined, as there are different possible choices for the element $g_{\sigma,\tau}$. However, if $g'_{\sigma,\tau}$ represents a different choice of element, then since both elements $g_{\sigma,\tau}, g'_{\sigma,\tau}$ map $\tilde{\tau}'$ to $\tilde{\tau}$, we see that the product $\big(g'_{\sigma,\tau}\big)\big(g_{\sigma,\tau}\big)^{-1}$ maps $\tilde{\tau}$ to itself, and hence we obtain the equality $g'_{\sigma,\tau} = h \cdot g_{\sigma,\tau}$, where $h \in G_{\tilde{\tau}}$. This implies that the map ϕ^{τ}_{σ} is well-defined, up to post-composition by an inner automorphism of $G_{\tilde{\tau}}$.

Consider the set $F(G)$ consisting of the disjoint union of the finite groups $G_{\tilde{v}}$ where v ranges over vertices in the 0-skeleton $(\underline{B}G)^{(0)}$ of $\underline{B}G$. Form the smallest equivalence relation $\sim$ on $F(G)$ with the property that:

(i) for each vertex $v \in (\underline{B}G)^{(0)}$, and elements $g, h \in G_{\tilde{v}}$ which are conjugate within $G_{\tilde{v}}$, we have $g \sim h$, and
(ii) for each edge $e \in (\underline{B}G)^{(1)}$ joining vertices $v, w \in (\underline{B}G)^{(0)}$, and element $g \in G_{\tilde{e}}$, we have $\phi^{v}_{e}(g) \sim \phi^{w}_{e}(g)$.

Note that, although the maps ϕ^{τ}_{σ} are not well-defined, the equivalence relation given above *is* well-defined. Indeed, for any given edge $e \in (\underline{B}G)^{(1)}$, the maps ϕ^{v}_{e}, ϕ^{w}_{e} are only well-defined up to inner automorphisms of $G_{\tilde{v}}, G_{\tilde{w}}$. In view of property (i), the resulting property (ii) is independent of the choice of representatives ϕ^{v}_{e}, ϕ^{w}_{e}.

For a finitely generated group, we let $eq(G)$ denote the number of $\sim$ equivalence classes on the corresponding set $F(G)$. We can now establish:

LEMMA 6. *For G an arbitrary finitely generated group, we have $cf(G) = eq(G)$.*

PROOF. Let us write $g \approx h$ if the elements g, h are conjugate in G. As each element in $F(G)$ is also an element in G, we now have the two equivalence relations $\sim, \approx$ on the set $F(G)$. It is immediate from the definition that $g \sim h$ implies $g \approx h$.

Next, we argue that, for elements $g, h \in F(G)$, $g \approx h$ implies $g \sim h$. To see this, assume that $k \in G$ is a conjugating element, so $g = khk^{-1}$. For the action on $\underline{E}G$, we know that g, h fix vertices $\tilde{v}, \tilde{w}$ (respectively) in the 0-skeleton $(\underline{E}G)^{(0)}$, which project down to vertices $v, w \in (\underline{B}G)^{(0)}$ (respectively). Since $g = khk^{-1}$, we also have that g fixes the vertex $k \cdot \tilde{w}$. The g fixed set $\underline{E}G^{g}$ is contractible, so we can find a path joining $\tilde{v}$ to $k \cdot \tilde{w}$ inside the subcomplex $\underline{E}G^{g}$. Within this subcomplex, we can push any path into the 1-skeleton, giving us a sequence of consecutive edges within the graph $\big(\underline{E}G^{g}\big)^{(1)} \subseteq (\underline{E}G)^{(1)}$ joining $\tilde{v}$ to $k \cdot \tilde{w}$. This projects down to a path in $(\underline{B}G)^{(1)}$ joining the vertex v to the vertex w (as $k \cdot \tilde{w}$ and $\tilde{w}$ lie in the same G-orbit, they have the same projection). Using property (ii), the projected path gives a sequence of elements $g = g_0 \sim g_1 \sim \cdots \sim g_k = h$, where each pair g_i, g_{i+1} are in the groups associated to consecutive vertices in the path.

So we now have that the two equivalence relations $\sim$ and $\approx$ coincide on the set $F(G)$, and in particular, have the same number of equivalence classes. Of course, the number of $\sim$ equivalence classes is precisely the number $eq(G)$. On the other hand, any element of finite order g in G must have non-trivial fixed set in $\underline{E}G$. Since the action is cellular, this forces the existence of a fixed vertex $\bar{v} \in (\underline{E}G)^{(0)}$ (which might not be unique). The vertex $\bar{v}$ has an image vertex $v \in (\underline{B}G)^{(0)}$ under the quotient map, and hence $g \approx \tilde{g}$ for some element $\tilde{g}$ in the set $F(G)$, corresponding to the subgroup $G_{\tilde{v}}$. This implies that the number of $\approx$ equivalence classes in $F(G)$ is equal to $cf(G)$, concluding the proof. □

Remark: The procedure we described in this section works for any model for $\underline{B}G$, and would compute the β_0 of the corresponding chain complex. On the other hand, if one has a model for $\underline{E}G$ with the property that the quotient $\underline{B}G$ has few vertices and edges, then it is fairly straightforward to calculate the number $eq(G)$ from the 1-skeleton of $\underline{B}G$. For the groups we are considering, we can use the model space X. The 1-skeleton of $\underline{B}G$ is then a quotient of the 1-skeleton of the polyhedron P. Along with Lemma 6, this allows us to easily compute the rank of $H_0(\mathcal{C}) \otimes \mathbb{Q}$ for the groups within our class.

3.3. Topology of X/G and the group $H_3(\mathcal{C})$. Our next step is to understand the rank of the group $H_3(\mathcal{C}) \otimes \mathbb{Q}$; this requires an understanding of the differentials appearing in the chain complex $\mathcal{C}$. In X, if we have a k-cell σ contained in the closure of a $(k+1)$-cell τ, then we have a natural inclusion of stabilizers $G_\tau \hookrightarrow G_\sigma$. Applying the functor $K_q\big(C_r^*(-)\big)$, where q is even, we get an induced morphism from the complex representation ring of G_τ to the complex representation ring of G_σ. Concretely, the image of a complex representation ρ of G_τ under this morphism is the induced complex representation $\rho\uparrow := \mathrm{Ind}_{G_\tau}^{G_\sigma}\rho$ in G_σ, with multiplicity given (as usual) by the degree of the attaching map from the boundary sphere $S^k = \partial\tau$ to the sphere $S^k = \sigma/\partial\sigma$. Note that conjugate representations induce up to the same representation.

In the chain complex, the individual terms are indexed by *orbits* of cells in X, rather than individual cells. To see what the chain map does, pick an orbit of $(k+1)$-cells, and fix an oriented representative τ. Then for each orbit of a k-cell, one can look at the k-cells in that oriented orbit that are incident to τ, call them $\sigma_1, \ldots, \sigma_r$. The stabilizer of each of the σ_i is a copy of the same group G_σ (where the identification between these groups is well-defined up to inner automorphisms). For each of these σ_i, the discussion in the previous paragraph allows us to obtain a map on complex representation rings. Finally, one identifies the groups G_{σ_i} with the group G_σ, and take the sum of the maps on the complex representation rings. This completes the description of the chain maps in the complex $\mathcal{C}$.

Consider a representative σ for the single 3-cell orbit in the G-CW-complex X (we can identify σ with the interior of the polyhedron P). The stabilizer of σ must be trivial (as any element stabilizing σ must stabilize all of X). We conclude that $\mathcal{C}_3 = \bigoplus_{\sigma \in (X/G)^{(3)}} K_q\big(C_r^*(G_\sigma)\big) \cong \mathbb{Z}$, and the generator for this group is given by the trivial representation of the trivial group. But inducing up the trivial representation of the trivial group always gives the left regular representation, which is just the sum of all irreducible representations. This tells us that, for each 2-cell in the boundary of σ, the corresponding map on the K-group is non-trivial.

Now when looking at the chain complex, the target of the differential is indexed by *orbits* of 2-cells, rather than individual 2-cells. Each 2-cell orbit has either one or two representatives lying in the boundary of σ. Whether there is one or two can be decided as follows: look at the G-translate σ' of σ which is adjacent to σ across the given boundary 2-cell τ. Since X is a *manifold* model for $\underline{E}G$, there is a unique such σ'. As the stabilizer of the 3-cell is trivial, there is a unique element $g \in G$ which takes σ to σ'. Let τ' denote the pre-image $g^{-1}(\tau)$, a 2-cell in the boundary of σ. Clearly g identifies together the cells τ, τ' in the quotient space X/G.

If $\tau = \tau'$, then the cell τ descends to a boundary cell in quotient space X/G, and the stabilizer of τ is isomorphic to $\mathbb{Z}_2$ (with non-trivial element given by g). On the other hand, if $\tau \neq \tau'$, then τ descends to an interior cell in the quotient space X/G, with trivial stabilizer.

Now if the 3-cell σ has a boundary 2-cell τ whose stabilizer is $\mathbb{Z}_2$, then the orbit of τ intersects the boundary of σ in precisely τ. Looking at the coordinate corresponding to the orbit of τ, we see that in this case the map $\mathbb{Z} \longrightarrow \bigoplus_{f \in (X/G)^{(2)}} K_q\big(C_r^*(G_f)\big)$ in the chain complex is an *injection*, and hence that $E^2_{3,q} = H_3(\mathcal{C}) = 0$ for all even q.

The other possibility is that *all* boundary 2-cells are pairwise identified, in which case the quotient space X/G is (topologically) a closed manifold. With respect to the induced orientation on the boundary of σ, if any boundary 2-cell τ is identified by an orientation *preserving* pairing to τ', then the quotient space X/G is a non-orientable manifold. Focusing on the coordinate corresponding to the orbit of τ, we again see that the map $\mathbb{Z} \longrightarrow \bigoplus_{f \in (X/G)^{(2)}} K_q\big(C_r^*(G_f)\big)$ in the chain complex is injective (the generator of $\mathbb{Z}$ maps to ± 2 in the τ-coordinate). So in this case we again conclude that $E^2_{3,q} = H_3(\mathcal{C}) = 0$ for all even q.

Finally, we have the case where all pairs of boundary 2-cells are identified together using orientation reversing pairings. Then the quotient space X/G is (topologically) a closed *orientable* manifold. In this case, the corresponding map $\mathbb{Z} \longrightarrow \bigoplus_{f \in X^{(2)}} K_q\big(C_r^*(G_f)\big)$ in the chain complex is just the zero map (the generator of $\mathbb{Z}$ maps to 0 in each τ-coordinate, due to the two occurrences with opposite orientations). We summarize our discussion in the following:

LEMMA 7. *For our groups G, the third homology group $H_3(\mathcal{C})$ is either (i) isomorphic to $\mathbb{Z}$, if the quotient space X/G is topologically a closed orientable manifold, or (ii) trivial in all remaining cases.*

Remark: In [**MV**, Lemma 3.21], it is shown that the comparison map from $H_i(\mathcal{C})$ to the ordinary homology of the quotient space $H_i(\underline{B}G; \mathbb{Z})$ is an isomorphism in all degrees $i > \dim(\underline{E}G^{\text{sing}})+1$, and injective in degree $i = \dim(\underline{E}G^{\text{sing}})+1$. Note that most of our Lemma 7 can also be deduced from this result. Indeed, our discussion shows that, in case (i), the singular set is 1-dimensional (i.e. all cells of dimension ≥ 2 have trivial stabilizer), and hence $H_3(\mathcal{C}) \cong H_3(X/G) \cong \mathbb{Z}$. If X/G is non-orientable, then [**MV**, Lemma 3.21] gives that $H_3(\mathcal{C})$ injects into $H_3(X/G) \cong \mathbb{Z}_2$, so our Lemma provides a bit more information. In the case where X/G has boundary, [**MV**, Lemma 3.21] implies that $H_3(\mathcal{C})$ injects into $H_3(X/G) \cong 0$, so again recovers our result. We chose to retain our original proof of Lemma 7, as a very similar argument will be subsequently used to calculate $H_2(\mathcal{C})$ (which does not follow from [**MV**, Lemma 3.21]).

3.4. 2-skeleton of X/G and the rank of $H_2(\mathcal{C})$. Now we turn our attention to the group $H_2(\mathcal{C})$. In order to describe this homology group, we will continue the analysis initiated in the previous section. Recall that we have an explicit (combinatorial) polyhedron P which serves as a fundamental domain for the G-action. We can view the quotient space X/G as obtained from the polyhedron P by identifying together certain faces of P. The CW-structure on X/G is induced from the natural (combinatorial) CW-structure on the polyhedron P. The quotient space X/G inherits the structure of a 3-dimensional orbifold. Note that, if we forget the orbifold structure and just think about the underlying topological space, then X/G is a compact manifold, with possibly non-empty boundary.

There is a close relationship between the *isotropy* of the cells in X/G, thought of as a 3-orbifold, and the *topology* of X/G, viewed as a topological manifold. Indeed, as was discussed in the previous Section 3.3, the stabilizer of any face σ of the polyhedron P is either (i) trivial, or (ii) is isomorphic to $\mathbb{Z}_2$. In the first case, there is an element in G which identifies the face σ with some other face of P. So at the level of the quotient space X/G, σ maps to a 2-cell which lies in the *interior* of the closed manifold X/G. In the second case, there are no other faces of the polyhedron P that lie in the G-orbit of σ, and hence σ maps to a boundary 2-cell of X/G. We summarize this analysis in the following

Fact 2: For any 2-cell σ in X/G, we have that:

i) σ lies in the boundary of X/G if and only if σ has isotropy $\mathbb{Z}_2$, and
ii) σ lies in the interior of X/G if and only if σ has trivial isotropy.

A similar analysis applies to 1-cells. Indeed, the stabilizer of any edge in the polyhedron P must either be (i) a finite cyclic group, or (ii) a finite dihedral group. But case (ii) can only occur if there is some orientation reversing isometry through one of the faces containing the edge. This would force the edge to lie in the boundary of the corresponding face, with the stabilizer of the face being $\mathbb{Z}_2$. In view of **Fact 2**, such an edge would have to lie in the boundary of X/G. Conversely, if one has an edge in the boundary of X/G, then it has two adjacent faces (which might actually coincide) in the boundary of X/G, each with stabilizer $\mathbb{Z}_2$, given by a reflection in the face. In most cases, these two reflections will determine a dihedral stabilizer for e; the exception occurs if the two incident faces have stabilizers which coincide in G. In that case, the stabilizer of e will also be a $\mathbb{Z}_2$, and will coincide with the stabilizers of the two incident faces. We summarize this discussion as our:

Fact 3: For any 1-cell e in X/G, we have that:

i) e lies in the interior of X/G if and only if e has isotropy a cyclic group, acting by rotations around the edge,
ii) if e has isotropy a dihedral group, then e lies in the boundary of X/G,
iii) the remaining edges in the boundary of X/G have stabilizer $\mathbb{Z}_2$, which coincides with the $\mathbb{Z}_2$ stabilizer of the incident boundary faces.

With these observations in hand, we are now ready to calculate $H_2(\mathcal{C}) \otimes \mathbb{Q}$. In order to understand this group, we need to understand the kernel of the morphism

$$\Phi : \bigoplus_{\sigma \in (X/G)^{(2)}} K_0\big(C_r^*(G_\sigma)\big) \to \bigoplus_{e \in (X/G)^{(1)}} K_0\big(C_r^*(G_e)\big).$$

Indeed, the group $H_2(\mathcal{C})$ is isomorphic to the quotient of $\ker(\Phi)$ by a homomorphic image of $K_0\big(C_r^*(G_\tau)\big) \cong \mathbb{Z}$, where τ is a representative for the unique 3-cell orbit. As such, we see that the rank of $H_2(\mathcal{C}) \otimes \mathbb{Q}$ either coincides with the rank of $\ker(\Phi)$, or is one less than the rank of $\ker(\Phi)$.

Our approach to analyzing $\ker(\Phi)$ is to split up this group into smaller pieces, which are more amenable to a geometric analysis. Let us introduce the notation Φ_e, where e is an edge, for the composition of the map Φ with the projection onto the summand $K_0\big(C_r^*(G_e)\big)$. The next Lemma analyzes the behavior of the map Φ in the vicinity of a boundary edge with stabilizer a dihedral group.

LEMMA 8. *Let e be a boundary edge, with stabilizer a dihedral group D_n. Then we have:*

(i) if σ is an incident interior face, then

$$\Phi_e\Big(K_0\big(C_r^*(G_\sigma)\big)\Big) \subseteq \mathbb{Z} \cdot \langle 1, 1, \ldots, 1, 1\rangle \leq K_0\big(C_r^*(D_n)\big),$$

(ii) if σ_1, σ_2 are the incident boundary faces, then

$$\Phi_e\Big(K_0\big(C_r^*(G_{\sigma_1})\big) \oplus K_0\big(C_r^*(G_{\sigma_2})\big)\Big) \cap \mathbb{Z} \cdot \langle 1, 1, \ldots, 1, 1\rangle = \langle 0, \ldots, 0\rangle.$$

Note that Lemma 8 tells us that, from the viewpoint of finding elements in $\ker(\Phi)$, boundary faces and interior faces that come together along an edge with dihedral stabilizer have *no interactions.*

PROOF. There are precisely *two* boundary faces which are incident to e, and some indeterminate number of interior faces which are incident to e. From **Fact 2**, the boundary faces each have corresponding $G_\sigma \cong \mathbb{Z}_2$, while the interior faces each have $G_\sigma \cong 1$. For the boundary faces, we have

$$K_0\big(C_r^*(G_\sigma)\big) = K_0\big(C_r^*(\mathbb{Z}_2)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$$

with generators given by the trivial representation and the sign representation of the group $\mathbb{Z}_2$. The interior faces have $K_0\big(C_r^*(G_\sigma)\big) \cong \mathbb{Z}$, generated by the trivial representation of the trivial group.

For each incidence of σ on e, the effect of Φ_e on the generator is obtained by inducing up representations. But the trivial representation of the trivial group always induces up to the left regular representation on the ambient group. The latter is the sum of all irreducible representations, hence corresponds to the element $\langle 1, \ldots, 1\rangle \leq K_0\big(C_r^*(\mathbb{Z}_n)\big)$. This tells us that, for each internal face, the image of Φ_e lies in the subgroup $\mathbb{Z} \cdot \langle 1, 1, \ldots, 1, 1\rangle$, establishing (i).

On the other hand, an easy calculation (see Appendix A) shows that, if $\sigma_1, \sigma_2 \in (X/G)^{(2)}$ are the two boundary faces incident to e, then in the e-coordinate we have

$$\Phi_e\Big(K_0\big(C_r^*(G_{\sigma_1})\big) \oplus K_0\big(C_r^*(G_{\sigma_2})\big)\Big) \cap \mathbb{Z} \cdot \langle 1, 1, \ldots, 1, 1\rangle = \langle 0, \ldots, 0\rangle,$$

which is the statement of (ii). □

To analyze $\ker(\Phi)$, we need to introduce some auxiliary spaces. Recall that X/G is topologically a closed 3-manifold, possibly with boundary. We introduce the following terminology for boundary components:

- a boundary component is *dihedral* if it has no edges with stabilizer $\mathbb{Z}_2$ (i.e. all its edges have stabilizers which are dihedral groups),

- a boundary component is *non-dihedral* if it is not dihedral (i.e. it contains at least one edge with stabilizer $\mathbb{Z}_2$),
- a dihedral boundary component is *odd* if it has no edges with stabilizer of the form D_{2k} (i.e. all its edges have stabilizers of the form D_{2k+1}), and
- a dihedral boundary component is *non-odd* if it contains an edge e with stabilizer of the form D_{2k} (i.e. an edge whose stabilizer has order a multiple of 4).

Let s denote the number of orientable non-odd dihedral boundary components, and let t denote the number of orientable odd dihedral boundary components. Note that it is straightforward to calculate the integers s, t from the polyhedral fundamental domain P for the G-action on X.

Next, form the 2-complex Y by taking the union of the closure of all interior faces of X/G, along with all the non-dihedral boundary components. We denote by $\partial Y \subset Y$ the subcomplex consisting of all non-dihedral boundary components. By construction, ∂Y consists precisely of the subcomplex generated by the 2-cells in $Y \cap \partial(X/G)$, so the choice of notation should cause no confusion. Let Z denote the union of all dihedral boundary components of X/G.

By construction, every 2-cell in X/G appears either in Y or in Z, but not in both. This gives rise to a decomposition of the indexing set $(X/G)^{(2)} = Y^{(2)} \coprod Z^{(2)}$, which in turn yields a splitting:

$$\bigoplus_{\sigma\in(X/G)^{(2)}} K_0\big(C_r^*(G_\sigma)\big) = \Big[\bigoplus_{\sigma\in Y^{(2)}} K_0\big(C_r^*(G_\sigma)\big)\Big] \oplus \Big[\bigoplus_{\sigma\in Z^{(2)}} K_0\big(C_r^*(G_\sigma)\big)\Big].$$

Let us denote by Φ_Y and Φ_Z the restrictions of Φ to the first and second summands described above. We then have the following:

LEMMA 9. *There is a splitting* $\ker(\Phi) = \ker(\Phi_Y) \oplus \ker(\Phi_Z)$.

PROOF. We clearly have the inclusion $\ker(\Phi_1) \oplus \ker(\Phi_2) \subseteq \ker(\Phi)$, so let us focus on the opposite containment. If we have some arbitrary element $v \in \ker(\Phi)$, we can decompose $v = v_Y + v_Z$, where we have $v_Y \in \bigoplus_{\sigma\in Y^{(2)}} K_0\big(C_r^*(G_\sigma)\big)$, and $v_Z \in \bigoplus_{\sigma\in Z^{(2)}} K_0\big(C_r^*(G_\sigma)\big)$. Let us first argue that $v_Z \in \ker(\Phi_Z)$, i.e. that $\Phi(v_Z) = 0$. This is of course equivalent to showing that for every edge e, we have $\Phi_e(v_Z) = 0$.

Since v_Z is supported on 2-cells lying in Z, it is clear that for any edge $e \not\subset Z$, we have $\Phi_e(v_Z) = 0$. For edges $e \subset Z$, we have:

$$0 = \Phi_e(v) = \Phi_e(v_Y + v_Z) = \Phi_e(v_Y) + \Phi_e(v_Z).$$

This tells us that $\Phi_e(v_Z) = \Phi_e(-v_Y)$ lies in the intersection

$$\Phi_e\Big(\bigoplus_{\sigma\in Y^{(2)}} K_0\big(C_r^*(G_\sigma)\big)\Big) \cap \Phi_e\Big(\bigoplus_{\sigma\in Z^{(2)}} K_0\big(C_r^*(G_\sigma)\big)\Big). \tag{2}$$

But $Y^{(2)}$ contains all the *interior* faces incident to e, while $Z^{(2)}$ contains all *boundary* faces incident to e. Since $e \subset Z$, and Z is the union of all dihedral boundary components of X/G, we have that the stabilizer G_e must be dihedral. Applying Lemma 8, we see that the intersection in equation (2) consists of just the zero vector, and hence $\Phi_e(v_Z) = 0$.

Since we have shown that $\Phi_e(v_Z) = 0$ holds for all edges e, we obtain that $v_Z \in \ker(\Phi_Z)$, as desired. Finally, we have that

$$\Phi(v_Y) = \Phi(v - v_Z) = \Phi(v) - \Phi(v_Z) = 0$$

as both v, v_Z are in the kernel of Φ. We conclude that $v_Y \in \ker(\Phi_Y)$, concluding the proof of the Lemma. □

We now proceed to analyze each of $\ker(\Phi_Y), \ker(\Phi_Z)$ separately. We start with:

LEMMA 10. *The group* $\ker(\Phi_Z)$ *is free abelian, of rank equal to* $s + 2t$.

Before establishing Lemma 10, recall that s, t counts the number of orientable dihedral boundary components of X/G which are non-odd and odd, respectively. From the definition of Z, we see that the number of connected components of the space Z is precisely $s + t$.

PROOF. It is obvious that $\ker(\Phi_Z)$ decomposes as a direct sum of the kernels of Φ restricted to the individual connected components of Z, which are precisely the dihedral boundary components of X/G. So we can argue one dihedral boundary component at a time. On a fixed dihedral boundary component, we have that each 2-cell contributes a $\mathbb{Z}\oplus\mathbb{Z}$ to the source of the map Φ, with canonical (ordered) basis given by the trivial representation and the sign representation on $\mathbb{Z}_2$. Fix a boundary edge e, and let σ_1, σ_2 be the two boundary faces incident to e. We assume that the two faces are equipped with compatible orientations, and let (a_i, b_i) be elements in the groups $K_0\big(C_r^*(G_{\sigma_i})\big) \cong \mathbb{Z} \oplus \mathbb{Z}$. Now assume that $\Phi_e\big((a_1, b_1 \mid a_2, b_2)\big) = 0$. Then an easy computation (see Appendix A) shows that:

a) if e has stabilizer of the form D_{2k+1}, then we must have $a_1 = a_2$ and $b_1 = b_2$,
b) if e has stabilizer of the form D_{2k}, then we must have $a_1 = a_2 = b_1 = b_2$

(and since Z consists of dihedral boundary components, there are no edges e in Z with stabilizer $\mathbb{Z}_2$). Note that reversing the orientation on one of the faces just changes the sign of the corresponding entries. We can now calculate the contribution of each boundary component to $\ker(\Phi_Z)$.

<u>Non-orientable components:</u> Any such boundary component contains an embedded Möbius band. Without loss of generality, we can assume that the sequence of faces $\sigma_1, \ldots, \sigma_r$ cyclically encountered by this Möbius band are all distinct. At the cost of flipping the orientations on σ_i, $2 \le i \le r$, we can assume that consecutive pairs are coherently oriented. Since we have a Möbius band, this forces the orientations of σ_1 and σ_r to be non-coherent along their common edge. So if we have an element lying in $\ker(\Phi_Z)$, the coefficients along the cyclic sequence of faces must satisfy (regardless of the edge stabilizers):

$$a_1 = a_2 = \ldots = a_k = -a_1$$

$$b_1 = b_2 = \ldots = b_k = -b_1$$

This forces $a_1 = b_1 = 0$. Regardless of the orientations and edge stabilizers, equations (a) and (b) imply that this propagates to force all coefficients to equal zero. We conclude that any element in $\ker(\Phi_Z)$ must have all zero coefficients in the 2-cells corresponding to any non-orientable boundary component.

<u>Orientable odd components:</u> Fix a coherent orientation of all the 2-cells in the boundary component. Then in view of equation (a) above, elements lying in $\ker(\Phi)$ must have all a_i-coordinates equal, and all b_i-coordinates equal (as one ranges over 2-cells within this fixed boundary component). This gives two degrees of freedom, and hence such a boundary component contributes a $\mathbb{Z}^2$ to $\ker(\Phi_Z)$.

Orientable non-odd components: Again, let us fix a coherent orientation of all the 2-cells in the boundary component. As in the odd component case, any element in $\ker(\Phi_Z)$ must have all a_i-coordinates equal, and all b_i-coordinates equal. However, the presence of a single edge with stabilizer of the form D_{2k} forces, for the two adjacent faces, to have corresponding a- and b-coordinates equal (see equation (b) above). This in turn propagates to yield that *all* the a- and b-coordinates must be equal. As such, we have one degree of freedom for elements in the kernel, and hence such a boundary component contributes a single $\mathbb{Z}$ to $\ker(\Phi_Z)$. This concludes the proof of Lemma 10. □

Next we focus on the group $\ker(\Phi_Y)$. We would like to relate $\ker(\Phi_Y)$ with the second homology of the space Y. Let $\mathcal{A}$ denote the cellular chain complex for the CW-complex Y, and let $d_Y : \mathcal{A}_2 \to \mathcal{A}_1$ denote the differentials in the cellular chain complex. Since Y is a 2-dimensional CW-complex, we have that $H_2(Y) = \ker(d_Y)$. Our next step is to establish:

Lemma 11. *There is a split surjection* $\phi : \ker(\Phi_Y) \to \ker(d_Y)$*, providing a direct sum decomposition* $\ker(\Phi_Y) \cong \ker(\phi) \oplus \ker(d_Y)$*.*

Proof. Let $\mathcal{D} \subset \mathcal{C}$ denote the subcomplex of our original chain complex determined by the subcollection of indices $Y^{(k)} \subset (X/G)^{(k)}$. By construction, the map Φ_Y we are interested in is the boundary operator $\Phi_Y : \mathcal{D}_2 \to \mathcal{D}_1$ appearing in the chain complex $\mathcal{D}$. We define the map

$$\hat{\phi} : \mathcal{D}_2 = \bigoplus_{\sigma \in Y^{(2)}} K_0\big(C_r^*(G_\sigma)\big) \to \bigoplus_{\sigma \in Y^{(2)}} \mathbb{Z} = \mathcal{A}_2$$

as the direct sum of maps $\hat{\phi}_\sigma : K_0\big(C_r^*(G_\sigma)\big) \to \mathbb{Z}$, where:

- if G_σ is trivial, then $\hat{\phi}_\sigma : \mathbb{Z} \to \mathbb{Z}$ takes the generator for $K_0\big(C_r^*(G_\sigma)\big) = \mathbb{Z}$ given by the trivial representation to the element $1 \in \mathbb{Z}$, and
- if $G_\sigma = \mathbb{Z}_2$, then $\hat{\phi}_\sigma : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is given by $\hat{\phi}_\sigma(\langle 1, 0\rangle) = 1, \hat{\phi}_\sigma(\langle 0, 1\rangle) = 0$, where, as usual, $\langle 1, 0\rangle, \langle 0, 1\rangle$ correspond to the trivial representation and the sign representation respectively.

For any element $z \in \ker(\Phi_Y)$, a computation shows that $(d_Y \circ \hat{\phi})(z) = 0$, and hence $\hat{\phi}$ restricts to a morphism $\phi : \ker(\Phi_Y) \to \ker(d_Y)$.

Next, we argue that the map $\phi : \ker(\Phi_Y) \to \ker(d_Y)$ is surjective. To see this, we construct a map $\bar{\phi} : \mathcal{A}_2 \to \mathcal{D}_2$ as a direct sum of maps $\bar{\phi}_\sigma : \mathbb{Z} \to K_0\big(C_r^*(G_\sigma)\big)$. In terms of our usual generating sets for the groups $K_0\big(C_r^*(G_\sigma)\big)$, the maps $\bar{\phi}_\sigma$ are given by:

- if G_σ is trivial, then $\bar{\phi}_\sigma : \mathbb{Z} \to \mathbb{Z}$ is defined by $\bar{\phi}_\sigma(1) = 1$, and
- if $G_\sigma = \mathbb{Z}_2$, then $\bar{\phi}_\sigma : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is defined by $\bar{\phi}_\sigma(1) = \langle 1, 1\rangle$.

We clearly have that $\hat{\phi} \circ \bar{\phi} : \mathcal{A}_2 \to \mathcal{A}_2$ is the identity, and an easy computation shows that if $z \in \ker(d_Y)$, then $\bar{\phi}(z) \in \ker(\Phi_Y)$. We conclude that the restriction $\phi : \ker(\Phi_Y) \to \ker(d_Y)$ is surjective, and that the restriction of $\bar{\phi}$ to $\ker(d_Y)$ provides a splitting of this surjection. Since the map ϕ is a split surjection, we see that $\ker(\Phi_Y) \cong \ker(d_Y) \oplus \ker(\phi)$, completing the proof of Lemma 11. □

So the last step is to identify $\ker(\phi)$. Recall that Y is a 2-complex which contains, as a subcomplex, the union of all boundary components of X/G which have an edge with stabilizer $\mathbb{Z}_2$. This subcomplex was denoted by $\partial Y \subset Y$. We can again call a connected component in ∂Y *odd* if all its edges have stabilizers of the form D_{2k+1}, and *non-odd* otherwise (i.e. some edge has stabilizer of the form D_{2k}). Let t' denote the number of orientable, odd connected components in ∂Y. Then we have:

LEMMA 12. *The group* $\ker(\phi)$ *is free abelian, of rank* $= t'$.

PROOF. From the definition of ϕ, it is easy to see what form an element in $\ker(\phi)$ must have: in terms of the splitting $\mathcal{D}_2 = \bigoplus_{\sigma\in Y^{(2)}} K_0\big(C_r^*(G_\sigma)\big)$, the element can only have non-zero terms in the coordinates corresponding to 2-cells in ∂Y. Moreover, in the coordinates $\sigma \in (\partial Y)^{(2)}$, the entries in the corresponding $K_0\big(C_r^*(G_\sigma)\big) \cong \mathbb{Z}\oplus\mathbb{Z}$ must lie in the subgroup $\mathbb{Z}\cdot\langle 0,1\rangle$. Finally, the fact that the elements we are considering lie in $\ker(\Phi_Y)$ means that, at each edge $e\in(\partial Y)^{(1)}$, with incident edges σ_1,σ_2, we must have that the corresponding coefficients $\langle 0,b_1\rangle\in K_0\big(C_r^*(G_{\sigma_1})\big)$ and $\langle 0,b_2\rangle\in K_0\big(C_r^*(G_{\sigma_2})\big)$ sum up to zero, i.e. that $b_1+b_2=0$. These properties almost characterize elements in $\ker(\phi)$. Clearly, we can again analyze the situation one connected component of ∂Y at a time. As in the argument for Lemma 10, there are cases to consider:

<u>Non-odd component:</u> In the case where an element $z\in\ker(\phi)$ is supported entirely on a non-odd boundary component, there is one additional constraint. For the two faces σ_1,σ_2 incident to the edge with stabilizer D_{2k}, the fact that $z\in\ker(\Phi)$ forces the corresponding coefficients to satisfy $b_1=b_2=a_1=a_2$ (see equation (b) in the proof of Lemma 10). Since $z\in\ker(\phi)$, we also have $a_1=a_2=0$. This implies that the coefficients $b_1=b_2$ must also vanish. But then all the b_i coefficients must vanish. We conclude that any element $z\in\ker(\phi)$ must have zero coefficients on all 2-cells contained in a non-odd component.

<u>Odd component:</u> In the case where an element $z\in\ker(\phi)$ is supported entirely on an odd boundary component, the conditions discussed above actually do characterize an element in $\ker(\phi)$. This is due to the fact that, at every edge, the b_i components are actually independent of the a_i components (see equation (a) in the proof of Lemma 10). But the description given above is just stating that the b_i form the coefficients for an (ordinary) 2-cycle in the boundary component. Such a 2-cycle can only exist if the boundary component is orientable, in which case there is a 1-dimensional family of such 2-cycles. We conclude that the orientable, odd components each contribute a $\mathbb{Z}$ to $\ker(\phi)$, while the non-orientable odd components make no contributions.

Since t' is the number of orientable, odd components in ∂Y, the Lemma follows. $\square$

We now have all the required ingredients to establish:

THEOREM 13. *The group* $\ker(\Phi)$ *is free abelian of rank* $s+t'+2t+\beta_2(Y)$.

PROOF. Lemma 9 provides us with a splitting $\ker(\Phi)=\ker(\Phi_Y)\oplus\ker(\Phi_Z)$. Lemma 10 shows that $\ker(\Phi_Z)$ is free abelian of rank $= s+2t$. Lemma 11 yields the splitting $\ker(\Phi_Y)\cong\ker(\phi)\oplus\ker(d_Y)$. Finally, Lemma 12 tells us that $\ker(\phi)$ is free abelian of rank $= t'$, while the fact that Y is a 2-complex tells us that $\ker(d_Y)$ is free abelian of rank $=\beta_2(Y)$. $\square$

As a consequence, we obtain the desired formula for $\beta_2(\mathcal{C})$.

Corollary 14. *For our groups G, we have that the rank of $H_2(\mathcal{C}) \otimes \mathbb{Q}$ is either:*

- $\beta_2(Y)$ *if X/G is a closed, oriented, 3-manifold, or*
- $s + t' + 2t + \beta_2(Y) - 1$ *otherwise.*

Remark: Corollary 14 gives us an algorithmically efficient method for computing $\beta_2(\mathcal{C})$, as it merely requires counting certain boundary components of X/G (to determine the integers s, t, t'), along with the calculation of the second Betti number of an explicit 2-complex (for the $\beta_2(Y)$ term).

3.5. Euler characteristic and the rank of $H_1(\mathcal{C})$. Using the procedure described in the previous section, we will now assume that the ranks $\beta_0(\mathcal{C}), \beta_2(\mathcal{C})$, and $\beta_3(\mathcal{C})$ have already been calculated. In order to compute the rank of $H_1(\mathcal{C})\otimes\mathbb{Q}$, we recall that any chain complex has an associated *Euler characteristic*. The latter is defined to be the alternating sum of the ranks of the groups appearing in the chain complex. It is an elementary exercise to verify that the Euler characteristic also coincides with the alternating sum of the ranks of the homology groups of the chain complex.

In our specific case, the Euler characteristic $\chi(\mathcal{C})$ of the chain complex $\mathcal{C}$ can easily be calculated from the various groups G_σ, where σ ranges over the cells in $\underline{B}G$. Each cell σ in $\underline{B}G$ contributes $(-1)^{\dim \sigma} c(G_\sigma)$, where $c(G_\sigma)$ is the number of conjugacy classes in the stabilizer G_σ of the cell. Since the homology groups $H_i(\mathcal{C})$ vanish when $i \neq 0, 1, 2, 3$, we also have the alternate formula

$$\chi(\mathcal{C}) = \beta_0(\mathcal{C}) - \beta_1(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C})$$

This allows us to solve for the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$, yielding

Lemma 15. *For our groups G, we have that the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$ coincides with $\beta_1(\mathcal{C}) = \beta_0(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C}) - \chi(\mathcal{C})$.*

4. Some examples

We illustrate our algorithm by computing the rational topological K-theory of several groups. The first two examples are classes of groups for which the topological K-theory has already been computed. Since our algorithm does indeed recover (rationally) the same results, these examples serve as a check on our method. The last three examples provide some new computations.

The first example considers the particular case where G is additionally assumed to be torsion-free. As a concrete special case, we deal with any semi-direct product of $\mathbb{Z}^2$ with $\mathbb{Z}$ (the integral computation for these groups can be found in the recent thesis of Isely [**I**]). The second example considers a finite extension of the integral Heisenberg group by $\mathbb{Z}_4$. The *integral* topological K-theory (and algebraic K- and L- theory) for this group has already been computed by Lück [**Lu3**].

The third and fourth classes of examples are hyperbolic Coxeter groups that have previously been considered by Lafont, Ortiz, and Magurn in [**LOM**, Example 7], and [**LOM**, Example 8] respectively (where their lower algebraic K-theory was computed). The fifth example is an affine split crystallographic group, whose algebraic K-theory has been studied by Farley and Ortiz [**FO**].

4.1. Torsion-free examples. In the special case where G is *torsion-free*, our algorithm becomes particularly simple, as we now proceed to explain.

Let G be a torsion-free group with a cocompact, 3-manifold model X for the classifying space $\underline{E}G = EG$. Firstly, recall that $\beta_0(\mathcal{C}) = cf(G)$, where $cf(G)$ denotes the number of conjugacy classes of elements of finite order in G (our Lemma 6 provides a way of computing this integer from the 1-skeleton of X/G). Since G is torsion-free, we obtain that $\beta_0(\mathcal{C}) = 1$.

Next, we consider the orbit space $M := X/G$. Recall that any boundary component in the 3-manifold M gives 2-cells with stabilizer $\mathbb{Z}_2$. Since G is torsion-free, the orbit space M has no boundary, hence is a *closed* 3-manifold. Then Lemma 7 tells us that

$$\beta_3(\mathcal{C}) = \begin{cases} 1 & \text{if } M \text{ orientable,} \\ 0 & \text{if } M \text{ non-orientable.} \end{cases}$$

To compute $\beta_2(\mathcal{C})$ we apply Corollary 14. The 2-simplex Y is just the 2-skeleton of M and, as $\partial M = \emptyset$, we obtain that

$$\beta_2(\mathcal{C}) = \begin{cases} \beta_2(Y) & \text{if } M \text{ orientable,} \\ \beta_2(Y) - 1 & \text{if } M \text{ non-orientable.} \end{cases}$$

Note that the 2$^{\text{nd}}$ Betti number of $Y = M^{(2)}$ can be deduced from that of M, as follows. Since M is obtained from Y by attaching a single 3-cell, the Mayer-Vietoris exact sequence gives

$$0 \longrightarrow H_3(M) \hookrightarrow H_2(S^2) \xrightarrow{g} H_2(Y) \oplus H_2(\mathbb{D}^3) \twoheadrightarrow H_2(M) \longrightarrow 0$$

(Here $\mathbb{D}^3$ is the attaching 3-disk.) Recall that $H_2(S^2) \cong \mathbb{Z}$ and $H_2(\mathbb{D}^3) = 0$. Hence if M is orientable, $H_3(M) \cong \mathbb{Z}$, the image of the map g is then torsion and tensoring with $\mathbb{Q}$ gives $\beta_2(Y) = \beta_2(M)$. If M is non-orientable, $H_3(M) = 0$, the map g is injective and we have $\beta_2(Y) - 1 = \beta_2(M)$. Hence in all cases we actually obtain that $\beta_2(\mathcal{C}) = \beta_2(M)$.

To compute $\beta_1(\mathcal{C})$ we should find $\chi(\mathcal{C})$. Since G is torsion-free all the isotropy groups are trivial and thus $\chi(\mathcal{C}) = \chi(M)$. Since M is a closed 3-manifold, $\chi(M)$ and therefore $\chi(\mathcal{C})$ are zero. Finally, Lemma 15 gives

$$\beta_1(\mathcal{C}) = \beta_0(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C}) - \chi(\mathcal{C}) = \beta_2(M) - \beta_3(\mathcal{C}) + 1,$$

which simplifies to two cases:

$$\beta_1(\mathcal{C}) = \begin{cases} \beta_2(M) & \text{if } M \text{ is orientable,} \\ \beta_2(M) + 1 & \text{if } M \text{ is not orientable.} \end{cases}$$

Finally applying Lemma 4, we deduce the:

Corollary 16. *Let G be a torsion-free group, and X be a cocompact 3-manifold model for $\underline{E}G = EG$. Assume that the quotient 3-manifold $M = X/G$ is geometrizable (this is automatic, for instance, if M is orientable). Then we have that*

$$\operatorname{rank}\left(K_q(C_r^*(G)) \otimes \mathbb{Q}\right) = \beta_2(M) + 1$$

holds for all q.

Remark: The number above is the sum of the even-dimensional Betti numbers of M (which coincides with the sum of the odd-dimensional Betti numbers of M, by Poincaré duality) — compare this with the Remark after Lemma 4.

Remark: Note that for G torsion-free, the dimension of the singular part is -1 and hence Lemma 3.21 in [**MV**] gives $H_i(\mathcal{C}) \cong H_i(M)$ for $i > 0$ and an injection $H_0(\mathcal{C}) \hookrightarrow H_0(M)$. From this it follows that $\beta_i(\mathcal{C}) = \beta_i(M)$ for $i = 1, 2, 3$ and $\beta_0(\mathcal{C}) = \beta_0(M)$ since $1 \leq \beta_0(\mathcal{C}) \leq \beta_0(M) = 1$. This is shown above by direct application of our algorithm.

Semi-direct product of $\mathbb{Z}^2$ and $\mathbb{Z}$. For a concrete example of the torsion-free case, consider a semi-direct product $G_\alpha = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \mathrm{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$. The automorphism α can be realized (at the level of the fundamental group) by an affine self diffeomorphism of the 2-torus $T^2 = S^1 \times S^1$, $f\colon T^2 \to T^2$. The mapping torus M_f of the map f yields a closed 3-manifold which is aspherical and satisfies $\pi_1(M_f) \cong G_\alpha$. Hence it is a model of BG_α and its universal cover a model of EG_α. Since G_α is torsion-free (as it is the semi-direct product of torsion-free groups), these spaces are also models of $\underline{B}G_\alpha$ respectively $\underline{E}G_\alpha$. In particular, these examples fall under the purview of Corollary 16, telling us that $\mathrm{rank}\,(K_q(C_r^*(G_\alpha)) \otimes \mathbb{Q}) = \beta_2(M_f) + 1$. To complete the calculation, we just need to compute the 2nd Betti number of the 3-manifold M_f. This follows from a straightforward application of the Leray-Serre spectral sequence. We have included the details in Appendix B and here we only quote the result

$$\beta_2(M_f) = \begin{cases} 3 & \text{if } \alpha = \mathrm{Id}, \\ 2 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) = 2, \alpha \neq \mathrm{Id}, \\ 1 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) \neq 2, \\ 1 & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) = 0, \\ 0 & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) \neq 0. \end{cases}$$

Adding 1 we obtain

$$K_q(C_r^*(G_\alpha)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}^4 & \text{if } \alpha = \mathrm{Id}, \\ \mathbb{Q}^3 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) = 2, \alpha \neq \mathrm{Id}, \\ \mathbb{Q}^2 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) \neq 2, \\ \mathbb{Q}^2 & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) = 0, \\ \mathbb{Q} & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) \neq 0. \end{cases}$$

These results agree with the integral computations in Isely's thesis [**I**, pp. 5-7], giving us a first check on our method.

4.2. Nilmanifold example. In the previous section, we discussed examples where the group was torsion-free, and hence the quotient space was a closed 3-manifold. In this next example, we have a group *with* torsion, but with quotient space again a closed 3-manifold.

The real Heisenberg group $\mathrm{Hei}(\mathbb{R})$ is the Lie group of upper unitriangular, 3×3 matrices with real entries. It is naturally homeomorphic to $\mathbb{R}^3$. The integral Heisenberg group $\mathrm{Hei}(\mathbb{Z})$ is the discrete subgroup consisting of matrices whose entries are in $\mathbb{Z}$. There is an automorphism $\sigma \in \mathrm{Aut}\big(\mathrm{Hei}(\mathbb{R})\big)$ of order 4 given by:

$$\sigma : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -y & z-xy \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

This automorphism restricts to an automorphism of the discrete subgroup $\mathrm{Hei}(\mathbb{Z})$, allowing us to define the group $G := \mathrm{Hei}(\mathbb{Z}) \rtimes \mathbb{Z}_4$. An explicit presentation of the group G is given by

$$G := \left\langle a, b, c, t \;\middle|\; \begin{array}{l} [a,c] = [b,c] = 1, \;\; [a,b] = c, \;\; t^4 = 1 \\ tat^{-1} = b, \;\; tbt^{-1} = a^{-1}, \;\; tct^{-1} = c \end{array} \right\rangle$$

where as usual, $[x, y]$ denotes the commutator of the elements x, y. In the above presentation, we are identifying the generators a, b, c with the matrices in $\mathrm{Hei}(\mathbb{Z})$ given by

$$T_a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; T_c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These generate the normal subgroup $\mathrm{Hei}(\mathbb{Z}) \lhd G$, while the conjugation by the last generator t acts via the automorphism $\sigma \in \mathrm{Aut}\big(\mathrm{Hei}(\mathbb{Z})\big)$.

The action of $\mathrm{Hei}(\mathbb{Z})$ on $\mathrm{Hei}(\mathbb{R})$ given by left multiplication and the action of $\mathbb{Z}_4$ on $\mathrm{Hei}(\mathbb{R})$ given by the automorphism σ fit together to give an action of the group G on $\mathrm{Hei}(\mathbb{R})$. It is shown in [**Lu3**, Lemma 2.4] that this action on $\mathrm{Hei}(\mathbb{R})$ provides a cocompact model for $\underline{E}G$, with orbit space $G\backslash\underline{E}G$ homeomorphic to S^3. In order to apply our algorithm, we need to identify a G-CW-structure on $\mathrm{Hei}(\mathbb{R})$. Let us identify $\mathbb{R}^3$ with $\mathrm{Hei}(\mathbb{R})$ via the map

$$(x, y, z) \leftrightarrow \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Via this identification, we will think of G as acting on $\mathbb{R}^3$.

The action of the index four subgroup $\mathrm{Hei}(\mathbb{Z}) \lhd G$ on $\mathbb{R}^3$

$$(n, m, l) \cdot (x, y, z) = (x + n, y + m, z + ny + l)$$

is free. The quotient space $\mathrm{Hei}(\mathbb{Z})\backslash\mathbb{R}^3$ can be identified in two steps. First, we quotient out by the normal subgroup $H := \langle T_b, T_c\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. On any hyperplane given by fixing the x-coordinate $x = x_0$, the subgroup H leaves the hyperplane invariant, with the generators T_b, T_c translating by one in the y and z coordinates respectively. Quotienting out by H, we obtain that $H\backslash\mathbb{R}^3$ is homeomorphic to $\mathbb{R} \times T^2$, where the T^2 refers to the standard torus obtained from the unit square (centered at the origin) by identifying the opposite sides. The quotient $\mathrm{Hei}(\mathbb{Z})\backslash\mathbb{R}^3$ can now be identified by looking at the action of the quotient group $\mathrm{Hei}(\mathbb{Z})/H$ on the space $\mathbb{R} \times T^2$. The generator for $\mathbb{Z} \cong \mathrm{Hei}(\mathbb{Z})/H$, being the image of the matrix $T_x \in \mathrm{Hei}(\mathbb{Z})$, acts by $(x, y, z) \mapsto (x + 1, y, z + y)$. Putting this together, we see that a fundamental domain for the $\mathrm{Hei}(\mathbb{Z})$-action on $\mathbb{R}^3$ is given by the unit cube $[-1/2, 1/2]^3$ centered at the origin. The quotient 3-manifold $M := \mathrm{Hei}(\mathbb{Z})\backslash\mathbb{R}^3$ can now be obtained from the cube via a suitable identification of the faces. The manifold M can also be thought of as the mapping torus of the map $\phi : T^2 \to T^2$ given by $(y, z) \mapsto (y, y + z) \pmod 1$.

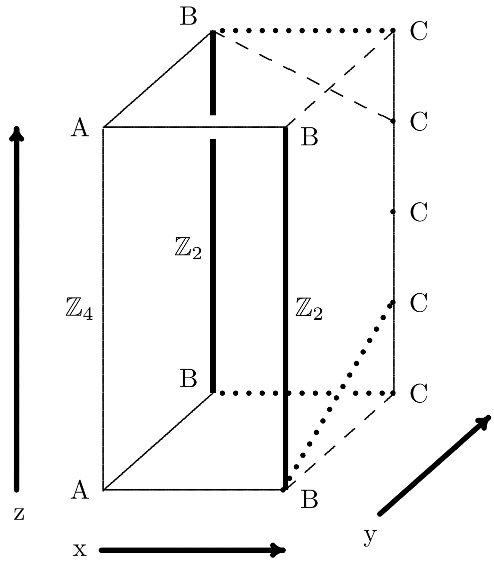

FIGURE 1. $P = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ is a fundamental polyhedron for the action of G on $\mathbb{R}^3$. In the quotient space $G\backslash\mathbb{R}^3$, vertices with the same label are identified, as are edges with the same endpoints and the same shading. The four edges with both endpoints labelled C are identified with the upward orientation. All faces with the same labels are identified in quotient space. Three edges (two in the quotient) have non-trivial isotropy, as indicated.

Next, we identify a fundamental domain for the G-action on $\mathbb{R}^3$. Observe that, since $\mathrm{Hei}(\mathbb{Z}) \triangleleft G$, there is an induced $G/\mathrm{Hei}(\mathbb{Z}) \cong \mathbb{Z}_4$ on M, and a natural identification between $G\backslash\mathbb{R}^3$ and $\mathbb{Z}_4\backslash M$. The manifold M naturally fibers over T^2, with fiber S^1, via the projection onto the (x, y)-plane. The $\mathbb{Z}_4$ action preserves the S^1-fibers, so induces an action on the 2-torus T^2. At the level of the fundamental domain $[-1/2, 1/2]^2 \subset \mathbb{R}^2$ in the (x, y)-plane, the $\mathbb{Z}_4$-action is given by $(x, y) \mapsto (-y, x)$. This tells us that a fundamental domain for the $\mathbb{Z}_4$-action can be obtained by restricting to the square $[0, 1/2] \times [0, 1/2]$. As far as the isotropy goes, there are four points in T^2 with non-trivial stabilizer: the images of points $(0, 0)$ and $(1/2, 1/2)$ both have stabilizer $\mathbb{Z}_4$, and the images of the points $(0, 1/2)$ and $(1/2, 0)$, both have stabilizer $\mathbb{Z}_2$ (and lie in the same σ-orbit).

We conclude that a fundamental domain for the G-action on $\mathbb{R}^3$ is given by the rectangular prism $P := [0, 1/2] \times [0, 1/2] \times [-1/2, 1/2] \subset \mathbb{R}^3$ (Figure 1). The interior of P gives the single 3-cell orbit for the equivariant polyhedral G-CW-structure on $\mathbb{R}^3$. For the isotropy groups, we just need to understand the action on the four vertical lines lying above each of the four points $(0, 0)$, $(1/2, 0)$, $(0, 1/2)$, and $(1/2, 1/2)$. It is easy to see that the vertical line $(0, 0, z)$ consists entirely of points with stabilizer $\mathbb{Z}_4$, while the vertical lines $(1/2, 0, z)$ and $(0, 1/2, z)$ both have stabilizer $\mathbb{Z}_2$. On the other hand, the action of the element of order 4 on the S^1-fiber above the point $(1/2, 1/2)$ can be calculated, and consists of a rotation by $\pi/4$ on the S^1-fiber. So the stabilizers for points on the line $(1/2, 1/2, z)$ are all trivial.

The last task remaining is to identify the gluings on the boundary of P. First, we have that the top and bottom squares of P are identified (via $T_z \in G$). Secondly, the two sides incident to the z-axis get "folded together" by $\sigma \in G$ (which rotates the front face $\pi/2$ radians to the left side face). Finally, the element $T_x \circ \sigma$ maps the hyperplane $y = 1/2$ (containing the back face) to the hyperplane $x = 1/2$ (containing the right side face). This element takes the line $(0, 1/2, z)$ to the line $(1/2, 0, z)$, identifying together the corresponding edges of P. On the line of intersection of these two hyperplanes, the element acts by $(1/2, 1/2, z) \mapsto (1/2, 1/2, z + 1/4)$. These give us the identifications between the faces of P, allowing us to obtain the description of $G\backslash\mathbb{R}^3$ shown in Figure 1.

EXAMPLE 17. *For the group* $G := \text{Hei}(\mathbb{Z}) \rtimes \mathbb{Z}_4$ *described above, we have that* $\text{rank}\Big(K_0\big(C_r^*(G)\big) \otimes \mathbb{Q}\Big) = 5$ *and* $\text{rank}\Big(K_1\big(C_r^*(G)\big) \otimes \mathbb{Q}\Big) = 5$.

Before establishing this result, we note that this is consistent with the computation by Lück, who showed that $K_n\big(C_r^*(G)\big) \cong \mathbb{Z}^5$ for all n (see [**Lu3**, Thm. 2.6]). This serves as a second check on our algorithm, and is, to the best of our knowledge, the only example in the literature of an explicit computation for the topological K-theory of a 3-orbifold group *with non-trivial torsion.*

PROOF. We apply our algorithm, using the polyhedron P described above. For the $\sim$ equivalence classes on $F(G)$, we note that the quotient space $G\backslash\mathbb{R}^3$ has three vertices, one each with stabilizer $\mathbb{Z}_4$ (vertex A), $\mathbb{Z}_2$ (vertex B), and the trivial group (vertex C). The edges joining distinct edges all have trivial stabilizer, allowing us to identify all the identity elements together. We conclude that there are precisely five $\sim$ equivalence classes, corresponding to the three non-trivial elements in the $\mathbb{Z}_4$ vertex stabilizer, the single non-trivial element in the $\mathbb{Z}_2$ vertex stabilizer, and the equivalence class combining all the trivial elements. This gives $\text{rank}(H_0(\mathcal{C}) \otimes \mathbb{Q}) = 5$.

Next we consider the quotient space $G\backslash\mathbb{R}^3$. The faces of P are pairwise identified, so the quotient space is a closed manifold. Moreover, with respect to the induced orientation on ∂P, the identifications between the faces are orientation *reversing*, so the quotient space is an orientable closed 3-manifold. Lemma 7 gives us that $H_3(\mathcal{C}) \cong \mathbb{Z}$, and hence that $\text{rank}(H_3(\mathcal{C}) \otimes \mathbb{Q}) = 1$. Note that, as mentioned earlier, [**Lu3**, Lemma 2.4] shows that the quotient space is actually a 3-sphere (but we do not need this fact for our computation).

The quotient space has empty boundary, so $s = t = t' = 0$. The 2-complex Y is just the 2-skeleton of the quotient space. This is the image of the boundary of P after performing the required identifications. As such, Y is constructed from two squares, a triangle, and a hexagon (see Figure 2). Note that the square corresponding to the front face of P (which also gets identified to the left face) folds up to a cylinder in Y, as its top and bottom edge get identified together (leftmost cylinder in Figure 2). The union of the hexagon and triangle, forming the back face of P (which also gets identified to the right face), similarly folds up to another cylinder in Y (rightmost cylinder in Figure 2). The two cylinders attach together along a common boundary loop (image of the edge BB) to form a single long cylinder. At one of the endpoints, the cylinder attaches to a single loop (image of the edge CC) by a degree four map. So, ignoring for the time being the last square, we

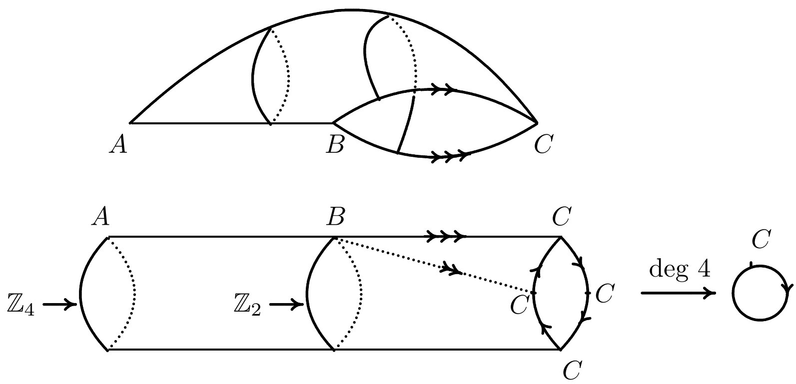

FIGURE 2. 2-skeleton of the quotient space $G\backslash\mathbb{R}^3$. The four side faces of the polyhedron P fold up into the two adjacent cylinders. On the right, the boundary circle of the cylinder gets attached to the circle by a degree 4 map. The top and bottom faces of P get identified into a single square, which attaches to the cylinder as indicated. The two loops in the cylinder based at A, B have isotropy $\mathbb{Z}_4$ and $\mathbb{Z}_2$ respectively. All remaining points have trivial isotropy.

have a subcomplex of Y which deformation retracts to S^1 (as it coincides with the mapping cylinder of the degree four map of S^1). Up to homotopy, we conclude that Y coincides with S^1, along with a single square attached. The square comes from the top face of P (which also gets identified with the bottom face), which, after composing with the homotopy to S^1, attaches to the S^1 via a degree one map of the boundary. This tells us that Y is homotopy equivalent to a 2-disk, and hence is contractible. By Corollary 14, we conclude that rank $\big(H_2(\mathcal{C}) \otimes \mathbb{Q}\big) = 0$.

Finally, we compute the Euler characteristic of $\mathcal{C}$. We have three vertices, one each with stabilizer $\mathbb{Z}_4$, $\mathbb{Z}_2$, and trivial. This gives an overall contribution of $+7$ to $\chi(\mathcal{C})$. We have six edges, one with stabilizer $\mathbb{Z}_4$, one with stabilizer $\mathbb{Z}_2$, and the remainder with trivial stabilizer. This contributes -10 to $\chi(\mathcal{C})$. There are four faces with trivial stabilizer, contributing $+4$ to to $\chi(\mathcal{C})$. There is one 3-cell with trivial stabilizer, contributing -1. Summing these up, we see that $\chi(\mathcal{C}) = 7-10+4-1 = 0$. From Lemma 15, we see that rank $\big(H_1(\mathcal{C})\otimes\mathbb{Q}\big) = 4$. Applying Lemma 4, we deduce that both the rational K-groups have rank $= 5$, as claimed. □

4.3. Hyperbolic reflection groups - I. Consider the groups Λ_n, $n \geq 5$, given by the following presentation:

$$\Lambda_n := \left\langle y, z, x_i,\ \ 1 \leq i \leq n \ \middle|\ \begin{array}{c} y^2, z^2, \\ x_i^2, (x_i x_{i+1})^2, (x_i z)^3, (x_i y)^3,\ \ 1 \leq i \leq n \end{array} \right\rangle$$

The groups Λ_n are Coxeter groups, and the presentation given above is in fact a Coxeter presentation of the group.

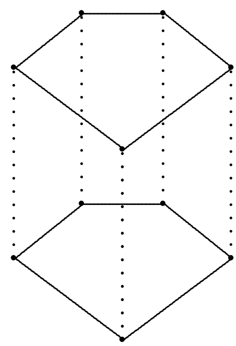

FIGURE 3. Hyperbolic polyhedron for Λ_5. Ordinary edges have internal dihedral angle $\pi/3$. Dotted edges have internal dihedral angle $\pi/2$.

EXAMPLE 18. *For the groups Λ_n whose presentations are given above,*

(1) the rank of $K_0\big(C_r^*(\Lambda_n)\big) \otimes \mathbb{Q}$ is equal to $3n+4$,
(2) the rank of $K_1\big(C_r^*(\Lambda_n)\big) \otimes \mathbb{Q}$ is equal to $n+1$.

PROOF. The groups Λ_n arise as hyperbolic reflection groups, with underlying polyhedron P the product of an n-gon with an interval. This polyhedron has exactly two faces which are n-gons, and the dihedral angle along the edges of these two faces is $\pi/3$. All the remaining edges have dihedral angle $\pi/2$. An illustration of the polyhedron associated to the group Λ_5 is shown in Figure 3. We will take the Λ_n action on $X := \mathbb{H}^3$, with fundamental polyhedron P, and quotient space X/Λ_n coinciding with P. Note that this action is a model for $\underline{E}\Lambda_n$, as finite subgroups F of Λ_n have non-empty fixed sets (the center of mass of any F-orbit will be a fixed point of F), which must be convex subsets (and hence contractible). Both of these last statements are consequences of the fact that the action is by isometries on a space of non-positive curvature.

Applying the argument detailed in Section 3, we compute $\beta_0(\mathcal{C})$ by counting equivalence classes on the set $F(\Lambda_n)$. Since $X/\Lambda_n = P$, the set $F(\Lambda_n)$ consists of $2n$ copies of the group S_4. Each individual S_4 has five conjugacy classes, given by the possible cycle structures of elements, with typical representatives: e, (12), (123), (1234), (12)(34). Next we consider how the edges identify the individual conjugacy classes to get the equivalence classes for $\sim$.

Firstly, all the individual identity elements will be identified together, yielding a single $\sim$ class. So we will henceforth focus on non-identity classes. Each of the edges on the top n-gon has stabilizer $D_3 \cong S_3$, which has three conjugacy classes, represented by e, (12), (123). Under the inclusion into each adjacent vertex stabilizers, representative elements for these classes map to representative elements with the same cycle structure. So we see that all of the 3-cycles in the stabilizers of the vertices in the top n-gon lie in the same $\sim$ class, and likewise for all of the 2-cycles. A similar analysis applies to the vertices in the bottom n-gon. Finally, each vertical edge has stabilizer D_2, and under the inclusion into the adjacent vertices, has image generated by the two permutations (12) and (34) (and hence identifies *three* conjugacy classes together). Putting all this together, we see that the $\sim$ equivalence classes consist of:

- one class consisting of all the identity elements in the individual vertex groups,
- n classes of elements of order $= 2$, coming from the identification of cycles of the form (12)(34) for each pair of vertices joined by a vertical edge,
- one class of elements of order $= 2$, coming from the cycles of the form (12) in all vertex stabilizers,
- two classes of elements of order $= 3$, each coming from the cycles of the form (123) in the top and bottom n-gon respectively, and
- $2n$ classes of elements of order $= 4$, each coming from the cycles of the form (1234) in each individual vertex stabilizer.

We conclude that the $\beta_0(\mathcal{C}) = \text{rank}\big(H_0(\mathcal{C}) \otimes \mathbb{Q}\big) = 3n + 4$.

Since our quotient space $X/\Lambda_n = P$ is not a closed orientable manifold, Lemma 7 tells us that $H_3(\mathcal{C}) = 0$. To calculate $\beta_2(\mathcal{C}) = \text{rank}\big(H_2(\mathcal{C}) \otimes \mathbb{Q}\big)$, we apply Corollary 14. There is a single boundary component for $X/\Lambda_n = P$, which is orientable and non-odd (it contains edges with stabilizer D_2), and contains no edges with stabilizer $\mathbb{Z}_2$, so $s = 1$, $t = 0$, and $t' = 0$. Also, there are no interior 2-cells, and the single boundary component is of dihedral type, so $Y = \emptyset$. By Corollary 14, we conclude that rank $\big(H_2(\mathcal{C}) \otimes \mathbb{Q}\big) = 0$.

To calculate rank $\big(H_1(\mathcal{C}) \otimes \mathbb{Q}\big)$, we need the Euler characteristic of the chain complex $\mathcal{C}$. There are $2n$ vertices, all with stabilizers S_4, which each have five conjugacy classes. There are a total of $3n$ edges, n of which have stabilizer D_2 (with four conjugacy classes), and $2n$ of which have stabilizer D_3 (with three conjugacy classes). There are $n + 2$ faces, with stabilizers $\mathbb{Z}_2$, which each have two conjugacy classes. There is one 3-cell, with trivial stabilizer, with a single conjugacy class. Putting this together, we have that

$$\chi(\mathcal{C}) = \big(5(2n)\big) - \big(3(2n) + 4(n)\big) + \big(2(n + 2)\big) - 1 = 2n + 3$$

Applying Lemma 15, we can now calculate:

$$\text{rank}\,\big(H_1(\mathcal{C}) \otimes \mathbb{Q}\big) = (3n + 4) - (2n + 3) = n + 1$$

Finally, applying Lemma 4, we obtain the desired result. □

4.4. Hyperbolic reflection groups - II. Next, let us consider a somewhat more complicated family of examples. For an integer $n \geq 2$, we consider the group Γ_n, defined by the following presentation:

$$\Gamma_n := \left\langle x_1, \ldots, x_6 \;\middle|\; \begin{array}{c} x_i^2, (x_1x_2)^n, (x_1x_5)^2, (x_1x_6)^2, (x_3x_4)^2, (x_2x_5)^2, (x_2x_6)^2 \\ (x_1x_4)^3, (x_2x_3)^3, (x_4x_5)^3, (x_4x_6)^3, (x_3x_5)^3, (x_3x_6)^3 \end{array} \right\rangle$$

Observe that the groups Γ_n are Coxeter groups, and that the presentation given above is in fact a Coxeter presentation of the group.

Example 19. *For the groups Γ_n whose presentations are given above, we have that:*

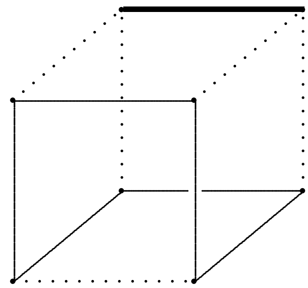

FIGURE 4. Hyperbolic polyhedron for Γ_n. Ordinary edges have internal dihedral angle $\pi/3$. Dotted edges have internal dihedral angle $\pi/2$. The thick edge has internal dihedral angle π/n.

$$\operatorname{rank}\Big(K_0\big(C_r^*(\Gamma_n)\big)\otimes\mathbb{Q}\Big)=\begin{cases}\frac{3}{2}(n-1)+12 & n \textit{ odd},\\ \frac{3}{2}n+14 & n \textit{ even},\end{cases}$$

$$\operatorname{rank}\Big(K_1\big(C_r^*(\Gamma_n)\big)\otimes\mathbb{Q}\Big)=\begin{cases}3 & n \textit{ odd},\\ 2 & n \textit{ even}.\end{cases}$$

PROOF. To verify the results stated in this example, we first observe that the Coxeter groups Γ_n arise as hyperbolic reflection groups, with underlying polyhedron P a combinatorial cube. The geodesic polyhedron associated to Γ_n is shown in Figure 4. Again, we set $X := \mathbb{H}^3$, with fundamental polyhedron P, and quotient space X/Γ_n coinciding with P. As in the previous example, X is a model for $\underline{E}G$.

To apply our procedure, we start by considering the equivalence relation $\sim$ on the set $F(\Gamma_n)$. Out of the eight vertices of the cube P, six have stabilizer isomorphic to S_4, while the remaining two have stabilizer $D_n \times \mathbb{Z}_2$. We will think of D_n as the symmetries of a regular n-gon, and let r_0, r_1 denote the reflection in a vertex, and in the midpoint of an adjacent side respectively (so r_0, r_1 are the standard Coxeter generators for D_n). Recall that the number of conjugacy classes of D_n depends on the parity of n: each rotation ϕ is only conjugate to its inverse ϕ^{-1}, while the reflections r_i fall into one or two conjugacy classes, depending on whether n is odd or even. Crossing with $\mathbb{Z}_2$, each of these conjugacy class in D_n gives rise to two conjugacy classes in $D_n \times \mathbb{Z}_2$: the image class under the obvious inclusion $D_n \hookrightarrow D_n \times \mathbb{Z}_2$, and its "flipped" image, obtained by composing with the non-trivial element τ in the $\mathbb{Z}_2$-factor. Next, we need to see how conjugacy classes in the individual vertex stabilizers get identified together by the edge stabilizers. After performing these identifications, we obtain that the $\sim$ equivalence classes consist of:

- one class consisting of all the identity elements in the individual vertex groups,

- six classes of elements of order $=4$, each coming from the cycles of the form (1234) in the six individual S_4 vertex stabilizers,
- one class of elements of order $=3$, coming from the cycles of the form (123) in the six S_4 vertex stabilizers (these classes get identified together via the edges with stabilizer D_3),
- one class of elements of order $=2$, comprised from the cycles of the form (12) in the six S_4 vertex stabilizers (identified via the edges with stabilizer D_3), along with the the three elements of the form $(r_0,1),(r_1,1),(1,\tau)$ in the two vertices with stabilizer $D_n\times\mathbb{Z}_2$ (identified via the edges with stabilizer D_2),
- one class of elements of order $=2$, consisting of the elements of cycle form (12)(34) in the two S_4 vertex stabilizers which are joined together by an edge with stabilizer D_2 (which identifies these elements together),
- two or four classes (according to the parity of n), coming from the two elements of the form (r_0,τ) or (r_1,τ) in the two vertices with stabilizer $D_n\times\mathbb{Z}_2$ (these two elements lie in the same conjugacy class when n odd), which are each identified to elements with cycle form (12)(34) in one of the two adjacent S_4 vertex stabilizers,
- $n-1$ or n conjugacy classes (according to n odd or even respectively), coming from elements of the form (ϕ_i,τ) in each of the two vertices with stabilizer $D_n\times\mathbb{Z}_2$, and
- $(n-1)/2$ or $n/2$ conjugacy classes (according to n odd or even respectively), coming from the elements of the form $(\phi_i,1)$ in the two vertices with stabilizer $D_n\times\mathbb{Z}_2$ (the elements in the two copies get identified together via the edge with stabilizer D_n).

Summing this up, we find that rank $\big(H_0(\mathcal{C})\otimes\mathbb{Q}\big)$ is $\frac{3}{2}(n-1)+12$ if n is odd, and $\frac{3}{2}n+14$ if n is even.

The quotient space $X/\Gamma_n=P$ is a 3-manifold with non-empty boundary, so Lemma 7 gives us that $H_3(\mathcal{C})=0$. The only boundary component is orientable and non-odd, and contains no edges with stabilizer $\mathbb{Z}_2$, so $s=1$ and $t=t'=0$. Moreover, there are no interior faces, so $Y=\emptyset$. By Corollary 14, we conclude that rank $\big(H_2(\mathcal{C})\otimes\mathbb{Q}\big)=0$.

Next, let us calculate the rank of $H_1(\mathcal{C})\otimes\mathbb{Q}$. To do this, we first compute the Euler characteristic $\chi(\mathcal{C})$. We have six vertices, four with stabilizer S_4 (having five conjugacy classes), and two with stabilizer $D_n\times\mathbb{Z}_2$ (having either $n+3$ or $n+6$ conjugacy classes, depending on whether n is odd or even). There are twelve edges, six with stabilizer D_3 (with three conjugacy classes), five with stabilizer D_2 (with four conjugacy classes), and one with stabilizer D_n (with $(n+3)/2$ or $(n+6)/2$ conjugacy classes, depending on whether n is odd or even). There are six faces, each with stabilizer $\mathbb{Z}_2$ (with two conjugacy classes each). Finally, there is one 3-cell with trivial stabilizer. Taking the alternating sum, we obtain that the Euler characteristic is

$$\chi(\mathcal{C})=\begin{cases}\frac{3}{2}(n-1)+9 & n \text{ odd},\\ \frac{3}{2}n+12 & n \text{ even}.\end{cases}$$

From Lemma 15, the difference between $\chi(\mathcal{C})$ and the rank of $H_0(\mathcal{C})\otimes\mathbb{Q}$ yields the rank of $H_1(\mathcal{C})\otimes\mathbb{Q}$, giving us that the latter is either 3 or 2 according to whether n is odd or even. Applying Lemma 4, we obtain the desired result. $\square$

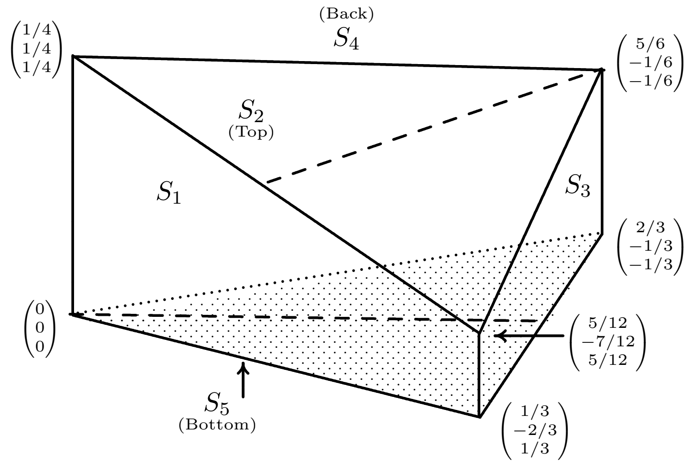

FIGURE 5. The polyhedron pictured here is an exact convex compact fundamental polyhedron for the action of G on $\mathbb{R}^3$. The dashed lines represent axes of rotation (through 180 degrees) for certain elements of G. Note that the base of the figure is an equilateral triangle, but the top is only isosceles.

4.5. Crystallographic group. Our next example is taken from the work of Farley and Ortiz [**FO**]. Consider the lattice $L \subset \mathbb{R}^3$ generated by the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

and let $G = \mathrm{Sym}(L)$ denote the subgroup of $\mathrm{Isom}(\mathbb{R}^3)$ which maps L to itself. The group G is one of the seven maximal split 3-dimensional crystallographic groups, and is discussed at length in [**FO**, Section 6.8].

A polyhedral fundamental domain P for the G-action on $\mathbb{R}^3$ is provided in Figure 5. Next we describe the stabilizers of the various faces, edges, and vertices of P (given in terms of the labeling in Figure 5).

Face stabilizers: The two triangles at the top (collectively labelled by S_2), and the two triangles at the bottom (labelled by S_5) have trivial stabilizer. The three quadrilateral sides (S_1, S_3, and S_4) each have stabilizer $\mathbb{Z}_2$, generated by the reflection in the 2-plane extending the corresponding side.

Edge stabilizers: The three vertical edges in Figure 5 each have stabilizer D_3, generated by the reflections in the two incident faces. The two dotted edges (in the middle of the faces S_2 and S_5) have stabilizer $\mathbb{Z}_2$, generated by a rotation by π centered on the edge. All remaining edges have stabilizer $\mathbb{Z}_2$, generated by the reflection in the (unique) incident face whose isotropy is non-trivial. Note that, when one passes to the quotient space X/G, the two triangles in the top face S_2 get identified together by the π-rotation in the dotted line (and similarly for the two triangles in the bottom face S_5).

Vertex stabilizers: The two vertices $(0,0,0)$ and $(5/6,-1/6,-1/6)$ have stabilizer $D_3 \times \mathbb{Z}_2$. The two vertices $(1/4,1/4,1/4)$ and $(2/3,-1/3,-1/3)$ have stabilizer D_3. Finally, the two vertices $(1/2,1/2,0)$ and $(1/3,-1/6,1/3)$, the midpoints of the edges at which the dotted lines terminate, have stabilizer D_2. The remaining vertices of P are in the same orbit as one of the six described above.

EXAMPLE 20. *For the split crystallographic group G described above, we have that* $\operatorname{rank}\Big(K_0\big(C_r^*(\Gamma_n)\big)\otimes\mathbb{Q}\Big) = 12$ *and* $\operatorname{rank}\Big(K_1\big(C_r^*(\Gamma_n)\big)\otimes\mathbb{Q}\Big) = 0$.

PROOF. We apply our algorithm, using the polyhedron P above. Our first step is to consider the $\sim$ equivalence relation on the set $F(G)$. The vertex and edge stabilizers for P have been described above, and the $\sim$ equivalence classes are given as follows:

- one class consisting of all the identity elements in the individual vertex groups,
- one class consisting of all the elements of order 3 in the individual vertex groups (these occur in the four vertices with stabilizer D_3 or $D_3\times\mathbb{Z}_2$, and are identified together via three consecutive edges with stabilizer D_3),
- one class of elements of order 2, consisting of elements of order two in the vertex groups isomorphic to D_3, along with elements of order two in the canonical D_3-subgroup within the vertex groups isomorphic to $D_3\times\mathbb{Z}_2$ (these are identified together via the three consecutive edges with stabilizer D_3), and the elements of the form $(1,0)$ in the two vertex groups isomorphic to $D_2\cong\mathbb{Z}_2\times\mathbb{Z}_2$ (identified together via the edges $S_1\cap S_2$ and $S_3\cap S_5$),
- two classes of elements of order 2, coming from each of the two dotted edges: the rotation by π in the edge identifies the element $(0,1)$ in one endpoint (vertex with stabilizer $D_2\cong\mathbb{Z}_2\times\mathbb{Z}_2$) with the element which is a product of a reflection in D_3 with a reflection in $\mathbb{Z}_2$ in the other endpoint (vertex with stabilizer $D_3\times\mathbb{Z}_2$),
- six remaining classes, two each in the vertices with stabilizer $D_3\times\mathbb{Z}_2$ and one each in those with stabilizer D_2 (these classes aren't identified to any others via the edges).

Summing this up, we see that $\operatorname{rank}(H_0(\mathcal{C})\otimes\mathbb{Q}) = 11$.

Next, we note that the quotient space X/G is obtained from the polyhedron P by "folding up" the top and bottom triangle along the dotted lines, resulting in $\mathbb{D}^3$, a 3-manifold with non-empty boundary. Lemma 7 gives us that $H_3(\mathcal{C}) = 0$. The only boundary component is orientable and odd, and contains edges with stabilizer $\mathbb{Z}_2$, so $s = t = 0$ and $t' = 1$. The 2-complex Y clearly deformation retracts to the boundary S^2, so $\beta_2(Y) = 1$. By Corollary 14, we conclude that $\operatorname{rank}\big(H_2(\mathcal{C})\otimes\mathbb{Q}\big) = 1$.

Next, we calculate the rank of $H_1(\mathcal{C})\otimes\mathbb{Q}$. As usual, we first calculate the Euler characteristic $\chi(\mathcal{C})$. We have six vertices, two with stabilizer D_2 (having four conjugacy classes), two with stabilizer D_3 (having three conjugacy classes), and two with stabilizer $D_3\times\mathbb{Z}_2$ (having six conjugacy classes), giving an overall contribution of $+26$. There are nine edges, six with stabilizer $\mathbb{Z}_2$ (with two conjugacy classes), and three with stabilizer D_3 (with three conjugacy classes), giving a contribution of -21. There are five faces, three with stabilizer $\mathbb{Z}_2$ (with two conjugacy classes each), and two with trivial stabilizer (with one conjugacy class each), giving a contribution of $+8$. There is one 3-cell with trivial stabilizer, contributing a -1. Summing up these contributions, we obtain that the Euler characteristic is $\chi(\mathcal{C}) =$

$26 - 21 + 8 - 1 = 12$. From Lemma 15, we see that the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$ is $= 0$. Applying Lemma 4, we obtain the desired result. □

5. Concluding remarks

The examples in the previous section were chosen to illustrate our algorithm on several different types of smooth 3-orbifold groups. As the reader can see, our algorithm is quite easy to apply, once one has a good description of the orbit space $G\backslash X$. There are several natural directions for further work.

For instance, in Section 4.5, we applied our algorithm to a specific 3-dimensional crystallographic group. It is known that, in dimension $= 3$, there are precisely 219 crystallographic groups up to isomorphism. One could in principle apply our algorithm to produce a complete table of the rational K-theory groups of all 219 groups. The essential difficulty in doing this lies in finding some convenient, systematic way to identify polyhedral fundamental domains for each of these groups. For the 73 *split* crystallographic groups, such fundamental domains can be found in the forthcoming paper of Farley and Ortiz [**FO**].

Another reasonable direction would be to focus on uniform arithmetic lattices Γ in the Lie group $PSL_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$. One could try to analyze the relationship (if any) between the rational K-theory of such a Γ and the underlying arithmetic structure. Again, the difficulty here lies in finding a good description of the polyhedral fundamental domain for the action (in terms of the arithmetic data).

In a different direction, one can consider *hyperbolic reflection groups.* These are groups generated by reflections in the boundary faces of a geodesic polyhedron $P \subset \mathbb{H}^3$. In this context, the polyhedron P serves as a polyhedral fundamental domain for the action, so one can readily apply our algorithm to compute the rational K-theory of the corresponding group (see the examples in Sections 4.3 and 4.4). One could try, in this special case, to refine our algorithm to produce expressions for the *integral* K-theory groups, in terms of the combinatorial data of the polyhedron P. This is the subject of an ongoing collaboration of the authors.

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Appendix A

In this Appendix, we provide the details for the computations used in some of the proofs in Section 3.4. Let $n \geq 2$ be an integer and D_n be the dihedral group with presentation

$$D_n = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n \rangle.$$

We will compute the map

$$\varphi\colon R_{\mathbb{C}}(\mathbb{Z}_2) \oplus R_{\mathbb{C}}(\mathbb{Z}_2) \longrightarrow R_{\mathbb{C}}(D_n) \tag{3}$$

given by induction between representation rings with respect to the subgroups $\langle s_1\rangle$ and $\langle s_2\rangle$ of D_n, both isomorphic to $\mathbb{Z}_2$, and opposite orientations. That is, $\varphi(\rho,\tau) = (\rho\uparrow) - (\tau\uparrow)$, where '$\uparrow$' means induction between the corresponding groups.

Recall from the main text (see Section 3.4, particularly Lemma 8) that if e is a boundary edge with stabilizer D_n and σ_1 and σ_2 are incident boundary faces, then $K_0(C_r^*(G_{\sigma_i})) \cong R_{\mathbb{C}}(\mathbb{Z}_2)$ and the relevant part of the Bredon chain complex at the edge e is the map given in equation (3).

The character table for D_n is given by

D_n	$(s_1s_2)^r$	$s_2(s_1s_2)^r$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	$(-1)^r$	$(-1)^r$
$\widehat{\chi_4}$	$(-1)^r$	$(-1)^{r+1}$
ϕ_p	$2\cos\left(\frac{2\pi pr}{m}\right)$	0

where $0 \le r \le n-1$, p varies between 1 and $n/2-1$ if n is even or $(n-1)/2$ if n is odd and the hat $\widehat{\ }$ denotes a character which appears only when n is even.
The character table for $\mathbb{Z}_2$ is given by

$\mathbb{Z}_2$	e	s_i
ρ_1	1	1
ρ_2	1	-1

To compute the induction homomorphism we will use Frobenius reciprocity. We first do the case $\langle s_1\rangle$. The characters of D_n restricted to this subgroup are

	e	s_1
$\chi_1\downarrow$	1	1
$\chi_2\downarrow$	1	-1
$\widehat{\chi_3}\downarrow$	1	-1
$\widehat{\chi_4}\downarrow$	1	1
$\phi_p\downarrow$	2	0

Multiplying with the rows of the character table of $\langle s_1\rangle \cong C_2$ we obtain the induced representations

$$\begin{aligned}\rho_1\uparrow &= \chi_1 + \widehat{\chi_4} + \textstyle\sum \phi_p,\\ \rho_2\uparrow &= \chi_2 + \widehat{\chi_3} + \textstyle\sum \phi_p.\end{aligned}$$

The case $\langle s_2\rangle$ is analogous, but note that the characters 3 and 4 must be interchanged in the even case:

	e	s_j
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	1	1
$\widehat{\chi_4}$	1	-1
ϕ_p	2	0

and

$$\begin{aligned}\rho_1\uparrow &= \chi_1 + \widehat{\chi_3} + \textstyle\sum \phi_p,\\ \rho_2\uparrow &= \chi_2 + \widehat{\chi_4} + \textstyle\sum \phi_p.\end{aligned}$$

As maps of free abelian groups we obtain

$$\begin{array}{rcll} \mathbb{Z}^2 & \to & \mathbb{Z}^{c(D_n)} & \\ (a,b) & \mapsto & (a,b,\widehat{b},\widehat{a},a+b,\dots,a+b) & \text{for } \langle s_1\rangle \hookrightarrow D_n, \\ (c,d) & \mapsto & (c,d,\widehat{c},\widehat{d},c+d,\dots,c+d) & \text{for } \langle s_2\rangle \hookrightarrow D_n. \end{array}$$

Finally, the map φ above is

$$\begin{array}{rcl} R_{\mathbb{C}}(\mathbb{Z}_2) \oplus R_{\mathbb{C}}(\mathbb{Z}_2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \to & \mathbb{Z}^{c(D_n)} \cong R_{\mathbb{C}}(D_n) \\ (a,b,c,d) & \mapsto & (a-c, b-d, \widehat{b-c}, \widehat{a-d}, S, \dots, S) \end{array}$$

where $S = a+b-c-d$.

As an immediate consequence of this computation, we see that if the element $\langle k,k,\dots,k\rangle$ lies in the image of ϕ, then one must have that

$$a-c = k = S = a+b-c-d.$$

Subtracting $a-c$ from both sides, we deduce that $0 = b-d = k$. In other words, the image of ϕ intersects the subgroup $\mathbb{Z}\cdot\langle 1,1,\dots,1\rangle$ only in the zero vector (as was stated in Lemma 8).

Another consequence is that it is easy to identify elements in the kernel of ϕ. The equation

$$0 = (a-c, b-d, \widehat{b-c}, \widehat{a-d}, S, \dots, S)$$

forces $a=c$ and $b=d$. If in addition, n is even, then we also have $a=d$, and hence all terms must be equal. This was used in the arguments for both Lemma 10 and Lemma 12.

Appendix B

In this Appendix we compute the 2$^{\text{nd}}$ Betti number of the 3-manifolds M_f appearing in the Remark at the end of Section 4.1. The manifold M_f, as a mapping torus, fibers over S^1 with fiber T^2. For this fibration, the Leray-Serre spectral sequence gives

$$E^2_{pq} = H_p(S^1, H_q(T^2)) \Rightarrow H_{p+q}(M_f).$$

Since S^1 is 1-dimensional, $E^2_{p,q} = 0$ unless $p = 0,1$. The differentials have bidegree $(-2,1)$ so the spectral sequence already collapses at the E^2-page. This implies that

$$H_2(M_f) \cong E^2_{0,2} \oplus E^2_{1,1} \cong H_0(S^1, H_2(T^2)) \oplus H_1(S^1, H_1(T^2)).$$

Recall that this is not ordinary homology but rather homology with local coefficient system given by the homology of the fiber.

The homology group $H_0(S^1, H_2(T^2))$ is obtained from the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{Id}-f_*} \mathbb{Z} \longrightarrow 0$$

where $f_*\colon \mathbb{Z} \to \mathbb{Z}$ is the map induced by the action of the gluing map f on the local coefficient $\mathbb{Z} = H_2(T^2)$. If $\det(\alpha) = 1$, f is orientation preserving and hence $f_* = \text{Id}$. This implies $H_0(S^1, H_2(T^2)) \cong \mathbb{Z}$. If $\det(\alpha) = -1$, f is orientation reversing and hence $f_* = -\text{Id}$. This implies $H_0(S^1, H_2(T^2)) \cong \mathbb{Z}_2$.

The homology group $H_1(S^1, H_1(T^2))$ is obtained from the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\mathrm{Id}-f_*} \mathbb{Z}^2 \longrightarrow 0$$

where now $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is induced by the action of the gluing map f on the local coefficient $\mathbb{Z}^2 = H_1(T^2)$. Note that by construction f acts on $\pi_1(T^2) \cong H_1(T^2)$ via the automorphism α. So the map above is $\mathrm{Id} - \alpha$ and hence $H_1(S^1, H_1(T^2)) \cong \ker(\mathrm{Id} - \alpha)$. Suppose that $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $\mathrm{Id} - \alpha = \left(\begin{smallmatrix} 1-a & -b \\ -c & 1-d \end{smallmatrix}\right)$. The kernel of this map has dimension 2 if and only if $\mathrm{Id} - \alpha = 0$, that is, $\alpha = \mathrm{Id}$. The dimension is at least 1 if and only if the determinant is zero, that is,

$$(1-a)(1-d) = bc \;\Leftrightarrow\; 1 - \mathrm{tr}(\alpha) + ad = bc \;\Leftrightarrow\; 1 + \det(\alpha) = \mathrm{tr}(\alpha).$$

This occurs if and only if $\det(\alpha) = 1$ and $\mathrm{tr}(\alpha) = 2$, or $\det(\alpha) = -1$ and $\mathrm{tr}(\alpha) = 0$. Altogether, this gives us

$$\beta_2(M_f) = \begin{cases} 3 & \text{if } \alpha = \mathrm{Id}, \\ 2 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) = 2, \alpha \neq \mathrm{Id}, \\ 1 & \text{if } \det(\alpha) = 1, \mathrm{tr}(\alpha) \neq 2, \\ 1 & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) = 0, \\ 0 & \text{if } \det(\alpha) = -1, \mathrm{tr}(\alpha) \neq 0. \end{cases}$$

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Automata Groups

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These notes present an introduction to the modern theory of groups generated by finite automata.

The class of automata groups contains several remarkable countable groups. Their study has led to the solution of a number of important problems in group theory. Its recent applications have extended to the fields of algebra, geometry, analysis and probability.

Rather than develop general theory we present solutions to some of the most important problems of group theory using automata groups. The reader will find simple constructions (with proofs) of infinite finitely generated torsion groups, groups of intermediate growth, groups of non-uniform exponential growth or exotic amenable groups.

The first section presents a definition and basic facts about automata groups.

Section 2 deals with spectral properties of groups generated by automata and is motivated by a question of Atiyah about L^2 Betti numbers of closed manifolds. It contains the simplest examples of interesting groups generated by finite automata like the lamplighter group.

In the following section we consider a group generated by a simple three state automaton. Its study shows that it is an amenable group with a very rich algebraic structure. This leads to a solution of some fundamental problems about amenable groups.

In Section 4 the reader can find very simple constructions of infinite finitely generated torsion groups (every element is of finite order). The question about their existence was asked by Burnside in 1902 and motivated some of the most important developments in group theory. This section introduces historically the first examples of groups generated by a finite automaton, namely the Aleshin group [**1**] from 1972. This group was rediscovered by Grigorchuk in 1980 [**22**]. Formally the two groups are isomorphic up to a finite index.

In Section 5 we consider a growth type for groups. It is a fundamental invariant for infinite finitely generated groups. The highlight is the presentation of historically the first example of the so-called intermediate growth group, which is the group from Section 4.

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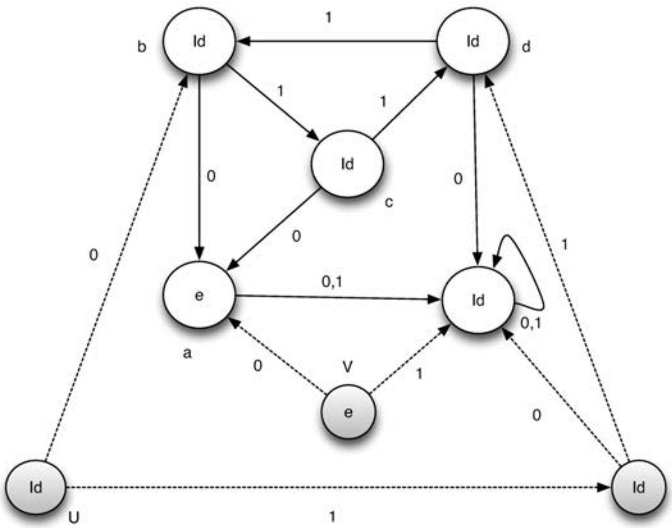

FIGURE 1. The Aleshin automaton from 1972

Finally, Section 6 deals with a group defined by Wilson and presents a solution to a problem of Gromov about groups of non-uniform exponential growth.

Some parts of the material presented here follow the author's Bourbaki seminar [**50**] on this subject.

1. Automata groups

1.1. Definition of groups generated by automata. The automata which we consider are finite, reversible and have the same input and output alphabets, say $D = \{0, 1, \ldots, d-1\}$ for a certain integer $d > 1$. To such an automaton A are associated a finite set of states Q, a transition function $\phi : Q \times D \to Q$ and the exit function $\psi : Q \times D \to D$. The automaton A is characterized by a quadruple (D, Q, ϕ, ψ).

The automaton A is invertible if, for every $q \in Q$, the function $\psi(q, \cdot) : D \to D$ is a bijection.

In this case, $\psi(q, \cdot)$ can be identified with an element σ_q of the symmetric group S_d on $d = |D|$ symbols.

There is a convenient way to represent a finite automaton by a marked graph $\Gamma(A)$ whose vertices correspond to elements of Q.

Two states $q, s \in Q$ are connected by an arrow labelled by $i \in D$ if $\phi(q, i) = s$; each vertex $q \in Q$ is labelled by a corresponding element σ_q of the symmetric group.

Figure 1 represents the Aleshin automaton from 1972 [**1**] which is historically the first example of an automaton defined to construct a group. In this example the alphabet consists of two letters 0 and 1. The elements of the symmetric group S_2 are denoted Id and ε. Aleshin defined a group associated to an automaton. This construction is explained below.

The automata we just defined are non-initial. To make them initial we need to mark some state $q \in Q$ as the initial state. The initial automaton $A_q =$

(D, Q, ϕ, ψ, q) acts on the right on the finite and infinite sequences over D in the following way. For every symbol $x \in D$ the automaton immediately gives $y = \psi(q, x)$ and changes its initial state to $\phi(q, x)$.

By joining the exit of A_q to the input of another automaton $B_s = (D, S, \alpha, \beta, s)$, we get a mapping which corresponds to the automaton called the composition of A_q and B_s and is denoted by $A_q \star B_s$.

This automaton is formally described as the automaton with a set of the states $Q \times S$ and the transition and exit functions Φ, Ψ defined by

$$\Phi((x, y), i) = (\phi(x, i), \alpha(y, \psi(x, i))),$$
$$\Psi((x, y), i) = \beta(y, \psi(x, i))$$

and the initial state (q, s).

The composition $A \star B$ of two non-initial automata is defined by the same formulas for input and output functions but without indicating the initial state.

Two initial automata are equivalent if they define the same mapping. There is an algorithm to minimize the number of states.

The automaton which produces the identity map on the set of sequences is called trivial. If A is invertible then for every state q the automaton A_q admits an inverse automaton A_q^{-1} such that $A_q \star A_q^{-1}$, $A_q^{-1} \star A_q$ are equivalent to the trivial one. The inverse automaton can be formally described as the automaton $(D, Q, \widetilde{\phi}, \widetilde{\psi}, q)$ were $\widetilde{\phi}(s, i) = \phi(s, \sigma_s(i))$, $\widetilde{\psi}(s, i) = \sigma_s^{-1}(i)$ for $s \in Q$. The equivalence classes of finite invertible automata over the alphabet D constitute a group called the group of finite automata which depends on D. Every set of finite automata generates a subgroup of this group.

Now let A be an invertible automaton. Let $Q = \{q_1, \ldots, q_t\}$ be the set of states of A and let $A_{q_1}, \ldots, A_{q_t}$ be the set of initial automata which can be obtained from A. The group $G(A) = \langle A_{q_1}, \ldots, A_{q_t} \rangle$ is called the group generated or determined by A.

1.2. Automata groups and wreath products. There is a relation between automata groups and wreath products. For a group of the form $G(A)$ one has the following interpretation.

Let $q \in Q$ be a state of A and let $\sigma_q \in S_d$ be the permutation associated to this state. For every symbol $i \in D$ we denote by $A_{q,i}$ the initial automaton having as the initial state $\phi(q, i)$ (then $A_{q,i}$ for $i = 0, 1, \ldots, d-1$ runs over the set of initial automata which are neighbors of A_q, i.e. such that the graph $\Gamma(A)$ has an arrow from A_q to $A_{q,i}$).

Let G and F be the groups of finite type such that F is a group of permutations of the set X (we are interested in the case where F is the symmetric group S_d and X is the set $\{0, 1, \ldots, d-1\}$). We define the wreath product $G \wr F$ of these groups as follows. The elements of $G \wr F$ are the couples (g, γ) where $g : X \to G$ is a function such that $g(x)$ is different from the identity element of G, denoted Id, only for a finite number of elements x of X, and where γ is an element of F. The multiplication in $G \wr F$ is defined by:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(x) = g_1(x)g_2(\gamma_1^{-1}(x)) \quad \text{for } x \in X.$$

We write the elements of the group $G \wr S_d$ as $(a_0, \dots, a_{d-1})\sigma$, where $a_0, \dots, a_{d-1} \in G$ and $\sigma \in S_d$.

The group $G = G(A)$ admits the embedding into a wreath product $G \wr S_d$ via the application

$$A_q \to (A_{q,0}, \dots, A_{q,d-1})\sigma_q,$$

where $q \in Q$. The right-hand expression is called a wreath decomposition of A. We write $A_q = (A_{q,0}, \dots, A_{q,d-1})\sigma_q$.

For simplicity we denote by a the generator of A_a of the group generated by the automaton A.

1.3. Action on the tree. The finite sequences over the alphabet $D = \{0, \dots, d-1\}$ are in bijection with the vertices of a rooted tree T_d of degree d (whose root corresponds to an empty sequence).

An initial automaton A_q acts on the sequences over D and thus acts on T_d by automorphisms. Therefore, for each group generated by an automaton, in particular for a group of the form $G(A)$, there exists a canonical action on a tree (for a theory of actions on non-rooted trees, see [**46**]).

For a group $G = G(A)$ acting by automorphisms on T, we denote by $\mathrm{St}_G(n)$ the subgroup of G made up of elements of G which act trivially on the level n of the tree T. In a similar way, for a vertex $u \in T$ we denote by $\mathrm{St}_G(u)$ the subgroup of G composed of the elements fixing u. The embedding of G into the wreath product $G \wr S_d$ induces $\phi : \mathrm{St}_G(1) \to G^d$ into the base group of the wreath product. This defines the canonical projections $\psi_i : \mathrm{St}_G(1) \to G$ $(i = 1, \dots, d)$ defined by $\psi_i(g) = \phi(g)|_i$ for $g \in \mathrm{St}_G(1)$.

The stabilizer $\mathrm{St}_G(n)$ of the n-th level is the intersection of the stabilizers of all vertices on this level. For a vertex $u \in T$ we can define the projection $\psi_u : \mathrm{St}_G(u) \to G$.

Definition 1.1. A group G is fractal if for every vertex u, we have $\psi_u(\mathrm{St}_G(u)) = G$ after the identification of the tree T with the subtree T_u issued from the vertex u.

The rigid stabilizer of the vertex u is a subgroup $\mathrm{Rist}_G(u)$ of the automorphisms of G which act trivially on $T \setminus T_u$. The rigid stabilizer of the n-th level $\mathrm{Rist}_G(n)$ is the subgroup generated by the rigid stabilizers on this level.

A group G acting on a rooted tree T is called spherically transitive if it acts transitively on each level. A spherically transitive group $G \leq \mathrm{Aut}(T)$ is branched if $\mathrm{Rist}_G(n)$ is a finite index subgroup for each $n \in \mathbb{N}$. A spherically transitive group $G \leq \mathrm{Aut}(T)$ is weakly branched if $|\mathrm{Rist}_G(n)| = \infty$ for all $n \in \mathbb{N}$.

If there is no risk of confusion we omit the index G in $\mathrm{St}_G(u)$, $\mathrm{Rist}_G(u)$, etc.

The embedding $G \to G \wr S_d$, $g \to (g_0, \dots, g_{d-1})\sigma$ defines the restriction g_i of g at the vertex i of the first level. The iteration of this procedure leads to the notion of restriction g_u of g at the vertex u.

Definition 1.2. We say that the group G is regularly weakly branched over a subgroup $K \neq \{1\}$ if $K \geq K \times \cdots \times K$ (direct product of d factors, each of them acting on the corresponding subtree T_u, $|u| = 1$).

We use the notations $x^y = y^{-1}xy, [x,y] = x^{-1}y^{-1}xy$ and denote by $\langle X\rangle^Y$ the normal closure of X in Y. The length of a word w and an element g are denoted by $|w|$ and $|g|$ respectively.

1.4. Classification of the automata groups on two states with the alphabet $\{0,1\}$. For the alphabet on two letters the automata with just one state produce only the trivial group or the group of order two.

We are going to analyze all groups generated by the automata on two states with the alphabet on two letters.

THEOREM 1.3 ([**24**]). *The only groups generated by the automata on two states over the alphabet on two letters are:*

- *the trivial group;*
- *the group of order two* $\mathbb{Z}/2\mathbb{Z}$*;*
- *the Klein group* $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$*;*
- *the infinite cyclic group* $\mathbb{Z}$*;*
- *the infinite dihedral group* $\mathbb{D}_\infty$*;*
- *the lamplighter group* $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$*.*

PROOF. We denote by a and b the two states of the automaton. If both states are labelled by the identity or both by e, then the group generated by the automaton is either trivial or $\mathbb{Z}/2\mathbb{Z}$.

Thus we can suppose that one state, say a, is labelled by the identity and the other by e. By exchanging if necessary 0 with 1, we can suppose that $a = (a,a)$ or $a = (b,b)$ or $a = (a,b)$.

(i) Case $a = (a,a)$.

In this case a corresponds to the identity in the group. The exchange of 0 and 1 (this does not change a) reduces b to three possibilities: $b = (b,b)\mathsf{e}$, $b = (a,b)\mathsf{e}$ or $b = (a,a)\mathsf{e}$.

The first case corresponds to $\mathbb{Z}/2\mathbb{Z}$, the second to $\mathbb{Z}$ and the third to $\mathbb{Z}/2\mathbb{Z}$.

(ii) Case $a = (b,b)$.

The exchange of 0 and 1 (this does not change a) reduces b to three possibilities: $b = (b,b)\mathsf{e}$, $b = (a,a)\mathsf{e}$ or $b = (a,b)\mathsf{e}$.

The first two possibilities correspond to the Klein group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed a and b are of order two and commute.

The third case corresponds to the infinite cyclic group. Indeed

$$ab = (ba, b^2)\mathsf{e},$$

$$ba = (ab, b^2)\mathsf{e},$$

so a and b commute. Secondly

$$b^2a = (b^2a, b^2a),$$

which implies the triviality of b^2a.

Therefore the group is cyclic. The preceding relation ensures that the order of a is twice the order of b. But a and b have the same order according to the relation $a = (b,b)$. As a and b are non-trivial it implies that the group is $\mathbb{Z}$.

(iii) Case $a = (a,b)$.

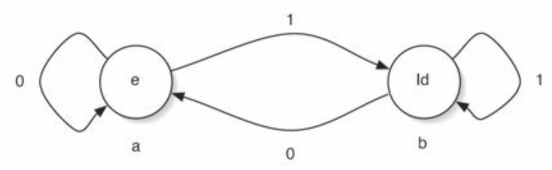

FIGURE 2. The automaton which generates the lamplighter group

By considering if necessary the inverse automaton (which generates the same group and does not change a) we can suppose that b satisfies one of three possibilities: $b = (b, b)\mathsf{e}$, $b = (a, b)\mathsf{e}$ or else $b = (a, a)\mathsf{e}$.

In the first case $b^2 = (b^2, b^2)$ so b is of order 2. As $a^2 = (a^2, b^2)$, a is also of order two. The relations $a^{-1}b = (a^{-1}b, 1)\mathsf{e}$ and $(a^{-1}b)^2 = (a^{-1}b, a^{-1}b)$ imply that $a^{-1}b$ is of infinite order. Therefore it is the infinite dihedral group $\mathbb{D}_\infty$.

The second case corresponds to the lamplighter group (see the next section).

The third case can be analyzed in a similar way.

1.5. Algorithmic problems. An important aspect of groups generated by finite automata is the existence of very effective algorithms. Here we concentrate on the word problem. This constitutes an important criterion for deciding if some groups are generated by automata.

The word problem has a solution for every group generated by a finite automaton due to the algorithm presented below.

PROPOSITION 1.4. *The word problem is solvable for automata groups.*

PROOF. Let w be a word over the alphabet composed of the labelings of the states of the automaton and their inverses.

1. Verify if $w \in \mathrm{St}_G(1)$ (otherwise $w \neq 1$ in G).
2. Compute $w = (w_0, \ldots, w_{d-1})$. Then

$$w = 1$$

in G iff $w_i = 1$ in G for $i = 0, \ldots, d-1$. Go to 1. by replacing w by w_i and proceeding with every w_i as with w.

If in some step we obtain a word which is not in $\mathrm{St}_G(1)$ then $w \neq 1$ in G. If in every step all the words $w_{i_1}, \ldots, w_{i_n}$ already appeared in the algorithm then $w = 1$ in G.

This algorithm converges because the lengths of w_0, …, w_{d-1} are at most the length of w and after sufficiently many steps there is repetition of the words.

1.6. Important examples. In the following sections we present important examples of automata groups, namely the lamplighter group, a group generated by a three state automaton, the Wilson group and the Aleshin group.

There are several other groups which played an important role in the development of the theory. Let us mention the Fabrykowski-Gupta group [**14**], the group of Sushchansky [**47**] and the group of Gupta-Sidki [**31**].

For a general theory of automata groups one can refer to [**2**], [**18**] and [**50**].

2. L^2 Betti numbers of closed manifolds

2.1. A question of Atiyah. In 1976 Atiyah [**3**] introduced for a closed Riemannian manifold (M, g) with the universal covering $\widetilde{M}$ the analytic L^2-Betti numbers $b^p_{(2)}(M, g)$ which measure the size of the space of harmonic square-integrable p-forms on $\widetilde{M}$. Let $k_p(x, y)$ be the (smooth) integral kernel of the orthogonal projection of all square integrable forms onto this subspace. On the diagonal, the fiberwise trace $\mathrm{tr}_x k_p(x, x)$ is defined and is invariant under deck transformations. It therefore defines a smooth function on M, and Atiyah sets $b^p_{(2)}(M, g) := \int_M \mathrm{tr}_x k_p(x, x)\, dx$. By a result of Dodziuk [**12**] this does not depend on the metric.

A priori, the L^2-Betti numbers are non-negative real numbers. However, we can express the Euler characteristic $\chi(M)$, an integer, in terms of the L^2-Betti numbers in the usual way:

$$\chi(M) = \sum_{p=0}^{\infty} (-1)^p b^p_{(2)}(M).$$

If $\pi = \pi_1(M)$ is a finite group, then the L^2-Betti numbers can be expressed in terms of ordinary Betti numbers as follows: $b^p_{(2)}(M) = \frac{1}{|\pi|} b^p(\widetilde{M})$.

Atiyah ended his paper with a question about the values of these numbers. Later this question gave rise to the so-called Atiyah conjecture.

For a group Γ we denote by $\mathrm{fin}^{-1}(\Gamma)$ the subgroup of $\mathbb{Q}$ generated by the inverses of the orders of finite subgroups of Γ. For a closed manifold M we denote by $b^i_{(2)}(M)$ its i-th L^2 Betti number.

Conjecture. — *Let M be a closed manifold whose fundamental group $\pi_1(M)$ is isomorphic to Γ. Then*

$$b^i_{(2)}(M) \in \mathrm{fin}^{-1}(\Gamma)$$

for every integer i.

The Atiyah conjecture can be equivalently formulated in terms of the dimension of the proper subspaces of the operators in $\mathbb{Z}[G]$ acting on $\ell^2(G)$ where $G = \pi_1(M)$. If G is a finitely presented group and A a random walk operator on G, there is a construction of a closed manifold M with fundamental group G and such that the third L^2 Betti number of M is equal to the von Neumann dimension of the kernel of the operator A.

This problem is closely related to the Kaplansky zero-divisor problem. Let us recall that conjecturally for any torsion-free group Γ, and for every $A, B \in \mathbb{Z}[\Gamma]$ such that

$$AB = 0, \tag{2.1}$$

either $A = 0$ or $B = 0$.

In the problem concerning L^2 Betti numbers we look at the equation

$$Ab = 0 \tag{2.2}$$

where $A \in \mathbb{Z}[\Gamma]$ and $b \in l^2(\Gamma)$ and one can ask whether this equation implies that either $A = 0$ or $b = 0$.

For some classes of groups it can be shown that the two questions are equivalent. For instance, for an amenable group the equation (2.2) for $b \neq 0$ implies that there exists $0 \neq B \in \mathbb{Z}[\Gamma]$ such that (2.1) holds.

Another result which one could mention concerns left-orderable groups. It is easy to see that such groups satisfy the Kaplansky zero-divisor conjecture. In [**37**] it was proven that this generalizes to the equation (2.2).

Let us mention that the L^2 condition is essential. For instance even for a free group $F_2 = \langle a, b \rangle$ there exists $f \neq 0$ such that

$$((a^{-1} + a + b^{-1} + b)^2 - 12)f = 0 \tag{2.3}$$

and $f \in l^{2+\varepsilon}(F_2)$ for every $\varepsilon > 0$.

Indeed, one can even write f explicitly, namely

$$f(\gamma) = \left(\frac{1}{\sqrt{3}}\right)^{|\gamma|}$$

satisfies the equation (2.3).

As the number of elements in Γ of norm n is equal to $4 \cdot 3^{n-1}$ we deduce that $f \in l^{2+\varepsilon}(F_2)$ for every $\varepsilon > 0$.

There are several texts presenting results about this conjecture; the most recent one is the book by Lück [**33**] and many results confirm different forms of the Atiyah conjecture.

The above conjecture is proved in many important cases, like the class $\mathcal{C}$ of Linnell which includes extensions of free groups with elementary amenable quotients, residually torsion-free elementary amenable groups and poly-free groups. It is known that the class of groups for which the Atiyah conjecture holds is closed under HNN-extensions, as long as $\mathrm{fin}^{-1}(\pi)$ is discrete. It follows that it holds for all subgroups of one-relator groups, and for all subgroups of right-angled Coxeter groups. It is also proven that the class of all torsion-free groups for which the Atiyah conjecture holds is closed under taking extension by groups in a certain large class, namely the smallest class which contains all the torsion-free, elementary amenable groups, and contains all the free groups, and is closed under taking subgroups, extensions, directed unions, amalgamated free products, and HNN-extensions.

However, we show that the strong version mentioned above is false [**27**].

THEOREM 2.1. *Let G be the group given by the presentation*

$$G = \langle a, t, s \mid a^2 = 1, [t, s] = 1, [t^{-1}at, a] = 1, s^{-1}as = at^{-1}at \rangle. \tag{2.4}$$

Every finite subgroup of G is an abelian 2-group, in particular the order of any finite subgroup of G is a power of 2. There exists a closed Riemannian manifold (M, g) of dimension 7 such that $\pi_1(M) = G$ for which the third L^2 Betti number is equal to

$$b^3_{(2)}(M, g) = \frac{1}{3}.$$

The computation of the spectral measure which is presented later should be compared with an approximation method for L^2 Betti numbers for residually finite groups.

Namely, let M be a closed manifold and let Γ be its fundamental group. We consider the case when Γ is residually finite, i.e. there exists a sequence of finite index subgroups Γ_n of Γ such that $\Gamma_n > \Gamma_{n+1}$ and $\bigcap \Gamma_n = id$.

Moreover, we ask for the property that Γ_i are normal subgroups of Γ (this condition can always be achieved for residually finite groups). Then

$$b^i_{(2)}(M) = \lim_{n\to\infty} \frac{b^i(\widetilde{M}/\Gamma_n)}{[\Gamma : \Gamma_n]}$$

where $\widetilde{M}$ is the universal cover of M, b^i denotes the usual i-th Betti number and $[\Gamma : \Gamma_n]$ is the index of Γ_n in Γ.

This result is due to Lück [**34**]. There is also a version of this result due to Farber [**16**] where Γ_n is not necessarily a normal subgroup of Γ but only in some asymptotic sense.

There are also approximation results for L^2 Betti numbers by some finite dimensional kernels for amenable groups. In this case the approximating sequence is defined using Følner sets (more information about amenable groups can be found in Section 3).

2.2. The lamplighter group as an automaton group. The automaton group from Figure 2 generates the lamplighter group [**24**]. This group can be defined as the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ or as a semi-direct product $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ with the action of $\mathbb{Z}$ on $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ by translation.

Let a and b be the generators of the lamplighter group $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ such that $a = (f_a, g_a)$, $b = (f_b, g_b)$, where $g_a = g_b \in \mathbb{Z}$ is a generator of $\mathbb{Z}$, $f_a \in \oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is the identity and $f_b = (\ldots, 0, 0, 1, 0, 0, \ldots) \in \oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is such that 1 is in position 1. There is an isomorphism between this group and the group generated by the automaton from Figure 2, where a and b correspond to the initial states of the automaton.

2.3. Operator reccurence. Let now Γ be the group generated by the automaton from Figure 2. We denote by $\partial T = E_0 \sqcup E_1$ the partition of the boundary ∂T associated to the subtrees T_0 et T_1 issuing from two vertices on the first level. We have the isomorphism $L^2(\partial T, \mu) \simeq L^2(E_0, \mu_0) \oplus L^2(E_1, \mu_1)$ where μ_i is the restriction of μ to E_i, as well as the isomorphism $L^2(\partial T, \mu) \simeq L^2(E_i, \mu_i)$, for $i = 0, 1$, coming from $T \simeq T_i$.

In this way we get an isomorphism between $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of infinite dimension. Thanks to this isomorphism, the operators $\pi(a)$, $\pi(b)$ (also denoted by a and b, respectively), where π is a representation as in Section 1.3, satisfy the following operator relations:

$$a = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

which correspond to the wreath product relations: $a = (a, b)\mathsf{e}$ and $b = (a, b)$.

Let π_n be a permutation representation of the group G induced by the action of G on the level n of the associated tree and let $\mathcal{H}_n$ be the space of functions on the n-th level. Let a_n and b_n be the matrices corresponding to generators for the representation π_n. Then $a_0 = b_0 = 1$ and

$$a_n = \begin{pmatrix} 0 & a_{n-1} \\ b_{n-1} & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix} \tag{2.5}$$

keeping in mind the natural isomorphism $\mathcal{H}_n \simeq \mathcal{H}_{n-1} \oplus \mathcal{H}_{n-1}$.

2.4. The lamplighter group and spectral measure. We are interested in the spectrum and the spectral measure of the Markov operator for the lamplighter group.

For a finite generating subset S of G which is symmetric ($S = S^{-1}$) we consider the simple random walk on the Cayley graph $\mathrm{Cay}(G, S)$. Then the random walk operator $A : \ell^2(G) \to \ell^2(G)$ is defined by

$$Af(g) = \frac{1}{|S|} \sum_{s \in S} f(sg),$$

where $f \in \ell^2(G)$ and $g \in G$.

As the operator A is bounded (we have $\|A\| \leq 1$) and self-adjoint, it admits a spectral decomposition

$$A = \int_{-1}^{1} \lambda \mathrm{d}E(\lambda),$$

where E is a spectral measure. This measure is defined on the Borel subsets of the interval $[-1, 1]$ and takes its values in the space of projectors of the Hilbert space $\ell^2(G)$. The Kesten spectral measure μ on the interval $[-1, 1]$ is defined by

$$\mu(B) = \langle E(B)\delta_{\mathrm{Id}}, \delta_{\mathrm{Id}} \rangle,$$

where B is a Borel subset of $[-1, 1]$ and $\delta_{\mathrm{Id}} \in \ell^2(G)$ is a function equal to 1 for the identity element and 0 elsewhere.

For a closed and G-invariant subspace H of $\ell^2(G)$ we define its von Neumann dimension $\dim(H)$ as

$$\dim(H) = \langle \mathrm{pr}_H \delta_{\mathrm{Id}}, \delta_{\mathrm{Id}} \rangle,$$

where pr_H is a projection of $\ell^2(G)$ on H.

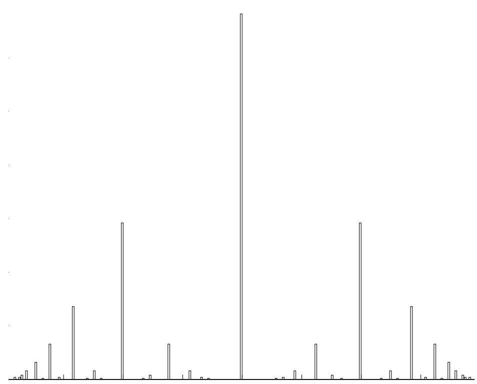

FIGURE 3. The histogram of the spectrum of $a + a^{-1} + b + b^{-1}$

For the lamplighter group we can compute this measure [**24**]:

THEOREM 2.2. *Let Γ be the group defined by the automaton from Figure 2, with generators a et b. The random walk operator A on $\ell^2(\Gamma)$ has the following eigenvalues:*

$$\cos\left(\frac{l}{q}\pi\right)$$

where $q = 2, 3, 4, \ldots$ *and* $l = 1, \ldots, q-1$.

The von Neumann dimension of the corresponding eigenspace is

$$\dim\left(\ker\left(A - \cos\left(\frac{l}{q}\pi\right)\right)\right) = \frac{1}{2^q - 1}$$

where $(l, q) = 1$.

In order to prove this theorem we use finite dimensional approximations π_n described above and first compute the eigenvalues of the matrices $a_n + a_n^{-1} + b_n + b_n^{-1}$.

Let us introduce the following matrix:

$$S_{n+1} = \begin{pmatrix} 0 & \mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & 0 \end{pmatrix}$$

for $n \geq 0$ and let $S_0 = \mathrm{Id}$. So $S_n = a_n^{-1} b_n = b_n^{-1} a_n$.

For $n \geq 0$ let us define

$$\Phi_n(\lambda, \mu) = \det(a_n + b_n + a_n^{-1} + b_n^{-1} - \lambda \mathrm{Id}_{2^n} - \mu S_n),$$

where λ and μ are complex parameters.

In particular

$$\Phi_0 = 4 - \lambda - \mu,$$

$$\Phi_1 = (\mu - \lambda)(4 - \lambda - \mu).$$

Proposition 2.3. *If* $n \geq 1$ *the following recursion holds:*

$$\Phi_{n+1}(\lambda, \mu) = (\mu - \lambda)^{2^n} \Phi_n\left(-\frac{\lambda^2 - \mu^2 - 2}{\mu - \lambda}, -\frac{2}{\mu - \lambda}\right).$$

Proof. In the proof we will use the following simple fact: let A, B, C and D be n by n matrices with complex coefficients such that $AC = CA$. Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Thus one obtains:

$$\begin{aligned}
\Phi_{n+1}(\lambda,\mu) &= \det(a_{n+1}+b_{n+1}+a_{n+1}^{-1}+b_{n+1}^{-1}-\lambda \mathrm{Id}_{2^{n+1}}-\mu S_{n+1}) \\
&= \det\begin{pmatrix} a_n+a_n^{-1}-\lambda & a_n+b_n^{-1}-\mu \\ b_n+a_n^{-1}-\mu & b_n+b_n^{-1}-\lambda \end{pmatrix} \\
&= \det\begin{pmatrix} a_n^{-1}-b_n^{-1}-\lambda+\mu & a_n+b_n^{-1}-\mu \\ a_n^{-1}-b_n^{-1}+\lambda-\mu & b_n+b_n^{-1}-\lambda \end{pmatrix} \\
&= \det\begin{pmatrix} -2\lambda+2\mu & a_n-b_n-\mu+\lambda \\ a_n^{-1}-b_n^{-1}+\lambda-\mu & b_n+b_n^{-1}-\lambda \end{pmatrix} \\
&= \det((-2\lambda+2\mu)(b_n+b_n^{-1}-\lambda) \\
&\quad -(a_n^{-1}-b_n^{-1}+\lambda-\mu)(a_n-b_n-\mu+\lambda)) \\
&= \det((-\lambda+\mu)(a_n+b_n+a_n^{-1}+b_n^{-1})+\lambda^2-\mu^2-2 \\
&\quad +(a_n^{-1}b_n+b_n^{-1}a_n)) \\
&= \det((\mu-\lambda)\mathrm{Id}_{2^n}) \\
&\quad \det\left(a_n+b_n+a_n^{-1}+b_n^{-1}+\frac{\lambda^2-\mu^2-2}{\mu-\lambda}\mathrm{Id}_{2^n}+\frac{2}{\mu-\lambda}S_n\right) \\
&= (\mu-\lambda)^{2^n}\Phi_n\left(-\frac{\lambda^2-\mu^2-2}{\mu-\lambda},-\frac{2}{\mu-\lambda}\right).
\end{aligned}$$

□

This implies the following expression for the determinant:

$$\begin{aligned}
& \det(a_n+b_n+a_n^{-1}+b_n^{-1}-4\cos z\cdot \mathrm{Id}_{2^n}) \\
=\; & \Phi_n(4\cos z,0) \\
=\; & (4-4\cos z)\left(\frac{1}{\sin(z)}\right)^{2^{n-1}} 2^n\prod_{k=2}^{n}(\sin(zk))^{2^{n-k}}\sin(z(n+1)).
\end{aligned}$$

From this it is easy to see that the spectrum of the operator $a_n+b_n+a_n^{-1}+b_n^{-1}$ is equal to

$$\begin{aligned}
& \mathrm{Sp}(a_n+b_n+a_n^{-1}+b_n^{-1}) \\
=\; & \left\{4\cup 4\cos\left(\frac{p}{q}\pi\right); q=2,\dots,n+1, 1\le p<q\right\}
\end{aligned}$$

The operators $a_n+b_n+a_n^{-1}+b_n^{-1}$ approximate the operator $a+b+a^{-1}+b^{-1}$ and in our situation from the above computation one can deduce Theorem 2.2.

For a general approach to the approximation results for automata groups in the setting of C^* algebras see [**45**].

2.5. Construction of a manifold. The computation of the spectral measure has several applications to random walks. In the following section we present an application of this computation to a problem of Atiyah about L^2 Betti numbers of closed manifolds.

The proof of Theorem 2.1 relies on the results explained earlier concerning the spectral measure of the random walk operator A on the lamplighter group, for which G is an HNN extension. The results imply

$$\dim\bigl(\ker(A)\bigr) = \frac{1}{3},$$

but the denominator 3 does not divide the powers of 2, which are the orders of finite subgroups of the lamplighter group. We present a construction of a manifold from [**28**].

The lamplighter group is not finitely presented. However, it admits a recursive presentation and therefore is a subgroup of a finitely presented group.

Let Γ denote the lamplighter group $(\oplus_{i\in\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\oplus_{i\in\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ by translation. Γ is generated by $t \in \mathbb{Z}$ and by $a = (\dots, 0, 1, 0, \dots) \in \oplus_{i\in\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ and has the presentation

$$\Gamma = \langle a, t \mid a^2 = 1, [t^{-k}at^k, t^{-n}at^n] = 1 \;\forall k, n \in \mathbb{Z}\rangle.$$

LEMMA 2.4. *Let $\alpha\colon \Gamma \to \Gamma$ be given by $\alpha(t) = t$ and $\alpha(a) = at^{-1}at$. This defines an injective group homomorphism, and G is the ascending HNN-extension of Γ along α. Moreover G' is isomorphic to a countable direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$.*

PROOF. The first assertion can be easily checked. The second part follows from the computation given below.

Let V be the HNN-extension of Γ along α. Then V has the presentation

$$\begin{aligned} V \;=\; &\langle a, t, s \mid a^2 = 1, [s,t] = 1, s^{-1}as = at^{-1}at = [a,t], \\ &[t^{-k}at^k, t^{-n}at^n] = 1 \;\forall k, n \in \mathbb{Z}\rangle. \end{aligned}$$

Obviously, we have an epimorphism of G onto V mapping a to a, s to s, and t to t. It only remains to show that every relation in the given presentation of V follows from the relations of G. Observe first in G that by conjugation with t^{-n}, $[t^{-k+n}at^{-n+k}, a] = 1$ implies $[t^{-k}at^k, t^{-n}at^n] = 1$. Moreover, in the relation of commutativity we can reverse the order of elements, i.e. $[t^{-k}at^k, t^{-n}at^n] = 1$ implies $[t^{-n}at^n, t^{-k}at^k] = 1$. Hence, it remains to prove $[t^{-n}at^n, a] = 1$ in G for $n > 1$. We will do this by induction on n. Assume therefore $t^{-j}at^j$ commutes with $t^{-l}at^l$ for $0 \le j \le l < n$. Conjugate the relation $[t^{-(n-1)}at^{n-1}, a] = 1$ with s. We obtain

$$1 = [t^{-(n-1)}at^{-1}att^{n-1}, at^{-1}at] = [(t^{-(n-1)}at^{n-1})(t^{-n}at^n), a(t^{-1}at)]. \tag{2.6}$$

Now observe that by induction a commutes with $a_1 := t^{-1}at$ and with $a_{n-1}t^{-(n-1)}at^{n-1}$. This second relation also implies (by conjugation with t^{-1}) that moreover, $t^{-1}at$ commutes with $a_n := t^{-n}at^n$. Therefore, we can simplify the commutator in (2.6) to the desired

$$\begin{aligned} 1 \;&=\; (a_n^{-1}a_{n-1}^{-1})(a_1^{-1}a^{-1})(a_{n-1}a_n)(aa_1) = a_n^{-1}(a_{n-1}^{-1}a_{n-1})(a_1^{-1}a_1)a^{-1}a_na \\ &=\; [t^{-n}at^n, a]. \end{aligned}$$

By induction we therefore see that $V = G$.

Using the presentation, we next check that the abelianization of V is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and s, t are mapped to two free generators, whereas a is mapped to zero. Therefore, G' is equal to the normal subgroup generated by a, which is generated by $s^{-l}t^{-k}at^ks^l$, $k, l \in \mathbb{Z}$, $l < 0$. All these elements are of order 2, and by conjugation with sufficiently high powers of s we see that they all commute. Therefore, G' is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with countably many generators, and therefore isomorphic to a countable direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$. Observe, however, that G' is quite different from the base of the HNN-extension Γ. The element sas^{-1} is a typical example which is not contained in Γ but in G'. □

Since, by Lemma 2.4, G is a two-step HNN-extension of $\oplus_{i\in\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$, it follows immediately that all finite subgroups of G are elementary abelian 2-groups. To prove Theorem 2.1, we need to construct M.

As a corollary of Theorem 2.2, we obtain:

Corollary 2.5. *There is an $A \in \mathbb{Z}G$ such that*

$$\dim_G \ker\big(A\colon l^2(G) \to l^2(G)\big) = \frac{1}{3}.$$

Proof. Observe that if A is induced from Γ, i.e. $A \in \mathbb{C}\Gamma$ (so we can view A also as an operator on $l^2(\Gamma)$), then essentially $\mathrm{pr}_\Gamma = \mathrm{pr}_G$ and we deduce that $\dim_\Gamma \ker(A) = \dim_G \ker(A)$. Therefore, it will be sufficient to find $A \in \mathbb{Z}\Gamma$ such that $\dim_\Gamma \ker A = 1/3$.

Take A of Theorem 2.2. Choosing $p = 1$ and $q = 2$, we see that 0 is in the spectrum of A, and that $\dim_\Gamma(\ker A) = 1/3$. □

Proposition 2.6. *There is a 3-dimensional finite CW-complex X with $\pi_1(X) = G$ and with $b_3^{(2)}(X) = \frac{1}{3}$.*

Proof. We perform a standard construction where one attaching map will be given by the A of Corollary 2.5.

Let X' be a finite 2-dimensional CW-complex with $\pi_1(X') = G$, e.g. the 2-complex of the finite presentation given above. Let X'' be the wedge product of X' and S^2. The corresponding map $\alpha\colon S^2 \to X''$ generates a free copy of $\mathbb{Z}[\pi_1(X'')] = \mathbb{Z}[G]$ inside $\pi_2(X'')$. Define now $X := X'' \cup_f D^3$, where $(f\colon S^2 \to X'') \in \pi_2(X'')$ is given by $A \in \mathbb{Z}[G]$ of Corollary 2.5, and where $\mathbb{Z}[G] \hookrightarrow \pi_2(X'')$ is given using α. Choosing an appropriate basis of cells, it follows that on the cellular L^2-chain complex $C_*^{(2)}(\widetilde{X}) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} l^2(G)$ of the universal covering $\widetilde{X}$ of X, the differential d_3

$$l^2(G) \cong C_3^{(2)}(\widetilde{X}) \xrightarrow{d_3} C_2^{(2)}(\widetilde{X}) \cong (l^2(G))^n$$

is given by the matrix $(A, 0, \ldots, 0)^t$, where t denotes transpose and n is the number of 2-cells in X. Since there are no 4-cells, d_4 is zero. Consequently,

$$b_3^{(2)}(X) = \dim_G(\ker d_3) = \dim_G(\ker A) = \frac{1}{3}.$$

We now can finish the proof of Theorem 2.1.

Choose a finite 3-dimensional simplicial complex Y homotopy equivalent to the CW-complex X of Proposition 2.6. Then embed Y into $\mathbb{R}^8$ and thicken Y to a homotopy equivalent 8-dimensional compact smooth manifold W with boundary M. By transversality, the inclusion $M \hookrightarrow W$ is a 4-equivalence. Therefore, $b_3^{(2)}(M) = b_3^{(2)}(W) = b_3^{(2)}(X) = \frac{1}{3}$ and $\pi_1(M) = \pi_1(X) = G$. If we choose a smooth Riemannian metric g on M, then by the L^2-Hodge de Rham theorem we also obtain $b_3^{(2)}(M, g) = \frac{1}{3}$.

Remark 2.7. The dimension of the manifold which is a counterexample to the strong Atiyah conjecture can be reduced to 6 as follows:

It is known that $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, Γ, G, and the direct limit of $\Gamma \xrightarrow{\alpha} \Gamma \xrightarrow{\alpha} \cdots$ all have vanishing L^2-Betti numbers in all degrees. Moreover, the zeroth and first L^2-Betti number of a space are equal to the first L^2-Betti number of its fundamental group, i.e. $b_1^{(2)}(G) = b_1^{(2)}(X) = b_1^{(2)}(M) = 0 = b_0^{(2)}(X) = b_0^{(2)}(M)$.

The CW-complex X has 1 zero-cell, 1 three-cell, and 3 one-cells and 5 two-cells (using the presentation of G given in Theorem 2.1). Consequently, $\chi(X) = 2$. Since

$$2 = \chi(X) = \sum_{k=0}^{3}(-1)^k b^k_{(2)}(X) = b^2_{(2)}(X) - b^3_{(2)}(X) b^2_{(2)}(X) - 1/3,$$

we have $b^2_{(2)}(X) = 7/3$.

Now we can do the same construction as in the proof of Theorem 2.1, but embed Y into $\mathbb{R}^7$ instead of $\mathbb{R}^8$. The inclusion of the boundary M' of the regular neighborhood W' into W' will now only be a 3-equivalence, but this is enough to conclude that $b^2_{(2)}(M') = b^2_{(2)}(W') = b^2_{(2)}(X) = 7/3$, and the denominator still is not a power of 2, giving the desired counterexample.

Using the Künneth formula and Poincaré duality for L^2-cohomology, one can on the other hand easily arrange that the dimension of a counterexample, as well as the degree of the Betti number which contradicts the strong Atiyah conjecture, is arbitrarily high.

Recently in [**4**] an uncountable family of lamplighter like groups was constructed for which one obtained different von Neumann dimensions of the kernel of some element in the group algebra, in particular some of these dimensions are irrational. A similar construction [**44**] (see also [**21**]) led to a construction of a closed manifold with irrational L^2 Betti numbers.

3. Exotic amenable groups

3.1. Amenability. In 1929 von Neumann [**41**] defined the notion of amenability, which became fundamental.

Definition 3.1. The group G is amenable if there is a measure μ defined on all subsets of G such that

- $\mu(G) = 1$;
- $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subset G$;
- $\mu(gA) = \mu(A)$ for every $g \in G$ and every $A \subset G$.

The origin of this definition is related to the Banach-Tarski paradoxical decomposition of the sphere.

Namely, a group Γ is non-amenable if and only if it admits a paradoxical decomposition, i.e. there exist $A_1, \ldots, A_n, B_1, \ldots, B_k \subset \Gamma$ and $g_1, \ldots, g_n, h_1, \ldots, h_k \in \Gamma$ such that

$$A_1 \dot\cup \cdots \dot\cup A_n \dot\cup B_1 \dot\cup \cdots \dot\cup B_k = \Gamma$$

and

$$g_1(A_1) \dot\cup \cdots \dot\cup g_n(A_n) = \Gamma$$

$$h_1(B_1) \dot\cup \cdots \dot\cup h_k(B_k) = \Gamma.$$

Such a decomposition clearly contradicts the existence of a left-invariant measure μ. For some groups one can construct it explicitly. For instance for a free group $F_2 = \langle a, b \rangle$ one can consider

$$\begin{aligned} A_1 &= \text{words which start with } a \\ A_2 &= \text{words which start with } a^{-1} \\ B_1 &= \text{words which start with } b \text{ and all powers of } b \\ B_2 &= \text{words which start with } b^{-1} \text{ but not powers of } b \end{aligned}$$

and $g_1 = \mathrm{id}$, $g_2 = a$, $h_1 = \mathrm{id}$ and $h_2 = b$.

This notion became fundamental in the theory of infinite groups. There are several important characterizations of amenability. One of the most important is in terms of so-called Følner sets:

THEOREM 3.2 ([**17**]). *A countable group G is amenable if and only if there is a sequence of finite subsets A_n of G such that for every $g \in G$*

$$\lim_{n\to\infty} |A_n \triangle g A_n| / |A_n| = 0,$$

where $\triangle$ denotes a symmetric difference.

From this it follows that the groups of subexponential growth are amenable and that this class is closed under the following elementary operations: extensions, quotients, subgroups and direct limits.

Before the construction of the group generated by an automaton from Figure 4, all groups known to be amenable could be obtained from groups of subexponential growth using the elementary operations described above. For the history of different conjectures concerning the class of amenable groups see [**26**]; the first reference is the paper of Day [**10**].

3.2. Group generated by an automaton on three states. We are interested in the group generated by the automaton on three states introduced in [**25**].

Namely we consider the following three state automaton $a = (c, b)$, $b = (c, a)\mathsf{e}$, $c = (c, c)$. The state c corresponds to the identity, so that the automaton group is defined by the following wreath product relations $a = (1, b)$, $b = (1, a)\mathsf{e}$.

This group appears also as the Galois group of iteration of the polynomial x^2-1 over finite fields (Pink) and as a monodromy group of the ramified covering of the Riemann sphere given by the polynomial $z^2 - 1$ (see [**40**]).

One of the remarkable properties of this group is related to the amenability.

3.3. Algebraic properties of G.

THEOREM 3.3 ([**25**]). *Let G be the group generated by the automaton from Figure 4.*

The group G has following properties:

a) *it is fractal;*

b) *it is regularly weakly branched over G';*

c) *it has no torsion;*

d) *the semigroup generated by a and b is free;*

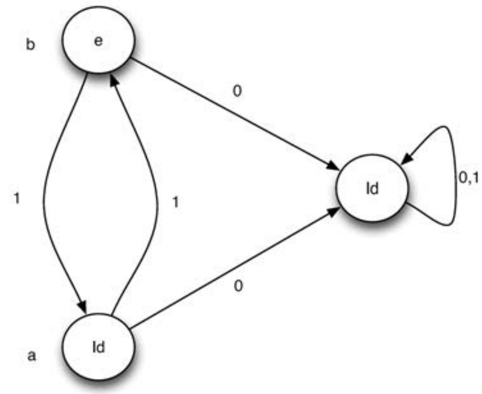

FIGURE 4. An automaton on three states

e) *it admits a presentation*

$$G = \langle a, b | \sigma^{\varepsilon}(\theta^m([a, a^b])) = 1, m = 0, 1, \dots, \varepsilon = 0, 1 \rangle,$$

where

$$\sigma : \begin{cases} a & \mapsto & b^2 \\ b & \mapsto & a \end{cases} \qquad \theta : \begin{cases} a & \mapsto & a^{b^2+1} \\ b & \mapsto & b. \end{cases}$$

We present here proofs of some algebraic properties of G mentioned in Theorem 3.3.

For $G = \langle a, b \rangle$, we have the relations $a = (1, b)$ and $b = (1, a)\mathsf{e}$.

PROPOSITION 3.4. *The group G is fractal.*

PROOF. We have

$$\mathrm{St}_G(1) = \langle a, a^b, b^2 \rangle.$$

But

$$\begin{array}{rcl} a & = & (1, b) \\ a^b & = & \mathsf{e}(1, a^{-1})(1, b)(1, a)\mathsf{e} = (b^a, 1) \\ b^2 & = & (a, a), \end{array} \tag{3.7}$$

and either of the images of the two projections of $\mathrm{St}_G(1)$ is G, i.e. G is fractal.

PROPOSITION 3.5. *The group G is regularly weakly branched over G', i.e.*

$$G' \geq G' \times G'.$$

PROOF. Indeed, as

$$[a, b^2] = (1, [b, a]),$$

using fractalness of G, we get $G' \geq \langle [a, b^2] \rangle^G \geq 1 \times \langle [b, a] \rangle^G = 1 \times G'$ and $(1 \times G')^b = G' \times 1$. Thus G' contains $G' \times G'$ and as $G' \neq 1$ the group G is regularly weakly branched over G'.

LEMMA 3.6. *The semigroup generated by a and b is free.*

Proof. Consider two different words $U(a,b)$ and $V(a,b)$ which represent the same element and such that $\rho = \max\{|U|,|V|\}$ is minimal. A direct verification shows that ρ cannot be either 0 or 1.

Suppose that $|U|_b$, the number of occurrences of b in U, is even (and thus $|V|_b$ is as well). If this is not the case we can consider the words bU and bV, thus increasing ρ by 1.

Now U and V are products of

$$a^m = (1, b^m)$$

and

$$ba^m b = (1,a)\mathsf{e}(1,b^m)(1,a)\mathsf{e} = (b^m a, a).$$

If one of these words has no b, say $U = a^m$, after projecting U and V on the first coordinate we get $1 = V_0$ where the projection V_0 is a non-empty word verifying $|V_0| < |V| \leq \rho$. This contradicts the minimality of U and V.

We consider now the situation where b appears in both words at least twice. If the number of occurrences of b in U and V were one, then by minimality they have to be equal to ba^n and $a^m b$. But $ba^n = (b^n, a)\mathsf{e}$ et $a^m b = (1, b^m a)\mathsf{e}$ which shows that these words are different and represent different elements.

Thus both words contain two b's and $|U|,|V| \leq \rho$, where one of them contains at least four b's and $|U|,|V| \leq \rho + 1$.

If we consider projections of U and V on the second coordinate, we get two different words (because of the minimality of U and V, they have to end with different letters) and of lengths which are shorter. This contradicts the minimality of ρ.

Lemma 3.7. *We have the following relation:*

$$\gamma_3(G) = (\gamma_3(G) \times \gamma_3(G)) \rtimes \langle [[a,b],b] \rangle$$

where $\gamma_3(G) = [[G,G],G]$.

Proof. We start with the relations

$$\gamma_3(G) = \langle [[a,b],a], [[a,b],b] \rangle^G,$$

$$[[a,b],a] = [(b^a, b^{-1}), (1,b)] = 1,$$

$$[[a,b],b] = (b^{-a}, b)\mathsf{e}(1,a^{-1})(b^a, b^{-1})(1,a)\mathsf{e} = (b^{-a}, b)(b^{-a}, b^a) = (b^{-2a}, bb^a). \quad (3.8)$$

The first two imply

$$\gamma_3(G) = \langle [[a,b],b] \rangle^G.$$

Thanks to the relation

$$[a, b^2] = (1, [a,b])$$

we have

$$[[a,b^2],a] = [(1,[a,b]),(1,b)] = (1, [[a,b],b]).$$

Let $\xi = [[a,b],b]$. Direct computations show that $\xi^a, \xi^{a^{-1}}, \xi^b, \xi^{b^{-1}} \in \langle \xi \rangle$ mod $\gamma_3(G) \times \gamma_3(G)$ and $\langle \xi \rangle \cap (\gamma_3(G) \times \gamma_3(G)) = 1$ because of (3.8) and $(bb^a)^n \in G'$ if and only if $n = 0$ and $\gamma_3(G) \leq G'$.

Lemma 3.8. *We have the following relation:*

$$G'' = \gamma_3(G) \times \gamma_3(G).$$

Proof. Let $f = (1, c) \in G$, where $c = [a, b]$. For $d = (b, b^{-1}) \in G'$ we get

$$[f, d^{-1}] = [(1, [a, b]), (b^{-1}, b)] = (1, [[a, b], b]) \in G''.$$

This implies that $G'' \geq 1 \times \gamma_3(G)$and thus $G'' \geq \gamma_3(G) \times \gamma_3(G)$. As $G'' \geq \gamma_3(G)$ according to Lemma 3.7 it is enough to show $\langle \xi \rangle \cap G'' = 1$.

One can easily show

$$G' = (G' \times G') \rtimes \langle c \rangle. \tag{3.9}$$

Using the relation (3.8) and the relation (3.9) we have

$$G'' = \langle [c, f] \rangle^G.$$

But

$$[c, f] = [(b^a, b^{-1}), (1, c)] = [1, [b^{-1}, [a, b]]] \in 1 \times \gamma_3(G).$$

This ends the proof.

Here is another general property of regular weakly branched groups which is easy to prove.

Proposition 3.9. *Let G be a regular weakly branched group over K. Then for every normal subgroup $N \lhd G$ there exists n such that*

$$K'_n < N$$

where $K_n = K \times \cdots \times K$ (direct product of d^n factors, each one acting on the corresponding subtree).

3.4. An interesting dynamical system. Computations of spectra of random walk operators for automata groups often lead to rational maps with interesting dynamical properties. In analogy to the case of the lamplighter group we consider the operators

$$M_n(\lambda, \eta) = \pi_n(a) + \pi_n(a^{-1}) + \lambda(\pi_n(b) + \pi_n(b^{-1})) - \eta I_n.$$

The operators M_n are given by $2^n \times 2^n$ matrices. Let $Q_n = \det M_n$. For instance

$$\begin{aligned}
Q_1(\lambda, \eta) &= 2\eta + 2 - \lambda, \\
Q_2(\lambda, \eta) &= -(2\eta - 2 + \lambda)(2\eta + 2 - \lambda), \\
Q_3(\lambda, \eta) &= (2\eta + 2 - \lambda)(2\eta - 2 + \lambda)(4\eta^2 - \lambda^2 + 4) \\
Q_4(\lambda, \eta) &= -(-2 + \lambda)(2\eta + 2 - \lambda)(2\eta - 2 + \lambda) \\
&\quad (2\lambda^2 - 8 - \lambda^3 + 4\lambda + 4\eta^2\lambda)(4\eta^2 - \lambda^2 + 4)
\end{aligned}$$

We have

Theorem 3.10 ([**26**]). *a) If $n \geq 1$ then*

$$Q_{n+1}(\lambda, \eta) = \lambda^{2^{n+1}} Q_n(F(\lambda, \eta))$$

where

$$F : \begin{cases} \lambda & \to \quad -2 - \frac{\lambda(2-\lambda)}{\eta^2} \\ \eta & \to \quad \frac{\lambda - 2}{\eta^2} \end{cases}.$$

b) The spectrum Σ of $M(\lambda, \eta)$, i.e. the set of pairs (λ, η) (including multiplicities) for which the operator $M(\lambda, \eta)$ is not invertible, is invariant with respect to the map $F : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $F^{-1}(\Sigma) = \Sigma$.

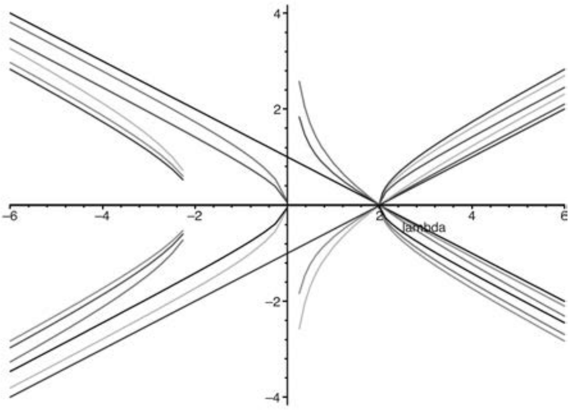

FIGURE 5. Zeros of $Q_n(\lambda, \eta)$

The zeros of $Q_n(\lambda, \eta)$ are given in Figure 5 while the set of accumulation points of the set $\cup_{n=1}^{\infty} F^{-n}(x_0, y_0)$ is shown on Figure 6, suggesting that the map F should have an attractor topology.

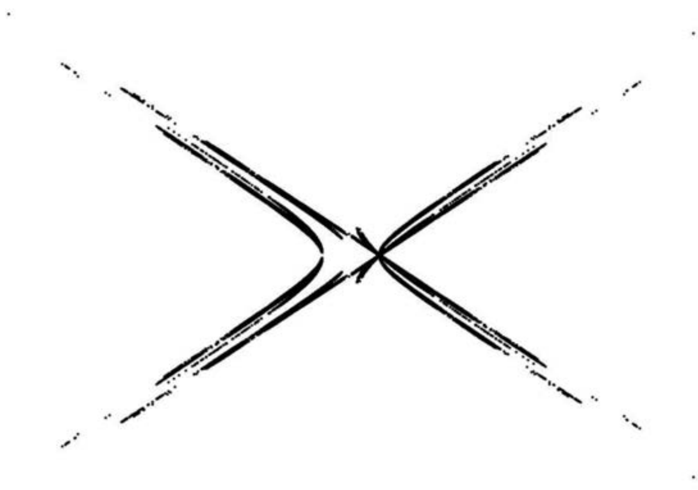

FIGURE 6. The attractor associated to the automaton from Figure 4

3.5. Amenability of the group generated by a three state automaton. Let SG_0 be the class of groups such that all finitely generated subgroups are of subexponential growth. Suppose that $\alpha > 0$ is an ordinal and we have defined SG_β for any ordinal $\beta < \alpha$. Now if α is a limit ordinal let

$$SG_\alpha = \bigcup_{\beta < \alpha} SG_\beta.$$

If α is not a limit ordinal let SG_α be the class of groups which can be obtained from groups in $SG_{\alpha-1}$ using either extensions or direct limits. Let

$$SG = \bigcup_\alpha SG_\alpha.$$

The groups in this class are called subexponentially amenable.

SG is the smallest class of groups which contains groups of subexponential growth and is closed under elementary operations. The classes SG_α are closed under taking subgroups and quotients.

PROPOSITION 3.11 ([**25**]). *The group G is not subexponentially amenable, i.e. $G \notin SG$.*

PROOF. We start with the following lemmas:

LEMMA 3.12. *We have the relation*

$$\psi_1(\gamma_3(G)) = \langle \gamma_3(G), b^{2a} \rangle.$$

PROOF. It is a consequence of Lemma 3.7 and the relation (3.8).

LEMMA 3.13. *We have*

$$\psi_1(\langle \gamma_3(G), b^{2a} \rangle) = \langle \gamma_3(G), b^{2a}, a \rangle.$$

PROOF. It is a consequence of the preceding lemma and the relation $b^{2a} = (a, a^b)$.

LEMMA 3.14. *For the projection on the second coordinate we have*

$$\psi_2(\langle \gamma_3(G), b^{2a}, a \rangle) = G.$$

PROOF. It follows from Lemma 3.7 and the relations $b^{2a} = (a, a^b)$ and $a = (1, b)$.

We can now prove Proposition 3.11. Suppose that $G \in SG_\alpha$ for α minimal. Then α cannot be 0 as G has exponential growth (the semigroup generated by a and b is free according to Lemma 3.6). Moreover, α is not a limit ordinal, for if $G \in SG_\alpha$ for a limit ordinal then $G \in SG_\beta$ for some ordinal $\beta < \alpha$. Also, G is not a direct limit (of an increasing sequence of groups) as it is finitely generated. Thus there exist N, $H \in SG_{\alpha-1}$ such that the following sequence is exact:

$$1 \to N \to G \to H \to 1.$$

Thanks to Proposition 3.9 there exists n such that $N > (\mathrm{Rist}_G(n))' \geq G'' \times \cdots \times G''$ (2^n fois). So $G'' \in SG_{\alpha-1}$ and then $\gamma_3(G) \in SG_{\alpha-1}$ according to Lemma 3.8. Each class SG_α is closed with respect to quotients and subgroups. From Lemmas 3.12, 3.13, 3.14 we deduce that $G \in SG_{\alpha-1}$. Contradiction.

To show amenability of G one uses a criterion of Kesten [**35**] concerning random walks on G.

Let μ be a symmetric probability measure supported on the generating set S of G, i.e. $G = \langle S \rangle$, $\mu(s) = \mu(s^{-1})$ for every $s \in S$ and $\mu(S) = 1$.

Let p_n be the probability of return to the identity after n steps of the random walk given by μ, i.e.

$$p_n(\mathrm{Id}, \mathrm{Id}) = \mu^{*n}(\mathrm{Id})$$

where μ^{*n} is the n-th power of the convolution of μ on G.

Theorem 3.15 (Kesten [**35**]). *The group G is amenable if and only if*

$$\lim_{n\to\infty} \sqrt[2n]{p_{2n}(\mathrm{Id},\mathrm{Id})} = 1.$$

Amenability of G was proven by Virag [**48**]. This proof was published in [**6**].

On G we consider the random walk Z_n given by a symmetric measure μ on $S = \{a, a^{-1}, b, b^{-1}\}$ with weights $\{1, 1, r, r\}$, i.e. $\mu(a^{-1}) = \mu(a) = \frac{1}{2r+2}$, $\mu(b^{-1}) = \mu(b) = \frac{r}{2r+2}$.

The image Z_n by the embedding of G in $G \wr S_2$ is denoted by

$$Z_n = (X_n, Y_n)\varepsilon_n$$

where $X_n, Y_n \in G$ and $\varepsilon_n \in S_2$.

We define the stopping times σ and τ:

$$\begin{aligned}
\sigma(0) &= 0\\
\sigma(m+1) &= \min\{n > \sigma(m) : \varepsilon_n = 1, X_n \neq X_{\sigma(m)}\}\\
\tau(0) &= \min\{n > 0 : \varepsilon_n = \mathsf{e}\}\\
\tau(m+1) &= \min\{n > \tau(m) : \varepsilon_n = \mathsf{e}, Y_n \neq Y_{\tau(m)}\}
\end{aligned}$$

A simple computation shows:

Lemma 3.16. *$X_{\sigma(m)}$ and $Y_{\tau(m)}$ are simple random walks on G according to the distribution $\mu'(a^{-1}) = \mu'(a) = \frac{r}{2r+4}$, $\mu'(b^{-1}) = \mu'(b) = \frac{1}{r+2}$.*

We remark that for $r = \sqrt{2}$ we get the same distribution on Z_n, $X_{\sigma(n)}$ and $Y_{\tau(n)}$.

One can also verify

Lemma 3.17. *Almost surely*

$$\lim_{m\to\infty} \frac{m}{\sigma(m)} = \lim_{m\to\infty} \frac{m}{\tau(m)} = \frac{2+r}{4+4r} < \frac{1}{2}.$$

To conclude we need to modify the distance on G, in order to control the norm of Z_n by the norms of X_n and Y_n.

Let T_n be a finite subtree of n level of T on which acts G. For $g \in G$ we define $|||\cdot|||_{T_n}$ by:

$$|||\, g \,|||_{T_n} = \sum_{\gamma\in\partial T_n} (|g_\gamma| + 1) - 1.$$

Finally we define the distance $|||\ \ |||$ on G :

$$|||\, g \,||| = \min_n |||\, g \,|||_{T_n}.$$

One checks that for $g = (g_0, g_1)\mathsf{e}^{0,1}$

$$|||g_0||| + |||g_1||| \le |||g||| \le |||g_0||| + |||g_1||| + 1$$

and that the growth with respect to the metric $|||\cdot|||$ is at most exponential, i.e. there exists $a > 1$ such that

$$|\{g : |||g||| \le n\}| \le a^n. \tag{3.10}$$

We have

Proposition 3.18. *Almost surely*

$$\lim_{n\to\infty} \frac{|||Z_n|||}{n} = 0.$$

Proof. The existence of the limit, which we denote by s, is a consequence of Kingman's ergodic theorem. Now

$$\frac{|||Z_n|||}{n} \leq \frac{|||X_n|||}{n} + \frac{|||Y_n|||}{n} + \frac{1}{n}.$$

But

$$\lim_{n\to\infty} \frac{|||X_n|||}{n} = \lim_{n\to\infty} \frac{|||X_{\sigma(n)}|||}{\sigma(n)} = \lim_{n\to\infty} \frac{|||X_{\sigma(n)}|||}{n} \lim_{n\to\infty} \frac{n}{\sigma(n)}$$

and similarly for Y_n. So for $r = \sqrt{2}$ if $s > 0$, by Lemma (3.17), $s < s\frac{1}{2} + s\frac{1}{2} = s$. This contradiction implies that $s = 0$.

Proposition 3.19. *The probability* $p(Z_{2n} = \mathrm{Id})$ *does not decay exponentially.*

Proof. For every $\varepsilon > 0$, we have

$$p(|||Z_{2n}||| \leq \varepsilon n) = \sum_{g\in G, |||g|||\leq \varepsilon n} p(Z_{2n} = g) \leq p(Z_{2n} = \mathrm{Id}) \times |\{g \in G; |||g||| \leq \varepsilon n\}|.$$

Thus according to (3.10)

$$p(Z_{2n} = \mathrm{Id}) \geq p\left(\frac{|||Z_n|||}{n} < \varepsilon\right) \cdot a^{-\varepsilon n}.$$

Following Proposition 3.18 and Kesten's criterion the group generated by the automaton from Figure 4 is amenable.

Using HNN extensions of the group G, one can construct amenable finitely presented groups which are subexponentially amenable. In [**26**] we show that the group

$$\widetilde{G} = \langle b, t | [b^{tb}, b^t] = 1, b^{t^2} = b^2 \rangle \tag{3.11}$$

has these properties.

The above proof can be adapted to prove amenability of a wide class of groups generated by automata as was shown in [**8**] and [**5**].

4. Non uniform exponential growth

We present a group constructed by Wilson to solve a problem of Gromov. To define it we use the language of wreath products (see Section 1.2).

4.1. Problem of Gromov. For groups of exponential growth, the growth function depends on the generating set. It is natural to ask if one can define an invariant independent of the generating set.

More precisely, for a group G generated by a finite set S one defines

$$h(G, S) = \lim_{n\to\infty} \sqrt[n]{|\{g \in G : |g|_S \leq n\}|}.$$

The entropy of the group G is then

$$h(G) = \inf_{S;\langle S\rangle = G} h(G, S).$$

In 1981, Gromov [**29**] asked if for every G of exponential growth

$$h(G) > 1,$$

i.e. whether it has uniform exponential growth, which means that there exists $a > 1$ such that for every generating set

$$|\{g \in G : |g|_S \leq n\}| \geq a^n.$$

This very natural problem leads to some interesting results. The answer is positive for several classes of groups like hyperbolic groups, finitely generated linear groups [**13, 7**] and elementary amenable groups of exponential growth [**43**].

The first group without uniform exponential growth was constructed by Wilson in 2003 [**49**].

4.2. Construction of Wilson. Let us consider A_{31}, the alternating subgroup of the symmetric group on 31 elements.

Theorem 4.1. *Let H be a perfect group of finite type which satisfies the property $H \simeq H \wr A_{31}$. Then there exists a sequence (x_n) of elements of order 2 and a sequence (y_n) of elements of order* 3 *such that*

(1) $\langle x_n, y_n \rangle = H$ *for every n* ;
(2) $\lim_{n\to\infty} h(H, \{x_n, y_n\}) = 1$.

Construction of H.

Let T_{31} be a rooted tree of degree 31. Let $x \in \mathrm{Aut}(T_{31})$ be such that it acts nontrivially only on the first level. We define $\overline{x} \in \mathrm{Aut}(T_{31})$ by its image in the wreath product

$$\overline{x} = (x, \overline{x}, \mathrm{Id}, \ldots, \mathrm{Id}).$$

Finally let

$$H = \langle x, \overline{x} | x \in A_{31} \rangle.$$

The group H is of finite type and H is perfect as A_{31} is.

Proposition 4.2. *We have*

$$H \simeq H \wr A_{31}.$$

Proof. Let $\sigma = (2,3,4)$, $\rho = (1,3,2) \in A_{31}$ and consider $x, y \in A_{31}$. Then $[\overline{x}, \sigma\overline{y}] = ([x,y], \mathrm{Id}, \ldots, \mathrm{Id})$. As A_{31} is perfect this shows that for every $x \in A_{31}$ we have $(x, \mathrm{Id}, \ldots, \mathrm{Id}) \in H$. Then $\rho(x, \mathrm{Id}, \ldots, \mathrm{Id})^{-1}\overline{x} = (\overline{x}, \mathrm{Id}, \ldots, \mathrm{Id})$. Thus H contains $\{(h, \mathrm{Id}, \ldots, \mathrm{Id}) | h \in H\}$ and using $x \in A_{31}$ we get $H \wr A_{31} \subseteq H$.

Now we explain what are the properties of the group A_{31} which we need.

Proposition 4.3. *The group A_{31} can be generated by an element of order* 2 *and an element of order* 3.

As $H \simeq H \wr A_{31}$ and H is perfect this implies that there exist $u, v \in H$ such that $u^2 = v^3 = \mathrm{id}$ and $H = \langle u, v \rangle$.

Proposition 4.4. *Let $H \simeq H \wr A_{31}$ be a perfect group generated by u and v such that $u^2 = v^3 = \mathrm{id}$. Then there exist $x, y \in A_{31}$ such that*

- *there exists $\alpha, \beta \in \{1, \ldots, 31\}$, $\alpha \neq \beta$*

$$x(\alpha) = x^y(\alpha) = \alpha$$
$$y(\beta) = \beta$$

- *the elements*

$$\widehat{x} = (\ldots, u, \ldots)x$$
$$\widehat{y} = (\ldots, v, \ldots)y,$$

where u is in position α and v in position β, satisfy $\widehat{x}^2 = \widehat{y}^3 = \mathrm{id}$ and $\langle \widehat{x}, \widehat{y} \rangle = H$.

PROOF. We easily verify this proposition with explicit x, y, α and β [**49**].

Now let

$$\begin{aligned}\gamma'(n) &= |\{w \in H : |w|_{\langle\widehat{x},\widehat{y}\rangle} \leq n\}|,\\ \gamma(n) &= |\{w \in H : |w|_{\langle u,v\rangle} \leq n\}|.\end{aligned}$$

PROPOSITION 4.5. *If we denote* $\lim\limits_{n\to\infty} \sqrt[n]{\gamma(n)} = c$ *and* $\lim\limits_{n\to\infty} \sqrt[n]{\gamma'(n)} = c'$, *then for* $s \geq 3$ *we have*

$$c' \leq \max\left(c^{1-\frac{1}{2s}}, (1+2/s)(s+2)^{2/s}\right).$$

PROOF. We start by explaining the second term. Consider $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Let

$$\rho_n = \{w \in \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}; |w|_{\langle x,y\rangle} \leq n \text{ et } \mid \{xy^{-1}xy \in w\} \mid \leq [n/s]\}.$$

Then $\lim\limits_{n\to\infty} \sqrt[n]{\rho_n} \leq (1+2/s)(s+2)^{2/s}$.

Now let

$$\begin{aligned}B(n) &= \{w \in \langle\widehat{x},\widehat{y}\rangle; |w| \leq n\}\\ B_+(n) &= \{w \in B(n); \mid \{\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y} \in w\} \mid \geq [n/s]\}\\ B_-(n) &= B(n) \setminus B_+(n).\end{aligned}$$

We get

$$\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y}\widehat{x} = (1, \ldots, 1, v^{-1}, \ldots, u, \ldots, v)xy^{-1}xyx$$

where v^{-1} is in position $xyx(\beta)$, u is in $yx(\alpha)$ and v in $x(\beta)$.

If $w \in B_+(n)$ then $\mid \{\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y}\widehat{x}\} \mid$ is at least $\frac{1}{2}[n/s] = r$. Thus

$$|B_+(n)| \leq |A_{31}| \sum_{n_1+\ldots+n_{31}\leq n-2r} \prod_{j=1}^{31} \gamma(n_i) \leq K(n)(c+\varepsilon)^{n-2r} = K(n)(c+\varepsilon)^{n(1-1/2s)}$$

where $K(n)$ is a polynomial in n. We get the desired estimate.

The proof of the theorem is thus reduced to the following elementary lemma:

LEMMA 4.6. *There exists a sequence* $s_n \to \infty$ *such that*

$$c_n \to 1$$

where $c_1 = 2$ *and* $c_n = \max\left(c_{n-1}^{1-\frac{1}{2s_n}}, (1+2/s_n)(s_n+2)^{2/s_n}\right)$ *for* $n \geq 2$.

Finally, to show that H has exponential growth, one shows that it contains a free semigroup.

In [**8**] there are constructions of groups which solve at the same time the problem about amenability explained in the previous section and the problem of Gromov.

5. Burnside problem

In 1902 Burnside asked if there is an infinite finitely generated group such that all elements are of finite order.

This problem inspired many developments in group theory in the twentieth century. It forced one to construct and understand groups which are far away from linear groups.

The most important result concerning the existence of such groups is the theorem of Adyan-Novikov [**42**].

It was announced in the mid-fifties but the proof appeared only in the late sixties. The length of the paper(s) (several hundred pages), which after all were only showing that some groups are infinite, was an indication how complicated some groups might be. In a way, since then group theory has became more a theory of examples than a general theory.

Before the paper of Adyan-Novikov was published, Golod and Shafarevich [**20**] presented another construction of infinite torsion groups, which unlike the Adyan-Novikov examples are of unbounded exponent.

Let $B(2,n)$ be a two-generated group given by the following presentation:

$$B(2,n) = \langle a, b | w(a,b)^n \rangle$$

where $w(a,b)$ are all possible words in a and b. Clearly elements of $B(2,n)$ are of order at most n.

THEOREM 5.1 (Adyan-Novikov). *If n is odd and $n \geq 665$ then $B(2,n)$ is infinite.*

The Aleshin group gives a very simple answer even if, unlike the groups of Adyan-Novikov, the orders of its elements are not uniformly bounded.

5.1. The Aleshin group. Let us consider the finite inversible automaton from Figure 1. The Aleshin group [**1**] is the group G generated by U and V.

Its study enabled one to give a particularly simple answer to the Burnside problem and to solve a problem of Milnor.

Aleshin [**1**] proved:

THEOREM 5.2. *The group generated by U and V is torsion and infinite.*

The original proof enables one to construct an uncountable family of infinite p-groups for every prime number p.

The Aleshin group is by definition of finite type.

Let G be the group generated by the states a, b, c and d of the automaton from Figure 1. It is easy to see that this group is commensurable with the group generated by the states U and V and with the group generated by all states of the automaton (i.e. these groups have finite index subgroups which are isomorphic).

We have relations:

$$a = (1,1)\mathsf{e}$$

$$\begin{aligned} b &= (a,c) \\ c &= (a,d) \\ d &= (1,b) \end{aligned} \tag{5.12}$$

and also

$$\begin{aligned} aba &= (c,a) \\ aca &= (d,a) \\ ada &= (b,1). \end{aligned} \tag{5.13}$$

PROPOSITION 5.3. *The group G is infinite.*

PROOF. Consider $\mathrm{St}_G(1)$ which is of index 2 in G. Its projection on the first coordinates contains a, b, c and d. Thus $\mathrm{St}_G(1)$ surjects onto G which shows that G is infinite.

LEMMA 5.4. *The group generated by b, c and d is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

PROOF. It is a simple verification.

For $g \in G$ we have $g^8 \in \mathrm{St}_G(3)$.

LEMMA 5.5. *For g^8 consider its image in G^8. Denote $g^8 = (g_1, \ldots, g_8)$. Then for $g \neq \mathrm{id}$*

$$|g_i| < |g|, \tag{5.14}$$

for the length with respect to generators a, b, c and d.

PROOF. As a is of order 2 and b, c, d are elements of the Klein group, every element $g \in G$ can be written as

$$g = \overline{a}k_1ak_2a \cdots ak_n\overline{a},$$

where $k_i \in \{b, c, d\}$ and the first and last a, denoted by $\overline{a}$, do not necessarily appear.

Consider the block $\gamma = k_iak_{i+1}a$. Thanks to relations (5.12) and (5.13) its image $\gamma = (\gamma_1, \gamma_2)$ in $G \times G$ verifies

$$|\gamma_i| \leq \frac{1}{2}|\gamma|$$

for $i = 1, 2$. If k_i or k_{i+1} is equal to d, one of these inequalities becomes strict:

$$|\gamma_i| < \frac{1}{2}|\gamma|. \tag{5.15}$$

The relations (5.12) and (5.13) show that the image of k_i or ak_ia in $G \times G$ gives c if $k_i = b$ and gives d if $k_i = c$. Thus if we iterate this procedure 3 times we are sure to be in the situation (5.15).

We have $g^2 \in \mathrm{St}_G(1)$, $g^4 \in \mathrm{St}_G(2)$, $g^8 \in \mathrm{St}_G(3)$ and thus

$$|g_i| < \frac{1}{8}|g^8| \leq |g|.$$

Therefore we have the inequality (5.14).

One shows by induction on the length that every element is of finite order.

The Aleshin group is not finitely presented.

Actually there are no known examples of finitely presented, infinite torsion groups.

6. Intermediate growth

The growth type for groups is one of the simplest invariants for infinite groups.

For a group G generated by a finite set S which we suppose to be symmetric (i.e. such that $S = S^{-1}$), we denote $|g|_S$ the minimal number of generators needed to represent g. The growth of the group G describes the asymptotic behavior of the function

$$b_G(n) = |\{g \in G : |g|_S \leq n\}|.$$

This type of growth is independent of the generating set. For instance, for nilpotent groups it is polynomial and for a group which contains a subgroup or even semigroup which is free the growth is exponential. For the history of this notion see [**29**]. Its systematic study starts with results of Milnor.

Actually, polynomial growth characterizes nilpotent groups, namely a group of polynomial growth contains a finite index subgroup which is nilpotent. This was proven by Gromov using a solution to Hilbert fifth problem which is a characterization of Lie groups among topological groups. An elementary proof of Gromov's result was given recently by Kleiner [**36**].

In [**39**] it was asked if there are other types of growth. We present a solution given in [**23**].

6.1. Growth of the Aleshin group.

PROPOSITION 6.1. *The group G is not of polynomial growth.*

PROOF. A polynomial growth group contains a finite index subgroup which is nilpotent (Gromov) and contains a finite index subgroup which is torsion free (Malcev). However, Aleshin's theorem show that G is infinite and torsion. One can also prove this proposition by a simple calculation.

Let us show that the Aleshin group has sub-exponential growth.

Let $\Gamma = \mathrm{St}_G(3)$. Then $[G : \Gamma] < \infty$.

LEMMA 6.2. *For $g \in \Gamma$ consider its image in G^8. Denote $g = (g_1, \ldots, g_8)$. Then*

$$\sum_{i=1}^{8} |g_i| \leq \frac{3}{4}|g| + 8, \tag{6.16}$$

for the length with respect to generators a, b, c and d.

PROOF. As in the proof of Lemma 5.5 any element $g \in G$ can be written as

$$g = \overline{a}k_1ak_2a \cdots ak_n\overline{a},$$

where $k_i \in \{b, c, d\}$ and the first and last a, denoted by $\overline{a}$, do not necessarily appear.

Consider the block $\gamma = k_iak_{i+1}a$. Thanks to relations (5.12) and (5.13) its image $\gamma = (\gamma_1, \gamma_2)$ in $G \times G$ verifies

$$|\gamma_1| + |\gamma_2| \leq |\gamma|.$$

If k_i or k_{i+1} is equal to d, this inequality becomes

$$|\gamma_1| + |\gamma_2| \leq \frac{3}{4}|\gamma|. \tag{6.17}$$

The relations (5.12) and (5.13) show that the image of k_i or ak_ia in $G \times G$ gives c if $k_i = b$ and gives d if $k_i = c$. Thus if we iterate this procedure 3 times we are sure to be in the situation (6.17). Therefore we have an inequality (6.16) (the term 8 is due to the fact that $|g|$ is not necessarily divisible by 8).

PROPOSITION 6.3. *The Aleshin group has sub-exponential growth.*

PROOF. The inequality (6.16) shows that

$$|b_\Gamma(k)| \leq \sum_{k_1+\ldots+k_8 \leq \frac{3}{4}k+8} |b_G(k_1)| \times \ldots \times |b_G(k_8)|. \tag{6.18}$$

It is important to compute the length with respect to a, b, c and d even if a does not belong to Γ. As Γ is of finite index in G, we have

$$\lim_{n\to\infty} \sqrt[n]{|b_G(n)|} = \lim_{n\to\infty} \sqrt[n]{|b_\Gamma(n)|} = \alpha. \tag{6.19}$$

For every $\varepsilon > 0$ there exists $c > 0$ such that for n sufficiently large, we get

$$|b_G(n)| \le c(\alpha+\varepsilon)^n.$$

The majoration (6.18) ensures there is c' such that

$$|b_\Gamma(n)| \le c' n^8 (\alpha+\varepsilon)^{\frac{3}{4}n+8}.$$

Thus $\lim_{n\to\infty} \sqrt[n]{|b_\Gamma(n)|} \le \alpha^{\frac{3}{4}}$ which together with (6.19), implies

$$\lim_{n\to\infty} \sqrt[n]{|b_G(n)|} = 1.$$

Therefore the Aleshin group has subexponential growth.

The exact asymptotic growth of this group is not known.

It is worth mentioning that in [**9**] one can find an example of a group of intermediate growth for which there are $0 < \alpha < \beta < 1$ such that for infinitely many n's the size of balls is less than e^{n^α} and for infinitely many n's more than e^{n^β}.

As was mentioned in the section concerning the Burnside problem, the Aleshin group is not finitely presented.

Moreover, there are no known examples of finitely presented groups of intermediate growth. The case of finitely presented groups of is particularly interesting as it relates to the growth of Riemannian manifolds. Namely, for a closed Riemannian manifold M, the growth type of the volume of balls on the universal cover $\widetilde{M}$ is equivalent to the growth type of the fundamental group of the manifold M. The only condition on a group to be a fundamental group of a closed manifold is to have a finite presentation. Therefore it is an open problem whether there are closed manifolds with universal cover of intermediate growth.

7. Finitely presented groups

In these notes we presented some spectacular advances in group theory using groups generated by finite automata. Many problems discussed here have very natural analogues in the geometric context, namely for closed manifolds. Then statements and conjectures about closed manifolds can be formulated in terms of the fundamental group. Clearly the fundamental group of a closed manifold is finitely presented. It is not very difficult to show that every finitely presented group can be realized as the fundamental group of a closed 4-dimensional manifold. The idea of this construction was explained in the section about L^2 Betti numbers although due to some other constraints it led to a 7-dimensional manifold.

The automata groups considered here were not finitely presented. However, they admit very special presentations which enables one to embed them into finitely presented groups and in some cases it provides solutions to geometric problems as well. In order to explain this we have to review the notion of a recursive presentation.

Let us recall that a subset S of the natural numbers is called recursive if there exists a totally computable function f such that $f(x) = 1$ if $x \in S$ and $f(x) = 0$ if $x \notin S$ and where a computable function is one for which there is an algorithm (Turing machine) telling how to compute the function.

Let S be a set and let F_S be the free group on S. Let R be a set of irreducible words on S and its inverses, i.e. a subset of F_S. The group defined by a presentation $\langle F_S \mid R\rangle$ is the quotient of F_S by the smallest normal subgroup of F_S containing R.

If S is indexed by a set consisting of all the natural numbers or a finite subset of them, then we consider a simple one to one coding $f : F_S \to \mathbb{N}$ from the free group on S to the natural numbers, such that we can find algorithms that, given $f(w)$, calculate w, and vice versa. We can then call a subset U of F_S recursive (respectively recursively enumerable) if $f(U)$ is recursive (respectively recursively enumerable). If S is indexed as above and R recursively enumerable, then the presentation is a recursive presentation and the corresponding group is recursively presented.

Examples of such presentations are given in sections about L^2 Betti numbers (the lamplighter group) and amenability (a group on three states). It appears that it gives a precise characterization of subgroups of finitely presented groups.

In the sixties Higman [**32**] proved that a group is a subgroup of a finitely presented group if and only if it admits a recursive presentation. Actually, there is an algorithm how to embed a group with a given recursive presentation into a finitely presented group. It consists of a series of HNN extensions.

This is exactly the procedure which was explained in detail in Section 2 to embed the lamplighter group, and HNN extensions are used to obtain a group with a finite presentation (2.4) and for a three-state automaton group we obtain a presentation (3.11).

It was important that these extensions preserved properties which were crucial for the problems which we considered. In the problem about L^2 Betti numbers it was important that there was no new torsion introduced by the extensions and for amenability it was important that the extensions that we considered preserved amenability.

However, these techniques failed to produce infinite, torsion, finitely presented groups or finitely presented groups of intermediate growth. The reason is that the extensions used to produce finitely presented groups introduce elements of infinite order and lead to groups of exponential growth.

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Spectral Triples and KK-Theory: A Survey

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Introduction

This survey covers the material in the forthcoming paper [**21**], which deals with the construction of a category of spectral triples that is compatible with the Kasparov product in KK-theory ([**18**]). These notes serve as an intuitive guide to the results described there, avoiding the necessary technical proofs. We will also add some background and a broader perspective on noncommutative geometry. The theory described shows that, by introducing a notion of smoothness on unbounded KK-cycles, the Kasparov product of such cycles can be defined directly, by an algebraic formula. This allows one to view such cycles as morphisms in a category whose objects are spectral triples.

We will consider all C^*-algebras to be equipped with a spectral triple that is sufficiently smooth. A smooth KK-cycle for a pair of such C^*-algebras (A, B) is a triple $(\mathcal{E}, S, \nabla)$, where the pair $(\mathcal{E}, S)$ is a KK-cycle in the sense of Baaj-Julg [**1**], satisfying some smoothness conditions compatible with the given spectral triples, and ∇ is a connection on the module, compatible with the operator S and the smooth structure on $\mathcal{E}$. Composition of such triples is defined by

$$(\mathcal{E}, S, \nabla) \circ (\mathcal{F}, T, \nabla') := (\mathcal{E}\tilde{\otimes}_B \mathcal{F}, S \otimes 1 + 1 \otimes_\nabla T, 1 \otimes_\nabla \nabla'),$$

and preserves all smoothness conditions. Moreover, it represents the Kasparov product of the KK-cycles $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$.

In particular this allows one to compute such products explicitly in terms of the operators and connection. This has possible applications to index problems, which are often defined in terms of the Kasparov product. Since Chern character formulas in cyclic homology are most easily computed for unbounded representatives (this is Connes' quantized calculus), explicit representatives of the Kasparov product are desirable in such problems.

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Viewing spectral triples as noncommutative metric spaces, the notion of morphism introduced here might shed light on the purely commutative problem of what the correct notion of morphism between metric spaces should be.

1. Spectral triples and noncommutative geometry

By the Gelfand-Naimark theorem, C^*-algebras can be viewed as noncommutative, locally compact Hausdorff topological spaces. This is the starting point for noncommutative geometry. A continuous map $f : X \to Y$ between compact Hausdorff spaces gives a *-homomorphism $f^* : C(Y) \to C(X)$ between the dual C^*-algebras. Noncommutative algebras, however, might not admit any nontrivial algebra homomorphisms. For intstance, the matrix algebra $M_n(\mathbb{C})$ is a such a *simple* algebra. A more flexible notion of morphism for noncommutative algebras is that of a suitable class of bimodules ${}_A\mathcal{E}_B$, with composition coming from the tensor product of bimodules.

Topological K-theory is the tool that generalizes in the most straightforward way from spaces to C^*-algebras. From the definition of K-theory it follows readily that $K_*(A) \cong K_*(M_n(A))$ for any C^*-algebra A. This is one of the reasons why one wants to regard these algebras as being equivalent. The notion of *Morita equivalence* formalizes this notion of equivalence and is compatible with the notion of bimodule morphism.

A Riemannian manifold M is a topological space with some finer structure defined on it. This can be encoded by considering some (pseudo) differential operators on the manifold, e.g. a Dirac operator (when M is Spin^c), or a signature type operator.

In the spin case, the Riemannian metric on M can be recoverd from the Dirac operator D by

$$d(x,y) = \sup\{\|f(x) - f(y)\| : \|[D, f]\| \leq 1\}.$$

The reader can consult [**10**] for a proof of this. Recently, some stronger reconstruction theorems have been announced [**11**].

This motivates the notion of spectral triple [**8**].

DEFINITION 1.1. A *spectral triple* $(A, \mathcal{H}, D)$ consists of a $\mathbb{Z}/2$-graded C^*-algebra A represented on a likewise graded Hilbert space $\mathcal{H}$, together with an odd, selfadjoint operator D, with compact resolvent, such that

$$\{a \in A : [D, a] \in B(\mathcal{H})\},$$

is dense in A.

Commutative examples are plentiful, mainly given by manifolds. Other examples come from groups, group actions, and foliations. Also, there are various extensions of the notion of spectral triple, notably in the type II and type III setting. Again we refer to [**7**] for these topics.

2. The noncommutative torus

The subject of these notes is a notion of *morphism* for spectral triples, a generalization of maps between manifolds. Let us first discuss an example to illustrate this. It will be a noncommutative geometry description of the *fibration* of the torus $S^1 \times S^1$ over the circle S^1. The projection $S^1 \times S^1 \to S^1$ on either of the coordinates is a smooth map, and the fiber over each point is again diffeomorphic to S^1. Of course this is a very simple fibration because it is just a direct product. However, its noncommutative analogue is very instructive in illustrating the general theory that follows.

The noncommutative torus A_θ is the C^*-algebra crossed product of the action of $\mathbb{Z}$ on the circle S^1 by a rotation over the angle $2\pi\theta$, denoted $x \mapsto \alpha_\theta(x)$. The algebra $C_c(S^1 \times \mathbb{Z})$ carries a convolution product

$$f * g(x,n) = \sum_{k\in\mathbb{Z}} f(x,k)g(\alpha_\theta^k(x), n-k),$$

defining a representation on $\mathcal{H} := L^2(S^1 \times \mathbb{Z})$, yielding the C^*-algebra A_θ.

Another way to describe A_θ is as the universal C^*-algebra generated by two unitaries u, v subject to the relation $uv = e^{2\pi i\theta} vu$. In this picture, elements of A_θ can be described as series

$$\sum_{n,m\in\mathbb{Z}} \lambda_{n,m} u^n v^m,$$

convergent in a certain norm, analogous to Fourier series.

The algebra A_θ carries two canonical unbounded derivations, defined on $C_c^\infty(S^1 \times \mathbb{Z})$ by

$$\partial_1 f(x,n) := nf(x,n), \quad \partial_2 f(x,n) := \frac{1}{2\pi i}\partial f(x,n).$$

In the u, v picture, these derivations are

$$\partial_1 u^n v^m = m u^n v^m, \quad \partial_1 u^n v^m = n u^n v^m.$$

On $\mathcal{H} \oplus \mathcal{H}$ this yields the operator

$$D := \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}.$$

giving the canonical spectral triple on A_θ.

3. "Fibration" over the circle

We now describe the structure that we think of as implementing the fibration of A_θ over the circle algebra $C(S^1)$. It consists of an $(A_\theta, C(S^1))$-bimodule, equipped with an unbounded operator and a connection. The precise structures present on these modules will be described later in these notes. The reader is encouraged to keep this example in mind.

Consider the module $\mathcal{E} = \ell^2(\mathbb{Z})\tilde{\otimes}C(S^1) \cong L^2(S^1)\tilde{\otimes}C(S^1)$. Here $\tilde{\otimes}$ denotes a certain completed tensor product. It carries an unbounded, $C(S^1)$ linear operator

$$S : e_n \otimes f \mapsto n e_n \otimes f.$$

The canonical spectral triple for the circle $(C(S^1), L^2(S^1), \frac{1}{2\pi i}\partial)$ defines a module of 1-"forms"

$$\Omega^1_\partial := \{\sum f_k[\frac{1}{2\pi i}\partial, g_k] : f_k, g_k \in \mathrm{Lip}^1(S^1)\} \subset B(L^2(S^1)),$$

where Lip^1 denotes the Lipschitz functions on S^1. The module $\mathcal{E}$ carries a densely defined connection

$$\nabla : e_n \otimes f \mapsto e_n \otimes [\frac{1}{2\pi i}\partial, f].$$

∇ is defined on a dense $\mathrm{Lip}^1(S^1)$-submodule $E^1 \subset \mathcal{E}$, and maps it into $E^1 \otimes_{\mathrm{Lip}^1} \Omega^1_{\partial}$. It satisfies $[\nabla, S] = 0$. The tensor product $\mathcal{E}\tilde{\otimes}_{C(S^1)} L^2(S^1)$ is isomorphic to $\mathcal{H}$. Under this identification, the derivation ∂_2 equals

$$e \otimes h \mapsto e \otimes \frac{1}{2\pi i}\partial h + \nabla(e)h.$$

This expression is well defined because ∇ satisfies a Leibniz rule

$$\nabla(ef) = \nabla(e)f + e \otimes [\frac{1}{2\pi i}\partial, f].$$

We denote it by $1 \otimes_\nabla \frac{1}{2\pi i}\partial$. We thus see that the canonical spectral triple on A_θ can be factorized as a graded tensor product

$$(A_\theta, \mathcal{H} \oplus \mathcal{H}, D) = (E, S, \nabla) \otimes (C(S^1), L^2(S^1), \frac{1}{2\pi i}\partial).$$

The tensor product on the right is to be interpreted as

$$\mathcal{E} \otimes_{C(S^1)} L^2(S^1) \oplus \mathcal{E} \otimes_{C(S^1)} L^2(S^1),$$

with operator

$$\begin{pmatrix} 0 & S \otimes 1 - i1 \otimes_\nabla \frac{1}{2\pi i}\partial \\ S \otimes 1 + i1 \otimes_\nabla \frac{1}{2\pi i}\partial & 0 \end{pmatrix}.$$

Thus, by choosing the right gradings, the triple $(\mathcal{E}, S, \nabla)$ can be viewed as a fibration of the noncommutative torus over the circle.

4. C^*-modules and regular operators

We now proceed by describing the modules, operators and connections involved in a more rigorous manner. Let (A, B) be a pair of separable, $\mathbb{Z}/2$-graded C^*-algebras. The reader who feels uneasy thinking about graded C^*-algebras, can think of trivially graded (i.e. ungraded) C^*-algebras. The reason for developing the theory for graded algebras is that one can treat the even and odd cases of K-theory at the same time. The standard reference for the theory of C^*-modules is [**20**].

Definition 4.2. A *C^*-module over B* is a right B-module $\mathcal{E}$ equipped with a positive definite B-valued inner product.

A positive definite B-valued inner product is a pairing $\mathcal{E} \times \mathcal{E} \to B$, satisfying

- $\langle e_1, e_2\rangle = \langle e_2, e_1\rangle^*$,
- $\langle e_1, e_2 b\rangle = \langle e_1, e_2\rangle b$,
- $\langle e, e\rangle \geq 0$ and $\langle e, e\rangle = 0 \Leftrightarrow e = 0$,
- $\mathcal{E}$ is complete in the norm $\|e\|^2 := \|\langle e, e\rangle\|$.

We use the notation $\mathcal{E} \leftrightharpoons B$ to indicate this structure.

The natural endomorphisms to consider in a C^*-module are the following:

$$\mathrm{End}^*_B(\mathcal{E}) := \{T : \mathcal{E} \to \mathcal{E} : \exists T^* : \mathcal{E} \to \mathcal{E}, \langle Te, f\rangle = \langle e, T^* f\rangle\}.$$

Operators in $\mathrm{End}^*_B(\mathcal{E})$ are automatically B-linear and bounded, and they form a C^*-algebra in the operator norm and the involution $T \mapsto T^*$.

There is a natural C^*-subalgebra, analogous to the compact operators on a Hilbert space. The algebra of *compact endomorphisms* $\mathbb{K}_B(\mathcal{E}) \subset \mathrm{End}^*_B(\mathcal{E})$ is the C^*-subalgebra generated by the operators $e \otimes f(g) := e\langle f, g\rangle$.

An (A,B)-*bimodule* is a C^*-module $\mathcal{E} \leftrightharpoons B$, together with a graded *-homomorphism $A \to \mathrm{End}^*_B(\mathcal{E})$.

A *regular operator in* $\mathcal{E}$ is a densely defined closed operator, $D : \mathfrak{Dom}\, D \to \mathcal{E}$, such that D^* is densely defined in $\mathcal{E}$ and $1 + D^*D$ has dense range. This condition is automatic in the Hilbert space setting, but needs to be imposed in C^*-modules, to avoid pathologies. The operator D is *selfadjoint* if it is symmetric on its domain, and $\mathfrak{Dom}\, D^* = \mathfrak{Dom}\, D$. An excellent reference for the theory of regular operators in C^*-modules is [**20**].

5. Unbounded KK-theory

KK-theory associates to a pair (A,B) of separable, $\mathbb{Z}/2$-graded C^*-algebras a $\mathbb{Z}/2$-graded abelian group $KK_*(A,B)$. Kasparov [**18**] originally constructed and described these groups using bounded *Fredholm operators* in C^*-modules.

A defining element of the group $KK_0(A,B)$ is a pair $(\mathcal{E}, F)$ consisting of an (A,B)-bimodule $\mathcal{E}$, together with an operator $F \in \mathrm{End}^*_B(\mathcal{E})$ satisfying

$$a(F^2 - 1), \quad a(F - F^*), \quad [F, a] \in \mathbb{K}_B(\mathcal{E}).$$

Subsequently one considers unitary equivalence classes of such pairs, and quotients by the relation of homotopy to obtain the abelian group $KK_0(A,B)$. The groups $KK_i(A,B)$ are defined as being $KK_0(A, B\tilde{\otimes}\mathbb{C}_i)$, where $\mathbb{C}_i$ is the i-th complex Clifford algebra. This is a graded C^*-algebra, and it is at this point that working with graded algebras comes in handy.

Kasparov's main achievement was the construction of an associative, distributive product

$$KK_i(A,B) \otimes_{\mathbb{Z}} KK_j(B,C) \to KK_{i+j}(A,C),$$

now known as the *Kasparov product.* The Kasparov product has remarkable properties. It allows one to view the KK-groups as the morphisms in a category $\mathfrak{KK}$ whose objects are C^*-algebras. Moreover, Cuntz [**12**] and Higson [**16**] showed that that KK-theory has a *universal property*, in the sense that any functor from C^*-algebras to abelian groups which is Morita invariant and split exact, factors through this category $\mathfrak{KK}$. Such functors are automatically homotopy invariant. In this sense KK-theory is the universal cohomology theory for C^*-algebras.

In the above Fredholm picture, the Kasparov product is very difficult to define, and we will refrain from doing so here. We will describe the product in a different picture, given below.

Definition 5.3 ([**1**]). The cycles for $KK_0(A,B)$ may also be described by pairs $(\mathcal{E}, D)$, where

- $\mathcal{E}$ is an (A,B)-bimodule.
- $D : \mathfrak{Dom}\, D \to \mathcal{E}$ is an odd selfadjoint regular operator.

- $\forall a \in A : a(1+D^2)^{-1} \in \mathbb{K}_B(\mathcal{E})$.
- The subalgebra
$$\mathcal{A}_1 := \{a \in A : [D,a] \in \mathrm{End}^*_B(\mathcal{E})\},$$
is dense in A.

Such pairs $(\mathcal{E}, D)$ are referred to as *unbounded KK-cycles.*

The relation between the bounded and the unbounded picture is given by a simple procedure. The following results are due to Baaj-Julg[**1**].

THEOREM 5.4 ([**1**]). *Let $F := D(1+D^2)^{-\frac{1}{2}} \in \mathrm{End}^*_B(\mathcal{E})$, the* bounded transform *of D.*

- *$(\mathcal{E}, F)$ is a* Kasparov module, *i.e. $F^* = F$ and*
$$\forall a \in A, a(F^2-1), [F,a] \in \mathbb{K}_B(\mathcal{E}).$$
- *Two unbounded modules are* equivalent *if their bounded transforms are homotopic. Any Kasparov module is homotopic to the bounded transform of an unbounded one.*

Their motivation for introducing the unbounded picture was that it simplifies another product structure in Kasparov's theory, the *external product*
$$KK_i(A,B) \otimes KK_j(A',B') \to KK_{i+j}(A\overline{\otimes}A', B\overline{\otimes}B'),$$
where A, A', B, B' are distinct C^*-algebras. Baaj and Julg proved the following

THEOREM 5.5 ([**1**]). *On unbounded cycles, the* external Kasparov product *is given by*
$$(\mathcal{E}, S) \times (\mathcal{F}, T) := (\mathcal{E}\overline{\otimes}\mathcal{F}, S \otimes 1 + 1 \otimes T),$$
where
$$1 \otimes T(e \otimes f) := (-1)^{\partial e} e \otimes Tf.$$

In the case $B = B' = \mathbb{C}$, this product corresponds to the direct product of manifolds. The case $A = A' = \mathbb{C}$ gives the external product in topological K-theory.

6. Algebraic intermezzo

When trying to define the *internal Kasparov product*
$$KK_i(A,B) \otimes KK_j(B,C) \to KK_{i+j}(A,C),$$
on unbounded cycles, we run into the following problem. In the Fredholm picture, Kasparov proved that on the module $\mathcal{E}\tilde{\otimes}_B\mathcal{F}$ one can always find an operator, unique up to homotopy, that defines the class of the Kasparov product. In the unbounded picture, as in the case of the external product, the natural guess for the operator is something like $S \otimes 1 + 1 \otimes T$. However, the expression $1 \otimes T$ does not make sense, since T does not commute with the elements of B, and we take a balanced tensor product. It turns out that there is a notion of *connection* which corrects for this problem. The algebraic theory of forms and connections is described in detail in [**13**].

For clarity, we first consider the following structure of a category on algebraic (A,B)-bimodules with odd operator (E,D).

DEFINITION 6.6. Let B be an algebra. The module of 1-*forms* of B is the kernel of the graded multiplication map

$$\Omega^1(B) := \ker(B \otimes B \xrightarrow{m} B)$$
$$b_1 \otimes b_2 \mapsto b_1\gamma(b_2),$$

where $\gamma \in \operatorname{Aut} B$ is the grading automorphism. The *universal derivation* $d : B \to \Omega^1(B)$ is given by

$$b \mapsto 1 \otimes b - \gamma(b) \otimes 1.$$

Any derivation $\delta : B \to M$ into a B-bimodule M factors through the bimodule $\Omega^1(B)$ in the following sense.

PROPOSITION 6.7 ([**13**]). *The bimodule $\Omega^1(B)$ is universal for derivations $\delta : B \to M$, where M is a B-bimodule. That is, for any such δ there is a unique map $j_\delta : \Omega^1(B) \to M$ such that $\delta = j_\delta \circ d$.*

The map j_δ is defined by setting $j_\delta(da) = \delta(a)$. This determines j_δ as a bimodule map, because the elements da generate $\Omega^1(B)$ as a bimodule.

DEFINITION 6.8. A *connection* on a right B-module E is a map

$$\nabla : E \to E \otimes_B \Omega^1(B),$$

satisfying

$$\nabla(eb) = \nabla(e)b + e \otimes db.$$

If a connection ∇ on E is given, F is a (B, C)-bimodule and $T \in \operatorname{End}_B(F)$, then the operator

$$1 \otimes_\nabla T(e \otimes f) := (-1)^{\partial e \partial T}(e \otimes Tf + \nabla_T(e)f),$$

is well defined on $E \otimes_B F$. Here $\partial e, \partial T \in \{0, 1\}$ denote the degree of the homogeneous elements e and T respectively. The connection $\nabla_T : E \to E \otimes_B \operatorname{End}_C(F)$ is the composition $j_\delta \circ \nabla$ with δ the derivation $b \mapsto [T, b]$. When a connection ∇' is given on F, we can apply the same trick and define a connection

$$1 \otimes_\nabla \nabla' : E \otimes_B F \to E \otimes_B F \otimes_C \Omega^1(C),$$

now by using the derivation $b \mapsto [\nabla, b]$. An *isomorphism* of triples (E, S, ∇) and (E', S', ∇') is a bimodule isomorphism $g : E \to F$ with the additional properties that

- $g^{-1}S'g = S$;
- $g^{-1}\nabla'g = \nabla$.

Of course, isomorphism of triples is an equivalence relation.

PROPOSITION 6.9 ([**21**]). *Let A, B, C be algebras, E, F $(A, B)-$ and $(B, C)-$ bimodules respectively. The composition law*

$$(E, S, \nabla) \circ (F, T, \nabla') := (E \otimes_B F, S \otimes 1 + 1 \otimes_\nabla T, 1 \otimes_\nabla \nabla'),$$

is associative up to isomorphism. Isomorphism classes of triples (E, S, ∇) are the morphisms in a category whose objects are pairs (E, D), where E is an (A, B)-bimodule and $D \in \operatorname{End}_B(E)$ an endomorphism.

REMARK 6.10. A morphism from (G, D) to (F, T) is a triple (E, S, ∇) such that $(E \otimes_B F, S \otimes 1 + 1 \otimes_\nabla T)$ is isomorphic to (G, D).

In this setting a spectral triple $(A, \mathcal{H}, D)$ is more conveniently denoted by just $(\mathcal{H}, D)$. In particular, these are $(A, \mathbb{C})$ bimodules. Unfortunately, the algebraic setting discussed above is not appropriate for dealing with spectral triples. It needs to be enriched to accommodate for the analytic phenomena governing them.

In order to construct a category of spectral triples (or unbounded bimodules) in which the morphisms are unbounded bimodues $(\mathcal{E}, D)$, with some notion of connection, several problems need to be addressed:

- Unbounded regular operators are not endomorphisms (i.e. not everywhere defined).
- The graded commutators $[D, a]$ are endomorphisms only for a in a dense subalgebra of A.
- An analytic version of $\Omega^1(B)$ and the notion of connection for dense subalgebras are needed.
- The product operator $S \otimes 1 + 1 \otimes_\nabla T$ should be selfadjoint, regular and have compact resolvent.

All these issues can be resolved by introducing an appropriate notion of smoothness for unbounded KK-cycles.

7. Operator algebras and modules

To overcome the aforementioned problems, we need to broaden our scope from C^*-algebras to operator spaces. The algebraic structures of algebras and modules will need operator space analogues as well. Operator space theory was developed by Effros and Ruan [**14**],[**22**], and many others.

Definition 7.11. An *operator space* is a closed subspace of some C^*-algebra.

The main feature of an operator space X is that it comes with canonical matrix norms, i.e. $M_n(X)$ carries a canonical norm. A map $\phi : X \to Y$ between operator spaces is *completely bounded* if $\|\phi\|_{cb} := \sup_n \|\phi_n\| < \infty$, where $\phi_n : M_n(X) \to M_n(Y)$ is the map induced by ϕ. It is *completely contractive* if $\|\phi\|_{cb} \leq 1$. The completely bounded maps form the natural class of maps between operator spaces.

Example 7.12. A $*$-homomorphism $\phi : A \to B$ between C^*-algebras is automatically completely bounded, as is an adjointable operator $T \in \mathrm{End}^*_B(\mathcal{E}, \mathcal{F})$ between C^*-modules.

The natural tensor product for operator spaces X and Y is the *Haagerup tensor product*, denoted by $X \tilde{\otimes} Y$. Its norm is given by

$$\|z\| := \inf\{\| \sum x_i x_i^* \|^{\frac{1}{2}} \| \sum y_i^* y_i \|^{\frac{1}{2}} : z = \sum x_i \otimes y_i\}.$$

Note that although x^* need not be an element of X, it does make sense in the containing C^*-algebra of X. The space $X \tilde{\otimes} Y$ is again an operator space. An *operator algebra* is an operator space $\mathcal{A}$ whose multiplication $\mathcal{A} \tilde{\otimes} \mathcal{A} \to \mathcal{A}$ is completely contractive. An *involutive operator algebra* is an operator algebra with an involution $a \mapsto a^*$ which is completely bounded. An *operator module* M over an operator algebra $\mathcal{B}$ is an operator space M, which is also a (say) right $\mathcal{B}$-module, such that the module action $M \tilde{\otimes} \mathcal{B} \to M$ is completely bounded. The *Haagerup module tensor product* $M \tilde{\otimes}_{\mathcal{B}} N$ of right and left $\mathcal{B}$ operator modules M and N, respectively, is the quotient of $M \tilde{\otimes} N$ by the closed subspace generated by $mb \otimes n - m \otimes bn$. The reader can consult [**6**] and [**5**] for many aspects of the theory of operator modules. Also, see [**15**] for a survey on operator space tensor products.

EXAMPLE 7.13. A C^*-module $\mathcal{E} \leftrightharpoons B$ is canonically an operator space by viewing it as the upper right corner of its *linking algebra* $\mathbb{K}_B(B \oplus \mathcal{E})$. As such it is an operator module and the Haagerup tensor product of C^*-modules is completely isometrically isomorphic to the C^*-module tensor product.

D.P. Blecher [**4**] observed that, when $\mathcal{E}$ is countably generated, by choosing an approximate unit

$$u_n = \sum_{1 \le |i| \le n} x_i \otimes x_i \in \mathbb{K}_B(\mathcal{E}),$$

(which is possible by Kasparov's stabilization theorem [**18**]) $\mathcal{E}$ can be written as an inductive limit of the canonical modules B^{2n}. This is done by considering the maps

$$\phi_n : (b_i) \mapsto \sum_{1 \le |i| \le n} x_i b_i, \quad \psi_n : e \mapsto (\langle x_i, e \rangle),$$

which are completely contractive, and $\phi_n \circ \psi_n \to 1$ strongly. The inner product in $\mathcal{E}$ can be recovered from these maps as

$$\langle e, f \rangle := \lim_{\alpha} \langle \psi_n(e), \psi_n(f) \rangle_n.$$

8. Stably rigged modules

Blecher used his observation to develop a theory of modules over operator algebras, that are in many ways similar to C^*-modules. In case the algebra is actually a C^*-algebra, this class of modules coincides with that of C^*-modules. See [**3**] for details.

DEFINITION 8.14 (Blecher). Let $\mathcal{B}$ be an operator algebra with contractive countable approximate identity. A *rigged module* over $\mathcal{B}$ is a right operator module E over $\mathcal{B}$ together with completely contractive module maps $\psi_n : E \to \mathcal{B}^{2n}$ and $\phi_n : \mathcal{B}^{2n} \to E$, such that $\phi_n \circ \psi_n$ converges strongly to 1, and ϕ_n is $\mathcal{B}$-essential. When ϕ_n, ψ_n and the approximate identity are merely completely bounded, E is an *stably rigged module.*

The difference between rigged and stably rigged modules might seem only formal at first sight. However, the contractivity assumption is a fairly strong one.

THEOREM 8.15 (Blecher). *A rigged module over a C^*-algebra is a C^*-module, and the Haagerup tensor product of (stably) rigged modules is again a (stably) rigged module.*

A rigged module over a C^*-algebra is completely isometrically isomorphic to a C^*-module. In the cb-setting such a theorem has not been established, and can definitely not be proven in a similar way. An important corollary of the above theorem is that for a rigged module E over an operator algebra $\mathcal{B}$, and a completely contractive homomorphism $\mathcal{B} \to \mathrm{End}^*_C(\mathcal{F})$, with $\mathcal{F} \leftrightharpoons C$ a C^*-module, the Haagerup tensor product $E \tilde{\otimes}_{\mathcal{B}} \mathcal{F}$ is a genuine C^*-module. This fact will be exploited when dealing with graphs of unbounded operators.

9. Sobolev modules

This section describes the construction of Sobolev modules and algebras as developed in [**21**]. They are the analogues of the usual Sobolev spaces that appear in Riemannian geometry, but we describe them in a more algebraic manner. For this reason we obtain only Sobolev spaces indexed by the natural numbers, as opposed to the positive real numbers. This is to avoid the use of functional calculus.

The *graph* of a regular operator D in $\mathcal{E}$ is the closed submodule

$$\mathfrak{G}(D) := \{(e, De) : e \in \mathfrak{Dom}\, D\} \subset \mathcal{E} \oplus \mathcal{E}.$$

If D is selfadjoint, we define

$$\mathfrak{Dom}\, D_2 := \{(e, De) \in \mathfrak{G}(D) : e \in \mathfrak{Dom}\, D^2\},$$

and

$$D_2 : (e, De) \mapsto (De, D^2 e).$$

LEMMA 9.16. *Let D be a selfadjoint regular operator in $\mathcal{E}$. The operator D_2 is selfadjoint and regular in $\mathfrak{G}(D)$.*

Iterating this construction gives the *Sobolev chain* of D:

$$\cdots \to \mathfrak{G}(D_{n+1}) \to \mathfrak{G}(D_n) \to \cdots \to \mathfrak{G}(D_2) \to \mathfrak{G}(D) \to \mathcal{E}.$$

In C^*-modules, not every closed submodule is the range of a projection in $\mathrm{End}^*_B(\mathcal{E})$. Modules with this property are called *complemented submodules.* The following theorem states that the graph of a regular operator in a C^*-module is a complemented submodule. The regularity condition on unbounded operators is imposed mainly for this reason.

THEOREM 9.17 ([**2**],[**20**],[**23**]). *Let D be a selfadjoint regular operator in $\mathcal{E}$. Then*

$$p_D := \begin{pmatrix} (1+D^2)^{-1} & D(1+D^2)^{-1} \\ D(1+D^2)^{-1} & D^2(1+D^2)^{-1} \end{pmatrix},$$

is a projection in $\mathrm{End}^*_B(\mathcal{E} \oplus \mathcal{E})$, and $p(\mathcal{E} \oplus \mathcal{E}) = \mathfrak{G}(D)$. Moreover

$$\mathfrak{G}(D) \oplus v\mathfrak{G}(D) \cong \mathcal{E} \oplus \mathcal{E},$$

is an orthogonal direct sum, where v is the unitary $v : (x, y) \mapsto (-y, x)$.

This result is attributed to several people, but Woronowicz explicitly mentions the projection p_D, which is why we refer to it as the Woronowicz projection. The Sobolev modules and Woronowicz projections can be used to construct a chain of subalgebras

$$\cdots \subset \mathcal{A}_{k+1} \subset \mathcal{A}_k \subset \cdots \subset \mathcal{A}_1 \subset A,$$

for any spectral triple or KK-cycle, in the following way. For a KK-cycle $(\mathcal{E}, D)$, we have a representation

$$\pi_1 : \mathcal{A}_1 \to \mathrm{End}^*_B(\mathcal{E} \oplus \mathcal{E}) \cong M_2(\mathrm{End}^*_B(\mathcal{E}))$$
$$a \mapsto \begin{pmatrix} a & 0 \\ [D, a] & (-1)^{\partial a} a \end{pmatrix}.$$

This gives a representation

$$\theta_1 : \mathcal{A}_1 \to \mathrm{End}^*_B(\mathfrak{G}(D)) \oplus \mathrm{End}^*_B(v\mathfrak{G}(D))$$
$$a \mapsto p_D\pi_1(a)p_D + p_D^{\perp}\pi_1(a)p_D^{\perp}.$$

The restriction χ_1 of θ_1 to $\mathfrak{G}(D)$ acts as

$$\chi_1(a) : \begin{pmatrix} e \\ De \end{pmatrix} \mapsto \begin{pmatrix} ae \\ Dae \end{pmatrix}.$$

This allows us to inductively define

$$\mathcal{A}_{n+1} := \{a \in \mathcal{A}_n : [D, \theta_n(a)] \in \mathrm{End}^*_B(\mathfrak{G}(D_n))\},$$
$$\pi_{n+1} : \mathcal{A}_n \to \mathrm{End}^*_B(\mathfrak{G}(D_n) \oplus \mathfrak{G}(D_n))$$

$$a \mapsto \begin{pmatrix} \theta_n(a) & 0 \\ [D, \theta_n(a)] & (-1)^{\partial a}\theta_n(a) \end{pmatrix}, \tag{9.1}$$

$$\theta_{n+1}(a) := p_{n+1}p_n\pi_{n+1}(a)p_np_{n+1} + p_{n+1}^{\perp}p_n^{\perp}\pi_{n+1}(a)p_n^{\perp}p_{n+1}^{\perp}. \tag{9.2}$$

Definition 9.18. The algebra $\mathcal{A}_n$ is the n-th *Sobolev subalgebra* of A. It allows for a completely contractive representation $\chi_n : \mathcal{A}_n \to \mathfrak{G}(D_n)$, which is *not* a *-homomorphism. When $A = \mathrm{End}^*_B(\mathcal{E})$, we write $\mathrm{Sob}_n(D)$ for $\mathcal{A}_n$.

The algebras $\mathcal{A}_n$ can also be characterized by a relative boundedness condition.

Proposition 9.19 ([**21**]). *We have $a \in \mathcal{A}_n$ if and only if*

$$(\mathrm{ad}(D))^n(a)(D \pm i)^{-n+1}, (\mathrm{ad}(D))^n(a^*)(D \pm i)^{-n+1} \in \mathrm{End}^*_B(\mathcal{E}).$$

The representations $\bigoplus_{j=0}^n \pi_j$ realize $\mathcal{A}_n$ a closed subspace of a C^*-algebra, i.e. as an *operator space.* Taking these representations as defining the topology on $\mathcal{A}_n$, the inclusions $\mathcal{A}_{n+1} \to \mathcal{A}_n$ become completely contractive *-homomorphisms.

Proposition 9.20 ([**21**]). *The involution on $\mathcal{A}_n$ is a complete anti-isometry.*

Thus, the Sobolev subalgebras are involutive operator algebras in their natural operator space topology.

10. Smoothness

Although the Sobolev subalgebras of a given KK-cycle always exist and contain the identity, in general we know very little about them. One of the conditions in the definition of KK-cycle is that the algebra $\mathcal{A}_1$ is dense in the C^*-algebra A. This can be interpreted as a smoothness condition.

Definition 10.21. A KK-cycle (E, D) is said to be *(left)* C^k if $\mathcal{A}_k$ is dense in A. It is said to be *(left) smooth* is it is (left) C^k for all k and $\mathscr{A} = \bigcap_k \mathcal{A}_k$ is dense in A.

This definition of smoothness is weaker then the one employed in [**8**]. In particular, spectral triples coming from manifolds are smooth in our sense. Indeed, for a C^k-cycle the Sobolev algebras have good properties.

Theorem 10.22 ([**21**]). *If (E, D) is C^k then the algebras $\mathcal{A}_i$ for $i \leq k$ are stable under holomorphic functional calculus in A.*

DEFINITION 10.23. A *smooth C^*-algebra* is an inverse system of involutive operator algebras

$$\cdots \to \mathcal{A}_{n+1} \to \mathcal{A}_n \to \cdots \to A,$$

coming from a spectral triple.

Smooth C^*-algebras should be thought of as the analogues of smooth manifolds. By holomorphic stability, any finitely generated projective module over a unital smooth C^*-algebra can be smoothened. In the case of countably generated modules, smoothness is not direct anymore, and needs to be imposed on the module.

DEFINITION 10.24. Let B be a C^k-algebra, and $\mathcal{E} \leftrightharpoons B$ a C^*-module. $\mathcal{E}$ is said to be C^k if there is an approximate unit

$$u_n := \sum_{1\leq|i|\leq n} x_i \otimes x_i \in \mathbb{K}_B(\mathcal{E}),$$

such that the norm of the infinite matrix

$$\|(\langle x_i, x_j\rangle)\|_k \leq C.$$

REMARK 10.25. The k-norm is the norm induced by the representation $\bigoplus_{j=0}^k \pi_j$. Since this is an operator norm, it gives norms for all matrix algebras.

The approximate unit, the existence of which is demanded, can be used to construct a chain of submodules

$$\cdots \subset E^{k+1} \subset E^k \subset \cdots \subset E^1 \subset \mathcal{E},$$

which correspond to higher order Lipschitz sections of a vector bundle. In the finitely generated unital case the approximate unit is an actual unit. It is no more than a choice of projection in the subalgebra, which, as mentioned above, is always possible.

PROPOSITION 10.26. *Let B be a smooth C^*-algebra and $\mathcal{E}$ a C^k-B-module. Then*

$$E^k := \{e \in \mathcal{E} : \langle x_i, e\rangle \in \mathcal{B}_k, \quad \sup_n \| \sum_{1\leq|i|\leq n} e_i\langle x_i^\alpha, e\rangle\|_k < \infty\},$$

is an stably rigged $\mathcal{B}_k$-module. Moreover, the inclusions $E^{k+1} \to E^k$ are completely contractive with dense range, and $E^{k+1}\tilde{\otimes}_{\mathcal{B}_{k+1}}\mathcal{B}_k \cong E^k$.

The E^k are stably rigged modules, but they are constructed in a very specific way. This allows us to say a lot more about them than for general stably rigged modules.

THEOREM 10.27. *Let $\mathcal{E}$ be a countably generated C^k-module over a C^k-algebra B. For all $i \leq k$, there are cb-isomorphisms $E^i \oplus \mathcal{H}_{\mathcal{B}_i} \cong \mathcal{H}_{\mathcal{B}_i}$, compatible with the C^k-structure. Consequently, a countably generated C^*-module is a C^k module if and only if it is completely isomorphic to a direct summand in a rigged module.*

For stably rigged modules, operator algebras $\mathrm{End}^*_{\mathcal{B}}(E)$ and $\mathbb{K}_{\mathcal{B}}(E)$ are defined [**3**]. For C^k-modules over a C^k-algebra, the definitions are the same as in the C^*-case.

THEOREM 10.28 ([**21**]). *The submodules $E^i \subset \mathfrak{E}$, $i \le k$, inherit a $\mathcal{B}_i$-valued inner product by restriction of the inner product on $\mathfrak{E}$. We have*

$$\mathrm{End}^*_{\mathcal{B}_i}(E^i) = \{T : E^i \to E^i \quad : \quad \exists T^* : E^i \to E^i, \quad \langle Te, f\rangle = \langle e, T^* f\rangle\},$$

and $\mathbb{K}_{\mathcal{B}_\rangle}(E^i)$ is the i-operator norm closure of the finite rank operators in $\mathrm{End}^*_{\mathcal{B}_i}(E^i)$. Moreover there is a cb-isomorphism

$$\mathbb{K}_{\mathcal{B}_i}(E^i) \cong E^i \tilde{\otimes}_{\mathcal{B}_i} E^{i*},$$

and

$$\mathbb{K}_{\mathcal{B}_i}(E^i) = \mathbb{K}_B(\mathfrak{E}) \cap \mathrm{End}^*_{\mathcal{B}_i}(E^i).$$

That is, they are those operators $T : E^i \to E^i$ that admit an adjoint with respect to the above inner product. Such operators are automatically completely bounded. The involution $T \mapsto T^*$ is in general not a complete isometry, but we have $\frac{1}{C}\|T\|_i \le \|T^*\|_i \le C\|T\|_i$ for some $C \ge 1$. In particular, unitaries are not necessarily isometries.

REMARK 10.29. Note that the topology on the E^i is not defined by the inner product, but by the approximate unit.

In case we have two smooth C^*-algebras A and B, we can now state what it means for a smooth module to be smooth as a bimodule.

DEFINITION 10.30. Let A, B be C^k-algebras. A C^k-module $\mathfrak{E} \leftrightharpoons B$ is a C^k bimodule if the A representation restricts to representations $\mathcal{A}_i \to \mathrm{End}^*_{\mathcal{B}_i}(E^i)$, for $i \le k$.

11. Transverse operators

The theory of regular operators can be developed for C^k-modules. Definitions and most of the essential results still hold true, but their proofs are quite different from the C^* setting. Thus, a selfadjoint operator $D : \mathfrak{Dom}\, D \to E^i$ is said to be regular if it is closed, its domain $\mathfrak{Dom}\, D \subset E^i$ is dense in E^i, and equals the domain of its adjoint and the range of the operators $D \pm i$ is all of E^i. A selfadjoint regular operator in E^k extends to a regular operator in E^i, $i \le k$, as $D \otimes 1$, by proposition 10.26. The main result on selfadjoint regular operators in C^k-modules is the existence of the Woronowicz projection.

THEOREM 11.31. *Let $\mathfrak{E}$ be a C^k-module over a C^k-algebra B, and D a selfadjoint regular operator in E^k. Then*

$$p_D := \begin{pmatrix} (1+D^2)^{-1} & D(1+D^2)^{-1} \\ D(1+D^2)^{-1} & D^2(1+D^2)^{-1} \end{pmatrix},$$

is a projection in $\mathrm{End}^*_{\mathcal{B}_k}(E^k \oplus E^k)$, and $p(E^k \oplus E^k) = \mathfrak{G}(D)$. Moreover

$$\mathfrak{G}(D) \oplus v\mathfrak{G}(D) \cong (E^k \oplus E^k),$$

is an orthogonal direct sum, where v is the unitary $v : (x, y) \mapsto (-y, x)$.

This implies that we get Sobolev subalgebras $\mathrm{Sob}^k_i(D) \subset \mathrm{End}^*_{\mathcal{B}_k}(E^k)$ for all i. We can use the same formulae (9.1),(9.2) to define the representations $\pi^k_i, \theta^k_i$of these Sobolev algebras. We get the same relative boundedness conditions as in proposition 9.19, but now for the i-norms.

Definition 11.32. A KK-cycle $(\mathcal{E}, D)$ over C^k-algebras (A, B) is said to be C^k if $\mathcal{E}$ is a C^k-bimodule, and D restricts to a regular operator in E^k.

D is said to be *transverse* C^k if $\mathcal{A}_i \to \mathrm{Sob}_i^i(D)$ completely boundedly, for all $i \leq k$ (transversality).

12. Smooth connections

The Haagerup tensor product linearizes the multiplication in an operator algebra continuously. Since the definition of connections and 1-forms in section 6 essentially only uses the multiplication in an algebra, these definitions carry over to operator algebras. We define

$$\Omega^1(\mathcal{B}) := \ker(\mathcal{B}\tilde{\otimes}\mathcal{B} \to \mathcal{B}).$$

Connections are defined as in the algebraic setting: A C^k-connection in a C^k-module $\mathcal{E}$ over a C^k-algebra B is a connection

$$\nabla : E^k \to E^k \tilde{\otimes}_{\mathcal{B}_k} \Omega^1(\mathcal{B}_k),$$

which is completely bounded for the present operator space topologies. Since our modules carry inner products, we now require the extra condition of being a $*$-*connection*. This means there is another connection

$$\nabla^* : E^k \to E^k \tilde{\otimes}_{\mathcal{B}_k} \Omega^1(\mathcal{B}_k),$$

such that

$$\langle e_1, \nabla(e_2)\rangle - \langle \nabla^*(e_1), e_2\rangle = d\langle e_1, e_2\rangle.$$

As usual, a $*$-connection is *Hermitian* when $\nabla = \nabla^*$, i.e.

$$\langle e_1, \nabla(e_2)\rangle - \langle \nabla(e_1), e_2\rangle = d\langle e_1, e_2\rangle.$$

We call two C^k-modules $\mathcal{E}$, $\mathcal{F}$ *topologically isomorphic* if there exists an invertible adjointable operator $g : E^k \to F^k$. Such g extends to a topological isomorphism between E^i and F^i for all $i \leq k$.

Theorem 12.33 ([**21**]). *Let B be a C^k-algebra, $\mathcal{E} \leftrightharpoons B$ a C^k-module, and $(\mathcal{F}, T)$ a transverse C^k KK-cycle for (B, C). If $\nabla : E^k \to E^k \tilde{\otimes}_{\mathcal{B}_k} \Omega^1(\mathcal{B}_k)$ is a Hermitian connection, then the operator*

$$1 \otimes_\nabla T : E^k \otimes \mathfrak{Dom}\, T \to E^k \tilde{\otimes}_{\mathcal{B}_k} F^k,$$

is essentially selfadjoint and regular in $E^k \tilde{\otimes}_{\mathcal{B}_k} F^k$. Morever, the graphs

$$\mathfrak{G}((1 \otimes_\nabla T)_i) \subset E^k$$

are topologically isomorphic to $E^k \tilde{\otimes}_{\mathcal{B}_k} \mathfrak{G}(T_i)$, for $i \leq k$.

The operator $1 \otimes_\nabla T$ is symmetric because ∇ is Hermitian. Note that in this theorem the transversality property enters to make sure that each $\mathfrak{G}(T_i)$ is a left $\mathcal{B}_k$-module for $i \leq k$. Also, it should be noted that the isomorphism $\mathfrak{G}(1 \otimes_\nabla T)_i) \to E^i \tilde{\otimes} \mathfrak{G}(T_i)$ is the identity in the first coordinate. As such it gives a description of the domain of the operator $(1 \otimes_\nabla T)_i$, see [**21**] for details. Transverse smoothness of connections is defined straightforwardly, again in an inductive way.

DEFINITION 12.34. A connection $\nabla : E^k \to E^k \tilde{\otimes}_{\mathcal{B}_k} \Omega^1(\mathcal{B}_k)$ on a C^k-cycle $(\mathcal{E}, D)$ is said to be a transverse C^k-connection if $[D, \theta_i(\nabla)]$ extends to a completely bounded operator $\mathfrak{G}(D_i) \to \mathfrak{G}(D_i)\tilde{\otimes}_{\mathcal{B}_i}\Omega^1(\mathcal{B}_i)$ for all $i \leq k$. Equivalently, if

$$(\mathrm{ad}(D))^n(\nabla)(D \pm i)^{-n+1}, \quad (D \pm i)^{-n+1}(\mathrm{ad}(D))^n(\nabla),$$

extend to completely bounded operators $E^k \to E^k\tilde{\otimes}_{\mathcal{B}_k}\Omega^1(\mathcal{B}_k)$, for $n \leq k$.

Note that a transverse C^k connection induces connections $\theta_i(\nabla) : \mathfrak{G}(D_i) \to \mathfrak{G}(D_i)\tilde{\otimes}_{\mathcal{B}_k}\Omega^1(\mathcal{B}_k)$ for all $i \leq k$. These connections are not Hermitian for the inner product on $\mathfrak{G}(D_i)$, but they are $*$-connections.

DEFINITION 12.35. Let A, B be C^k-algebras. A *geometric correspondence* is a C^k-cycle with connection. That is, it is a triple $(\mathcal{E}, D, \nabla)$, where $\mathcal{E}$ is a C^k-bimodule, D a C^k operator, and ∇ a transverse C^k-connection.

13. The product construction

Geometric correspondences can be composed according to the algebraic procedure described in Proposition 6.9. The smoothness conditions imposed on the cycles make sure that this algebraic procedure preserves all the desired analytic properties. In particular, the smoothness conditions themselves are preserved.

THEOREM 13.36 ([**21**]). *Let A, B, C be C^k-algebras, with $k \geq 1$, and $(\mathcal{E}, S, \nabla)$, $(\mathcal{F}, T, \nabla')$ C^k-cycles with connection. Then*

$$(\mathcal{E}\tilde{\otimes}_B \mathcal{F}, S \otimes 1 + 1 \otimes_\nabla T, 1 \otimes_\nabla \nabla'),$$

is a C^k-cycle with connection. It represents the Kasparov product of $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$.

REMARK 13.37. The condition $k \geq 1$ is needed to guarantee that the operator $S \otimes 1 + 1 \otimes_\nabla T$ is selfadjoint. Commutator conditions are direct. A result of Kucerovsky [**19**] on unbounded Kasparov products then gives the last assertion.

We can view geometric correspondences as morphisms of spectral triples. A morphism between C^k spectral triples $(A, \mathcal{H}, D)$ and $(B, \mathcal{H}', T)$ is a C^k-bimodule with connection $(\mathcal{E}, S, \nabla)$ such that the spectral triple

$$(A, \mathcal{E}\tilde{\otimes}_B\mathcal{H}', S \otimes 1 + 1 \otimes_\nabla T),$$

is C^k unitarily isomorphic to $(A, \mathcal{H}, D)$. There is a category of spectral triples for each degree of smoothness. If we denote the category of k-smooth spectral triples by Ψ^k, then Theorem 13.36 says that the bounded transform

$$\mathfrak{b} : (\mathcal{E}, D, \nabla) \mapsto [(\mathcal{E}, D(1 + D^2)^{-\frac{1}{2}})],$$

is a functor $\Psi^k \to \mathfrak{KK}$.

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Deformations of the Canonical Spectral Triples

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Abstract. Deformations of the canonical spectral triples over the $n-$dimensional torus are considered. These deformations have a discrete dimension spectrum consisting of non-integer values less than n. The differential algebra corresponding to these spectral triples is studied. No junk forms appear for non-vanishing deformation parameter. The action of a scalar field in these spaces is considered, leading to non-trivial extra structure compared to the integer dimensional cases, which does not involve a loss of covariance.

1. Introduction

The dimension of a space is a basic concept of particular relevance both in nature and in mathematics. Non-commutative geometry[**1**][**2, 3, 4**] provides a generalization of classical geometry. In particular, it includes a definition of dimension that allows for complex non-integer values[**5**]. A motivation for this definition and a series of very interesting examples of geometries with non-integer dimensions has been given in relation to the study of fractal sets in this geometrical setting([**1**],[**6**] and references therein).

The motivation for this work comes from a different subject. In the realm of quantum field theory(QFT), the widely employed dimensional regularization technique[**7**] provides a hint that non-integer dimensional spaces could be of relevance there. This technique is employed in QFT as a means to regularize divergent integrals appearing in perturbation theory, being preferred in the regularization of gauge theories since it preserves gauge invariance. The technique essentially consists in considering the analytical continuation in the number of dimensions for the area of a d-dimensional sphere, a quantity that appears in the calculation of the above-mentioned integrals. The general question to be addressed in this work is whether a suitable well-defined differential geometry can be found that makes sense for non-integer dimensions and reduces to the canonical one for the integer case[1]. In the affirmative case the natural question to ask is, what does a field theory defined in such a space look like?. More precisely, the idea is to take a field theory defined purely in geometrical terms and repeat the construction in the deformed

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[1]A preliminary study of this question in the 1-dimensional case appears in [**8**]

case. The output of that procedure is by no means obvious since, as will be seen in subsequent sections, the differential algebra is qualitatively different between the integer and non-integer cases and such a change reflects directly in the action of the field theory. For the case of the field theory of a scalar field considered in section 6, the resulting theory is of a novel type. This theory, in spite of reflecting its non-commutative origin, does not involve a breakdown of covariance, as happens in the so-called non-commutative field theories[**9**].

The salient features and results of this work are summarized as follows,

- Spectral triples are considered that differ from the canonical ones in the choice of the Dirac operator.
- The dimension spectrum of these triples consists of a discrete set of real values less than the dimension of the canonical triple.
- The differential of a zero form is not a multiplicative operator.
- There are no junk forms for a non-zero deformation parameter.
- The action of a scalar field contains derivatives of any order and involves an integration over the cosphere.
- In spite of the "non-commutativity" of the differential algebra, there is no loss of covariance involved in the field theory mentioned above.

This paper is organized as follows. Section 2 describes the spectral triple to be considered. In section 3 the corresponding dimension spectrum is computed. The differential of a 0-form is considered in section 4. Section 5 deals with junk forms. Section 6 considers the calculation of the action for a complex scalar field. In addition an Appendix is included which contains the calculation of the Wodzicki residue involved in the definition of the above-mentioned action.

2. The Dirac operator

The differential algebra derived from the canonical spectral triple involving functions over a manifold M reduces to the usual exterior differential algebra over M. The spectral triples to be considered in this work differ from the canonical ones only in the choice of the Dirac operator. More precisely, the triples $(\mathcal{A}, \mathcal{H}, D_\alpha)$ are considered, where,

- $\mathcal{A}$ is the commutative C^*-algebra of smooth functions over the n-dimensional torus T^n $n \in \mathbb{N}$.
- $\mathcal{H}$ is the Hilbert space of square integrable sections of a spinor bundle over T^n.
- $D_\alpha : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator to be defined below.

The usual Dirac operator over an n-dimensional torus T^n is given by,

$$D = i\gamma \cdot \partial = i\gamma_\mu \partial_\mu \;\; , \gamma_\mu = \gamma_\mu^\dagger \, , \;\; \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \; , \mu, \nu = 1, \cdots, n$$

this operator is not positive definite. Indeed since,

$$D^2 = -\Delta = -\partial_\mu \partial_\mu$$

denoting by $\lambda \geq 0$ an eigenvalue of D^2, then $\pm\sqrt{\lambda}$ will be eigenvalues of D.

In this work the usual Dirac operator will be replaced by D_α given below. One of the motivations for this choice is to obtain a dimension spectrum with non-integer real values. This could be done in many ways, for example, by choosing,

$$D_a = D|D^2|^{-\frac{(1-a)}{2}} \;\; , a \in \mathbb{R}, \; 1 > a > 0$$

this operator leads to a dimension spectrum[2] which consists in a single value given by $z = \frac{n}{a}$. However, it is not well-behaved in the infrared. In order to improve its infrared properties and have the same behavior in the ultraviolet, the following operator will be considered in this work,

$$D_\alpha = D(1 + D^2)^{-\alpha},\ \alpha > 0$$

the power appearing in this last equation being defined by,

$$(1 + D^2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} e^{-\tau(1+D^2)} \tag{2.1}$$

Thus the Dirac operator to be considered is,

$$D_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} D(\tau) \ \ , D(\tau) = e^{-\tau(1+D^2)} D$$

this operator is self-adjoint in $\mathcal{H}$, with compact resolvent, and such that the differential of any $a \in \mathcal{A}$ is bounded. This last condition is ensured by the choice $\alpha \geq 0$, as can be readily shown using the expression for the differential of section 4. Therefore, the triple fulfills all the properties required for it to be a spectral triple.

3. Dimension spectrum

The definition of dimension spectrum of a spectral triple is briefly reviewed.

Definition 1. [Connes-Moscovici] *Discrete dimension spectrum.* A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has discrete dimension spectrum Sd if $Sd \subset \mathbb{C}$ is discrete and for any element b in the algebra[3] $\mathcal{B}$ the function,

$$\zeta_b^D(z) = Tr[\pi(b)\, |D|^{-z}] \tag{3.3}$$

extends holomorphically to $\mathbb{C}/Sd$.

The interpretation of these poles is that each of them gives the dimension of a certain piece of the whole space.

In order to apply this definition to the spectral triples considered in this work, it is useful to note that,

$$\begin{aligned} |D_\alpha|^{-z} &= |D|^{-z}(1 + |D|^2)^{\alpha z} = |D|^{-z} \sum_{k=0}^{\infty} \begin{pmatrix} \alpha z \\ k \end{pmatrix} |D|^{2(\alpha z - k)} \\ &= \sum_{k=0}^{\infty} \begin{pmatrix} \alpha z \\ k \end{pmatrix} |D|^{2((\alpha - \frac{1}{2})z - k)} \end{aligned} \tag{3.4}$$

---

[2]See the next section for the definition of dimension spectrum[**5**].

[3]The definition of the algebra $\mathcal{B}$ is the following. Let δ denote the derivation $\delta : L(\mathcal{H}) \to L(\mathcal{H})$ defined by,

$$\delta(T) = [|D|, T] \qquad , T \in L(\mathcal{H}) \tag{3.1}$$

The algebra $\mathcal{B}$ is generated by the elements,

$$\delta^n(\pi(a)),\ a \subset \mathcal{A},\ \ n \geq 0\ (\delta^0(\pi(a)) = \pi(a)) \tag{3.2}$$

where Newton's binomial formula has been employed. From the definitions above it is clear that,

$$\zeta_b^{D_\alpha}(z) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha z \\ k \end{pmatrix} \zeta_b^{D}(2(k-(\alpha-\frac{1}{2})z)) \tag{3.5}$$

where the binomial coefficients are given by,

$$\begin{pmatrix} \alpha z \\ k \end{pmatrix} = \frac{\alpha z(\alpha z-1)\cdots(\alpha z-k+1)}{k!} \quad , \begin{pmatrix} \alpha z \\ 0 \end{pmatrix} = 1$$

The zeta functions appearing in the r.h.s. of (3.5) are the ones corresponding to the canonical spectral triple. Thus, since for the canonical spectral triples the corresponding zeta functions have a single simple pole at its argument equal to n, then $\zeta_b^{D_\alpha}(z)$ has simple poles at,

$$z = \frac{n-2k}{1-2\alpha} \quad , k = 0, 1, 2, \cdots$$

these values of z are therefore the dimension spectrum of the spectral triple considered in this work.

4. The differential

The differential of a 0-form f is given by,

$$df = [D_\alpha, f] = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \, \tau^{\alpha-1} df(\tau) \tag{4.1}$$

$$df(\tau) = [D(\tau), f] \quad .D(\tau) = U(\tau)D \quad , U(\tau) = e^{-\tau(1+D^2)} \tag{4.2}$$

thus when applied to an element ϕ of $\mathcal{H}$, $df(\tau)$ is given by,

$$\begin{aligned} df(\tau)\,\phi &= [D(\tau), f]\,\phi = U(\tau)[(Df)\,\phi + f D\phi] - f\,U(\tau)D\phi \\ &= [U(\tau)(Df) + [U(\tau)f - f\,U(\tau)]D]\,\phi \\ &= U(\tau)\,[(Df) + [f - U(-\tau)f\,U(\tau)]D]\,\phi \end{aligned} \tag{4.3}$$

the second term in the parenthesis of the r.h.s. can be expressed as,

$$e^{\tau(1+D^2)} f(x)\, e^{-\tau(1+D^2)} = f(x - 2\tau\partial) \tag{4.4}$$

this can be easily derived using an analogy with quantum mechanics. This is done noting that $e^{-\tau(1+D^2)}$ is, up to a constant, the imaginary time evolution operator for a free particle of mass $m = 1/2$. Thus,

$$df(\tau) = U(\tau)\,[(Df) - [f(x) - f(x-2\tau\partial)]D]$$

integrating the second line in (4.3) as in (4.1) leads to,

$$df = (1+D^2)^{-\alpha}(Df) + [(1+D^2)^{-\alpha} f - f(1+D^2)^{-\alpha}] i\gamma\cdot\partial$$

which clearly shows that when $\alpha \to 0$, $df \to i\gamma\cdot\partial f$, which is the corresponding expression in the canonical case. It is worth remarking that, as the last equations indicate, this differential is a non-multiplicative operator for any value of $\alpha \neq 0$. As the next section shows, this fact plays an important role in the issue of junk forms.

5. Junk Forms

In this section it is shown that there are no junk forms for a non-zero deformation parameter. To show this it is noted that a generic 1-form can be written as,

$$\omega^{(1)} = \sum_{I,J} \alpha_{JI} f_J df_I$$

where the summation is over a complete basis $B = \{f_I\}$ for $\mathcal{A}$ and the α_{IJ} are numerical coefficients. Replacing (4.1) in the last equation leads to,

$$\omega^{(1)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} \sum_{I,J} \alpha_{JI} f_J \, [U(\tau)(Df_I) + (U(\tau)f_I - f_I\, U(\tau))\, D] \tag{5.1}$$

Junk 2-forms $\omega^{(2)}$ are such that they can be written as the differential of a vanishing 1-form, i.e.,

$$\omega^{(2)} = d\omega^{(1)} \ , \ \ \omega^{(1)} = 0$$

thus the general expression for a vanishing 1-form is sought. From eq.(5.1) this leads to the operatorial equation,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} \sum_{I,J} \alpha_{JI} f_J \, [U(\tau)(Df_I) + (U(\tau)f_I - f_I\, U(\tau))\, D]$$

this equation when applied to a constant spinor leads to,

$$0 = \sum_{I,J} \alpha_{JI} f_J \, Df_I \tag{5.2}$$

which is the same relation that appears for the $\alpha = 0$ case. The general solution is given by,

$$\omega^{(1)} = \sum_{I,J} \beta_{JI} f_J \, (2 f_I df_I - d(f_I^2)) \tag{5.3}$$

for arbitrary numerical coefficients β_{JI}. Using (4.3) gives,

$$\begin{aligned} d(f_I^2) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} d(f_I^2)(\tau) \ \ , \ d(f_I^2)(\tau) \\ &= U(\tau) 2 f_I (Df_I) - [U(\tau), f_I^2] D \\ 2 f_I df_I &= \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} d(f_I^2)(\tau) \ \ , \ 2 f_I df_I(\tau) \\ &= 2 f_I \, \{U(\tau)(Df_I) - [U(\tau), f_I] D\} \end{aligned}$$

thus,

$$2 f_I df_I - d(f_I^2)(\tau) \ = \ [U(\tau), f_I^2] D - 2 f_I [U(\tau), f_I] D - 2 [U(\tau), f_I](Df_I)$$

Applying equation (5.3) to a constant spinor ψ_0 shows that in that case only the last term in the previous equation contributes, therefore the equation $\omega^{(1)} = 0$ leads to,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} \sum_{I,J} \beta_{JI} f_J [U(\tau), f_I](Df_I)\psi_0$$

the linear independence of the basis $B = \{f_J\}$ implying that,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} \sum_{I} \beta_{JI} [U(\tau), f_I](Df_I)\psi_0 \ \ , \forall J \tag{5.4}$$

Next the expansion of the quantity $[U(\tau), f_I](Df_I)$ in the basis B is considered, i.e.,

$$[U(\tau), f_I](Df_I) = \sum_K \alpha_K^I(\tau)\, f_K$$

the coefficients α_K^I being given by,

$$\alpha_K^I(\tau) = \int_x f_K^*[U(\tau), f_I](Df_I)$$

At this stage it is convenient to use the Fourier basis $f_I = e^{iI\cdot x}\ \ , I \in \mathbb{Z}^n$, which are eigenstates of D^2. Noting that,

$$e^{iI\cdot(x-2\tau\partial)} = e^{iI\cdot x} e^{-i2\tau I\cdot\partial} e^{\tau I\cdot I}$$

leads to,

$$\begin{aligned}
\alpha_K^I(\tau) &= \int_x e^{-iK\cdot x}(e^{iI\cdot(x-2\tau\partial)} - 1)U(\tau)(De^{iI\cdot x}) \\
&= \int_x e^{-iK\cdot x}(e^{iI\cdot(x-2\tau\partial)} - 1)e^{iI\cdot x}e^{-\tau(1+I^2)}(-\gamma\cdot I) \\
&= \delta(K-2I)C(\tau, I) \qquad (5.5)
\end{aligned}$$

where the matrix $C(\tau, I)$ is given by,

$$C(\tau, I) = (e^{-3\tau I^2} - 1)e^{-\tau(1+I^2)}(-\gamma\cdot I)$$

replacing in (5.4) leads to,

$$0 = \frac{1}{\Gamma(\alpha)}\int_0^\infty d\tau\, \tau^{\alpha-1}\sum_{I,K}\beta_{JI}\alpha_K^I(\tau)\, f_K\psi_0 \ \ , \forall j$$

which taking into account the linear independence of the basis B and replacing (5.5), implies that,

$$\begin{aligned}
0 &= \frac{1}{\Gamma(\alpha)}\int_0^\infty d\tau\, \tau^{\alpha-1}\beta_{JI}C(\tau, I) \ \ , \forall\, I, J \\
&= \beta_{JI}\left[(1+4I^2)^\alpha - (1+I^2)^\alpha\right](-\gamma\cdot I)
\end{aligned}$$

which is solved by,

$$\beta_{IJ} = 0 \ \ , \forall\, J, I \neq 0$$

The case $\beta_{0J} \neq 0$ is trivial since anyhow in that case $\omega^{(1)} = 0$.

6. The scalar field

In this section the part of this space corresponding to the highest pole will be considered, i.e. for $d = \frac{n}{1-2\alpha}$. The action for a free scalar field propagating in this space is taken to be,

$$S = \frac{1}{2} < d\phi, d\phi >$$

where ϕ is a 0-form and the norm in the space of forms is given by[4],

$$< \omega, \omega > \ = \ tr_\omega[\omega\omega^\dagger|D_\alpha|^{-d}] \qquad (6.1)$$

thus,

$$S = -tr_\omega[d\phi d\phi^*|D_\alpha|^{-d}]$$

[4]See for example ref.[**3**]

where it was used that $d\phi^\dagger = -d\phi^*$ and tr_ω denotes the Diximier trace. In the evaluation of this trace it is important to note that replacing $d = \frac{n}{1-2\alpha}$ in (3.4) leads to,

$$|D_\alpha|^{-d} = |D_\alpha|^{-\frac{n}{1-2\alpha}} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{\alpha n}{1-2\alpha} \\ k \end{pmatrix} |D|^{-n-2k} \tag{6.2}$$

Therefore S is given by,

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \begin{pmatrix} \frac{\alpha n}{1-2\alpha} \\ k \end{pmatrix} S_k \\ S_k &= -tr_\omega[d\phi(\tau)d\phi(\tau')^*|D|^{-n-2k}] \end{aligned}$$

Noting that,

$$\begin{aligned} d\phi &= [U_\alpha D, \phi(x)], \quad U_\alpha = (1+D^2)^{-\alpha} \\ d\phi^* &= [U_\alpha D, \phi^*(x)] \end{aligned}$$

leads to,

$$S_k = tr_\omega \left\{ [U_\alpha D, \phi][DU_\alpha, \phi^*]|D|^{-n-2k} \right\}$$

Thus, replacing the expression obtained in the appendix for S_k leads to,

$$S = -\frac{2^{[\frac{n}{2}]} V_{S^{n-1}}}{n(2\pi)^n} \int_{T^n} \phi(D^2 + \frac{\alpha n}{1-2\alpha})(1+D^2)^{-2\alpha}\phi^* \tag{6.3}$$

where $V_{s^{n-1}} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the $n-1$-dimensional sphere. It is worth noting that in spite of starting with an action involving no mass term, the fact of working on a non-integer dimensional space effectively generates such a term as shown by (6.3), with a coefficient that vanishes in the integer case($\alpha = 0$). In that case (6.3) reduces to the usual action of a massless complex scalar field, i.e.,

$$S_{can} = \lim_{\alpha \to 0} S = \frac{2^{[\frac{n}{2}]} V_{s^{n-1}}}{(2\pi)^n} \int_{T^n} \left(\frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi^*(x) \right)$$

It is remarked that the approach presented in this work differs significantly from the so-called non-commutative field theory[**9**]. No non-commutativity of the coordinates is assumed. On the contrary, non-commutativity enters at the level of the differential algebra through the deformed choice of the Dirac operator. This difference implies that this non-commutativity does not spoil the covariance of suitably chosen field theories on these spaces. From the point of view of physics this feature, together with its eventual connection with dimensional regularization, are believed to be of interest and justify further research on these theories.

Appendix : Evaluation of the Diximiers trace

As shown in the previous section the actions to be evaluated are,

$$\begin{aligned} S_k &= tr_\omega \left\{ [U_\alpha D, \phi][DU_\alpha, \phi^*]|D|^{-n-2k} \right\} \ , \ U_\alpha = (1+D^2)^{-\alpha} \\ &= tr_\omega \left\{ \left(2\phi D U_\alpha \phi^* U_\alpha D - \phi\phi^*(U_\alpha D)^2 - \phi(U_\alpha D)^2\phi^* \right) |D|^{-n-2k} \right\} = tr_\omega \left\{ A_k \right\} \end{aligned}$$

these Diximier traces will be evaluated using their expression as Wodzicki residues,

$$S_k = \frac{1}{n(2\pi)^n} \int_{S^*T^n} tr\, \sigma_{-n}^{A_k}(x, \xi)$$

where $\sigma_{-n}^{A_k}(x,\xi)$ denotes the term of order $-n$ of the symbol of the operator A_k , (x,ξ) denote coordinates on the unit co-sphere on the cotangent bundle of T^n, so that $\int_{S^*T^n} = \int_x \int_\xi d\Omega_{n-1}$where $d\Omega_{n-1}$ is the volume element of the sphere S^{n-1}. The trace is taken over the spin space and the symbol is defined by,

$$\sigma^{A_k}(x,\xi) = e^{-ix\cdot\xi} A_k \, e^{ix\cdot\xi}$$

so that,

$$\begin{aligned}\sigma^{A_k}(x,\xi) = (\; 2e^{-ix\cdot\xi}\phi D U_\alpha \phi^* e^{ix\cdot\xi}(-\gamma\cdot\xi)(1+|\xi|^2)^{-\alpha} \\ - \phi\phi^*(1+|\xi|^2)^{-\alpha}|\xi|^2 - \phi(DU_\alpha)^2\phi^* \;) \, |\xi|^{-n-2k}\end{aligned}$$

using that,

$$e^{-ix\cdot\xi} D U_\alpha e^{ix\cdot\xi} = (D-\gamma\cdot\xi)(1+(D-\gamma\cdot\xi)^2)^{-\alpha}$$

ignoring terms that vanish when integrating over $|\xi| = 1$ and evaluating the trace, leads to,

$$\begin{aligned} tr\sigma^{A_k}(x,\xi) &= 2^{[\frac{n}{2}]} \big(2\phi(1+(D-\xi)^2)^{-\alpha}\phi^*(1+|\xi|^2)^{-\alpha}|\xi|^{-n-2k+2} \\ &\quad -\phi\phi^*(1+|\xi|^2)^{-2\alpha}|\xi|^{-n-2k+2} \\ &\quad -\phi[D^2+\xi^2](1+(D-\xi)^2)^{-2\alpha}\phi^*|\xi|^{-n-2k}\big) \end{aligned}$$

for $\alpha \neq 0$ the first two terms do not contribute to the term of order $-n$ of the symbol because they include the factor,

$$(1+|\xi|^2)^{-\alpha} = \sum_{k=0}^{\infty} \left(\begin{array}{c} -\alpha \\ k \end{array} \right) |\xi|^{-2(\alpha+k)}$$

which decreases the order of the corresponding terms by at least -2α. The last term gives two non-vanishing contributions. One coming from the term D^2 inside the square bracket, which contributes to the term of order $-n$ of the symbol only when $k = 0$. And the other from the ξ^2 inside the square bracket, which contributes to the term of order $-n$ of the symbol only when $k = 1$. Thus,

$$\begin{aligned} \sigma_{-n}^{A_0}(x,1) &= -\phi D^2(1+D^2)^{-2\alpha}\phi^* \\ \sigma_{-n}^{A_1}(x,1) &= -\phi(1+D^2)^{-2\alpha}\phi^* \end{aligned}$$

and,

$$\begin{aligned} S_0 &= -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\int_{T^n}\phi D^2(1+D^2)^{-2\alpha}\phi^* \\ S_1 &= -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\frac{\alpha n}{1-2\alpha}\int_{T^n}\phi(1+D^2)^{-2\alpha}\phi^* \end{aligned}$$

leading to,

$$S = -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\int_{T^n}\phi(D^2+\frac{\alpha n}{1-2\alpha})(1+D^2)^{-2\alpha}\phi^*$$

where $V_{S^{n-1}}$ denotes the area of the sphere S^{n-1} given by,

$$V_{S^{n-1}} = \int d\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

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Twisted Bundles and Twisted K-Theory

Max Karoubi

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Introduction

Many papers have been devoted recently to twisted K-theory as originally defined in [**15**] and [**29**]. See for instance the references [**2**], [**23**] and the very accessible paper [**30**]. We offer here a more direct approach based on the notion of "twisted vector bundles". This is not an entirely new idea, since we find it in [**4**], [**6**], [**7**], [**8**] and [**9**] for instance, under different names and from various viewpoints. However, a careful look at this notion shows that we may interpret such bundles as modules over suitable algebra bundles. More precisely, the category of twisted vector bundles is equivalent to the category of vector bundles which are modules over algebra bundles with fibre $\mathrm{End}(V)$, where V is a finite dimensional vector space. This notion was first explored in [**15**] in order to define twisted K-theory. In the same vein, twisted Hilbert bundles may be used to define extended twisted K-groups, following [**14**] and [**29**].

More generally, we also analyse the notion of "twisted principal bundles" with structural group G. Under favourable circumstances, we show that the associated category is equivalent to the category of locally trivial fibrations, with an action of a bundle of groups with fibre G, which is simply transitive on each fibre. Such

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bundles are classically called "torsors" in the literature. When the bundle of groups is trivial, we recover the usual notion of principal G-bundle.

As is well known, twisted K-theory is a graded group, indexed essentially by the third cohomology[1] of the base space X, namely $H^3(X;\mathbb{Z})$. The twisted vector bundles we define in this paper are also indexed by elements of the same group up to isomorphism. Roughly speaking, twisted K-theory appears as the Grothendieck group of the category of twisted vector bundles. This provides a geometric description of this theory, very close in spirit to Steenrod's definition of coordinate bundles [**31**]. The more subtle notion of graded twisted K-theory, indexed by $H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z})$, may also be analyzed in this framework.

The usual operations on vector bundles (exterior powers, Adams operations,...) are easily extended to twisted vector bundles, in a way parallel to the operations defined in [**2**]. We have also added a section on cup-products, in order to show that the various ways to define them coincide up to isomorphism. This is essentially relevant in the last section of the paper, where we define an analog of the Chern character.

In this section, we define connections on twisted vector bundles in the finite and infinite dimensional cases, very much in the spirit of [**26**, pg. 78], [**6**], [**22**, Chapitre 1], in a quite elementary way. It is also described in [**4**] and [**11**] with a different method. From this analog of Chern-Weil theory, we deduce a "Chern character" from twisted K-theory to twisted cohomology. This character is defined in a much more elaborate way in [**3**], [**8**], [**27**] and [**32**] in the general framework of the "Connes-Karoubi Chern character" [**12**], [**22**], except in [**3**]. In the paper of Atiyah and Segal [**3**], classical topology tools are used to show that the twisted Chern character is essentially unique. Therefore, it coincides with the character defined by our elementary approach in this paper.

Finally, in a detailed appendix divided into three subsections, we study carefully the relation between Čech cohomology with coefficients in S^1 and de Rham cohomology. We also discuss more deeply multiplicative structures and the functorial aspects of twisted K-theory and of the Chern character.

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1. Twisted principal bundles

Let G be a topological group and let $\mathcal{U} = (U_i), i \in I$, be an open covering of a topological space X. The Čech cohomology set $H^1(\mathcal{U};G)$ is well known (see [**31**], [**18**] for instance). One starts with "non-abelian" 1-cocycles g, i.e. a set of continuous maps (also called "transition functions")

$$g_{ji} : U_i \cap U_j \longrightarrow G,$$

such that $g_{kj}\cdot g_{ji} = g_{ki}$ on $U_i \cap U_j \cap U_k$. Two cocycles g and h are equivalent if there are continuous maps

$$u_i : U_i \longrightarrow G,$$

[1] More precisely, it is indexed by 3-cocycles. Two cohomologous cocycles give twisted K-groups which are isomorphic (non-canonically). This technical point is discussed in Appendix 8.3.

such that

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i. \tag{1.1}$$

The set of equivalence classes is denoted by $H^1(\mathcal{U}; G)$. A covering $\mathcal{V} = (V_s), s \in S$, is a refinement of $\mathcal{U}$ if there is a map $\tau : S \longrightarrow I$ such that $V_s \subset U_{\tau(s)}$. We then have a "restriction map"

$$R_\tau : H^1(\mathcal{U}; G) \longrightarrow H^1(\mathcal{V}; G),$$

assigning to the $g's$ the functions $k = \tau^*(g)$ defined by

$$k_{s,r} = g_{\tau(s),\tau(r)}.$$

It is shown in [**18**, pg. 48] for instance that the map R_τ is in fact independent of the choice of τ. We then define

$$H^1(X; G) = \operatorname*{Colim}_{\mathcal{U}} H^1(\mathcal{U}; G),$$

where $\mathcal{U}$ runs over the "set" of coverings of X.

Now let Z be a subgroup of the centre of G and let $\lambda = (\lambda_{kji})$ be a completely normalized 2-cocycle of $\mathcal{U}$ with values in Z. This normalization condition on the cocycle means that $\lambda = 1$ if two of the three indices k, j, i are equal and that

$$\lambda_{\sigma(k)\sigma(j)\sigma(i)} = (\lambda_{kji})^{\varepsilon(\sigma)},$$

where σ is a permutation of the indices (k, j, i), with signature $\varepsilon(\sigma)$.

Remark 1.1. One can prove (see [**21**] for instance) that a Čech cocycle in any dimension is cohomologous to a completely normalized one. Moreover, if every open subset of X is paracompact, any cohomology class may be represented by a completely normalized Čech cocycle.

A λ-twisted 1-cocycle (simply called twisted cocycle if λ is implicit) is then given by transition functions $g = (g_{ji})$ as above, such that

$$g_{ii} = 1, g_{ji} = (g_{ij})^{-1}$$

and

$$g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji}$$

on $U_i \cap U_j \cap U_k$. If we compute the product $g_{lk} \cdot g_{kj} \cdot g_{ji}$ in two different ways using associativity, we indeed find that λ should be a 2-cocycle. On the other hand, one can easily show that the function $g_{ij} \cdot g_{jk} \cdot g_{ki}$ is invariant under a cyclic permutation of the indices and is changed to its inverse if we permute i and k. Since we have $\lambda_{kjk} = 1$, the cocycle λ should be completely normalized.

Two twisted cocycles g and h are equivalent if there are continuous maps $u_i : U_i \longrightarrow G$, such that we have a condition analogous to the above:

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i \qquad (1.1)$$

We define the twisted (non-abelian) cohomology $H^1_\lambda(\mathcal{U}; G)$ as the set of equivalence classes.

Proposition 1.2. *Let μ be a 2-cocycle cohomologous to λ, i.e. such that we have the relation*

$$\mu_{kji} = \lambda_{kji} \cdot \eta_{ji} \cdot \eta_{ki}^{-1} \cdot \eta_{kj},$$

for some $\eta = (\eta_{ji})$ with $\eta_{ji} = (\eta_{ij})^{-1}$ and $\eta_{ii} = 1$. Then the map

$$\Theta : H^1_\lambda(\mathcal{U}; G) \longrightarrow H^1_\mu(\mathcal{U}; G),$$

sending (g) *to the twisted cocycle* (g') *given by* $g'_{ji} = g_{ji} \cdot \eta_{ji}$*, is an isomorphism.*

Proof. If we compute $g'_{kj} \cdot g'_{ji}$ we indeed find

$$g'_{kj} \cdot g'_{ji} = g'_{ki} \cdot \lambda_{kji} \cdot \eta_{kj} \cdot \eta_{ji} \cdot (\eta_{ki})^{-1} = g'_{ki} \cdot \mu_{kji}.$$

This shows that the map Θ is well defined. The inverse map is of course given by the correspondence $(g'_{ji}) \longmapsto (g'_{ji} \cdot \eta_{ji}^{-1})$. □

From the previous considerations one may define the following category. The objects are λ-twisted bundles on a covering $\mathcal{U}$, the morphisms between (g_{ji}) and (h_{ji}) being continuous maps (u_i), with the compatibility condition (1.1). In this category the covering $\mathcal{U}$ is fixed together with the 2-cocycle λ.

However, this category is too rigid for our purposes, since we want to consider covering refinements. The covering $\mathcal{V} = (V_s), s \in S$, is a refinement of $\mathcal{U} = (U_i), i \in I$ if there is a map $\tau : S \longrightarrow I$ such that $V_s \subset U_{\tau(s)}$.This map τ induces a morphism

$$\Theta_\tau : H^1_\lambda(\mathcal{U}; G) \longrightarrow H^1_\mu(\mathcal{V}; G)$$

which is not necessarily an isomorphism. Starting with a twisted cocycle (g_{ji}), its image by Θ_τ is the cocycle (h_{sr}) given by the formula

$$h_{sr} = g_{\tau(s)\tau(r)}.$$

The 2-cocycle associated to h is

$$\mu_{tsr} = g_{\tau(t)\tau(s)} \cdot g_{\tau(s)\tau(r)} \cdot g_{\tau(r)\tau(t)} = \lambda_{\tau(t)\tau(s)\tau(r)}.$$

Proposition 1.3. *Let* τ *and* τ' *be two maps from* S *to* I *such that* $V_s \subset U_{\tau(s)}$ *and* $V_s \subset U_{\tau'(s)}$ *and let* x *be an element of the set* $H^1_\lambda(\mathcal{U}; G)$*. Then* $\Theta_\tau(x)$ *and* $\Theta_{\tau'}(x)$ *are related through an isomorphism*

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G),$$

made explicit in the proof below. This isomorphism does not depend on x *and depends only on* τ, τ' *and the 2-cocycle* λ*.*

Proof. Let h' be the following transition functions:

$$h'_{sr} = g_{\tau'(s)\tau'(r)}.$$

We may write

$$h'_{sr} = g_{\tau'(s)\tau(s)} \cdot h_{sr} \cdot g_{\tau(r)\tau'(r)} \cdot \sigma_{sr},$$

where

$$\sigma_{sr} = \lambda_{\tau'(s)\tau(r)\tau(s)} \cdot \lambda_{\tau'(s)\tau'(r)\tau(r)}.$$

Since we have $h_{rs} = (h_{sr})^{-1}, h_{rr} = 1$ and the same properties for h', it follows that $\sigma_{rs} = (\sigma_{sr})^{-1}$ and $\sigma_{rr} = 1$. Therefore, 1) the twisted 1-cocycles (h_{sr}) and $(\overline{h}_{sr})$, where

$$\overline{h}_{sr} = g_{\tau'(s)\tau(s)} \cdot h_{sr} \cdot g_{\tau(r)\tau'(r)},$$

are isomorphic in the category of twisted bundles over $\mathcal{V}$ with the same twist. 2) the twisted bundles defined by the 1-cocycles $(\overline{h}_{sr})$ and (h'_{sr}) are also isomorphic through the isomorphism

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G)$$

defined in the previous proposition. We note that μ' is the following 2-cocycle with values in Z:

$$\mu'_{tsr} = \overline{h}_{ts} \cdot \overline{h}_{sr}.\overline{h}_{rt} = h_{ts} \cdot h_{sr} \cdot h_{rt} \cdot \sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt} = \mu_{tsr} \cdot \sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt},$$

which is of course cohomologous to μ. It remains to show that the isomorphism

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G)$$

only depends on τ and τ' and not on the specific element x. The previous identity indeed shows that the 2-cocycles μ and μ' are cohomologous through the completely normalized 1-cochain σ which is a function of λ only. □

Remark 1.4. Although we don't need it in the proof, this computation showing that μ and μ' are cohomologous is based on the existence of a twisted 1-cocycle (g_{ji}) associated to a 2-completely normalised cocycle λ. Unfortunately, this is not true in general. However, when X has the homotopy type of a CW-complex, we may also argue as follows in greater generality. First we may assume that X is pathwise connected, so that we can choose a base point on X. Now let PX be the path space of X and let

$$\pi : PX \to X$$

be the canonical map associating to a path starting at the base point its end point. In order to check that $\sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt} = \mu'_{tsr} \cdot (\mu_{tsr})^{-1}$, we consider the covering of PX defined by the pullback $\pi^*(\mathcal{U})$ of the covering $\mathcal{U}$ of X. Since the nerve of $\pi^*(\mathcal{U})$ is contractible, there is a completely normalized 1-cochain $\overline{g}$, with values in the subgroup Z of G, such that λ_{tsr} is the associated twist, i.e. its coboundary. This enables us to perform the previous computations on PX (with $\overline{g}$ as our 1-twisted cocycle) and hence on X, since the pullback of functions from X to PX by the map π is injective.

2. Relation with torsors

There is another interpretation of twisted principal bundles in some favourable circumstances and which is more familiar. For this, we observe that G acts on itself by inner automorphisms and that the kernel of the map

$$G \to \mathrm{Aut}(G)$$

is the centre of G. We now assume that the map

$$G \to G/Z$$

is a locally trivial fibration. In the applications we have in mind, G is a Lie group or a Banach Lie group and it is well known that this condition is fulfilled if Z is a closed subgroup of the centre.

On the other hand, we notice that if P is a twisted principal bundle associated to a covering $\mathcal{U}$ with transition functions g_{ji}, we may define a bundle of groups $\mathrm{AUT}(P)$ with fiber G as follows. Its transition functions are defined over $U_i \cap U_j$ by

$$g \longmapsto g_{ji} \cdot g \cdot (g_{ji})^{-1} = g_{ji} \cdot g \cdot g_{ij}.$$

Proposition 2.1. *Let $\widetilde{G}$ be a bundle of groups with fibre G and with structural group G/Z, acting by inner automorphisms on G. Then, if the covering $\mathcal{U} = (U_i)$ is fine enough, there is a twisted principal bundle P such that $\widetilde{G}$ is isomorphic to the bundle of groups $\mathrm{AUT}(P)$ defined above.*

Proof. The bundle of groups $\widetilde{G}$ is given by transition functions

$$\gamma_{ji} : U_i \cap U_j \to G/Z,$$

where

$$\gamma_{ii} = \mathrm{Id} \quad \text{and} \quad \gamma_{ij} = (\gamma_{ji})^{-1}.$$

According to our assumptions, the fibration $G \to G/Z$ is locally trivial. Therefore, if the covering $\mathcal{U}$ is fine enough, we can find continuous functions

$$g_{ji} : U_i \cap U_j \to G$$

such that the class of g_{ji} is γ_{ji}, and moreover $g_{ii} = \mathrm{Id}$, $g_{ij} = (g_{ji})^{-1}$. From these identities, it follows that the following continuous function defined on $U_i \cap U_j \cap U_k$:

$$\lambda_{kji} = g_{kj} \cdot g_{ji} \cdot g_{ik},$$

is a completely normalized 2-cocycle with values in Z. Therefore, it defines a twisted principal bundle P with transition functions (g_{ji}). Moreover, according to the previous considerations, the bundle of groups $\widetilde{G}$ is canonically isomorphic to $\mathrm{AUT}(P)$, with transition functions

$$u \longmapsto \overline{g}_{ji}(u) = g_{ji} \cdot u \cdot g_{ij}.$$

□

This proposition enables us to relate the category of twisted principal bundles to more classical mathematical objects. We notice that if P and Q are twisted principal bundles with transition functions g_{ji} and h_{ji} respectively (with the same twist λ), we can define a locally trivial bundle $\mathrm{ISO}(P, Q)$ with fibre G, the transition functions being automorphisms of the underlying space G defined by

$$u \longmapsto h_{ji} \cdot u \cdot g_{ij} = \theta_{ji}(u).$$

Since we have $g_{kj} \cdot g_{ji} \cdot g_{ik} = h_{kj} \cdot h_{ji} \cdot h_{ik} = \lambda_{kji}$, the 1-cocycle condition is satisfied for the bundle $\mathrm{ISO}(P, Q)$, i.e. we have the relation

$$\theta_{kj} \cdot \theta_{ji} = \theta_{ki}.$$

In particular, if $P = Q$, we get the previous bundle of groups $\mathrm{AUT}(P)$.

Moreover, there is a bundle map

$$\mathrm{ISO}(P, Q) \times \mathrm{AUT}(P) \to \mathrm{ISO}(P, Q).$$

It is defined by

$$(u, v) \longmapsto u \circ v,$$

or by $(u_i, v_i) \mapsto u_i \circ v_i$ in local coordinates. Therefore, the bundle $\mathrm{ISO}(P, Q)$ inherits a right fibrewise $\mathrm{AUT}(P)$-action which is simply transitive on each fibre. In classical terminology[2], the bundle $\mathrm{ISO}(P, Q)$ is a "torsor" over the bundle of groups $\mathrm{AUT}(P)$, acting on the right.

[2]It is not the purpose of this paper to develop the theory of torsors. Roughly speaking, this notion is a generalization of the definition of a principal bundle P. Instead of having a topological group G acting on P as usual, we have a bundle of groups $\widetilde{G}$ acting fiberwise on P in a way which is simply transitive on each fiber. In our situation, the structural group of $\widetilde{G}$ is G/Z, acting on G by inner automorphisms.

THEOREM 2.2. *Let $\widetilde{G}$ be a bundle of groups with fibre G and structural group G/Z acting on G by inner automorphisms. We assume the existence of a covering $\mathcal{U} = (U_i)$ such that $\widetilde{G}$ may be written as* $\mathrm{AUT}(P)$*, where P is a λ-twisted principal bundle. Then, any torsor M over $\widetilde{G}$ may be written as* $\mathrm{ISO}(P, Q)$*, where Q is a λ-twisted principal bundle. More precisely, the correspondence $Q \longmapsto$* $\mathrm{ISO}(P, Q)$ *induces an equivalence between the category of λ-twisted principal bundles and the category of $\widetilde{G}$-torsors.*

PROOF. Let γ_{ji} be the transition functions of M with fibre G and let g_{ji} be the transition functions of P. Then the transition functions of $\mathrm{AUT}(P)$ are given by $\overline{g}_{ji}(u) = g_{ji} \cdot u \cdot g_{ij}$. Now we claim that the transition functions of M should be of type

$$\gamma_{ji}(u) = h_{ji} \cdot u \cdot g_{ij},$$

for some continuous functions h_{ji}. In order to prove this, we use the action of $\widetilde{G}$ on the right by writing

$$\gamma_{ji}(u) = \gamma_{ji}(1 \cdot u) = \gamma_{ji}(1) \cdot \overline{g}_{ji}(u) = \gamma_{ji}(1) \cdot g_{ji}.u \cdot g_{ij}.$$

We then put $h_{ji} = \gamma_{ji}(1) \cdot g_{ji}$. The fact that $\gamma_{kj} \cdot \gamma_{ji} = \gamma_{ki}$ implies the identity

$$h_{kj} \cdot h_{ji} \cdot u \cdot g_{ij} \cdot g_{jk} = h_{ki} \cdot u \cdot g_{ik}.$$

Since $g_{ij} \cdot g_{jk} = g_{ik} \cdot \lambda_{ijk}$, this implies that $h_{kj} \cdot h_{ji} = h_{ki} \cdot (\lambda_{ijk})^{-1} = h_{ki} \cdot \lambda_{kji}$; therefore the (h_{ji}) are the transition functions of a λ-twisted principal bundle. We have to check the coherence of the action of $\widetilde{G}$ on the right, i.e. the identity

$$\gamma_{ji}(u \cdot v) = \gamma_{ji}(u) \cdot \overline{g}_{ji}(v).$$

This follows from the simple calculation in local coordinates

$$\gamma_{ji}(u \cdot v) = h_{ji} \cdot (u \cdot v) \cdot g_{ij} = (h_{ji} \cdot u \cdot g_{ij}) \cdot (g_{ji} \cdot v \cdot g_{ij}) = \gamma_{ji}(u) \cdot \overline{g}_{ji}(v).$$

The previous computations show that we can define a functor backwards from the category of $\widetilde{G}$-torsors to the category of λ-twisted principal bundles. It remains to prove that the map

$$\mathrm{Hom}(Q, Q') \to \mathrm{Hom}(\mathrm{ISO}(P, Q), \mathrm{ISO}(P, Q'))$$

is an isomorphism. For this, we analyse the morphisms

$$\mathrm{ISO}(P, Q) \to \mathrm{ISO}(P, Q')$$

which are compatible with the structure of $\mathrm{AUT}(P)$-torsor. Such a morphism

$$\mathrm{ISO}(P, Q) \to \mathrm{ISO}(P, Q')$$

is given in local coordinates by the formula

$$\Phi : u \longmapsto \beta_i \cdot u \cdot \alpha_i,$$

where (α_i) (resp. (β_i)) is associated to $\mathrm{AUT}(P)$ (resp. $\mathrm{ISO}(Q, Q')$). We notice the formula

$$h'_{ji} \cdot \beta_i \cdot u \cdot \alpha_i \cdot g_{ij} = \beta_j \cdot h_{ji} \cdot u \cdot g_{ij} \cdot \alpha_j,$$

where h'_{ji} are the coordinate functions of Q'. In the same way, an element of $\mathrm{AUT}(P)$ is given in local coordinates by

$$\Upsilon : g \longmapsto g \cdot \alpha_i.$$

Therefore, the equation

$$\Phi(u \cdot g) = \Phi(u) \cdot \Upsilon(g)$$

may be written

$$\beta_i \cdot (u \cdot g) \cdot \alpha_i = (\beta_i \cdot u \cdot \alpha_i) \cdot (g \cdot \alpha_i),$$

which is only possible if $\alpha_i = 1$. □

Remark 2.3. An analog of this theorem in the framework of vector bundles will be proved in the next section (Theorem 3.5).

3. Twisted vector bundles

One of the main aims of this paper is the theory of "twisted" vector bundles[3]. We essentially studied it in Section 1, with the structural group $G = GL_n(\mathbb{C})$. However, to keep track of the linear structure and because we want the "fibres" not to have the same dimension on each connected component of X, we slightly change the general definition as follows.

We start as before with a covering $\mathcal{U} = (U_i), i \in I$, together with a finite dimensional vector space E_i "over" U_i. Another piece of information is a completely normalized 2-cocycle λ_{kji} with values in $\mathbb{C}^*$. A λ-twisted vector bundle E on X is then defined by transition functions

$$g_{ji} : U_i \cap U_j \to \mathrm{Iso}(E_i, E_j),$$

such that

$$g_{ii} = 1, g_{ji} = (g_{ij})^{-1}$$

and

$$g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji},$$

as in the previous section. There is however a slight change for the definition of morphisms from a twisted vector bundle E to another one F, with the same twist λ. They are defined as continuous maps

$$u_i : U_i \to \mathrm{Hom}(E_i, F_i),$$

such that

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i.$$

The point is that we no longer require the u_i to be isomorphisms.

More generally, let E be a λ-twisted vector bundle on a covering $\mathcal{U}$ with transition functions (g_{ji}) and let F be a μ-twisted vector bundle on the same covering with transition functions (h_{ji}). We define a $\lambda^{-1} \cdot \mu$-twisted vector bundle in the following way: over each U_i we take as "fibre" $\mathrm{Hom}(E_i, F_i)$ and as transition functions the isomorphisms

$$\mathrm{Hom}(E_i, F_i) \to \mathrm{Hom}(E_j, F_j),$$

defined by

$$\theta_{ji} : f_i \longmapsto h_{ji} \circ f_i \circ g_{ij} = f_j.$$

We denote this twisted vector bundle by $\mathrm{HOM}(E, F)$. An interesting case is when E and F are associated to the same 2-cocycle λ. Then $\mathrm{HOM}(E, F)$ is a genuine vector bundle associated to the vector space of morphisms $\mathrm{Hom}(E, F)$ by the following proposition.

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[3]For simplicity's sake, we shall only consider complex vector bundles. The theory for real or quaternionic vector bundles follows the same pattern. More generally, we may also consider vector bundles with fibres finitely generated projective modules over a Banach algebra. This remark will be useful in the next section for $\mathcal{A}$-bundles.

PROPOSITION 3.1. *Let E and F be two λ-twisted vector bundles. Then the vector space of morphisms from E to F, i.e. $Hom(E,F)$, may be canonically identified with the vector space of sections of the vector bundle* $\mathrm{HOM}(E,F)$.

PROOF. A section of this vector bundle is defined by elements f_i of $\mathrm{Hom}(E_i,F_i)$ such that

$$\theta_{ji}(f_i) = f_j.$$

This relation is translated as

$$h_{ji} \circ f_i = f_j \circ g_{ji},$$

which is exactly the definition of morphisms from E to F. □

An interesting case of the previous proposition is when $E = F$, so that $\mathrm{HOM}(E,E) = \mathrm{END}(E)$ is an algebra bundle A. The following theorem relates algebra bundles to twisted vector bundles.

THEOREM 3.2. *Any algebra bundle A with fibre* $\mathrm{End}(V)$, *where V is a finite dimensional vector space of positive dimension, is isomorphic to some* $\mathrm{END}(E)$, *where E is a twisted vector bundle on a suitably fine covering of X.*

PROOF. Let $V = \mathbb{C}^n$. According to the Skolem-Noether Theorem, the structural group of A is $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times$, where $\mathrm{PGL}_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by inner automorphisms. We may describe this bundle A by transition functions

$$\gamma_{ji} : U_i \cap U_j \to \mathrm{PGL}_n(\mathbb{C}),$$

for a suitable covering $\mathcal{U} = (U_i)$ of X. Without loss of generality, we may assume that $\gamma_{ii} = 1$ and that $\gamma_{ji} = (\gamma_{ij})^{-1}$. On the other hand, the principal fibration

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

admits local continuous sections. Therefore, if we choose the covering $\mathcal{U} = (U_i)$ fine enough, we can lift these γ_{ji} to continuous functions

$$g_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C}).$$

Moreover, we may choose the g_{ji} such that $g_{ii} = 1, g_{ij} = (g_{ji})^{-1}$. Therefore, we have the identity $g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji}$, where

$$\lambda_{kji} : U_i \cap U_j \cap U_k \to \mathbb{C}^\times$$

is de facto a completely normalized 2-cocycle. If E is the twisted vector bundle associated to the g_{ji} we see that the algebra bundle $\mathrm{END}(E)$ has transition functions which are

$$f \longmapsto g_{ji} \circ f \circ (g_{ji})^{-1},$$

i.e. the inner automorphisms associated to the g_{ji}. □

REMARK 3.3. We shall assume from now on that the coverings $\mathcal{U}$ we are considering are "good". This means that $\mathcal{U}$ has a finite number of elements and that all possible intersections of elements of $\mathcal{U}$ are either empty or contractible. This is always possible if X is for instance a compact manifold [**5**], [**25**]. In the previous theorem, we are then able to replace the words "suitably fine" by "good" since the fibration

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

has the homotopy lifting property. In this case, we also have

$$H^*(X) \cong H^*(N(\mathcal{U})), K(X) \cong K(N(\mathcal{U})),$$

etc., where $N(\mathcal{U})$ is the nerve of the covering $\mathcal{U}$. Note that its geometric realization has the homotopy type of X.

Remark 3.4. For most spaces we are considering, good coverings are cofinal: any open covering has a good refinement. This is the case for finite polyedra and, more geometrically, for compact riemannian manifolds with open geodesic coverings [**5**].

The previous considerations also show that the cohomology class in

$$H^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z})$$

associated to a twisted vector bundle is a torsion class (assuming that the covering is good as in Remark 3.3). To prove this, we consider the commutative diagram

$$\begin{array}{ccccccc} 1 \to & \mu_n & \to & \mathrm{U}(n) & \to & \mathrm{PU}(n) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \mathbb{C}^* & \to & \mathrm{GL}_n(\mathbb{C}) & \to & \mathrm{PGL}_n(\mathbb{C}) & \to 1 \end{array}$$

The non-abelian cohomologies $H^1(X;\mathrm{PU}(n))$ and $H^1(X;\mathrm{PGL}_n(C))$ are isomorphic and the coboundary map

$$H^1(X;\mathrm{PU}(n)) \cong H^1(X;\mathrm{PGL}_n(\mathbb{C})) \to H^2(X;\mathbb{C}^\times) \cong H^3(X;\mathbb{Z})$$

factors through $H^2(X;\mu_n)$ (also see Appendix 8.1). Therefore the cocycle (λ_{kji}) defines a torsion class in $H^3(X;\mathbb{Z})$. It is a theorem of Serre [**17**] that such an element comes from an algebra bundle such as we have described. Later on, we shall show how we can recover the full cohomology group $H^3(X;\mathbb{Z})$ from algebra bundles of infinite dimension, as it was observed by Rosenberg [**29**].

The following theorem is important for our dictionary relating twisted vector bundles to modules over suitable algebra bundles.

Theorem 3.5. *Let A be an algebra bundle which may be written as* $\mathrm{END}(E)$, *where E is a twisted vector bundle associated to a covering $\mathcal{U}$, transition functions g_{ji} and a completely normalized 2-cocycle λ with values in $\mathbb{C}^*$. Let $\mathcal{E}_\lambda(\mathcal{U})$ be the category of λ-twisted vector bundles and $\mathcal{E}^A(\mathcal{U})$ be the category of finite dimensional vector bundles trivialized by the covering $\mathcal{U}$, which are right A-modules. Then the functor*

$$\psi : \mathcal{E}_\lambda(\mathcal{U}) \to \mathcal{E}^A(\mathcal{U})$$

defined by

$$F \mapsto \mathrm{HOM}(E,F)$$

is an equivalence of categories.

Proof. We first notice that if M, N and P are finite dimensional vector spaces with $M \neq 0$ and if $\Lambda = \mathrm{End}(M)$, the obvious map

$$\mathrm{Hom}(N,P) \to \mathrm{Hom}_\Lambda(\mathrm{Hom}(M,N),\mathrm{Hom}(M,P))$$

is an isomorphism. Since N is a direct summand of some M^r, it is enough to check the statement for $N = M$, in which case it is obvious. This functorial isomorphism at the level of vector spaces may be translated into the framework of twisted vector bundles by the isomorphism

$$\mathrm{Hom}(F,G) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{HOM}(E,F),\mathrm{HOM}(E,G)).$$

This shows that the functor Ψ is fully faithful. On the other hand, we have a canonical isomorphism of vector spaces

$$\mathrm{Hom}(M, N) \otimes_A M \to N,$$

defined by $(f, x) \longmapsto f(x)$ which can also be translated into the framework of twisted vector bundles. This shows that if we start with a bundle L which is a right A-module, where A is some $\mathrm{END}(E)$, we can associate to it a twisted vector bundle F by the formula

$$F = L \otimes_A E = \Psi'(L).$$

Since $\mathrm{HOM}(E, F) \otimes_A E$ is canonically isomorphic to F, ψ' induces a functor going backwards

$$\psi' : \mathcal{E}^A(\mathcal{U}) \to \mathcal{E}_\lambda(\mathcal{U}).$$

Finally, there is an obvious isomorphism

$$L \to \mathrm{HOM}(E, L \otimes_A E) = \Psi(\Psi'(L))$$

This shows that the functor Ψ is essentially surjective. □

This module interpretation enables us to prove the following Theorem.

THEOREM 3.6. *Let $\mathcal{U} = (U_i), i \in I$, be a good covering of X as in Remark 3.3 and let $\mathcal{V} = (V_s)$, $s \in S$, be a refinement of $\mathcal{U}$ which is also good. Then for any $\tau : S \to I$ such that $V_s \subset U_{\tau(s)}$, the associated restriction map*

$$R_\tau : H^1_\lambda(\mathcal{U}; GL_n(\mathbb{C})) \to H^1_{\tau^*(\lambda)}(\mathcal{V}; GL_n(\mathbb{C}))$$

is a bijection.

PROOF. Since $\mathcal{U}$ is good, for any completely normalized cocycle λ, one can find a twisted vector bundle E of rank m on $\mathcal{U}$ and $\mathcal{V}$, such that $A = \mathrm{END}(E)$ is a bundle of algebras associated to λ. According to the previous equivalence of categories, the sets $H^1_\lambda(\mathcal{U}; GL_n(\mathbb{C}))$ and $H^1_{\tau^*(\lambda)}(\mathcal{V}; GL_n(\mathbb{C}))$ are in bijective correspondence with the set of A-modules which are locally of type $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. With this identification, the restriction map R is just an automorphism of this set. □

REMARK 3.7. We may prove the homotopy invariance of the category of twisted vector bundles thanks to this dictionary (at least if X is compact): a twisted vector bundle may be interpreted as a bundle of A-modules, or as a finitely generated projective module over the Banach algebra $\Lambda = \Gamma(X, A)$ of continuous sections of A. It is easy to show that modules over $\Lambda\,[0, 1]$ can be extended from Λ (see e.g. [**24**]).

4. Twisted K-theory

Let $\mathcal{U}$ be a good covering (Remark 3.3) of a space X and let λ_{kji} be a completely normalized 2-cocycle with values in $\mathbb{C}^*$. We consider the category of twisted vector bundles associated to $\mathcal{U}$ and to the cocycle λ. This is clearly an additive category which is moreover pseudo-abelian (every projection operator has a kernel). We denote by $K_\lambda(\mathcal{U})$ its Grothendieck group, which is also the K-group of the category of A-modules over X, where A= $\mathrm{END}(E)$, as explained at the end of the previous section. Since this definition is independent of $\mathcal{U}$ up to a non-canonical isomorphism (see Appendix 8.3), we shall also call it $K_\lambda(X)$: this is the classical definition of (ungraded) twisted K-theory as detailed in many references, e.g. [**15**], [**2**], [**23**].

In this situation, the cocycle λ has a cohomology class $[\lambda]$ in the torsion subgroup of

$$H^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z}).$$

as we saw in Section 2. When $[\lambda]$ is not necessarily a torsion class, we should consider "twisted Hilbert bundles" which are defined in the same way as twisted vector bundles but with a fibre which is an infinite dimensional Hilbert space[4] H. It is also more convenient to use the unitary group $\mathrm{U}(H)$ instead of the general linear group as our basic structural group. In other words, the (g_{ji}) in Sections 1 and 2 are now elements of $\mathrm{U}(H)$. The 2-cocycle (λ_{kji}) takes its values in the topological group S^1.

From the fibration

$$S^1 \to \mathrm{U}(H) \to \mathrm{PU}(H)$$

and the contractibility of $\mathrm{U}(H)$ (Kuiper's theorem), we see that $\mathrm{PU}(H)$ is a model of the Eilenberg-Mac Lane space $K(\mathbb{Z},3)$. On the other hand, since $\mathrm{PU}(H)$ acts on $\mathcal{L}(H) = \mathrm{End}(H)$ by inner automorphisms, we deduce that any 2-cocycle $\lambda = (\lambda_{kji})$ defines an algebra bundle $\underline{\mathcal{L}}_\lambda$ with fibre $\mathcal{L}(H)$ which is well defined up to isomorphism. Therefore, as in the finite dimensional case, we have the following theorem.

Theorem 4.1. *Let $\underline{\mathcal{L}}_\lambda$ be the bundle of algebras with fibre $\mathcal{L}(H)$ associated to the cocycle λ. Then, if the covering $\mathcal{U}$ is good as in Remark 3.3, $\underline{\mathcal{L}}_\lambda$ may be written as* $\mathrm{END}(E)$, *where E is a λ-twisted Hilbert bundle.*

Proof. We just copy the proof of Theorem 3.2 in the infinite dimensional case. In a more precise way, the structural group of $\underline{\mathcal{L}}_\lambda$ is $\mathrm{PU}(H) = \mathrm{U}(H)/S^1$ acting on $\mathcal{L}(H)$ by inner automorphisms. Therefore, we may describe the principal bundle by transition functions

$$\gamma_{ji} : U_i \cap U_j \to \mathrm{PU}(H)$$

for a good covering $\mathcal{U} = (U_i)$ of X and we have $\gamma_{ii} = 1, \gamma_{ji} = (\gamma_{ij})^{-1}$. Since the principal fibration

$$\pi : \mathrm{U}(H) \to \mathrm{PU}(H)$$

is locally trivial and the $U_i \cap U_j$ are contractible (if non-empty), there are continuous maps

$$g_{ji} : U_i \cap U_j \to \mathrm{U}(H),$$

such that $\pi \circ g_{ji} = \gamma_{ji}$. The proof now ends as the proof of Theorem 3.2. □

Theorem 4.2. *Let $\underline{\mathcal{L}}_\lambda$ be the algebra bundle* $\mathrm{END}(E)$, *where E is a λ-twisted Hilbert bundle on a covering $\mathcal{U}$. Let $\mathcal{E}_\lambda(\mathcal{U})$ be the category of λ-twisted Hilbert bundles with fibre H and, finally, let $\mathcal{E}^{\underline{\mathcal{L}}_\lambda}(\mathcal{U})$ be the category of bundles which are right $\underline{\mathcal{L}}_\lambda$-modules[5], trivialized over the elements of $\mathcal{U}$. Then, the functor*

$$\Psi : \mathcal{E}_\lambda(\mathcal{U}) \to \mathcal{E}^{\underline{\mathcal{L}}_\lambda}(\mathcal{U}),$$

defined by the formula

$$F \longmapsto \mathrm{HOM}(E,F).$$

[4]For simplicity's sake, we assume H to be separable, i.e. isomorphic to the classical l^2 space.

[5]More precisely, we assume that locally the module is isomorphic to $\mathcal{L}(H)$, with its standard $\mathcal{L}(H)$-module structure.

is an equivalence of categories.

PROOF. It is also completely analogous to the proof of Theorem 3.5. In a more precise way, instead of considering all finite dimensional vector spaces, we take Hilbert spaces $M, N, P, etc.$ of the same cardinality, i.e. isomorphic to the classical l^2-space. For instance, the isomorphism used in the proof of Theorem 3.5

$$\mathrm{Hom}(F,G) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{HOM}(E,F), \mathrm{HOM}(E,G))$$

is a consequence of the fact that it is true at the level of Hilbert spaces since $\mathrm{Hom}(M,N)$ is isomorphic to $\mathrm{End}(M) = \mathcal{L}(H)$. The proof of the theorem again ends as in the case of finite dimensional vector spaces. □

For $[\lambda] \in H^3(X;\mathbb{Z}) = H^1(X;\mathrm{PU}(H))$ which is not necessarily a torsion class, we may define the associated twisted K-theory in many ways. The first definition is due to Rosenberg [**29**]: the class $[\lambda]$ is represented up to isomorphism by a principal bundle P with structural group $\mathrm{PU}(H)$. Since $\mathrm{PU}(H)$ is acting on the ideal of compact operators $\mathcal{K}$ in $\mathcal{L} = \mathcal{L}(H)$ by inner automorphisms, we get an associated bundle $\underline{\mathcal{K}}_\lambda$ of C*-algebras. The twisted K-theory is then the usual K-theory of the algebra of sections of $\underline{\mathcal{K}}_\lambda$. An equivalent way to define $\underline{\mathcal{K}}_\lambda$ is to consider a twisted Hilbert bundle E associated to the cocycle λ (it is unique up to isomorphism). Then, $\underline{\mathcal{K}}_\lambda$ is the subalgebra of sections of the bundle $\underline{\mathcal{L}}_\lambda = \mathrm{END}(E)$ which belong to $\mathcal{K}(H)$ over each open set of $\mathcal{U}$.

One unpleasant aspect of this definition is the non-existence of a unit element in $\underline{\mathcal{K}}_\lambda$, which makes its K-theory slightly complicated to handle. However, we may replace $\mathcal{K}$ by the subalgebra $\mathcal{A}$ of $\mathcal{L} \times \mathcal{L}$ consisting of couples of operators (f,g) such that $f - g \in \mathcal{K}$. The group $\mathrm{PU}(H)$ is acting on $\mathcal{A}$, so that we may also twist the algebra $\mathcal{A}$ by λ in order to get an algebra bundle $\underline{\mathcal{A}}_\lambda$. The obvious exact sequence of C*-algebras

$$0 \to \mathcal{K} \to \mathcal{A} \to \mathcal{L} \to 0$$

induces an exact sequence of algebra bundles

$$0 \to \underline{\mathcal{K}}_\lambda \to \underline{\mathcal{A}}_\lambda \to \underline{\mathcal{L}}_\lambda \to 0.$$

Here and elsewhere, using a variation of the Serre-Swan theorem, we shall often use the same terminology for an algebra bundle and its associated algebra of continuous sections. In particular the K-theory of $\underline{\mathcal{A}}_\lambda$ is canonically isomorphic to the K-theory of $\underline{\mathcal{K}}_\lambda$ since $\underline{\mathcal{L}}_\lambda$ is a flabby algebra[6] (in particular its K-groups are trivial).

A comment is in order to make our previous definition more functorial: the λ-twisted K-theory is defined precisely as the K-theory of bundles with fibres $\mathcal{A}$-modules which are finitely generated and projective but twisted by the cocycle λ. How this depends only on the cohomology class $[\lambda]$ is discussed in Appendix 8.3. Our Section 3 on twisted vector bundles may now be rewritten by replacing the field of complex numbers $\mathbb{C}$ by the C*-algebra $\mathcal{A}$ and the finite dimensional bundles by "$\mathcal{A}$-bundles" as above. Theorem 3.5 adapted to this situation shows that the category of λ-twisted $\mathcal{A}$-bundles is equivalent to the category of $\underline{\mathcal{A}}_\lambda$-modules if the covering $\mathcal{U}$ of X is good. This shows in particular that the theory of twisted $\mathcal{A}$-bundles is homotopically invariant (at least if X is compact).

[6]A Banach algebra A is called flabby if there is a topological A-bimodule M which is projective of finite type as a right module, such that $M \oplus A$ is isomorphic to M. This is equivalent to saying that the Banach category $\mathcal{C} = \mathcal{P}(A)$ is "flabby": there is a linear continuous functor τ from $\mathcal{C}$ to itself such that $\tau \oplus \mathrm{Id}_{\mathcal{C}}$ is isomorphic to τ.

One has to point out a main difference between $\mathbb{C}$-modules and $\mathcal{A}$-modules: a priori, the fibres of $\mathcal{A}$-bundles are not necessarily free[7]. However, since $K(\mathcal{A})$ is canonically isomorphic to $\mathbb{Z}$, each $\mathcal{A}$-bundle E induces a locally constant function (called the "rank")

$$Rk : X \to \mathbb{Z},$$

obtained by applying the K-functor to each fibre. This correspondence defines a group map

$$Ch_{(0)} : K(\underline{\mathcal{A}}_\lambda) \to H^0(X;\mathbb{Z}).$$

In Section 7 we shall see how to define "higher Chern characters" $Ch_{(m)}$, starting from this elementary step.

In the spirit of Section 1, we may also consider twisted principal G-bundles, where G is the group of invertible elements in the algebra $\mathcal{A}$. We note that the elements of G are couples of invertible operators (g, h) in a Hilbert space such that $g - h$ is compact. We get elements in the centre by considering $g = h \in \mathbb{C}^*$. More accurately, one should replace $\mathcal{A}$ by the subalgebra $\mathrm{End}(P)$, where P is a finitely generated projective $\mathcal{A}$-module which is the fibre of the bundles we are considering (assuming the base is connected; otherwise the fibre P may vary). Then G is not exactly $\mathcal{A}^*$ but the subgroup $\mathrm{Aut}(P)$ of $\mathcal{A}^*$. This point of view will be exploited in Section 7 for the definition of the Chern character, whose target is twisted cohomology.

Finally, there is a third definition of twisted K-theory in terms of Fredholm operators, following the ideas in [**1**], [**19**] and [**15**]. We consider the set of homotopy classes of triples

$$(E_0, E_1, D),$$

where E_0 and E_1 are λ-twisted Hilbert bundles on a good covering $\mathcal{U}$ and D is a family of Fredholm operators[8] from E_0 to E_1. With the operation induced by the direct sum of triples, we get a group denoted by $K_\lambda(\mathcal{U})$. We note that $K_\lambda(\mathcal{U})$ is a module over $K(\mathcal{U})$. Here $K(\mathcal{U})$ is a shorthand notation for the usual K-theory of the nerve of $\mathcal{U}$. If $\mathcal{U}$ is good as in Remark 3.3, this group is isomorphic to the classical topological K-group $K(X)$.

In order to prove that this last definition is consistent with the previous ones, we consider the Banach category of λ-twisted Hilbert bundles. It is equivalent to the category of bundles of $\underline{\mathcal{L}}_\lambda$-modules, where $\underline{\mathcal{L}}_\lambda$ is the algebra bundle above with fibre $\mathcal{L}(H)$ twisted by λ. Let $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$ be the quotient bundle with fibre the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$.

Lemma 4.3. *Let $\overline{D}$ be the class of D as a morphism between the associated $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$-modules. Then two triples (E_0, E_1, D) and (E_0', E_1', D') are homotopic if and only if the associated triples $(E_0, E_1, \overline{D})$ and $(E_0', E_1', \overline{D}')$ are homotopic.*

Proof. In general, let us denote also by $\overline{M}$ the class of M as an $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$-module. We have a continuous map

$$\mathcal{F}(E_0, E_1) \to \mathrm{Iso}(\overline{E}_0, \overline{E}_1),$$

[7]However, we shall show in Section 7 that the fibres are free modules if the restriction of the cohomology class of λ to every connected component of X is of infinite order.

[8]We note that $\mathrm{HOM}(E, F)$ is an ordinary bundle with fibre $\mathrm{Hom}(H, H) = \mathcal{L}(H)$. The space of Fredholm operators "from E to F" is the subspace of sections of $\mathrm{HOM}(E, F)$ which are Fredholm over each point of X.

where the notation $\mathcal{F}$ stands for continuous families of Fredholm maps. According to a classical theorem on Banach spaces, this map admits a continuous section. Therefore, we get a trivial fibration with contractible fibre which is the Banach space of sections of the bundle $\underline{\mathcal{K}}_\lambda$. The proposition follows immediately. □

The philosophy of the lemma is that our third definition of twisted K-theory is equivalent to the Grothendieck group of the Banach functor

$$\varphi : \mathcal{P}'(\underline{\mathcal{L}}_\lambda) \to \mathcal{P}'(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda),$$

as defined in [**24**, Section II]. Here the category $\mathcal{P}'(\underline{\mathcal{L}}_\lambda)$ (resp. $\mathcal{P}'(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda)$) is equivalent to the category of free modules over $\underline{\mathcal{L}}_\lambda$ (resp. $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$). Since $K_0(\mathcal{P}'(\underline{\mathcal{L}}_\lambda)) = 0$, this Grothendieck group is canonically isomorphic to $K_0(\underline{\mathcal{K}}_\lambda)$ which is precisely our first definition since, as already mentioned, $\underline{\mathcal{K}}_\lambda$ is the algebra bundle with fibre $\mathcal{K}(H)$ associated to the cocycle λ.

Remark 4.4. Instead of the Grothendieck group of the functor φ, we could as well consider the group $K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda)$ which is isomorphic to $K(\varphi)$, since $\underline{\mathcal{L}}_\lambda$ is a flabby ring. We shall use this equivalent description of twisted K-theory in Appendix 8.2.

Remark 4.5. If λ is of finite order, the Fredholm definition of twisted K-theory is detailed in [**15**, pg. 18]. If $\lambda = 1$, we recover the theorem of Atiyah and Jänich [**1**], [**19**], in a slightly weaker form.

As is shown in [**15**] and [**23**], there is a $\mathbb{Z}/2$-graded version of twisted K-theory. This version is needed for the Thom isomorphism in the general case of an arbitrary real vector bundle V (which is not necessarily oriented). It is also needed for the Poincaré pairing applied to arbitrary manifolds. We shall concentrate on the case of non-torsion classes $[\lambda]$ in the third cohomology group of X. The case when $[\lambda]$ is a torsion class in $H^3(X;\mathbb{Z})$ has been extensively studied in [**15**].

The essential idea is to replace the previous structural group $\mathrm{U}(H)$ by the group $\Gamma(H)$ of matrices in $\mathrm{U}(H \oplus H)$ of type

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & h_1 \\ h_2 & 0 \end{pmatrix}.$$

The point here is that $\Gamma(H)$ acts by inner automorphisms on $\mathcal{L}(H \oplus H)$ with a degree shift which is either 0 or 1, the first copy of H being of degree 0 and the second one of degree 1. As in the previous Section, we may give a $\mathbb{Z}/2$-graded module interpretation of twisted Hilbert bundles modelled on $\Gamma(H)$. If E is such a graded twisted Hilbert bundle, $A = \mathrm{END}(E)$ is a bundle of graded algebras with fibre $\mathcal{L}(H \oplus H)$. Conversely, for any bundle of graded algebras A with fibre $\mathcal{L}(H \oplus H)$, there is a twisted Hilbert bundle E with structural group $\Gamma(H)$ such that A is isomorphic to $\mathrm{END}(E)$. According to [**15**], [**23**] and our previous computations, these graded algebras are classified by the following cohomology group:

$$H^1(X;\mathbb{Z}/2) \times H^3(X;\mathbb{Z}),$$

with a twisted addition rule, as explained in [**15**, p. 10]. The first invariant in $H^1(X;\mathbb{Z}/2)$ is induced by the map

$$\Gamma(H) \to \mathbb{Z}/2,$$

which describes the type of matrices in $\Gamma(H)$ (diagonal or antidiagonal). The second invariant is defined as before for the underlying ungraded twisted Hilbert bundle.

If we consider the graded tensor product of the twisted Hilbert bundle E by the Clifford algebra $C^{0,1} = \mathbb{C}\,[x]\,/(x^2-1)$, we get another type of structural group we might call $\Gamma_1(H)$ which is simply $\mathrm{U}(H) \times \mathrm{U}(H)$. The elements of degree 0 are of type (g, g), while the ones of degree 1 are of type $(g, -g)$. Algebraically, this reflects the fact that over the complex numbers there are two types of $\mathbb{Z}/2$-graded Azumaya algebra, up to graded Morita equivalence, which are $\mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$. For simplicity's sake, in the following discussion, we shall restrict ourselves to the first case which is the group $\Gamma(H)$ above. We note however that, for real graded vector bundles, there are eight types of graded algebras (up to graded Morita equivalence) to consider instead of two, as noticed in [**15**]. They correspond to the Clifford algebras $C^{0,n}$ for $n = 0, 1, ..., 7$, over the real numbers.

If E and F are two graded twisted Hilbert bundles of structural group $\Gamma(H)$, a morphism (g_i) is of degree 0 (resp. 1) if it is represented locally by a matrix of type

$$\begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix} 0 & u_i \\ v_i & 0 \end{pmatrix}.$$

From the previous category equivalences and the definitions in [**23**], we deduce the following theorem.

THEOREM 4.6. *Let λ be a graded twist defined by two cocycles, with classes in $H^1(X; \mathbb{Z}/2)$ and $H^3(X; \mathbb{Z})$ respectively. We consider the set of homotopy classes of couples (E, ∇), where E is a λ-twisted graded Hilbert bundle and ∇ a family of self-adjoint Fredholm operators on E which are of degree one. With the operation given by the direct sum of couples, the group obtained is isomorphic to the λ-twisted graded K-theory defined in* [**23**].

REMARK 4.7. One should point out that there is a variant of this Fredholm definition of twisted K-theory on a base X which is locally compact: the family of Fredholm operators ∇ must be an isomorphism outside a compact set (see e.g. [**1**] or [**24**]). This remark will be important for the definition of the Thom isomorphism in Section 6.

REMARK 4.8. Whatever definition of graded or ungraded twisted K-theory we choose, the group we obtain, denoted by $K_\lambda(X)$ in all cases, may be "derived". One nice way to see this is to notice that we are considering a K-group of special Banach algebras (or $\mathbb{Z}/2$-graded Banach algebras, see [**23**]), for instance $A = \underline{\mathcal{K}}_\lambda$. We then define $K_\lambda^{-n}(X)$ as $K_n(A)$. By Bott periodicity for complex Banach algebras, we have $K_\lambda^{-n}(X) \cong K_\lambda^{-n-2}(X)$. According to general theorems on K-theory, one shows that

$$K_\lambda^{-n}(X) \cong \mathrm{Coker}(K_\lambda(X) \to K_{\pi^*\lambda}(X \times S^n)),$$

where $\pi : X \times S^n \to X$ is the canonical projection. We note here that the smash product $X \wedge S^n$ cannot be used to define $K_\lambda^{-n}(X)$, since there is no associated twist in the cohomology of $X \wedge S^n$ in general.

As a consequence, we may apply Mayer-Vietoris arguments to the direct sum $K_\lambda(X) \oplus K_\lambda^{-1}(X)$, as for the K-theory of general Banach algebras.

5. Multiplicative structures

Since we have defined twisted K-theory in three ways (at least in the non-graded case), we should investigate the possible multiplicative structure from these different viewpoints and show that they coincide up to isomorphism. These multiplicative structures were also investigated in a more general framework in [**20**].

The end result is a "cup-product"

$$K_\lambda(X) \times K_\mu(X) \to K_{\lambda\mu}(X),$$

where λ and μ are two 2-cocycles[9] with values in S^1. Since $K_\lambda(X)$ is the K-theory of the Banach algebra $\underline{\mathcal{K}}_\lambda$ in general, it is enough to define a continuous bilinear pairing between nonunital Banach algebras

$$\varphi : \underline{\mathcal{K}}_\lambda \times \underline{\mathcal{K}}_\mu \to \mathcal{K}_{\lambda\mu},$$

such that $\varphi(aa', bb') = \varphi(a,b)\varphi(a',b')$. The implication that such a φ induces a pairing between K-groups is not completely obvious and relies on excision in K-theory.

To define the pairing φ, we observe that if E_λ is a twisted Hilbert bundle with twist λ and F_μ another one with twist μ, then $E\widehat{\otimes}F$ is a twisted Hilbert bundle with twist $\lambda\mu$. Here, the fibres of $E_\lambda\widehat{\otimes}F_\mu$ are the Hilbert tensor product of the fibres of E and F respectively (we implicitly identify the Hilbert tensor product of $H \otimes H$ with H since it is infinite dimensional). Therefore, we have a pairing between Banach bundles

$$\mathrm{END}(E_\lambda) \times \mathrm{END}(F_\mu) \to \mathrm{END}(E_\lambda\widehat{\otimes}F_\mu),$$

which is bilinear and continuous. If we take continuous sections, we deduce the map φ required. We note that φ also induces a continuous ring map

$$\underline{\mathcal{K}}_\lambda\widehat{\otimes}\underline{\mathcal{K}}_\mu \to \mathcal{K}_{\lambda\mu}.$$

where the symbol $\widehat{\otimes}$ denotes the completed projective tensor product of Grothendieck. However, this map is not an isomorphism.

This cup-product is much simpler to define if $[\lambda]$ and $[\mu]$ are torsion classes in the cohomology. According to Section 3, we may then assume that E and F are finite dimensional twisted vector bundles. The cup-product is then the usual one[10]

$$K(\underline{A}) \times K(\underline{B}) \to K(\underline{A \otimes B})$$

where $A = \mathrm{END}(E_\lambda)$ and $B = \mathrm{END}(F_\mu)$ are bundles of finite dimensional algebras, with matrix algebras as fibres.

Coming back to the general case, we now use our second definition of twisted K-theory in order to get a cup-product between K-groups of unital rings. According to Section 4, we have exact sequences of Banach algebras

$$0 \to \underline{\mathcal{K}}_\lambda \to \underline{\mathcal{A}}_\lambda \to \underline{\mathcal{L}}_\lambda \to 0$$
$$0 \to \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\mu \to \underline{\mathcal{L}}_\mu \to 0,$$

---

[9] See Appendix 8.3 for a possible pairing if we replace λ and μ by their cohomology classes in $H^2(X; S^1) \cong H^3(X; \mathbb{Z})$.

[10] As often, we underline the algebra of sections of the algebra bundles involved.

Therefore, we deduce another exact sequence by taking completed projective tensor products (since the previous exact sequences split as exact sequences of Banach spaces):

$$0 \to \underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu \to \mathcal{D}_{\lambda,\mu} \to 0.$$

Here the Banach algebra $\mathcal{D}_{\lambda,\mu}$ is the following fibre product

$$\begin{array}{ccc} \mathcal{D}_{\lambda,\mu} & \to & \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{L}}_\mu \\ \downarrow & & \downarrow \\ \underline{\mathcal{L}}_\lambda \widehat{\otimes} \underline{\mathcal{A}} & \to & \underline{\mathcal{L}}_\lambda \widehat{\otimes} \underline{\mathcal{L}}_\mu \end{array}.$$

Since the algebras $\underline{\mathcal{L}}_\lambda$ and $\underline{\mathcal{L}}_\mu$ are flabby, the algebra $\mathcal{D}_{\lambda,\mu}$ has trivial K-groups. It follows that the map

$$\underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu$$

is a K-theory equivalence. Therefore, we may also define a cup-product

$$K(\underline{\mathcal{A}}_\lambda) \times K(\underline{\mathcal{A}}_\mu) \to K(\underline{\mathcal{A}}_{\lambda\mu}),$$

as the following composition

$$K(\underline{\mathcal{A}}_\lambda) \times K(\underline{\mathcal{A}}_\mu) \to K(\underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu) \cong K(\underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu) \to K(\underline{\mathcal{K}}_{\lambda\mu}) \cong K(\underline{\mathcal{A}}_{\lambda\mu}).$$

We now come to the third definition of the cup-product in terms of Fredholm operators. As is well known (see e.g. [**1**], [**15**] or [**23**]), one advantage of this definition of twisted K-theory (for $[\lambda]$ of finite or infinite order) is a handy description of the cup-product. In the ungraded case, it is more convenient still to view $E = E_1 \oplus E_1$ as a $\mathbb{Z}/2$-graded twisted bundle and replace[11] $D : E_0 \to E_1$ by the following operator ∇ which is self-adjoint and of degree 1:

$$\nabla = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.$$

The cup-product of (E, ∇) with another couple of the same type (E', ∇') is simply defined by the formula

$$(E, \nabla) \smile (E', \nabla') = (E \widehat{\otimes} E', \nabla \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla').$$

Here the symbol $\widehat{\otimes}$ denotes the graded and Hilbert tensor product. We notice that if E is associated to the twist λ, E' to the twist λ', the cup-product is associated to the twist $\lambda \cdot \lambda'$, a cocycle whose cohomology class is the sum of the two related cohomology classes in $H^2(X; S^1)$.

It is not completely obvious that this third definition of the cup-product is equivalent to the previous one with the bundles $\underline{\mathcal{K}}_\lambda$ or $\underline{\mathcal{A}}_\lambda$. In order to prove this technical point, we use the results of Appendix 8.2 describing explicitly the isomorphism between $K(\underline{\mathcal{K}}_\lambda)$ and $K_1(\underline{\mathcal{L}}_\lambda / \underline{\mathcal{K}}_\lambda)$. In fact, any element of $K_1(\underline{\mathcal{L}}_\lambda / \underline{\mathcal{K}}_\lambda)$ is the cup-product of an element u of $K(\underline{\mathcal{K}}_\lambda)$ by a generator τ of

$$K_1(\mathcal{L}/\mathcal{K}) \cong \mathbb{Z}.$$

This generator is classically defined by the shift (as a Fredholm operator). Moreover, we may assume that u is induced by a self-adjoint involution on $M_2((\underline{\mathcal{K}}_\lambda)^+)$, where $(\underline{\mathcal{K}}_\lambda)^+$ is the algebra $\underline{\mathcal{K}}_\lambda$ with a unit added. On the other hand, both $K_*(\underline{\mathcal{K}}_\lambda)$ and $K_{1+*}(\underline{\mathcal{L}}_\lambda / \underline{\mathcal{K}}_\lambda)$ may be considered as (twisted) cohomology theories on X and we have a pairing

$$K_*(\underline{\mathcal{K}}_\lambda) \times K_1(\mathcal{L}/\mathcal{K}) \to K_{1+*}(\underline{\mathcal{L}}_\lambda / \underline{\mathcal{K}}_\lambda)$$

---

[11] More correctly, we should write D as a section of the bundle $\mathrm{HOM}(E_0, E_1)$.

Therefore, the formula above for the cup-product with Fredholm operators has to be compared with the previous one only when X is reduced to a point, a case which is obvious.

This Fredholm multiplicative setting has the advantage that it may be extended to the graded version of twisted K-theory by the same formula

$$(E,\nabla) \smile (E',\nabla') = (E\widehat{\otimes}E', \nabla\widehat{\otimes}1 + 1\widehat{\otimes}\nabla').$$

If $([\lambda_1], [\lambda_3])$ and $([\lambda'_1], [\lambda'_3])$ are the twists of E and E' respectively, the twist of $E\widehat{\otimes}E'$ in cohomology is $([\mu_1], [\mu_3])$, where

$$[\mu_1] = [\lambda_1] + [\lambda'_1]$$

and

$$[\mu_3] = [\lambda_3] + [\lambda'_3] + \beta([\lambda_1]\cdot[\lambda'_1]).$$

Here $\beta : H^2(X;\mathbb{Z}/2) \to H^3(X;\mathbb{Z})$ is the Bockstein homomorphism (compare with [**15**, p. 10]). Thanks to the Thom isomorphism which is proved in [**23**] (see also the next section and [**10**]), this graded cup-product is compatible with the ungraded one defined on the Thom space of the orientation bundle determined by the graded twist.

6. Thom isomorphism and operations in twisted K-theory

This Section is just a short rewriting of the sections 4 and 7 of [**23**], with the point of view of twisted Hilbert bundles. It is added here for completeness' sake.

In order to define the Thom isomorphism in twisted K-theory, as in [**23**] and [**10**] with our new point of view, we need to consider twisted Hilbert bundles E with a Clifford module structure. Such a structure is given by a finite dimensional real vector bundle V on X, provided with a positive metric q and an action of V on E, such that $(v)^2 = q(v)\cdot 1$. Now let λ be a graded twist, given by a covering $\mathcal{U} = (U_i)$ together with a couple (λ_1, λ_3) consisting of a 1-cocycle with values in $\mathbb{Z}/2$ and a 2-cocycle with values in S^1. We define the Grothendieck group $K^V_\lambda(X)$ from the set of homotopy classes of couples

$$(E,\nabla),$$

as follows: E is a $\mathbb{Z}/2$-graded twisted Hilbert bundle which is also a graded $C(V)$-module, V acting by self-adjoint endomorphisms of degree 1. Moreover, the family of Fredholm operators ∇ must satisfy the following properties

1) ∇ is self-adjoint and of degree 1, as in the previous section,

2) ∇ anticommutes with the elements v in V.

This group is not entirely new. Using our dictionary relating twisted Hilbert bundles and module bundles, we described it in great detail in [**23**, § 4]. We should also notice that this structure of $C(V)$-module may be integrated into the twist λ: if $w_1 = w_1(V)$ and $w_2 = w_2(V)$ are the first two Stiefel-Whitney classes of V, one has to replace λ by the sum of λ and $C(V)$ in the graded Brauer group (this was one of the main motivations for the paper [**15**]). More precisely, the resulting cohomology classes are

$$[\lambda_1] + w_1(V)$$

in degree one and

$$[\lambda_3] + \beta([\lambda_1]\cdot w_1) + \beta(w_2)$$

in degree 3.

Using our previous reference [**23**], we are now able to define the Thom isomorphism

$$t : K_\lambda^V(X) \to K_{\pi^*\lambda}(V)$$

in simpler terms. If π denotes the projection $V \to X$, and if (E, ∇) defines an element of the group $K_\lambda^V(X)$, we define $t(E, \nabla)$ as the couple $(\pi^*(E), \nabla')$, where ∇' is defined over a point v of V, with projection x, by the formula

$$\nabla'_v = v + \nabla_x.$$

We recognize here the formula already given in [**23**]: we have just replaced module bundles by twisted Hilbert bundles.

Operations on twisted K-theory have already been defined in many references [**15**], [**2**], [**23**]. Twisted Hilbert bundles give a nice framework to redefine them. For simplicity's sake, we restrict ourselves to ungraded twisted K-groups.

If we start with an element (E, ∇) defining an element of $K_\lambda(X)$ as at the end of Section 4, its k^{th} power[12]

$$(E^{\widehat{\otimes}k}, \nabla\widehat{\otimes}...\widehat{\otimes}1 + \cdots + 1\widehat{\otimes}...\widehat{\otimes}\nabla)$$

has an obvious action of the symmetric group S_k. We should notice that the twist of the k^{th} power is λ^k. According to Atiyah's philosophy [**1**], the k^{th} power defines a map

$$K_\lambda(X) \to K_{\lambda^k}(X) \otimes_{\mathbb{Z}} R(S_k),$$

where $R(S_k)$ denotes the complex representation ring of S_k. Therefore, any $\mathbb{Z}$-homomorphism

$$R(S_k) \to \mathbb{Z}$$

gives rise to an operation in twisted K-theory. In particular, the Grothendieck exterior powers and the Adams operations may be defined in twisted K-theory, using Atiyah's method.

As an interesting $\mathbb{Z}$-homomorphism from $R(S_k)$ to $\mathbb{Z}$, one may choose the map which associates to a complex representation ρ the trace of $\rho(c_k)$, where c_k is the cycle $(1, 2, ..., k)$, a trace which is in fact an integer. The resulting homomorphism

$$K_\lambda(X) \to K_{\lambda^k}(X)$$

is quite explicit. It associates to $F = (E, \nabla)$ the "Gauss sum"

$$\sum (F^{\widehat{\otimes}k})_n \otimes \omega^n$$

in the group $K_{\lambda^k}(X) \otimes_{\mathbb{Z}} \Omega_k$, where Ω_k is the ring of k-cyclotomic integers. In this sum, ω is a primitive k^{th} root of unity. The element $(F^{\widehat{\otimes}k})_n$ is the eigenmodule associated to the eigenvalue ω^n of a generator of the cyclic group C_k acting on $F^{\widehat{\otimes}k}$. This sum belongs in fact to $K_{\lambda^k}(X)$, as a subgroup of $K_{\lambda^k}(X) \otimes \Omega_k$. As shown by Atiyah [**1**], we get this way a nice alternative definition of the Adams operation Ψ^k.

[12] where the symbol $\widehat{\otimes}$ again denotes the graded Hilbert tensor product.

REMARK 6.1. If the class of λ in $H^3(X;\mathbb{Z})$ is of finite order, it is not necessary to consider twisted Hilbert bundles and Fredholm operators. One just deals with finite dimensional twisted vector bundles as in Section 3.

REMARK 6.2. One should notice that operations are much more delicate to define in graded twisted K-theory, even for coefficients $[\lambda]$ of finite order in $H^3(X;\mathbb{Z})$. This was pointed out in [**15**] and recalled in [**23**]. Fredholm operators were already introduced in [**15**] in order to deal with this problem, before subsequent works on twisted K-theory.

7. Connections and the Chern homomorphism

Let us now assume that X is a manifold. The previous definitions make sense in the differential category. The fact that we get the same K-groups is more or less standard and relies on arguments going back to Steenrod [**31**]. As an illustrative example, the Čech cohomologies $H^1(X;\mathrm{GL}_n(\mathbb{C}))$ and $H^1(X;\mathrm{PGL}_n(\mathbb{C}))$ may be computed with differential cochains. Therefore the classification of topological algebra bundles (with fibre $M_n(\mathbb{C})$) is the same in the differential category. This general result is also true for module bundles and therefore for twisted K-theory, if we choose differential 2-cocycles λ with values in S^1 to parametrize the twisted K-groups.

In the differential category, the definition of the Chern homomorphism between twisted K-theory and "twisted cohomology" was given in many papers [**3**], [**27**], [**8**], [**32**], [**11**], and [**4**]. Our method is more elementary and is based on the classical definitions of Chern-Weil theory applied to twisted bundles[13]. We start with twisted finite dimensional bundles which are easier to handle. However, as we shall see later on, the same method may be applied to infinite dimensional bundles in the spirit of Section 4.

Let E be a twisted vector bundle of rank[14] n, defined on a covering $\mathcal{U} = (U_i)$ by transition functions (g_{ji}), with the twisted cocycle condition

$$g_{ki} = g_{kj} \cdot g_{ji} \cdot \lambda_{kji},$$

as in Section 3. We assume that all functions are of class C^∞, which does not change the classification problem for twisted bundles as we have seen previously.

DEFINITION 7.1. A connection Γ on E is given by $(n \times n)$-matrices Γ_i of differential 1-forms on U_i such that on $U_i \cap U_j$ we have the relation

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.$$

Here ω_{ji} is a differential 1-form related to the λ_{kji} by the following relation:

$$\omega_{ji} - \omega_{ki} + \omega_{kj} = \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Moreover, from the relation above with the Γ's, we deduce that $\omega_{ij} = -\omega_{ji}$. If we take the differential of the previous relation, we also get

$$d\omega_{ji} - d\omega_{ki} + d\omega_{kj} = 0.$$

[13]For the classical computations, we refer to the books [**26**, pg. 78] and [**22**] for instance.

[14]The rank may vary above different connected component of X.

In the applications below, ω will be a differential form with values in $i\mathbb{R}$, where[15] $i = \sqrt{-1}$ (if the g_{ji} are unitary operators).

Example 7.2. (which shows the existence of such connections). Let (α_k) be a partition of unity associated to the covering $\mathcal{U}$. We then consider the "barycentric connection" defined by the formula

$$\Gamma_i = \sum_k \alpha_k \cdot g_{ki}^{-1} \cdot dg_{ki}.$$

Since $g_{ki} = g_{kj} \cdot g_{ji} \cdot \lambda_{kji}$, we have the following expansion:

$$g_{ki}^{-1} \cdot dg_{ki} = g_{ji}^{-1} \cdot (g_{kj}^{-1} \cdot dg_{kj}) \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Therefore, on $U_i \cap U_j$ we have the expected identity

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji} \cdot 1,$$

where

$$\omega_{ji} = \sum_k \alpha_k \cdot \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Remark 7.3. It is clear from the definition that the space of connections on E is an affine space: if Γ and ∇ are two connections on E, for any real number t, $(1-t)\Gamma + t\nabla$ is also a connection.

We have choosen a definition of a connection in terms of "local coordinates". However, we have to check how connections correspond when we change them. In other terms, let (α) be an isomorphism from the coordinate bundle (h) to (g) as in Section 1. According to Formula(1.1), we then have the relation

$$g_{ji} \cdot \alpha_i = \alpha_j \cdot h_{ji}$$

Associated to this morphism, we define the pullback $\alpha^*(\Gamma)$ of the connection (Γ) as locally defined on the coordinate bundle (h) by the formula

$$\nabla_i = \alpha_i^{-1} \cdot \Gamma_i \cdot \alpha_i + \alpha_i^{-1} \cdot d\alpha_i$$

In order for this to make sense, we have to check the relation

$$\nabla_i = h_{ji}^{-1} \cdot \nabla_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} + \omega_{ji}.1.$$

which is slightly tedious. We start from the formula

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1,$$

where we replace g_{ji} by $\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1}$. We also replace dg_{ji} by

$$dg_{ji} = d\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} + \alpha_j \cdot dh_{ji} \cdot \alpha_i^{-1} - \alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} \cdot d\alpha_i \cdot \alpha_i^{-1}.$$

We then get

$$\begin{gathered}\nabla_i = \alpha_i^{-1} \cdot (g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.) \cdot \alpha_i + \alpha_i^{-1} \cdot d\alpha_i \\ = \alpha_i^{-1} \cdot (g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji}) \cdot \alpha_i \\ +\alpha_i^{-1} \cdot g_{ji}^{-1} \cdot (d\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} + \alpha_j \cdot dh_{ji} \cdot \alpha_i^{-1} - \alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} \cdot d\alpha_i \cdot \alpha_i^{-1}) \cdot \alpha_i \\ +\omega_{ji}.1.\end{gathered}$$

[15]The two different meanings of the symbol "i" are clear from the context.

$$= h_{ji}^{-1} \cdot (\nabla_j - \alpha_j^{-1} \cdot d\alpha_j) \cdot h_{ji}$$
$$+ h_{ji}^{-1} \cdot \alpha_j^{-1} \cdot d\alpha_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} - \alpha_i^{-1} \cdot d\alpha_i + \alpha_i^{-1} \cdot d\alpha_i + \omega_{ji}.1.$$
$$= h_{ji}^{-1} \cdot \nabla_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} + \omega_{ji}.1,$$

which is the expected formula.

The "local curvatures" R_i associated to the Γ_i are given by the usual formula[16]

$$R_i = d\Gamma_i + (\Gamma_i)^2.$$

Unfortunately, the traces of these local curvatures do not agree on $U_i \cap U_j$, since a simple computation as above leads to the relation

$$R_i = g_{ji}^{-1} \cdot R_j \cdot g_{ji} + d\omega_{ji}.1.$$

However, using a partition of unity (α_i), as in the case of the barycentric connection, we may define a family of "twisted curvatures" by the following formula, where $m = 1, 2, ...$:

$$R_{(m)} = \sum_i \alpha_i \cdot (R_i)^m.$$

We now define a family of "Chern characters" $Ch_{(m)}(E, \Gamma)$ as

$$Ch_{(m)}(E, \Gamma) = \mathrm{Tr}(R_{(m)}).$$

We should notice that $Ch_{(m)}(E, \Gamma)$ belongs to the vector space of differential forms with values in $(i)^m\mathbb{R}$, since the g_{kl} are unitary matrices. By convention, we put

$$Ch_{(0)}(E, \Gamma) = n.$$

The differential of $Ch_{(1)}$ is

$$d(Ch_{(1)}(E, \Gamma)) = \sum_i \alpha_i \cdot \mathrm{Tr}(dR_i) + \sum_i d\alpha_i \cdot \mathrm{Tr}(R_i).$$

It is well known (and easy to prove) that

$$\mathrm{Tr}(dR_i) = \mathrm{Tr}(d\Gamma_i \cdot \Gamma_i - \Gamma_i \cdot d\Gamma_i) = 0.$$

On the other hand, the relation between R_i and R_j above leads to the following identity between differential forms on U_j :

$$\sum_i d\alpha_i \cdot \mathrm{Tr}(R_i) = (\sum_i d\alpha_i \cdot \mathrm{Tr}(R_j)) + n \sum_i d\alpha_i \cdot d\omega_{ji} = n \sum_i d\alpha_i \cdot d\omega_{ji}.$$

The differential 3-form $\theta_j = \sum_k d\alpha_k \cdot d\omega_{jk}$ is clearly closed on U_j. Moreover, on $U_i \cap U_j$ we have

$$\theta_j - \theta_i = \sum_k d\alpha_k \cdot (d\omega_{jk} - d\omega_{ik}) = \sum_k d\alpha_k \cdot d\omega_{ji} = 0,$$

according to the relation above between various $d\omega_{ji}$. Therefore, the $\{\theta_i\}$ define a global differential 3-form θ on the manifold X with values in $i\mathbb{R}$. This 3-cohomology class is the opposite of the image of λ by the connecting homomorphism[17]:

$$H^2(X; S^1) \to H^3(X; 2\pi i\mathbb{Z}),$$

[16]As in classical Chern-Weil theory, one may also write $\frac{1}{2}[\Gamma_i, \Gamma_i]$ instead of $(\Gamma_i)^2$.

[17]See Appendix 8.1 for a proof of this statement..

associated to the classical exact sequence of sheaves:

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow i\mathbb{R} \xrightarrow{\exp} \mathbb{S}^1 \to 0,$$

followed by the map[18]

$$H^3(X; 2\pi i\mathbb{Z}) \to H^3(X; i\mathbb{R}^\delta),$$

deduced from the inclusion $\mathbb{Z} \subset \mathbb{R}^\delta$. Summarizing the above discussion, we get our first relation

$$d(Ch_{(1)}(E,\Gamma)) = n \cdot \theta.$$

Analogous computations can be made with $R_{(2)}, R_{(3)}$, etc.. For an arbitrary m,

$$d(Ch_{(m)}(E,\Gamma)) = \sum_i \alpha_i \cdot \mathrm{Tr}(d(R_i)^m) + \sum_i d\alpha_i \cdot \mathrm{Tr}(R_i)^m.$$

Since $\mathrm{Tr}(d(R_i)^m) = 0$ for the same reasons as above, we have

$$d(Ch_{(m)}(E,\Gamma)) = \sum_i d\alpha_i \cdot \mathrm{Tr}(R_i)^m.$$

On the other hand, from the relation

$$\mathrm{Tr}(R_i)^m = \mathrm{Tr}(R_j)^m + m\mathrm{Tr}(R_j)^{m-1} \cdot d\omega_{ji},$$

we deduce the following identity between differential forms on U_j :

$$\sum_i d\alpha_i \cdot \mathrm{Tr}(R_i)^m = \sum_i d\alpha_i \cdot \mathrm{Tr}(R_j)^m + \sum_i m \cdot d\alpha_i \cdot \mathrm{Tr}(R_j)^{m-1} \cdot .d\omega_{ji}$$

$$= m \cdot \mathrm{Tr}(R_j)^{m-1} \cdot \theta.$$

Therefore,

$$\sum_i d\alpha_i \cdot \mathrm{Tr}(R_i)^m = \sum_j \alpha_j \sum_i d\alpha_i \cdot \mathrm{Tr}(R_i)^m$$

$$= \sum_j m \cdot \alpha_j \cdot \mathrm{Tr}(R_j)^m \cdot \theta = m \cdot Ch_{(m-1)}(E,\Gamma) \cdot \theta.$$

Summarizing again, we get the relation

$$d(Ch_{(m)}(E,\Gamma)) = m \cdot Ch_{(m-1)}(E,\Gamma) \cdot \theta.$$

We now define the total Chern character of (E,Γ) with values in the even de Rham forms[19]

$$\Omega^0(X) \oplus \Omega^2(X) \oplus ... \oplus \Omega^{2m}(X) \oplus ...$$

by the following formula:

$$Ch(E,\Gamma) = Ch_{(0)}(E,\Gamma) + Ch_{(1)}(E,\Gamma) + \frac{1}{2!}Ch_{(2)}(E,\Gamma) + ...$$

$$+\frac{1}{m!}Ch_{(m)}(E,\Gamma) + ...$$

We have chosen the coefficients in front of the $Ch_{(m)}$ such that $Ch(E,\Gamma)$ is a cycle in the even/odd de Rham complex (see footnote Nr 24) with the differential given by $D = d - .\theta$, where $.\theta$ is the map defined by the cup-product with θ.

In Appendix 8.3, we prove by classical considerations that this total Chern character is well defined as a twisted cohomology class and does not depend on the

[18]Here $\mathbb{R}^\delta$ denotes now the field $\mathbb{R}$ with the discrete topology.

[19]More precisely, $\Omega^{2k}(X)$ is the vector space of $2k$-differential forms with values in $(i)^k\mathbb{R}$.

connection Γ and on the partition of unity. This remark is also valid in the infinite dimensional case which will be studied later on.

For the time being, since we consider finite dimensional bundles, the class θ in $H^3(X; i\mathbb{R}^\delta)$ is reduced to 0. Therefore, by classical considerations on complexes using exponentials of even forms [**3**], we see that the target of this special Chern character reduces to the classical one. Moreover, we may also consider twisted bundles over $X \times S^1$, which enables us to define a Chern character from odd twisted K-groups to odd twisted cohomology. From standard Mayer-Vietoris arguments and Bott periodicity, we deduce that the Chern character induces an isomorphism between $K_\lambda(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $H^{even}(X; \mathbb{R})$.

We want to extend the previous considerations to the case when the cohomology class $[\lambda]$ is of infinite order. For this, we use the second definition of twisted K-theory in terms of twisted principal bundles associated to the group G of couples (g, h) such that g and h are invertible operators in $\mathcal{L}(H)$ with $g - h$ compact. However, in order to be able to take traces, we have to slightly modify this group by assuming moreover that $g - h$ is a trace class operator, i.e. belongs to L^1. By abuse of notation, we still call $\mathcal{A}$ the algebra[20] of couples $(g, h) \in \mathcal{L} \times \mathcal{L}$ such that $g - h \in L^1$. Using the classical density theorem in topological K-theory [**24**, pg. 109], it is easy to show that we get the same twisted K-theory as for $g - h$ compact. We may also choose the transition functions to be C^∞, as we did in the finite dimensional case.

The computations in the finite dimensional case may now be easily transposed into this framework if we consider transition functions (g_{ji}, h_{ji}) in the group[21] $G = \mathcal{A}^*$ and take "supertraces" instead of traces. We just have to be careful since the fibres of our bundles are not necessarily free[22]. Concretely, we define a rank map

$$\mathrm{Rk} = Ch_{(0)} : K_\lambda(X) \to H^0(X; \mathbb{Z})$$

as follows: if E is a finitely generated projective module over $\underline{\mathcal{A}}_\lambda$, it is defined by a family of two projection operators (p_0, p_1) in the algebra $\underline{\mathcal{A}}_\lambda$. Then the trace of $p_0 - p_1$ is a locally constant integer, defining the rank function, since

$$K(\mathcal{A}) \cong K(\mathcal{K}) \cong K(\mathbb{C}) = \mathbb{Z}.$$

If we look at E as a twisted $\mathcal{A}$-bundle over X with fibre P (which is a finitely generated projective $\mathcal{A}$-module), we may consider $\mathrm{End}(P)$ as included in $M_n(\mathcal{A}) \cong \mathcal{A}$ and restrict the supertrace defined on $\mathcal{A}$ to $\mathrm{End}(P)$. For instance, the supertrace of the identity on P is just the rank of P. By abuse of notations we shall identify $\mathrm{End}(P)$ and its image in $\mathcal{A}$.

We now define a connection on E as a family of differential forms $\Gamma_i = (\Gamma_i^0, \Gamma_i^1)$ with values in $\mathrm{End}(P) \subset \mathcal{A} \subset \mathcal{L} \times \mathcal{L}$, such that $\Gamma_i^0 - \Gamma_i^1$ is a differential form with values in L^1, satisfying the same compatibility condition as above:

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.$$

[20] The norm of an element (g, h) is the sum of the operator norm on g and the L^1-norm on $g - h$.

[21] More precisely, in the group $\mathrm{Aut}(P) \subset G = \mathcal{A}^*$; see below.

[22] As we mentioned already in Section 4, the fibres should be free if λ does not define a torsion class in the cohomology of each connected component of X; see below.

We choose the transition functions $g_{ji} = (g^0_{ji}, g^1_{ji})$ to be in $\mathrm{Aut}(P)$ rather than $\mathrm{GL}_n(\mathbb{C})$. Such connections exist, for instance the barycentric connection considered in the finite dimensional case

$$\Gamma_i = \sum \alpha_k \cdot g_{ki}^{-1} \cdot dg_{ki},$$

where (α_k) is a partition of unity associated to the covering. The only difference with the finite dimensional case is that n is replaced by $\mathrm{Rk}(E) = Ch_{(0)}(E)$ and the usual trace by the supertrace[23]. If we denote by str this supertrace, we have define:

$$Ch(E, \Gamma) = Ch_{(0)}(E) + \sum_{m=1}^{\dim(X)/2} \frac{1}{m!} \mathrm{str}(\sum_i \alpha_i (R_i)^m),$$

where the R_i are the local curvatures as functions of the Γ_i defined above, and where (α_i) is a partition of unity associated to the given covering $\mathcal{U}$.

The computations made before in the finite dimensional case show as well that $Ch(E, \Gamma)$ is a cocycle for the differential $D = d - .\theta$. As in the finite dimensional case, standard homotopy arguments also show that the cohomology class of $Ch(E, \Gamma)$ is independent of the connection Γ and of the partition of unity (α_i) (see Appendix 8.3 for the details).

Therefore, for any λ, the Chern character induces an isomorphism between $K_\lambda(X) \otimes_Z \mathbb{R}$ and the twisted cohomology which is the cohomology of the even part of the even/odd de Rham complex[24] with the twisted differential $D = d - .\theta$. It is proved in [**3**], in a computation involving again the exponential of even forms, that this twisted cohomology depends only on the class of θ in the cohomology group $H^3(X; i\mathbb{R})$.

Summarizing the previous discussion, we get the following theorem:

THEOREM 7.4. *Let $\mathcal{U}$ be a good covering of X, λ be a completely normalized 2-cocycle with values in S^1 associated to this covering. Let (α_i) be a partition of unity associated to this covering and let θ be the differential 3-form associated to $-\lambda$, according to Appendix 8.1. Then the Chern character*

$$Ch : K_\lambda(X) \to H^{ev}_\theta(X; \mathbb{R})$$

from twisted K-theory to even twisted cohomology induces an isomorphism

$$K_\lambda(X) \otimes_Z \mathbb{R} \cong H^{ev}_\theta(X; \mathbb{R}).$$

REMARK 7.5. The functoriality of the Chern character is discussed in Appendix 8.3. Its multiplicative properties will be studied later (Theorem 7.2).

REMARK 7.6. One may also normalize the Chern character by putting a factor $(1/2\pi i)^r$ in front of $Ch_{(r)}(E, \Gamma)$ and replace θ by $\theta/2\pi i$. Then we have to consider the usual de Rham complex, contrarily to our convention in the footnote Nr 24.

If the space X is formal in the sense of rational homotopy theory [**16**], we may replace the de Rham complex by its cohomology viewed as a graded vector space

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[23]Note again that the supertrace of "1" is the rank of P which is positive or negative.

[24]Note that $\Omega^{2k}(X)$ and $\Omega^{2k+1}(X)$ are the real vector spaces of differential forms of degree $2k$ or $2k+1$ with values in $(i)^k\mathbb{R}$. The differential is the usual one d on $\Omega^{2k}(X)$ and id on $\Omega^{2k+1}(X)$.

(with the differential reduced to 0). In that case, the (even) twisted cohomology is isomorphic to the even part of the cohomology of the complex

$$\left[\oplus H^{2k}(X;(i)^k\mathbb{R})\right] \oplus \left[\oplus H^{2k+1}(X;(i)^k\mathbb{R})\right],$$

with the differential given by the cup-product with the cohomology class of θ in $H^3(X;i\mathbb{R})$. By a well known and deep theorem of Deligne, Griffiths, Morgan and Sullivan [**13**], this computation is valid when X is a simply connected compact Kähler manifold.

In the particular case when θ is not 0 in all the cohomology groups $H^3(X_r;i\mathbb{R})$, where the X_r are the connected components of X, we see by a direct computation that $Ch_{(0)}(E,\Gamma)$ is necessarily 0, which implies that the fibres of E should be free $\mathcal{A}$-modules. This also implies that $Ch_{(1)}(E,\Gamma)$ is a closed differential form. Therefore, for any λ, one can define the first Chern character[25] $Ch_{(1)}(E,\Gamma)$ in the (non twisted) cohomology group $H^2(X;i\mathbb{R})$. However, we need the twisted differential cycles for the total Chern character of E.

Let now $\mathcal{U}=(U_i)$ and $\mathcal{V}=(V_j)$ be coverings of X and Y respectively. Let (α_i) (resp. (β_j)) be a partition of unity associated to $\mathcal{U}$ (resp $\mathcal{V}$). The products $(\alpha_i\cdot\beta_j)$ define a partition of unity associated to the covering $\mathcal{W}=(U_i\times V_j)$ of $X\times Y$.

Theorem 7.7. *Let E be a λ-twisted $\mathcal{A}$-bundle on X and let F be a μ-twisted $\mathcal{A}$-bundle on Y. Here λ and μ are explicit Čech cocycles λ_{tsr} and μ_{wvu} with values in S^1, associated to the coverings $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\overline{\lambda}$ and $\overline{\mu}$ be the closed differential forms defined on each $U_i\times V_j$ by the formulas*

$$\overline{\lambda}=\sum_{t,s} d\alpha_t\cdot d\alpha_s\cdot\lambda_{tsi}^{-1}\cdot d\lambda_{tsi}$$

$$\overline{\mu}=\sum_{w,v} d\beta_w\cdot d\beta_v\cdot\mu_{wvj}^{-1}\cdot d\mu_{wvj},$$

as in Appendix 8.1. Then we have the commutative diagram[26]

$$\begin{array}{ccc} K_\lambda(X)\times K_\mu(Y) & \to & K_{\lambda\mu}(X\times Y)\\ \downarrow & & \downarrow\\ H^{ev}_{\overline{\lambda}}(X)\times H^{ev}_{\overline{\mu}}(Y) & \longrightarrow & H^{ev}_{\overline{\lambda}+\overline{\mu}}(X\times Y)\end{array}$$

Proof. Let $\Gamma=(\Gamma_i)$ (resp. $\nabla=(\nabla_j)$) be a connection on E (resp. F). Then $\Delta=\Gamma\otimes 1+1\otimes\nabla$ is a connection on $E\otimes F$. Therefore, if R_E (resp. R_F) is the curvature associated to Γ (resp. ∇), then

$$R_{E\otimes F}=R_E\otimes 1+1\otimes R_F$$

is the curvature associated to Δ over each open subset $U_i\times V_j$ of $X\times Y$. Using the partition of unity $(\alpha_i\cdot\beta_j)$ associated to the covering $(U_i\times V_j)$ and the binomial identity, we find the relation

$$\frac{1}{m!}Ch_{(m)}(E\otimes F,\Delta)=\sum_{p+q=m}\frac{1}{p!q!}Ch_{(p)}(E,\Gamma)Ch_{(q)}(F,\nabla),$$

from which the theorem follows. □

[25] which is also a Chern class.

[26] According to the computations in Section 5, we map the K-theory of $\mathcal{A}_\lambda\widehat{\otimes}\mathcal{A}_\mu$ to the K-theory of $\mathcal{A}_{\lambda\mu}$. However, in these computations, one has to replace $\mathcal{K}$ by the ideal L^1 of trace class operators.

Finally, we should add a few words concerning graded twisted K-theory which is indexed essentially by elements

$$[\widetilde{\lambda}] \in H^1(X;\mathbb{Z}/2) \times H^2(X;S^1).$$

If we apply Theorem 4.4 of [**23**], this group (at least rationally) is isomorphic to the ungraded twisted K-theory of Y, where Y is the Thom space of the orientation real line bundle L. This L corresponds to the image of $[\widetilde{\lambda}]$ in $H^1(X;\mathbb{Z}/2)$. In more precise terms, the graded twisted K-group tensored with the field of real numbers is isomorphic to the odd twisted relative cohomology group of the pair (P,X). Here $P = \mathbb{P}(L\oplus 1)$ denotes the real projective bundle of $L\oplus 1$ (with fibre $P^1 \cong S^1$), and the 3-dimensional cohomology twist is induced by the projection $P \to X$ from the one on X. This (graded) twisted cohomology is different in general from the twisted cohomology associated to the image of $[\widetilde{\lambda}]$ in $H^3(X;i\mathbb{R})$.This is not surprising since the usual real cohomology of a manifold with a coefficient system in $H^1(X;\mathbb{Z}/2)$ also depends on this system.

Remark 7.8. If A is not a commutative Banach algebra, there is no internal product

$$K_n(A)\times K_p(A) \to K_{n+p}(A)$$

in general. Therefore, it is remarkable that such a product exists for twisted K-groups which are $K_*(\underline{\mathcal{K}}_\lambda)$, where $\underline{\mathcal{K}}_\lambda$ is a noncommutative Banach algebra..

8. Appendix

8.1. Relation between Čech cohomology with coefficients in S^1 and de Rham cohomology. This section does not claim any originality. It may be easily deduced from the classical books [**5**], [**25**] for instance, the basic ideas going back to André Weil. It is added for completeness' sake and normalization purposes.

Our first task is to make more explicit the cohomology isomorphism

$$H^r(\mathcal{U})\cong H^r_{dR}(X),$$

where $\mathcal{U}$ is a good covering of X. The Čech and de Rham cohomologies are here taken with coefficients in a real vector space of finite dimension V.

Let us denote by $\Omega^r(X)$ the vector space of differential forms on X with values in V and let (α_i) be a partition of unity associated to the covering $\mathcal{U}$. We define a morphism[27]

$$f_r : C^r(\mathcal{U};V) \to \Omega^r(X)$$

in the following way. For $r=0$, we send a cochain (c_i) to the C^∞ function

$$x \longmapsto \sum \alpha_i(x)\cdot c_i,$$

which we simply write $\sum\limits_i \alpha_i \cdot c_i$. For general $r>0$, we send the r-cochain $(c_{i_0 i_1 \ldots i_r})$ to the sum

$$\sum_{(i_0,\ldots,i_r)} \alpha_{i_0}.d\alpha_{i_1}\cdot \ldots d\alpha_{i_r}\cdot c_{i_0 i_1\ldots i_r}.$$

We have to check that this correspondence is compatible with the coboundaries, i.e. that

$$f_{r+1}(\partial c) = d(f_r(c)).$$

[27]With V provided with the discrete topology.

The cochain ∂c, which we call v, is defined by the usual formula

$$v_{i_0 i_1 \ldots i_{r+1}} = \sum_{m=0}^{r+1} (-1)^m c_{i_0 \ldots \widehat{i_m} \ldots i_{r+1}}.$$

Therefore,

$$f_{r+1}(v) = \sum_{(i_0,\ldots,i_{r+1})} \alpha_{i_0}.d\alpha_{i_1} \cdot \ldots d\alpha_{i_{r+1}} \cdot \sum_{m=0}^{r+1} (-1)^m c_{i_0 \ldots \widehat{i_m} \ldots i_{r+1}}.$$

In the previous sum, the terms corresponding to an index $m > 0$ are reduced to 0 since the sum of the corresponding $d\alpha$ is 0. The previous identity may then be written

$$\begin{aligned} f_{r+1}(v) &= \sum_{(i_0,\ldots,i_{r+1})} \alpha_{i_0}.d\alpha_{i_1} \cdot \ldots d\alpha_{i_{+1}} \cdot c_{i_1 \ldots i_{r+1}} \\ &= \sum_{(i_1,\ldots,i_{r+1})} d\alpha_{i_1} \cdot \ldots d\alpha_{i_{r+1}} \cdot c_{i_1 \ldots i_{r+1}}, \end{aligned}$$

which is $d(f_r(c))$, if we reindex the components of this sum: notice that the $c_{i_1 \ldots i_m}$ are constant functions.

The maps (f_r) define a morphism of complexes which is a quasi-isomorphism over any intersection of the U_i since the covering $\mathcal{U}$ is good. Therefore, by a classical Mayer-Vietoris argument, they induce an isomorphism between the Čech and de Rham cohomologies.

We take a step further and now compare the Čech cohomology $H^{r-1}(X : S^1)$ with $H^r_{dR}(X)$ via a map

$$H^{r-1}(\mathcal{U}; S^1) \to H^r(\mathcal{U}; V) \cong H^r_{dR}(X).$$

This is the coboundary map associated to the exact sequence

$$0 \to 2\pi i\mathbb{Z} \to i\mathbb{R} \xrightarrow{e} S^1 \to 0,$$

where e is the exponential function and V the real vector space $i\mathbb{R}$. If $\lambda_{i_0 i_1 .. i_{r-1}} \in Z^{r-1}(\mathcal{U}; S^1)$, there is a cochain $u = u_{i_0 i_1 .. i_{r-1}}$ such that $e(u) = \lambda$. The classical definition of the coboundary map

$$H^{r-1}(\mathcal{U}; S^1) \to H^r(\mathcal{U}; 2\pi i\mathbb{Z})$$

is as follows. We first consider the coboundary of u in $C^r(\mathcal{U})$, which we look as a cocycle with values in $2\pi i\mathbb{Z}$, defined by

$$c_{i_0 i_1 \ldots i_r} = \sum_{m=0}^{r} (1)^m u_{i_0 \ldots \widehat{i_m} \ldots i_r}.$$

According to the previous considerations, the associated de Rham class with values in $i\mathbb{R} = V$ is defined by

$$\omega = \sum_{(i_0,\ldots,i_r)} \alpha_{i_0}.d\alpha_{i_1} \cdot \ldots d\alpha_{i_r} \cdot c_{i_0 \ldots i_r}$$

$$= \sum_{(i_0,\ldots,i_r)} \alpha_{i_0}.d\alpha_{i_1} \cdot \ldots d\alpha_{i_r}. \sum_{m=0}^{r} (1)^m u_{i_0 \ldots \widehat{i_m} \ldots i_r}.$$

Using the same argument as above, this sum may be written

$$\omega = \sum_{(i_1,\ldots,i_r)} d\alpha_{i_1} \cdot \ldots d\alpha_{i_r} \cdot u_{i_1\ldots i_r}.$$

We notice that ω is a closed form since $c_{i_0 i_1\ldots i_r} \in 2\pi i\mathbb{Z}$. On the other hand, it is cohomologous up to the sign $(-1)^r$ to the form

$$\theta = \sum_{(i_1,\ldots,i_r)} \alpha_{i_1} \cdot d\alpha_{i_2} \ldots d\alpha_{i_r} \cdot du_{i_1 i_2\ldots i_r}.$$

Using again the fact that $c_{i_0 i_1\ldots i_r} \in 2i\pi\mathbb{Z}$, we see that θ is equal on U_{i_0} to the following differential form

$$\sum_{(i_1,\ldots,i_r)} \alpha_{i_1} d\alpha_{i_2} \ldots d\alpha_{i_r} \cdot du_{i_1 i_2\ldots i_r} = \sum_{(i_2,\ldots,i_r)} d\alpha_{i_2} \ldots d\alpha_{i_r} \cdot du_{i_0 i_2\ldots i_r}.$$

We observe that $du_{i_0 i_2\ldots i_r}$ is the logarithmic differential of $\lambda_{i_0 i_2\ldots i_r}$. Therefore, if we change the indices and r to $r+1$, we get the following theorem.

Theorem 8.1. *Let $\lambda_{i_0\ldots i_r}$ be an r-cocycle on a good covering $\mathcal{U}$ with values in S^1 and let (α_i) be a partition of unity associated to $\mathcal{U}$. Then the closed de Rham form ω of degree $r+1$ with values in $V = i\mathbb{R}$ which is associated to λ by the coboundary map*[28]

$$H^r(\mathcal{U}; S^1) \to H^{r+1}(\mathcal{U}; 2\pi i\mathbb{Z}) \to H^{r+1}(\mathcal{U}; i\mathbb{R})$$

is given by the following formula on each open set U_{i0} :

$$\omega = (-1)^{r+1} \sum_{(i_1,\ldots,i_r)} d\alpha_{i_1 \ldots} d\alpha_r \cdot (\lambda_{i_0\ldots i_r})^{-1} \cdot d\lambda_{i_0\ldots i_r}.$$

Example 8.2. If we choose $r = 2$ as in our paper, those formulas may be simply written as

$$\omega = \sum_{(i,j,k)} d\alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot c_{ijk}$$

which is cohomologous to

$$-\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ijk}.$$

On the other hand, for a fixed l, if we consider the sequence (l, i, j, k), and the fact that

$$c_{ijk} - c_{ljk} + c_{lik} - c_{lij} \in 2i\pi\mathbb{Z},$$

we may replace dc_{ijk} by $dc_{ljk} - dc_{lik} + dc_{lij}$. Therefore, the restriction of ω to U_l may be written as

$$\omega = -\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ljk} + \sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{lik}$$
$$- \sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{lij}.$$

$$= -\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ljk} = -\sum_{(j,k)} d\alpha_j \cdot d\alpha_k \cdot dc_{ljk} = -\sum_{(j,k)} d\alpha_j \cdot d\alpha_k \cdot \lambda_{ijk}^{-1} d\lambda_{ljk},$$

[28]Note that $\mathbb{R}$ is provided with the discrete topology.

as a differential form on U_l. If we assume the cocycle λ completely normalized, we find the explicit formula given in Section 7.

8.2. Some key isomorphisms between various definitions of twisted K-groups. We want to make more explicit the isomorphisms between the various definitions of twisted K-theory given in Section 4. This is especially relevant to the proof of the multiplicativity of the Chern character in Section 7.

With the notations of Section 4, the most basic one is probably the following:

$$K(\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

We recall that the first group $K(\underline{\mathcal{K}}_\lambda)$ is the original definition of Rosenberg [**29**]. The second group may be interpreted as the Fredholm definition of twisted K-theory as in [**2**] (or [**15**] if λ defines a torsion class in $H^3(X;\mathbb{Z})$). More precisely, if E is a λ-twisted Hilbert bundle and if $\mathcal{F}(E)$ is the space of Fredholm maps in $\mathrm{END}(E)$, the map

$$\mathcal{F}(E) \to (\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda)^*$$

is a locally trivial fibration with contractible fibres, as we pointed out in Section 4. Therefore, we have the identifications

$$K_\lambda(X) \cong K(\underline{\mathcal{K}}_\lambda) \cong K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

THEOREM 8.3. *Let τ be the generator of $K_1(\mathcal{L}/\mathcal{K}) \cong \mathbb{Z}$, associated to the Fredholm operator given by the shift. Then the cup-product with τ induces an isomorphism*

$$\varphi : K(\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

PROOF. In this statement, we implicitly identify the Hilbert tensor product $H \otimes H$ with H. If we forget the twisting, there is a well defined ring map

$$\mathcal{K} \otimes \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}.$$

For the same reasons, there is a ring map

$$\underline{\mathcal{K}}_\lambda \widehat{\otimes} \mathcal{L}/\mathcal{K} \to \underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda.$$

When the base space X varies, the cup-product with the element τ induces a morphism between the (twisted) K_*-theories associated to $\underline{\mathcal{K}}_\lambda$ and $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$ respectively (with a shift for the second one). By a standard Mayer-Vietoris argument and Bott periodicity, we reduce the theorem to the case when X is contractible, a case which is obvious. $\square$

Although we don't really need it in this paper, it might be interesting to define explicitly the backwards isomorphism:

$$\psi : K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K(\underline{\mathcal{K}}_\lambda) \cong K(\mathcal{A}_\lambda).$$

Such a map ψ is simply the connecting homomorphism in the Mayer-Vietoris exact sequence in K-theory associated to the cartesian square

$$\begin{array}{ccc} \underline{\mathcal{A}}_\lambda & \to & \underline{\mathcal{L}}_\lambda \\ \downarrow & & \downarrow \\ \underline{\mathcal{L}}_\lambda & \to & \underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda \end{array}.$$

In more detail: if α is an invertible element in the ring $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$, we consider the 2×2 matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

By the Whitehead lemma (or analytic considerations: see below), this matrix may be lifted as an invertible 2×2 matrix with coefficients in $\underline{\mathcal{L}}_\lambda$, say γ. Let ε be the matrix defining the obvious grading

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the couple $(\varepsilon, \gamma \cdot \varepsilon \cdot \gamma^{-1})$ defines an involution J on $M_2(\underline{\mathcal{A}}_\lambda) \cong \underline{\mathcal{A}}_\lambda$, hence a finitely generated projective module over $\underline{\mathcal{A}}_\lambda$ which is simply the image of $(J+1)/2$. It is easy to show that the class in $K(\underline{\mathcal{A}}_\lambda)$ is independent of the choice of the lifting γ : this is the classical definition of the connecting homomorphism ψ (see e.g. [**28**]).

Instead of working with invertible elements α, we may as well consider families of Fredholm maps D mapping to α, which are already in $\underline{\mathcal{L}}_\lambda$. Without loss of generality, we may also assume α unitary which implies that a lifting of α^{-1} may be choosen to be the adjoint D^*. We now write the identity

$$\begin{pmatrix} D & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we define ∇_D as

$$\nabla_D = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$$

in general, we see that we may choose the element γ above to be $\exp(\pi\nabla_D/2) \cdot \nabla_{-1}$. Therefore,

$$\begin{aligned} \gamma \cdot \varepsilon \cdot \gamma^{-1} &= \exp(\pi\nabla_D/2) \cdot \nabla_{-1}.\varepsilon \cdot \nabla_1 \cdot \exp(-\pi\nabla_D/2) \\ &= -\exp(\pi\nabla_D/2).\varepsilon \cdot \exp(-\pi\nabla_D/2). \end{aligned}$$

On the other hand, it is clear that ∇_D and ε anticommute. Therefore, the previous formula may be written as

$$\gamma \cdot \varepsilon \cdot \gamma^{-1} = \exp(\pi\nabla_D).\varepsilon.$$

The couple

$$J = (\varepsilon, \exp(\pi\nabla_D).\varepsilon)$$

defines the required element of $K(\underline{\mathcal{A}}_\lambda)$. By construction, we see that J also defines an element of the relative group associated to the augmentation map

$$(\underline{\mathcal{K}}_\lambda)^+ \to \mathbb{C}.$$

Here $(\underline{\mathcal{K}}_\lambda)^+$ is the ring $\underline{\mathcal{K}}_\lambda$ with a unit added and the relative K-group is the usual one:

$$K(\underline{\mathcal{K}}_\lambda) = Ker(K((\underline{\mathcal{K}}_\lambda)^+) \to K(\mathbb{C}) = \mathbb{Z})$$

which is canonically isomorphic to $K(\mathcal{A}_\lambda)$.

8.3. Some functorial properties of twisted K-theory and of the Chern character. In this paper, we have indexed twisted K-theory by completely normalized 2-cocycles λ with values in S^1. Of course, such a cocycle determines a cohomology class $[\lambda]$ in $H^2(X;S^1) \cong H^3(X;2\pi i\mathbb{Z})$ as we have seen in 8.1 and we would like to index twisted K-theory by elements of this smaller group. There is an obstruction to doing so, however, as we shall see. If we apply Proposition 1.2 to $\mathbb{C}$-bundles (if $[\lambda]$ is a torsion class) or to $\mathcal{A}$-bundles in general, we see that if μ is cohomologous to λ, the equivalence Θ in this last proposition, between the categories of λ-twisted bundles and μ-twisted bundles, depends on the choice of a cochain η such that

$$\mu_{kji} = \lambda_{kji} \cdot \eta_{ji} \cdot \eta_{ki}^{-1} \cdot \eta_{kj}.$$

If η' is another choice, $\eta_{ji}.\eta'^{-1}_{ji}$ is a one-dimensional cocycle with values in S^1. Since a one-dimensional coboundary does not change λ, we see that the ambiguity in the definition of the previous category equivalence lies in the cohomology group[29] $H^1(\mathcal{U};S^1) \cong H^2(X;2\pi i\mathbb{Z}) \cong H^2(X;\mathbb{Z})$. In particular, the definition of twisted K-theory with coefficients in $H^3(X;\mathbb{Z})$ has a well-defined meaning only if $H^2(X;\mathbb{Z}) = 0$.

This remark is also important for the definition of the product

$$K_\lambda(X) \times K_\mu(X) \to K_{\lambda\mu}(X)$$

which is detailed in many ways in Section 5. The Hilbert bundle E_λ, defined at the beginning of this section, depends on the cocycle λ. It depends on its cohomology class $[\lambda]$ up to a non-canonical isomorphism as we have just seen (except if $H^2(X;\mathbb{Z}) = 0$). Therefore, strictly speaking, we cannot define in a functorial way a cup-product

$$K_{[\lambda]}(X) \times K_{[\mu]}(X) \to K_{[\lambda\mu]}(X).$$

Another remark is the choice of a good covering in order to define twisted K-theory via twisted bundles. There is also a functorial problem since many choices are possible. One way to deal with this is to show that the categories of twisted bundles associated to different coverings give the same twisted K-theory if we choose two Čech cocycles which are cohomologous. This is again included in the contents of Proposition 1.2. As we already pointed out, this identification is not canonical, except if $H^2(X;\mathbb{Z}) = 0$.

Let us now turn our attention to the definition of the Chern character. If we fix the good covering $\mathcal{U}$, our definition depends heavily on the choice of a partition of unity (α_i). If (β_i) is another choice, there is a homotopy between them which is $t \longmapsto (1-t)\alpha_i + t\beta_i$. If λ is a completely normalized 2-cocycle with values in S^1, the associated closed differential forms θ_α and θ_β are homotopic and therefore cohomologous: they define the same class in $H^3(X;i\mathbb{R})$. However, it is not completely obvious that the associated twisted cohomologies $H^{ev}_{\theta_\alpha}(X)$ and $H^{ev}_{\theta_\beta}(X)$ are isomorphic in a way compatible with the Chern character. One way to deal with this problem is to consider λ-twisted bundles over $X \times [0,1]$ with the partition of unity given by $(1-t)\alpha_i + t\beta_i$ as above. We then have a commutative diagram where

[29]We assume the covering good as in 3.3.

the horizontal arrows are isomorphisms

$$\begin{array}{ccccc} K_\lambda(X\times\{0\}) & \longleftarrow & K_\lambda(X\times[0,1]) & \longrightarrow & K_\lambda(X\times\{1\}) \\ \downarrow & & \downarrow & & \downarrow \\ H^{ev}_{\theta_\alpha}(X\times\{0\}) & \longleftarrow & H^{ev}_{\theta}(X\times[0,1] & \longrightarrow & H^{ev}_{\theta_\beta}(X\times\{1\}). \end{array}$$

This diagram shows that the Chern character does not depend on the choice of partition of unity up to canonical isomorphisms given by the horizontal arrows.

We cannot expect the Chern character to be functorial with respect to the cohomology class of λ in $H^3(X;\mathbb{Z})$. However, it is "partially functorial" in the following sense: if we choose a good refinement $\mathcal{V}=(V_s)$ of $\mathcal{U}=(U_i)$ as in Section 1, any restriction map of type

$$\Theta_\tau : K_\lambda(\mathcal{U}) \to K_\mu(\mathcal{V})$$

(where $V_s \subset U_{\tau(s)}$) is an isomorphism. This isomorphism is not unique and depends on τ, as was pointed out in the proof of Proposition 1.3. If (β_s) is a partition of unity associated to the covering $\mathcal{V}$ and (α_i) a partition of unity associated to the covering $\mathcal{U}$, the functions $(\alpha_i\cdot\beta_s)$ define a partition of unity associated to $\mathcal{U}\cap\mathcal{V}$ which is just a reindexing of the covering $\mathcal{V}$. On the other hand, we may also reindex $\mathcal{U}$ in such a way that the functions $(\alpha_i\cdot\beta_s)$ define also a partition of unity of $\mathcal{U}$. Since the twisted cohomology is homotopically invariant, it follows that the "restriction map"

$$H^{ev}_{\overline{\lambda}}(X) \to H^{ev}_{\overline{\mu}}(X)$$

is also well defined and that the diagram

$$\begin{array}{ccc} K_\lambda(\mathcal{U}) & \to & K_\mu(\mathcal{V}) \\ \downarrow & & \downarrow \\ H^{ev}_{\overline{\lambda}}(X) & \to & H^{ev}_{\overline{\mu}}(X) \end{array}$$

is commutative (with the notation of Theorem 7.2).

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A Guided Tour Through the Garden of Noncommutative Motives

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Abstract. These are the extended notes of a survey talk on noncommutative motives given at the 3^era *Escuela de Inverno Luis Santaló-CIMPA Research School: Topics in Noncommutative Geometry*, Buenos Aires, July 26 to August 6, 2010.

Contents

Acknowledgments: I would like to start by thanking all the organizers of this winter school and specially professor Guillermo Cortiñas for kindly giving me the opportunity to present my work. I am also very grateful to the Clay Mathematics Institute for financially supporting my participation. A special thanks goes also to Eugenia Ellis who made me discover the magic of the *Argentine Tango* as well as the fascinating city of Buenos Aires. Finally, I would like to thank the Department of Mathematics of MIT for a warm reception.

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In order to make these notes accessible to a broad audience, I have decided to emphasize the conceptual ideas behind the theory of noncommutative motives rather than its technical aspects. I will start by stating two foundational questions. One concerns *higher algebraic K-theory* (**Question A**) and the other one concerns *noncommutative algebraic geometry* (**Question B**). One of the main goals of this guided

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tour will be not only to provide precise answers to these distinct questions but moreover to explain what is the relation between the corresponding answers.

1. Higher algebraic K-theory

Algebraic K-theory goes back to Grothendieck's work [**20**] on the Riemann-Roch theorem. Given a commutative ring R (or more generally an algebraic variety), he introduced what is nowadays called the *Grothendieck group* $K_0(R)$ of R. Later, in the sixties, Bass [**2**] defined $K_1(R)$ as the abelianization of the general linear group $\mathrm{GL}(R)$. These two abelian groups, whose applications range from arithmetic to surgery of manifolds, are very well understood from a conceptual and computational point of view; see Weibel's survey [**57**]. After Bass' work, it became clear that these groups should be part of a whole family of *higher* algebraic K-theory groups. After several attempts made by several mathematicians, it was Quillen who devised an elegant *topological* construction; see [**39**]. He introduced what is nowadays called Quillen's *plus construction* $(-)^+$, by which we simplify the fundamental group of a space without changing its (co-)homology groups. By applying this construction to the classifying space $\mathrm{BGL}(R)$ (where simplification in this case means abelianization), he defined the higher algebraic K-theory groups as

$$K_n(R) := \pi_n(\mathrm{BGL}(R)^+ \times K_0(R)) \qquad n \geq 0 \,.$$

Since Quillen's foundational work, higher algebraic K-theory has found extraordinary applications in a wide range of research fields; consult [**18**]. However, Quillen's mechanism for manufacturing these higher algebraic K-theory groups remained rather mysterious until today. Hence, the following question is of major importance:

Question A: *How to conceptually characterize higher algebraic K-theory ?*

2. Noncommutative algebraic geometry

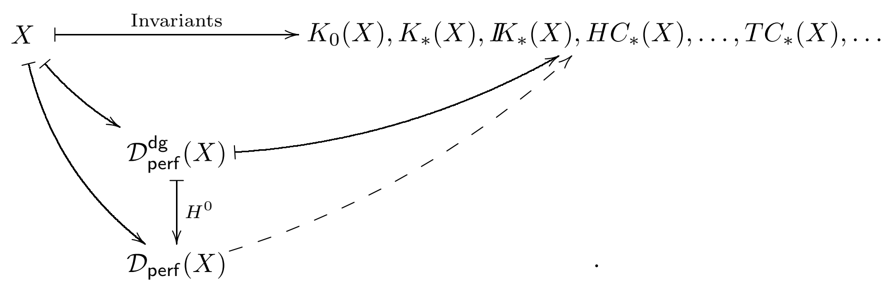
Noncommutative algebraic geometry goes back to Bondal-Kapranov's work [**7, 8**] on exceptional collections of coherent sheaves. Since then, Drinfeld, Kaledin, Kontsevich, Orlov, Van den Bergh, and others, have made important advances; see [**9, 10, 16, 17, 25, 30, 31, 32, 33**]. Let X be an algebraic variety. In order to study it, we can proceed in two distinct directions.

In one direction, we can associate to X several (functorial) invariants like the Grothendieck group (K_0), the higher K-theory groups (K_*), the negative K-theory groups ($I\!\!K_*$), the cyclic homology groups (HC_*) and all its variants (Hochschild, periodic, negative, ...), the topological cyclic homology groups (TC_*), etc. Each one of these invariants encodes a particular arithmetic/geometric feature of the algebraic variety X.

In the other direction, we can associate to X its derived category $\mathcal{D}_{\mathsf{perf}}(X)$ of perfect complexes of $\mathcal{O}_X$-modules. The importance of this triangulated category relies on the fact that any correspondence between X and X' which induces an equivalence between the derived categories $\mathcal{D}_{\mathsf{perf}}(X)$ and $\mathcal{D}_{\mathsf{perf}}(X')$ also induces an isomorphism on all the above invariants. Hence, it is natural to ask if the above invariants of X can be recovered directly from $\mathcal{D}_{\mathsf{perf}}(X)$. This can be done in very particular cases (*e.g.* the Grothendieck group) but not in full generality. The reason is that when we pass from X to $\mathcal{D}_{\mathsf{perf}}(X)$ we lose too much information concerning X. We should therefore "stop somewhere in the middle". In order to formalize this insight, Bondal and Kapranov introduced the following notion.

Definition 2.1. (Bondal-Kapranov [**7, 8**]) A *differential graded (=dg) category* $\mathcal{A}$, over a (fixed) base commutative ring k, is a category enriched over complexes of k-modules (morphism sets $\mathcal{A}(x,y)$ are complexes) in such a way that composition fulfills the Leibniz rule: $d(f\circ g) = d(f)\circ g + (-1)^{\deg(f)} f\circ d(g)$. A *differential graded (=dg) functor* is a functor which preserves the differential graded structure; consult Keller's ICM adress [**28**] for further details. The category of (small) dg categories (over k) is denoted by dgcat.

Associated to the algebraic variety X there is a natural dg category $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)$ which enhances[1] the derived category $\mathcal{D}_{\mathsf{perf}}(X)$, *i.e.* the latter category is obtained from the former one by applying the 0^{th}-cohomology group functor at each complex of morphisms. By considering $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)$ instead of $\mathcal{D}_{\mathsf{perf}}(X)$ we solve many of the (technical) problems inherent to triangulated categories like the non-functoriality of the cone. More importantly, we are able to recover all the above invariants of X directly out of $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)$. This circle of ideas is depicted in the following diagram:



From the point of view of the invariants, there is absolutely no difference between the algebraic variety X and the dg category $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)$. This is the main idea behind noncommutative algebraic geometry: given a dg category, we should consider it as being the dg derived category of perfect complexes over a hypothetical noncommutative space and try to do "algebraic geometry" directly on it. Citing Drinfeld [**17**], noncommutative algebraic geometry can be defined as: *"the study of dg categories and their homological invariants "*.

Example 2.2. (Beilinson [**3**]) Suppose that X is the n dimensional projective space $\mathbb{P}^n$. Then, there is an equivalence of dg categories

$$\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(\mathbb{P}^n) \simeq \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(B)\,,$$

where B is the algebra $\mathrm{End}(\mathcal{O}(0)\oplus\mathcal{O}(1)\oplus\cdots\oplus\mathcal{O}(n))^{\mathsf{op}}$. Note that the abelian category of quasi-coherent sheaves on $\mathbb{P}^n$ is far from being the category of modules over an algebra. Beilinson's remarkable result show us that this situation changes radically when we pass to the derived setting. Intuitively speaking, the n dimensional projective space is an "affine object" in noncommutative algebraic geometry since it is described by a single (noncommutative) algebra.

In the commutative world, Grothendieck envisioned a theory of *motives* as a gateway between algebraic geometry and the assortment of the classical Weil

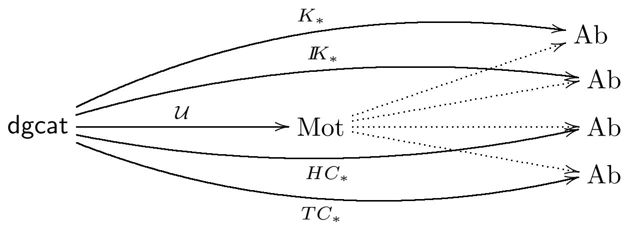
[1] Consult Lunts-Orlov [**35**] for the uniqueness of this enhancement.

cohomology theories (de Rham, Betti, l-adic, crystalline, and others); consult the monograph [**23**].

In the noncommutative world we can envision a similiar picture. The role of the algebraic varieties and of the classical Weil cohomologies is played, respectively, by the dg categories and the numerous (functorial) invariants[2]

$$\mathsf{dgcat} \xrightarrow{K_*,\, I\!K_*,\, HC_*, \ldots, TC_*, \ldots} \mathrm{Ab} \tag{2.3}$$

The Grothendieckian idea of motives consists then on combing this skein of invariants in order to isolate the truly fundamental one:



The gateway category Mot, through which all invariants factor uniquely, should then be called the category of *noncommutative motives* and the functor $\mathcal{U}$ the *universal invariant*. Note that, in this yoga, the different invariants are simply different representations of the motivic category Mot. In particular, any result which holds in Mot holds everywhere. This beautiful circle of ideas leads us to the following down-to-earth question:

Question B: *Is there a well-defined category of noncommutative motives ?*

3. Derived Morita equivalences

Note first that all the classical constructions which can be performed with k-algebras can also be performed with dg categories; consult [**28**]. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *derived Morita equivalence* if the induced restriction of scalars functor $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories. Thanks to the work of Blumberg-Mandell, Keller, Schlichting, and Thomason-Trobaugh, all the invariants (2.3) invert derived Morita equivalences; see [**6, 29, 40, 53**]. Intuitively speaking, although defined at the "dg level", these invariants only depend on the underlying derived category. Hence, it is crucial to understand dg categories up to derived Morita equivalence. The following result is central in this direction.

THEOREM 3.1. *(*[**42, 48**]*) The category* dgcat *carries a (cofibrantly generated) Quillen model structure*[3] *whose weak equivalences are the derived Morita equivalences.*

The homotopy category obtained is denoted by Hmo. Theorem 3.1 allows us to study the purely algebraic setting of dg categories using ideas, techniques, and insights of topological nature. Here are some examples:

[2]In order to simplify the (graphical) exposition, we have decided to forget the k-linear structure of the cyclic homology groups HC_*.

[3]An analogous model structure in the setting of spectral categories was developed in [**45**].

Bondal-Kapranov's pre-triangulated envelope. Using "one-sided twisted complexes", Bondal and Kapranov constructed in [**7**] a pre-triangulated envelope $\mathcal{A}^{\mathsf{pre\text{-}tr}}$ of every dg category $\mathcal{A}$. Intuitively speaking, their construction consists in formally adding to $\mathcal{A}$ (de-)suspensions, cones, cones of morphisms between cones, etc. Thanks to Theorem 3.1, this involved contribution can be conceptually characterized as being simply a functorial fibrant resolution functor; see [**42**].

Drinfeld's DG quotient. The most useful operation which can be performed on triangulated categories is the passage to a Verdier quotient. Recently, through a very elegant construction (reminiscent of the Dwyer-Kan localization), Drinfeld [**16**] lifted this operation to the world of dg categories. Although very elegant, this construction didn't seem to satisfy any obvious universal property. Theorem 3.1 allowed us to complete this aspect of Drinfeld's work by characterizing the dg quotient as a homotopy cofiber construction; see [**44**].

Kontsevich's saturated dg categories. Kontsevich understood precisely how to express smoothness and properness in the noncommutative world.

Definition 3.2. (Kontsevich [**30, 31**]) A dg category $\mathcal{A}$ is called:

- *smooth* if it is perfect as a bimodule over itself;
- *proper* if its complexes of k-modules $\mathcal{A}(x, y)$ are perfect;
- *saturated* if it is smooth and proper.

Definition 3.2 is justified by the following fact: given a quasi-compact and quasi-separated scheme X, the dg category $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)$ is smooth and proper if and only if X is smooth and proper in the sense of classical algebraic geometry. Other examples of saturated dg categories appear in the study of Deligne-Mumford stacks, quantum projective varieties, Landau-Ginzburg models, etc.

Now, note that the tensor product of k-algebras extends naturally to dg categories. By deriving it (with respect to derived Morita equivalences), we obtain then a symmetric monoidal structure on Hmo. Making use of it, the saturated dg categories can be conceptually characterized as being precisely the dualizable (or rigid) objects in the symmetric monoidal category Hmo; see [**12**]. As in any symmetric monoidal category, we can define the Euler characteristic of a dualizable object. In topology, for instance, the Euler characteristic of a finite CW-complex is the alternating sum of the number of cells. In Hmo, we have the following result.

Proposition 3.3. *(Cisinski & Tabuada* [**12**]*) Let $\mathcal{A}$ be a saturated dg category. Then its Euler characteristic $\chi(\mathcal{A})$ in Hmo is the Hochschild homology*[4] *complex $HH(\mathcal{A})$ of $\mathcal{A}$.*

Proposition 3.3 illustrates the Grothendieckian idea of combing the skein of invariants (2.3) "as far as possible" in order to understand, directly on Mot, their conceptual nature. By simply inverting the class of derived Morita equivalences, Hochschild homology can be conceptually understood as the Euler characteristic.

[4]More generally, the trace of an endomorphisms is given by Hochschild homology with coefficients.

4. Noncommutative pure motives

In order to answer **Question B** we need to start by identifying the properties common to all the invariants (2.3). In the previous section we have already observed that they are *derived Morita invariant*, *i.e.* they send derived Morita equivalences to isomorphisms. In this section, we identify another common property. An *upper triangular matrix* M is given by

$$M \quad := \quad \begin{pmatrix} \mathcal{A} & X \\ 0 & \mathcal{B} \end{pmatrix},$$

where $\mathcal{A}$ and $\mathcal{B}$ are dg categories and X is an $\mathcal{A}$-$\mathcal{B}$-bimodule. The totalization $|M|$ of M is the dg category whose set of objects is the disjoint union of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$, and whose morphisms are given by: $\mathcal{A}(x,y)$ if $x,y \in \mathcal{A}$; $\mathcal{B}(x,y)$ if $x,y \in \mathcal{B}$; $X(x,y)$ if $x \in \mathcal{A}$ and $y \in \mathcal{B}$; 0 if $x \in \mathcal{B}$ and $y \in \mathcal{A}$. Composition is induced by the composition operation on $\mathcal{A}$ and $\mathcal{B}$, and by the $\mathcal{A}$-$\mathcal{B}$-bimodule structure of X. Note that we have two natural inclusion dg functors $\iota_{\mathcal{A}} : \mathcal{A} \to |M|$ and $\iota_{\mathcal{B}} : \mathcal{B} \to |M|$.

Definition 4.1. Let $E : \mathsf{dgcat} \to \mathsf{A}$ be a functor with values in an additive category. We say that E is an *additive invariant of dg categories* if it is derived Morita invariant and satisfies the following condition: for every upper triangular matrix M, the inclusion dg functors $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ induce an isomorphism

$$E(\mathcal{A}) \oplus E(\mathcal{B}) \xrightarrow{\sim} E(|M|)\,.$$

It follows from the work of Blumberg-Mandell, Keller, Schlichting, and Thomason-Trobaugh, that all the invariants (2.3) satisfy additivity, and hence are additive invariants of dg categories; see [**6, 29, 40, 53**]. The universal additive invariant of dg categories was constructed in [**42**]. It can be described[5] as follows: let Hmo_0 be the category whose objects are the dg categories and whose morphisms are given by $\mathsf{Hom}_{\mathsf{Hmo}_0}(\mathcal{A},\mathcal{B}) := K_0\mathsf{rep}(\mathcal{A},\mathcal{B})$, where $\mathsf{rep}(\mathcal{A},\mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ is the full triangulated subcategory of those $\mathcal{A}$-$\mathcal{B}$-bimodules X such that $X(a,-) \in \mathcal{D}_{\mathsf{perf}}(\mathcal{B})$ for every object $a \in \mathcal{A}$. Composition is induced by the tensor product of bimodules. Note that we have a natural functor

$$\mathcal{U}_{\mathsf{A}} : \mathsf{dgcat} \longrightarrow \mathsf{Hmo}_0$$

which is the identity on objects and which maps a dg functor to the class (in the Grothendieck group) of the naturally associated bimodule. The category Hmo_0 is additive and the functor $\mathcal{U}_{\mathsf{A}}$ is additive in the sense of Definition 4.1. Moreover, it is characterized by the following universal property.

Theorem 4.2. *([**42**]) Given an additive category* A*, we have an induced equivalence of categories*

$$(\mathcal{U}_{\mathsf{A}})^* : \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0, \mathsf{A}) \xrightarrow{\sim} \mathrm{Fun}_{\mathsf{additivity}}(\mathsf{dgcat}, \mathsf{A})\,,$$

where the left hand side denotes the category of additive functors and the right hand side the category of additive invariants in the sense of Definition 4.1.

The additive category Hmo_0 (and $\mathcal{U}_{\mathsf{A}}$) is our first answer to **Question B**. A second answer will be described in Section 5. Note that by Theorem 4.2, all the invariants (2.3) factor uniquely through Hmo_0. This motivic category has enabled several (tangential) applications. Here are two examples:

[5] A similar construction in the setting of spectral categories was developed in [**43**].

EXAMPLE 4.3. (Chern characters) The Chern character maps are one of the most important working tools in mathematics. Although they admit numerous different constructions, they were not fully understood at the conceptual level. Making use of the additive category Hmo_0 and of Theorem 4.2 we have bridged this gap by characterizing the Chern character maps, from the Grothendieck group to the (negative) cyclic homology groups, in terms of simple universal properties; see [**47**].

EXAMPLE 4.4. (Fundamental theorem) The fundamental theorems in homotopy algebraic K-theory and periodic cyclic homology, proved respectively by Weibel [**56**] and Kassel [**26**], are of major importance. Their proofs are not only very different but also quite involved. Making use of the additive category Hmo_0 and of Theorem 4.2, we have given a simple, unified and conceptual proof of these fundamental theorems; see [**46**].

Noncommutative Chow motives. By restricting himself to saturated dg categories, which morally are the "noncommutative smooth projective varieties", Kontsevich introduced the following category.

DEFINITION 4.5. (Kontsevich [**30, 33**]; [**52**]) Let F be a field of coefficients. The category NChow_F of *noncommutative Chow motives* (over the base ring k and with coefficients in F) is defined as follows: first consider the F-linearization $\mathsf{Hmo}_{0;F}$ of the additive category Hmo_0. Then, pass to its idempotent completion $\mathsf{Hmo}_{0;F}^\natural$. Finally, take the idempotent complete full subcategory of $\mathsf{Hmo}_{0;F}^\natural$ generated by the saturated dg categories.

The precise relation between the classical category of Chow motives and the category of noncommutative Chow motives is the following: recall that the category $\mathrm{Chow}_\mathbb{Q}$ of Chow motives (with rational coefficients) is $\mathbb{Q}$-linear, additive and symmetric monoidal. Moreover, it is endowed with an important $\otimes$-invertible object, namely the Tate motive $\mathbb{Q}(1)$. The functor $- \otimes \mathbb{Q}(1)$ is an automorphism of $\mathrm{Chow}_\mathbb{Q}$ and so we can consider the associated orbit category $\mathrm{Chow}(k)_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)}$; consult [**52**] for details. Informally speaking, Chow motives which differ from a Tate twist become isomorphic in the orbit category.

THEOREM 4.6. *(Kontsevich* [**30, 33**]*;* [**52**]*) There exists a fully-faithful,* $\mathbb{Q}$*-linear, additive, and symmetric monoidal functor* R *making the diagram*

$$
\begin{array}{ccc}
\mathsf{SmProj}^{\mathrm{op}} & \xrightarrow{\ \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(-)\ } & \mathsf{dgcat} \\
{\scriptstyle M}\downarrow & & \downarrow{\scriptstyle \mathcal{U}_\mathsf{A}} \\
\mathrm{Chow}_\mathbb{Q} & & \mathsf{Hmo}_0 \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle (-)^\natural_\mathbb{Q}} \\
\mathrm{Chow}_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)} & \xrightarrow[R]{} & \mathrm{NChow}_\mathbb{Q} \subset \mathsf{Hmo}^\natural_{0;\mathbb{Q}}
\end{array}
\tag{4.7}
$$

commute (up to a natural isomorphism).

Intuitively speaking, Theorem 4.6 formalizes the conceptual idea that the commutative world can be embedded into the noncommutative world after factorizing out by the action of the Tate motive. The above diagram (4.7) opens new horizons and opportunities of research by enabling the interchange of results, techniques, ideas, and insights between the commutative and the noncommutative world. This yoga was developed in [**52**] in what regards Schur and Kimura finiteness, motivic measures, and motivic zeta functions.

Noncommutative numerical motives. In order to formalize and solve "counting problems", such as counting the number of common points to two planar curves in general position, the classical category of Chow motives is not appropriate as it makes use of a very refined notion of equivalence. Motivated by these "counting problems", Grothendieck developed in the sixties the category Num_F of numerical motives; see [**23**]. Its noncommutative analogue can be described as follows: let $\mathcal{A}$ and $\mathcal{B}$ be two saturated dg categories and $\underline{X} = [\sum_i a_i X_i] \in \mathsf{Hom}_{\mathrm{NChow}_F}(\mathcal{A}, \mathcal{B})$ and $\underline{Y} = [\sum_j b_j Y_j] \in \mathsf{Hom}_{\mathrm{NChow}_F}(\mathcal{B}, \mathcal{A})$ two *noncommutative correspondences.* Their *intersection number* is given by the formula

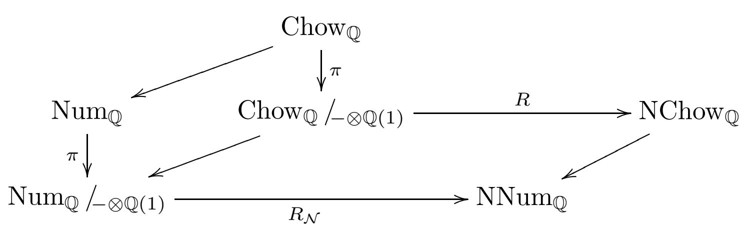
$$\langle \underline{X} \cdot \underline{Y} \rangle := \sum_{i,j,n} (-1)^n \, a_i \cdot b_j \cdot \mathrm{rk}\, HH_n(\mathcal{A}; X_i \otimes^{\mathbb{L}}_{\mathcal{B}} Y_j) \in F \,, \tag{4.8}$$

where $\mathrm{rk}\, HH_n(\mathcal{A}; X_i \otimes_{\mathcal{B}} Y_j)$ denotes the rank of the n^{th} Hochschild homology group of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-$\mathcal{A}$-bimodule $X_i \otimes^{\mathbb{L}}_{\mathcal{B}} Y_j$. A noncommutative correspondence $\underline{X}$ is *numerically equivalent to zero* if for every noncommutative correspondence $\underline{Y}$ the intersection number $\langle \underline{X} \cdot \underline{Y} \rangle$ is zero. As proved in [**36**], these correspondences form a $\otimes$-ideal of $\mathrm{NChow}(k)_F$, which we denote by $\mathcal{N}$.

Definition 4.9. (*Marcolli* & *Tabuada* [**36**]) The category NNum_F of *noncommutative numerical motives* (over the base ring k and with coefficients in F) is the idempotent completion of the quotient category $\mathrm{NChow}_F / \mathcal{N}$.

The relation between Chow motives and noncommutative motives described in diagram (4.7) admits the following numerical analogue.

Theorem 4.10. *(Marcolli & Tabuada* [**36**]*) There exists a fully faithful, $\mathbb{Q}$-linear, additive, and symmetric monoidal functor $R_{\mathcal{N}}$ making the diagram*



commute (up to natural isomorphism).

Intuitively speaking, Theorem 4.10 formalizes the conceptual idea that Hochschild homology is the correct way to express "counting" in the noncommutative world. In the commutative world, Grothendieck conjectured that the category of numerical motives Num_F was abelian semi-simple. Jannsen [**24**], thirty years latter, proved this conjecture without the use of any of the standard conjectures. Recently, we gave a further step forward by proving that Grothendieck's conjecture holds more broadly in the noncommutative world.

Theorem 4.11. *(Marcolli & Tabuada* [**36**]*) Assume one of the following two conditions:*

(i) *The base ring k is local (or more generally that $K_0(k) = \mathbb{Z}$) and F is a k-algebra; a large class of examples is given by taking $k = \mathbb{Z}$ and F an arbitrary field.*

(ii) *The base ring k is a field extension of F; a large class of examples is given by taking $F = \mathbb{Q}$ and k a field of characteristic zero.*

Then the category NNum_F is abelian semi-simple. Moreover, if $\mathcal{J}$ is a $\otimes$-ideal in NChow_F for which the idempotent completion of the quotient category $\mathrm{NChow}_F/\mathcal{J}$ is abelian semi-simple, then $\mathcal{J}$ agrees with $\mathcal{N}$.

Roughly speaking, Theorem 4.11 shows that the unique way to obtain an abelian semi-simple category out of NChow_F is through the use of the above "counting formula" (4.8), defined in terms of Hochschild homology. Among other applications, Theorem 4.11 allowed us to obtain an alternative proof of Jannsen's result; see [**36**].

Kontsevich's noncommutative numerical motives. Making use of a well-behaved bilinear form on the Grothendieck group of saturated dg categories, Kontsevich introduced in [**30**] a category NCNum_F of noncommutative numerical motives. Via duality arguments, the authors proved the following agreement result.

THEOREM 4.12. *(Marcolli & Tabuada [**37**]) The categories NCNum_F and NNum_F are equivalent.*

By combining Theorem 4.12 with Theorem 4.11, we then conclude that NCNum_F is abelian semi-simple. Kontsevich conjectured this latter result in the particular case where $F = \mathbb{Q}$ and k is of characteristic zero. We observe that Kontsevich's beautiful insight not only holds much more generally, but moreover it does not require the assumption of any (polarization) conjecture.

5. Noncommutative mixed motives

Up to now, we have been considering invariants with values in additive categories. From now on we will consider "richer invariants", taking values not in additive categories but in "highly structured" triangulated categories. In order to make this precise we will use the language of *Grothendieck derivators*, a formalism which allows us to state and prove precise universal properties; the reader who is unfamiliar with this language is invited to consult Appendix A at this point. Recall from Drinfeld [**16**] that a sequence of dg functors $\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C}$ is called *exact* if the induced sequence of derived categories $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{C})$ is exact in the sense of Verdier. For example, if X is quasi-compact and quasi-separated scheme, $U \subset X$ a quasi-compact open subscheme and $Z := X \backslash U$ the closed complement, then the sequence of dg functors

$$\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X)_Z \longrightarrow \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(X) \longrightarrow \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(U)$$

is exact; see Thomason-Trobaugh [**53**]. An exact sequence of dg functors is called *split-exact* if there exist dg functors $R : \mathcal{B} \to \mathcal{A}$ and $S : \mathcal{C} \to \mathcal{B}$, right adjoints to I and P, respectively, such that $R \circ I \simeq \mathrm{Id}$ and $P \circ S \simeq \mathrm{Id}$ via the adjunction morphisms; consult [**41**] for details.

DEFINITION 5.1. Let $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ be a filtered homotopy colimit preserving morphism of derivators, from the derivator associated to the Quillen model structure of Theorem 3.1, to a strong triangulated derivator. We say that E is a *localizing invariant* if it sends exact sequences to distinguished triangles

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \quad \mapsto \quad E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C}) \longrightarrow E(\mathcal{A})[1]$$

in the base category $\mathbb{D}(e)$ of $\mathbb{D}$. We say that E is an *additive invariant* if it sends split exact sequences to direct sums

$$\mathcal{A} \rightleftarrows \mathcal{B} \rightleftarrows \mathcal{C} \quad \mapsto \quad E(\mathcal{A}) \oplus E(\mathcal{C}) \xrightarrow{\sim} E(\mathcal{B})\,.$$

Clearly, every localizing invariant is additive. Here are some classical examples.

Example 5.2. (Connective K-theory) As explained in [**41**], connective K-theory gives rise to an additive invariant

$$K : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

with values in the triangulated derivator associated to the (stable) Quillen model category of spectra. Quillen's higher K-theory groups K_* can then be obtained from this spectrum by taking stable homotopy groups. This invariant, although additive, is *not* localizing. The following example corrects this default.

Example 5.3. (Nonconnective K-theory) As explained in [**41**], nonconnective K-theory gives rise to a localizing invariant

$$I\!K : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

As in the previous example, Bass' negative algebraic K-theory groups $I\!K_*$ can be obtained from this spectrum by taking (negative) stable homotopy groups.

Example 5.4. (Mixed complex) Following Kassel [**26**], let Λ be the dg algebra $k[\epsilon]/\epsilon^2$ where ϵ is of degree -1 and $d(\epsilon) = 0$. Under this notation, a *mixed complex* is simply a right dg Λ-module. As explained in [**41**], the mixed complex construction gives rise to a localizing invariant

$$C : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\Lambda\text{-Mod})$$

with values in the triangulated derivator associated to the (stable) Quillen model category of right dg Λ-modules. Cyclic homology and all its variants (Hochschild, periodic, negative, ...) can be obtained from this mixed complex construction by simple procedures; see [**26**].

Example 5.5. (Topological cyclic homology) As explained by Blumberg and Mandell in [**6**] (see also [**50**]), topological cyclic homology gives rise to a localizing invariant

$$TC : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

The topological cyclic homology groups TC_* can be obtained from this spectrum by taking stable homotopy groups.

In order to simultaneously study all the above classical examples, the universal additive and localizing invariants

$$\mathcal{U}^{\mathsf{add}}_{\mathsf{dg}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}} \qquad \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$$

were constructed[6] in [**41**]. They are characterized (in the 2-category of Grothendieck derivators) by the following universal property.

[6] A similar approach in the setting of ∞-categories was developed by Blumberg, Gepner and the author in [**4**]. Besides algebraic and geometric examples, the authors also studied topological examples like A-theory.

THEOREM 5.6. *([**41**]) Given a strong triangulated derivator $\mathbb{D}$, we have induced equivalences of categories*

$$\begin{aligned}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}, \mathbb{D}) &\stackrel{\sim}{\longrightarrow} \underline{\mathsf{Hom}}_{\mathsf{add}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}) \\ (\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}, \mathbb{D}) &\stackrel{\sim}{\longrightarrow} \underline{\mathsf{Hom}}_{\mathsf{loc}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}),\end{aligned}$$

where the right-hand sides denote, respectively, the categories of additive and localizing invariants.

REMARK 5.7. (Quillen model) The additive and the localizing motivator admit natural Quillen models given in terms of a Bousfield localization of presheaves of (symmetric) spectra; consult [**41**] for details.

Because of these universal properties, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ is called the *additive motivator*, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ the *localizing motivator*, $\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}$ the *universal additive invariant*, $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$ the *universal localizing invariant*, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ the *triangulated category of noncommutative additive motives*, and $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ the *triangulated category of noncommutative localizing motives.* Note that since localization implies additivity, we have a well-defined (homotopy colimit preserving) morphism of derivators $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}} \to \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$. The triangulated category $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ (and $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$) is our second answer to **Question B**. Note that by Theorem 5.6, all the invariants of Examples 5.2-5.5 factor uniquely through $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$. Since the composed functor

$$\mathsf{dgcat} \longrightarrow \mathsf{Hmo} \stackrel{\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(e)}{\longrightarrow} \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$$

is an additive invariant of dg categories in the sense of Definition 4.1, we obtain by Theorem 4.2 an induced additive functor $\mathsf{Hmo}_0 \to \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$, which turns out to be fully faithful. Intuitively speaking, our second answer to **Question B** contains the first one. In other words, the world of noncommutative pure motives is contained in the world of noncommutative mixed motives. As we will see in the next section, the latter world is much richer than the former one.

In Example 2.2, we observed that the dg category $\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(\mathbb{P}^n)$ is derived Morita equivalent to the algebra $\mathrm{End}(\mathcal{O}(0) \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n))^{\mathsf{op}}$. By passing to the triangulated category of noncommutative additive motives, we obtain the following splitting:

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(\mathbb{P}^n)) \simeq \underbrace{\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k) \oplus \cdots \oplus \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k)}_{(n+1)\text{-copies}}.$$

The reason behind this phenomenon is a semi-orthogonal decomposition of the triangulated category $\mathcal{D}_{\mathsf{perf}}(X)$. Intuitively speaking, the noncommutative additive motive of the n^{th} projective space consists simply of $n+1$ "points".

The motivic category $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ has enabled several (tangential) applications, Here is one illustrative example:

EXAMPLE 5.8. (Farrell-Jones isomorphism conjectures) The Farrell-Jones isomorphism conjectures are important driving forces in current mathematical research and imply well-known conjectures due to Bass, Borel, Kaplansky, Novikov; see Lück-Reich's survey in [**18**]. Given a group G, they predict the value of algebraic K- and L-theory of the group ring $k[G]$ in terms of its values on the virtually cyclic subgroups of G. In addition, the literature contains many variations on this theme, obtained by replacing the K- and L-theory functors by other functors like

Hochschild homology, topological cyclic homology, etc. During the last few decades each one of these isomorphism conjectures has been proved for large classes of groups using a variety of different methods. Making use of Theorem 5.6, Balmer and the author organized this exuberant herd of conjectures by explicitly describing the fundamental isomorphism conjecture; see [**1**]. It turns out that this fundamental conjecture, which implies all the existing isomorphism conjectures on the market, can be described solely in terms of algebraic K-theory. More precisely, it is a simple "coefficient variant" of the classical Farrell-Jones conjecture in algebraic K-theory.

6. Co-representability

As in any triangulated derivator, the additive and localizing motivators are canonically enriched over spectra. Let us denote by $\mathbb{R}\mathsf{Hom}(-,-)$ their spectra of morphisms; see Appendix A. Connective algebraic K-theory is an example of an additive invariant while nonconnective algebraic K-theory is an example of a localizing invariant. Therefore, by Theorem 5.6, they descend to the additive and localizing motivator, respectively. The following result show us that they become co-representable by the noncommutative motive associated to the base ring.

Theorem 6.1. *(*[**41**]*; Cisinski & Tab.* [**11**]*) Given a dg category $\mathcal{A}$, we have natural equivalences of spectra*

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k),\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\mathcal{A})) \simeq K(\mathcal{A}) \qquad \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(k),\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \simeq I\!\!K(\mathcal{A})\,.$$

In the triangulated categories of noncommutative motives, we have natural isomorphisms of abelian groups

$$\begin{aligned} \mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k),\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\mathcal{A})[-n]) &\simeq K_n(\mathcal{A}) \qquad n \geq 0 \\ \mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(k),\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})[-n]) &\simeq I\!\!K_n(\mathcal{A}) \qquad n \in \mathbb{Z}\,. \end{aligned}$$

Example 6.2. (Schemes) By taking $\mathcal{A} = \mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(X)$ in Theorem 6.1, with X a quasi-compact and quasi-separated scheme, we recover the connective $K(X)$ and nonconnective $I\!\!K(X)$ K-theory spectrum of X.

Remark 6.3. (Bivariant K-theory) Theorem 6.1 is in fact richer. In what concerns the additive motivator, the base ring k can be replaced by any homotopically finitely presented dg category $\mathcal{B}$ (the homotopical version of the classical notion of finite presentation) and $K(\mathcal{A})$ by the bivariant K-theory of $\mathcal{B}$-$\mathcal{A}$-bimodules. In what concerns the localizing motivator, the base ring k can be replaced by any saturated dg category $\mathcal{B}$ and $I\!\!K(\mathcal{A})$ by the spectrum $I\!\!K(\mathcal{B}^{\mathsf{op}} \otimes \mathcal{A})$; consult [**11, 12, 41**].

Remark 6.4. (Bivariant cyclic homology) Classical theories like bivariant cyclic cohomology (and the associated Connes' bilinear pairings) can also be expressed as morphism sets in the category of noncommutative motives; see [**51**].

Theorem 6.1 is our answer to **Question A**. Note that while the right-hand sides are, respectively, connective and nonconnective algebraic K-theory, the left-hand sides are defined solely in terms of precise universal properties: algebraic K-theory is never used (or even mentioned) in their construction. Hence, the equivalences of Theorem 6.1 provide us with a conceptual characterization of higher algebraic K-theory. To the best of the author's knowledge, this is the first conceptual characterization of algebraic K-theory since Quillen's foundational work. We can even take these equivalences as the very definition of higher algebraic K-theory. The precise relation between the answers to **Questions A** and **B** is by now clear. Intuitively speaking, connective (resp. nonconnective) algebraic K-theory is the additive (resp. localizing) invariant co-represented by the noncommutative motive associated to the base ring, which as explained in the next section is simply the $\otimes$-unit object.

7. Symmetric monoidal structure

The tensor product of k-algebras extends naturally to dg categories, giving rise to a symmetric monoidal structure on $\mathsf{HO}(\mathsf{dgcat})$. The $\otimes$-unit is the base ring k (considered as a dg category). Making use of a derived version of Day's convolution product, the authors proved the following result.

THEOREM 7.1. *(Cisinski & Tabuada [**12**]) The additive and localizing motivators carry a canonical symmetric monoidal structure making the universal additive and localizing invariants symmetric monoidal. Moreover, these symmetric monoidal structures preserve homotopy colimits in each variable and are characterized by the following universal property: given any strong triangulated derivator $\mathbb{D}$, endowed with a symmetric monoidal structure, we have induced equivalence of categories:*

$$(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}})^* : \underline{\mathsf{Hom}}_!^{\otimes}(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathsf{add}}^{\otimes}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$$
$$(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}})^* : \underline{\mathsf{Hom}}_!^{\otimes}(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathsf{loc}}^{\otimes}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}) .$$

Kontsevich's noncommutative mixed motives. In [**30, 33**], Kontsevich introduced a category KMM of *noncommutative mixed motives* (over the base ring k). Roughly speaking, KMM is obtained by taking a formal idempotent completion of the triangulated envelope of the category of saturated dg categories (with bivariant algebraic K-theory spectra as morphism sets). Making use Theorem 7.1, the category KMM can be "realized" inside the triangulated category of noncommutative motives.

PROPOSITION 7.2. *(Cisinski & Tabuada [**12**]) There is a natural fully faithful embedding (enriched over spectra) of Kontsevich's category* KMM *of noncommutative mixed motives into the triangulated category* $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ *of noncommutative localizing motives. The essential image is the thick triangulated subcategory spanned by the noncommutative motives of saturated dg categories.*

REMARK 7.3. (Relation with Voevodsky's motives) In the same vein as Theorem 4.6, Voevodsky's triangulated category DM of motives [**54**] relates to (an $\mathbb{A}^1$-homotopy variant of) Kontsevich's category KMM of noncommutative mixed motives. The author and Cisinski are currently in the process of writing up this result.

Products in algebraic K-theory. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. On one hand, following Waldhausen [**55**], we have a classical algebraic K-theory pairing

$$K(\mathcal{A}) \wedge K(\mathcal{B}) \longrightarrow K(\mathcal{A} \otimes \mathcal{B}) . \tag{7.4}$$

On the other hand, by combining the co-representability Theorem 6.1 with Theorem 7.1, we obtain another well-defined algebraic K-theory pairing

$$K(\mathcal{A}) \wedge K(\mathcal{B}) \longrightarrow K(\mathcal{A} \otimes \mathcal{B}) . \tag{7.5}$$

THEOREM 7.6. *([**49**]) The pairings* (7.4) *and* (7.5) *agree up to homotopy; a similar result holds for nonconnective K-theory.*

Example 7.7. (Commutative algebras) Let $\mathcal{A} = \mathcal{B} = A$, where A is a *commutative* k-algebra. Then, by composing the pairing (7.5) with the multiplication map

$$K(A \otimes A) \simeq \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k), \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(A \otimes A)) \longrightarrow \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k), \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(A)) \simeq K(A)$$

we recover inside $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ the algebraic K-theory pairing on $K(A)$ constructed originally by Waldhausen. In particular, we recover the (graded commutative) multiplicative structure on $K_*(A)$ constructed originally by Loday [**34**].

Example 7.8. (Schemes) When $\mathcal{A} = \mathcal{B} = \mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(X)$, with X a quasi-compact and quasi-separated k-scheme, an argument similar to the one of the above example allows us to recover inside $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ the algebraic K-theory pairing on X constructed originally by Thomason-Trobaugh [**53**].

Theorem 7.6 (and Examples 7.7-7.8) offers an elegant conceptual characterization of the algebraic K-theory products. Intuitively speaking, while Theorem 6.1 shows us that connective algebraic K-theory is the additive invariant co-represented by the $\otimes$-unit of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$, Theorem 7.6 shows us that the classical algebraic K-theory products are simply the operations naturally induced by the symmetric monoidal structure on $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$.

8. Higher Chern characters

Higher algebraic K-theory is a very powerful and subtle invariant whose calculation is often out of reach. In order to capture some of its information, Connes-Karoubi, Dennis, Goodwillie, Hood-Jones, Kassel, McCarthy, and others, constructed higher Chern characters towards simpler theories by making use of a variety of highly involved techniques; see [**14, 15, 19, 22, 27, 38**].

Making use of the theory of noncommutative motives, these higher Chern characters can be constructed, and conceptually characterized, in a simple and elegant way; see [**11, 12, 41, 49, 50**]. Let us now illustrate this in a particular case: choose your favorite additive invariant E with values in the derivator associated to spectra. A classical example is given by connective algebraic K-theory. Thanks to Theorem 5.6, we then obtain (homotopy colimit preserving) morphisms of derivators

$$\overline{K}, \overline{E} : \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}} \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

such that $\overline{K} \circ \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}} = K$ and $\overline{E} \circ \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}} = E$. Recall from Theorem 6.1 that the functor $\overline{K}$ is co-represented by the noncommutative additive motive $\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(k)$. Hence, the enriched Yoneda lemma furnishes us a natural equivalence of spectra $\mathbb{R}\mathsf{Nat}(\overline{K}, \overline{E}) \simeq \overline{E}(k)$, where $\mathbb{R}\mathsf{Nat}$ denotes the spectrum of natural transformations. Using Theorem 5.6 again, we obtain a natural equivalence $\mathbb{R}\mathsf{Nat}(K, E) \simeq E(k)$. By passing to the 0^{th}-homotopy group, we conclude that there is a natural bijection between the natural transformation (up to homotopy) from K to E and $\pi_0 E(k)$. In sum, the theory of noncommutative motives allows us to fully classify in simple and elegant terms all possible natural transformation from connective K-theory to any additive invariant; a similar result holds for nonconnective K-theory.

Example 8.1. (Chern character) Let E be the cyclic homology HC additive functor (promoted to an invariant taking values in spectra). Then, we have the following identifications:

$$\mathsf{Nat}(K, HC) \stackrel{\sim}{\to} k \simeq HC_0(k) \qquad \{\text{Chern character}\} \mapsto \mathbf{1}\,.$$

Example 8.1 provides a conceptual characterization of the Chern character as being precisely the unit among all possible natural transformations. A similar characterization of the cyclotomic trace map, in the setting of ∞-categories, was recently developed by Blumberg, Gepner and the author in [**5**].

Appendix A. Grothendieck derivators

The original reference for the theory of derivators is Grothendieck's manuscript [**21**]. See also a short account by Cisinski and Neeman in [**13**]. Derivators originate in the problem of higher homotopies in derived categories. For a triangulated category $\mathcal{T}$ and for X a small category, it essentially never happens that the diagram category $\mathrm{Fun}(X, \mathcal{T}) = \mathcal{T}^X$ remains triangulated; it already fails for the category of arrows in $\mathcal{T}$, that is, for $X = (\bullet \to \bullet)$. Now, very often, our triangulated category $\mathcal{T}$ appears as the homotopy category $\mathcal{T} = \mathsf{Ho}(\mathcal{M})$ of some Quillen model $\mathcal{M}$. In this case, we can consider the category $\mathrm{Fun}(X, \mathcal{M})$ of diagrams in $\mathcal{M}$, whose homotopy category $\mathsf{Ho}(\mathrm{Fun}(X, \mathcal{M}))$ is often triangulated and provides a reasonable approximation for $\mathrm{Fun}(X, \mathcal{T})$. More importantly, one can let X vary. This nebula of categories $\mathsf{Ho}(\mathrm{Fun}(X, \mathcal{M}))$, indexed by small categories X, and the various functors and natural transformations between them is what Grothendieck formalized into the concept of *derivator*.

A derivator $\mathbb{D}$ consists of a strict contravariant 2-functor from the 2-category of small categories to the 2-category of all categories

$$\mathbb{D} : \mathsf{Cat}^{\mathsf{op}} \longrightarrow \mathsf{CAT},$$

subject to certain conditions; consult [**13**] for details. The essential example to keep in mind is the derivator $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ associated to a (cofibrantly generated) Quillen model category $\mathcal{M}$ and defined for every small category X by

$$\mathsf{HO}(\mathcal{M})(X) = \mathsf{Ho}(\mathrm{Fun}(X^{\mathsf{op}}, \mathcal{M}))\,.$$

We denote by e the 1-point category with one object and one identity morphism. Heuristically, the category $\mathbb{D}(e)$ is the basic "derived" category under consideration in the derivator $\mathbb{D}$. For instance, if $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ then $\mathbb{D}(e) = \mathsf{Ho}(\mathcal{M})$. Let us now recall two slightly technical properties of derivators.

- A derivator $\mathbb{D}$ is called *strong* if for every finite free category X and every small category Y, the natural functor $\mathbb{D}(X \times Y) \longrightarrow \mathrm{Fun}(X^{\mathsf{op}}, \mathbb{D}(Y))$ is full and essentially surjective.
- A derivator $\mathbb{D}$ is called *triangulated* (or *stable*) if it is pointed and if every global commutative square in $\mathbb{D}$ is cartesian exactly when it is cocartesian. A source of examples is provided by the derivators $\mathsf{HO}(\mathcal{M})$ associated to *stable* Quillen model categories $\mathcal{M}$.

Recall from [**13**] that given any triangulated derivator $\mathbb{D}$ and small category X, the category $\mathbb{D}(X)$ has a canonical triangulated structure. In particular, the category $\mathbb{D}(e)$ is triangulated. Recall also from [**11**] that any triangulated derivator $\mathbb{D}$ is canonically enriched over spectra, *i.e.* we have a well-defined morphism of derivators

$$\mathbb{R}\mathsf{Hom}(-,-) : \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

Finally, given derivators $\mathbb{D}$ and $\mathbb{D}'$, we denote by $\underline{\mathsf{Hom}}(\mathbb{D},\mathbb{D}')$ the category of all morphisms of derivators and by $\underline{\mathsf{Hom}}_!(\mathbb{D},\mathbb{D}')$ the category of morphisms of derivators which preserve arbitrary homotopy colimits.

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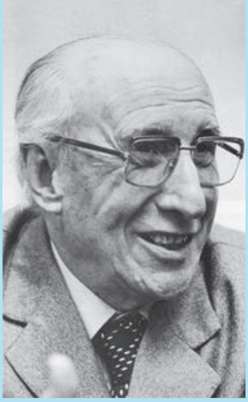

Professor Luis Santaló

Luis Santaló Winter Schools are organized yearly by the Mathematics Department and the Santaló Mathematical Research Institute of the School of Exact and Natural Sciences of the University of Buenos Aires (FCEN). This volume contains the proceedings of the third Luis Santaló Winter School which was devoted to noncommutative geometry and held at FCEN July 26–August 6, 2010.

Topics in this volume concern noncommutative geometry in a broad sense, encompassing various mathematical and physical theories that incorporate geometric ideas to the study of noncommutative phenomena. It explores connections with several areas including algebra, analysis, geometry, topology and mathematical physics.

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