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Grassmannians, Moduli Spaces and Vector Bundles

Clay Mathematics Institute Workshop
Moduli Spaces of Vector Bundles,
with a View Towards Coherent Sheaves
October 6–11, 2006
Cambridge, Massachusetts

Clay Mathematics Institute, Cambridge, MA

P. E. Newstead

David A. Ellwood
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Introduction

In 2006, Peter E. Newstead paid his first academic visit to North America after the 1960s, and the occasion originated a number of workshops and conferences in his honor. The editors of this volume, together with Montserrat Teixidor i Bigas, organized a Clay Mathematics Institute workshop, "Moduli spaces of vector bundles, with a view toward coherent sheaves" (October 6-10, 2006). The experts convened produced a vigorous confluence of so many different techniques and discussed such deep connections that we felt a proceedings volume would be a valuable asset to the community of mathematicians and physicists; when a participant was not available to write for this volume, state-of-the-art coverage of the topic was provided through the generosity of an alternative expert.

Peter E. Newstead earned his Ph.D. from the University of Cambridge in 1966; both John A. Todd and Michael F. Atiyah supervised his doctoral work. From the beginning of his career, he was interested in topological properties of classification spaces of vector bundles. Geometric Invariant Theory was re-invigorated around that time by the refinement of concepts of (semi-)stability and projective models, and Newstead contributed some of the more original and deeper constructions, for example the projective models for rank-2 bundles of fixed determinant of odd degree over a curve of genus two, a quadratic complex (obtained also, using a different method, by M.S. Narasimhan with S. Ramanan), and a topological proof for non-existence of universal bundles in certain cases. He played a prominent role in the development of a major area by focusing on moduli of vector bundles over an algebraic curve, was a main contributor to a Brill-Noether geography for these spaces, and his topological results led him to make major contributions (cf. [N1-2] and [KN]) to the description, by generators and relations, of the rational cohomology algebra $H^*(\mathcal{SU}_X(2,L))$ for the moduli space $\mathcal{SU}_X(2,L)$ of stable rank-2 bundles over X with fixed determinant L, where X is a compact Riemann surface of genus $g \geq 2$, and L a line bundle of odd degree over X (the higher-rank case was then settled in [EK]). The main concerns of his current work are coherent systems on algebraic curves and Picard bundles. This bird's-eye view of Newstead's work omits several topics, ranging from invariants of group action to algebraic geometry over the reals, conic bundles and other special projective varieties defined by quadrics, and compactifications of moduli spaces, and is merely intended to serve as orientation for the readers of the present volume.

This volume of cutting-edge contributions provides a collection of problems and methods that are greatly enriching our understanding of moduli spaces and their applications. It should be accessible to non-experts, as well as further the interaction among researchers specializing in various aspects of these spaces. Indeed, we hope this volume will impress the reader with the diversity of ideas and techniques that are brought together by the nature of these varieties.

In brief and non-technical terms, the volume covers the following areas. An aspect of moduli spaces that recently emerged is the disparate set of dualities that parallel the classical Hecke correspondence of number theory. In modern terms, such a pairing of two variables (or two categories) is a Fourier-Mukai-Laumon transform; implications go under the heading of geometric Langlands program, the area of Kamnitzer's article. Pareschi and Popa offer original techniques in the derived-theoretic study of regularity and generic vanishing for coherent sheaves on abelian varieties, with applications to the study of vector bundles, as well as that of linear series on irregular varieties. Also on the theme of moduli spaces, Aprodu and Farkas on the one hand, Jeffrey on the other, provide techniques to analyze different properties: respectively, applications of Koszul cohomology to the study of various moduli spaces, and symplectic-geometric methods for intersection cohomology over singular moduli spaces. Teixidor treats moduli spaces of vector bundles over reducible curves, a delicate issue with promising applications to integrable systems. Arcara and Bertram work towards a concept of stability for bundles over surfaces and conjecturally over threefolds. Using the Brauer group, Lieblich relates the geometry of moduli spaces to the properties of certain non-commutative algebras and to arithmetic local-to-global principles. Andersen and Gammelgaard's paper addresses a quantization of the moduli space: the fibration of the moduli space of curves given by the moduli of bundles admits a projective connection whose associated operator generalizes the heat equation – it was defined independently by N. Hitchin, and by S. Axelrod with S. Della Pietra and E. Witten. Finally, the workshop's guest of honor, Peter Newstead himself, offers a cutting-edge overview of his current area of work, coherent systems over algebraic curves; may we salute it as the Brill-Noether theory of the XXI century?

We hope you enjoy the book and find it as inspiring as we do.

David A. Ellwood, Cambridge
Emma Previato, Boston

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Hitchin's Projectively Flat Connection, Toeplitz Operators and the Asymptotic Expansion of TQFT Curve Operators

Jørgen Ellegaard Andersen and Niels Leth Gammelgaard

Abstract. In this paper, we will provide a review of the geometric construction, proposed by Witten, of the SU(n) quantum representations of the mapping class groups which are part of the Reshetikhin-Turaev TQFT for the quantum group $U_q(sl(n, \mathbb{C}))$. In particular, we recall the differential geometric construction of Hitchin's projectively flat connection in the bundle over Teichmüller space obtained by push-forward of the determinant line bundle over the moduli space of rank n, fixed determinant, semi-stable bundles fibering over Teichmüller space. We recall the relation between the Hitchin connection and Toeplitz operators which was first used by the first named author to prove the asymptotic faithfulness of the SU(n) quantum representations of the mapping class groups. We further review the construction of the formal Hitchin connection, and we discuss its relation to the full asymptotic expansion of the curve operators of Topological Quantum Field Theory. We then go on to identify the first terms in the formal parallel transport of the Hitchin connection explicitly. This allows us to identify the first terms in the resulting star product on functions on the moduli space. This is seen to agree with the first term in the star-product on holonomy functions on these moduli spaces defined by Andersen, Mattes and Reshetikhin.

1. Introduction

Witten constructed, via path integral techniques, a quantization of Chern-Simons theory in $2+1$ dimensions, and he argued in [**Wi**] that this produced a TQFT, indexed by a compact simple Lie group and an integer level k. For the group SU(n) and level k, let us denote this TQFT by $Z_k^{(n)}$. Witten argues in [**Wi**] that the theory $Z_k^{(2)}$ determines the Jones polynomial of a knot in S^3. Combinatorially, this theory was first constructed by Reshetikhin and Turaev, using the representation theory of $U_q(\mathrm{sl}(n, C))$ at $q = e^{(2\pi i)/(k+n)}$, in [**RT1**] and [**RT2**]. Subsequently, the TQFT's $Z_k^{(n)}$ were constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [**BHMV1**], [**BHMV2**] and [**B1**].

The two-dimensional part of the TQFT $Z_k^{(n)}$ is a modular functor with a certain label set. For this TQFT, the label set $\Lambda_k^{(n)}$ is a finite subset (depending on k) of the set of finite dimensional irreducible representations of SU(n). We use the

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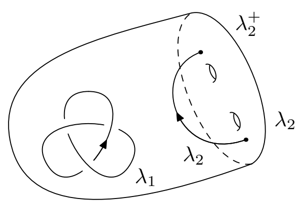
usual labeling of irreducible representations by Young diagrams, so in particular $\square \in \Lambda_k^{(n)}$ is the defining representation of $\mathrm{SU}(n)$. Let further $\lambda_0^{(d)} \in \Lambda_k^{(n)}$ be the Young diagram consisting of d columns of length k. The label set is also equipped with an involution, which is simply induced by taking the dual representation. The trivial representation is a special element in the label set which is clearly preserved by the involution.

$$Z_k^{(n)} : \quad \left\{\begin{array}{l}\text{Category of (ex-}\\ \text{tended) closed}\\ \text{oriented surfaces}\\ \text{with } \Lambda_k^{(n)}\text{-labeled}\\ \text{marked points with}\\ \text{projective tangent}\\ \text{vectors}\end{array}\right\} \to \left\{\begin{array}{l}\text{Category of finite}\\ \text{dimensional vector}\\ \text{spaces over } \mathbb{C}\end{array}\right\}$$

The three-dimensional part of $Z_k^{(n)}$ is an association of a vector,

$$Z_k^{(n)}(M, L, \lambda) \in Z_k^{(n)}(\partial M, \partial L, \partial\lambda),$$

to any compact, oriented, framed 3–manifold M together with an oriented, framed link $(L, \partial L) \subseteq (M, \partial M)$ and a $\Lambda_k^{(n)}$-labeling $\lambda : \pi_0(L) \to \Lambda_k^{(n)}$.



This association has to satisfy the Atiyah-Segal-Witten TQFT axioms (see e.g. [**At**], [**Se**] and [**Wi**]). For a more comprehensive presentation of the axioms, see Turaev's book [**T**].

The geometric construction of these TQFTs was proposed by Witten in [**Wi**] where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two-dimensional part of the theory. Further, he proposed an alternative construction of the two-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno and Yamada in [**TUY**] provided the major geometric constructions and results needed. In [**BK**], their results were used to show that the category of integrable highest weight modules of level k for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further, in [**BK**], this result is combined with the work of Kazhdan and Lusztig [**KL**] and the work of Finkelberg [**Fi**] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two-dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [**AU1**], [**AU2**], [**AU3**] and [**AU4**], we have given a proof, based mainly on the results of [**TUY**], that the TUY-construction of the WZW-conformal field theory, after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full $(2+1)$-dimensional TQFT resulting from this is isomorphic to the aforementioned one, constructed by BHMV via skein theory. Combining this with the theorem of Laszlo [**La1**], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [**Wi**].

Part of this TQFT is the quantum $\mathrm{SU}(n)$ representations of the mapping class groups. Namely, if Σ is a closed oriented surfaces of genus g, Γ is the mapping class group of Σ, and p is a point on Σ, then the modular functor induces a representation

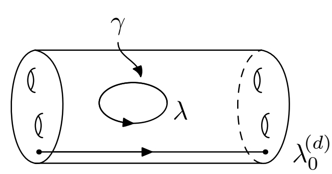
$$Z_k^{(n,d)} : \Gamma \to \mathbb{P}\,\mathrm{Aut}\big(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\big). \tag{1}$$

For a general label of p, we would need to choose a projective tangent vector $v_p \in T_p\Sigma/\mathbb{R}_+$, and we would get a representation of the mapping class group of (Σ, p, v_p). But for the special labels $\lambda_0^{(d)}$, the dependence on v_p is trivial and in fact we get a representation of Γ. Furthermore, the curve operators are also part of any TQFT: For $\gamma \subseteq \Sigma - \{p\}$ an oriented simple closed curve and any $\lambda \in \Lambda_k^{(n)}$, we have the operators

$$Z_k^{(n,d)}(\gamma, \lambda) : Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \to Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}), \tag{2}$$

defined as

$$Z_k^{(n,d)}(\gamma, \lambda) = Z_k^{(n,d)}(\Sigma \times I, \gamma \times \{\tfrac{1}{2}\} \coprod \{p\} \times I, \{\lambda, \lambda_0^{(d)}\}).$$



The curve operators are natural under the action of the mapping class group, meaning that the following diagram,

$$\begin{array}{ccc} Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \xrightarrow{Z_k^{(n,d)}(\gamma,\lambda)} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \\ {\scriptstyle Z_k^{(n,d)}(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle Z_k^{(n,d)}(\phi)} \\ Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \xrightarrow{Z_k^{(n,d)}(\phi(\gamma),\lambda)} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}), \end{array}$$

is commutative for all $\phi \in \Gamma$ and all labeled simple closed curves $(\gamma, \lambda) \subset \Sigma - \{p\}$.

For the curve operators, we can derive an explicit formula using factorization: Let Σ' be the surface obtained from cutting Σ along γ and identifying the two boundary components to two points, say $\{p_+, p_-\}$. Here p_+ is the point corresponding to the "left" side of γ. For any label $\mu \in \Lambda_k^{(n)}$, we get a labeling of the ordered points (p_+, p_-) by the ordered pair of labels $(\mu, \mu^\dagger)$.

Since $Z_k^{(n)}$ is also a modular functor, one can factor the space $Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})$ as a direct sum, 'along' γ, over $\Lambda_k^{(n)}$. That is, we get an isomorphism

$$Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \cong \bigoplus_{\mu\in\Lambda_k^{(n)}} Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)}). \tag{3}$$

Strictly speaking, we need a base point on γ to induce tangent directions at $p_\pm$. However, the corresponding subspaces of $Z^{(k)}(\Sigma, p, \lambda_0^{(d)})$ do not depend on the choice of base point. The isomorphism (3) induces an isomorphism

$$\mathrm{End}(Z^{(k)}(\Sigma, p, \lambda_0^{(d)})) \cong \bigoplus_{\mu\in\Lambda_k^{(n)}} \mathrm{End}(Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)})),$$

which also induces a direct sum decomposition of $\mathrm{End}(Z^{(k)}(\Sigma, p, \lambda_0^{(d)}))$, independent of the base point.

The TQFT axioms imply that the curve operator $Z^{(k)}(\gamma, \lambda)$ is diagonal with respect to this direct sum decomposition along γ. One has the formula

$$Z^{(k)}(\gamma, \lambda) = \bigoplus_{\mu\in\Lambda_k^{(n)}} S_{\lambda,\mu}(S_{0,\mu})^{-1}\,\mathrm{Id}_{Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^\dagger, \lambda_0^{(d)})}.$$

Here $S_{\lambda,\mu}$ is the S-matrix[1] of the theory $Z_k^{(n)}$. See e.g. [**B1**] for a derivation of this formula.

Let us now briefly recall the geometric construction of the representations $Z_k^{(n,d)}$ of the mapping class group, as proposed by Witten, using geometric quantization of moduli spaces.

We assume from now on that the genus of the closed oriented surface Σ is at least two. Let M be the moduli space of flat $\mathrm{SU}(n)$ connections on $\Sigma - p$ with holonomy around p equal to $\exp(2\pi i d/n)\,\mathrm{Id} \in \mathrm{SU}(n)$. When (n, d) are coprime, the moduli space is smooth. In all cases, the smooth part of the moduli space has a natural symplectic structure ω. There is a natural smooth symplectic action of the mapping class group Γ of Σ on M. Moreover, there is a unique prequantum line bundle $(\mathcal{L}, \nabla, (\cdot,\cdot))$ over (M, ω). The Teichmüller space $\mathcal{T}$ of complex structures on Σ naturally, and Γ-equivariantly, parametrizes Kähler structures on (M, ω). For $\sigma \in \mathcal{T}$, we denote by M_σ the manifold (M, ω) with its corresponding Kähler structure. The complex structure on M_σ and the connection ∇ in $\mathcal{L}$ induce the structure of a holomorphic line bundle on $\mathcal{L}$. This holomorphic line bundle is simply the determinant line bundle over the moduli space, and it is an ample generator of the Picard group [**DN**].

By applying geometric quantization to the moduli space M, one gets, for any positive integer k, a certain finite rank bundle over Teichmüller space $\mathcal{T}$ which we will call the *Verlinde bundle* $\mathcal{V}_k$ at level k. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{V}_{k,\sigma} = H^0(M_\sigma, \mathcal{L}^k)$. We observe that there is a natural Hermitian structure $\langle\cdot,\cdot\rangle$ on $H^0(M_\sigma, \mathcal{L}^k)$ by restricting the L_2-inner product on global L_2 sections of $\mathcal{L}^k$ to $H^0(M_\sigma, \mathcal{L}^k)$.

The main result pertaining to this bundle is:

Theorem 1 (Axelrod, Della Pietra and Witten; Hitchin). *The projectivization of the bundle $\mathcal{V}_k$ supports a natural flat Γ-invariant connection $\hat{\nabla}$.*

[1] The S-matrix is determined by the isomorphism that a modular functor induces from two different ways of glueing an annulus to obtain a torus. For its definition, see e.g. [**MS**], [**Se**], [**Wa**] or [**BK**] and references therein. It is also discussed in [**AU3**].

This is a result proved independently by Axelrod, Della Pietra and Witten [**ADW**] and by Hitchin [**H**]. In section 2, we review our differential geometric construction of the connection $\hat{\nabla}$ in the general setting discussed in [**A6**]. We obtain as a corollary that the connection constructed by Axelrod, Della Pietra and Witten projectively agrees with Hitchin's.

Definition 1. *We denote by* $Z_k^{(n,d)}$ *the representation*

$$Z_k^{(n,d)} : \Gamma \to \mathbb{P}\operatorname{Aut}\big(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\big),$$

obtained from the action of the mapping class group on the covariant constant sections of $\mathbb{P}(\mathcal{V}_k)$ *over* $\mathcal{T}$.

The projectively flat connection $\hat{\nabla}$ induces a *flat* connection $\hat{\nabla}^e$ in $\operatorname{End}(\mathcal{V}_k)$. Let $\operatorname{End}_0(\mathcal{V}_k)$ be the subbundle consisting of traceless endomorphisms. The connection $\hat{\nabla}^e$ also induces a connection in $\operatorname{End}_0(\mathcal{V}_k)$, which is invariant under the action of Γ.

In [**A3**], we proved

Theorem 2 (Andersen). *Assume that* n *and* d *are coprime or that* $(n, d) = (2, 0)$ *when* $g = 2$. *Then, we have that*

$$\bigcap_{k=1}^{\infty} \ker(Z_k^{(n,d)}) = \begin{cases} \{1, H\} & g = 2,\ n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise,} \end{cases}$$

where H *is the hyperelliptic involution.*

The main ingredient in the proof of this Theorem is the *Toeplitz operators* associated to smooth functions on M. For each $f \in C^\infty(M)$ and each point $\sigma \in \mathcal{T}$, we have the Toeplitz operator,

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \to H^0(M_\sigma, \mathcal{L}_\sigma^k),$$

which is given by

$$T_{f,\sigma}^{(k)} = \pi_\sigma^{(k)}(fs)$$

for all $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$. Here $\pi_\sigma^{(k)}$ is the orthogonal projection onto $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ induced from the L_2-inner product on $C^\infty(M, \mathcal{L}^k)$. We get a smooth section of $\operatorname{End}(\mathcal{V}^{(k)})$,

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \operatorname{End}(\mathcal{V}^{(k)})),$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ (see [**A3**]). See section 3 for further discussion of the Toeplitz operators and their connection to deformation quantization.

The sections $T_f^{(k)}$ of $\operatorname{End}(\mathcal{V}^{(k)})$ over $\mathcal{T}$ are not covariant constant with respect to Hitchin's connection $\hat{\nabla}^e$. However, they are asymptotically so as k goes to infinity. This will be made precise when we discuss the formal Hitchin connection below.

As a further application of TQFT and the theory of Toeplitz operators together with the theory of coherent states, we recall the first author's solution to a problem in geometric group theory, which has been around for quite some time (see e.g. Problem (7.2) in Chapter 7, "A short list of open questions", of [**BHV**]): In [**A8**], Andersen proved that

Theorem 3 (Andersen). *The mapping class group of a closed oriented surface, of genus at least two, does not have Kazhdan's property (T).*

Returning to the geometric construction of the Reshetikhin-Turaev TQFT, let us recall the geometric construction of the curve operators. First of all, the decomposition (3) is geometrically obtained as follows (see [**A7**] for the details):

One considers a one parameter family of complex structures $\sigma_t \in \mathcal{T}$, $t \in \mathbb{R}_+$, such that the corresponding family in the moduli space of curves converges in the Mumford-Deligne boundary to a nodal curve, which topologically corresponds to shrinking γ to a point. By the results of [**A1**], the corresponding sequence of complex structures on the moduli space M converges to a non-negative polarization on M whose isotropic foliation is spanned by the Hamiltonian vector fields associated to the holonomy functions of γ. The main result of [**A7**] is that the covariant constant sections of $\mathcal{V}^{(k)}$ along the family σ_t converge to distributions supported on the Bohr-Sommerfeld leaves of the limiting non-negative polarization as t goes to infinity. The direct sum of the geometric quantization of the level k Bohr-Sommerfeld levels of this non-negative polarization is precisely the left-hand side of (3). A sewing construction, inspired by conformal field theory (see [**TUY**]), is then applied to show that the resulting linear map from the right-hand side of (3) to the left-hand side is an isomorphism. This is described in detail in [**A7**].

In [**A7**], we further prove the following important asymptotic result. Let $h_{\gamma,\lambda} \in C^\infty(M)$ be the holonomy function obtained by taking the trace in the representation λ of the holonomy around γ.

Theorem 4 (Andersen). *For any one-dimensional oriented submanifold γ and any labeling λ of the components of γ, we have that*

$$\lim_{k\to\infty} \| Z_k^{(n,d)}(\gamma,\lambda) - T_{h_{\gamma,\lambda}}^{(k)} \| = 0.$$

Let us here give the main idea behind the proof of Theorem 4 and refer to [**A7**] for the details. One considers the explicit expression for the S-matrix, as given in formula (13.8.9) in Kac's book [**Kac**],

$$S_{\lambda,\mu}/S_{0,\mu} = \lambda(e^{-2\pi i \frac{\check{\mu}+\check{\rho}}{k+n}}), \tag{4}$$

where ρ is half the sum of the positive roots and $\check{\nu}$ (ν any element of Λ) is the unique element of the Cartan subalgebra of the Lie algebra of $\mathrm{SU}(n)$ which is dual to ν with respect to the Cartan-Killing form $(\cdot,\cdot)$.

From the expression (4), one sees that under the isomorphism $\check{\mu} \mapsto \mu$, the expression $S_{\lambda,\mu}/S_{0,\mu}$ makes sense for any $\check{\mu}$ in the Cartan subalgebra of the Lie algebra of $\mathrm{SU}(n)$. Furthermore, one finds that the values of this sequence of functions (depending on k) are asymptotic to the set of values of the holonomy function $h_{\gamma,\lambda}$ at the level k Bohr-Sommerfeld sets of the limiting non-negative polarizations discussed above (see [**A1**]). From this, one can deduce Theorem 4. See again [**A7**] for details.

Let us now consider the general setting treated in [**A6**]. Thus, we consider, as opposed to only considering the moduli spaces, a general prequantizable symplectic manifold (M,ω) with a prequantum line bundle $(L,(\cdot,\cdot),\nabla)$. We assume that $\mathcal{T}$ is a complex manifold which holomorphically and rigidly (see Definition 5) parameterizes Kähler structures on (M,ω). Then, the following theorem, proved in [**A6**], establishes the existence of the Hitchin connection under a mild cohomological condition.

Theorem 5 (Andersen). *Suppose that I is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold (M,ω) which satisfies that there* *exists an* $n \in \mathbb{Z}$ *such that the first Chern class of* (M, ω) *is* $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$ *and* $H^1(M, \mathbb{R}) = 0$*. Then, the Hitchin connection* $\hat{\nabla}$ *in the trivial bundle* $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ *preserves the subbundle* $H^{(k)}$ *with fibers* $H^0(M_\sigma, \mathcal{L}^k)$*. It is given by*

$$\hat{\nabla}_V = \hat{\nabla}^t_V + \frac{1}{4k+2n}\left\{\Delta_{G(V)} + 2\nabla_{G(V)dF} + 4kV'[F]\right\},$$

where $\hat{\nabla}^t$ *is the trivial connection in* $\mathcal{H}^{(k)}$*, and* V *is any smooth vector field on* $\mathcal{T}$*.*

In section 4, we study the formal Hitchin connection which was introduced in [**A6**]. Let $\mathcal{D}(M)$ be the space of smooth differential operators on M acting on smooth functions on M. Let C_h be the trivial $C^\infty_h(M)$-bundle over $\mathcal{T}$.

Definition 2. *A formal connection* D *is a connection in* C_h *over* $\mathcal{T}$ *of the form*

$$D_V f = V[f] + \tilde{D}(V)(f),$$

where $\tilde{D}$ *is a smooth one-form on* $\mathcal{T}$ *with values in* $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$*,* f *is any smooth section of* C_h*,* V *is any smooth vector field on* $\mathcal{T}$ *and* $V[f]$ *is the derivative of* f *in the direction of* V*.*

Thus, a formal connection is given by a formal series of differential operators

$$\tilde{D}(V) = \sum_{l=0}^{\infty} \tilde{D}^{(l)}(V)h^l.$$

From Hitchin's connection in $H^{(k)}$, we get an induced connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(H^{(k)})$. As previously mentioned, the Teoplitz operators are not covariant constant sections with respect to $\hat{\nabla}^e$, but asymptotically in k they are. This follows from the properties of the formal Hitchin connection, which is the formal connection D defined through the following theorem (proved in [**A6**]).

Theorem 6. *(Andersen) There is a unique formal connection* D *which satisfies that*

$$\hat{\nabla}^e_V T^{(k)}_f \sim T^{(k)}_{(D_V f)(1/(k+n/2))} \tag{5}$$

for all smooth sections f *of* C_h *and all smooth vector fields on* $\mathcal{T}$*. Moreover,*

$$\tilde{D} = 0 \mod h.$$

Here $\sim$ *means the following: For all* $L \in \mathbb{Z}_+$ *we have that*

$$\left\| \hat{\nabla}^e_V T^{(k)}_f - \left(T^{(k)}_{V[f]} + \sum_{l=1}^{L} T^{(k)}_{\tilde{D}^{(l)}_V f} \frac{1}{(k+n/2)^l} \right) \right\| = O(k^{-(L+1)}),$$

uniformly over compact subsets of $\mathcal{T}$*, for all smooth maps* $f : \mathcal{T} \to C^\infty(M)$*.*

Now fix an $f \in C^\infty(M)$ which does not depend on $\sigma \in \mathcal{T}$, and notice how the fact that $\tilde{D} = 0$ mod h implies that

$$\left\| \hat{\nabla}^e_V T^{(k)}_f \right\| = O(k^{-1}).$$

This expresses the fact that the Toeplitz operators are asymptotically flat with respect to the Hitchin connection.

We define a mapping class group equivariant formal trivialization of D as follows.

Definition 3. *A formal trivialization of a formal connection D is a smooth map $P : \mathcal{T} \to \mathcal{D}_h(M)$ which modulo h is the identity, for all $\sigma \in \mathcal{T}$, and which satisfies*

$$D_V(P(f)) = 0,$$

for all vector fields V on $\mathcal{T}$ and all $f \in C_h^\infty(M)$. Such a formal trivialization is mapping class group equivariant if $P(\phi(\sigma)) = \phi^* P(\sigma)$ for all $\sigma \in \mathcal{T}$ and $\phi \in \Gamma$.

Since the only mapping class group invariant functions on the moduli space are the constant ones (see [**Go1**]), we see that, in the case where M is the moduli space, such a P, if it exists, must be unique up to multiplication by a formal constant.

Clearly, if D is not flat, such a formal trivialization cannot exist even locally on $\mathcal{T}$. However, if D is flat and its zero-order term is just given by the trivial connection in C_h, then a local formal trivialization exists, as proved in [**A6**].

Furthermore, it is proved in [**A6**] that flatness of the formal Hitchin connection is implied by projective flatness of the Hitchin connection. As was proved by Hitchin in [**H**], and stated above in Theorem 1, this is the case when M is the moduli space. Furthermore, the existence of a formal trivialization implies the existence of a unique (up to formal scale) mapping class group equivariant formal trivialization, provided that $H^1_\Gamma(\mathcal{T}, D(M)) = 0$. The first steps towards proving that this cohomology group vanishes have been taken in [**AV1, AV2, AV3, Vi**]. In this paper, we prove that

Theorem 7. *The mapping class group equivariant formal trivialization of the formal Hitchin connection exists to first order, and we have the following explicit formula for the first order term of P;*

$$P_\sigma^{(1)}(f) = \frac{1}{4}\Delta_\sigma(f) + i\nabla_{X_F''}(f),$$

where X_F'' denotes the (0,1)-part of the Hamiltonian vector field for the Ricci potential.

For the proof of the theorem, see section 4. We will make the following conjecture.

Conjecture 1. *The mapping class group equivariant formal trivialization of the formal Hitchin connection exists, and for any one-dimensional oriented submanifold γ and any labeling λ of the components of γ, we have the full asymptotic expansion*

$$Z_k^{(n,d)}(\gamma, \lambda) \sim T^{(k)}_{P(h_{\gamma,\lambda})},$$

which means that for all L and all $\sigma \in \mathcal{T}$, we have that

$$\|Z_k^{(n,d)}(\gamma, \lambda) - \sum_{l=0}^{L} T^{(k)}_{P_\sigma^{(l)}(h_{\gamma,\lambda})} \frac{1}{(k+n/2)^l}\| = O(k^{L+1}).$$

It is very likely that the techniques used in [**A7**] to prove Theorem 4 can be used to prove this conjecture.

When we combine this conjecture with the asymptotics of the product of two Toeplitz operators (see Theorem 11), we get the full asymptotic expansion of the product of two curve operators:

$$Z_k^{(n,d)}(\gamma_1, \lambda_1) Z_k^{(n,d)}(\gamma_2, \lambda_2) \sim T^{(k)}_{P(h_{\gamma_1,\lambda_1}) \tilde{\star}^{BT}_\sigma P(h_{\gamma_2,\lambda_2})},$$

where $\tilde{\star}^{BT}_\sigma$ is very closely related to the Berezin-Toeplitz star product for the Kähler manifold (M_σ, ω), as first defined in [**BMS**]. See section 3 for further details regarding this.

Suppose that we have a mapping class group equivariant formal trivialization P of the formal Hitchin connection D. We can then define a new smooth family of star products parametrized by $\mathcal{T}$ as follows:

$$f \star_\sigma g = P_\sigma^{-1}(P_\sigma(f) \tilde{\star}^{\mathrm{BT}}_\sigma P_\sigma(g))$$

for all $f, g \in C^\infty(M)$ and all $\sigma \in \mathcal{T}$. Using the fact that P is a trivialization, it is not hard to prove that $\star_\sigma$ is independent of σ, and we simply denote it $\star$. The following theorem is proved in section 4.

THEOREM 8. *if The star product $\star$ has the form*

$$f \star g = fg - \frac{i}{2}\{f, g\}h + O(h^2).$$

We observe that this formula for the first-order term of $\star$ agrees with the first-order term of the star product constructed by Andersen, Mattes and Reshetikhin in [**AMR2**], when we apply the formula in Theorem 8 to two holonomy functions h_{γ_1,λ_1} and h_{γ_2,λ_2}:

$$h_{\gamma_1,\lambda_1} \star h_{\gamma_2,\lambda_2} = h_{\gamma_1\gamma_2,\lambda_1\cup\lambda_2} - \frac{i}{2} h_{\{\gamma_1,\gamma_2\},\lambda_1\cup\lambda_2} + O(h^2).$$

We recall that $\{\gamma_1, \gamma_2\}$ is the Goldman bracket (see [**Go2**]) of the two simple closed curves γ_1 and γ_2.

A similar result was obtained for the abelian case, i.e. in the case where M is the moduli space of flat $U(1)$-connections, by the first author in [**A2**], where the agreement between the star product defined in differential geometric terms and the star product of Andersen, Mattes and Reshetikhin was proved to all orders. We conjecture that the two star products also *agree*[2] in the non-abelian case.

We also remark that the constructions presented here seems to explicitly realized, to first order, some of the constructions contemplated by Gukov and Witten in [**GW**] in the part concerned with Chern-Simons theory.

We would finally also like to recall that the first named author has shown that the Nielsen-Thurston classification of mapping classes is determined by the Reshetikhin-Turaev TQFTs. We refer to [**A5**] for the full details of this.

Warm thanks are due to the editor of this volume for her persistent encouragements towards the completion of this contribution.

2. The Hitchin connection

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [**A6**]. This approach is close in spirit to Axelrod, Della Pietra and Witten's in [**ADW**], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [**H**] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin's connection agrees with Axelrod, Della Pietra and Witten's.

[2] By agree we don't just mean agree up to equivalence, but that the two star products of any two functions exactly agree.

Hence, we start in the general setting and let (M,ω) be any compact symplectic manifold.

Definition 4. *A prequantum line bundle* $(\mathcal{L},(\cdot,\cdot),\nabla)$ *over the symplectic manifold* (M,ω) *consist of a complex line bundle* $\mathcal{L}$ *with a Hermitian structure* $(\cdot,\cdot)$ *and a compatible connection* ∇ *whose curvature is*

$$F_\nabla(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} = -i\omega(X,Y).$$

We say that the symplectic manifold (M,ω) *is prequantizable if there exists a prequantum line bundle over it.*

Recall that the condition for the existence of a prequantum line bundle is that $[\frac{\omega}{2\pi}] \in \mathrm{Im}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}))$. Furthermore, the inequivalent choices of prequantum line bundles (if they exist) are parametrized by $H^1(M,U(1))$ (see e.g. [**Wo**]).

We shall assume that (M,ω) is prequantizable and fix a prequantum line bundle $(\mathcal{L},(\cdot,\cdot),\nabla)$.

Assume that $\mathcal{T}$ is a smooth manifold which smoothly parametrizes Kähler structures on (M,ω). This means that we have a smooth[3] map $I : \mathcal{T} \to C^\infty(M,\mathrm{End}(TM))$ such that (M,ω,I_σ) is a Kähler manifold for each $\sigma \in \mathcal{T}$.

We will use the notation M_σ for the complex manifold (M,I_σ). For each $\sigma \in \mathcal{T}$, we use I_σ to split the complexified tangent bundle $TM_\mathbb{C}$ into the holomorphic and the anti-holomorphic parts. These we denote by

$$T_\sigma = E(I_\sigma,i) = \mathrm{Im}(\mathrm{Id} - iI_\sigma)$$

and

$$\bar{T}_\sigma = E(I_\sigma,-i) = \mathrm{Im}(\mathrm{Id} + iI_\sigma)$$

respectively.

The real Kähler metric g_σ on (M_σ,ω), extended complex linearly to $TM_\mathbb{C}$, is by definition

$$g_\sigma(X,Y) = \omega(X,I_\sigma Y), \tag{6}$$

where $X,Y \in C^\infty(M,TM_\mathbb{C})$.

The divergence of a vector field X is the unique function $\delta(X)$ determined by

$$\mathcal{L}_X\omega^m = \delta(X)\omega^m. \tag{7}$$

It can be calculated by the formula $\delta(X) = \Lambda d(i_X\omega)$, where Λ denotes contraction with the Kähler form. Even though the divergence only depends on the volume, which is independent of particular Kähler structure, it can be expressed in terms of the Levi-Civita connection on M_σ by $\delta(X) = \mathrm{Tr}\,\nabla_\sigma X$.

Inspired by this expression, we define the divergence of a symmetric bivector field $B \in C^\infty(M,S^2(TM_\mathbb{C}))$ by

$$\delta_\sigma(B) = \mathrm{Tr}\,\nabla_\sigma B.$$

Notice that the divergence on bivector fields does depend on the point $\sigma \in \mathcal{T}$.

[3] Here a smooth map from $\mathcal{T}$ to $C^\infty(M,W)$, for any smooth vector bundle W over M, means a smooth section of $\pi_M^*(W)$ over $\mathcal{T}\times M$, where π_M is the projection onto M. Likewise, a smooth p-form on $\mathcal{T}$ with values in $C^\infty(M,W)$ is, by definition, a smooth section of $\pi_\mathcal{T}^*\Lambda^p(\mathcal{T}) \otimes \pi_M^*(W)$ over $\mathcal{T}\times M$. We will also encounter the situation where we have a bundle $\tilde{W}$ over $\mathcal{T}\times M$ and then we will talk about a smooth p-form on $\mathcal{T}$ with values in $C^\infty(M,\tilde{W}_\sigma)$ and mean a smooth section of $\pi_\mathcal{T}^*\Lambda^p(\mathcal{T}) \otimes \tilde{W}$ over $\mathcal{T}\times M$.

Suppose V is a vector field on $\mathcal{T}$. Then, we can differentiate I along V and we denote this derivative by $V[I] : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM_{\mathbb{C}}))$. Differentiating the equation $I^2 = -\mathrm{Id}$, we see that $V[I]$ anti-commutes with I. Hence, we get that

$$V[I]_\sigma \in C^\infty(M, (T^*_\sigma \otimes \bar{T}_\sigma) \oplus (\bar{T}^*_\sigma \otimes T_\sigma))$$

for each $\sigma \in \mathcal{T}$. Let

$$V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma$$

be the corresponding decomposition such that $V[I]'_\sigma \in C^\infty(M, \bar{T}^*_\sigma \otimes T_\sigma)$ and $V[I]''_\sigma \in C^\infty(M, T^*_\sigma \otimes \bar{T}_\sigma)$.

Now we will further assume that $\mathcal{T}$ is a complex manifold and that I is a holomorphic map from $\mathcal{T}$ to the space of all complex structures on M. Concretely, this means that

$$V'[I]_\sigma = V[I]'_\sigma$$

and

$$V''[I]_\sigma = V[I]''_\sigma$$

for all $\sigma \in \mathcal{T}$, where V' means the $(1,0)$-part of V and V'' means the $(0,1)$-part of V over $\mathcal{T}$.

Let us define $\tilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ by

$$V[I] = \tilde{G}(V)\omega,$$

and define $G(V) \in C^\infty(M, T_\sigma \otimes T_\sigma)$ such that

$$\tilde{G}(V) = G(V) + \overline{G}(V)$$

for all real vector fields V on $\mathcal{T}$. We see that $\tilde{G}$ and G are one-forms on $\mathcal{T}$ with values in $C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ and $C^\infty(M, T_\sigma \otimes T_\sigma)$, respectively. We observe that

$$V'[I] = G(V)\omega,$$

and $G(V) = G(V')$.

Using the relation (6), one checks that

$$\tilde{G}(V) = -V[g^{-1}],$$

where $g^{-1} \in C^\infty(M, S^2(TM))$ is the symmetric bivector field obtained by raising both indices on the metric tensor. Clearly, this implies that $\tilde{G}$ takes values in $C^\infty(M, S^2(TM_{\mathbb{C}}))$ and therefore that G takes values in $C^\infty(M, S^2(T_\sigma))$.

On $\mathcal{L}^k$, we have the smooth family of $\bar{\partial}$-operators $\nabla^{0,1}$ defined at $\sigma \in \mathcal{T}$ by

$$\nabla^{0,1}_\sigma = \frac{1}{2}(1 + iI_\sigma)\nabla.$$

For every $\sigma \in \mathcal{T}$, we consider the finite-dimensional subspace of $C^\infty(M, \mathcal{L}^k)$ given by

$$H^{(k)}_\sigma = H^0(M_\sigma, \mathcal{L}^k) = \{s \in C^\infty(M, \mathcal{L}^k) | \nabla^{0,1}_\sigma s = 0\}.$$

Let $\hat{\nabla}^t$ denote the trivial connection in the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$, and let $\mathcal{D}(M, \mathcal{L}^k)$ denote the vector space of differential operators on $C^\infty(M, \mathcal{L}^k)$. For any smooth one-form u on $\mathcal{T}$ with values in $\mathcal{D}(M, \mathcal{L}^k)$, we have a connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ given by

$$\hat{\nabla}_V = \hat{\nabla}^t_V - u(V)$$

for any vector field V on $\mathcal{T}$.

LEMMA 1. *The connection* $\hat{\nabla}$ *in* $\mathcal{H}^{(k)}$ *preserves the subspaces* $H^{(k)}_\sigma \subset C^\infty(M, \mathcal{L}^k)$, *for all* $\sigma \in \mathcal{T}$, *if and only if*

$$\frac{i}{2} V[I]\nabla^{1,0} s + \nabla^{0,1} u(V) s = 0 \tag{8}$$

for all vector fields V *on* $\mathcal{T}$ *and all smooth sections* s *of* $H^{(k)}$.

This result is not surprising. See [**A6**] for a proof this lemma. Observe that, if this condition holds, we can conclude that the collection of subspaces $H^{(k)}_\sigma \subset C^\infty(M, \mathcal{L}^k)$, for all $\sigma \in \mathcal{T}$, form a subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$.

We observe that $u(V'') = 0$ solves (8) along the anti-holomorphic directions on $\mathcal{T}$ since

$$V''[I]\nabla^{1,0} s = 0.$$

In other words, the $(0,1)$-part of the trivial connection $\hat{\nabla}^t$ induces a $\bar{\partial}$-operator on $H^{(k)}$ and hence makes it a holomorphic vector bundle over $\mathcal{T}$.

This is of course not in general the situation in the $(1,0)$ direction. Let us now consider a particular u and prove that it solves (8) under certain conditions.

On the Kähler manifold (M_σ, ω), we have the Kähler metric and we have the Levi-Civita connection ∇ in T_σ. We also have the Ricci potential $F_\sigma \in C^\infty_0(M, \mathbb{R})$. Here

$$C^\infty_0(M, \mathbb{R}) = \left\{ f \in C^\infty(M, \mathbb{R}) \mid \int_M f \omega^m = 0 \right\},$$

and the Ricci potential is the element of $F_\sigma \in C^\infty_0(M, \mathbb{R})$ which satisfies

$$\mathrm{Ric}_\sigma = \mathrm{Ric}^H_\sigma + 2i\partial_\sigma \bar{\partial}_\sigma F_\sigma,$$

where $\mathrm{Ric}_\sigma \in \Omega^{1,1}(M_\sigma)$ is the Ricci form and Ric^H_σ is its harmonic part. We see that we get in this way a smooth function $F : \mathcal{T} \to C^\infty_0(M, \mathbb{R})$.

For any symmetric bivector field $B \in C^\infty(M, S^2(TM))$ we get a linear bundle map

$$B\colon T^*M \to TM$$

given by contraction. In particular, for a smooth function f on M, we get a vector field $Bdf \in C^\infty(M, TM)$.

We define the operator

$$\begin{aligned} \Delta_B \quad : \quad & C^\infty(M, \mathcal{L}^k) \stackrel{\nabla}{\to} C^\infty(M, T^*M \otimes \mathcal{L}^k) \stackrel{B\otimes \mathrm{Id}}{\to} C^\infty(M, TM \otimes \mathcal{L}^k) \\ & \stackrel{\nabla_\sigma \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla}{\to} C^\infty(M, T^*M \otimes TM \otimes \mathcal{L}^k) \stackrel{\mathrm{Tr}}{\to} C^\infty(M, \mathcal{L}^k). \end{aligned}$$

Let's give a more concise formula for this operator. Define the operator

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y},$$

which is tensorial and symmetric in the vector fields X and Y. Thus, it can be evaluated on a symmetric bivector field and we have

$$\Delta_B = \nabla^2_B + \nabla_{\delta(B)}.$$

Putting these constructions together, we consider, for some $n \in \mathbb{Z}$ such that $2k + n \neq 0$, the operator

$$u(V) = \frac{1}{k + n/2} o(V) - V'[F], \tag{9}$$

where

$$o(V) = -\frac{1}{4}(\Delta_{G(V)} + 2\nabla_{G(V)dF} - 2nV'[F]). \tag{10}$$

The connection associated to this u is denoted by $\hat{\nabla}$, and we call it the *Hitchin connection* in $\mathcal{H}^{(k)}$.

DEFINITION 5. *We say that the complex family I of Kähler structures on (M,ω) is* rigid *if*

$$\bar{\partial}_\sigma(G(V)_\sigma) = 0$$

for all vector fields V on $\mathcal{T}$ and all points $\sigma \in \mathcal{T}$.

We will assume our holomorphic family I is rigid.

THEOREM 9 (Andersen). *Suppose that I is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold (M,ω) which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M,ω) is $n[\frac{\omega}{2\pi}] \in H^2(M,\mathbb{Z})$ and $H^1(M,\mathbb{R}) = 0$. Then u given by (9) and (10) satisfies (8), for all k such that $2k+n \neq 0$.*

Hence, the Hitchin connection $\hat{\nabla}$ preserves the subbundle $H^{(k)}$ under the stated conditions. Theorem 9 is established in [**A6**] through the following three lemmas.

LEMMA 2. *Assume that the first Chern class of (M,ω) is $n[\frac{\omega}{2\pi}] \in H^2(M,\mathbb{Z})$. For any $\sigma \in \mathcal{T}$ and for any $G \in H^0(M_\sigma, S^2(T_\sigma))$, we have the following formula:*

$$\nabla^{0,1}_\sigma(\Delta_G(s) + 2\nabla_{GdF_\sigma}(s)) = -i(2k+n)\omega G\nabla(s) + 2ik(GdF_\sigma)\omega + ik\delta_\sigma(G)\omega)s,$$

for all $s \in H^0(M_\sigma, \mathcal{L}^k)$.

LEMMA 3. *We have the following relation:*

$$4i\bar{\partial}_\sigma(V'[F]_\sigma) = 2(G(V)dF)_\sigma\omega + \delta_\sigma(G(V))_\sigma\omega,$$

provided that $H^1(M,\mathbb{R}) = 0$.

LEMMA 4. *For any smooth vector field V on $\mathcal{T}$, we have that*

$$2(V'[\mathrm{Ric}])^{1,1} = \partial(\delta(G(V))\omega). \tag{11}$$

Let us here recall how Lemma 3 is derived from Lemma 4. By the definition of the Ricci potential

$$\mathrm{Ric} = \mathrm{Ric}^H + 2id\bar{\partial}F,$$

where $\mathrm{Ric}^H = n\omega$ by the assumption $c_1(M,\omega) = n[\frac{\omega}{2\pi}]$. Hence

$$V'[\mathrm{Ric}] = -dV'[I]dF + 2id\bar{\partial}V'[F],$$

and therefore

$$4i\partial\bar{\partial}V'[F] = 2(V'[\mathrm{Ric}])^{1,1} + 2\partial V'[I]dF.$$

From the above, we conclude that

$$(2(G(V)dF)\omega + \delta(G(V))\omega - 4i\bar{\partial}V'[F])_\sigma \in \Omega^{0,1}_\sigma(M)$$

is a ∂_σ-closed one-form on M. From Lemma 2, it follows that it is also $\bar{\partial}_\sigma$-closed, whence it must be a closed one-form. Since we assume that $H^1(M,\mathbb{R}) = 0$, we see that it must be exact. But then it in fact vanishes since it is of type $(0,1)$ on M_σ.

From the above we conclude that

$$u(V) = \frac{1}{k+n/2}o(V) - V'[F] = -\frac{1}{4k+2n}\left\{\Delta_{G(V)} + 2\nabla_{G(V)dF} + 4kV'[F]\right\}$$

solves (8). Thus we have established Theorem 9 and hence Theorem 5.

In [**AGL**] we use half-forms and the metaplectic correction to prove the existence of a Hitchin connection in the context of half-form quantization. The assumption that the first Chern class of (M,ω) is $n[\frac{\omega}{2\pi}] \in H^2(M,\mathbb{Z})$ is then just replaced by the vanishing of the second Stiefel-Whitney class of M (see [**AGL**] for more details).

Suppose Γ is a group which acts by bundle automorphisms of $\mathcal{L}$ over M preserving both the Hermitian structure and the connection in $\mathcal{L}$. Then there is an induced action of Γ on (M,ω). We will further assume that Γ acts on $\mathcal{T}$ and that I is Γ-equivariant. In this case we immediately get the following invariance.

LEMMA 5. *The natural induced action of* Γ *on* $\mathcal{H}^{(k)}$ *preserves the subbundle* $H^{(k)}$ *and the Hitchin connection.*

We are actually interested in the induced connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(H^{(k)})$. Suppose Φ is a section of $\mathrm{End}(H^{(k)})$. Then for all sections s of $H^{(k)}$ and all vector fields V on $\mathcal{T}$, we have that

$$(\hat{\nabla}^e_V\Phi)(s) = \hat{\nabla}_V\Phi(s) - \Phi(\hat{\nabla}_V(s)).$$

Assume now that we have extended Φ to a section of $\mathrm{Hom}(\mathcal{H}^{(k)}, H^{(k)})$ over $\mathcal{T}$. Then

$$\hat{\nabla}^e_V\Phi = \hat{\nabla}^{e,t}_V\Phi + [\Phi, u(V)], \tag{12}$$

where $\hat{\nabla}^{e,t}$ is the trivial connection in the trivial bundle $\mathrm{End}(\mathcal{H}^{(k)})$ over $\mathcal{T}$.

3. Toeplitz operators and Berezin-Toeplitz deformation quantization

We shall in this section discuss the Toeplitz operators and their asymptotics as the level k goes to infinity. The properties we need can all be derived from the fundamental work of Boutet de Monvel and Sjöstrand. In [**BdMS**], they did a microlocal analysis of the Szegő projection which can be applied to the asymptotic analysis in the situation at hand, as was done by Boutet de Monvel and Guillemin in [**BdMG**] (in fact in a much more general situation than the one we consider here) and others following them. In particular, the applications developed by Schlichenmaier and further by Karabegov and Schlichenmaier to the study of Toeplitz operators in the geometric quantization setting is what will interest us here. Let us first describe the basic setting.

For each $f \in C^\infty(M)$, we consider the prequantum operator, namely the differential operator $M_f^{(k)} : C^\infty(M,L^k) \to C^\infty(M,L^k)$ given by

$$M_f^{(k)}(s) = fs$$

for all $s \in H^0(M,L^k)$.

These operators act on $C^\infty(M,\mathcal{L}^k)$ and therefore also on the bundle $\mathcal{H}^{(k)}$; however, they do not preserve the subbundle $H^{(k)}$. In order to turn these operators into operators which act on $H^{(k)}$ we need to consider the Hilbert space structure.

Integrating the inner product of two sections against the volume form associated to the symplectic form gives the pre-Hilbert space structure on $C^\infty(M, \mathcal{L}^k)$

$$\langle s_1, s_2\rangle = \frac{1}{m!}\int_M (s_1, s_2)\omega^m.$$

We think of this as a pre-Hilbert space structure on the trivial bundle $\mathcal{H}^{(k)}$, which of course is compatible with the trivial connection in this bundle. This pre-Hilbert space structure induces a Hermitian structure $\langle\cdot,\cdot\rangle$ on the finite rank subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$. The Hermitian structure $\langle\cdot,\cdot\rangle$ on $H^{(k)}$ also induces the operator norm $\|\cdot\|$ on $\operatorname{End}(H^{(k)})$.

Since $H^{(k)}_\sigma$ is a finite-dimensional subspace of $C^\infty(M, \mathcal{L}^k) = \mathcal{H}^{(k)}_\sigma$ and therefore closed, we have the orthogonal projection $\pi^{(k)}_\sigma \colon \mathcal{H}^{(k)}_\sigma \to H^{(k)}_\sigma$. Since $H^{(k)}$ is a smooth subbundle of $\mathcal{H}^{(k)}$, the projections $\pi^{(k)}_\sigma$ form a smooth map $\pi^{(k)}$ from $\mathcal{T}$ to the space of bounded operators on the L_2-completion of $C^\infty(M, \mathcal{L}^k)$. The easiest way to see this is to consider a local frame $(s_1, \ldots, s_{\operatorname{Rank} H^{(k)}})$ of $H^{(k)}$. Let $h_{ij} = \langle s_i, s_j\rangle$, and let h^{-1}_{ij} be the inverse matrix of h_{ij}. Then

$$\pi^{(k)}_\sigma(s) = \sum_{i,j}\langle s, (s_i)_\sigma\rangle (h^{-1}_{ij})_\sigma (s_j)_\sigma. \tag{13}$$

From these projections, we can construct the Toeplitz operators associated to any smooth function $f \in C^\infty(M)$. They are the operators $T^{(k)}_{f,\sigma} : \mathcal{H}^{(k)}_\sigma \to H^{(k)}_\sigma$ defined by

$$T^{(k)}_{f,\sigma}(s) = \pi^{(k)}_\sigma(fs)$$

for any element s in $\mathcal{H}^{(k)}_\sigma$ and any point $\sigma \in \mathcal{T}$. We observe that the Toeplitz operators are smooth sections $T^{(k)}_f$ of the bundle $\operatorname{Hom}(\mathcal{H}^{(k)}, H^{(k)})$ and restrict to smooth sections of $\operatorname{End}(H^{(k)})$.

Remark 1. *Similarly, for any pseudo-differential operator A on M with coefficients in $\mathcal{L}^k$ (which may even depend on $\sigma \in \mathcal{T}$), we can consider the associated Toeplitz operator $\pi^{(k)}A$ and think of it as a section of* $\operatorname{Hom}(\mathcal{H}^{(k)}, H^{(k)})$. *However, whenever we consider asymptotic expansions of such or consider their operator norms, we implicitly restrict them to $H^{(k)}$ and consider them as sections of* $\operatorname{End}(H^{(k)})$ *or equivalently assume that they have been precomposed with $\pi^{(k)}$.*

Suppose that we have a smooth section $X \in C^\infty(M, T_\sigma)$ of the holomorphic tangent bundle of M_σ. We then claim that the operator $\pi^{(k)}\nabla_X$ is a zero-order Toeplitz operator. Supposing that $s_1 \in C^\infty(M, \mathcal{L}^k)$ and $s_2 \in H^0(M_\sigma, \mathcal{L}^k)$, we have that

$$X(s_1, s_2) = (\nabla_X s_1, s_2).$$

Now, calculating the Lie derivative along X of $(s_1, s_2)\omega^m$ and using the above, one obtains after integration that

$$\langle \nabla_X s_1, s_2\rangle = -\langle \delta(X) s_1, s_2\rangle,$$

Thus

$$\pi^{(k)}\nabla_X = -T^{(k)}_{\delta(X)}, \tag{14}$$

as operators from $C^\infty(M, L^k)$ to $H^0(M, L^k)$.

Iterating (14), we find for all $X_1, X_2 \in C^\infty(M, T_\sigma)$ that

$$\pi^{(k)} \nabla_{X_1} \nabla_{X_2} = T^{(k)}_{\delta(X_2)\delta(X_1) + X_2[\delta(X_1)]}, \tag{15}$$

again as operators from $C^\infty(M, \mathcal{L}^k)$ to $H^0(M_\sigma, \mathcal{L}^k)$.

We calculate the adjoint of ∇_X for any complex vector field $X \in C^\infty(M, TM_\mathbb{C})$. For $s_1, s_2 \in C^\infty(M, \mathcal{L}^k)$, we have that

$$\bar{X}(s_1, s_2) = (\nabla_{\bar{X}} s_1, s_2) + (s_1, \nabla_X s_2).$$

Computing the Lie derivative along $\bar{X}$ of $(s_1, s_2)\omega^m$ and integrating, we get that

$$\langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle (\nabla_X)^* s_1, s_2 \rangle = -\langle \delta(\bar{X}) s_1, s_2 \rangle.$$

Hence, we see that

$$(\nabla_X)^* = -\nabla_{\bar{X}} - \delta(\bar{X}) \tag{16}$$

as operators on $C^\infty(M, \mathcal{L}^k)$. In particular, if $X \in C^\infty(M, T_\sigma)$ is a section of the holomorphic tangent bundle, we see that

$$\pi^{(k)} (\nabla_X)^* \pi^{(k)} = -T^{(k)}_{\delta(\bar{X})}|_{H^0(M_\sigma, \mathcal{L}^k)}, \tag{17}$$

again as operators on $H^0(M_\sigma, \mathcal{L}^k)$.

The product of two Toeplitz operators associated to two smooth functions will in general not again be the Toeplitz operator associated to a smooth function. But, by the results of Schlichenmaier [**Sch**], there is an asymptotic expansion of the product in terms of such Toeplitz operators on a compact Kähler manifold.

Theorem 10 (Schlichenmaier). *For any pair of smooth functions $f_1, f_2 \in C^\infty(M)$, we have an asymptotic expansion*

$$T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} \sim \sum_{l=0}^{\infty} T^{(k)}_{c^{(l)}_\sigma(f_1,f_2),\sigma} k^{-l},$$

where $c^{(l)}_\sigma(f_1, f_2) \in C^\infty(M)$ are uniquely determined since $\sim$ means the following: For all $L \in \mathbb{Z}_+$ we have that

$$\| T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} - \sum_{l=0}^{L} T^{(k)}_{c^{(l)}_\sigma(f_1,f_2),\sigma} k^{-l} \| = O(k^{-(L+1)})$$

uniformly over compact subsets of $\mathcal{T}$. Moreover, $c^{(0)}_\sigma(f_1, f_2) = f_1 f_2$.

Remark 2. *It will be useful for us to define new coefficients $\tilde{c}^{(l)}_\sigma(f, g) \in C^\infty(M)$ which correspond to the expansion of the product in $1/(k + n/2)$ (where n is some fixed integer):*

$$T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} \sim \sum_{l=0}^{\infty} T^{(k)}_{\tilde{c}^{(l)}_\sigma(f_1,f_2),\sigma} (k + n/2)^{-l}.$$

For future reference, we note that the first three coefficients are given by $\tilde{c}^{(0)}_\sigma(f_1, f_2) = c^{(0)}_\sigma(f_1, f_2)$, $\tilde{c}^{(1)}_\sigma(f_1, f_2) = c^{(1)}_\sigma(f_1, f_2)$ and $\tilde{c}^{(2)}_\sigma(f_1, f_2) = c^{(2)}_\sigma(f_1, f_2) + \frac{n}{2} c^{(1)}_\sigma(f_1, f_2)$.

Theorem 10 is proved in [**Sch**] where it is also proved that the formal generating series for the $c^{(l)}_\sigma(f_1, f_2)$'s gives a formal deformation quantization of the symplectic manifold (M, ω).

We recall the definition of a formal deformation quantization. Introduce the space of formal functions $C^\infty_h(M) = C^\infty(M)[[h]]$ as the space of formal power series in the variable h with coefficients in $C^\infty(M)$. Let $\mathbb{C}_h = \mathbb{C}[[h]]$ denote the formal constants.

Definition 6. *A deformation quantization of* (M, ω) *is an associative product* $\star$ *on* $C^\infty_h(M)$ *which respects the* $\mathbb{C}_h$*-module structure. For* $f, g \in C^\infty(M)$*, it is defined as*

$$f \star g = \sum_{l=0}^{\infty} c^{(l)}(f, g) h^l,$$

through a sequence of bilinear operators

$$c^{(l)} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M),$$

which must satisfy

$$c^{(0)}(f, g) = fg \qquad \text{and} \qquad c^{(1)}(f, g) - c^{(1)}(g, f) = -i\{f, g\}.$$

The deformation quantization is said to be differential if the operators $c^{(l)}$ *are bidifferential operators. Considering the symplectic action of* Γ *on* (M, ω)*, we say that a star product is* Γ*-invariant if*

$$\gamma^*(f \star g) = \gamma^*(f) \star \gamma^*(g)$$

for all $f, g \in C^\infty(M)$ *and all* $\gamma \in \Gamma$.

Theorem 11 (Karabegov & Schlichenmaier). *The product* $\star^{BT}_\sigma$ *given by*

$$f \star^{BT}_\sigma g = \sum_{l=0}^{\infty} c^{(l)}_\sigma(f, g) h^l,$$

where $f, g \in C^\infty(M)$ *and* $c^{(l)}_\sigma(f, g)$ *are determined by Theorem 10, is a differentiable deformation quantization of* (M, ω).

Definition 7. *The Berezin-Toeplitz deformation quantization of the compact Kähler manifold* (M_σ, ω) *is the product* $\star^{BT}_\sigma$.

Remark 3. *Let* Γ_σ *be the* σ*-stabilizer subgroup of* Γ*. For any element* $\gamma \in \Gamma_\sigma$*, we have that*

$$\gamma^*(T^{(k)}_{f,\sigma}) = T^{(k)}_{\gamma^* f, \sigma}.$$

This implies the invariance of $\star^{BT}_\sigma$ *under the* σ*-stabilizer* Γ_σ.

Remark 4. *Using the coefficients from Remark 2, we define a new star product by*

$$f \tilde{\star}^{BT}_\sigma g = \sum_{l=0}^{\infty} \tilde{c}^{(l)}_\sigma(f, g) h^l.$$

Then

$$f \tilde{\star}^{BT}_\sigma g = \left((f \circ \phi^{-1}) \star^{BT}_\sigma (g \circ \phi^{-1}) \right) \circ \phi$$

for all $f, g \in C^\infty_h(M)$*, where* $\phi(h) = \frac{2h}{2+nh}$.

4. The formal Hitchin connection

In this section, we study the the formal Hitchin connection. We assume the conditions on (M,ω) and I of Theorem 9, thus providing us with a Hitchin connection $\hat{\nabla}$ in $H^{(k)}$ over $\mathcal{T}$ and the associated connection $\hat{\nabla}^e$ in $\mathrm{End}(H^{(k)})$.

Recall from the introduction the definition of a formal connection in the trivial bundle of formal functions. Theorem 6 establishes the existence of a unique formal Hitchin connection, expressing asymptotically the interplay between the Hitchin connection and the Toeplitz operators.

We want to give an explicit formula for the formal Hitchin connection in terms of the star product $\tilde{\star}^{BT}$. We recall that in the proof of Theorem 6, given in [**A6**], it is shown that the formal Hitchin connection is given by

$$\tilde{D}(V)(f) = -V[F]f + V[F]\tilde{\star}^{\mathrm{BT}}f + h(E(V)(f) - H(V)\tilde{\star}^{\mathrm{BT}}f), \tag{18}$$

where E is the one-form on $\mathcal{T}$ with values in $\mathcal{D}(M)$ such that

$$T^{(k)}_{E(V)f} = \pi^{(k)}o(V)^*f\pi^{(k)} + \pi^{(k)}fo(V)\pi^{(k)}, \tag{19}$$

and H is the one-form on $\mathcal{T}$ with values in $C^\infty(M)$ such that $H(V) = E(V)(1)$. Thus, we must find an explicit expression for the operator $E(V)$.

The following lemmas will prove helpful.

Lemma 6. *The adjoint of Δ_B is given by*

$$\Delta_B^* = \Delta_{\bar{B}},$$

for any (complex) symmetric bivector field $B \in C^\infty(M, S^2(TM_{\mathbb{C}}))$.

Proof. First, we write $B = \sum_r^R X_r \otimes Y_r$. Then

$$\Delta_B = \sum_r^R \nabla_{X_r}\nabla_{Y_r} + \nabla_{\delta(X_r)Y_r}.$$

Now, using (16), we get that

$$\begin{aligned}(\nabla_{X_r}\nabla_{Y_r})^* &= (\nabla_{Y_r})^*(\nabla_{X_r})^* = (\nabla_{\bar{Y}_r} + \delta(\bar{Y}_r)(\nabla_{\bar{X}_r} + \delta(\bar{X}_r))\\ &= \nabla_{\bar{Y}_r}\nabla_{\bar{X}_r} + \nabla_{\bar{Y}_r}\delta(X_r) + \delta(\bar{Y}_r)\nabla_{\bar{X}_r} + \delta(\bar{Y}_r)\delta(\bar{X}_r),\end{aligned}$$

and

$$\begin{aligned}(\nabla_{\delta(X_r)Y_r})^* &= -\nabla_{\delta(\bar{X})\bar{Y}_r} - \delta(\delta(\bar{X}_r)\bar{Y}_r)\\ &= -\delta(\bar{X}_r)\nabla_{\bar{Y}_r} - \bar{Y}_r[\delta(\bar{X}_r)] - \delta(\bar{X}_r)\delta(\bar{Y}_r)\\ &= -\nabla_{\bar{Y}_r}\delta(\bar{X}_r) - \delta(\bar{X}_r)\delta(\bar{Y}_r),\end{aligned}$$

so we conclude that

$$\Delta_B^* = \sum_r^R \nabla_{\bar{Y}_r}\nabla_{\bar{X}_r} + \delta(\bar{Y}_r)\nabla_{\bar{X}_r} = \Delta_{\bar{B}},$$

since B is symmetric. □

Lemma 7. *The operator Δ_B satisfies*

$$\pi^{(k)}\Delta_B s = 0,$$

for any section $s \in C^\infty(M, \mathcal{L}^k)$ and any symmetric bivector field B.

PROOF. Again, we write $B = \sum_r^R X_r \otimes Y_r$ and recall from (15) that

$$\pi^{(k)} \nabla_{X_r} \nabla_{Y_r} s = \pi^{(k)} (\delta(X_r)\delta(Y_r) + Y_r[\delta(X_r)])s.$$

On the other hand, we have that

$$\pi^{(k)} \nabla_{\delta(X_r)Y_r} s = -\pi^{(k)} \delta(\delta(X_r)Y_r)s = -\pi^{(k)} (\delta(X_r)\delta(Y_r) + Y_r[\delta(X_r)])s,$$

and it follows immediately that

$$\pi^{(k)} \Delta_B s = \pi^{(k)} \sum_r^R (\nabla_{X_r} \nabla_{Y_r} + \nabla_{\delta(X_r)Y_r})s = 0,$$

which proves the lemma. □

Finally, it will prove useful to observe that

$$\delta(Bdf) = \Delta_B(f), \tag{20}$$

for any function f and any bivector field B.

Now, the adjoint of $o(V)$ is given by

$$o(V)^* = -\frac{1}{4}(\Delta_{\bar{G}(V)} - 2\nabla_{\bar{G}(V)dF} - 2\Delta_{\bar{G}(V)}(F) - 2nV''[F]),$$

where we used (20). Furthermore, we observe that $o(V)^*$ differentiates in anti-holomorphic directions only, which implies that

$$\begin{aligned}\pi^{(k)} o(V)^* f \pi^{(k)} &= \pi^{(k)} o(V)^*(f) \pi^{(k)} \\ &= -\frac{1}{4}\pi^{(k)} (\Delta_{\bar{G}(V)}(f) - 2\nabla_{\bar{G}(V)dF}(f) - 2\Delta_{\bar{G}(V)}(F)f - 2nV''[F]f)\pi^{(k)}.\end{aligned}$$

This gives an explicit formula for the first term of (19).

To determine the second term of (19), we observe that

$$\Delta_{G(V)} fs = f\Delta_{G(V)} s + \Delta_{G(V)}(f)s + 2\nabla_{G(V)df} s.$$

Projecting both sides onto the holomorphic sections and applying Lemma 7 and the formula (20), we get that

$$\pi^{(k)} f \Delta_{G(V)} = -\pi^{(k)} (\Delta_{G(V)}(f) + 2\nabla_{G(V)df}) = \pi^{(k)} \Delta_{G(V)}(f).$$

Furthermore, observe that

$$\begin{aligned}\pi^{(k)} f \nabla_{G(V)dF} &= \pi^{(k)} (\nabla_{G(V)dF} f - \nabla_{G(V)dF}(f)) \\ &= -\pi^{(k)} (\nabla_{G(V)dF}(f) + \Delta_{G(V)}(F)f),\end{aligned}$$

where we once again used (20) for the last equality. Thus, we get that

$$\pi^{(k)} f o(V) \pi^{(k)} = -\frac{1}{4}\pi^{(k)} (\Delta_{G(V)}(f) - 2\nabla_{G(V)dF}(f) - 2\Delta_{G(V)}(F)f - 2nV'[F]f)\pi^{(k)},$$

which gives an explicit formula for the second term of (19). Finally, we can conclude that

$$E(V)(f) = -\frac{1}{4}(\Delta_{\tilde{G}(V)}(f) - 2\nabla_{\tilde{G}(V)dF}(f) - 2\Delta_{\tilde{G}(V)}(F)f - 2nV[F]f),$$

satisfies (19) and hence (18). Also, we note that

$$H(V) = E(V)(1) = \frac{1}{2}(\Delta_{\tilde{G}(V)}(F) + nV[F]).$$

Summarizing the above, we have proved the following

THEOREM 12. *The formal Hitchin connection is given by*

$$D_V f = V[f] - \frac{1}{4}h\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}h\nabla_{\tilde{G}(V)dF}(f) + V[F]\tilde{\star}^{BT}f - V[F]f$$
$$- \frac{1}{2}h(\Delta_{\tilde{G}(V)}(F)\tilde{\star}^{BT}f + nV[F]\tilde{\star}^{BT}f - \Delta_{\tilde{G}(V)}(F)f - nV[F]f)$$

for any vector field V and any section f of C_h.

The next lemma is also proved in [**A6**], and it follows basically from the fact that

$$\hat{\nabla}^e_V(T^{(k)}_f T^{(k)}_g) = \hat{\nabla}^e_V(T^{(k)}_f)T^{(k)}_g + T^{(k)}_f\hat{\nabla}^e_V(T^{(k)}_g).$$

We have

LEMMA 8. *The formal operator D_V is a derivation for $\tilde{\star}^{BT}_\sigma$ for each $\sigma \in \mathcal{T}$, i.e.*

$$D_V(f\tilde{\star}^{BT}g) = D_V(f)\tilde{\star}^{BT}g + f\tilde{\star}^{BT}D_V(g)$$

for all $f, g \in C^\infty(M)$.

If the Hitchin connection is projectively flat, then the induced connection in the endomorphism bundle is flat and hence so is the formal Hitchin connection by Proposition 3 of [**A6**].

Recall from Definition 3 in the introduction the definition of a formal trivialization. As mentioned there, such a formal trivialization will not exist even locally on $\mathcal{T}$, if D is not flat. However, if D is flat, then we have the following result.

PROPOSITION 1. *Assume that D is flat and that $\tilde{D} = 0$ mod h. Then locally around any point in $\mathcal{T}$ there exists a formal trivialization. If $H^1(\mathcal{T}, \mathbb{R}) = 0$, then there exists a formal trivialization defined globally on $\mathcal{T}$. If further $H^1_\Gamma(\mathcal{T}, D(M)) = 0$, then we can construct P such that it is Γ-equivariant.*

In this proposition, $H^1_\Gamma(\mathcal{T}, D(M))$ simply refers to the Γ-equivariant first de Rham cohomology of $\mathcal{T}$ with coefficients in the real Γ-vector space $D(M)$.

Now suppose we have a formal trivialization P of the formal Hitchin connection D. We can then define a new smooth family of star products, parametrized by $\mathcal{T}$, by

$$f \star_\sigma g = P^{-1}_\sigma(P_\sigma(f)\tilde{\star}^{\mathrm{BT}}_\sigma P_\sigma(g))$$

for all $f, g \in C^\infty(M)$ and all $\sigma \in \mathcal{T}$. Using the fact that P is a trivialization, it is not hard to prove the following:

PROPOSITION 2. *The star products $\star_\sigma$ are independent of $\sigma \in \mathcal{T}$.*

Then, we have the following which is proved in [**A6**].

THEOREM 13 (Andersen). *Assume that the formal Hitchin connection D is flat and*

$$H^1_\Gamma(\mathcal{T}, D(M)) = 0,$$

then there is a Γ-invariant trivialization P of D and the star product

$$f \star g = P^{-1}_\sigma(P_\sigma(f)\tilde{\star}^{BT}_\sigma P_\sigma(g))$$

is independent of $\sigma \in \mathcal{T}$ and Γ-invariant. If $H^1_\Gamma(\mathcal{T}, C^\infty(M)) = 0$ and the commutant of Γ in $D(M)$ is trivial, then a Γ-invariant differential star product on M is unique.

We calculate the first term of the equivariant formal trivialization of the formal Hitchin connection. Let f be any function on M and suppose that $P(f) = \sum_l \tilde{f}_l h^l$ is parallel with respect to the formal Hitchin connection. Thus, we have that

$$0 = D_V P(f) = h(V[\tilde{f}_1] - \frac{1}{4}\Delta_{\tilde{G}(V)}(\tilde{f}_0) + \frac{1}{2}\nabla_{\tilde{G}(V)dF}(\tilde{f}_0) + c^{(1)}(V[F], \tilde{f}_0)) + O(h^2).$$

But $\tilde{f}_0 = f$, and so we get in particular

$$0 = V[\tilde{f}_1] - \frac{1}{4}\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}\nabla_{\tilde{G}(V)dF}(f) + c^{(1)}(V[F], f). \tag{21}$$

By the results of [**KS**], $\star^{\mathrm{BT}}$ is a differential star product with separation of variables, in the sense that it only differentiates in holomorphic directions in the first entry and antiholomorphic directions in the second. As argued in [**Kar**], all such star products have the same first order coefficient, namely

$$c^{(1)}(f_1, f_2) = -g(\partial f_1, \bar{\partial} f_2) = i\nabla_{X''_{f_1}}(f_2) \tag{22}$$

for any functions $f_1, f_2 \in C^\infty(M)$. From this, it is easily seen that

$$V[c^{(1)}](f_1, f_2) = \frac{1}{2} df_1 \tilde{G}(V) df_2 = \frac{1}{2}\nabla_{\tilde{G}(V)df_1} f_2.$$

Applying this to (21), we see that

$$V[\tilde{f}_1] = \frac{1}{4}\Delta_{\tilde{G}(V)}(f) - V[c^{(1)}(F, f)]. \tag{23}$$

But the variation of the Laplace-Beltrami operator is given by

$$V[\Delta]f = V[\delta(g^{-1}df)] = \delta(V[g^{-1}]df) = -\delta(\tilde{G}(V)df) = -\Delta_{\tilde{G}(V)} f,$$

and so we conclude that

$$V[\tilde{f}_1] = -V[\frac{1}{4}\Delta f + c^{(1)}(F, f)].$$

We have thus proved

Proposition 3. *When it exists, the equivariant formal trivialization of the formal Hitchin connection has the form*

$$P = \mathrm{Id} - h(\frac{1}{4}\Delta + i\nabla_{X''_F}) + O(h^2).$$

Using this proposition, one easily calculates that

$$\begin{aligned} P(f_1)\tilde{\star}^{\mathrm{BT}} P(f_2) = f_1 f_2 - h(&\frac{1}{4} f_1 \Delta f_2 + \frac{1}{4} f_2 \Delta f_1 + i\nabla_{X''_F} f_1 + i\nabla_{X''_F} f_2) \\ &+ hc^{(1)}(f_1, f_2) + O(h^2). \end{aligned}$$

Finally, using the explicit formula (22) for $c^{(1)}$, we get that

$$\begin{aligned} P^{-1}(P(f_1)\tilde{\star}^{\mathrm{BT}} P(f_2)) &= f_1 f_2 - hg(\partial f_1, \bar{\partial} f_2) + \frac{1}{2} hg(df_1, df_2) + O(h^2) \\ &= f_1 f_2 - h\frac{1}{2}(g(\partial f_1, \bar{\partial} f_2) - g(\bar{\partial} f_1, \partial f_2)) + O(h^2) \\ &= f_1 f_2 - ih\frac{1}{2}\{f_1, f_2\} + O(h^2). \end{aligned}$$

This proves Theorem 8.

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Koszul Cohomology and Applications to Moduli

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1. Introduction

One of the driving problems in the theory of algebraic curves in the past two decades has been *Green's Conjecture* on syzygies of canonical curves. Initially formulated by M. Green [**Gr84a**], it is a deceptively simple vanishing statement concerning Koszul cohomology groups of canonical bundles of curves: If $C \stackrel{|K_C|}{\longrightarrow} \mathbb{P}^{g-1}$ is a smooth canonically embedded curve of genus g and $K_{i,j}(C, K_C)$ are the Koszul cohomology groups of the canonical bundle on C, Green's Conjecture predicts the equivalence

$$K_{p,2}(C, K_C) = 0 \Longleftrightarrow p < \mathrm{Cliff}(C), \tag{1}$$

where $\mathrm{Cliff}(C) := \min\{\deg(L) - 2r(L) : L \in \mathrm{Pic}(C),\ h^i(C, L) \geq 2,\ i = 0, 1\}$ denotes the *Clifford index* of C. The main attraction of Green's Conjecture is that it links the *extrinsic* geometry of C encapsulated in $\mathrm{Cliff}(C)$ and all the linear series $\mathfrak{g}^r_d$ on C, to the *intrinsic* geometry (equations) of the canonical embedding. In particular, quite remarkably, it shows that one can read the Clifford index of any curve off the equations of its canonical embedding. Hence, in some sense, a curve has no other interesting line bundle apart from the canonical bundle and its powers[1].

One implication in (1), namely that $K_{p,2}(C, K_C) \neq 0$ for $p \geq \mathrm{Cliff}(C)$, having been immediately established in [**GL84**], see also Theorem 2.4 in this survey, the converse, that is, the vanishing statement

$$K_{p,2}(C, K_C) = 0 \ \text{ for } p < \mathrm{Cliff}(C),$$

attracted a great deal of effort and resisted proof despite an amazing number of attempts and techniques devised to prove it, see [**GL84**], [**Sch86**], [**Sch91**], [**Ein87**], [**Ei92**], [**PR88**], [**Tei02**], [**V93**]. The major breakthrough came around 2002 when Voisin [**V02**], [**V05**], using specialization to curves on $K3$ surfaces, proved that Green's Conjecture holds for a general curve $[C] \in \mathcal{M}_g$:

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[1] "The canonical bundle is not called canonical for nothing"- Joe Harris

THEOREM 1.1. *For a general curve $[C] \in \mathcal{M}_{2p+3}$ we have that $K_{p,2}(C, K_C) = 0$. For a general curve $[C] \in \mathcal{M}_{2p+4}$ we have that $K_{p,2}(C, K_C) = 0$. It follows that Green's Conjecture holds for general curves of any genus.*

Combining the results of Voisin with those of Hirschowitz and Ramanan [**HR98**], one finds that Green's Conjecture is true for *every* smooth curve $[C] \in \mathcal{M}_{2p+3}$ of maximal gonality $\text{gon}(C) = p + 3$. This turns out to be a remarkably strong result. For instance, via a specialization argument, Green's Conjecture for *arbitrary* curves of maximal gonality implies Green's Conjecture for *general* curves of genus g and *arbitrary* gonality $2 \leq d \leq [g/2] + 2$. One has the following more precise result, cf. [**Ap05**], see Theorem 4.5 for a slightly different proof:

THEOREM 1.2. *We fix integers $2 \leq d \leq [g/2] + 1$. For any smooth d-gonal curve $[C] \in \mathcal{M}_g$ satisfying the condition*

$$\dim W^1_{g-d+2}(C) \leq g - 2d + 2,$$

we have that $K_{d-3,2}(C, K_C) = 0$. In particular C satisfies Green's Conjecture.

Dimension theorems from Brill-Noether theory due to Martens, Mumford, and Keem, cf. [**ACGH85**], indicate precisely when the condition appearing in the statement of Theorem 1.2 is verified. In particular, Theorem 1.2 proves Green's Conjecture for general d-gonal curves of genus g for any possible gonality $2 \leq d \leq [g/2] + 2$ and offers an alternate, unitary proof of classical results of Noether, Enriques-Babbage-Petri as well as of more recent results due to Schreyer and Voisin. It also implies the following new result which can be viewed as a proof of statement (1) for 6-gonal curves. We refer to Subsection 4.1 for details:

THEOREM 1.3. *For any curve C with* $\text{Cliff}(C) \geq 4$, *we have $K_{3,2}(C, K_C) = 0$. In particular, Green's Conjecture holds for arbitrary* 6*-gonal curves.*

Theorem 1.1 can also be applied to solve various related problems. For instance, using precisely Theorem 1.1, the *Green-Lazarsfeld Gonality Conjecture* [**GL86**] was verified for general d-gonal curves, for any $2 \leq d \leq (g+2)/2$, cf. [**ApV03**], [**Ap05**]. In a few words, this conjecture states that the gonality of any curve can be read off the Koszul cohomology with values in line bundles of large degree, such as the powers of the canonical bundle. We shall review all these results in Subsection 4.2.

Apart from surveying the progress made on Green's and the Gonality Conjectures, we discuss a number of new conjectures for syzygies of line bundles on curves. Some of these conjectures have already appeared in print (e.g. the *Prym-Green Conjecture* [**FaL08**], or the syzygy conjecture for special line bundles on general curves [**Fa06a**]), whereas others like the *Minimal Syzygy Conjecture* are new and have never been formulated before.

For instance we propose the following refinement of the Green-Lazarsfeld Gonality Conjecture [**GL86**]:

CONJECTURE 1.4. *Let C be a general curve of genus $g = 2d - 2 \geq 6$ and $\eta \in \text{Pic}^0(C)$ a general line bundle. Then $K_{d-4,2}(C, K_C \otimes \eta) = 0$.*

Conjecture 1.4 is the sharpest vanishing statement one can make for general line bundles of degree $2g - 2$ on a curve of genus g. Since

$$\dim K_{d-4,2}(C, K_C \otimes \eta) = \dim K_{d-3,1}(C, K_C \otimes \eta),$$

it follows that the failure locus of Conjecture 1.4 is a virtual divisor in the universal degree 0 Picard variety $\mathfrak{Pic}^0_g \to \mathcal{M}_g$. Thus it predicts that the non-vanishing locus

$$\{[C,\eta] \in \mathfrak{Pic}^0_g : K_{d-4,2}(C, K_C \otimes \eta) \neq 0\}$$

is an "honest" divisor on $\mathfrak{Pic}^0_g$. Conjecture 1.4 is also sharp in the sense that from the Green-Lazarsfeld Non-Vanishing Theorem 2.8 it follows that

$$K_{d-2,1}(C, K_C \otimes \eta) \neq 0.$$

Similarly, always $K_{d-3,2}(C, K_C \otimes \eta) \neq 0$ for all $[C,\eta] \in \mathfrak{Pic}^0_g$. A yet stronger conjecture is the following vanishing statement for l-roots of trivial bundles on curves:

CONJECTURE 1.5. *Let C be a general curve of genus $g = 2d-2 \geq 6$. Then for every prime l and every line bundle $\eta \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$ satisfying $\eta^{\otimes l} = \mathcal{O}_C$, we have that $K_{d-4,2}(C, K_C \otimes \eta) = 0$.*

In order to prove Conjecture 1.5 it suffices to exhibit a single pair $[C,\eta]$ as above, for which $K_C \otimes \eta \in \mathrm{Pic}^{2g-2}(C)$ satisfies property (N_{d-4}). The case most studied so far is that of level $l = 2$, when one recovers the *Prym-Green Conjecture* [**FaL08**] which has been checked using Macaulay2 for $g \leq 14$. The Prym-Green Conjecture is a subtle statement which for small values of g is equivalent to the Prym-Torelli Theorem, again see [**FaL08**]. Since the condition $K_{d-4,2}(C, K_C \otimes \eta) \neq 0$ is divisorial in moduli, Conjecture 1.5 is of great help in the study of the birational geometry of the compactification $\overline{\mathcal{R}}_{g,l} := \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_l)$ of the moduli space $\mathcal{R}_{g,l}$ classifying pairs $[C,\eta]$, where $[C] \in \mathcal{M}_g$ and $\eta \in \mathrm{Pic}^0(C)$ satisfies $\eta^{\otimes l} = \mathcal{O}_C$.

Concerning both Conjectures 1.4 and 1.5, it is an open problem to find an analogue of the Clifford index of the curve, in the sense that the classical Green Conjecture is not only a Koszul cohomology vanishing statement but also allows one to read off the Clifford index from a non-vanishing statement for $K_{p,2}(C, K_C)$. It is an interesting open question to find a *Prym-Clifford index* playing the same role as the original $\mathrm{Cliff}(C)$ in (1) and to describe it in terms of the corresponding Prym varieties: Is there a geometric characterization of those Prym varieties $\mathcal{P}_g([C,\eta]) \in \mathcal{A}_{g-1}$ corresponding to pairs $[C,\eta] \in \mathcal{R}_g$ with $K_{p,2}(C, K_C \otimes \eta) \neq 0$?

Another recent development on syzygies of curves came from a completely different direction, with the realization that loci in the moduli space $\mathcal{M}_g$ consisting of curves having exceptional syzygies can be used effectively to answer questions about the birational geometry of the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g, cf. [**FaPo05**], [**Fa06a**], [**Fa06b**], in particular, to produce infinite series of effective divisors on $\overline{\mathcal{M}}_g$ violating the Harris-Morrison Slope Conjecture [**HaM90**]. We recall that the slope $s(D)$ of an effective divisor D on $\overline{\mathcal{M}}_g$ is defined as the smallest rational number a/b with $a, b \geq 0$, such that the class $a\lambda - b(\delta_0 + \cdots + \delta_{[g/2]}) - [D] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ is an effective $\mathbb{Q}$-combination of boundary divisors. The Slope Conjecture [**HaM90**] predicts a lower bound for the slope of effective divisors on $\overline{\mathcal{M}}_g$,

$$s(\overline{\mathcal{M}}_g) := \inf_{D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) \geq 6 + \frac{12}{g+1},$$

with equality precisely when $g+1$ is composite; the quantity $6 + 12/(g+1)$ is the slope of the Brill-Noether divisors on $\overline{\mathcal{M}}_g$, in case such divisors exist. A first counterexample to the Slope Conjecture was found in [**FaPo05**]: The locus

$$\mathcal{K}_{10} := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}$$

can be interpreted as being set-theoretically equal to the locus of curves $[C] \in \mathcal{M}_{10}$ carrying a linear series $L \in W^4_{12}(C)$ such that the multiplication map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is not an isomorphism, or equivalently $K_{0,2}(C, L) \neq 0$. The main advantage of this Koszul-theoretic description is that it provides a characterization of the $K3$ divisor $\mathcal{K}_{10}$ in a way that makes no reference to $K3$ surfaces and can be easily generalized to other genera. Using this characterization one shows that $s(\overline{\mathcal{K}}_{10}) = 7 < 78/11$, that is, $\overline{\mathcal{K}}_{10} \in \mathrm{Eff}(\overline{\mathcal{M}}_{10})$ is a counterexample to the Slope Conjecture.

Koszul cohomology provides an effective way of constructing cycles on $\mathcal{M}_g$. Under suitable numerical conditions, loci of the type

$$\mathcal{Z}_{g,2} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \text{ such that } K_{p,2}(C, L) \neq 0\}$$

are virtual divisors on $\mathcal{M}_g$, that is, degeneracy loci of morphisms between vector bundles of the same rank over $\mathcal{M}_g$. The problem of extending these vector bundes over $\overline{\mathcal{M}}_g$ and computing the virtual classes of the resulting degeneracy loci is in general daunting, but has been solved successfully in the case $\rho(g, r, d) = 0$, cf. [**Fa06b**]. Suitable vanishing statements of the Koszul cohomology for general curves (e.g. Conjectures 1.4, 5.4) show that, when applicable, these virtual Koszul divisors are actual divisors and they are quite useful in specific problems such as the Slope Conjecture or showing that certain moduli spaces of curves (with or without level structure) are of general type, see [**Fa08**], [**FaL08**]. A picturesque application of the Koszul technique in the study of parameter spaces is the following result about the birational type of the moduli space of Prym varieties $\overline{\mathcal{R}}_g = \overline{\mathcal{R}}_{g,2}$, see [**FaL08**]:

Theorem 1.6. *The moduli space $\overline{\mathcal{R}}_g$ is of general type for $g \geq 13$ and $g \neq 15$.*

The proof of Theorem 1.6 depends on the parity of g. For $g = 2d - 2$, it boils down to calculating the class of the compactification in $\overline{\mathcal{R}}_g$ of the failure locus of the Prym-Green Conjecture, that is, of the locus

$$\{[C, \eta] \in \mathcal{R}_{2d-2} : K_{d-4,2}(C, K_C \otimes \eta) \neq 0\}.$$

For odd $g = 2d - 1$, one computes the class of a "mixed" Koszul cohomology locus in $\overline{\mathcal{R}}_g$ defined in terms of Koszul cohomology groups of K_C with values in $K_C \otimes \eta$.

The outline of the paper is as follows. In Section 2 we review the definition of Koszul cohomology as introduced by M. Green [**Gr84a**] and discuss basic facts. In Section 3 we recall the construction of (virtual) Koszul cycles on $\overline{\mathcal{M}}_g$ following [**Fa06a**] and [**Fa06b**] and explain how their cohomology classes can be calculated. In Section 4 we discuss a number of conjectures on syzygies of curves, starting with Green's Conjecture and the Gonality Conjecture and continuing with the Prym-Green Conjecture. We end by proposing in Section 5 a strong version of the Maximal Rank Conjecture.

Some results stated in [**Ap04**], [**Ap05**] are discussed here in greater detail. Other results are new (see Theorems 4.9 and 4.16).

2. Koszul cohomology

2.1. Syzygies. Let V be an n-dimensional complex vector space, $S := S(V)$ the symmetric algebra of V, and $X \subset \mathbb{P}V^\vee := \mathrm{Proj}(S)$ a non-degenerate subvariety, and denote by $S(X)$ the homogeneous coordinate ring of X. To the embedding of X in $\mathbb{P}V^\vee$, one associates the *Hilbert function*, defined by

$$h_X(d) := \dim_{\mathbb{C}} \left(S(X)_d\right)$$

for any positive integer d. A remarkable property of h_X is its polynomial behavior for large values of d. It is a consequence of the existence of a *graded minimal resolution* of the S-module $S(X)$, which is an exact sequence

$$0 \to E_s \to \cdots \to E_2 \to E_1 \to S \to S(X) \to 0$$

with

$$E_p = \bigoplus_{j>p} S(-j)^{\beta_{pj}(X)}.$$

The Hilbert function of X is then given by

$$h_X(d) = \sum_{p,j} (-1)^p \beta_{pj}(X) \left(\begin{array}{c} n+d-j \\ n \end{array} \right), \tag{2}$$

where

$$\left(\begin{array}{c} t \\ n \end{array} \right) = \frac{t(t-1)\cdots(t-n+1)}{n\,!}, \text{ for any } t \in \mathbf{R},$$

and note that the expression on the right-hand side is polynomial for large d. This reasoning leads naturally to the definition of *syzygies of* X, which are the graded components of the graded S-modules E_p. The integers

$$\beta_{pj}(X) = \dim_{\mathbb{C}} \mathrm{Tor}^j(S(X), \mathbb{C})_p$$

are called the *graded Betti numbers of* X and determine completely the Hilbert function, according to formula (2). Sometimes we also write $b_{i,j}(X) := \beta_{i,i+j}(X)$.

One main difficulty in developing syzygy theory was to find effective geometric methods for computing these invariants. In the eighties, M. Green and R. Lazarsfeld published a series of papers, [**Gr84a**], [**Gr84b**], [**GL84**], [**GL86**] that shed a new light on syzygies. Contrary to the classical point of view, they look at integral closures of the homogeneous coordinates rings, rather than at the rings themselves. This approach, using intensively the language of Koszul cohomology, led to a number of beautiful geometrical results with numerous applications in classical algebraic geometry as well as moduli theory.

2.2. Definition of Koszul cohomology. Throughout this paper, we follow M. Green's approach to Koszul cohomology [**Gr84a**]. The general setup is the following. Suppose X is a complex projective variety, $L \in \mathrm{Pic}(X)$ a line bundle, $\mathcal{F}$ is a coherent sheaf on X, and $p, q \in \mathbb{Z}$. The canonical contraction map

$$\bigwedge^{p+1} H^0(X,L) \otimes H^0(X,L)^\vee \to \bigwedge^{p} H^0(X,L),$$

acting on tensors as

$$(s_0 \wedge \cdots \wedge s_p) \otimes \sigma \mapsto \sum_{i=0}^{p} (-1)^i \sigma(s_i)(s_0 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_p),$$

and the multiplication map

$$H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^{q-1})\to H^0(X,\mathcal{F}\otimes L^q),$$

define together a map

$$\bigwedge^{p+1} H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^{q-1})\to \bigwedge^{p} H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^q).$$

In this way, we obtain a complex (called the *Koszul complex*)

$$\wedge^{p+1}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q-1})\to \wedge^{p}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q})\to \wedge^{p-1}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q+1}),$$

whose cohomology at the middle term is denoted by $K_{p,q}(X,\mathcal{F},L)$. In the particular case $\mathcal{F}\cong\mathcal{O}_X$, to ease the notation, one drops $\mathcal{O}_X$ and writes directly $K_{p,q}(X,L)$ for Koszul cohomology.

Here are some samples of direct applications of Koszul cohomology:

Example 2.1. If L is ample, then L is normally generated if and only if $K_{0,q}(X,L)=0$ for all $q\geq 2$. This fact follows directly from the definition.

Example 2.2. If L is globally generated with $H^1(X,L)=0$, then

$$H^1(X,\mathcal{O}_X)\cong K_{h^0(L)-2,2}(X,L)$$

see [**ApN08**]. In particular, the genus of a curve can be read off (vanishing of) Koszul cohomology with values in non-special bundles. More generally, under suitable vanishing assumptions on L, all the groups $H^i(X,\mathcal{O}_X)$ can be computed in a similar way, and likewise $H^i(X,\mathcal{F})$ for an arbitrary coherent sheaf $\mathcal{F}$, cf. [**ApN08**].

Example 2.3. If L is very ample the *Castelnuovo-Mumford regularity* can be recovered from Koszul cohomology. Specifically, if $\mathcal{F}$ is a coherent sheaf on X, then

$$\mathrm{reg}_L(\mathcal{F})=\min\{m : K_{p,m+1}(X,\mathcal{F},L)=0, \text{ for all } p\}.$$

As a general principle, any invariant that involves multiplication maps is presumably related to Koszul cohomology.

Very surprisingly, Koszul cohomology interacts much closer with the geometry of the variety than might have been expected. This phenomenon was discovered by Green and Lazarsfeld [**Gr84a**, Appendix]:

Theorem 2.4 (Green-Lazarsfeld). *Suppose X is a smooth variety and consider a decomposition $L=L_1\otimes L_2$ with $h^0(X,L_i)=r_i+1\geq 2$ for $i\in\{1,2\}$. Then $K_{r_1+r_2-1,1}(X,L)\neq 0$.*

In other words, non-trivial geometry implies non-trivial Koszul cohomology. We shall discuss the case of curves, which is the most relevant, and then some consequences in Section 4.

Many problems in the theory of syzygies involve vanishing/nonvanishing of Koszul cohomology. One useful definition is the following.

Definition 2.5. An ample line bundle L on a projective variety is said to satisfy the property (N_p) if and only if $K_{i,q}(X,L)=0$ for all $i\leq p$ and $q\geq 2$.

From the geometric viewpoint, the property (N_p) means that L is normally generated, the ideal of X in the corresponding embedding is generated by quadrics, and all the syzygies up to order p are linear. In many cases, for example canonical curves, or 2-regular varieties, the property (N_p) reduces to the single condition $K_{p,2}(X, L) = 0$, see e.g. [**Ein87**]. This phenomenon justifies the study of various loci given by the nonvanishing of $K_{p,2}$, see Section 4.

2.3. Kernel bundles. The proofs of the facts discussed in examples 2.2 and 2.3 use the kernel bundles description which is due to Lazarsfeld [**La89**]: Consider L a globally generated line bundle on the projective variety X, and set

$$M_L := \operatorname{Ker}\left(H^0(X, L) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} L\right).$$

Note that M_L is a vector bundle on X of rank $h^0(X, L) - 1$. For any coherent sheaf $\mathcal{F}$ on X, and integers $p \geq 0$, and $q \in \mathbb{Z}$, we have a short exact sequence on X

$$0 \to \wedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1} \to \wedge^{p+1} H^0(L) \otimes \mathcal{F} \otimes L^{q-1} \to \wedge^p M_L \otimes \mathcal{F} \otimes L^q \to 0.$$

Taking global sections, we remark that the Koszul differential factors through the map

$$\wedge^{p+1} H^0(X, L) \otimes H^0(X, \mathcal{F} \otimes L^{q-1}) \to H^0(X, \wedge^p M_L \otimes \mathcal{F} \otimes L^q),$$

hence we have the following characterization of Koszul cohomology, [**La89**]:

Theorem 2.6 (Lazarsfeld). *Notation as above. We have*

$$\begin{aligned} K_{p,q}(X, \mathcal{F}, L) &\cong \operatorname{coker}\big(\wedge^{p+1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q-1}) \to H^0(\wedge^p M_L \otimes \mathcal{F} \otimes L^q)\big) \\ &\cong \ker\big(H^1(X, \wedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1}) \to \wedge^{p+1} H^0(L) \otimes H^1(\mathcal{F} \otimes L^{q-1})\big). \end{aligned}$$

Theorem 2.6 has some nice direct consequences. The first one is a duality theorem which was proved in [**Gr84a**].

Theorem 2.7. *Let L be a globally generated line bundle on a smooth projective variety X of dimension n. Set $r := \dim |L|$. If*

$$H^i(X, L^{q-i}) = H^i(X, L^{q-i+1}) = 0, \quad i = 1, \ldots, n-1$$

then

$$K_{p,q}(X, L)^\vee \cong K_{r-n-p, n+1-q}(X, K_X, L).$$

Another consequence of Theorem 2.6, stated implicitly in [**Fa06b**] without proof, is the following:

Theorem 2.8. *Let L be a non-special globally generated line bundle on a smooth curve C of genus $g \geq 2$. Set $d = \deg(L)$, $r = h^0(C, L) - 1$, and consider $1 \leq p \leq r$. Then*

$$\dim\, K_{p,1}(C, L) - \dim\, K_{p-1,2}(C, L) = p \cdot \binom{d-g}{p} \left(\frac{d+1-g}{p+1} - \frac{d}{d-g}\right).$$

In particular, if

$$p < \frac{(d+1-g)(d-g)}{d} - 1,$$

then $K_{p,1}(C, L) \neq 0$, and if

$$\frac{(d+1-g)(d-g)}{d} \leq p \leq d - g,$$

then $K_{p-1,2}(C, L) \neq 0$.

Proof. Since we work with a spanned line bundle on a curve, Theorem 2.7 applies, hence we have

$$K_{p,1}(C,L)\cong K_{r-p-1,1}(C,K_C,L)^\vee$$

and

$$K_{p-1,2}(C,L)\cong K_{r-p,0}(C,K_C,L)^\vee.$$

Set, as usual, $M_L = \mathrm{Ker}\{H^0(C,L)\otimes\mathcal{O}_C \to L\}$, and consider the Koszul complex

$$0\to \wedge^{r-p}H^0(C,L)\otimes H^0(C,K_C)\to H^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)\to 0. \tag{3}$$

Since L is non-special, $K_{r-p,0}(C,K_C,L)$ is isomorphic to the kernel of the differential appearing in (3), hence the difference which we wish to compute coincides with the Euler characteristic of the complex (3).

Next we determine $h^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)$. Note that $\mathrm{rk}(M_L)=r$ and $\wedge^{r-p-1}M_L\otimes L\cong\wedge^{p+1}M_L^\vee$. In particular, since $H^0(C,\wedge^{p+1}M_L)\cong K_{p+1,0}(C,L)=0$, we obtain

$$h^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)=-\chi(C,\wedge^{p+1}M_L).$$

Observe that

$$\deg(\wedge^{p+1}M_L)=\deg(M_L)\binom{r-1}{p}=-d\binom{r-1}{p}$$

and

$$\mathrm{rk}(\wedge^{p+1}M_L)=\binom{r}{p+1}$$

From the Riemann-Roch Theorem it follows that

$$-\chi(C,\wedge^{p+1}M_L)=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}$$

and hence

$$\dim K_{p,1}(C,L)-\dim K_{p-1,2}(C,L)=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}-g\binom{r+1}{p+1}.$$

The formula is obtained by replacing r by $d-g$. □

Remark 2.9. A full version of Theorem 2.8 for special line bundles can be obtained in a similar manner by adding alternating sums of other groups $K_{p-i,i+1}$. For example, if $L^{\otimes 2}$ is non-special, then from the complex

$$0\to\wedge^{r-p+1}H^0(C,L)\otimes H^0(C,K_C\otimes L^{-1})\to\wedge^{r-p}H^0(C,L)\otimes H^0(C,K_C)\to$$

$$\to H^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)\to 0$$

we obtain the following conclusion:

$$\dim K_{p,1}(C,L)-\dim K_{p-1,2}(C,L)+\dim K_{p-2,3}(C,L)=$$

$$=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}-g\binom{r+1}{p+1}+\binom{r+1}{p}(r-d+g).$$

2.4. Hilbert schemes. Suppose X is a smooth variety, and consider $L \in \mathrm{Pic}(X)$. A novel description of the Koszul cohomology of X with values in L was provided in [**V02**] via the Hilbert scheme of points on X.

Denote by $X^{[n]}$ the Hilbert scheme parameterizing zero-dimensional length n subschemes of X, let $X^{[n]}_{\mathrm{curv}}$ be the open subscheme parameterizing curvilinear length n subschemes, and let

$$\Xi_n \subset X^{[n]}_{\mathrm{curv}} \times X$$

be the incidence subscheme with projection maps $p : \Xi_n \to X$ and $q : \Xi_n \to X^{[n]}_{\mathrm{curv}}$. For a line bundle L on X, the sheaf $L^{[n]} := q_* p^* L$ is locally free of rank n on $X^{[n]}_{\mathrm{curv}}$, and the fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$ is isomorphic to $H^0(\xi, L \otimes \mathcal{O}_\xi)$.

There is a natural map

$$H^0(X, L) \otimes \mathcal{O}_{X^{[n]}_{\mathrm{curv}}} \to L^{[n]},$$

acting on the fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$, by $s \mapsto s|_\xi$. In [**V02**] and [**EGL**] it is shown that, by taking wedge powers and global sections, this map induces an isomorphism:

$$\wedge^n H^0(X, L) \cong H^0(X^{[n]}_{\mathrm{curv}}, \det\, L^{[n]}).$$

Voisin proves that there is an injective map

$$H^0(\Xi_{p+1}, \det\, L^{[p+1]} \boxtimes L^{q-1}) \to \wedge^p H^0(X, L) \otimes H^0(X, L^q)$$

whose image is isomorphic to the kernel of the Koszul differential. This eventually leads to the following result:

THEOREM 2.10 (Voisin [**V02**]). *For all integers p and q, the Koszul cohomology $K_{p,q}(X, L)$ is isomorphic to the cokernel of the restriction map*

$$H^0(X^{[p+1]}_{\mathrm{curv}} \times X, \det\, L^{[p+1]} \boxtimes L^{q-1}) \to H^0(\Xi_{p+1}, \det\, L^{[p+1]} \boxtimes L^{q-1}|_{\Xi_{p+1}}).$$

In particular,

$$K_{p,1}(X, L) \cong \mathrm{coker}(H^0(X^{[p+1]}_{\mathrm{curv}}, \det\, L^{[p+1]}) \xrightarrow{q^*} H^0(\Xi_{p+1}, q^*\det\, L^{[p+1]}|_{\Xi_{p+1}})).$$

REMARK 2.11. The group $K_{p,q}(X, \mathcal{F}, L)$ is obtained by replacing L^{q-1} by $\mathcal{F} \otimes L^{q-1}$ in the statement of Theorem 2.10.

The main application of this approach is the proof of the generic Green Conjecture [**V02**]; see Subsection 4.1 for a more detailed discussion on the subject. The precise statement is the following.

THEOREM 2.12 (Voisin [**V02**] and [**V05**]). *Consider a smooth projective $K3$ surface S, such that $\mathrm{Pic}(S)$ is isomorphic to $\mathbb{Z}^2$, and is freely generated by L and $\mathcal{O}_S(\Delta)$, where Δ is a smooth rational curve such that $\deg(L_{|\Delta}) = 2$, and L is a very ample line bundle with $L^2 = 2g - 2$, $g = 2k + 1$. Then $K_{k+1,1}\big(S, L \otimes \mathcal{O}_S(\Delta)\big) = 0$ and*

$$K_{k,1}(S, L) = 0. \tag{4}$$

Voisin's result, apart from settling the Generic Green Conjecture, offers the possibility (via the cohomological calculations carried out in [**Fa06b**], see also Section 3), to give a much shorter proof of the Harris-Mumford Theorem [**HM82**] on the Kodaira dimension of $\overline{\mathcal{M}}_g$ in the case of odd genus. This proof does not use intersection theory on the stack of admissible coverings at all and is considerably shorter than the original proof. This approach has been described in full detail in [**Fa08**].

3. Geometric cycles on the moduli space

3.1. Brill-Noether cycles. We recall a few basic facts from Brill-Noether theory; see [**ACGH85**] for a general reference.

For a smooth curve C and integers $r, d \geq 0$ one considers the Brill-Noether locus

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}$$

as well as the variety of linear series of type $\mathfrak{g}^r_d$ on C, that is,

$$G^r_d(C) := \{(L, V) : L \in W^r_d(C), V \in \mathbf{G}(r + 1, H^0(L))\}.$$

The locus $W^r_d(C)$ is a deteminantal subvariety of $\mathrm{Pic}^d(C)$ of expected dimension equal to the *Brill-Noether number* $\rho(g, r, d) = g - (r + 1)(g - d + r)$. According to the Brill-Noether Theorem for a general curve $[C] \in \mathcal{M}_g$, both $W^r_d(C)$ and $G^r_d(C)$ are irreducible varieties of dimension

$$\dim\, W^r_d(C) = \dim\, G^r_d(C) = \rho(g, r, d).$$

In particular, $W^r_d(C) = \emptyset$ when $\rho(g, r, d) < 0$. By imposing the condition that a curve carry a linear series $\mathfrak{g}^r_d$ when $\rho(g, r, d) < 0$, one can define a whole range of geometric subvarieties of $\mathcal{M}_g$.

We introduce the Deligne-Mumford stack $\sigma : \mathfrak{G}^r_d \to \mathbf{M}_g$ classifying pairs $[C, l]$ where $[C] \in \mathcal{M}_g$ and $l = (L, V) \in G^r_d(C)$ is a linear series $\mathfrak{g}^r_d$, together with the projection $\sigma[C, l] := [C]$. The stack $\mathfrak{G}^r_d$ has a determinantal structure inside a Grassmann bundle over the universal Picard stack $\mathfrak{Pic}^d_g \to \mathbf{M}_g$. In particular, each irreducible component of $\mathfrak{G}^r_d$ has dimension at least $3g-3+\rho(g, r, d)$, cf. [**AC81b**]. We define the *Brill-Noether cycle*

$$\mathcal{M}^r_{g,d} := \sigma_*(\mathfrak{G}^r_d) = \{[C] \in \mathcal{M}_g : W^r_d(C) \neq \emptyset\},$$

together with the substack structure induced from the determinantal structure of $\mathfrak{G}^r_d$ via the morphism σ. A result of Steffen [**St98**] guarantees that each irreducible component of $\mathcal{M}^r_{g,d}$ has dimension at least $3g - 3 + \rho(g, r, d)$.

When $\rho(g, r, d) = -1$, Steffen's result coupled with the Brill-Noether Theorem implies that the cycle $\mathcal{M}^r_{g,d}$ is pure of codimension 1 inside $\mathcal{M}_g$. One has the following more precise statement due to Eisenbud and Harris [**EH89**]:

THEOREM 3.1. *For integers g, r and d such that $\rho(g, r, d) = -1$, the locus $\mathcal{M}^r_{g,d}$ is an irreducible divisor on $\mathcal{M}_g$. The class of its compactification $\overline{\mathcal{M}}^r_{g,d}$ inside $\overline{\mathcal{M}}_g$ is given by the following formula:*

$$\overline{\mathcal{M}}^r_{g,d} \equiv c_{g,d,r} \left((g + 3)\lambda - \frac{g + 1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g - i)\delta_i \right) \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

The constant $c_{g,d,r}$ has a clear intersection-theoretic interpretation using Schubert calculus. Note that remarkably, the slope of all the Brill-Noether divisors on $\overline{\mathcal{M}}_g$ is independent of d and r and

$$s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g + 1}$$

for all $r, d \geq 1$ satisfying $\rho(g, r, d) = -1$. For genera g such that $g+1$ is composite, one has as many Brill-Noether divisors on $\overline{\mathcal{M}}_g$ as ways of non-trivially factoring $g+1$. It is natural to raise the following:

PROBLEM 3.2. Construct an explicit linear equivalence between various Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ on $\overline{\mathcal{M}}_g$ for different integers $r, d \geq 1$ with $\rho(g, r, d) = -1$.

The simplest case is $g = 11$ when there exist two (distinct) Brill-Noether divisors $\overline{\mathcal{M}}^1_{11,6}$ and $\overline{\mathcal{M}}^2_{11,9}$ and $s(\overline{\mathcal{M}}^1_{11,6}) = s(\overline{\mathcal{M}}^2_{11,9}) = 7$. These divisors can be understood in terms of Noether-Lefschetz divisors on the moduli space $\overline{\mathcal{F}}_{11}$ of polarized $K3$ surfaces of degree $2g-2 = 20$. We recall that there exists a rational $\mathbb{P}^{11}$-fibration

$$\phi : \overline{\mathcal{M}}_{11} \dashrightarrow \overline{\mathcal{F}}_{11}, \quad \phi[C] := [S, \mathcal{O}_S(C)],$$

where S is the unique $K3$ surface containing C, see [**M94**]. Noting that $\mathcal{M}^1_{11,6} = \mathcal{M}^5_{11,14}$ it follows that

$$\mathcal{M}^1_{11,6} = \phi^*_{|\mathcal{M}_{11}}(NL_1),$$

where NL_1 is the Noether-Lefschetz divisor on $\mathcal{F}_{11}$ of polarized $K3$ surfaces S with Picard lattice $\mathrm{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)] + \mathbb{Z} \cdot [C]$, where $C^2 = 20$ and $C \cdot c_1(\mathcal{O}_S(1)) = 14$. Similarly, by Riemann-Roch, we have an equality of divisors $\mathcal{M}^2_{11,9} = \mathcal{M}^3_{11,11}$, and then

$$\mathcal{M}^2_{11,9} = \phi^*_{|\mathcal{M}_{11}}(NL_2),$$

with NL_2 being the Noether-Lefschetz divisor whose general point corresponds to a quartic surface $S \subset \mathbb{P}^3$ with $\mathrm{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)] + \mathbb{Z} \cdot [C]$, where $C^2 = 20$ and $C \cdot c_1(\mathcal{O}_S(1)) = 11$. It is not clear whether NL_1 and NL_2 should be linearly equivalent on $\mathcal{F}_{11}$.

The next interesting case is $g = 23$, see [**Fa00**]: The three (distinct) Brill-Noether divisors $\overline{\mathcal{M}}^1_{23,12}$, $\overline{\mathcal{M}}^2_{23,17}$ and $\overline{\mathcal{M}}^1_{23,20}$ are multicanonical in the sense that there exist explicitly known integers $m, m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ and an effective boundary divisor $E \equiv \sum_{i=1}^{11} c_i \delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{23})$ such that

$$m_1 \cdot \overline{\mathcal{M}}^1_{23,12} + E \equiv m_2 \cdot \overline{\mathcal{M}}^2_{23,17} + E = m_3 \cdot \overline{\mathcal{M}}^3_{23,20} + E \in |mK_{\overline{\mathcal{M}}_{23}}|.$$

QUESTION 3.3. For a genus g such that $g+1$ is composite, is there a good geometric description of the stable base locus

$$\mathbf{B}\big(\overline{\mathcal{M}}_g, |\overline{\mathcal{M}}^r_{g,d}|\big) := \bigcap_{n \geq 0} \mathrm{Bs}(\overline{\mathcal{M}}_g, |n\overline{\mathcal{M}}^r_{g,d}|)$$

of the Brill-Noether linear system? It is clear that $\mathbf{B}(\overline{\mathcal{M}}_g, |\overline{\mathcal{M}}^r_{g,d}|)$ contains important subvarieties of $\overline{\mathcal{M}}_g$ like the hyperelliptic and trigonal locus, cf. [**HaM90**].

Of the higher codimension Brill-Noether cycles, the best understood are the d-gonal loci

$$\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \cdots \subset \mathcal{M}^1_{g,d} \subset \cdots \subset \mathcal{M}_g.$$

Each stratum $\mathcal{M}^1_{g,d}$ is an irreducible variety of dimension $2g + 2d - 5$. The gonality stratification of $\mathcal{M}_g$, apart from being essential in the statement of Green's Conjecture, has often been used for cohomology calculations or for bounding the cohomological dimension and the affine covering number of $\mathcal{M}_g$.

3.2. Koszul cycles. Koszul cohomology behaves like the usual cohomology in many regards. Notably, it can be computed in families, see [**BG85**], or the book [**ApN08**]:

THEOREM 3.4. *Let $f : X \to S$ be a flat family of projective varieties, parameterized by an integral scheme, $L \in \mathrm{Pic}(X/S)$ a line bundle and $p, q \in \mathbb{Z}$. Then there exists a coherent sheaf $\mathcal{K}_{p,q}(X/S, L)$ on X and a nonempty Zariski open subset $U \subset S$ such that for all $s \in U$ one has that $\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$.*

In the statement above, the open set U is precisely determined by the condition that all $h^i(X_s, L_s)$ are minimal.

By Theorem 3.4, Koszul cohomology can be used to construct effective determinantal cycles on the moduli spaces of smooth curves. This works particularly well for Koszul cohomology of canonical curves, as h^i remain constant over the whole moduli space. More generally, Koszul cycles can be defined over the relative Picard stack over the moduli space. Under stronger assumptions, the canonically defined determinantal structure can be given a better description. To this end, one uses the description provided by Lazarsfeld kernel bundles.

In many cases, for example canonical curves, or 2-regular varieties, the property (N_p) reduces to the single condition $K_{p,2}(X, L) = 0$, see for instance [**Ein87**] Proposition 3. This phenomenon justifies the study of various loci given by the non-vanishing of $K_{p,2}$. Note however that extending this determinantal description over the boundary of the moduli stack (especially over the locus of reducible stable curves) poses considerable technical difficulties, see [**Fa06a**], [**Fa06b**]. We now describe a general set-up used to compute Koszul non-vanishing loci over a partial compactification $\widetilde{\mathcal{M}}_g$ of the moduli space $\mathcal{M}_g$ inside $\overline{\mathcal{M}}_g$. As usual, if $\mathbf{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its associated coarse moduli space.

We fix integers $r, d \geq 1$, such that $\rho(g, r, d) = 0$ and denote by $\mathbf{M}_g^0 \subset \mathbf{M}_g$ the open substack classifying curves $[C] \in \mathcal{M}_g$ such that $W^r_{d-1}(C) = \emptyset$ and $W^{r+1}_d(C) = \emptyset$. Since $\rho(g, r+1, d) \leq -2$ and $\rho(g, r, d-1) = -r-1 \leq -2$, it follows from [**EH89**] that $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}_g^0, \mathcal{M}_g) \geq 2$. We further denote by $\Delta_0^0 \subset \Delta_0 \subset \overline{\mathcal{M}}_g$ the locus of nodal curves $[C_{yq} := C/y \sim q]$, where $[C] \in \mathcal{M}_{g-1}$ is a curve that satisfies the Brill-Noether Theorem and $y, q \in C$ are *arbitrary* distinct points. Finally, $\Delta_1^0 \subset \Delta_1 \subset \overline{\mathcal{M}}_g$ denotes the open substack classifying curves $[C \cup_y E]$, where $[C] \in \mathcal{M}_{g-1}$ is Brill-Noether general, $y \in C$ is an arbitrary point and $[E, y] \in \overline{\mathcal{M}}_{1,1}$ is an arbitrary elliptic tail. Note that every Brill-Noether general curve $[C] \in \mathcal{M}_{g-1}$ satisfies

$$W^r_{d-1}(C) = \emptyset, \quad W^{r+1}_d(C) = \emptyset \ \text{ and } \ \dim W^r_d(C) = \rho(g-1, r, d) = r.$$

We set $\widetilde{\mathbf{M}}_g := \mathbf{M}_g^0 \cup \Delta_0^0 \cup \Delta_1^0 \subset \overline{\mathbf{M}}_g$ and we regard it as a partial compactification of $\mathbf{M}_g$. Then following [**EH86**] we consider the Deligne-Mumford stack

$$\sigma_0 : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathbf{M}}_g$$

classifying pairs $[C, l]$ with $[C] \in \widetilde{\mathcal{M}}_g$ and l a limit linear series of type $\mathfrak{g}^r_d$ on C. We remark that for any curve $[C] \in \mathcal{M}_g^0 \cup \Delta_0^0$ and $L \in W^r_d(C)$, we have that $h^0(C, L) = r + 1$ and that L is globally generated. Indeed, for a smooth curve $[C] \in \mathcal{M}_g^0$ it follows that $W^{r+1}_d(C) = \emptyset$, so necessarily $W^r_d(C) = G^r_d(C)$. For a point $[C_{yq}] \in \Delta_0^0$ we have the identification

$$\sigma_0^{-1}[C_{yq}] = \{L \in W_d^r(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r\},$$

where we note that since the normalization $[C] \in \mathcal{M}_{g-1}$ is assumed to be Brill-Noether general, any sheaf $L \in \sigma_0^{-1}[C_{yq}]$ satisfies

$$h^0(C, L \otimes \mathcal{O}_C(-y)) = h^0(C, L \otimes \mathcal{O}_C(-q)) = r$$

and $h^0(C, L) = r+1$. Furthermore, $\overline{W}_d^r(C_{yq}) = W_d^r(C_{yq})$, where the left-hand-side denotes the closure of $W_d^r(C_{yq})$ inside the variety $\overline{\text{Pic}}^d(C_{yq})$ of torsion-free sheaves on C_{yq}. This follows because a non-locally free torsion-free sheaf in $\overline{W}_d^r(C_{yq}) - W_d^r(C_{yq})$ is of the form $\nu_*(A)$, where $A \in W_{d-1}^r(C)$ and $\nu : C \to C_{yq}$ is the normalization map. But we know that $W_{d-1}^r(C) = \emptyset$, because $[C] \in \mathcal{M}_{g-1}$ satisfies the Brill-Noether Theorem. The conclusion of this discussion is that $\sigma : \widetilde{\mathfrak{G}}_d^r \to \widetilde{\mathbf{M}}_g$ is proper. Since $\rho(g, r, d) = 0$, by general Brill-Noether theory there exists a unique irreducible component of $\mathfrak{G}_d^r$ which maps onto $\mathbf{M}_g^0$.

In [**Fa06b**], a universal Koszul non-vanishing locus over a partial compactification of the moduli space of curves is introduced. Precisely, one constructs two locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ over $\widetilde{\mathfrak{G}}_d^r$ such that for a point $[C, l]$ corresponding to a *smooth* curve $[C] \in \mathcal{M}_g^0$ and a (necessarily complete and globally generated linear series) $l = (L, H^0(C, L)) \in G_d^r(C)$ inducing a map $C \xrightarrow{|L|} \mathbb{P}^r$, we have the following description of the fibres:

$$\mathcal{A}(C, L) = H^0\big(\mathbb{P}^r, \wedge^p M_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(2)\big) \text{ and } \mathcal{B}(C, L) = H^0\big(C, \wedge^p M_L \otimes L^{\otimes 2}\big).$$

There is a natural vector bundle morphism $\phi : \mathcal{A} \to \mathcal{B}$ given by restriction. From Grauert's Theorem it follows that both $\mathcal{A}$ and $\mathcal{B}$ are vector bundles over $\mathfrak{G}_d^r$ and from Bott's Theorem (in the case of $\mathcal{A}$) and Riemann-Roch (in the case of $\mathcal{B}$) respectively, we compute their ranks

$$\text{rank}(\mathcal{A}) = (p+1)\binom{r+2}{p+2} \text{ and } \text{rank}(\mathcal{B}) = \binom{r}{p}\Big(-\frac{pd}{r} + 2d + 1 - g\Big).$$

Note that M_L is a stable vector bundle (again, one uses that $[C] \in \mathcal{M}_g^0$), hence $H^1(C, \wedge^p M_L \otimes L^{\otimes 2}) = 0$ and then $\text{rank}(\mathcal{B}) = \chi(C, \wedge^p M_L \otimes L^{\otimes 2})$ can be computed from Riemann-Roch. We have the following result, cf. [**Fa06b**] Theorem 2.1:

THEOREM 3.5. *The cycle*

$$\mathcal{U}_{g,p} := \{(C, L) \in \mathfrak{G}_d^r : K_{p,2}(C, L) \neq 0\},$$

is the degeneracy locus of the vector bundle map $\phi : \mathcal{A} \to \mathcal{B}$ over $\mathfrak{G}_d^r$.

Under suitable numerical restrictions, when $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B})$, the cycle constructed in Theorem 3.5 is a virtual divisor on $\mathfrak{G}_d^r$. This happens precisely when

$$r := 2s + sp + p,\ g := rs + s \text{ and } d := rs + r.$$

for some $p \geq 0$ and $s \geq 1$. The first remarkable case occurs when $s = 1$. Set $g = 2p+3$, $r = g-1 = 2p+2$, and $d = 2g-2 = 4p+4$. Note that, since the canonical bundle is the only $\mathfrak{g}_{2g-2}^{g-1}$ on a curve of genus g, the Brill-Noether stack is isomorphic to $\mathbf{M}_g$. The notable fact that the cycle in question is an actual divisor follows directly from Voisin's Theorem 2.12 and from Green's Hyperplane Section Theorem [**V05**].

Hence $\mathcal{Z}_{g,p} := \sigma_*(\mathcal{U}_{g,p})$ is a virtual divisor on $\mathcal{M}_g$ whenever

$$g = s(2s + sp + p + 1).$$

In the next section, we explain how to extend the morphism $\phi : \mathcal{A} \to \mathcal{B}$ to a morphism of locally free sheaves over the stack $\widetilde{\mathfrak{G}}^r_d$ of limit linear series and reproduce the class formula proved in [**Fa06a**] for the degeneracy locus of this morphism.

3.3. Divisors of small slope. In [**Fa06a**] it was shown that the determinantal structure of $\mathcal{Z}_{g,p}$ can be extended over $\overline{\mathcal{M}}_g$ in such a way that whenever $s \geq 2$, the resulting virtual slope violates the Harris-Morrison Slope Conjecture. One has the following general statement:

Theorem 3.6. *If $\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\boldsymbol{M}}_g$ denotes the compactification of $\mathfrak{G}^r_d$ given by limit linear series, then there exists a natural extension of the vector bundle morphism $\phi : \mathcal{A} \to \mathcal{B}$ over $\widetilde{\mathfrak{G}}^r_d$ such that $\overline{\mathcal{Z}}_{g,p}$ is the image of the degeneracy locus of ϕ. The class of the pushforward to $\widetilde{\boldsymbol{M}}_g$ of the virtual degeneracy locus of ϕ is given by*

$$\sigma_*(c_1(\mathcal{G}_{p,2} - \mathcal{H}_{p,2})) \equiv a\lambda - b_0\delta_0 - b_1\delta_1 - \cdots - b_{[\frac{g}{2}]}\delta_{[\frac{g}{2}]},$$

where $a, b_0, \ldots, b_{[\frac{g}{2}]}$ are explicitly given coefficients such that $b_1 = 12b_0 - a$, $b_i \geq b_0$ for $1 \leq i \leq [g/2]$ and

$$s\big(\sigma_*(c_1(\mathcal{G}_{p,2} - \mathcal{H}_{p,2}))\big) = \frac{a}{b_0} = 6\frac{f(s,p)}{(p+2)\ sh(s,p)}, \ \textit{with}$$

$$f(s,p) = (p^4 + 24p^2 + 8p^3 + 32p + 16)s^7 + (p^4 + 4p^3 - 16p - 16)s^6 - (p^4 + 7p^3 + 13p^2 - 12)s^5 - (p^4 + 2p^3 + p^2 + 14p + 24)s^4 + (2p^3 + 2p^2 - 6p - 4)s^3 + (p^3 + 17p^2 + 50p + 41)s^2 + (7p^2 + 18p + 9)s + 2p + 2$$

and

$$h(s,p) = (p^3 + 6p^2 + 12p + 8)s^6 + (p^3 + 2p^2 - 4p - 8)s^5 - (p^3 + 7p^2 + 11p + 2)s^4 - (p^3 - 5p)s^3 + (4p^2 + 5p + 1)s^2 + (p^2 + 7p + 11)s + 4p + 2.$$

Furthermore, we have that

$$6 < \frac{a}{b_0} < 6 + \frac{12}{g+1}$$

whenever $s \geq 2$. If the morphism ϕ is generically non-degenerate, then $\overline{\mathcal{Z}}_{g,p}$ is a divisor on $\overline{\mathcal{M}}_g$ which gives a counterexample to the Slope Conjecture for $g = s(2s + sp + p + 1)$.

A few remarks are necessary. In the case $s = 1$ and $g = 2p + 3$, the vector bundles $\mathcal{A}$ and $\mathcal{B}$ exist not only over a partial compactification of $\widetilde{\mathbf{M}}_g$ but can be extended (at least) over the entire stack $\mathbf{M}_g \cup \Delta_0$ in such a way that $\mathcal{B}(C, \omega_C) = H^0(C, \wedge^p M_{\omega_C} \otimes \omega_C^2)$ for any $[C] \in \mathcal{M}_g \cup \Delta_0$. Theorem 3.6 reads in this case, see also [**Fa08**] Theorem 5.7:

$$(5)\quad [\overline{\mathcal{Z}}_{2p+3,p}]^{virt} = c_1(\mathcal{B} - \mathcal{A}) = \frac{1}{p+2}\binom{2p}{p}\Big(6(p+3)\lambda - (p+2)\delta_0 - 6(p+1)\delta_1 - \cdots\Big),$$

in particular $s([\overline{\mathcal{Z}}_{2p+3,p}]^{virt}) = 6 + 12/(g+1)$.

Particularly interesting is the case $p = 0$ when the condition $K_{0,2}(C, L) = 0$ for $[C, L] \in \mathfrak{G}^r_d$, is equivalent to the multiplication map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

not being an isomorphism. Note that $\nu_2(L)$ is a linear map between vector spaces of the same dimension and $\mathcal{Z}_{g,0}$ is the failure locus of the Maximal Rank Conjecture:

COROLLARY 3.7. *For $g = s(2s+1), r = 2s, d = 2s(s+1)$ the slope of the virtual class of the locus of those $[C] \in \overline{\mathcal{M}}_g$ for which there exists $L \in W^r_d(C)$ such that the embedded curve $C \overset{|L|}{\hookrightarrow} \boldsymbol{P}^r$ sits on a quadric hypersurface is equal to*

$$s(\overline{\mathcal{Z}}_{s(2s+1),0}) = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}.$$

4. Conjectures on Koszul cohomology of curves

4.1. Green's Conjecture. In what follows, we consider (C, K_C) a smooth canonical curve of genus $g \geq 2$. In this case, the Duality Theorem 2.7 applies, and the distribution of the numbers $b_{p,q} := \dim K_{p,q}(C, K_C)$ organized in a table (the *Betti table*) is the following:

$$\begin{bmatrix} b_{0,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & b_{1,1} & b_{2,1} & \dots & b_{g-3,1} & b_{g-2,1} \\ b_{0,2} & b_{1,2} & b_{2,2} & \dots & b_{g-3,2} & 0 \\ 0 & 0 & 0 & \dots & 0 & b_{g-2,3} \end{bmatrix}$$

The Betti table is symmetric with respect to its center, that is, $b_{i,j} = b_{g-2-i,3-j}$ and all the other entries not marked here are zero.

Trying to apply the Non-Vanishing Theorem 2.4 to the canonical bundle K_C, we obtain one *condition* and one *quantity.* The condition comes from the hypothesis that for a decomposition $K_C = L_1 \otimes (K_C \otimes L_1^\vee)$, Theorem 2.4 is applicable whenever

$$r_1 + 1 := h^0(C, L_1) \geq 2 \text{ and } r_2 + 1 := h^1(C, L_1) \geq 2. \tag{6}$$

A line bundle L_1 satisfying (6) is said to *contribute to the Clifford index of C.*

The quantity that appears in Theorem 2.4 is the *Clifford index* itself. More precisely

$$r_1 + r_2 - 1 = g - \text{Cliff}(L_1) - 2,$$

where

$$\text{Cliff}(L_1) := \deg(L_1) - 2h^0(L_1) + 2.$$

Clifford's Theorem [**ACGH85**] says that $\text{Cliff}(L_1) \geq 0$, and $\text{Cliff}(L_1) > 0$ unless L_1 is a $\mathfrak{g}^1_2$. Following [**GL86**] we define the *Clifford index of C* as the quantity

$$\text{Cliff}(C) := \min\{\text{Cliff}(L_1) : \ L_1 \text{ contributes to the Clifford index of } C\}.$$

In general, the Clifford index will be computed by minimal pencils. Specifically, a general d-gonal curve $[C] \in \mathcal{M}^1_{g,d}$ (recall that the gonality strata are irreducible) will have $\text{Cliff}(C) = d-2$. However, this equality is not valid for all curves, that is, there exist curves $[C] \in \mathcal{M}_g$ with $\text{Cliff}(C) < \text{gon}(C)-2$, basic examples being plane curves, or exceptional curves on $K3$ surfaces. Even in these exotic cases, Coppens and Martens [**CM91**] established the precise relation $\text{Cliff}(C) = \text{gon}(C) - 3$.

Theorem 2.4 implies the following non-vanishing,

$$K_{g-\text{Cliff}(C)-2,1}(C, K_C) \neq 0,$$

and Green's Conjecture predicts optimality of Theorem 2.4 for canonical curves:

CONJECTURE 4.1. *For any curve $[C] \in \mathcal{M}_g$ we have the vanishing $K_{p,1}(C, K_C) = 0$ for all $p \geq g - \text{Cliff}(C) - 1$.*

In the statement of Green's Conjecture, it suffices to prove the vanishing of $K_{g-\mathrm{Cliff}(C)-1,1}(C, K_C)$ or, by duality, that $K_{\mathrm{Cliff}(C)-1,2}(C, K_C) = 0$.

We shall analyze some basic cases:

EXAMPLE 4.2. Looking at the group $K_{0,2}(C, K_C)$, Green's Conjecture predicts that it is zero for all non-hyperelliptic curves. Or, the vanishing of $K_{0,2}(C, K_C)$ is equivalent to the projective normality of the canonical curve. This is precisely the content of the classical Max Noether Theorem [**ACGH85**], p. 117.

EXAMPLE 4.3. For a non-hyperelliptic curve, we know that $K_{1,2}(C, K_C) = 0$ if and only if the canonical curve $C \subset \mathbb{P}^{g-1}$ is cut out by quadrics. Green's Conjecture predicts that $K_{1,2}(C, K_C) = 0$ unless the curve is hyperelliptic, trigonal or a smooth plane quintic. This is precisely the Enriques-Babbage-Petri Theorem, see [**ACGH85**], p. 124.

Thus Conjecture 4.1 appears as a sweeping generalization of two famous classical theorems. Apart from these classical results, strong evidence has been found for Green's Conjecture (and one should immediately add that not a shred of evidence has been found suggesting that the conjecture might fail even for a single curve $[C] \in \mathcal{M}_g$). For instance, the conjecture is true for general curves in *any* gonality stratum $\mathcal{M}^1_{g,d}$, see [**Ap05**], [**Tei02**] and [**V02**]. The proof of this fact relies on semicontinuity. Since $\mathcal{M}^1_{g,d}$ is irreducible, it suffices to find one example of a d-gonal curve that satisfies the conjecture, for any $2 \le d \le (g+2)/2$; here we also need the fact mentioned above, that the Clifford index of a general d-gonal curve is $d-2$. The most important and challenging case, solved by Voisin [**V05**], was the case of curves of odd genus $g = 2d-1$ and maximal gonality $d+1$. Following Hirschowitz and Ramanan [**HR98**] one can compare the Brill-Noether divisor $\mathcal{M}^1_{g,d}$ of curves with a $\mathfrak{g}^1_d$ and the virtual divisor of curves $[C] \in \mathcal{M}_g$ with $K_{d-1,1}(C, K_C) \neq 0$. The non-vanishing Theorem 2.4 gives a set-theoretic inclusion $\mathcal{M}^1_{g,d} \subset \mathcal{Z}_{g,d-2}$. Now, we compare the class $[\mathcal{Z}_{g,d-2}]^{virt} \in \mathrm{Pic}(\mathcal{M}_g)$ of the virtual divisor $\mathcal{Z}_{g,d-2}$ to the class $[\mathcal{M}^1_{g,d}]$ computed in [**HM82**]. One finds the following relation

$$[\mathcal{Z}_{g,d-2}]^{virt} = (d-1)[\mathcal{M}^1_{g,d}] \in \mathrm{Pic}(\mathcal{M}_g)$$

cf. [**HR98**]; Theorem 3.6 in the particular case $s = 1$ provides an extension of this equality to a partial compactification of $\mathcal{M}_g$. Green's Conjecture for general curves of odd genus [**V05**] implies that $\mathcal{Z}_{g,d-2}$ is a genuine divisor on $\mathcal{M}_g$. Since a general curve $[C] \in \mathcal{M}^1_{g,d}$ satisfies

$$\dim K_{d-1,1}(C, K_C) \ge d-1,$$

cf. [**HR98**], one finds the set-theoretic equality $\mathcal{M}^1_{g,d} = \mathcal{Z}_{g,d-2}$. In particular we obtain the following strong characterization of curves of odd genus and maximal gonality:

THEOREM 4.4 (Hirschowitz-Ramanan, Voisin). *If C is a smooth curve of genus $g = 2d-1 \ge 7$, then $K_{d-1,1}(C, K_C) \neq 0$ if and only if C carries a $\mathfrak{g}^1_d$.*

Voisin proved Theorem 2.12, using Hilbert scheme techniques, then she applied Green's Hyperplane Section Theorem [**Gr84a**] to obtain the desired example of a curve $[C] \in \mathcal{M}_g$ satisfying Green's Conjecture.

Starting from Theorem 4.4, all the other generic d-gonal cases are obtained in the following refined form, see [**Ap05**]:

THEOREM 4.5. *We fix integers g and $d \geq 2$ such that $2 \leq d \leq [g/2]+1$. For any smooth curve $[C] \in \mathcal{M}_g$ satisfying the condition*

$$\dim W^1_{g-d+2}(C) \leq g-2d+2 = \rho(g,1,g-d+2), \tag{7}$$

we have that $K_{g-d+1,1}(C,K_C)=0$. In particular, C satisfies Green's Conjecture.

Note that the condition $d \leq [g/2]+1$ excludes the case already covered by Theorem 4.4. The proof of Theorem 4.5 relies on constructing a singular stable curve $[C'] \in \overline{\mathcal{M}}_{2g+3-2d}$ of maximal gonality $g+3-d$ (that is, $[C'] \notin \overline{\mathcal{M}}^1_{2g+3-d,g+2-d}$), starting from any smooth curve $[C] \in \mathcal{M}_g$ satisfying (7). The curve C' is obtained from C by gluing together $g+3-2d$ pairs of general points of C, and then applying an analogue of Theorem 4.4 for singular stable curves, [**Ap05**], see Section 4.2. The version in question is the following, cf. [**Ap05**] Proposition 7. The proof we give here is however slightly different:

THEOREM 4.6. *For any nodal curve $[C'] \in \mathcal{M}_{g'} \cup \Delta_0$, with $g' = 2d'-1 \geq 7$ such that $K_{d'-1,1}(C',\omega_{C'}) \neq 0$, it follows that $[C'] \in \overline{\mathcal{M}}^1_{g',d'}$.*

PROOF. By duality, we obtain the following equality of cycles on $\widetilde{\mathcal{M}}_{g'}$:

$$\{[C']:\ K_{d'-1,1}(C',\omega_{C'}) \neq 0\} = \{[C']:\ K_{d'-2,2}(C',\omega_{C'}) \neq 0\} =: \overline{\mathcal{Z}}_{g',d'-2}.$$

Theorem 3.6 shows that this locus is a virtual divisor on $\widetilde{\mathcal{M}}_{g'}$ whose class is given by formula (5) and Theorem 2.12 implies that $\overline{\mathcal{Z}}_{g',d'-2}$ is actually a divisor. Comparing its class against the class of the Hurwitz divisor $\overline{\mathcal{M}}^1_{g',d'}$ [**HM82**], we find that

$$\overline{\mathcal{Z}}_{g',d'-2} \equiv (d'-1)\overline{\mathcal{M}}^1_{g',d'} \in \mathrm{Pic}(\widetilde{\mathbf{M}}_{g'}).$$

Note that this is a stronger statement than [**HR98**] Proposition 3.1, being an equality of codimension 1 cycles on the compactified moduli space $\widetilde{\mathcal{M}}_{g'}$, rather than on $\mathcal{M}_{g'}$. The desired statement follows immediately since for any curve $[C'] \in \mathcal{M}^1_{g',d'}$ one has $\dim K_{d'-1,1}(C',\omega_{C'}) \geq d'-1$, hence the degeneracy locus $\overline{\mathcal{Z}}_{g',d'-2}$ contains $\overline{\mathcal{M}}^1_{g',d'}$ with multiplicity at least $d'-1$. □

We return to the discussion on Theorem 4.5 (the proof will be resumed in the next subsection). By duality, the vanishing in the statement above can be rephrased as

$$K_{d-3,2}(C,K_C)=0.$$

The condition (7) is equivalent to a string of inequalities

$$\dim W^1_{d+n}(C) \leq n$$

for all $0 \leq n \leq g-2d+2$, in particular $\mathrm{gon}(C) \geq d$. This condition is satisfied for a general d-gonal curve, cf. [**Ap05**]. More generally, if $[C] \in \mathcal{M}^1_{g,d}$ is a general d-gonal curve then any irreducible component

$$Z \neq W^1_d(C) + W_{n-d}(C)$$

of $W^1_n(C)$ has dimension $\rho(g,1,n)$. In particular, for $\rho(g,1,n)<0$ it follows that $W^1_n(C) = W^1_d(C) + W_{n-d}(C)$ which of course implies (7). For $g=2d-2$, the inequality (7) becomes necessarily an equality and it reads: the curve C carries finitely many $\mathfrak{g}^1_d$'s of minimal degree.

We make some comments regarding condition (7). Let us suppose that C is non-hyperelliptic and $d \geq 3$. From Martens' Theorem [**ACGH85**] p.191, it follows that $\dim W^1_{g-d+2}(C) \leq g-d-1$. Condition (7) requires the better bound $g-2d+2 \leq g-d-1$. However, for $d=3$, the two bounds are the same, and Theorem 4.5 shows that $K_{0,0}(C,K_C)=0$, for any non-hyperelliptic curve, which is Max Noether's Theorem, see also Example 4.2. Applying Mumford's Theorem [**ACGH85**] p.193, we obtain the better bound $\dim W^1_{g-d+2}(C) \leq g-d-2$ for $d \geq 4$, unless the curve is trigonal, a smooth plane quintic or a double covering of an elliptic curve. Therefore, if C is not one of the three types listed above, then $K_{1,2}(C,K_C)=0$, and we recover the Enriques-Babbage-Petri Theorem, see also Example 4.3 (note, however, the exception made for bielliptic curves).

Keem has improved the dimension bounds for $W^1_{g-d+2}(C)$. For $d \geq 5$ and C a curve that has neither a $\mathfrak{g}^1_4$ nor is a smooth plane sextic, one has the inequality $\dim W^1_{g-d+2}(C) \leq g-d-3$, cf. [**Ke90**] Theorems 2.1 and 2.3. Consequently, Theorem 4.5 implies the following result which is a complete solution to Green's Conjecture for 5-gonal curves:

THEOREM 4.7 (Voisin [**V88**], Schreyer [**Sch91**]). *If $K_{2,2}(C,K_C) \neq 0$, then C is hyperelliptic, trigonal, tetragonal or a smooth plane sextic, that is,* $\mathrm{Cliff}(C) \leq 2$.

Geometrically, the vanishing of $K_{2,2}(C,K_C)$ is equivalent to the ideal of the canonical curve being generated by quadrics, and the minimal relations among the generators being linear.

Theorem 3.1 from [**Ke90**] gives the next bound $\dim W^1_{g-d+2}(C) \leq g-d-4$, for $d \geq 6$ and C with $\mathrm{gon}(C) \geq 6$ which does not admit a covering of degree two or three on another curve, and which is not a plane curve. The following improvement of Theorem 4.7 is then obtained directly from Theorem 4.5 and [**Ap05**] Theorem 3.1:

THEOREM 4.8. *If $g \geq 12$ and $K_{3,2}(C,K_C) \neq 0$, then C is one of the following: hyperelliptic, trigonal, tetragonal, pentagonal, double cover of an genus 3 curve, triple cover of an elliptic curve, smooth plane septic. In other words, if* $\mathrm{Cliff}(C) \geq 4$ *then $K_{3,2}(C,K_C)=0$.*

Theorem 4.8 represents the solution to Green's Conjecture for hexagonal curves. Likewise, Theorem 4.5 can be used together with the Brill-Noether theory to prove Green's Conjecture for any gonality d and large genus. The idea is to apply Coppens' results [**Co83**].

THEOREM 4.9. *If $g \geq 10$ and $d \geq 5$ are two integers such that $g > (d-2)(2d-7)$, and C is any d-gonal curve of genus g which does not admit any morphism of degree less than d onto another smooth curve, then* $\mathrm{Cliff}(C) = d-2$ *and Green's Conjecture is verified for C, i.e. $K_{g-d+1,1}(C,K_C)=0$.*

The statement of Conjecture 4.1 (meant as a vanishing result), is empty for hyperelliptic curves is, hence the interesting cases begin with $d \geq 3$.

It remains to verify Green's Conjecture for curves which do not verify (7). One result in this direction was proved in [**ApP06**].

THEOREM 4.10. *Let S be a $K3$ surface with* $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus Z \cdot \ell$, *with H very ample, $H^2 = 2r-2 \geq 4$, and $H \cdot \ell = 1$. Then any smooth curve in the linear system $|2H+\ell|$ verifies Green's conjecture.*

Smooth curves in the linear system $|2H + \ell|$ count among the few known examples of curves whose Clifford index is not computed by pencils, i.e. $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$, [**ELMS89**] (other obvious examples are plane curves, for which Green's Conjecture was verified before, cf. [**Lo89**]). Such curves are the most special ones in the moduli space of curves from the point of view of the Clifford dimension. Hence, this case may be considered as opposite to that of a general curve of fixed gonality. Note that these curves carry a one-parameter family of pencils of minimal degree, hence the condition (7) is not satisfied.

4.2. The Gonality Conjecture. The Green-Lazarsfeld Gonality Conjecture [**GL86**] predicts that the gonality of a curve can be read off the Koszul cohomology with values in any sufficiently positive line bundle.

CONJECTURE 4.11 (Green-Lazarsfeld). *Let C be a smooth curve of gonality d, and L a sufficiently positive line bundle on C. Then*

$$K_{h^0(L)-d,1}(C, L) = 0.$$

Theorem 2.8 applied to L written as a sum of a minimal pencil and the residual bundle yields

$$K_{h^0(L)-d-1,1}(C, L) \neq 0.$$

Note that if L is sufficiently positive, then the Green-Lazarsfeld Nonvanishing Theorem is optimal when applied for a decomposition where one factor is a pencil. Indeed, consider any decomposition $L = L_1 \otimes L_2$ with $r_1 = h^0(C, L_1) - 1 \geq 2$, and $r_2 = h^0(C, L_2) - 1 \geq 2$. Since L is sufficiently positive, the linear system $|K_C^{\otimes 2} \otimes L^\vee|$ is empty, and finiteness of the addition map of divisors shows that at least one of the two linear systems $|K_C \otimes L_i^\vee|$ is empty. Suppose $|K_C \otimes L_2^\vee| = \emptyset$, choose a point $x \in C - \mathrm{Bs}(|L_1|)$ and consider a new decomposition $L = L_1' \otimes L_2'$, with $L_1' = L_1 \otimes \mathcal{O}_C(-x)$, and $L_2' = L_2 \otimes \mathcal{O}_C(x)$. Denoting as usual $r_i' = h^0(C, L_i') - 1$, we find that $r_1' + r_2' - 1 = r_1 + r_2 - 1$, whereas $r_1' = r_1 - 1$, and L_2' is again non-special. We can apply an inductive argument until r_1 becomes 1. Hence the Gonality Conjecture predicts the optimality of the Green-Lazarsfeld Nonvanishing Theorem. However, one major disadvantage of this statement is that "sufficiently positive" is not a precise condition. It was proved in [**Ap02**] that by adding effective divisors to bundles that verify the Gonality Conjecture we obtain again bundles that verify the conjecture. Hence, in order to find a precise statement for Conjecture 4.11 one has to study the edge cases.

In most generic cases (general curves in gonality strata, to be more precise), the Gonality Conjecture can be verified for line bundles of degree $2g$, see [**ApV03**] and [**Ap05**]. The test bundles are obtained by adding two generic points to the canonical bundle.

THEOREM 4.12 ([**Ap05**]). *For any d-gonal curve $[C] \in \mathcal{M}_g$ with $d \leq [g/2] + 2$ which satisfies the condition (7), and for general points $x, y \in C$, we have that $K_{g-d+1,1}\big(C, K_C \otimes \mathcal{O}_C(x + y)\big) = 0$.*

The case not covered by Theorem 4.12 is slightly different. A general curve C of odd genus carries infinitely many minimal pencils, hence a bundle of type $K_C \otimes \mathcal{O}_C(x+y)$ can never verify the vanishing predicted by the Gonality Conjecture. Indeed, for any two points x and y there exists a minimal pencil L_1 such that $H^0(C, L_1(-x - y)) \neq 0$, and we apply Theorem 2.4. However, adding *three* points to the canonical bundle solves the problem, cf. [**Ap04**], [**Ap05**].

THEOREM 4.13. *For any curve* $[C] \in \mathcal{M}_{2d-1}$ *of maximal gonality* $\mathrm{gon}(C) = d+1$ *and for general points* $x, y, z \in C$, *we have that* $K_{d,1}\big(C, K_C \otimes \mathcal{O}_C(x+y+z)\big) = 0$.

The proofs of Theorems 4.5, 4.12 and 4.13 are all based on the same idea. We start with a smooth curve C and construct a stable curve of higher genus out of it, in such a way that the Koszul cohomology does not change. Then we apply a version of Theorem 4.4 for singular curves.

Proof of Theorems 4.5 and 4.12. We start with $[C] \in \mathcal{M}_g$ satisfying the condition (7). We claim that if we choose $\delta := g + 3 - 2d$ pairs of general points $x_i, y_i \in C$ for $1 \leq i \leq \delta$, then the resulting stable curve

$$\big[C' := \frac{C}{x_1 \sim y_1, \ldots, x_\delta \sim y_\delta}\big] \in \overline{\mathcal{M}}_{2g+3-2d}$$

is a curve of maximal gonality, that is, $g + 3 - d$. Indeed, otherwise $[C'] \in \overline{\mathcal{M}}^1_{2g+3-2d, g+2-d}$ and this implies that there exists a degree $g + 2 - d$ admissible covering $f : \tilde{C} \to R$ from a nodal curve $\tilde{C}$ that is semi-stably equivalent to C', onto a genus 0 curve R. The curve C is a subcurve of $\tilde{C}$ and if $\deg(f_{|C}) = n \leq g+2-d$, then it follows that $f_{|C}$ induces a pencil $\mathfrak{g}^1_n$ such that $f_{|C}(x_i) = f_{|C}(y_i)$ for $1 \leq i \leq \delta$. Since the points $x_i, y_i \in C$ are general, this implies that dim $W^1_n(C) + \delta \geq 2\delta$, which contradicts (7).

To conclude, apply Theorem 4.6 and use the following inclusions, [**V02**], [**ApV03**]:

$$K_{g-d+1,1}(C, K_C) \subset K_{g-d+1,1}(C, K_C(x+y)) \subset K_{g-d+1,1}(C', \omega_{C'}).$$

□

REMARK 4.14. The proofs of Theorems 4.5, 4.6 and 4.12 indicate an interesting phenomenon, completely independent of Voisin's proof of the generic Green Conjecture. They show that Green's Conjecture for general curves of genus $g = 2d-1$ and maximal gonality $d+1$ is *equivalent* to the Gonality Conjecture for bundles of type $K_C \otimes \mathcal{O}_C(x+y)$ for general pointed curves $[C, x, y] \in \mathcal{M}_{2d-2,2}$. We refer to [**Ap02**] and [**ApV03**] for further implications between the two conjectures, in both directions.

Proof of Theorem 4.13. For C as in the hypothesis, and for general points $x, y, z \in C$, we construct a stable curve $[C'] \in \mathcal{M}_{2d+1}$ by adding a smooth rational component passing through the points x, y and z. Using admissible covers one can show, as in the proofs of Theorems 4.5 and 4.12, that C' is of maximal gonality, that is $d+2$. From Theorem 4.6, we obtain $K_{d,1}(C', \omega_{C'}) = 0$. The conclusion follows from the observation that $K_{d,1}(C, K_C \otimes \mathcal{O}_C(x+y+z)) \cong K_{d,1}(C', \omega_{C'})$. □

It is natural to ask the following:

QUESTION 4.15. For a curve C and points $x, y \in C$, can one give explicit conditions on Koszul cohomology ensuring that $x + y$ is contained in a fiber of a minimal pencil?

We prove here the following result, which can be considered as a precise version of the Gonality Conjecture for generic curves.

THEOREM 4.16. *Let* $[C] \in \mathcal{M}_{2d-2}$, *and* $x, y \in C$ *arbitrarily chosen distinct points. Then* $K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y)) \neq 0$ *if and only if there exists* $A \in W^1_d(C)$ *such that* $h^0(C, A \otimes \mathcal{O}_C(-x-y)) \neq 0$.

Proof. Suppose there exists $A \in W^1_d(C)$ such that $h^0(C, A(-x-y)) \neq 0$. Theorem 2.8 applied to the decomposition $K_C \otimes \mathcal{O}_C(x+y) = A \otimes B$, with $B = K_C(x+y) \otimes A^\vee$ produces nontrivial classes in the group $K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y))$.

For the converse, we consider C', the stable curve obtained from C by gluing together the points x and y and denote by $\nu : C \to C'$ the normalization morphism. Clearly $[C'] \in \overline{\mathcal{M}}_{2d-1}$. We observe that

$$K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y)) \cong K_{d-1,1}(C', \omega_{C'}).$$

From Theorem 4.6, it follows that $[C'] \in \overline{\mathcal{M}}^1_{2d-1,d}$, hence there exists a map

$$f : \widetilde{C} \xrightarrow{d:1} R$$

from a curve $\widetilde{C}$ semistably equivalent to C' onto a rational nodal curve R. The curve C is a subcurve of $\widetilde{C}$ and $f_{|C}$ provides the desired pencil. □

As mentioned above, the lower possible bound for explicit examples of line bundles that verify the Gonality Conjecture found so far was $2g$. One can raise the question whether this bound is optimal or not and the sharpest statement one can make is Conjecture 1.4 discussed in the introduction of this paper.

5. The Strong Maximal Rank Conjecture

Based mainly on work carried out in [**Fa06a**] and [**Fa06b**] we propose a conjecture predicting the resolution of an embedded curve with general moduli. This statement unifies two apparently unrelated deep results in the theory of algebraic curves: The *Maximal Rank Conjecture* which predicts the number of hypersurfaces of each degree containing a general embedded curve $C \subset \mathbb{P}^r$ and *Green's Conjecture* on syzygies of canonical curves.

We begin by recalling the statement of the classical *Maximal Rank Conjecture*. The modern formulation of this conjecture is due to Harris [**H82**] p. 79, even though it appears that traces of a similar statement can be found in the work of Max Noether. We fix integers g, r and d such that $\rho(g, r, d) \geq 0$ and denote by $\mathfrak{I}_{d,g,r}$ the unique component of the Hilbert scheme $\mathrm{Hilb}_{d,g,r}$ of curves $C \subset \mathbb{P}^r$ with Hilbert polynomial $h_C(t) = dt + 1 - g$ containing curves with general moduli. In other words, the variety $\mathfrak{I}_{d,g,r}$ is characterized by the following properties:
(1) The general point $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ corresponds to a smooth curve $C \subset \mathbb{P}^r$ with $\deg(C) = d$ and $g(C) = g$.
(2) The moduli map $m : \mathfrak{I}_{d,g,r} \dashrightarrow \mathcal{M}_g$, $m([C \hookrightarrow \mathbb{P}^r]) := [C]$ is dominant.

Conjecture 5.1. (Maximal Rank Conjecture) *A general embedded smooth curve* $[C \hookrightarrow \boldsymbol{P}^r] \in \mathfrak{I}_{d,g,r}$ *is of maximal rank, that is, for all integers* $n \geq 1$ *the restriction maps*

$$\nu_n(C) : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(C, \mathcal{O}_C(n))$$

are of maximal rank, that is, either injective or surjective.

Thus if a curve $C \subset \mathbb{P}^r$ lies on a hypersurface of degree d, then either hypersurfaces of degree d cut out the complete linear series $|\mathcal{O}_C(d)|$ on the curve, or else C is special in its Hilbert scheme. Since C can be assumed to be a Petri general curve, it follows that $H^1(C, \mathcal{O}_C(n)) = 0$ for $n \geq 2$, so $h^0(C, \mathcal{O}_C(n)) = nd + 1 - g$ and Conjecture 5.1 amounts to knowing the Hilbert function of $C \subset \mathbb{P}^r$, that is, the value of $h^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n))$ for all n.

Example 5.2. We consider the locus of curves $C \subset \mathbb{P}^3$ with $\deg(C) = 6$ and $g(C) = 3$ that lie on a quadric surface, that is, $\nu_2(C)$ fails to be an isomorphism. Such curves must be of type $(2, 4)$ on the quadric, in particular, they are hyperelliptic. This is a divisorial condition on $\mathfrak{I}_{6,3,3}$, that is, for a general $[C \hookrightarrow \mathbb{P}^3] \in \mathfrak{I}_{6,3,3}$ the map $\nu_2(C)$ is an isomorphism.

Conjecture 5.1 makes sense of course for any component of $\mathfrak{I}_{d,g,r}$ but is known to fail outside the Brill-Noether range, see [**H82**]. The Maximal Rank Conjecture is known to hold in the non-special range, that is when $d \geq g + r$, due to work of Ballico and Ellia relying on the *méthode d'Horace* of Hirschowitz, see [**BE87**]. Voisin has also proved cases of the conjecture when $h^1(C, \mathcal{O}_C(1)) = 2$, cf. [**V92**]. Finally, Conjecture 5.1 is also known in the case $\rho(g, r, d) = 0$ when it has serious implications for the birational geometry of $\overline{\mathcal{M}}_g$. This case can be reduced to the case when $\dim \mathrm{Sym}^n H^0(C, \mathcal{O}_C(1)) = \dim H^0(C, \mathcal{O}_C(n))$, that is,

$$\binom{n+r}{n} = nd + 1 - g,$$

when Conjecture 5.1 amounts to constructing one smooth curve $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ such that $H^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$. In this situation, the failure locus of Conjecture 5.1 is precisely the virtual divisor $\mathcal{Z}_{g,0}$ on $\mathcal{M}_g$ whose geometry has been discussed in Section 3, Corollary 3.7. The most interesting case (at least from the point of view of slope calculations) is that of $n = 2$. One has the following result [**Fa06b**] Theorem 1.5:

Theorem 5.3. *For each $s \geq 1$ we fix integers*

$$g = s(2s+1), \quad r = 2s \ \textit{and} \ \ d = 2s(s+1),$$

hence $\rho(g, r, d) = 0$. The locus

$$\mathcal{Z}_{g,0} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \textit{ such that } \nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \stackrel{\ncong}{\to} H^0(C, L^{\otimes 2})\}$$

is an effective divisor on $\mathcal{M}_g$. In particular, a general curve $[C] \in \mathcal{M}_g$ satisfies the Maximal Rank Conjecture with respect to all *linear series $L \in W^r_d(C)$.*

For $s = 1$ we have the equality $\mathcal{Z}_{3,0} = \mathcal{M}^1_{3,2}$, and we recover the hyperelliptic locus on $\mathcal{M}_3$. The next case, $s = 2$ and $g = 10$, has been treated in detail in [**FaPo05**]. One has a scheme-theoretic equality $\mathcal{Z}_{10,0} = 42 \cdot \mathcal{K}_{10}$ on $\mathcal{M}_{10}$, where $42 = \#(W^4_{12}(C))$ is the number of minimal pencils $\mathfrak{g}^1_6 = K_C(-\mathfrak{g}^4_{12})$ on a general curve $[C] \in \mathcal{M}_{10}$. Thus a curve $[C] \in \mathcal{M}_{10}$ fails the Maximal Rank Conjecture for a linear series $L \in W^4_{12}(C)$ if and only if it fails it for all the 42 linear series $\mathfrak{g}^4_{12}$! This incarnation of the $K3$ divisor $\mathcal{K}_{10}$ is instrumental in being able to compute the class of $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$, cf. [**FaPo05**].

In view of Theorem 5.3 it makes sense to propose a much stronger form of Conjecture 5.1, replacing the generality assumption of $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ by a generality assumption of $[C] \in \mathcal{M}_g$ with respect to moduli and asking for the maximal rank of the curve with respect to all linear series $\mathfrak{g}^r_d$.

We fix positive integers g, r, d such that $g - d + r \geq 0$ and satisfying

$$0 \leq \rho(g, r, d) < r - 2.$$

We also fix a general curve $[C] \in \mathcal{M}_g$. The numerical assumptions imply that all the linear series $l \in G^r_d(C)$ are complete (the inequality $\rho(g, r+1, d) < 0$ is satisfied), as well as very ample. For each (necessarily complete) linear series $l = (L, H^0(C,L)) \in G^r_d(C)$ and integer $n \geq 2$, we denote by

$$\nu_n(L) : \mathrm{Sym}^n H^0(C,L) \to H^0(C, L^{\otimes n})$$

the multiplication map of global sections. We then choose a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$ and construct two vector bundles $\mathcal{E}_n$ and $\mathcal{F}_n$ over $G^r_d(C)$ with $\mathrm{rank}(\mathcal{E}_n) = \binom{r+n}{n}$ and $\mathrm{rank}(\mathcal{F}_n) = h^0(C, L^{\otimes n}) = nd + 1 - g$, together with a bundle morphism $\phi_n : \mathcal{E}_n \to \mathcal{F}_n$, such that for $L \in G^r_d(C)$ we have that

$$\mathcal{E}_n(L) = \mathrm{Sym}^n H^0(C,L) \text{ and } \mathcal{F}_n(L) = H^0(C, L^{\otimes n})$$

and $\nu_n(L)$ is the map given by multiplication of global sections.

Conjecture 5.4. (Strong Maximal Rank Conjecture) *We fix integers $g, r, d \geq 1$ and $n \geq 2$ as above. For a general curve $[C] \in \mathcal{M}_g$, the determinantal variety*

$$\Sigma^r_{n,g,d}(C) := \{L \in G^r_d(C) : \nu_n(L) \textit{ is not of maximal rank}\}$$

has expected dimension, that is,

$$\dim\, \Sigma^r_{n,g,d}(C) = \rho(g,r,d) - 1 - |\mathrm{rank}(\mathcal{E}_n) - \mathrm{rank}(\mathcal{F}_n)|,$$

where by convention, negative dimension means that $\Sigma^r_{n,g,d}(C)$ is empty.

For instance, in the case $\rho(g,r,d) < nd + 2 - g - \binom{r+n}{n}$, the conjecture predicts that for a general $[C] \in \mathcal{M}_g$ we have that $\Sigma^r_{n,g,d}(C) = \emptyset$, that is,

$$H^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$$

for *every* embedding $C \overset{|L|}{\hookrightarrow} \mathbb{P}^r$ given by $L \in G^r_d(C)$.

When $\rho(g,r,d) = 0$ (and in particular whenever $r \leq 3$), using a standard monodromy argument showing the uniqueness the component $\mathfrak{I}_{d,g,r}$, the Strong Maximal Rank Conjecture is equivalent to Conjecture 5.1, and it states that $\nu_n(L)$ is of maximal rank for a general $[C \overset{L}{\hookrightarrow} \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$.

For $\rho(g,r,d) \geq 1$ however, Conjecture 5.4 seems to be a more difficult question than Conjecture 5.1 because one requires a way of seeing *all* linear series $L \in G^r_d(C)$ at once.

Remark 5.5. The bound $\rho(g,r,d) < r - 2$ in the statement of Conjecture 5.4 implies that all linear series $L \in G^r_d(C)$ on a general curve C, are very ample.

Remark 5.6. We discuss Conjecture 5.4 when $r = 4$ and $\rho(g,r,d) = 1$. The conjecture is trivially true for $g = 1$. The first interesting case is $g = 6$ and $d = 9$. For a general curve $[C] \in \mathcal{M}_6$ we observe that there is an isomorphism $C \cong W^4_9(C)$ given by $C \ni x \mapsto K_C \otimes \mathcal{O}_C(-x)$. Since $\mathrm{rank}(\mathcal{E}_2) = 15$ and $\mathrm{rank}(\mathcal{F}_2) = 13$, Conjecture 5.4 predicts that,

$$\nu_2(K_C(-x)) : \mathrm{Sym}^2 H^0\big(C, K_C \otimes \mathcal{O}_C(-x)\big) \twoheadrightarrow H^0\big(C, K_C^{\otimes 2} \otimes \mathcal{O}_C(-2x)\big),$$

for all $x \in C$, which is true (use the Base-Point-Free Pencil Trick).

The next case is $g = 11, d = 13$, when the conjecture predicts that the map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$$

is injective for all $L \in W^4_{13}(C)$. This follows (non-trivially) from [**M94**]. Another case that we checked is $r = 5, g = 14$ and $d = 17$, when $\rho(14,5,17) = 2$.

Proposition 5.7. *The Strong Maximal Rank Conjecture holds for general non-special curves, that is, when $r = d - g$.*

Proof. This is an immediate application of a theorem of Mumford's stating that for any line bundle $L \in \text{Pic}^d(C)$ with $d \geq 2g+1$, the map $\nu_2(L)$ is surjective, see e.g. [**GL86**]. The condition $\rho(g,r,d) < r-2$ forces in the case $r = d-g$ the inequality $d \geq 2g+3$. Since the expected dimension of $\Sigma_{n,g,d}^{d-g}(C)$ is negative, the conjecture predicts that $\Sigma_{n,g,d}^{d-g}(C) = \emptyset$. This is confirmed by Mumford's result. □

5.1. The Minimal Syzygy Conjecture. Interpolating between Green's Conjecture for generic curves (viewed as a vanishing statement) and the Maximal Rank Conjecture, it is natural to expect that the Koszul cohomology groups of line bundles on a general curve $[C] \in \mathcal{M}_g$ should be subject to the vanishing suggested by the determinantal description provided by Theorem 3.5. For simplicity we restrict ourselves to the case $\rho(g,r,d) = 0$:

Conjecture 5.8. *We fix integers $r, s \geq 1$ and set $d := rs + r$ and $g := rs + s$, hence $\rho(g,r,d) = 0$. For a general curve $[C] \in \mathcal{M}_g$ and for every integer*

$$0 \leq p \leq \frac{r-2s}{s+1}$$

we have the vanishing $K_{p,2}(C,L) = 0$, for every linear series $L \in W_d^r(C)$.

As pointed out in Theorem 3.5, in the limiting case $p = \frac{r-2s}{s+1} \in \mathbb{Z}$, Conjecture 5.8 would imply that the failure locus

$$\mathcal{Z}_{g,p} := \{[C] \in \mathcal{M}_g : \exists L \in W_d^r(C) \text{ such that } K_{p,2}(C,L) \neq \emptyset\}$$

is an effective divisor on $\mathcal{M}_g$ whose closure $\overline{\mathcal{Z}}_{g,p}$ violates the Slope Conjecture.

Conjecture 5.8 generalizes Green's Conjecture for generic curves: When $s = 1$, it reads like $K_{p,2}(C, K_C) = 0$ for $g \geq 2p+3$, which is precisely the main result from [**V05**]. Next, in the case $p = 0$, Conjecture 5.8 specializes to Theorem 5.3. The conjecture is also known to hold when $s = 2$ and $g \leq 22$ (cf. [**Fa06a**] Theorems 2.7 and 2.10).

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Reider's Theorem and Thaddeus Pairs Revisited

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1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n equipped with an ample line bundle L and a subscheme $Z \subset X$ of length d. Serre duality provides a natural isomorphism of vector spaces (for each $i = 0, ..., n$)

$$(*) \quad \mathrm{Ext}^i(L \otimes \mathcal{I}_Z, \mathcal{O}_X) \cong \mathrm{H}^{n-i}(X, K_X \otimes L \otimes \mathcal{I}_Z)^\vee$$

Thaddeus pairs and Reider's theorem concern the cases $i = 1$ and $n = 1, 2$. In these cases one associates a rank two torsion-free coherent sheaf E_ϵ to each *extension class* $\epsilon \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$ via the short exact sequence

$$(**) \quad \epsilon : 0 \to \mathcal{O}_X \to E_\epsilon \to L \otimes \mathcal{I}_Z \to 0$$

and the Mumford stability (or instability) of E_ϵ allows one to distinguish among extension classes. The ultimate aim of this paper is to show how a new notion of Bridgeland stability can similarly be used to distinguish among higher extension classes, leading to a natural higher-dimensional generalization of Thaddeus pairs as well as the setup for a higher-dimensional Reider's theorem.

Reider's theorem gives numerical conditions on an ample line bundle L on a surface S that guarantee the vanishing of the vector spaces $\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_Z)$ which in turn implies the base-point-freeness (the $d = 1$ case) and very ampleness (the $d = 2$ case) of the adjoint line bundle $K_S \otimes L$.

In the first part of this note we will revisit Reider's Theorem in the context of Bridgeland stability conditions. Reider's approach, following Mumford, uses the Bogomolov inequality for Mumford-stable coherent sheaves on a surface to argue (under suitable numerical conditions on L) that no exact sequence $(**)$ can produce a Mumford stable sheaf E_ϵ, and then uses the Hodge Index Theorem to argue that all the exact sequences $(**)$ that produce non-stable sheaves must split. Thus one concludes that $\mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S) = 0$ and $\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_Z) = 0$, as desired.

Here, we will regard an extension class in $(**)$ as a morphism $\epsilon : L \otimes \mathcal{I}_Z \to \mathcal{O}_S[1]$ to the *shift* of $\mathcal{O}_S$ in one of a family of *tilts* $\mathcal{A}_s$ $(0 < s < 1)$ of the abelian category of coherent sheaves on X within the bounded derived category $\mathcal{D}(X)$ of complexes of

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coherent sheaves on X. Reider's argument for a surface S is essentially equivalent to ruling out non-trivial extensions by determining that:

- ϵ is neither injective nor surjective and
- if neither injective nor surjective, then $\epsilon = 0$ (using Hodge Index).

This way of looking at Reider's argument allows for some minor improvements, but more importantly leads to the notion of *Bridgeland stablility conditions*, which are stability conditions, not on coherent sheaves, but rather on objects of $\mathcal{A}_s$.

In [**AB**], it was shown that the Bogomolov Inequality and Hodge Index Theorem imply the existence of such stability conditions on arbitrary smooth projective surfaces S (generalizing Bridgeland's stability conditions for K-trivial surfaces [**Bri08**]). Using these stability conditions, we investigate the stability of objects of the form $L \otimes \mathcal{I}_Z$ and $\mathcal{O}_S[1]$ with a view toward reinterpreting the vanishing

$$\mathrm{Hom}(L \otimes \mathcal{I}_Z, \mathcal{O}_S[1]) = 0$$

as a consequence of an inequality $\mu(L \otimes \mathcal{I}_Z) > \mu(\mathcal{O}_S[1])$ of Bridgeland slopes. Since this is evidently a stronger condition than just the vanishing of the Hom, it is unsurprising that it should require stronger numerical conditions. This reasoning easily generalizes to the case where $\mathcal{O}_S$ is replaced by $\mathcal{I}_W^\vee$, the derived dual of the ideal sheaf of a finite length subscheme $W \subset S$.

The Bridgeland stability of the objects $L \otimes \mathcal{I}_Z$ and $\mathcal{I}_W^\vee[1]$ is central to a new generalization of Thaddeus pairs from curves to surfaces. A Thaddeus pair on a curve C is an extension of the form:

$$\epsilon : 0 \to \mathcal{I}_W^\vee \to E_\epsilon \to L \otimes \mathcal{I}_Z \to 0$$

where L is a line bundle and $Z, W \subset C$ are effective divisors. Normally we would write this

$$\epsilon : 0 \to \mathcal{O}_C(W) \to E_\epsilon \to L(-Z) \to 0$$

since finite length subschemes of a curve are effective Cartier divisors. The *generic* such extension determines a Mumford-stable vector bundle E_ϵ on C whenever

$$\deg(L(-Z)) > \deg(\mathcal{O}_C(W)) \quad (\text{and} \quad C \neq \mathbb{P}^1)$$

or, equivalently, whenever the Mumford slope of $L(-Z)$ exceeds that of $\mathcal{O}_C(W)$ (both line bundles are trivially Mumford-stable). Moreover, the Mumford-unstable vector bundles arising in this way are easily described in terms of the secant varieties to the image of C under the natural linear series map:

$$\phi : C \to \mathbb{P}(\mathrm{H}^0(C, K_C \otimes L(-Z-W))^\vee) \cong \mathbb{P}(\mathrm{Ext}^1(L(-Z), \mathcal{O}_C(W)))$$

since an unstable vector bundle E_ϵ can only be destabilized by a sub-line bundle $L(-Z') \subset L(-Z)$ that lifts to a sub-bundle of E_ϵ:

$$\begin{array}{ccccccccc} & & & & & & L(-Z') & & \\ & & & & & \swarrow & \downarrow & & \\ (\dagger) \quad 0 & \to & \mathcal{O}_C(W) & \to & E_\epsilon & \to & L(-Z) & \to & 0 \end{array}$$

In the second part of this paper, we note that Thaddeus pairs naturally generalize to surfaces as extensions of the form

$$\epsilon : 0 \to \mathcal{I}_W^\vee[1] \to E_\epsilon^\bullet \to L \otimes \mathcal{I}_Z \to 0$$

in the categories $\mathcal{A}_s$ under appropriate Bridgeland stability conditions for which both $L \otimes \mathcal{I}_Z$ and $\mathcal{I}_W^\vee[1]$ are Bridgeland stable and their Bridgeland slopes satisfy $\mu(L \otimes \mathcal{I}_Z) > \mu(\mathcal{I}_W^\vee[1])$. Note that $E_\epsilon^\bullet$ is never a coherent sheaf.

This is a very satisfying generalization of Thaddeus pairs since

$$\mathrm{Ext}^1_{\mathcal{A}_s}(L \otimes \mathcal{I}_Z, \mathcal{I}_W^\vee[1]) \cong \mathrm{H}^0(S, K_S \otimes L \otimes \mathcal{I}_Z \otimes \mathcal{I}_W)^\vee$$

by Serre duality. In this case, however, there are subobjects

$$K \subset L \otimes \mathcal{I}_Z$$

not of the form $L \otimes \mathcal{I}_{Z'}$ that may destabilize $E_\epsilon^\bullet$, as in (†). These subobjects are necessarily coherent sheaves, but may be of higher rank than one, and therefore not subsheaves of $L \otimes \mathcal{I}_Z$ in the usual sense. This leads to a much richer geometry for the locus of "unstable" extensions than in the curve case.

We will finally discuss the moduli problem for families of Bridgeland stable objects with the particular invariants

$$[E] = [L \otimes \mathcal{I}_Z] + [\mathcal{I}_W^\vee[1]] = [L \otimes \mathcal{I}_Z] - [\mathcal{I}_W^\vee]$$

in the Grothendieck group (or cohomology ring) of S, and finish by describing wall-crossing phenomena of (some of) these moduli spaces in the K-trivial case, following [**AB**].

This line of reasoning suggests a natural question for *threefolds* X. Namely, might it be possible to prove a Reider theorem for L and $Z \subset X$ by ruling out non-trivial extensions of the form

$$\epsilon : 0 \to \mathcal{O}_X[1] \to E_\epsilon^\bullet \to L \otimes \mathcal{I}_Z \to 0$$

in some tilt $\mathcal{A}_s$ of the category of coherent sheaves on X via a version of the Bogomolov Inequality and Hodge Index Theorem for objects of $\mathcal{A}_s$ on threefolds?

We do not know versions of these results that would allow a direct application of Reider's method of proof, but this seems a potentially fruitful direction for further research, and ought to be related to the current active search for examples of Bridgeland stability conditions on complex projective threefolds.

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2. The Original Reider

Fix an ample divisor H on a smooth projective variety X over $\mathbb{C}$ of dimension n. A non-zero torsion-free coherent sheaf E on X has Mumford slope

$$\mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rk}(E) H^n}$$

and E is H-*Mumford-stable* if $\mu_H(K) < \mu_H(E)$ for all non-zero subsheaves $K \subset E$ with the property that $Q = E/K$ is supported in codimension ≤ 1.

Bogomolov Inequality: Suppose E is an H-Mumford-stable torsion-free coherent sheaf on X and $n \geq 2$. Then

$$\mathrm{ch}_2(E) \cdot H^{n-2} \leq \frac{c_1^2(E) \cdot H^{n-2}}{2\mathrm{rk}(E)}$$

(in case $X = S$ is a surface, the conclusion is independent of the choice of H).

Application 2.1: For an ample line bundle $L = \mathcal{O}_S(H)$ on a smooth projective surface S and a finite subscheme $Z \subset S$, consider $\epsilon \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)$ and suppose

$$\epsilon : 0 \to \mathcal{O}_S \to E \to L \otimes \mathcal{I}_Z \to 0$$

yields an H-Mumford-stable sheaf E. Then $H^2 = c_1^2(L) \leq 4d$.

Proof: If E is H-Mumford stable, then by the Bogomolov inequality,

$$\mathrm{ch}_2(E) = \frac{c_1^2(L)}{2} - d \leq \frac{c_1^2(E)}{4} = \frac{c_1^2(L)}{4}.$$

Hodge Index Theorem: Let D be an arbitrary divisor on X. Then

$$\left(D^2 \cdot H^{n-2}\right)(H^n) \leq (D \cdot H^{n-1})^2$$

and equality holds if and only if there exists a (rational) number k with the property that $(D \cdot H^{n-2}) \cdot E = (kH^{n-1}) \cdot E$ for all divisors E.

Application 2.2: For $S, H = c_1(L)$ and Z as in Application 2.1, suppose an extension class $\epsilon \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)$,

$$\epsilon : 0 \to \mathcal{O}_S \to E \to L \otimes \mathcal{I}_Z \to 0$$

yields a sheaf E that is **not** H-Mumford-stable. Then either $\epsilon = 0$ or else there is an effective curve $C \subset S$ such that:

$$\text{(a)} \quad C \cdot c_1(L) \leq \frac{1}{2}c_1^2(L) \quad \text{and} \quad \text{(b)} \quad C \cdot c_1(L) \leq C^2 + d$$

and it follows that $-d < C^2 \leq d$. Moreover,

$$\text{(c)} \quad c_1^2(L) > 4d \Rightarrow C^2 < d \quad \text{and} \quad \text{(d)} \quad c_1^2(L) > (d+1)^2 \Rightarrow C^2 \leq 0.$$

Proof: By definition of (non)-stability, there is a rank-one subsheaf $K \subset E$ such that $c_1(K) \cdot c_1(L) \geq \frac{1}{2}c_1(E) \cdot c_1(L) = \frac{1}{2}c_1^2(L)$. We may assume that the quotient $Q = E/K$ is torsion-free, replacing Q by Q/Q_{tors} and K by the kernel of the map $E \to Q/Q_{\mathrm{tors}}$ if necessary (which can only increase $c_1(K) \cdot c_1(L)$). It follows that the induced map $K \to L \otimes \mathcal{I}_Z$ is non-zero and either K splits the sequence, or else $K \subset L \otimes \mathcal{I}_Z$ is a proper subsheaf. In the latter case, $K = L(-C) \otimes \mathcal{I}_W$ for some effective curve C and zero-dimensional $W \subset S$, and (a) now follows immediately.

The inequality (b) is seen by computing the second Chern character of E in two different ways. The quotient Q has the form $\mathcal{O}_S(C) \otimes \mathcal{I}_V$ for some $V \subset S$, necessarily of dimension zero since $c_1(K) + c_1(Q) = c_1(L)$, and in particular

$$\mathrm{ch}_2(E) = \frac{c_1^2(L)}{2} - d = \frac{(c_1(L) - C)^2}{2} - l(W) + \frac{C^2}{2} - l(V) \leq \frac{c_1^2(L)}{2} + C^2 - C \cdot c_1(L)$$

which gives (b).

Next, applying Hodge Index and (a) and (b) give

$$C^2 c_1^2(L) \leq (C \cdot c_1(L))^2 \leq \frac{1}{2}c_1^2(L)\left(C^2 + d\right)$$

from which we conclude that $C^2 \leq d$. That $C^2 > -d$ follows immediately from (b) and the fact that L is ample. Finally, suppose $C^2 = d - k$ for $0 \leq k < d$ and apply Hodge Index and (b) to conclude that:

$$(d-k)c_1^2(L) \leq (C \cdot c_1(L))^2 \leq (2d-k)^2.$$

In particular, $c_1^2(L) \le 4d + \frac{k^2}{d-k}$ and then (the contrapositives of) (c) and (d) follow from the cases $k = 0$ and $k \le d-1$, respectively.

All of this gives as an immediate corollary a basic version of

Reider's Theorem: If L is an ample line bundle on a smooth projective surface S such that $c_1^2(L) > (d+1)^2$ and $C \cdot c_1(L) > C^2 + d$ for all effective divisors C on S satisfying $C^2 \le 0$, then "$K_S + L$ separates length d subschemes of S," i.e.

$$\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_Z) = 0$$

for all subschemes $Z \subset S$ of length d (or less).

Corollary (Fujita's Conjecture for Surfaces): If L is an ample line bundle on a smooth projective surface S, then $K_S + (d+2)L$ separates length d subschemes.

Note: For other versions of Reider's theorem, see e.g. [**Laz97**].

3. Reider Revisited

A torsion-free coherent sheaf E is H-*Mumford semi-stable* (for X and H as in §2) if

$$\mu_H(K) \le \mu_H(E)$$

for all subsheaves $K \subseteq E$ (where μ_H is the Mumford slope from §2). A Mumford H-semi-stable sheaf E has a *Jordan-Hölder filtration*

$$F_1 \subset F_2 \subset \cdots \subset F_M = E,$$

where the F_{i+1}/F_i are Mumford H-stable sheaves all of the same slope $\mu_H(E)$. Although the filtration is not unique, in general, the associated graded coherent sheaf $\oplus F_i = \mathrm{Ass}_H(E)$ is uniquely determined by the semi-stable sheaf E (and H).

The Mumford H-slope has the following additional crucial property:

Harder-Narasimhan Filtration: Every coherent sheaf E on X admits a uniquely determined (finite) filtration by coherent subsheaves

$$0 \subset E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_N = E \qquad \text{such that}$$

- E_0 is the torsion subsheaf of E and
- Each E_i/E_{i-1} is H-semi-stable of slope μ_i with $\mu_1 > \mu_2 > \cdots > \mu_N$.

Harder-Narasimhan filtrations for a fixed ample divisor class H give rise to a family of "torsion pairs" in the category of coherent sheaves on X:

Definition: A pair $(\mathcal{F}, \mathcal{T})$ of full subcategories of a fixed abelian category $\mathcal{A}$ is a *torsion pair* if:

(a) For all objects $T \in \mathrm{ob}(\mathcal{T})$ and $F \in \mathrm{ob}(\mathcal{F})$, $\mathrm{Hom}(T, F) = 0$.

(b) Each $A \in \mathrm{ob}(\mathcal{A})$ fits into a (unique) extension $0 \to T \to A \to F \to 0$ for some (unique up to isomorphism) objects $T \in \mathrm{ob}(\mathcal{T})$ and $F \in \mathrm{ob}(\mathcal{F})$.

Application 3.1: For each real number s, let $\mathcal{T}_s$ and $\mathcal{F}_s$ be full subcategories of the category $\mathcal{A}$ of coherent sheaves on X that are closed under extensions and which are generated by, respectively:

$\mathcal{F}_s \supset \{$torsion-free H-stable sheaves of H-slope $\mu \le s\}$

$\mathcal{T}_s \supset \{$torsion-free H-stable sheaves of H-slope $\mu > s\} \cup \{$torsion sheaves$\}$

Then $(\mathcal{F}_s, \mathcal{T}_s)$ is a torsion pair of $\mathcal{A}$.

Proof: Part (a) of the definition follows from the fact that $\mathrm{Hom}(T, F) = 0$ if T, F are H-stable and $\mu_H(T) > \mu_H(F)$, together with the fact that $\mathrm{Hom}(T, F) = 0$ if T is torsion and F is torsion-free.

A coherent sheaf E is either torsion (hence in $\mathcal{T}_s$ for all s) or else let $E(s) := E_i$ be the largest subsheaf in the Harder-Narasimhan of E with the property that $\mu(E_i/E_{i-1}) > s$. Then $0 \to E(s) \to E \to E/E(s) \to 0$ is the desired short exact sequence for (b) of the definition.

Theorem (Happel-Reiten-Smalø) [HRS96]: Given a torsion pair $(\mathcal{T}, \mathcal{F})$, then there is a t-structure on the bounded derived category $\mathcal{D}(\mathcal{A})$ defined by:

$$\mathrm{ob}(\mathcal{D}^{\geq 0}) = \{E^\bullet \in \mathrm{ob}(\mathcal{D}) \mid \mathrm{H}^{-1}(E^\bullet) \in \mathcal{F}, \mathrm{H}^i(E^\bullet) = 0 \text{ for } i < -1\}$$

$$\mathrm{ob}(\mathcal{D}^{\leq 0}) = \{E^\bullet \in \mathrm{ob}(\mathcal{D}) \mid \mathrm{H}^0(E^\bullet) \in \mathcal{T}, \mathrm{H}^i(E^\bullet) = 0 \text{ for } i > 0\}$$

In particular, the heart of the t-structure:

$$\mathcal{A}_{(\mathcal{F},\mathcal{T})} := \{E^\bullet \mid \mathrm{H}^{-1}(E^\bullet) \in \mathcal{F}, \mathrm{H}^0(E^\bullet) \in \mathcal{T}, \mathrm{H}^i(E^\bullet) = 0 \text{ otherwise}\}$$

is an abelian category (referred to as the "tilt" of $\mathcal{A}$ with respect to $(\mathcal{F}, \mathcal{T})$).

Notation: We will let $\mathcal{A}_s$ denote the tilt with respect to $(\mathcal{F}_s, \mathcal{T}_s)$ (for fixed H).

In practical terms, the category $\mathcal{A}_s$ consists of:

- Extensions of torsion and H-stable sheaves T of slope $> s$
- Extensions of shifts $F[1]$ of H-stable sheaves F of slope $\leq s$
- Extensions of a sheaf T by a shifted sheaf $F[1]$.

Extensions in $\mathcal{A}_s$ of coherent sheaves T_1, T_2 in $\mathcal{T}$ or shifts of coherent sheaves $F_1[1], F_2[1]$ in $\mathcal{F}$ are given by extension classes in $\mathrm{Ext}^1_{\mathcal{A}}(T_1, T_2)$ or $\mathrm{Ext}^1_{\mathcal{A}}(F_1, F_2)$, which are first extension classes in the category of coherent sheaves.

However, an extension of a coherent sheaf T by a shift $F[1]$ in $\mathcal{A}_s$ is quite different. It is given by an element of $\mathrm{Ext}^1_{\mathcal{A}_s}(T, F[1])$ by definition, but:

$$\mathrm{Ext}^1_{\mathcal{A}_s}(T, F[1]) = \mathrm{Ext}^2_{\mathcal{O}_X}(T, F)$$

and this observation will allow us to associate objects of $\mathcal{A}_s$ to certain "higher" extension classes of coherent sheaves in $\mathcal{A}$ just as coherent sheaves are associated to first extension classes of coherent sheaves.

For a general introduction to derived categories, we find the reference [**M**] to be quite useful. A quick introduction to some important notions is contained in [**AB**]. For this paper, we will need to recall that the shift $F[1]$ of a coherent sheaf is the unique object of the derived category of coherent sheaves on X satisfying $\mathrm{H}^{-1}(F[1]) = F$ and $\mathrm{H}^i(F[1])) = 0$ for $i \neq -1$, that each short exact sequence:

$$0 \to K^\bullet \to E^\bullet \to Q^\bullet \to 0$$

of objects in each of the abelian categories $\mathcal{A}_s$ induces a long exact sequence

$$0 \to \mathrm{H}^{-1}(K^\bullet) \to \mathrm{H}^{-1}(E^\bullet) \to \mathrm{H}^{-1}(Q^\bullet) \to$$

$$\to \mathrm{H}^0(K^\bullet) \to \mathrm{H}^0(E^\bullet) \to \mathrm{H}^0(Q^\bullet) \to 0$$

of coherent sheaves on X, and finally that in the Grothendieck group $K(\mathcal{D})$ of the derived category, the class of each $E^\bullet$ in each $\mathcal{A}_s$ satisfies

$$[E^\bullet] = [\mathrm{H}^0(E^\bullet)] - [\mathrm{H}^{-1}(E^\bullet)].$$

Next, recall that the **rank** is an integer-valued linear function

$$r : K(\mathcal{D}) \to \mathbb{Z}$$

with the property that $r([E]) \geq 0$ for all classes of coherent sheaves E.

Always fixing H, we may define an analogous rank function for each $s \in \mathbb{R}$:

$$r_s : K(\mathcal{D}) \to \mathbb{R};\ r_s([E^\bullet]) = c_1(E^\bullet) \cdot H^{n-1} - s \cdot r(E^\bullet)H^n$$

which by definition has the property that $r_s([E^\bullet]) \geq 0$ for all objects $E^\bullet$ of $\mathcal{A}_s$ and in addition $r_s([T]) > 0$ for all coherent sheaves in $\mathcal{T}_s$ that are supported in codimension ≤ 1. This rank is evidently rational-valued if $s \in \mathbb{Q}$.

Now consider the objects $\mathcal{O}_X[1]$ and $L \otimes \mathcal{I}_Z$ of $\mathcal{A}_s$ for $0 \leq s < 1$ ($H = c_1(L)$) where $Z \subset X$ is any closed subscheme supported in codimension ≥ 2.

Sub-objects of $\mathcal{O}_X[1]$ in $\mathcal{A}_s$:

An exact sequence $0 \to K^\bullet \to \mathcal{O}_X[1] \to Q^\bullet \to 0$ of objects of $\mathcal{A}_s$ (for any $s \geq 0$) induces a long exact sequence of cohomology sheaves:

$$0 \to \mathrm{H}^{-1}(K^\bullet) \to \mathcal{O}_X \to \mathrm{H}^{-1}(Q^\bullet) \xrightarrow{\delta} \mathrm{H}^0(K^\bullet) \to 0$$

Since $\mathrm{H}^{-1}(Q^\bullet)$ is torsion-free and δ is not a (non-zero) isomorphism, then either:

(*i*) $\mathrm{H}^{-1}(K^\bullet) = \mathcal{O}_X, Q^\bullet = 0, \delta = 0$ and $K^\bullet = \mathcal{O}_X[1]$,

(*ii*) $\mathrm{H}^{-1}(K^\bullet) = 0, \delta = 0$ and $Q^\bullet = \mathcal{O}_X[1]$, or else

(*iii*) $\mathrm{H}^{-1}(K^\bullet) = 0$ and $\delta \neq 0$. Then $\mathrm{H}^0(K^\bullet)$ has no torsion subsheaf supported in codimension two (since it would lift to a torsion subsheaf of $\mathrm{H}^{-1}(Q^\bullet)$). Moreover, $Q^\bullet = \mathrm{H}^{-1}(Q^\bullet)$ is the shift $Q[1]$ of a non-zero torsion-free sheaf Q that satisfies:

$$0 \leq r_s(Q[1]) = -c_1(Q) \cdot H^{n-1} + s \cdot r(Q)H^n < r_s(\mathcal{O}_X[1]) = sH^n$$

hence in particular the (ordinary) rank $r(Q) > 1$, and:

$$s\left(1 - \frac{1}{r(Q)}\right) < \mu_H(Q) \leq s.$$

Moreover, if E is a stable coherent sheaf appearing in the associate graded of a semi-stable coherent sheaf in the Harder-Narasimhan filtration of Q, then the same inequality holds for $\mu_H(E)$ (because the r_s rank is additive).

Sub-objects of $L \otimes \mathcal{I}_Z$ in $\mathcal{A}_s$:

An exact sequence: $0 \to K'^\bullet \to L \otimes \mathcal{I}_Z \to Q'^\bullet \to 0$ in $\mathcal{A}_s$ (for any $s < 1$) induces a long exact sequence of cohomology sheaves

$$0 \to \mathrm{H}^{-1}(Q'^\bullet) \to \mathrm{H}^0(K'^\bullet) \to L \otimes \mathcal{I}_Z \to \mathrm{H}^0(Q'^\bullet) \to 0$$

from which it follows that $K' := \mathrm{H}^0(K'^\bullet)$ is a torsion-free sheaf, and either:

(*i'*) $K' = 0$ and $Q'^\bullet = L \otimes \mathcal{I}_Z$,

(*ii'*) $r(K') = 1$, so that $K' = L \otimes \mathcal{I}_{Z'}$ and $Q'^\bullet = \mathrm{H}^0(Q'^\bullet) \cong L \otimes (\mathcal{I}_Z/\mathcal{I}_{Z'})$ for some closed subscheme Z' containing (and possibly equal to) Z, or else

(*iii'*) $r(K') > 1$ and $\mathrm{H}^{-1}(Q'^\bullet) \neq 0$.

In cases (ii') and (iii'), we have the inequality:

$$s < \mu_H(K') \le s + \frac{(1-s)}{r(K')}$$

and the same inequality when K' is replaced by any E' appearing in the associated graded of a semi-stable coherent sheaf in the Harder-Narasimhan filtration of K'.

Corollary 3.2: The alternatives for a non-zero homomorphism

$$f \in \mathrm{Hom}_{\mathcal{A}_s}(L \otimes \mathcal{I}_Z, \mathcal{O}_X[1]) = \mathrm{Ext}^1_{\mathcal{O}_X}(L \otimes \mathcal{I}_Z, \mathcal{O}_X) \text{ for some fixed } 0 < s < 1$$

are as follows:

(a) f is injective, with quotient $Q^\bullet = Q[1]$,

$$0 \to L \otimes \mathcal{I}_Z \xrightarrow{f} \mathcal{O}_X[1] \to Q[1] \to 0,$$

which in particular implies that $1/2 = \mu_H(Q) \le s$ and, more generally, that each stable E in the Harder-Narasimhan filtration of Q has Mumford-slope $\mu_H(E) \le s$.

(b) f is surjective, with kernel $(K')^\bullet = K'$,

$$0 \to K' \to L \otimes \mathcal{I}_Z \xrightarrow{f} \mathcal{O}_X[1] \to 0,$$

which in particular implies that $1/2 = \mu_H(K') > s$ and, more generally, that each stable E' in the Harder-Narasimhan filtration of K' has Mumford-slope $\mu_H(E') > s$.

(c) f is neither injective nor surjective, inducing a long exact sequence

$$0 \to L(-D) \otimes \mathcal{I}_W \to L \otimes \mathcal{I}_Z \xrightarrow{f} \mathcal{O}_X[1] \to (\mathcal{O}_X(D) \otimes \mathcal{I}_V)[1] \to 0$$

for some effective divisor D satisfying $D \cdot H^{n-1} \le sH^n$ and $D \cdot H^{n-1} < (1-s)H^n$, as well as subschemes $V, W \subset X$ supported in codim ≥ 2.

Proof: Immediate from the considerations above.

Example: At $s = 1/2$, we nearly get the same dichotomy as in §2. Here

$$f \text{ is injective in } \mathcal{A}_{1/2} \Leftrightarrow E \text{ is } H\text{-semistable}$$

where E is the rank two sheaf of slope $\mu_H(E) = \frac{1}{2}$ defined by the extension class determined by f. If E is H-semistable, then $E[1]$ is an object of $\mathcal{A}_{1/2}$, and the sequence of (a) is the tilt of the extension. Conversely, if f is injective in $\mathcal{A}_{1/2}$, then $E = Q$ in (a) and is H-semi-stable since non-semistable sheaves of slope $\frac{1}{2}$ do not belong to $\mathcal{F}_{\frac{1}{2}}$. If f is surjective, then $E = K'$ in (b), but this yields a contradiction (so f cannot be surjective), and if f is neither surjective nor injective, then E is an extension of $\mathcal{O}_X(D) \otimes \mathcal{I}_V$ by $L(-D) \otimes \mathcal{I}_W$ in (c) which is visibly not H-semistable.

Next, recall that the ordinary **degree** is an integer-valued linear function

$$d : K(\mathcal{D}) \to \mathbb{Z}; \quad d([E]) = c_1(E) \cdot H^{n-1}$$

(depending upon H) with the property that for all coherent sheaves E,

$$r(E) = 0 \Rightarrow \big(d(E) \ge 0 \text{ and } d(E) = 0 \Leftrightarrow E \text{ is supported in codim} \ge 2\big).$$

There is an analogous two-parameter family of degree functions ($s \in \mathbb{R}, t > 0$),

$$d_{(s,t)} : K(\mathcal{D}) \to \mathbb{R}; \quad d_{(s,t)}([E^\bullet]) =$$

$$\mathrm{ch}_2(E^\bullet) \cdot H^{n-2} - sc_1(E^\bullet) \cdot H^{n-1} + \left(\frac{s^2 - t^2}{2}\right) r(E^\bullet) H^n$$

(i.e. there is a ray of degree functions parametrized by t for each rank r_s).

Note that if $E^\bullet$ is an object of $\mathcal{A}_s$ and $r_s([E^\bullet]) = 0$, then $\mathrm{H}^0(E^\bullet)$ is a torsion sheaf supported in codimension ≥ 2, $\mathrm{H}^{-1}(E^\bullet)$ is an H-semistable coherent sheaf with $\mu_H(\mathrm{H}^{-1}(E^\bullet)) = s$, and the "cohomology" sequence:

$$0 \to \mathrm{H}^{-1}(E^\bullet)[1] \to E^\bullet \to \mathrm{H}^0(E^\bullet) \to 0$$

exhibits $E^\bullet$ as an extension in $\mathcal{A}_s$ of a torsion sheaf by the shifted semistable sheaf.

Proposition 3.3: Suppose $r_s([E^\bullet]) = 0$ for an object $E^\bullet$ of $\mathcal{A}_s$. Then for all $t > 0$,

$$d_{(s,t)}([E^\bullet]) \geq 0 \text{ and } d_{(s,t)}([E^\bullet]) = 0 \Leftrightarrow E^\bullet \text{ is a sheaf, supported in codim} \geq 3.$$

Proof: Because $d_{(s,t)}$ is linear, it suffices to prove the Proposition for torsion sheaves T supported in codimension ≥ 2 and for shifts $F[1]$ of H-*stable* torsion-free sheaves of slope s. In the former case

$$d_{(s,t)}(T) = \mathrm{ch}_2(T) \cdot H^{n-2} \geq 0 \text{ with equality} \Leftrightarrow T \text{ is supported in codim} \geq 3.$$

In the latter case

$$d_{(s,t)}(F[1]) = -\mathrm{ch}_2(F) \cdot H^{n-2} + sc_1(F) \cdot H^{n-1} - \left(\frac{s^2 - t^2}{2}\right) r(F)H^n$$

and $\mu_H(F) = s$ implies $(c_1(F) - sr(F)H) \cdot H^{n-1} = 0$, which in turn implies that $(c_1(F) - sr(F)H)^2 H^{n-2} \leq 0$ by the Hodge Index Theorem. It follows from the Bogomolov inequality that

$$d_{(s,t)}(F[1]) \geq -\left(\frac{c_1^2(F)}{2r(F)}\right) \cdot H^{n-2} + sc_1(F) \cdot H^{n-1} - \left(\frac{s^2}{2}\right) r(F)H^n + \left(\frac{t^2}{2}\right) r(F)H^n$$

$$= -\left(\frac{1}{2r(F)}\right) (c_1(F) - sr(F)H)^2 \cdot H^{n-2} + \left(\frac{t^2}{2}\right) r(F)H^n > 0. \quad \square$$

Corollary 3.4: If $X = S$ is a surface, then the complex linear function

$$Z_{s+it} := (-d_{(s,t)} + itr_s) : K(\mathcal{D}) \to \mathbb{C}; \;\; s \in \mathbb{R}, t > 0, i^2 = -1$$

has the property that $Z_{s+it}(E^\bullet) \neq 0$ for all nonzero objects $E^\bullet$ of $\mathcal{A}_s$, and:

$$0 < \arg(Z_{s+it}(E^\bullet)) \leq 1 \text{ (where } \arg(re^{i\pi\rho}) = \rho)$$

i.e. Z_{s+it} takes values in the (extended) upper half plane.

In higher dimensions, the Corollary holds modulo coherent sheaves supported in codimension ≥ 3, just as the ordinary H-degree and rank lead to the same conclusion modulo torsion sheaves supported in codimension ≥ 2.

Remark: The "central charge" Z_{s+it} has the form

$$Z_{s+it}(E) = -d_{(s,t)}(E) + itr_s(E) = -\int_S e^{-(s+it)H} \mathrm{ch}(E) H^{n-2},$$

which is a much more compact (and important) formulation.

Corollary 3.5: Each "slope" function

$$\mu := \mu_{s+it} = \frac{d_{(s,t)}}{tr_s} = -\frac{\mathrm{Re}(Z_{s+it})}{\mathrm{Im}(Z_{s+it})}$$

has the usual properties of a slope function on the objects of $\mathcal{A}_s$. That is, given an exact sequence of objects of $\mathcal{A}_s$:

$$0 \to K^\bullet \to E^\bullet \to Q^\bullet \to 0$$

then $\mu(K^\bullet) < \mu(E^\bullet) \Leftrightarrow \mu(E^\bullet) < \mu(Q^\bullet)$ and $\mu(K^\bullet) = \mu(E^\bullet) \Leftrightarrow \mu(E^\bullet) = \mu(Q^\bullet)$.

Also, when we make the usual

Definition: $E^\bullet$ is μ-stable if $\mu(K^\bullet) < \mu(E^\bullet)$ whenever $K^\bullet \subset E^\bullet$ and the quotient has nonzero central charge (i.e. is not a torsion sheaf supported in codim ≥ 3),

then $\mathrm{Hom}(E^\bullet, F^\bullet) = 0$ whenever $E^\bullet, F^\bullet$ are μ-stable and $\mu(E^\bullet) > \mu(F^\bullet)$.

Proof (of the Corollary): Simple arithmetic.

Example: In dimension $n \geq 2$

$$\mu_{s+it}(\mathcal{O}_X[1]) = \frac{t^2 - s^2}{2st} \quad \text{and} \quad \mu_{s+it}(L \otimes \mathcal{I}_Z) = \frac{(1-s)^2 - t^2 - \frac{2d}{H^n}}{2t(1-s)}$$

where $d = [Z] \cap H^{n-2}$ is the (codimension two) degree of the subscheme $Z \subset X$. Thus $\mu_{s+it}(L \otimes \mathcal{I}_Z) > \mu_{s+it}(\mathcal{O}_X[1])$ if and only if

$$t^2 + \left(s - \left(\frac{1}{2} - \frac{d}{H^n}\right)\right)^2 < \left(\frac{1}{2} - \frac{d}{H^n}\right)^2 \text{ and } t > 0.$$

This describes a nonempty subset (interior of a semicircle) of $\mathbb{R}^2$ if $H^n > 2d$.

Proposition 3.6: For all smooth projective varieties X of dimension ≥ 2 (and L)

(a) $\mathcal{O}_X[1]$ is a μ_{s+it}-stable object of $\mathcal{A}_s$ for all $s \geq 0$ and $t > 0$.

(b) L is a μ_{s+it}-stable object of $\mathcal{A}_s$ for all $s < 1$ and $t > 0$.

Proof: (a) Suppose $0 \neq K^\bullet \subset \mathcal{O}_X[1]$, and let E be an H-stable torsion-free sheaf in the associated graded of Q, where $Q[1]$ is the quotient object. Recall that $0 < \mu_H(E) \leq s$. The Proposition follows once we show $\mu_{s+it}(\mathcal{O}_X[1]) < \mu_{s+it}(E[1])$ for all E with these properties. We compute

$$\mu_{s+it}(E[1]) = \frac{-2\mathrm{ch}_2(E)H^{n-2} + 2sc_1(E)H^{n-1} - \left(s^2 - t^2\right) r(E)H^n}{2t(-c_1(E)H^{n-1} + sr(E)H^n)}$$

and we conclude (using the computation of $\mu_{s+it}(\mathcal{O}_X[1])$ above) that

$$\mu_{s+it}(\mathcal{O}_X[1]) > \mu_{s+it}(E[1]) \Leftrightarrow (s^2 + t^2)c_1(E)H^{n-1} > (2s)\mathrm{ch}_2(E)H^{n-2}.$$

But by the Bogomolov Inequality

$$(2s)\mathrm{ch}_2(E)H^{n-2} \leq s(c_1^2(E)H^{n-2})/r(E),$$

and by the Hodge Index Theorem and the inequality $c_1(E) \cdot H^{n-1} \leq sr(E)H^n$,

$$sc_1^2(E)H^{n-2}/r(E) \leq s^2 c_1(E)H^{n-1}$$

The desired inequality follows from the fact that $t > 0$ and $c_1(E) \cdot H^{n-1} > 0$.

The proof of (b) proceeds similarly. Suppose $0 \neq (K')^\bullet \subset L$ in $\mathcal{A}_s$, and let E' be an H-stable coherent sheaf in the Harder-Narasimhan filtration of $K' = H^0(K'^\bullet)$. Then $s < \mu_H(E') < 1$, and we need to prove that $\mu_{s+it}(E') < \mu_{s+it}(L)$. This follows as in (a) from the Bogomolov Inequality and Hodge Index Theorem. □

Corollary 3.7 (Special case of Kodaira vanishing): If $\dim X = n > 1$, then

$$\mathrm{H}^{n-1}(X, K_X + L) = 0.$$

Proof: Within the semicircle $\{(s,t) \mid t^2 + \left(s - \frac{1}{2}\right)^2 < \frac{1}{4} \text{ and } t > 0\}$ the inequality $\mu_{s+it}(L) > \mu_{s+it}(\mathcal{O}_X[1])$ holds. But L and $\mathcal{O}_X[1]$ are always μ_{s+it}-stable, hence

$$0 = \operatorname{Hom}_{\mathcal{A}_s}(L, \mathcal{O}_X[1]) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(L, \mathcal{O}_X) \cong \mathrm{H}^{n-1}(X, K_X + L)^\vee$$

□

Remark: The Bogomolov Inequality and Hodge Index Theorem are trivially true in dimension one. However, the computation of $\mu_{s+it}(L)$ is different in dimension one, and indeed in that case the inequality $\mu_{s+it}(L) > \mu_{s+it}(\mathcal{O}_X[1])$ never holds (and evidently the corollary is false in dimension one)!

Restrict attention to $X = S$ a surface for the rest of this section, and consider

$$\mathcal{I}_W^\vee[1],$$

the shifted derived dual of the ideal sheaf of a subscheme $W \subset S$ of length d. Since

$$\mathrm{H}^{-1}(\mathcal{I}_W^\vee[1]) = \mathcal{O}_S \text{ and } \mathrm{H}^0(\mathcal{I}_W^\vee[1]) \text{ is a torsion sheaf, supported on } W,$$

it follows that $\mathcal{I}_W^\vee[1]$ is in $\mathcal{A}_s$ for all $s \geq 0$.

Every quotient object $\mathcal{I}_W^\vee[1] \to Q^\bullet$ in $\mathcal{A}_s$ satisfies:

- $\mathrm{H}^0(Q^\bullet)$ is supported in codimension two (on the scheme W, in fact).
- Let $Q = \mathrm{H}^{-1}(Q^\bullet)$ (a torsion-free sheaf). Then every H-stable term E in the Harder-Narasimhan filtration of Q satisfies:

$$0 \leq r_s(E[1]) = -c_1 H + rsH^2 < r_s(\mathcal{I}_W^\vee[1]) = sH^2,$$

where $r = r(E)$ and $c_1 = c_1(E)$ (the second inequality follows because the r_s-rank of the kernel of $\mathcal{I}_W^\vee[1] \to Q^\bullet$ in $\mathcal{A}_s$ is positive). Therefore,

$$(r-1)sH^2 < c_1 H \leq rsH^2.$$

Proposition 3.8: For subschemes $Z, W \subset S$ of the same length d (and $H = c_1(L)$):

(a) If $H^2 > 8d$, then $\mu_{s+it}(L \otimes \mathcal{I}_Z) > \mu_{s+it}(\mathcal{I}_W^\vee[1])$ for all (s,t) in the semicircle

$$C(d, H^2) := \left\{ (s,t) \mid t^2 + \left(s - \frac{1}{2}\right)^2 < \frac{1}{4} - \frac{2d}{H^2} \text{ and } t > 0 \right\}$$

centered at the point $(1/2, 0)$ (and the semicircle is nonempty!).

(b) If $H^2 > 8d$ and $\mathcal{I}_W^\vee[1]$ or $L \otimes \mathcal{I}_Z$ is not stable at $(s,t) = (\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}})$, then there is a divisor D on S and an integer $r > 0$ such that

$$\frac{r-1}{2} H^2 < D \cdot H \leq \frac{r}{2} H^2, \text{ and } \frac{D}{r} \cdot H < \frac{D^2}{r^2} + 2d.$$

Proof: Part (a) is immediate from

$$\mu_{s+it}(L \otimes \mathcal{I}_Z) = \frac{(1-s)^2 - t^2 - \frac{2d}{H^2}}{2(1-s)t} \text{ and } \mu_{s+it}(\mathcal{I}_W^\vee[1]) = \frac{t^2 - s^2 + \frac{2d}{H^2}}{2st}.$$

We prove part (b) for $\mathcal{I}_W^\vee[1]$ (the proof for $L \otimes \mathcal{I}_Z$ is analogous).

Let $\mathcal{I}_W^\vee[1] \to Q^\bullet$ be a surjective map in the category $\mathcal{A}_s$ and let $Q = \mathrm{H}^{-1}(Q^\bullet)$. Since $\mathrm{H}^0(Q^\bullet)$ is torsion, supported on W, it follows that $\mu_{s+it}(Q[1]) \leq \mu_{s+it}(Q^\bullet)$ with equality if and only if $\mathrm{H}^0(Q^\bullet) = 0$.

Thus if $\mathcal{I}_W^\vee[1]$ is not μ_{s+it}-stable, then $\mu_{s+it}(\mathcal{I}_W^\vee[1]) \geq \mu_{s+it}(Q[1])$ for some torsion-free sheaf Q satisfying $(r-1)sH^2 < c_1(Q)\cdot H \leq rsH^2$, and moreover, the same set of inequalities hold for (at least) one of the stable torsion-free sheaves E appearing in the Harder-Narasimhan filtration of Q. We let $D = c_1(E)$ and $r = \mathrm{rk}(E)$. Then $\mu_{s+it}(\mathcal{I}_W^\vee[1]) \geq \mu_{s+it}(E[1])$ if and only if

$$(t^2+s^2)(D\cdot H) \leq (2s)\mathrm{ch}_2(E) + \frac{2d}{H^2}(rsH^2 - D\cdot H)$$

and by the Bogomolov inequality, $(2s)\mathrm{ch}_2(E) \leq s\frac{D^2}{r}$. Setting $(s,t) = (\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}})$, we obtain the desired inequalities. □

Corollary 3.9: (a) If $L = \mathcal{O}_S(H)$ is ample on S and satisfies $H^2 > (2d+1)^2$ and

$$\mathrm{H}^1(S, K_S \otimes L \otimes I_W \otimes I_Z) \neq 0$$

for a pair $Z, W \subset S$ of length d subschemes, then there is a divisor D on S satisfying $D^2 \leq 0$ and $0 < D\cdot H \leq D^2 + 2d$.

(b) (Fujita-type result) If L is an ample line bundle on S, then

$$\mathrm{H}^1(S, K_S \otimes L^{\otimes(2d+2)} \otimes I_W \otimes I_Z) = 0$$

for all subschemes $Z, W \subset S$ of length d (or less).

Proof: Part (b) immediately follows from (a). By Serre duality,

$$\mathrm{H}^1(S, K_S \otimes L \otimes I_W \otimes I_Z) \cong \mathrm{Hom}_{\mathcal{A}_{\frac{1}{2}}}(L\otimes \mathcal{I}_Z, \mathcal{I}_W^\vee[1])^\vee.$$

Since $(2d+1)^2 \geq 8d+1$ for $d \geq 1$, the non-vanishing of H^1 and Proposition 3.8(a) imply that either $L\otimes\mathcal{I}_Z$ or $\mathcal{I}_W^\vee[1]$ must not be stable at $(\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}})$, and by Proposition 3.8(b) there is a divisor D and integer $r \geq 1$ such that the $\mathbb{Q}$-divisor $C = D/r$ satisfies

$$(1-\frac{1}{r})\frac{H^2}{2} < C\cdot H \leq \frac{H^2}{2} \;\text{ and }\; C\cdot H \leq C^2 + 2d$$

(similar to Application 2.2). The result now follows as in Application 2.2 once we prove that $C^2 \geq 1$ whenever $r > 1$.[1] To this end, note:

(i) $r \geq 3 \Rightarrow C^2 + 2d \geq C\cdot H > \frac{H^2}{3} > \frac{8d+1}{3} \Rightarrow C^2 > \frac{2d+1}{3} \geq 1$.

(ii) $r = 2 \Rightarrow C^2 + 2d \geq C\cdot H > \frac{H^2}{4} > 2d + \frac{1}{4} \Rightarrow \frac{H^2}{4} \geq 2d + \frac{1}{2}, C\cdot H \geq 2d+1$, and $C^2 \geq 1$, since C is of the form $D/2$ for an "honest" divisor D.

Thus either $C^2 \leq 0$, in which case $r = 1$ and $C = D$ is an "honest" divisor, or else $C^2 \geq 1$. Furthermore, by the Hodge index theorem,

$$C^2H^2 \leq (C\cdot H)^2 \leq \frac{H^2}{2}\left(C^2+2d\right) \Rightarrow C^2 \leq 2d$$

and if $C^2 = \kappa$ for $1 \leq \kappa \leq 2d$, then $\kappa^2H^2 \leq (C\cdot H)^2 \leq (\kappa+2d)^2$ and $H^2 \leq \left(1+\frac{2d}{\kappa}\right)^2$. This is a decreasing function, giving us $H^2 \leq (2d+1)^2$, contradicting $H^2 > (2d+1)^2$.

Remark: This variation resembles other variations of Reider's theorem, e.g. [**Lan99**], though the authors do not see how to directly obtain this result from the others.

[1] The authors thank Valery Alexeev for pointing out the embarrassing omission of this step in the original version of the paper.

In a special case, Proposition 3.8 can be made even stronger, as noted in [**AB**].

Proposition 3.10: If $\mathrm{Pic}(S) = \mathbb{Z}$, generated by $c_1(L) = H$, then the two objects $L \otimes \mathcal{I}_Z$ and $\mathcal{I}_W^\vee[1]$ are $\mu_{(\frac{1}{2},t)}$-stable for all $t > 0$ and any degree of Z (and W).

Proof: Again we do this for $\mathcal{I}_W^\vee[1]$, the proof for $L \otimes \mathcal{I}_Z$ being analogous. Consider again the condition on every subbundle $E \subset Q$, where $Q = \mathrm{H}^{-1}(Q^\bullet)$, and $Q^\bullet$ is a quotient object of $\mathcal{I}_W^\vee[1]$:

$$(r(E) - 1)(\frac{1}{2})H^2 < c_1(E) \cdot H \leq r(E)(\frac{1}{2})H^2$$

Since $c_1(E) = kH$ is an integer multiple of H, by assumption, it follows immediately that Q is itself of even rank and H-stable, satisfying $c_1(Q) = (r(Q)/2)\, H$. But in that case, $Q[1]$ has "Bridgeland rank" $r_{\frac{1}{2}}(Q[1]) = 0$, hence has maximal phase (infinite slope), and thus cannot destabilize $\mathcal{I}_W^\vee[1]$. □

Remark: The Proposition is not generally true for pairs (s, t) when $s \neq \frac{1}{2}$.

Corollary 3.11: If $\mathrm{Pic}(S) = \mathbb{Z}H$ and $H^2 > 8d$, then

$$\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_Z) = 0$$

for all pairs of subschemes $Z, W \subset S$ of length d.

4. Thaddeus Pairs Revisited

Suppose S is a surface with ample line bundle $L = \mathcal{O}_S(H)$ and that $\mathrm{Pic}(S) = \mathbb{Z}H$. Consider the objects of $\mathcal{A}_s$ $(0 < s < 1)$ appearing as extensions

$$\epsilon : 0 \to \mathcal{O}_S[1] \to E_\epsilon^\bullet \to L \to 0$$

parametrized by

$$\epsilon \in \mathrm{Ext}^1_{\mathcal{A}_s}(L, \mathcal{O}_S[1]) = \mathrm{Ext}^2_{\mathcal{O}_S}(L, \mathcal{O}_S) \cong \mathrm{H}^0(S, K_S \otimes L)^\vee$$

As we saw in Proposition 3.6 and the preceding calculation, $\mathcal{O}_S[1]$ and L are both μ_{s+it} stable for all (s, t). Moreover, $\mu_{s+it}(\mathcal{O}_S[1]) < \mu_{s+it}(L)$ inside the semicircle

$$C := \left\{ (s, t) \mid t^2 + \left(s - \frac{1}{2}\right)^2 < \frac{1}{4} \text{ and } t > 0 \right\}$$

Remark: Here and earlier, we are using the notion of stability a little bit loosely. The correct definition, given by Bridgeland [**Bri08**], requires the existence of finite-length Harder-Narasimhan filtrations for all objects of $\mathcal{A}_s$. This is straightforward to prove when (s, t) are both rational numbers (following Bridgeland), but much more subtle in the irrational case. For the purposes of this paper, the rational values will suffice.

We investigate the dependence of the $\mu_{\frac{1}{2}+it}$-stability of $E_\epsilon^\bullet$ upon the extension class ϵ for $\frac{1}{2} + it$ inside the semicircle S. If $E_\epsilon^\bullet$ is $\mu_{\frac{1}{2}+it}$-unstable, destabilized by

$$K^\bullet \subset E_\epsilon^\bullet; \text{ then:}$$

(i) $K^\bullet = \mathrm{H}^0(K^\bullet) =: K$ is a coherent sheaf with $\mu_{\frac{1}{2}+it}(K) > 0$.

(ii) K is H-stable of odd rank r and $c_1(K) = ((r + 1)/2)H$.

(iii) The induced map $K \to L$ is injective (in the category $\mathcal{A}_{\frac{1}{2}}$).

Thus as in the curve case, $E^\bullet_\epsilon$ can only be destabilized by lifting subobjects $K \subset L$ (in the category $\mathcal{A}_{\frac{1}{2}}$) of positive $\mu_{\frac{1}{2}+it}$-slope to subobjects of $E^\bullet_\epsilon$:

$$\begin{array}{ccccccccc} & & & & & & K & & \\ & & & & & \swarrow & \downarrow & & \\ (\dagger) \quad 0 & \to & \mathcal{O}_S[1] & \to & E^\bullet_\epsilon & \to & L & \to & 0 \end{array}$$

That is, the unstable objects $E^\bullet_\epsilon$ correspond to extensions in the kernel of the map

$$\mathrm{Ext}^2_{\mathcal{O}_S}(L, \mathcal{O}_S) \to \mathrm{Ext}^2_{\mathcal{O}_S}(K, \mathcal{O}_S)$$

for some mapping of coherent sheaves $K \to L$ with K satisfying (i) and (ii).

Proof (of (i)-(iii)): The $d_{(\frac{1}{2},t)}$-degree of $E^\bullet_\epsilon$ is

$$\mathrm{ch}_2(E^\bullet_\epsilon) - \frac{c_1 \cdot H}{2} + \frac{(\frac{1}{4} - t^2)}{2} rH^2 = 0$$

since $\mathrm{ch}_2(E^\bullet_\epsilon) = H^2/2, c_1 = H$ and $r = 0$. Thus the slope (equivalently, the degree) of any destabilizing $K^\bullet \subset E^\bullet_\epsilon$ is positive, by definition. Moreover the "ranks"

$$r_{\frac{1}{2}}(\mathcal{O}_S[1]) = r_{\frac{1}{2}}(L) = \frac{H^2}{2}$$

are the minimal possible (as in the curve case) without being zero; hence, as in the curve case, $K^\bullet \subset E^\bullet_\epsilon$ must also have minimal "rank" $r_{\frac{1}{2}}(K^\bullet) = \frac{H^2}{2}$ (if it had the next smallest "rank" $H^2 = r_{\frac{1}{2}}(E^\bullet_\epsilon)$, it would fail to destabilize). The presentation of $E^\bullet_\epsilon$ gives $\mathrm{H}^{-1}(E^\bullet_\epsilon) = \mathcal{O}_S$ and $\mathrm{H}^0(E^\bullet_\epsilon) = L$, hence if we let $Q^\bullet = E^\bullet_\epsilon / K^\bullet$, then

$$0 \to \mathrm{H}^{-1}(K^\bullet) \to \mathcal{O}_S \to \mathrm{H}^{-1}(Q^\bullet) \to \mathrm{H}^0(K^\bullet) \to L \to \mathrm{H}^0(Q^\bullet) \to 0$$

and, as usual, either $\mathrm{H}^{-1}(K^\bullet) = 0$ or $\mathrm{H}^{-1}(K^\bullet) = \mathcal{O}_S$. The latter is impossible, since in that case the rank consideration would give $K^\bullet = \mathcal{O}_S[1]$, which doesn't destabilize for $(\frac{1}{2}, t) \in C$. Thus $K^\bullet = K$ is a coherent sheaf. This gives (i).

Next, the condition that $r_{\frac{1}{2}}(K)$ be minimal implies that there can only be one term in the Harder-Narasimhan filtration of K (i.e. K is H-stable), and that

$$r_{\frac{1}{2}}(K) = c_1(K)H - \frac{r(K)H^2}{2} = \frac{H^2}{2}.$$

Since $c_1(K) = kH$ for some k, this gives (ii).

Finally, (iii) follows again from the minimal rank condition since any kernel of the induced map to L would be a torsion-free sheaf, of positive $r_{\frac{1}{2}}$-rank. $\square$

Suppose now that K satisfies (i) and (ii). By the Bogomolov inequality,

$$d_{(\frac{1}{2},t)}(K) \leq \frac{1}{2r}\left(c_1(K) - \frac{r}{2}H\right)^2 - \frac{rt^2H^2}{2} = \frac{H^2}{2r}\left(\frac{1}{4} - r^2t^2\right)$$

so in particular, $t \leq \frac{1}{2r}$, or in other words, we have shown:

Proposition 4.1: Given a positive integer r_0, if $t > \frac{1}{2r_0}$ and if $\mu_{\frac{1}{2}+it}(K) < 0$ for all $K \subset L$ (in $\mathcal{A}_{\frac{1}{2}}$) of odd ordinary rank $r \leq r_0$, then $E^\bullet_\epsilon$ is $\mu_{\frac{1}{2}+it}$-stable.

Special Case: Suppose $t > \frac{1}{6}$. Because $H = c_1(L)$ generates $\mathrm{Pic}(S)$ it follows that the only rank one subobjects $K \subset L$ in $\mathcal{A}_{\frac{1}{2}}$ are the subsheaves $L \otimes \mathcal{I}_Z$ for $Z \subset S$ of finite length. Thus $E^\bullet_\epsilon$ only fails to be $\mu_{\frac{1}{2}+it}$-stable if **both**

$$d_{(\frac{1}{2},t)}(L \otimes \mathcal{I}_Z) = \frac{1}{2}\left(\frac{1}{4} - t^2\right) H^2 - d \geq 0 \left(\Leftrightarrow t^2 \leq \frac{1}{4} - \frac{2d}{H^2}\right)$$

and $\epsilon \in \ker(\mathrm{Ext}^2(L, \mathcal{O}_S) \to \mathrm{Ext}^2(L \otimes \mathcal{I}_Z, \mathcal{O}_S))$, so that $L \otimes \mathcal{I}_Z \subset L$ lifts to a subobject of $E^\bullet_\epsilon$. As in the curve case, it can be shown using Serre duality that the image of such a (non-zero) extension in the projective space

$$\mathbb{P}(\mathrm{H}^0(S, K \otimes L)^\vee)$$

is a point of the *secant* $(d-1)$*-plane* spanned by $Z \subset S$ under the linear series map:

$$\phi_{K+L} : S - - > \mathbb{P}(\mathrm{H}^0(S, K \otimes L)^\vee).$$

By Corollary 3.11, this inequality on t guarantees $\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_Z)^\vee = 0$ for all subschemes $Z, W \subset S$ of length d, hence in particular, the $(d-1)$-secant planes spanned by $Z \subset S$ are well-defined.

Thus there are "critical points" or "walls" at $t = \sqrt{\frac{1}{4} - \frac{2d}{H^2}} > \frac{1}{6}$, i.e. $d < \frac{2H^2}{9}$ on the line $s = \frac{1}{2}$, where the objects $E^\bullet_\epsilon$ corresponding to points of the secant variety

$$\left(\mathrm{Sec}^{d-1}(S) - \mathrm{Sec}^{d-2}(S)\right) \subset \mathbb{P}(\mathrm{H}^0(S, K \otimes L)^\vee),$$

change from μ-stable to μ-unstable as t crosses the wall.

Moduli. The Chern class invariants of each $E^\bullet_\epsilon$ are:

$$\mathrm{ch}_2 = \frac{H^2}{2}, \quad c_1 = H, \quad r = 0$$

Thus it is natural to ask for the set of all $\mu_{\frac{1}{2}+it}$-stable objects with these invariants, and further to ask whether they have (projective) moduli that are closely related (by flips or flops) as t crosses over a critical point. In one case, this is clear:

Proposition 4.2: For $t > \frac{1}{2}$, the $\mu_{\frac{1}{2}+it}$-stable objects with Chern class invariants above are precisely the (Simpson)-stable coherent sheaves with these invariants, i.e. sheaves of pure dimension one and rank one on curves in the linear series $|H|$.

Proof: Suppose $E^\bullet$ has the given invariants and is not a coherent sheaf. Then the cohomology sequence

$$0 \to \mathrm{H}^{-1}(E^\bullet)[1] \to E^\bullet \to \mathrm{H}^0(E^\bullet) \to 0$$

destabilizes $E^\bullet$ for $t > \frac{1}{2}$ for the following reason. Let $E = \mathrm{H}^{-1}(E^\bullet)$. If $c_1(E) = kH$, then $k \leq \frac{r}{2}$ is required in order that $E[1] \in \mathcal{A}_{\frac{1}{2}}$. Moreover, since $r_{\frac{1}{2}}(E^\bullet) = H^2$ and $\mathrm{H}^0(E^\bullet)$ has positive (ordinary) rank, hence also positive $r_{\frac{1}{2}}$-rank, it follows that either $r_{\frac{1}{2}}(E[1]) = 0$ or $r_{\frac{1}{2}}(E[1]) = \frac{H^2}{2}$. But $r_{\frac{1}{2}}(E[1]) = 0$ implies $E[1]$ has maximal (infinite) slope, and then $E^\bullet$ is unstable (for all t). It follows similarly that if $r_{\frac{1}{2}}(E[1]) = \frac{H^2}{2}$, then $E^\bullet$ is unstable for all t unless E is H-stable, of rank $r = 2k+1$. In that case, by the Bogomolov inequality,

$$d_{(\frac{1}{2},t)}(E[1]) = -\mathrm{ch}_2(E) + \frac{c_1(E)H}{2} - \frac{(\frac{1}{4} - t^2)rH^2}{2} \geq \frac{H^2}{2}\left(t^2 r - \frac{1}{4r}\right)$$

and this is positive if $t > \frac{1}{2}$. □

In fact, at $t = \frac{1}{2}$ only $\mathrm{H}^{-1}(E^\bullet) = \mathcal{O}_S$ (the rank one case, moreover matching the Bogomolov bound) would fail to destabilize a non-sheaf $E^\bullet$, and conversely, among the coherent sheaves T with these invariants, only those fitting into an exact sequence (of objects of $\mathcal{A}_{\frac{1}{2}}$)

$$0 \to L \to T \to \mathcal{O}_S[1] \to 0$$

become unstable as t crosses $\frac{1}{2}$, and they are replaced by the "Thaddeus" extensions

$$0 \to \mathcal{O}_S[1] \to E^\bullet \to L \to 0.$$

The moduli of Simpson-stable coherent sheaves

$$M_S\left(0, H, \frac{H^2}{2}\right)$$

is known to be projective by a geometric invariant theory construction [**Sim94**]. It is the moduli of $\mu_{\frac{1}{2}+it}$-stable objects of $\mathcal{A}_{\frac{1}{2}}$ for $t > \frac{1}{2}$. The wall crossing at $t = \frac{1}{2}$ removes

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_S[1], L)) = \mathbb{P}(\mathrm{H}^0(S, L)) \subset M_S\left(0, H, \frac{H^2}{2}\right)$$

and replaces it with $\mathbb{P}(\mathrm{H}^0(S, K_S+L)^\vee)$ in what we conjecture to be a new projective birational model. In the case $K_S = 0$, the Simpson moduli spaces are holomorphic symplectic varieties, this new birational model is a Mukai flop of the moduli of stable sheaves, and the further wall crossings (up to $t = \frac{1}{6}$, when rank three bundles appear) all replace extensions of the form

$$0 \to L \otimes \mathcal{I}_Z \to (T \text{ or } E^\bullet) \to \mathcal{I}_W^\vee[1] \to 0$$

with

$$0 \to \mathcal{I}_W^\vee[1] \to E^\bullet \to L \otimes \mathcal{I}_Z \to 0.$$

This is achieved globally by Mukai flops, replacing projective bundles over the product $\mathrm{Hilb}^d(S) \times \mathrm{Hilb}^d(S)$ of Hilbert schemes with their dual bundles,

$$\mathbb{P}(\mathrm{H}^0(S, L \otimes \mathcal{I}_W \otimes \mathcal{I}_Z)) \leftrightarrow \mathbb{P}(\mathrm{H}^0(S, L \otimes \mathcal{I}_W \otimes \mathcal{I}_Z)^\vee)$$

This was constructed in detail in [**AB**].

General questions regarding moduli of Bridgeland-stable objects remain fairly wide open, however. Toda [**Tod08**] has shown that when S is a $K3$ surface, then the Bridgeland semistable objects of fixed numerical class are represented by an Artin stack of finite type. One expects the isomorphism classes of Bridgeland-stable objects, at least in special cases as above, to be represented by a proper scheme when (s, t) is not on a "wall." However:

Question 1: When are the isomorphism classes of Bridgeland-stable objects of fixed numerical type represented by a (quasi)-projective scheme of finite type?

Question 2: Conversely, is there an example where the isomorphism classes are represented by a proper algebraic space which is not a projective scheme? (The examples produced in [**AB**] are proper algebraic spaces. It is unknown whether they are projective.)

For each $t < \frac{1}{2}$, we make the following provisional

Definition: The space of t-stable Thaddeus pairs (given S and ample L) is the proper transform of the projective space of extensions $\mathbb{P}(\mathrm{Ext}^1(L, \mathcal{O}_S[1]))$ under the natural rational embedding in the moduli space of (isomorphism classes of) $\mu_{\frac{1}{2}+it}$-stable objects with invariants $(0, H, H^2/2)$.

Remark: Note that for $t < \frac{1}{6}$, this will contain objects that have no analogue in the curve case, corresponding to destabilizing Mumford-stable torsion-free sheaves K of higher odd rank r and first Chern class $c_1(K) = \frac{r+1}{2}H$.

Question 3. Can stable Thaddeus pairs, as a function of t (inside the moduli of $\mu_{\frac{1}{2}+it}$-stable objects of the same numerical class) be defined as a moduli problem? If so, what are its properties? Is it projective? Smooth? What happens as $t \downarrow 0$?

5. Reider in Dimension Three? Let X be a smooth projective threefold, with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ and $H = c_1(L)$ for an ample line bundle L. Consider

$$\epsilon : 0 \to \mathcal{O}_X[1] \to E^\bullet_\epsilon \to L \otimes \mathcal{I}_Z \to 0$$

for subschemes $Z \subset X$ of finite length d, taken within the tilted category $\mathcal{A}_{\frac{1}{2}}$.

Question 4: Are there bounds d_0 and t_0 such that *all* objects $E^\bullet_\epsilon$ formed in this way are $\mu_{\frac{1}{2}+it}$-unstable when $d > d_0$ and $t < t_0$? If so, does this follow from a more general Bogomolov-type codimension three inequality for the numerical invariants of μ-stable objects?

As we have already discussed in the surface case, a destabilizing subobject of an *unstable* such $E^\bullet_\epsilon$ would be exhibited by lifting

$$\begin{array}{ccccccccc} & & & & & \swarrow & K \\ & & & & & & \downarrow \\ (\dagger) & 0 & \to & \mathcal{O}_S[1] & \to & E^\bullet_\epsilon & \to & L \otimes \mathcal{I}_Z & \to & 0 \end{array}$$

where $K \subset L \otimes \mathcal{I}_Z$ is a subobject in $\mathcal{A}_{\frac{1}{2}}$. By requiring $\mathrm{Pic}(X) = \mathbb{Z}$, we guarantee that $L(-D)$ does not belong to $\mathcal{A}_{\frac{1}{2}}$ for any effective divisor D, and thus that K does not factor through any $L(-D)$. The two cases to consider are therefore

- $K = L \otimes \mathcal{I}_C$ where $C \subset X$ is a curve, and
- K is an H-stable torsion-free sheaf of odd rank $r > 1$ and $c_1(K) = \frac{r+1}{2}H$.

Question 5: Assuming Question 4, are there examples of threefolds where the bounds of Question 4 are satisfied (hence all $E^\bullet$ are μ-unstable), but for which there nevertheless exist non-zero extensions?

And the last question is whether the two cases above can be numerically eliminated (by some form of the Hodge Index Theorem) for non-zero extensions, leading to a proof of a form of Fujita's conjecture for threefolds.

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Intersection Pairings in Singular Moduli Spaces of Bundles

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Abstract. This article summarizes results from [**13**] about intersection theory of moduli spaces of holomorphic bundles on Riemann surfaces, together with background material on partial desingularisation and intersection cohomology.

1. Introduction

Let $M(n,d)$ be the moduli space of bundles of rank n and degree d with fixed determinant on a Riemann surface Σ of genus $g \geq 2$. We do not assume n and d are coprime, so this space may be singular. The space $M(n,d)$ may also be described as the space of representations of a central extension of the fundamental group of a Riemann surface Σ into $SU(n)$ (where the generator of the center is sent to the element $e^{2\pi i d/n}I$, where I is the identity matrix) modulo the action of $SU(n)$ by conjugation. From this point of view the singularities of the space are determined by the stabilizers of the conjugation action. For example $M(2,0)$ is the space of conjugacy classes of representations of the fundamental group of a surface Σ into $SU(2)$. The most singular points are those that send the entire fundamental group to the center of $SU(2)$ (the identity matrix and minus the identity matrix); at these points the stabilizer is all of $SU(2)$. The other singular stratum is the set of those points where the stabilizer is conjugate to $U(1)$: these are the representations which send the entire fundamental group to a subgroup conjugate to $U(1)$.

We shall study intersection pairings in $M(n,d)$ using

- the ordinary cohomology of the partial desingularization $\widetilde{M(n,d)}$ (see Sections 2 and 3)
- intersection homology (see Sections 4 and 6)

We note (see [**19**] and [**20**] for the details) that $M(n,d)$ is obtained as a geometric invariant theory (GIT) quotient of a finite dimensional variety $R(n,d)$ equipped with a holomorphic action of $SL(p)$ for

$$p = d - n(g-1).$$

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We blow up the loci with the highest dimensional stabilizers, so the action of $SL(p)$ extends to the blowup of $R(n,d)$. After blowing up, the dimension of the highest dimensional stabilizer has been reduced. We repeat this process to finally reach a space where all the stabilizers are finite so the quotient $\widetilde{M(n,d)}$ is an orbifold. This procedure (*partial desingularization*) is described in [**18**]. We use the notation of [**24**], where the GIT quotient of a manifold M with an action of a reductive group G is defined as M^{ss}/G, where M^{ss} is the set of semistable points. The set of stable points is denoted by M^s. For the definition of 'stable' and 'semistable' we refer the reader to [**24**].

Note that rather than using the description of $M(n,d)$ as a GIT quotient of $R(n,d)$ by $SL(p)$, we often use the description of it as a symplectic quotient of the "extended moduli space" M^{ext} [**8**] by a symplectic action of $SU(n)$. The extended moduli space is a finite dimensional space with a symplectic structure on an open dense set with a Hamiltonian action of $SU(n)$ for which the symplectic quotient is $M(n,d)$. An infinite dimensional construction of the extended moduli space can be given [**8**] as the symplectic quotient of the space of all connections on a 2-manifold by the based gauge group. The extended moduli space can be written as the fiber product

$$\begin{array}{ccc} M^{\mathrm{ext}} & \xrightarrow{\mu} & \mathrm{Lie}(K) \\ \downarrow & & \downarrow \exp \\ K^{2g} & \xrightarrow{\Phi} & K \end{array}$$

Here $K = SU(n)$ and Φ is the map

$$\Phi(k_1,\ldots,k_{2g}) = \prod_{j=1}^{g} k_{2j-1}k_{2j}k_{2j-1}^{-1}k_{2j}^{-1},$$

while exp is the exponential map. An open dense set of M^{ext} is smooth and is equipped with a symplectic structure, for which μ is the moment map for the action of K. In algebraic geometry the closest analogue is Seshadri's moduli space of bundles with level structure [**25**]; see also the monograph by Huybrechts and Lehn [**7**].

2. Partial desingularization

Let $\mu : M \to \mathbf{k}^*$ be the moment map for a Hamiltonian action of a compact Lie group K (with maximal torus T) on a symplectic manifold M. The Lie algebra of K is denoted by $\mathbf{k}$. If the symplectic quotient $\mu^{-1}(0)/K$ is singular, there is a way to blow up M (by reverse induction on the dimension of the stabilizer) for which the partial desingularization $\widetilde{M}$ inherits a Hamiltonian K action with moment map $\tilde{\mu} : \widetilde{M} \to \mathbf{k}^*$ and its symplectic quotient $\tilde{\mu}^{-1}(0)/K$ is an orbifold. The details are given in [**18**].

The partial desingularization construction may be applied to the extended moduli space M^{ext} (see for example Section 8 of [**13**]). The classes in $H_K^*(M^{\mathrm{ext}})$ pull back to $H_K^*(\widetilde{M^{\mathrm{ext}}})$.

In this article all cohomology groups are assumed to be with complex coefficients unless specified otherwise. We introduce the map $\kappa_{\widetilde{M}}^{K,0} : H_K^*(\widetilde{M}) \to H^*(\widetilde{\mu}^{-1}(0)/K)$ (the Kirwan map), which is the composition of the restriction map from $H^*_K(\widetilde{M})$ to $H^*_K(\widetilde{\mu}^{-1}(0))$ with the isomorphism

$$H^*_K(\widetilde{\mu}^{-1}(0)) \cong H^*(\widetilde{\mu}^{-1}(0)/K).$$

The latter isomorphism is valid because K acts locally freely on $\widetilde{\mu}^{-1}(0)$. There is a similar map $\kappa_M^{T,c} : H^*_T(M) \to H^*(\mu_T^{-1}(c)/T)$ where μ_T is the moment map for the action of the maximal torus T and c is an element in the Lie algebra of the maximal torus which is a regular value for μ_T. The class $\mathcal{D}$ is the class in $H^*_T(\mathrm{pt})$ given by the product of the positive roots – its image under the Kirwan map is the Poincaré dual of K/T, so it enables us to replace integration over $\widetilde{\mu}^{-1}(0)/K$ by $1/n!$ times integration over $\widetilde{\mu}^{-1}(0)/T$.

Throughout this article, if X is a topological space we shall use the notation $\int_X$ to mean pairing with the fundamental class of X. Let $\alpha, \beta \in H^*_K(\widetilde{M^{\mathrm{ext}}})$. With the above notation,

$$\int_{\tilde{\mu}^{-1}(0)/T} \kappa(\alpha\beta\mathcal{D}) = A + B \tag{1}$$

where

$$A = \int_{\tilde{\mu}^{-1}(\epsilon)/T} \kappa(\alpha\beta\mathcal{D})$$

and

$$B = \left(\int_{\tilde{\mu}^{-1}(0)/T} \kappa(\alpha\beta\mathcal{D}) - \int_{\tilde{\mu}^{-1}(\epsilon)/T} \kappa(\alpha\beta\mathcal{D}) \right).$$

Here we have chosen a small regular value ϵ of $\tilde{\mu}$. The term A is evaluated by studying "wall crossings", components of the fixed point set of a subtorus $T' \subset T$ which intersect $\mu^{-1}(0)$. This first term is evaluated using the results of Guillemin, Kalkman and Martin (see Theorem 6.1). The term B is the same as the pairing for nonsingular symplectic manifolds (see [**9**]).

3. Partial desingularization of $M(2,0)$

In this section we give the results of the partial desingularization for $M(2,0)$. These were computed in Section 8 of [**13**], where detailed proofs may be found. *In this section only*, we use the notation $M(2,0)$ to refer to the moduli space of semistable bundles of rank 2 and degree 0 *without fixing the determinant*, since this was the case treated in Section 8 of [**13**] and this was the notation used there. Let M^{ext} be the extended moduli space with the group $U(2)$ on which $K = SU(2)$ acts by conjugation. Let T be the maximal torus of K.

From (1) we have (see ([**13**], (8.2)) that

$$\kappa^{K,0}_{\widetilde{M}^{\mathrm{ext}}}\big(\eta e^{\bar{\omega}}\big)[\widetilde{\mathcal{M}}(2,0)] = -\frac{1}{2}\int_{\mu_T^{-1}(\varepsilon)/T} \kappa^{T,\varepsilon}_{M^{\mathrm{ext}}}\big(\eta e^{\bar{\omega}}\mathcal{D}^2\big) \tag{2}$$
$$-\frac{1}{2}\left(\int_{\tilde{\mu}_T^{-1}(0)/T} \kappa^{T,0}_{\widetilde{M}^{\mathrm{ext}}}\big(\eta e^{\bar{\omega}}\mathcal{D}^2\big) - \int_{\tilde{\mu}_T^{-1}(\varepsilon)/T} \kappa^{T,\varepsilon}_{\widetilde{M}^{\mathrm{ext}}}\big(\eta^{\bar{\omega}}\mathcal{D}^2\big)\right).$$

Here Y is the generator of $H^*_T(\mathrm{pt})$ and $\bar{\omega}$ is $\bar{\omega}(Y) = \tilde{\omega} + \mu_T Y$, the equivariantly closed extension of the symplectic form $\tilde{\omega}$ on $\widetilde{M}$. The element η is a class in $H^*_K(\widetilde{M^{\mathrm{ext}}})$. The expression $e^{\bar{\omega}}$ denotes the formal sum $\sum_{m=0}^{\infty} \bar{\omega}^m/m!$, where only a finite number of terms have nonzero contributions because all the manifolds are finite dimensional. The notation is as in Section 2. Here μ_T and $\tilde{\mu_T}$ are moment maps for the T-action on $M^{\rm ext}$ and its partial desingularization $\widetilde{M^{\rm ext}}$. The first term of (2) can be computed using periodicity, as in [**11**] (see Section 6 below).

To compute the final term, we need to examine the walls crossed in passing from 0 to ε in $\mathbf{t}^* = \mathbb{R}$. The components of the fixed point set that are relevant for us are those that meet the exceptional divisors in $\widetilde{M}^{\rm ext}$.

We first describe the wall crossing term from the first blow-up. See Section 8.1 of [**13**]. To form $\widetilde{M}^{\rm ext}$ from $M^{\rm ext}$, we first blow up along the set Δ consisting of the points with stabilizer $K = SU(2)$; this set is $(S^1)^{2g}$, where S^1 represents the center of $U(2)$.

The component with positive moment map of the T-fixed point set in the exceptional divisor is the projectivization

$$\mathbb{P}\left(T_{\rm Jac} \otimes \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \cong \mathbb{P}T_{\rm Jac} \cong \mathbb{P}^{g-1} \times {\rm Jac}.$$

of the tangent bundle $T_{\rm Jac}$ of the Jacobian ${\rm Jac} \cong (S^1)^{2g}$. The normal bundle to this component of the fixed point set is

$$\mathcal{O}_{\mathbb{P}^{g-1}}(-1) \oplus \left[T_{\rm Jac} \otimes \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathcal{O}_{\mathbb{P}^{g-1}}(1)\right] \tag{3}$$

$$\oplus \left[T_{\rm Jac} \otimes \mathbb{R}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times \mathcal{O}_{\mathbb{P}^{g-1}}(1)\right].$$

Here the 2×2 matrices are elements of the Lie algebra of $GL(2, \mathbb{C})$. The first summand is normal to the exceptional divisor and the last two are normal in the exceptional divisor.

The torus T (the maximal torus of K) acts on the three summands of (3) with weights $2, -2, -4$ respectively. As $T_{\rm Jac}$ is trivial, the equivariant Euler class of the normal bundle to Jac is

$$(-y + 2Y)(y - 2Y)^g (y - 4Y)^g$$

where $y := c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(1))$ is the generator of $H^*(\mathbb{P}^{g-1})$ and $Y \in H^2_T({\rm pt})$ is the generator of $H^*_T({\rm pt})$.

The wall crossing term from the first blow-up in the partial desingularization process is

$$W = -\frac{1}{2}\int_{\mathbb{P}^{g-1}\times{\rm Jac}} {\rm Res}_{Y=0} \frac{\eta e^{\bar\omega}|_\Delta \mathcal{D}^2}{(-y + 2Y)(y - 2Y)^g (y - 4Y)^g}$$

$$= -\frac{1}{2^{3g}}\int_{\rm Jac} {\rm Res}_{y=0}{\rm Res}_{Y=0} \frac{\eta e^{\bar\omega}|_\Delta}{y^g Y^{2g-1}} \left(1 - \frac{y}{2Y}\right)^{-g-1} \left(1 - \frac{y}{4Y}\right)^{-g}.$$

This implies that the classes a_2, b_2^j, f_2 (in the notation of [**13**]) restrict to Y^2, Yd^j and -2γ respectively, where the cohomology of Jac is generated by classes $d^j \in H^1({\rm Jac})$ ($j = 1, \ldots, 2g$) and we define $\gamma = \sum_{j=1}^g d^j d^{j+g}$. Therefore, we can explicitly compute the restriction of $\eta e^{\bar\omega}$ to Δ in terms of Y and d^j.

The last expression above is nonzero only when

$$\eta e^{\bar\omega}|_\Delta \in H^*_T(\Delta) = H^*_T({\rm Jac})$$

is a constant multiple of the product of Y^{3g-3} and the fundamental class $\frac{\gamma^g}{g!}$ of Δ, in which case the wall crossing term is computed as follows:

$$\begin{aligned} W &= -\frac{1}{2^{3g}}\int_\Delta \mathrm{Res}_{y=0}\mathrm{Res}_{Y=0}\frac{\frac{\gamma^g}{g!}Y^{3g-3}}{y^g Y^{2g-1}}(1-\frac{y}{2Y})^{-g-1}(1-\frac{y}{4Y})^{-g} \\ &= -\frac{1}{2^{3g}}\mathrm{Res}_{y=0}\mathrm{Res}_{Y=0}\frac{Y^{g-2}}{y^g}(1-\frac{y}{2Y})^{-g-1}(1-\frac{y}{4Y})^{-g} \qquad (4) \\ &= -\frac{1}{2^{3g}}\mathrm{Coeff}_{y^{g-1}Y^{-g+1}}(1-\frac{y}{2Y})^{-g-1}(1-\frac{y}{4Y})^{-g} \\ &= -\frac{1}{2^{3g}}\mathrm{Coeff}_{t^{g-1}}(1-\frac{t}{2})^{-g-1}(1-\frac{t}{4})^{-g}. \end{aligned}$$

Here, t is a variable used to define a power series (t represents y/Y) and $\mathrm{Coeff}_{t^{g-1}}$ denotes the coefficient of t^{g-1} in the power series.

The next step in this program is to compute the wall-crossing term for the second blowup. Let Γ denote the set of points in M^{ext} fixed by the action of T, and let $\widehat{\Gamma}$ be the proper transform after the first blow-up. To get the partial desingularization we blow up along $G\widehat{\Gamma}$ for $G = K^{\mathbb{C}}$. The details are given in [**13**], Section 8.2.

4. Intersection cohomology

Intersection cohomology shares some properties with the ordinary cohomology of smooth manifolds, notably

- The Lefschetz hyperplane theorem: If X is a complex projective algebraic variety of complex dimension n and H is a general hyperplane section, the inclusion map $IH_i(X\cap H)\to IH_i(X)$ is an isomorphism for $i < n-1$ and a surjection for $i = n-1$.
- The hard Lefschetz theorem: the map $\omega : IH^i(X)\to IH^{i+2}(X)$ (multiplication by the Kähler form) is injective for $i < n$ and surjective for $i > n-2$. This implies $\omega^i : IH^{n-i}(X)\to IH^{n+i}(X)$ is an isomorphism.
- Poincaré duality: The cup product map
$$IH_j(X)\otimes IH_{2n-j}(X)\to\mathbb{R}$$
is a perfect pairing if $\dim_{\mathbb{R}} X = 2n$.

Intersection homology has the pairing given by Poincaré duality, but in general it does not have a ring structure. Intersection cohomology was originally defined by Goresky and MacPherson [**3, 4**]. A reference is the book [**22**] by Kirwan and Woolf.

5. Results of Y.-H. Kiem on equivariant cohomology and intersection homology

Definition 5.1. *Let G be a connected reductive group and let its maximal compact subgroup be K, so $G = K^{\mathbb{C}}$. Let G act linearly on a complex vector space A. A point $\beta\in A$ will be defined to be in the set $\mathcal{B}$ if β is the closest point to 0 in the convex hull of some subset of weights of the action of the maximal torus.*

(A point $\beta\in\mathcal{B}$ parametrizes a stratum S_β of the gradient flow of the Yang-Mills functional $|\mu|^2$ on $\mathbb{P}A$, where μ is the moment map for the action of K.) For each $\beta\in\mathcal{B}$, $n(\beta)$ is defined as the number of weights α such that $\langle\alpha,\beta\rangle < \langle\beta,\beta\rangle$.

Definition 5.2. [**15**] *In the notation of Definition 5.2, the action of G on A is linearly balanced if for all $\beta \in \mathcal{B}$,*

$$2n(\beta) \geq \frac{1}{2} \dim A.$$

Here, if H is the identity component of $\mathrm{Stab}(p)$ *for a point*

$$p \in Z := \mu^{-1}(0),$$

we define W_p as the complement of the H fixed point set $\widehat{W_p}^H$ in the symplectic slice [**26**]

$$\widehat{W_p} := (T_p(G \cdot p))^\omega / T_p(G \cdot p),$$

(where V^ω is the symplectic annihilator of V) so that the symplectic slice $\widehat{W_p}$ decomposes as $\widehat{W_p} = \widehat{W_p}^H \oplus W_p$.

For example, a linear action of $G = \mathbb{C}^*$ on a vector space is linearly balanced if the number of positive weights equals the number of negative weights.

Definition 5.3. *(*[**15**]*, Definition 4.2) The action of G on a Hamiltonian space U (with moment map $\mu : U \to \mathbf{g}^*$ and $Z := \mu^{-1}(0)$) is balanced if*

- *For all $p \in Z$ the action of the identity component H of the stabilizer of G acting on W_p is linearly balanced*
- *For a subgroup $K \subset H$ which also appears as the identity component of the stabilizer of a point in Z, the $N^K \cap H/K$ action on the K fixed point subspace $W_p^{\mathbf{k}}$ is also linearly balanced, where $\mathbf{k}$ is the Lie algebra of K. Here N^K is the normalizer of K, and the space W_p was introduced in Definition 5.2.*

If G is a reductive group acting on M, then the Kirwan map is defined as the composition ([**13**], (2.2))

$$\kappa_M : H^*_G(M) \to H^*_G(\widetilde{M}) \to H^*(\widetilde{M}//G) \to IH^*(M//G). \tag{5}$$

The map κ_M in (5) is surjective. This map is the composition of the restriction map from $H^*_K(M)$ to $H^*_K(M^{ss})$ and a surjection $\kappa_M^{ss} : H^*_K(M^{ss}) \to IH^*(M//G)$ (see [**21**], Theorem 2.5). The proofs of these assertions are given in [**21**], [**17**] and [**27**]. The final map in the composition (5) exists (and is surjective) because the intersection cohomology $IH^*(M//G)$ is a direct summand of the cohomology of the partial resolution of singularities $H^*(\widetilde{M}//G)$. Here $\widetilde{M}$ is the partial desingularization of M, which was discussed in Section 2 above– see [**18**]. The second map in the composition (5) is the usual Kirwan map for the orbifold $\widetilde{M}$ so it is surjective.

Theorem 5.4. *(*[**14**]*, Section 7) If the action of G is balanced then there is a vector subspace V_M of $H^*_G(M^{ss})$ such that*

$$\kappa_M^{ss} : V_M \to IH^*(M//G)$$

is a bijection.

Remark 1: If $\sigma_1, \sigma_2 \in V_M$ are of complementary degrees, i.e. the sum of their degrees equals the real dimension of $M//G$, then also $\sigma_1 \cup \sigma_2 \in V_M$. In fact we have the following

THEOREM 5.5. *([**12**] Theorem 14) If σ_1 and σ_2 are both in V_M and have complementary degrees (as in Remark 1) then*

$$\sigma_1 \cup \sigma_2 = \tau \langle \sigma_1, \sigma_2 \rangle$$

where τ is the top degree class in V_M and $\langle \cdot, \cdot \rangle$ is the pairing in intersection cohomology.

This result shows that the pairing in intersection cohomology is given by the cup product in equivariant cohomology.

There is a weaker condition than Definition 5.3, called a *weakly balanced action* [**14**], under which Theorems 5.4 and 5.5 hold true. As mentioned in the Introduction, the space $M(n,d)$ is the GIT quotient by $SL(p)$ (for $p = d - n(g-1)$) of a smooth quasi-projective variety $R(n,d)^{ss}$ (the semistable points of $R(n,d)$). See [**19**] and [**20**].

THEOREM 5.6. *([**16**], Proposition 7.4) The $SL(p)$ action on $R(n,d)^{ss}$ is weakly balanced.*

Remark 2: ([**15**]) In the case of $M(n,d)$, the space V_M is characterized by decomposing the equivariant cohomology of components of the fixed point set of subtori and taking only those terms where the factor involving the equivariant cohomology of a point has degree less than a specific upper bound. The details are given in [**15**], (4.2) and [**14**], Section 7. More precisely

$$V_M = \{\xi \in H^*_G(M^{ss}) : \Phi_H(\xi) \in H^*_{N_0^H/H}(Y_H) \otimes H_H^{<n_H}(\mathrm{pt}) \ \forall H\} \tag{6}$$

Here $H \subset G$ is the identity component of the stabilizer of points in $Z = \mu^{-1}(0)$ for μ the moment map for a G action on a Hamiltonian space U, and Y_H is the subset of points in Z fixed by H. Also, $n_H = \frac{1}{2}\dim(W_p//H)$ for a point $p \in Z$ whose stabilizer has H as its identity component. (Here W_p was defined in Definition 5.2.) The group N_0^H is the identity component of the normalizer N^H of H. The map

$$\Phi_H : H^*_G(Z) \to H^*_G(GY_H) \to H^*_{N^H}(Y_H)$$

is the map defined in [**15**] (4.1) associated to the composition of the obvious map

$$G \times_{N^H} Y_H \to GY_H$$

with the inclusion $GY_H \to Z$.

Remark 3: If $K = SU(2)$, the K equivariant cohomology of the open subset

$$U = \mu^{-1}(B) \subset M^{\mathrm{ext}}$$

of M^{ext} (where B is a G-invariant open ball containing 0) is given in [**16**] (Theorem 1, "Structure Theorem"), see also [**15**], Theorem 5.3. If the action of K had been locally free (which is of course not the case) this equivariant cohomology would be isomorphic to the ordinary cohomology of $\mathrm{Hom}(\pi, K)/K$.

$$H^*_{SU(2)}(U) \cong \bigoplus_{l=0}^{g} \mathrm{Prim}_l \otimes \mathbb{Q}[\alpha, \beta, \xi]/I_{g-l}. \tag{7}$$

Here α, β, ξ are generators of degrees 2, 4, 6 in integral cohomology. The ideals I_n are generated by three elements $c_{n+1}, c_{n+2}, c_{n+3}$ satisfying the recursion relation

$$nc_n = \alpha c_{n-1} + (n-2)\beta c_{n-2} + 2\gamma c_{n-3} \tag{8}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \alpha^2/2$. Here $\gamma = -2\sum_{i=1}^{g} \psi_i\psi_{i+g}$ and $\xi = \alpha\beta + 2\gamma$ (see [**16**] p. 254), where the ψ_i are cohomology classes of degree 3. Finally Prim_l is defined on [**16**] p. 255 before Lemma 1: it is the primitive part of the l-th exterior power $\wedge^l(\psi_1, \ldots, \psi_{2g})$ with respect to the action of the symplectic group $Sp(g)$.

THEOREM 5.7. *([**15**], Corollary 5.4) Let α and β be as in Remark 3. (For details see the first and second paragraphs of [**15**], Section 5). Then $\alpha^i\beta^j \in V_U$ only if $j < g - 1$.*

6. Results on intersection cohomology of $M(n, d)$

Assume $\alpha, \beta \in H^*_G(M)$ and their restrictions to M^{ss} are in V_M. Then if $\alpha\beta$ is represented by a top-degree form η compactly supported in the stable part $M(n, d)^s$ of $M(n, d)$, Theorem 5.5 tells us that the pairing $\langle\kappa(\alpha), \kappa(\beta)\rangle$ is given by $\int_{M(n,d)^s} \eta$.

Martin's theorem [**23**] asserts that if μ is the moment map for the action of $SU(n)$ on M where $M(n, d)$ is the symplectic quotient of M by $SU(n)$, then

$$\int_{M(n,d)^s} \kappa_M^{K,0}(\alpha\beta) = \frac{1}{n!}\int_{\mu^{-1}(0)^s/T} \kappa_M^{T,0}(\alpha\beta\mathcal{D}) \tag{9}$$

where $\mathcal{D}$ is the element of equivariant cohomology represented by the product of all the positive roots (for some choice of positive roots) or the Poincaré dual of $\mu^{-1}(0)$ in $\mu_T^{-1}(0)$.

Let $\epsilon \in \mathbf{t}$ be a regular value of μ close to 0. There is a surjective map $\mu^{-1}(\epsilon)/T \to \mu^{-1}(0)/T$ which is a diffeomorphism on $\mu^{-1}(0)^s/T$ (this map is defined using the gradient flow of $|\mu|^2$). So we find that

$$\int_{\mu^{-1}(0)^s/T} \kappa_M^{T,0}(\alpha\beta\mathcal{D}) = \int_{\mu^{-1}(\epsilon)^s/T} \kappa_M^{T,\epsilon}(\alpha\beta\mathcal{D}).$$

The right hand side can be evaluated using the methods of [**11**]. Specifically, we have

$$\frac{1}{n!}\int_{\mu^{-1}(\epsilon)^s/T} \kappa_M^{T,\epsilon}(\eta e^{\bar{\omega}}\mathcal{D}) = \mathrm{Res}\Big(\sum_F e^{\langle\mu(F),X\rangle}\int_F \frac{\eta e^{\omega}\mathcal{D}^2}{e_F}\Big). \tag{10}$$

Here F are the components of the fixed point set of the action of T and e_F is the equivariant Euler class of the normal bundle to F for a variable X in $\mathbf{t}$.

The map Res from $H^*_T(M)$ to $\mathbb{C}$ is defined as follows.[1] If $\beta_j \in \mathbf{t}^*$ are weights of T, and $\bar{\beta} = \{\beta_1, \ldots, \beta_n\}$, define $h_{\bar{\beta}} : \mathbf{t} \to \mathbb{C}$ by

$$h_{\bar{\beta}}(X) = \frac{1}{\prod_j \beta_j(X)}. \tag{11}$$

According to the Duistermaat-Heckman theorem [**2**], if M is a compact smooth manifold equipped with a Hamiltonian group action with moment map μ the integral $\int_M e^{\omega+\mu_X}$ (or "Duistermaat-Heckman oscillatory integral" – see [**1**]) is a linear combination of such functions $h_{\bar{\beta}}$. This integral is a smooth function of X and has a Fourier transform, although the individual $h_{\bar{\beta}}$ are singular at 0. Nonetheless

[1]This map has been referred to as a "residue map" only because it can be written in terms of residues of meromorphic functions at 0 – see [**10**], Proposition 3.2. This description is most obvious in the case $T = U(1)$, as in the example of the rotation action of the circle on S^2 presented below.

in [**6**] Guillemin, Lerman and Sternberg showed how to make sense of the Fourier transform of the $h_{\bar{\beta}}$ so that the appropriate linear combination of these Fourier transforms is the pushforward of Liouville measure under the moment map (the Fourier transform of the Duistermaat-Heckman oscillatory integral). They define the Fourier transform $\mathcal{F}h_{\bar{\beta}}(\cdot)$ to be the pushforward of Lebesgue measure from $\mathbb{C}^n$ to $\mathbf{t}^*$ under the map $(y_1, \ldots, y_n) \mapsto \sum_j \beta_j y_j$; in other words $\mathcal{F}h_{\bar{\beta}}(\xi)$ is the volume of the set

$$\{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid \sum_j \beta_j y_j = \xi\}.$$

We determine the signs of the β_j by choosing a polarization: we choose some cone $\Lambda \subset \mathbf{t}$ for which none of the β_j vanish anywhere on Λ, and change the signs of the β_j so that all $\beta_j(X) > 0$ for any $X \in \Lambda$. Then the residue is defined as the value at $\xi = 0$ of the Fourier transform of a certain class of meromorphic functions on $\mathbf{t} \otimes \mathbb{C}$: if f is such a function, then

$$\operatorname{Res}(f) = \Big(\mathcal{F}f\Big)(0).$$

For more details on the residue see Section 3 of [**10**].

For example, if S^2 is acted on by $T = S^1$ by rotation around the vertical axis, there are two fixed points F_+ and F_- (the north and south poles). The moment map is the height function and the values at the fixed points are $\mu(F_\pm) = \pm 1$. We find that the function in (10) is

$$h(X) = \frac{e^X - e^{-X}}{X} = 2\frac{\sinh(X)}{X}.$$

This function does have a Fourier transform (it is smooth at $X = 0$). According to Guillemin-Lerman-Sternberg [**6**], we should write h as $h(X) = h_+(X) - h_-(X)$ where

$$h_\pm(X) = e^{\pm X}/X.$$

According to the recipe in [**6**], we should define

$$\mathcal{F}h_+(\xi) = H(\xi - 1)$$

and

$$\mathcal{F}h_-(\xi) = H(\xi + 1),$$

where H is the Heaviside function (the characteristic function of the positive real line). Then we find that the Fourier transform of h is the characteristic function of the interval from -1 to $+1$, as one would expect since this interval is the image of S^2 under the moment map.

THEOREM 6.1. *(Guillemin-Kalkman* [**5**]*; Martin* [**23**]*) If a and b are regular values for the moment map μ of a circle action and $a < b$, then (defining the corresponding Kirwan maps $\kappa_M^{T,a}$ and $\kappa_M^{T,b}$) we have*

$$\int_{\mu^{-1}(a)/T} \kappa_M^{T,a}(\eta e^{\bar{\omega}}) - \int_{\mu^{-1}(b)/T} \kappa_M^{T,b}(\eta e^{\bar{\omega}}) = \sum_{F \,\mid\, a<\mu(F)<b} \operatorname{Res} \int_F \frac{e^{\mu(F)X} \eta e^{\omega}}{e_F(X)}$$

where Res *is as defined above. Here F are the components of the fixed point set of the circle action.*

Remark 6.2. *In the situation of Section 1 (for $K = SU(n)$ acting on the extended moduli space $M^{\rm ext}$) it is possible to show that $\mu^{-1}(a)$ is diffeomorphic to $\mu^{-1}(a+1)$. This is the 'periodicity' alluded to in (2). So Theorem 6.1 gives*

$$\int_{\mu^{-1}(a)/T} \kappa_M^{T,a}(\eta e^{\bar{\omega}}) = \sum_{F \ | \ a<\mu(F)<a+1} \mathrm{Res} \frac{e^{\mu(F)X}\eta(X)}{e_F(X)(1-e^X)}.$$

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The Beilinson-Drinfeld Grassmannian and Symplectic Knot Homology

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Abstract. Seidel-Smith and Manolescu constructed knot homology theories using symplectic fibrations whose total spaces were certain varieties of matrices. These knot homology theories were associated to $SL(n)$ and tensor products of the standard and dual representations. In this paper, we place their geometric setups in a natural, general framework. For any complex reductive group and any sequence of minuscule dominant weights, we construct a fibration of affine varieties over a configuration space. The middle cohomology of these varieties is isomorphic to the space of invariants in the corresponding tensor product of representations. Our construction uses the Beilinson-Drinfeld Grassmannian and the geometric Satake correspondence.

1. Introduction

Let G be a complex reductive group. For any n-tuple $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant weights of G, consider the space of invariants ${}_0V^{\underline{\lambda}} := (V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_n})^G$. This representation carries an action of the symmetric group by permuting the tensor factors. More precisely, the group which acts is the stabilizer $\Sigma_{\underline{\lambda}}$ of $\underline{\lambda}$ under the action of the symmetric group Σ_n (so that if all λ_i are equal, then $\Sigma_{\underline{\lambda}} = \Sigma_n$).

These representations of symmetric groups are of interest from the point of view of knot invariants. Via quantum groups and R-matrices, these representations can be deformed to representations of braid groups. This construction leads to knot invariants, such as the Jones polynomial. Roughly speaking, the invariant of a knot K is the trace of a braid (in such a representation) whose closure is K.

Khovanov [**K**] has proposed categorifying these representations. This means finding a triangulated category ${}_0\mathcal{D}^{\underline{\lambda}}$, with an action of an appropriate braid group, such that the Grothendieck group of ${}_0\mathcal{D}^{\underline{\lambda}}$ is isomorphic to ${}_0V^{\underline{\lambda}}$ with the above action of $\Sigma_{\underline{\lambda}}$. From these categorifications, Khovanov explained how to obtain more refined knot invariants by computing an appropriate categorical "trace". (Actually Khovanov has proposed categorifying the braid group representations coming from quantum groups. However, in this paper, we will just focus on categorifying the symmetric group representations. Restricting to the categorification of these symmetric group representations is sufficient to get non-trivial braid group representations and knot invariants.)

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The construction of these categorifications has thus far proceeded in a rather "ad-hoc" manner in different special cases, most notably when $G = SL_2$ or SL_m.

One approach was taken by Seidel-Smith [**SS**] and then extended by Manolescu [**M**]. They considered the cases when $G = SL_2$ [**SS**] or $G = SL_m$ [**M**] and where $\underline{\lambda} = (\omega_1, \dots, \omega_1, \omega_{m-1}, \dots, \omega_{m-1})$, a tensor product of standard and dual representations. In this case, they constructed a symplectic fibration $S \to \mathbb{C}^n_{\mathrm{reg}}/\Sigma_{\underline{\lambda}}$. Here S is a certain variety of $nm \times nm$ matrices. They proved that there is a well-defined (up to Hamiltonian isotopy) action of the braid group $\pi_1(\mathbb{C}^n_{\mathrm{reg}}/\Sigma_{\underline{\lambda}})$ on Lagrangian submanifolds in a fixed fibre. The braid group acts by parallel transport through the fibration.

The Lagrangians are objects in the derived Fukaya category. So philosophically, this means that they constructed an action of the braid group on the derived Fukaya category and this category can be considered as ${}_0\mathcal{D}^{\underline{\lambda}}$ (recently, this perspective has been pursued by Reza Rezazadegan [**R**]). As supporting evidence, the middle cohomology of a fibre is isomorphic to ${}_0V^{\underline{\lambda}}$ as a representation of $\Sigma_{\underline{\lambda}}$ (see [**M**, section 3.1]).

The purpose of this paper is to place the geometric setup of Seidel-Smith and Manolescu in a more general and more natural framework. For any complex reductive group G and any sequence of minuscule dominant weights $\underline{\lambda}$, we construct a fibration of smooth complex affine varieties (and hence symplectic manifolds) ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \to \mathrm{Conf}_{\underline{\lambda}}$, where $\mathrm{Conf}_{\underline{\lambda}} = {\mathbb{P}^1}^n_{\mathrm{reg}}/\Sigma_{\underline{\lambda}}$ is a coloured configuration space of points on $\mathbb{P}^1$.

The family ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \to \mathrm{Conf}_{\underline{\lambda}}$ is a type of Beilinson-Drinfeld Grassmannian [**BD**]. It is a moduli space of Hecke modifications of the trivial principal $G^\vee$-bundle P_0 on $\mathbb{P}^1$, where $G^\vee$ denotes the Langlands dual group. The fibre over a point $x = [x_1, \dots, x_n]$ is defined by

$$\begin{aligned}{}_0\mathrm{Gr}^{\underline{\lambda}}_x = \Big\{ (P,\phi) : \;& P \text{ is a principal } G^\vee\text{-bundle on } \mathbb{P}^1, \\ & \phi : P_0|_{X \smallsetminus \{x_1,\dots,x_n\}} \to P|_{X \smallsetminus \{x_1,\dots,x_n\}} \text{ is an isomorphism,} \\ & \phi \text{ is of Hecke type } \lambda_i \text{ at } x_i, \text{ for each } i, \\ & \text{and } P \text{ is isomorphic to the trivial bundle.} \Big\}\end{aligned}$$

We prove the following facts about this fibration.

(i) The action (by monodromy) of $\pi_1(\mathrm{Conf}_{\underline{\lambda}})$ on the cohomology of a fibre factors through the group $\Sigma_{\underline{\lambda}}$. Moreover there is a $\Sigma_{\underline{\lambda}}$ equivariant isomorphism $H^{\mathrm{mid}}({}_0\mathrm{Gr}^{\underline{\lambda}}_x) \cong {}_0V^{\underline{\lambda}}$ (Proposition 2.8).

(ii) When two points come together in the base corresponding to dual weights, there is a local statement entirely analogous to lemmas in [**M**] and [**SS**] (Lemma 3.2).

(iii) In the case when $G = SL_m$ and $(\lambda_1, \dots, \lambda_n) = (1, \dots, 1, m-1, \dots, m-1)$, then the portion of the fibration lying over points in $\mathbb{C}$ is isomorphic to the fibration studied by Manolescu, which in turn reduces to the fibration studied by Seidel-Smith when $m = 2$ (Theorem 4.1).

We prove statement (i) in section 2 as a consequence of the geometric Satake correspondence of Mirković-Vilonen [**MVi**]. In section 3, we prove statement (ii) as a consquence of the factorization property of the Beilinson-Drinfeld Grassmannian. It is interesting to see how this key technical lemma of [**M**] and [**SS**] follows extremely naturally and easily in this setting. In section 4, we prove statement (iii) following ideas of Mirković-Vybornov [**MVy**] and Ngo [**N**].

In a future work, we hope to use this setup to define an action of the braid group on the Fukaya category of the fibres (categorifying the action of $\Sigma_{\underline{\lambda}}$ on the middle cohomology) and then to construct homological knot invariants, following the approach of Seidel-Smith and Manolescu.

There is a close connection between this paper and the algebraic geometry approach to knot homology pursued jointly with Sabin Cautis in [**CK**]. We expect that the two constructions are related by hyperKähler rotation. The hyperKähler structure of the varieties ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\underline{x}}$ is described by Kapustin-Witten in sections 10.2 and 10.3 of [**KM**].

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2. The fibration

We begin by reviewing different versions of the Beilinson-Drinfeld Grassmannian and exploring their properties using the geometric Satake correspondence. These varieties were introduced by Beilinson-Drinfeld in [**BD**].

2.1. The affine Grassmannian. Let G be a complex reductive group and let $G^\vee$ be its Langlands dual group. Let Λ denote the set of weights of G, which is the same as the set of coweights of $G^\vee$. Let Λ_+ denote the subset of dominant weights.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. The affine Grassmannian Gr of $G^\vee$ is defined as $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. The $G^\vee(\mathcal{O})$ orbits on Gr are labelled by dominant weights of G. We write Gr^λ for the $G^\vee(\mathcal{O})$ orbit through t^λ, for $\lambda \in \Lambda_+$. Let $L_0 = t^0$ denote the identity coset in Gr. These orbits are closed in Gr (and hence projective) if and only if λ is a minuscule weight. More generally, we have that $\overline{\mathrm{Gr}^\lambda} = \bigcup_{\mu \le \lambda} \mathrm{Gr}^\mu$, where $\mu \le \lambda$ means that μ is a dominant weight and $\lambda - \mu$ is a sum of positive roots. The smooth locus of $\overline{\mathrm{Gr}^\lambda}$ is exactly Gr^λ. In particular, $\overline{\mathrm{Gr}^\lambda}$ is smooth iff λ is minuscule.

Similarly, the $G^\vee(\mathcal{K})$ orbits on Gr $\times$ Gr are also labelled by Λ_+ and we write $L_1 \xrightarrow{\lambda} L_2$ if (L_1, L_2) is in the same orbit as (L_0, t^λ).

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a k-tuple of dominant weights of G. Then we define the local convolution Grassmannian as

$$\widetilde{\mathrm{Gr}}^{\underline{\lambda}} = \{(L_1, \dots, L_k) \in \mathrm{Gr}^k : L_0 \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{k-1}} L_{k-1} \xrightarrow{\lambda_k} L_k\}$$

There is a map $m_{\underline{\lambda}} : \widetilde{\mathrm{Gr}}^{\underline{\lambda}} \to \mathrm{Gr}$ with $(L_1, \dots, L_k) \mapsto L_k$. The image of $m_{\underline{\lambda}}$ is $\overline{\mathrm{Gr}^{\lambda_1 + \cdots + \lambda_k}}$.

The following result follows from the geometric Satake correspondence [**MVi**].

THEOREM 2.1. *Assume that all λ_i are minuscule. There are canonical isomorphisms*

(i) $H^*(\mathrm{Gr}^\lambda) \cong V_\lambda$,
(ii) $H^*(\widetilde{\mathrm{Gr}}^{\underline{\lambda}}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, *and*
(iii) $H^{\mathrm{top}}(m_{\underline{\lambda}}^{-1}(L_0)) \cong (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$.

Moreover, the isomorphisms in (ii), (iii) are compatible in the sense that the diagram

$$\begin{array}{ccc} H^*(\widetilde{\mathrm{Gr}}^{\underline{\lambda}}) & \xrightarrow{\sim} & V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \\ \downarrow & & \downarrow \\ H^{\mathrm{top}}(m_{\underline{\lambda}}^{-1}(L_0)) & \xrightarrow{\sim} & (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G \end{array}$$

commutes, where the left vertical map comes from the inclusion $m_{\underline{\lambda}}^{-1}(L_0) \to \widetilde{\mathrm{Gr}}^{\underline{\lambda}}$.

PROOF. The geometric Satake correspodence gives us an equivalence of tensor categories between the category of $G^\vee(\mathcal{O})$ equivariant perverse sheaves on Gr (called the spherical Hecke category) and the category of representations of G, compatible with the fibre functors to the category of vector spaces [**MVi**, Theorem 7.3]. The fibre functor on the spherical Hecke category is derived global sections.

This equivalence takes $\mathbb{C}_{\mathrm{Gr}^\lambda}[\dim \mathrm{Gr}^\lambda]$ to V_λ when λ is minuscule. This immediately gives us (i). Also from the definition of the tensor product on the spherical Hecke category [**MVi**, Section 4], we see that the equivalence takes $(m_{\underline{\lambda}})_* \mathbb{C}_{\widetilde{\mathrm{Gr}}^{\underline{\lambda}}}[\dim \widetilde{\mathrm{Gr}}^{\underline{\lambda}}]$ to $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. This gives us (ii) and (iii). The compatibility statement is clear by construction. □

2.2. The notion of Hecke type. Let us now fix a smooth curve X. Later we will take $X = \mathbb{P}^1$.

For any $x \in X$, and any open set $U \subset X$ containing x, we define

$$\mathrm{Gr}_{U,x} := \big\{(P, \phi) : P \text{ is a principal } G^\vee\text{-bundle on } U \\ \text{and } \phi : P_0|_{U \smallsetminus x} \to P|_{U \smallsetminus x} \text{ is an isomorphism}\big\}.$$

here and below, P_0 denotes the trivial $G^\vee$-bundle on U.

Picking a coordinate at x gives an isomorphism $\mathrm{Gr}_{U,x} \cong \mathrm{Gr}$. The isomorphism is well-defined up to the action of $\mathrm{Aut}(\mathcal{O})$ on Gr. Since Gr^λ is $\mathrm{Aut}(\mathcal{O})$ invariant, we may consider the locus $\mathrm{Gr}^\lambda_{U,x}$ obtained as the preimage of Gr^λ under any isomorphism. An element $(P, \phi) \in \mathrm{Gr}^\lambda_{U,x}$ is said to have Hecke type λ. Note that $G^\vee(U)$ acts on $\mathrm{Gr}_{U,x}$ preserving its stratification into Hecke types.

We need to generalize our notion of Hecke type. Again fix $x \in X$ and suppose we have two $G^\vee$-bundles (P_1, P_2) on X and an isomorphism ϕ between them over $X \smallsetminus x$. Let us pick an open neighbourhood U of x on which P_1 trivializes and pick a trivialization of P_1 on this neighbourhood. This gives us a point $L = (P_2|_U, \phi') \in \mathrm{Gr}_{U,x}$, where $\phi' : P_0|_{U \smallsetminus x} \to P_2|_{U \smallsetminus x}$ is the composition of this trivialization and ϕ. We say that ϕ has Hecke type λ at x if L has Hecke type λ.

Changing the trivialization of P_1 on U will change L by the action of $G^\vee(U)$ and hence will not change its Hecke type. Also, changing the open set U also does not change λ, as can be seen by shrinking the open set U. Hence the Hecke type of ϕ is independent of the choices made in its definition.

For $X = \mathbb{P}^1$, there is an alternate characterization of Hecke type. First, we should note that if P is a principal G-bundle on $\mathbb{P}^1$, then we can consider its topological type which will be an element of $\pi_1(G^\vee) = \Lambda/Q$, where Q denotes the coroot lattice of $G^\vee$. Using this notion, we have the following result, due to Finkelberg-Mirković [**FM**, Prop 10.2].

PROPOSITION 2.2. *Let $x, X = \mathbb{P}^1, \phi, P_1, P_2$ be as above. ϕ has Hecke type $\leq \lambda$ at x if and only if*

(i) *for all irreps V_β of $G^\vee$, the map ϕ induces the inclusions*

$$V_\beta^{P_1}(-\langle \beta, \lambda \rangle x) \subset V_\beta^{P_2} \subset V_\beta^{P_1}(\langle \beta, \lambda \rangle x)$$

(here V_β^P denotes the associated vector bundle $P \times_{G^\vee} V_\beta$) and

(ii) *the topological types of P_2 and P_1 differ by $[\lambda] \in \pi_1(G^\vee) = \Lambda/Q$.*

(Note that the second condition here is vacuous when $G^\vee$ is simply connected. Also, note that a similar characterization works for any complete curve. On the other hand, it is not clear to the author what to do with the second condition when considering open curves.)

2.3. The global convolution Grassmannian. Now fix $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ an n-tuple of dominant minuscule weights of G. We now consider the global convolution Grassmannian, which is the variety

$$\begin{aligned}\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n} := \{&((x_1, \dots, x_n), (P_1, \dots, P_n), (\phi_1, \dots, \phi_n)) : x_i \in X, \\ &P_i \text{ is a principal } G^\vee\text{-bundle on } X, \\ &\phi_i : P_{i-1}|_{X \smallsetminus x_i} \to P_i|_{X \smallsetminus x_i} \text{ is an isomorphism of Hecke type } \lambda_i \text{ at } x_i.\}\end{aligned}$$

We have the projection $\tilde{p} : \widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n} \to X^n$. For any subset $A \subset X^n$, let $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_A = \tilde{p}^{-1}(A)$ be the preimage of A under this map.

Consider the following two loci in X^n: the locus of regular points $X^n_{\mathrm{reg}} := \{(x_1, \dots, x_n) : x_i \neq x_j\}$ and the small diagonal $X = \{(x, \dots, x)\} \subset X^n$. The fibres of $\tilde{p}$ over regular points in $X^n_{\mathrm{reg}} := \{(x_1, \dots, x_n) : x_i \neq x_j\}$ are isomorphic to $\mathrm{Gr}^{\lambda_1} \times \cdots \times \mathrm{Gr}^{\lambda_n}$. The fibres over points on the small diagonal are the local convolution products $\mathrm{Gr}^{\lambda_1} \tilde{\times} \dots \tilde{\times} \mathrm{Gr}^{\lambda_n}$.

A proof of the following result can be found in the proof of Lemma 6.1 of [**MVi**].

PROPOSITION 2.3. *The pushforwards $R^k \tilde{p}_* \mathbb{C}_{\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n}}$ are constant sheaves.*

2.4. The global singular Grassmannian. Now, we can introduce the singular version of the above space. We define the global singular Grassmannian to be

$$\begin{aligned}\mathrm{Gr}^{\underline{\lambda}}_{X^n} := \Big\{ &\big((x_1, \dots, x_n), P, \phi\big) : x_i \in X, P \text{ is a principal } G^\vee\text{-bundle on } X, \\ &\phi : P_0|_{X \smallsetminus \{x_1, \dots, x_n\}} \to P|_{X \smallsetminus \{x_1, \dots, x_n\}} \text{ is an isomorphism} \\ &\text{and for each } x \in X, \ \phi \text{ has Hecke type} \leq \sum_{i \,:\, x_i = x} \lambda_i \text{ at } x \Big\}\end{aligned}$$

Again this is a family over X^n.

PROPOSITION 2.4. *The fibres of this family are described as follows.*

(i) *The fibres over regular points are still* $\mathrm{Gr}^{\lambda_1} \times \cdots \times \mathrm{Gr}^{\lambda_n}$*. In fact* $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n_{\mathrm{reg}}} \cong \mathrm{Gr}^{\underline{\lambda}}_{X^n_{\mathrm{reg}}}$.

(ii) *The fibres over points on the small diagonal are the (usually singular) varieties* $\overline{\mathrm{Gr}^{\lambda_1+\cdots+\lambda_n}}$.

There is an obvious map $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n} \to \mathrm{Gr}^{\underline{\lambda}}_{X^n}$. This map is 1-1 over the locus in $\mathrm{Gr}^{\underline{\lambda}}_{X^n}$ which sits over X^n_{reg} and coincides with the map $m_{\underline{\lambda}}$ on a fibre over a point in the small diagonal in X^n.

The biggest subgroup of Σ_n which acts on $\mathrm{Gr}^{\underline{\lambda}}_{X^n}$ is denoted $\Sigma_{\underline{\lambda}}$. More precisely, $\Sigma_{\underline{\lambda}}$ is the stabilizer of $\underline{\lambda}$ inside Σ_n (so if all λ_i are equal, then $\Sigma_{\underline{\lambda}} = \Sigma_n$ and if all λ_i are different, then $\Sigma_{\underline{\lambda}} = \{1\}$).

The quotient of X^n by $\Sigma_{\underline{\lambda}}$ is denoted by $X^{\underline{\lambda}}$. Since all λ_i are minuscule, distinct λ_i are linearly independent. Hence we can identify $X^{\underline{\lambda}}$ with the space of "Λ_+-coloured divisors" of total weight $\sum \lambda_i$, i.e. functions $D : X \to \Lambda_+$ such that $\sum_{x \in \mathbb{P}^1} D(x) = \sum_i \lambda_i$.

The quotient of $\mathrm{Gr}^{\underline{\lambda}}_{X^n}$ by $\Sigma_{\underline{\lambda}}$ is denoted by $\mathrm{Gr}^{\underline{\lambda}}_{X^{\underline{\lambda}}}$ and we can describe its points as follows.

$$\begin{aligned}\mathrm{Gr}^{\underline{\lambda}}_{X^{\underline{\lambda}}} := \Big\{ &(D, P, \phi) : D \in X^{\underline{\lambda}}, P \text{ is a principal } G^\vee\text{-bundle on } X, \\ &\phi : P_0|_{\mathbb{P}^1 \smallsetminus \mathrm{supp}(D)} \to P|_{\mathbb{P}^1 \smallsetminus \mathrm{supp}(D)} \text{ is an isomorphism} \\ &\text{and for each } x \in X,\ \phi \text{ has Hecke type} \leq D(x) \text{ at } x \Big\}\end{aligned}$$

Let $\mathrm{Conf}_{\underline{\lambda}} := X^n_{\mathrm{reg}}/\Sigma_{\underline{\lambda}}$. We have the smooth family $\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \to \mathrm{Conf}_{\underline{\lambda}}$ and the following Cartesian square:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n_{\mathrm{reg}}} & \longrightarrow & \mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X^n_{\mathrm{reg}} & \longrightarrow & \mathrm{Conf}_{\underline{\lambda}} \end{array} \tag{1}$$

We can consider the monodromy action of $\pi_1(\mathrm{Conf}_{\underline{\lambda}})$ on the cohomology of fibres in the family p. There is a short exact sequence of groups

$$1 \to \pi_1(X^n_{\mathrm{reg}}) \to \pi_1(\mathrm{Conf}_{\underline{\lambda}}) \to \Sigma_{\underline{\lambda}} \to 1.$$

Since the pushforwards $R^k \tilde{p}_* \mathbb{C}_{\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{X^n}}$ are constant by Proposition 2.3, the monodromy action of $\pi_1(X^n_{\mathrm{reg}})$ on the cohomology of the fibres of $\tilde{p}$ is trivial. So, because of the Cartesian square (1), this means that the monodromy action of $\pi_1(\mathrm{Conf}_{\underline{\lambda}})$ factors through $\Sigma_{\underline{\lambda}}$. Moreover the following fact is true.

Proposition 2.5. *Assume that all λ_i are minuscule and let $x \in \mathrm{Conf}_{\underline{\lambda}}$. There is an isomorphism $H^*(\mathrm{Gr}^{\underline{\lambda}}_x) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, compatible with the actions of $\Sigma_{\underline{\lambda}}$ on both sides (by monodromy and by permuting tensor factors).*

Proof. The isomorphism comes from using Proposition 2.5, Proposition 2.4, and Theorem 2.1. The compatibility with the actions of $\Sigma_{\underline{\lambda}}$ is an immediate consequence of the construction of the commutativity constraint for the spherical Hecke category (see [**MVi**, Section 5]). □

2.5. The open global convolution/singular Grassmannian. So far we have dealt with arbitrary curves X. Now we specialize to $X = \mathbb{P}^1$.

Let $\underline{\lambda}$ be such that $\lambda_1 + \cdots + \lambda_n$ is in the root lattice of G. This condition is necessary for $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$ to be non-empty.

We introduce the open subvariety ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}}$ of $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}}$, which consists of the locus where we impose the constraint that P_n is isomorphic to the trivial $G^\vee$-bundle on $\mathbb{P}^1$. This subvariety would be empty if we did not impose the above condition that $\lambda_1 + \cdots + \lambda_n$ is in the root lattice. We call this the open global convolution Grassmannian.

The significance of this open subvariety is given by the following result.

Proposition 2.6. (i) *For any $y \in \mathbb{P}^1$, ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\ldots,y)}$ retracts onto the half-dimensional locus $m_{\underline{\lambda}}^{-1}(L_0)$.*

(ii) *There is an isomorphism $H^{\mathrm{mid}}({}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\ldots,y)}) \cong (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$.*

Proof. Assume that $y = 0 \in \mathbb{P}^1$. We can identify $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\ldots,y)}$ with the convolution product $\mathrm{Gr}^{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^{\lambda_n}$. A point L in Gr corresponds to a trivial $G^\vee$-bundle on $\mathbb{P}^1$ if and only if L is in the $G[z^{-1}]$-orbit of L_0, which we denote by ${}_0\mathrm{Gr}$. So ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\ldots,y)}$ is identified with $m_{\underline{\lambda}}^{-1}({}_0\mathrm{Gr})$.

The open subset ${}_0\mathrm{Gr}$ of the affine Grassmannian retracts onto L_0 via the loop rotation action of the group $\mathbb{C}^\times$ (see for example [**MVi**, Section 2]). This action extends to the convolution product and retracts $m_{\underline{\lambda}}^{-1}({}_0\mathrm{Gr})$ onto $m_{\underline{\lambda}}^{-1}(L_0)$ as desired.

We have $\dim m_{\underline{\lambda}}^{-1}(L_0) = \frac{1}{2} \dim m_{\underline{\lambda}}^{-1}({}_0\mathrm{Gr})$ because the map $m_{\underline{\lambda}}$ is semismall (see the proof of Lemma 4.4 in [**MVi**]).

Hence we have

$$H^{\mathrm{mid}}({}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\ldots,y)}) \cong H^{\mathrm{top}}(m_{\underline{\lambda}}^{-1}(L_0)) \cong (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$$

where the last isomorphism follows from Theorem 2.1.(iii). $\square$

Analogously, we also define ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1n}}$ and ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1\underline{\lambda}}}$, the open global singular Grassmannian. So

$${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1\underline{\lambda}}} = \{(D, P, \phi) \in \mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1\underline{\lambda}}} : P \text{ is isomorphic to the trivial bundle } \}$$

We have a Cartesian square identical to (1):

$$\begin{array}{ccc} {}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}_{\mathrm{reg}}} & \longrightarrow & {}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \\ \tilde{p}\big\downarrow & & \big\downarrow p \\ \mathbb{P}^{1n}_{\mathrm{reg}} & \longrightarrow & \mathrm{Conf}_{\underline{\lambda}} \end{array} \tag{2}$$

Proposition 2.7. *The pushforwards $R^k\tilde{p}_*\mathbb{C}_{{}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}}}$ are constant sheaves on $\mathbb{P}^{1n}$.*

We would like to thank Alexander Braverman for suggesting the following proof.

Proof. Since $\mathbb{P}^{1n}$ is simply connected, it suffices to show that the push forwards are locally constant. Since the category of locally constant sheaves is an abelian category, it suffices to find a stratification $Y_{\underline{\mu}}$ of ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}}$ such that each $R^k\tilde{p}_*\mathbb{C}_{Y_{\underline{\mu}}}$ is locally constant.

Recall that isomorphism classes of $G^\vee$-bundles on $\mathbb{P}^1$ are given by dominant weights μ of G (this is because every $G^\vee$-bundle on $\mathbb{P}^1$ admits a reduction to $T^\vee$). Hence for $\underline{\mu} = (\mu_1, \dots, \mu_n = 0)$, we define

$$Y_{\underline{\mu}} := \{(P_1, \dots, P_n) \in {}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{P}^{1n}} : P_i \text{ has isomorphism type } \mu_i \text{ for } i = 1, \dots, n\}.$$

The family $Y_{\underline{\mu}} \to \mathbb{P}^{1^n}$ is trivial and thus the pushforwards $R^k \tilde{p}_* \mathbb{C}_{Y_{\underline{\mu}}}$ are constant sheaves. Hence the result follows. □

By similar reasoning as in the paragraph before Proposition 2.5, we can use Proposition 2.7 to see that the monodromy action of $\pi_1(\mathrm{Conf}_{\underline{\lambda}})$ on the cohomology of the fibres of $p : {}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}} \to \mathrm{Conf}_{\underline{\lambda}}$ factors through $\Sigma_{\underline{\lambda}}$. We obtain the following description of the monodromy action on the middle homology of the fibres of ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathrm{Conf}_{\underline{\lambda}}}$.

Proposition 2.8. *Let $x \in \mathbb{P}^{1^n}_{\mathrm{reg}}$. There is an isomorphism $H_{\mathrm{mid}}({}_0\mathrm{Gr}^{\underline{\lambda}}_x) \cong (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$, which is compatible with the actions of $\Sigma_{\underline{\lambda}}$ on both sides.*

Proof. Using Proposition 2.7 and Proposition 2.6, we obtain the isomorphism

$$H_{\mathrm{mid}}({}_0\mathrm{Gr}^{\underline{\lambda}}_x) = H_{\mathrm{mid}}({}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_x) \cong H_{\mathrm{mid}}({}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{(y,\dots,y)})$$

where $y \in \mathbb{P}^1$.

By the last statement of Proposition 2.1 we obtain a commutative square

$$\begin{array}{ccc} H^*(\mathrm{Gr}^{\underline{\lambda}}_x) & \xrightarrow{\sim} & V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \\ \downarrow & & \downarrow \\ H^{mid}({}_0\mathrm{Gr}^{\underline{\lambda}}_x) & \xrightarrow{\sim} & (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G \end{array}$$

By Proposition 2.5, the top horizontal arrow is $\Sigma_{\underline{\lambda}}$ equivariant. The right vertical arrow is clearly $\Sigma_{\underline{\lambda}}$ equivariant and the left vertical arrow is clearly $\pi_1(\mathrm{Conf}_{\underline{\lambda}})$ equivariant. Hence we conclude that the bottom horizontal arrow is $\Sigma_{\underline{\lambda}}$ equivariant as desired. □

Remark 2.9. For μ a dominant weight of G, we can also define the subvariety ${}_\mu\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1\underline{\lambda}}}$ consisting of those tuples (D, P, ϕ) such that P has isomorphism class μ. These varieties will be related to $Hom(V_\mu, V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})$.

Remark 2.10. It would be interesting to generalize the construction of ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{P}^{1n}}$ by replacing $\mathbb{P}^1$ by arbitrary smooth curves X. To do so, one needs to impose a condition on a bundle P. The two natural choices are P is trivial and P is semistable. Imposing that P is trivial would give us too small a space and imposing that P is semistable seems to give too big a space (I believe we lose the fact that the fibres are affine).

3. Semilocal geometry

The purpose of this section is to establish the behaviour of our fibration when two points come together. The basic tool is the factorization property of the Grassmannians. We begin by stating this property, which is due to Beilinson-Drinfeld.

THEOREM 3.1. [**BD**, section 5.3.10] *Let X be any smooth curve. Let J_1, J_2 be a disjoint decomposition of $(1, \dots, n)$. Let U denote the set of points $[x_1, \dots, x_n]$ in $X^{\underline{\lambda}}$ such that $x_{j_1} \neq x_{j_2}$ if $j_1 \in J_1$ and $j_2 \in J_2$. Then there is an isomorphism*

$$\mathrm{Gr}^{\underline{\lambda}}_U \cong (\mathrm{Gr}^{\underline{\lambda}^1}_{X^{J_1}} \times \mathrm{Gr}^{\underline{\lambda}^2}_{X^{J_2}})|_U$$

where $\underline{\lambda}^i$ consists of those λ_j with $j \in J_i$. This isomorphism is compatible with the two projections to U.

Note that we have already implicitly used a special case of this Theorem in Proposition 2.4.(i).

3.1. Semilocal geometry in the global singular Grassmannian. For $\lambda \in X_+$, let $\lambda^\vee = -w_0(\lambda)$, where w_0 is the long element of the Weyl group. Equivalently, we have $V^{\lambda^\vee} = (V^\lambda)^\vee$.

Let us assume that $\mu = \lambda_i = \lambda_{i+1}^\vee$. Consider $x = [x_1, \dots, x_n] \in \mathbb{P}^{1\underline{\lambda}}$ with $y = x_i = x_{i+1}$ and $x_j \neq x_k$ otherwise. By the factorization property of the Grassmannians, we have that $\mathrm{Gr}^{\underline{\lambda}}_x \cong \mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)} \times \mathrm{Gr}^{(\mu,\mu^\vee)}_{(y,y)}$ where d_i deletes the i and $i+1$ entries of a list.

The variety $\mathrm{Gr}^{(\mu,\mu^\vee)}_{(y,y)}$ is isomorphic to $\overline{\mathrm{Gr}^{\mu+\mu^\vee}}$. It is singular with a stratum for each ν such that V_ν appears in $V_\mu \otimes V_\mu^\vee$. In particular, we have a distinguished point which corresponds to the trivial bundle with trivial isomorphism. This gives a copy of $\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ embedded in $\mathrm{Gr}^{\underline{\lambda}}_x$ as the locus of (D, P, ϕ) where the isomorphism ϕ extends over y (equivalently, has Hecke type 0 at y).

3.2. Semilocal geometry in the open singular Grassmannian. Now, let us pass to the open singular Grassmannian (the main object of study). The singular fibre ${}_0\mathrm{Gr}^{\underline{\lambda}}_x$ does not factor as a product in this case. However, we do have a copy of ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ embedded in ${}_0\mathrm{Gr}^{\underline{\lambda}}_x$ as the locus of (D, P, ϕ) where ϕ extends over y. More generally we have the following "semilocal geometry" which matches Lemma 3.5 of [**M**] and Lemma 21 of [**SS**].

Let r be chosen such that $r \leq |x_j - y|$ for all j. Let $B \subset \mathbb{P}^{1\underline{\lambda}}$ be a disc corresponding to the points of the form $[x_1, \dots, y-u, y+u, \dots, x_n]$, where $|u| < r$ and let $B' \subset (\mathbb{P}^1)^{(\mu,\mu^\vee)}$ be the points of the form $[y-u, y+u]$, for $|u| < r$.

LEMMA 3.2. *There exists an open neighbourhood of ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ in ${}_0\mathrm{Gr}^{\underline{\lambda}}_B$ and an isomorphism ψ of this neighbourhood with a neighbourhood of ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ in ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)} \times {}_0\mathrm{Gr}^{(\mu,\mu^\vee)}_{B'}$ such that the following diagram commutes*

$$\begin{array}{ccc} {}_0\mathrm{Gr}^{\underline{\lambda}}_B & \xrightarrow{\psi} & {}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)} \times {}_0\mathrm{Gr}^{(\mu,\mu^\vee)}_{B'} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & B' \end{array}$$

In [**SS**] and [**M**], the corresponding Lemma is used to construct a Lagrangian M' in ${}_0\mathrm{Gr}^{\underline{\lambda}}_x$ from a Lagrangian M in ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ via a relative vanishing cycle construction (see [**M**, section 4.4]). In these cases the Lagrangian M' was diffeomorphic to $\mathbb{P}^{m-1} \times M$. In our case, we expect that a similar construction will exist with $\mathbb{P}^{m-1}$

replaced by $m^{-1}_{(\mu,\mu^\vee)}(L_0) = \mathrm{Gr}^\mu$, which is a cominuscule partial flag variety for the group $G^\vee$.

This construction can be interpreted as a functor from the derived Fukaya category of ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ to that of ${}_0\mathrm{Gr}^{\underline{\lambda}}_x$ (see [**R**]). This functor should categorify the map

$${}_0V^{d_i(\underline{\lambda})} = (V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_n})^G \to (V^{\lambda_1} \otimes \cdots \otimes V^{\mu} \otimes V^{\mu^\vee} \otimes \cdots \otimes V^{\lambda_n})^G = {}_0V^{\underline{\lambda}}.$$

Proof. By the factorization property, we have, as above, an isomorphism

$$\psi : \mathrm{Gr}^{\underline{\lambda}}_B \cong \mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)} \times \mathrm{Gr}^{(\mu,\mu^\vee)}_{B'}.$$

Now, we let $V \subset \mathrm{Gr}^{\underline{\lambda}}_B$ be defined as the intersection of ${}_0\mathrm{Gr}^{\underline{\lambda}}_B$ with $\psi^{-1}\big({}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)} \times {}_0\mathrm{Gr}^{(\lambda,\lambda^\vee)}_{B'}\big)$. By construction V is a neighbourhood of ${}_0\mathrm{Gr}^{d_i(\underline{\lambda})}_{d_i(x)}$ on each side. The restriction of ψ to V fulfills the hypotheses. □

4. Comparison with resolution of slices

4.1. Slices and their resolutions. Fix a positive integer $N = mk$. We will study slices inside $\mathfrak{gl}_N$.

Consider the nilpotent matrix $E_{m,k}$ which consists of $k-1$ copies of the $m \times m$ identity matrix arranged below the diagonal.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}$$

This is the matrix for the linear operator z acting on $\mathbb{C}[z]^m/z^k\mathbb{C}[z]^m$ with respect to the usual basis $e_1, \dots, e_m, ze_1, \dots, z^{k-1}e_m$.

Let $F_{m,k}$ be the matrix completing it to a Jacobson-Morozov triple. Let $S_{m,k} = E_{m,k} + ker(\cdot F_{m,k})$. So $S_{m,k}$ is the set of block matrices consisting of identity matrices below the diagonal and arbitrary matrices on the right column. In particular, it is an affine space.

$$\begin{pmatrix} 0 & 0 & 0 & * \\ I & 0 & 0 & * \\ 0 & I & 0 & * \\ 0 & 0 & I & * \end{pmatrix}$$

Here each block is of size $m \times m$.

Let $\pi = (\pi_1, \dots, \pi_n)$ a sequence of integers with $1 \le \pi_i \le m-1$ and $\pi_1 + \cdots + \pi_n = N$. We can consider the partial flag variety $\mathcal{F}_\pi$ of type π (that means that the jumps are given by π). We now define the partial Grothendieck resolution $\widetilde{\mathfrak{gl}}^\pi_N \subset \mathcal{F}_\pi \times \mathfrak{gl}_N$ by

$$\widetilde{\mathfrak{gl}}^\pi_N := \{(Y, W_\bullet) : YW_i \subset W_i \text{ and } Y \text{ acts as a scalar on } W_i/W_{i-1}\}.$$

By recording the scalars with which Y acts on each successive quotient, we obtain a morphism $\widetilde{\mathfrak{gl}}^\pi_N \to \mathbb{C}^n$.

Of course there is also a morphism $\widetilde{\mathfrak{gl}}^\pi_N \to \mathfrak{gl}_N$. Let $\widetilde{S}^\pi_{m,k}$ denote the preimage of $S_{m,k}$ inside $\widetilde{\mathfrak{gl}}^\pi_N$.

Let $\mathfrak{gl}_N^\pi$ denote the image of $\widetilde{\mathfrak{gl}}_N^\pi$ inside $\mathfrak{gl}_N$. It consists of those matrices such that π refines their partition of eigenvalues. In particular any matrix in $\mathfrak{gl}_N^\pi$ has at most n distinct eigenvalues.

Let Σ_π denote the stabilizer of π in Σ_n and let $\mathbb{C}^\pi := \mathbb{C}^n/\Sigma_\pi$. Let $S_{m,k}^\pi$ denote the following variety:

$$S_{m,k}^\pi := \big\{(Y, [x_1, \dots, x_n]) : Y \in S_{m,k} \cap \mathfrak{gl}_N^\pi, [x_1, \dots, x_n] \in \mathbb{C}^\pi \\ \text{and } \{x_1\}^{\cup \pi_1} \cup \dots \cup \{x_n\}^{\cup \pi_n} \text{ are the eigenvalues of } Y\big\}$$

4.2. Examples and relation with [M], [SS]. Consider the case $m = 2$. This forces $\pi = (1, \dots, 1)$ and $n = N$. So $\widetilde{\mathfrak{gl}}_N^\pi = \widetilde{\mathfrak{gl}}_N$ is the ordinary Grothendieck resolution. This places no restriction on the matrices, so $\mathfrak{gl}_N^\pi = \mathfrak{gl}_N$. Also $\Sigma_\pi = \Sigma_n$ and so $\mathbb{C}^\pi = \mathbb{C}^n/\Sigma_n$. We also see that $S_{m,k}^\pi = S_{m,k}$. This is precisely the slice considered by Seidel-Smith which maps down to the space of eigenvalues $\mathbb{C}^n/\Sigma_n$.

Now, consider the case m arbitrary and $\pi = (1, \dots, 1, m-1, \dots, m-1)$ where there are k 1s and k m-1s. Then $\Sigma_\pi = \Sigma_k \times \Sigma_k$ and $\mathbb{C}^\pi = \mathbb{C}^{2k}/\Sigma_k \times \Sigma_k$ is the space of collections of “thin” and “thick” eigenvalues, to use the terminology of Manolescu. The regular locus is the bipartite configuration space $BConf_k$ studied by Manolescu. The fibres of $S_{m,k}^\pi \to BConf_k$ is the fibration studied by Manolescu.

Manolescu also develops the geometry in the same general framework that we do. To continue the comparison of terminology, our π is his π, our $\widetilde{\mathfrak{gl}}_N^\pi$ is his $\mathfrak{g}^\pi$, our $\mathfrak{gl}_N^\pi$ is his $\mathfrak{g}^\pi$, our Σ_π is his W^π, our $\mathbb{C}^n$ is his $\mathfrak{h}^\pi = \mathbb{C}^{s-1}$, our $\mathbb{C}^\pi$ is his $\mathfrak{h}^\pi/W^\pi$. We are unable to reconcile his map $\mathfrak{g}^\pi \to \mathfrak{h}^\pi/W^\pi$. We don't believe that such a map exists and that is why we constructed the more complicated $S_{m,k}^\pi$.

4.3. Comparison theorem. Now we take $G = SL_m$, so $G^\vee = PGL_m$. We have the usual labelling $\omega_1, \dots, \omega_{m-1}$ of minuscule weights for SL_m.

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ be such that $\lambda_1 + \dots + \lambda_n$ is in the root lattice and each λ_i is minuscule. For each i, let $\pi_i \in \{1, \dots, m-1\}$ be such that $\lambda_i = \omega_{\pi_i}$. Then, $N := \pi_1 + \dots + \pi_n$ is divisible by m (so that $\lambda_1 + \dots + \lambda_n$ is in the root lattice), so $N = mk$ for some positive integer k. Note that $\Sigma_{\underline{\lambda}} = \Sigma_\pi$ and $\mathbb{C}^{\underline{\lambda}} = \mathbb{C}^\pi$ (here $\mathbb{C}^{\underline{\lambda}}$ denotes the subvariety of $\mathbb{P}^{1\underline{\lambda}}$ where all points are in $\mathbb{C}$).

The following result is due to Mirković-Vybornov [**MVy**, Theorem 5.3] and Ngo [**N**, Lemma 2.3.1].

Theorem 4.1. *There exist isomorphisms* ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{C}^n} \cong \widetilde{S}^\pi_{m,k}$ *and* ${}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{C}^{\underline{\lambda}}} \cong S^\pi_{m,k}$ *which are compatible with the following two squares:*

$$\begin{array}{ccc} {}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{C}^n} & \longrightarrow & {}_0\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{C}^{\underline{\lambda}}} \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \longrightarrow & \mathbb{C}^{\underline{\lambda}} \end{array} \qquad \begin{array}{ccc} \widetilde{S}^\pi_{m,k} & \longrightarrow & S^\pi_{m,k} \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \longrightarrow & \mathbb{C}^\pi \end{array}$$

Though the theorem has already been proved by Ngo and Mirković-Vybornov, we provide a sketch proof for the convenience of the reader.

PROOF. We will show that both ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{C}^n}$ and $\widetilde{S}^{\pi}_{m,k}$ are isomorphic to the following space.

$$\begin{aligned}{}_0\mathcal{L}^{\underline{\lambda}} := \big\{ &L_0 = \mathbb{C}[z]^m \supset L_1 \supset \cdots \supset L_n : L_i \text{ are free } \mathbb{C}[z] \text{ submodules of rank } m, \\ &\dim(L_{i-1}/L_i) = \pi_i, z \text{ acts by a scalar on } L_{i-1}/L_i, \text{ and} \\ &[e_1], \dots [e_m], [ze_1], \dots [ze_m], \dots, [z^{k-1}e_1], \dots, [z^{k-1}e_m] \\ &\text{is a basis for } \mathbb{C}[z]^m/L_n \big\}\end{aligned}$$

First, suppose P_1, P_2 are two principal PGL_m-bundles on $\mathbb{P}^1$ and ϕ is an isomorphism between them away from a point $x \neq \infty$ of Hecke type ω_j. Then there exist rank m vector bundles V_1, V_2 representing P_1, P_2 such that $L_2 := \Gamma(\mathbb{C}, V_2) \subset L_1 := \Gamma(\mathbb{C}, V_1)$ and $\dim(L_1/L_2) = j$. Moreover, we see that $(z - x)L_1 \subset L_2$. (This is the PGL_m version of Proposition 2.2.)

This allows us to define an isomorphism $\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{C}^n} \to \mathcal{L}^{\underline{\lambda}}$, where $\mathcal{L}^{\underline{\lambda}}$ is just like ${}_0\mathcal{L}^{\underline{\lambda}}$ except without the last condition.

Hence it suffices to show that if a vector bundle V on $\mathbb{P}^1$ has space of sections $L = \Gamma(\mathbb{C}, V) \subset \mathbb{C}[z]^m$, then $[e_1], \dots, [z^{k-1}e_m]$ is a basis if and only if $V \cong \mathcal{O}(k)^{\oplus m}$ (this latter condition implies that $\mathbb{P}(V)$ is trivial and in this case it is in fact equivalent to it).

To prove this claim, let us define an isomorphism $V \to \mathcal{O}(k)^{\oplus m}$. To do so it suffices to define an isomorphism ψ of $\mathbb{C}[z]$-modules $\psi : L \to z^k\mathbb{C}[z]^{\oplus m}$ which extends over $\mathbb{P}^1$. Let B be the set $\{e_1, \dots, z^{k-1}e_m\}$. For each i define p_i to be the unique element of $\mathrm{Span}(B)$ such that $p_i - z^k e_i \in L$ (such a p_i exists by the "basis for quotient" assumption). Note that $\{z^k e_1 - p_1, \dots, z^k e_m - p_m\}$ forms a basis for L as a free $\mathbb{C}[z]$-module. Now, we define ψ to take $z^k e_i - p_i$ to $z^k e_i$. Since it takes one basis to another, ψ is a isomorphism of $\mathbb{C}[z]$-modules. Finally, a simple calculation shows that ψ extends to a isomorphism over $\mathbb{P}^1$.

Thus, we have proven that ${}_0\widetilde{\mathrm{Gr}}^{\underline{\lambda}}_{\mathbb{C}^n}$ is isomorphic to ${}_0\mathcal{L}^{\underline{\lambda}}$. So it remains to construct the isomorphism between ${}_0\mathcal{L}^{\underline{\lambda}}$ and $\widetilde{S}^{\pi}_{m,k}$.

Given $(L_0, \dots, L_n) \in \mathcal{L}^{\underline{\lambda}}$, we consider the linear operator Y defined as the action of z on the quotient L_0/L_n. Since we have a preferred basis for L_0/L_n, we may identify L_0/L_n with $\mathbb{C}^N$. The sequence $L_n/L_n, L_{n-1}/L_n, \dots, L_0/L_n$ gives us a flag $W_\bullet$ of type π in $\mathbb{C}^N$ such that Y preserves each subspace and acts as a scalar on each quotient. Thus $(Y, W_\bullet)$ is a point in $\widetilde{\mathfrak{gl}}^{\pi}_N$. Moreover, we see that $Y[z^i e_j] = [z^{i+1}e_j]$ for each $0 \le i \le k-2$ and all j. Thus $Y \in S_{m,k}$ as desired. (Note that the condition that $L_n = z^k\mathbb{C}[z]^m$ is equivalent to the condition $Y = E_{m,k}$.) Hence $(Y, W_\bullet)$ lies in $\widetilde{S}^{\pi}_{m,k}$. An inverse map can easily be constructed by an explicit formula (see [**MVy**, section 4.5]). □

5. Affine nature of the fibres

As a final step, we would like to prove the following result.

PROPOSITION 5.1. *Let $x \in {\mathbb{P}^1}^n_{\mathrm{reg}}$. Then ${}_0\mathrm{Gr}^{\underline{\lambda}}_x$ is an affine variety.*

We do not know a truly satisfactory proof of this result. Here is a slightly ad hoc approach which uses the results of the previous section.

PROOF. We begin with the case of $G^\vee = PSL_m$. We may assume that $x = (x_1, \dots, x_n) \in \mathbb{C}^n_{\mathrm{reg}}$, since the fibre does not depend on x. Then by Theorem 4.1,

${}_0\mathrm{Gr}_x^{\underline{\lambda}}$ is the variety of matrices in $S_{m,k}$ whose set of eigenvalues is $\{x_1\}^{\cup\pi_1} \cup \cdots \cup \{x_n\}^{\cup\pi_n}$. Since $S_{m,k}$ is an affine space and imposing this set of eigenvalues is a closed condition, we see that ${}_0\mathrm{Gr}_x^{\underline{\lambda}}$ is an affine variety. The varieties ${}_0\mathrm{Gr}_x^{\underline{\lambda}}$ for $G^\vee = GL_m$ are the same as those for $G^\vee = PSL_m$, so this establishes that case too.

Now suppose that $G^\vee$ is arbitrary. Pick a faithful representation $\rho : G^\vee \to GL_m$ which takes the maximal torus T of $G^\vee$ into the maximal torus T_m. This gives us a map $\rho : Bun_{G^\vee}(\mathbb{P}^1) \to Bun_{GL_m}(\mathbb{P}^1)$ and a closed embedding $\rho : \mathrm{Gr}_{G^\vee,x}^{\underline{\lambda}} \to \mathrm{Gr}_{GL_m,x}^{\rho(\underline{\lambda})}$ (here $\rho(\underline{\lambda})$ denotes the result of transforming $\underline{\lambda}$ into a tuple of dominant coweights for GL_m via the map $T \to T_m$). By Lemma 5.2 below, we see that $\rho^{-1}({}_0\mathrm{Gr}_{GL_m,x}^{\rho(\underline{\lambda})}) = {}_0\mathrm{Gr}_{G^\vee,x}^{\underline{\lambda}}$ and hence ${}_0\mathrm{Gr}_{G^\vee,x}^{\underline{\lambda}}$ is a closed subvariety of ${}_0\mathrm{Gr}_{GL_m,x}^{\rho(\underline{\lambda})}$ and hence is affine. □

LEMMA 5.2. *Let $\rho : G^\vee \to GL_m$ be as above. Let P be a principal $G^\vee$-bundle on $\mathbb{P}^1$. Then P is trivial if and only if $\rho(P)$ is trivial.*

PROOF. By Grothendieck's theorem, every $G^\vee$-bundle on $\mathbb{P}^1$ admits a reduction to T. Hence our bundle P comes from a T-bundle P'. Note that P is trivial iff P' is trivial. We may first turn P' into a T_m-bundle $\rho_T(P')$ and then into the GL_m-bundle $\rho(P)$.

Now, T-bundles are determined by maps $X(T) \to \mathbb{Z}$. The bundle $\rho_T(P')$ is determined by the transformed map $X(T_m) \to X(T) \to \mathbb{Z}$. Since $T \to T_m$ is an embedding, $X(T_m) \to X(T)$ is onto.

Hence we deduce

$$\begin{aligned} P \text{ is trivial } &\iff P' \text{ is trivial } \iff \text{ the map } X(T) \to \mathbb{Z} \text{ is zero} \\ &\iff \text{ the map } X(T_m) \to X(T) \to \mathbb{Z} \text{ is zero} \\ &\iff \rho_T(P') \text{ is trivial} \iff \rho(P) \text{ is trivial.} \end{aligned}$$

□

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Arithmetic Aspects of Moduli Spaces of Sheaves on Curves

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Abstract. We describe recent work on the arithmetic properties of moduli spaces of stable vector bundles and stable parabolic bundles on a curve over a global field. In particular, we describe a connection between the period-index problem for Brauer classes over the function field of the curve and the Hasse principle for rational points on étale forms of such moduli spaces, refining classical results of Artin and Tate.

Contents

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Introduction

The goal of this paper is to introduce the reader to recent work on some basic arithmetic questions about moduli spaces of vector bundles on curves. In particular, we will focus on the correspondence between rational-point problems on (étale) forms of such moduli spaces and classical problems on the Brauer groups of function fields. This might be thought of as a non-abelian refinement of the following result of Artin and Tate (which we state in a special case).

Fix a global field K with scheme of integers S. (If $\operatorname{char} K > 0$, we assume that S is proper so that it is unique.) Let $f : X \to S$ be a proper flat morphism of relative dimension 1 with a section $D \subset X$ and smooth generic fiber X_η. Let $\operatorname{Br}_\infty(X)$ denote the kernel of the restriction map $\operatorname{Br}(X) \to \prod_{\nu|\infty} \operatorname{Br}(X \otimes K_\nu)$, i.e., the Brauer classes which are trivial at the fibers over the Archimedean places. (If $\operatorname{char} K > 0$ or K is totally imaginary, $\operatorname{Br}_\infty(X) = \operatorname{Br}(X)$.)

THEOREM (Artin-Tate). *There is a natural isomorphism*

$$\operatorname{Br}_\infty(X) \xrightarrow{\sim} Ш^1(\operatorname{Spec} K, \operatorname{Jac}(X_\eta)).$$

From a modern point of view, this isomorphism arises by sending a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ to the relative moduli space of invertible twisted sheaves of degree 0. (We have provided a brief review of gerbes, moduli of twisted sheaves, and the Brauer group in the form of an appendix.) Our goal will be to study the properties of higher-rank moduli of twisted sheaves. Each moduli space carries an adelic point (i.e., is in the analogue of the Tate-Shafarevich group), but most of the moduli spaces are geometrically rational (or at least rationally connected) with trivial Brauer-Manin obstruction, so that the Hasse principle is conjectured to hold. This yields connections between information about the complexity of Brauer classes on arithmetic surfaces and conjectures on the Hasse principle for geometrically rational varieties.

As the majority of work on moduli spaces of vector bundles on curves is done in a geometric context, we give in Section 1 a somewhat unconventional introduction to the study of forms of moduli. This is done primarily to fix notation and introduce some basic constructions. We also use this section to introduce a central idea: forms of the moduli problem naturally give forms of the stack and not merely of the coarse moduli space, and the relation between these forms captures cohomological information which ultimately is crucial for making the arithmetic connections. In this geometric section, we use this philosophy to prove a silly "non-abelian Torelli theorem".

Starting with Section 2 we turn our attention to arithmetic problems. In particular, after asking some basic questions in Section 2, we link a standard conjecture on the Hasse principle for 0-cycles of degree 1 (Conjecture 1.5(a) of [**4**]) to recent conjectures on the period-index problem in Section 3. We also recast some well-known results due to Lang and de Jong on the Brauer group in terms of rational points on moduli spaces of twisted sheaves. This serves to create a tight link between the Hasse principle and the period-index problem for unramified Brauer classes on arithmetic surfaces. In particular, we get the following refinement of the Artin-Tate isomorphism.

Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe which is trivial on geometric fibers of f and over infinite places, and let $\alpha \in \operatorname{Br}_\infty(X)$ denote its associated Brauer class. As explained in Proposition 1.6, the stack $\mathscr{M}_{\mathscr{X}/S}(n, \mathscr{O}(D))$ of flat families of stable $\mathscr{X}$-twisted sheaves of rank n and determinant $\mathscr{O}(D)$ is a form of the stack $\mathscr{M}_{X/S}(n, \mathscr{O}(D))$ of (non-twisted) stable sheaves with the same discrete invariants. We write $M_{\mathscr{X}/S}(n, \mathscr{O}(D))$ for the coarse moduli space of $\mathscr{M}_{\mathscr{X}/S}(n, \mathscr{O}(D))$ and $M_{\mathscr{X}_K/K}(n, \mathscr{O}(D_K))$ for its generic fiber over S.

THEOREM (L). *The class α satisfies $\operatorname{per}(\alpha) = \operatorname{ind}(\alpha)$ if and only if the Hasse principle for 0-cycles of degree 1 holds for the smooth projective geometrically rational K-scheme $M_{\mathscr{X}_K/K}(n, \mathscr{O}(D_K))$.*

Since $\operatorname{Pic}(M_{\mathscr{X}_{\overline{K}}/\overline{K}}(n, \mathscr{O}(D_{\overline{K}}))) = \mathbf{Z}$, the condition of the theorem is equivalent to the statement that the Brauer-Manin obstruction is the only obstruction to the Hasse principle for this particular variety. Just as the element of $Ш^1(\operatorname{Spec} K, X_\eta)$ measures non-triviality of α, here the success (or failure) of the Hasse principle for $M_{\mathscr{X}_K/K}(n, \mathscr{O}(D))$ measures the complexity of the division algebra over $K(X)$ with Brauer class α. As an example for the reader unfamiliar with the period and index (reviewed in the appendix), this says that if the Brauer-Manin obstruction is the only one for 0-cycles of degree 1 on geometrically rational varieties then any class of order 2 in $\operatorname{Br}_\infty(X)$ must be the class associated to a generalized quaternion algebra (a, b) over $K(X)$.

When $\operatorname{char} K = p > 0$, one can prove that any $\alpha \in \operatorname{Br}(X)$ whose period is relatively prime to p satisfies $\operatorname{per}(\alpha) = \operatorname{ind}(\alpha)$ (see [**20**]). The proof uses the geometry of moduli spaces of twisted sheaves on surfaces over finite fields, and therefore contributes nothing to our understanding of the mixed-characteristic situation.

In Sections 4 and 5 we start to consider how one might generalize the Theorem to the case of ramified Brauer classes. Things here are significantly more complicated – for example, we know that there are ramified classes whose period and index are unequal. A nice collection of such examples is provided in [**15**] in the form of biquaternion algebras over $\mathbf{Q}(t)$ with Faddeev index 4 such that for all places ν of $\mathbf{Q}$ the restriction to $\mathbf{Q}_\nu(t)$ has index 2. Can we view such algebras as giving a violation of some sort of Hasse principle? As we discuss in Section 5, there is a canonical rational-point problem associated to such an algebra, but it fails to produce a counterexample to the Hasse principle because it lacks a local point at some place! In other words, while these algebras seem to violate some sort of "Hasse principle," the canonically associated moduli problems actually have *local* obstructions. This phenomenon is intriguing and demands further investigation.

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Notation and assumptions

As in the introduction, given an arithmetic surface $X \to S$ with S the scheme of integers of K, we will write $\operatorname{Br}_\infty(X)$ for the subgroup of $\operatorname{Br}(X)$ of classes whose restriction to $X \otimes K_\nu$ is trivial for all Archimedean places of K.

We assume familiarity with the theory of algebraic stacks, as explained in [**16**]. Given a stack $\mathscr{X}$, we will denote the inertia stack by

$$\mathscr{I}(\mathscr{X}) := \mathscr{X} \times_{\Delta, \mathscr{X} \times \mathscr{X}, \Delta} \mathscr{X};$$

this is the fiber product of the diagonal with itself, and represents the sheaf of groups which assigns to any object its automorphisms. The notation related to moduli is explained in the Appendix. While gerbes, twisted sheaves, and the Brauer group are briefly reviewed in the Appendix, the reader completely unfamiliar with them will probably benefit from consulting [**10, 18, 20**]

Most of the material described here is derived from the more extensive treatments in [**19, 20**]. We have thus felt free to merely sketch proofs. The material of Sections 4 and 5 is new, but is not in its final form. Again, we have not spelled out all of the details; they will appear once a more satisfactory understanding of the phenomena described there has been reached.

1. Forms of moduli and an isotrivial Torelli theorem

Let C/K be a curve over a field with a K-rational point p and L an invertible sheaf on C. The central objects of study in this paper will be forms of the moduli stack $\mathscr{M}_{C/K}(r, L)$ of stable vector bundles on C of rank r and determinant L. In this section we give a geometric introduction to the existence and study of these forms. It turns out that the most interesting forms of $\mathscr{M}_{C/K}(r, L)$ are those which arise from $\boldsymbol{\mu}_r$-gerbes $\mathscr{C} \to C$. The reader unfamiliar with gerbes should think of these as stacky forms of C; the curve C itself corresponds to the trivial gerbe $\mathsf{B}\boldsymbol{\mu}_r \times C$. As a warm-up for the rest of the paper, we give an amusing Torelli-type theorem for such stacky forms of C.

We begin by discussing forms of the coarse space $M_C(r, L)$. From a geometric point of view, forms of a variety are simply isotrivial families.

DEFINITION 1.1. Given an algebraically closed field k, a morphism of k-schemes $f : X \to Y$ is *isotrivial* if there is a faithfully flat morphism $g : Z \to Y$, a k-scheme W, and isomorphism $X \times_Y Z \xrightarrow{\sim} W \times Z$.

Isotrivial families arise naturally from geometry, as the following example shows.

EXAMPLE 1.2. Suppose C_ϵ is an infinitesimal deformation of C over $k[\epsilon]$ and L_ϵ and L'_ϵ are two infinitesimal deformations of L. We claim that the moduli spaces $M_{C_\epsilon/k[\epsilon]}(r, L_\epsilon)$ and $M_{C_\epsilon/k[\epsilon]}(r, L'_\epsilon)$ are isomorphic. To see this, note that $L'_\epsilon \otimes L_\epsilon^{-1}$ is an infinitesimal deformation of $\mathscr{O}_C$ and is thus an rth power in $\mathrm{Pic}(C_\epsilon)$. Twisting by an rth root gives an isomorphism of moduli problems. This obviously generalizes to deformations of C over Artinian local k-algebras A. As a consequence, if R is a complete local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k and $\mathcal{L}$ is an invertible sheaf of degree d on $C \otimes R$ with reduction L over the residue field, we see that $M_{C \otimes R/R}(r, \mathcal{L}) \cong M_{C/k}(r, L) \otimes R$. Indeed, since $M_{C \otimes R/R}(r, \mathcal{L})$ and $M_{C/k}(r, L)$ are proper, the Grothendieck existence theorem shows that the natural map

$$\mathrm{Isom}_R(M_{C \otimes R/R}(r, \mathcal{L}), M_{C/k}(r, L) \otimes R) \to \varprojlim \mathrm{Isom}_{R_n}(M_{C \otimes R_n/R_n}(r, \mathcal{L}_{R_n}), M_{C/k}(r, L) \otimes R_n)$$

is an isomorphism, where $R_n := R/\mathfrak{m}^{n+1}$. Since the identity map over the residue field lifts to each infinitesimal level, we conclude that there is an isomorphism over R.

Now, given r and d as above, let $M_C(r,d)$ denote the moduli space of all stable vector bundles of rank r and degree d. The determinant defines a morphism $\det : M_C(r,d) \to \mathrm{Pic}^d_{C/k}$. By the argument of the previous paragraph, all of the fibers of $\det$ are mutually isomorphic. This is a basic example of an isotrivial family which arises naturally from geometry.

How can we classify isotrivial families? The classification is familiar from descent theory. Let us illustrate how this works using Example 1.2. To simplify notation, let M denote the moduli space $M_C(r,L)$ for some fixed $L \in \mathrm{Pic}^d_{C/k}$, and write $X = C \times \mathrm{Pic}^d_{C/k}$. Consider the $\mathrm{Pic}^d_{C/k}$-scheme

$$T = \mathrm{Isom}_{\mathrm{Pic}^d_{C/k}}(M(r,d), M \times \mathrm{Pic}^d_{C/k});$$

this is a left torsor under the scheme of automorphisms $\mathrm{Aut}(M)$. We now make one simplifying assumption which will tie the geometry to algebraic constructions we will do later.

HYPOTHESIS 1.3. Suppose that the curve C has trivial automorphism group, and that both r and d are odd.

The purpose of Hypothesis 1.3 is to ensure that the automorphism group of $M_C(r,L)$ is given entirely by the natural image of $\mathrm{Pic}_C[r]$ in $\mathrm{Aut}(M)$ under the map sending an invertible sheaf $N \in \mathrm{Pic}_C[r]$ to the automorphism given by twisting by N (see [**13**]).

Given Hypothesis 1.3, the torsor T corresponds to a class

$$\overline{\alpha} \in \mathrm{H}^1(\mathrm{Pic}^d_{C/k}, \mathbf{R}^1\mathrm{pr}_{2*}\boldsymbol{\mu}_r),$$

where pr_2 is the second projection of the product $C \times \mathrm{Pic}^d_{C/k}$. The cohomology class thus arising belongs to a convergent in the Leray spectral sequence for $\boldsymbol{\mu}_r$ with respect to the morphism $\mathrm{pr}_2 : C \times \mathrm{Pic}^d_{C/k} \to \mathrm{Pic}^d_{C/k}$. In fact, the Leray spectral sequence gives a surjection

$$\mathrm{H}^2(C \times \mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r) \to \mathrm{H}^1(\mathrm{Pic}^d_{C/k}, \mathbf{R}^1\mathrm{pr}_{2*}\boldsymbol{\mu}_r).$$

(This is surjective because the edge map $\mathrm{H}^3(\mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r) \to \mathrm{H}^3(C \times \mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r)$ is split by any k-point $p \in C$.) There is a Chern class map

$$\mathrm{Pic}(C \times \mathrm{Pic}^d_{C/k}) \to \mathrm{H}^2(C \times \mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r)$$

arising in the usual way from the Kummer sequence.

Since C has a k-point, there is a tautological sheaf $\mathscr{L}$ on $C \times \mathrm{Pic}^d_{C/k}$.

CLAIM 1.4. *The class $\overline{\alpha}$ is the image of $\mathscr{L} \otimes L^{-1}$ under the composition of the above maps, where L is the invertible sheaf corresponding to the base point parameterizing M.*

To prove this claim, we give a geometric interpretation of the maps

$$\mathrm{Pic}(C \times \mathrm{Pic}^d_{C/k}) \to \mathrm{H}^2(C \times \mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r) \tag{1}$$

and

$$\mathrm{H}^2(C \times \mathrm{Pic}^d_{C/k}, \boldsymbol{\mu}_r) \to \mathrm{H}^1(\mathrm{Pic}^d_{C/k}, \mathrm{Pic}_{C/k}[r]). \tag{2}$$

To interpret (1): given an invertible sheaf $\mathscr{N}$ on $C \times \mathrm{Pic}^d /C/k$, we get a $\boldsymbol{\mu}_r$-gerbe $[\mathscr{N}]^{1/r}$ over X. This is a stack, i.e., a moduli problem, which in this case is very explicit. Given an X-scheme $T \to X$, an object of $[\mathscr{N}]^{1/r}$ is given by a pair (Λ, ψ) with Λ an invertible sheaf on T and $\psi : \Lambda^{\otimes r} \to \mathscr{N}$ an isomorphism.

To interpret (2): given a $\boldsymbol{\mu}_r$-gerbe $\mathscr{X} \to X$, we can consider the pushforward stack $\mathrm{pr}_2\, \mathscr{X} \to \mathrm{Pic}^d_{C/k}$.

CLAIM 1.5. *With the above notation, given a morphism $f : X \to Y$ the stack $f_*\mathscr{X}$ is an $f_*\boldsymbol{\mu}_r$-gerbe over a $\mathbf{R}^1 f_*\boldsymbol{\mu}_r$-pseudotorsor. If $\mathscr{X} \times_Y U$ is a neutral gerbe over $X \times_Y U$ for some étale surjection $U \to Y$, the sheafification of $f_*\mathscr{X}$ is an étale torsor.*

SKETCH OF PROOF. By definition, an object of $f_*\mathscr{X}$ over a Y-scheme $S \to Y$ is an object of $\mathscr{X}$ over $S \times_Y X$. It immediately follows that $f_*\mathscr{X}$ is a gerbe banded by $f_*\boldsymbol{\mu}_r$. Let the sheafification of $f_*\mathscr{X}$ be denoted $\Theta \to Y$. Twisting by torsors gives an action $\mathbf{R}^1 f_*\boldsymbol{\mu}_r \times \Theta \to \Theta$. One checks that this makes Θ into a pseudotorsor; the hypothesis of the last sentence of the claim transparently makes Θ have local sections in the étale topology, and therefore a torsor. □

Sending $\mathscr{X}$ to the sheafification of $\mathrm{pr}_2\, \mathscr{X}$ gives an interpretation of (2).

PROOF OF CLAIM 1.4. Let $\mathscr{X} \to X$ be the $\boldsymbol{\mu}_r$-gerbe $[\mathscr{L} \otimes L^{-1}]^{1/r}$ defined above, and let Θ denote the sheafification of $\mathscr{X}$ (as a $\mathrm{Pic}^d_{C/k}$-stack). Tensor product defines a morphism of $\mathrm{Pic}^d_{C/k}$-stacks

$$\mathrm{pr}_{2*}\mathscr{X} \times \mathscr{M}(r, L)_{\mathrm{Pic}^d_{C/k}} \to \mathscr{M}(r, d),$$

where $\mathscr{M}$ is used in place of M to denote the stack instead of the coarse moduli space. Passing to sheafifications yields a map

$$\Theta \times M(r, L) \to M(r, d)$$

which is compatible with the natural $\mathbf{R}^1\mathrm{pr}_{2*}\boldsymbol{\mu}_r$-actions. By adjunction, this gives a $\mathrm{Pic}(C)[r]$-equivariant map

$$\Theta \to \mathrm{Isom}(M(r, L), M(r, d)),$$

yielding the desired result. □

This analysis of Example 1.2 fits into a general picture. Let $f : C \to S$ be a proper smooth relative curve of genus $g \geq 2$. Suppose $\mathscr{C} \to C$ is a $\boldsymbol{\mu}_r$-gerbe whose associated cohomology class $[\mathscr{C}] \in \mathrm{H}^2(C, \boldsymbol{\mu}_r)$ is trivial on all geometric fibers of f.

PROPOSITION 1.6. *There is an étale surjection $U \to S$ and an isomorphism of stacks $\tau : \mathscr{M}_{\mathscr{C}_U/U}(r, L) \xrightarrow{\sim} \mathscr{M}_{C_U/U}(r, L)$.*

In other words, the stack of stable $\mathscr{C}$-twisted sheaves of rank r determinant L is a form of the stack of stable sheaves on C of rank r and determinant L. (As the reader will note from the proof, it is essential that the cohomology class be trivial on geometric fibers for this to be true.)

SKETCH OF PROOF. The proper and smooth base change theorems and the compatibility of the formation of étale cohomology with limits show that there is an étale surjection $U \to S$ such that $[\mathscr{C}]_U = 0$. Thus, it suffices to show that if $[\mathscr{C}] = 0$ (in other words, if $\mathscr{C} \cong \mathsf{B}\boldsymbol{\mu}_{r,C}$) then there is an isomorphism

$\mathscr{M}_{\mathscr{C}/S}(r,L) \xrightarrow{\sim} \mathscr{M}_{C/S}(r,L)$. Under this assumption, there is an invertible $\mathscr{C}$-sheaf χ such that $\chi^{\otimes r}$ is isomorphic to $\mathscr{O}$. Tensoring with χ^{-1} gives the isomorphism in question. □

The isomorphism τ of Proposition 1.6 yields a coarse isomorphism $\overline{\tau}$: $M_{\mathscr{C}_U/U}(r,L) \xrightarrow{\sim} M_{C_U/U}(r,L)$, so that $M_{\mathscr{C}/S}(r,L)$ is a form of $M_{C/S}(r,L)$. Just as above, we can give the cohomology class corresponding to this form (by descent theory): it is precisely the image of $[\mathscr{C}]$ under the edge map $\epsilon : \mathrm{H}^2(C,\boldsymbol{\mu}_r) \to \mathrm{H}^1(S,\mathbf{R}^1 f_*\boldsymbol{\mu}_r)$ in the Leray spectral sequence. There is one mildly interesting consequence of this fact. Since ϵ has (in general) a kernel, we see that the coarse moduli space $M_{\mathscr{C}/S}(r,L)$ does not (in general) characterize the "curve" $\mathscr{C}$. In other words, it appears that there is no "stacky Torelli theorem". It is perhaps illuminating to give an example of the failure.

LEMMA 1.7. *If $f : X \to S$ is a proper morphism with geometrically connected fibers such that $\mathrm{Pic}_{X/S} = \mathbf{Z}$, then the natural pullback map $\mathrm{H}^2(S,\boldsymbol{\mu}_r) \to \mathrm{H}^2(X,\boldsymbol{\mu}_r)$ is injective.*

PROOF. The Leray spectral sequence shows that the kernel of the map is the image of $\mathrm{H}^0(S,\mathbf{R}^1 f_*\boldsymbol{\mu}_r) = \mathrm{H}^1(X,\mathrm{Pic}_{X/S}[r]) = 0$. □

EXAMPLE 1.8. If $\mathscr{C} = C \times \mathscr{S}$ for a $\boldsymbol{\mu}_r$-gerbe $\mathscr{S} \to S$, then (for example, by the above Leray spectral sequence calculation) there is an isomorphism b : $M_{\mathscr{C}/S}(r,L) \xrightarrow{\sim} M_{C/S}(r,L)$. However, the stacks $\mathscr{M}_{\mathscr{C}/S}(r,L)$ and $\mathscr{M}_{C/S}(r,L)$ are not isomorphic unless $\mathscr{S}$ is isomorphic to $\mathsf{B}\boldsymbol{\mu}_{r,S}$. In fact, viewing (via b) both of these stacks as $\boldsymbol{\mu}_r$-gerbes over $M_{C/S}(r,L)$, we have an equation

$$[\mathscr{M}_{\mathscr{C}/S}(r,L)] - [\mathscr{M}_{C/S}(r,L)] = [\mathscr{S}_{M_{C/S}(r,L)}]$$

as described in [**14**]. By Lemma 1.7, we see that $[\mathscr{S}] = 0$ if and only if $[\mathscr{S}_{M_{C/S}(r,L)}] = 0$, as desired.

On the other hand, if we keep track of the stacky structure, we have the following silly "Torelli" theorem. Let $f : C \to S$ be a proper smooth relative curve of genus $g \geq 2$ with a section whose geometric generic fiber has no nontrivial automorphisms. Suppose L is an invertible sheaf on C of degree d on each geometric fiber of f and r is a positive integer relatively prime to d such that rd is odd. Given stacks $\mathscr{X}$ and $\mathscr{Y}$ with inclusions $\boldsymbol{\mu}_r \hookrightarrow \mathscr{I}(\mathscr{X})$ and $\boldsymbol{\mu}_r \hookrightarrow \mathscr{I}(\mathscr{Y})$, the notation $\mathscr{X} \cong_r \mathscr{Y}$ will mean that there is an isomorphism $\iota : \mathscr{X} \xrightarrow{\sim} \mathscr{Y}$ such that the composition $\iota^*\boldsymbol{\mu}_r \xrightarrow{\sim} \boldsymbol{\mu}_r \to \mathscr{I}(\mathscr{X}) \to \iota^*\mathscr{I}(\mathscr{Y})$ is the pullback under ι of the given inclusion $\boldsymbol{\mu}_r \to \mathscr{I}(\mathscr{Y})$. We will call such an isomorphism "$\boldsymbol{\mu}_r$-linear".

THEOREM 1.9 (Isotrivial Torelli). *With the above notation, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are $\boldsymbol{\mu}_r$-gerbes on C whose restrictions to geometric fibers of f are trivial, then $\mathscr{C}_1 \cong_r \mathscr{C}_2$ if and only if $\mathscr{M}_{\mathscr{C}_1/S}(r,L) \cong_r \mathscr{M}_{\mathscr{C}_2/S}(r,L)$.*

PROOF. The assumption that f has a section leads (via pullback and the relative cohomology class of the section) to a natural splitting

$$\mathrm{H}^2(C,\boldsymbol{\mu}_r) = \mathrm{H}^2(S,\boldsymbol{\mu}_r) \oplus \mathrm{H}^1(S,\mathbf{R}^1 f_*\boldsymbol{\mu}_r) \oplus \mathrm{H}^0(S,\mathbf{R}^2 f_*\boldsymbol{\mu}_r)$$

such that the first two summands correspond to classes which are trivial on geometric fibers of f. As discussed above, the image of $[\mathscr{C}_i]$ in $\mathrm{H}^1(S,\mathbf{R}^1 f_*\boldsymbol{\mu}_r)$ is the class associated to $M_{\mathscr{C}_i/S}(r,L)$. If $\mathscr{M}_{\mathscr{C}_1/S}(r,L)$ is isomorphic to $\mathscr{M}_{\mathscr{C}_2/S}(r,L)$

(say, by an isomorphism φ) then certainly the same is true for the coarse moduli spaces (isomorphic via $\overline{\varphi}$), from which we conclude that $[\mathscr{C}_1]$ and $[\mathscr{C}_2]$ have the same image in $\mathrm{H}^1(S, \mathbf{R}^1 f_* \boldsymbol{\mu}_r)$. Thus, there exists some $\boldsymbol{\mu}_r$-gerbe $\mathscr{S} \to S$ such that $[\mathscr{C}_1] - [\mathscr{C}_2] = [\mathscr{S}]|_C$ in $\mathrm{H}^2(C, \boldsymbol{\mu}_r)$. Using Giraud's theory [**10**, §IV.2.4], it follows that we can write $\mathscr{C}_1 = \mathscr{C}_2 \wedge \mathscr{S}_C$. In this situation, there is a canonical isomorphism $b : M_{\mathscr{C}_1/S}(r, L) \xrightarrow{\sim} M_{\mathscr{C}_2/S}(r, L)$ with the property that $[\mathscr{M}_{\mathscr{C}_1/S}(r, L)] - b^*[\mathscr{M}_{\mathscr{C}_2/S}(r, L)] = [\mathscr{S}]_{M_{\mathscr{C}_1/S}(r,L)}$ in $\mathrm{H}^2(M_{\mathscr{C}_1/S}(r, L), \boldsymbol{\mu}_r)$.

By Hypothesis 1.3, any automorphism of $M_{\mathscr{C}_1/S}(r, L)$ lifts to a ($\boldsymbol{\mu}_r$-linear) automorphism of $\mathscr{M}_{\mathscr{C}_1/S}(r, L)$. Applying this to $b \circ \overline{\varphi}^{-1}$, we see that b lifts to an isomorphism $\mathscr{M}_{\mathscr{C}_1/S}(r, L) \xrightarrow{\sim} \mathscr{M}_{\mathscr{C}_2/S}(r, L)$. We thus conclude that $\mathscr{S}_{M_{\mathscr{C}_1/S}(r,L)}$ is a trivial gerbe. By Lemma 1.7, we see that $[\mathscr{S}] = 0$, so $\mathscr{C}_1$ and $\mathscr{C}_2$ are isomorphic $\boldsymbol{\mu}_r$-gerbes. □

2. Some arithmetic questions about Brauer groups and rational points on varieties over global fields

Let C/k be a curve over a field. In this section we describe how the forms arising in the preceding section are related to basic questions about the arithmetic of function fields. The linkage is provided by an interpretation of the group $\mathrm{H}^2(C, \boldsymbol{\mu}_r)$. The inclusion $\boldsymbol{\mu}_r \to \mathbf{G}_m$ yields classes in $\mathrm{H}^2(C, \mathbf{G}_m)$, which is equal to the Brauer group of C. The reader is referred to the appendix for a review of basic facts about the Brauer group of a scheme.

Given a field K, there are several questions about the arithmetic properties of K in which the Brauer group plays a central role.

(1) The period-index problem: given $\alpha \in \mathrm{Br}(K)$, what is the minimal g such that $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^g$?
(2) The index-reduction problem: given a field extension L/K and a class $\alpha \in \mathrm{Br}(K)$, how can we characterize the number $\mathrm{ind}(\alpha)/\mathrm{ind}(\alpha|_L)$?
(3) The Brauer-Manin obstruction to the Hasse principle: is the Brauer-Manin obstruction the only obstruction to the existence of 0-cycles of degree 1?

In this paper we will focus on questions (1) and (3). Question (2) also has close ties to the geometry of moduli spaces; we refer the reader to [**14**] for details.

As the third question is the most technical, let us briefly review what it means. Suppose K is a global field with adèle ring A and X is a proper geometrically connected K-scheme. For example, X could be a smooth quadric hypersurface. For smooth quadric hypersurfaces, a classical theorem of Hasse and Minkowski says that $X(K) \neq \emptyset$ if and only if $X(K_\nu) \neq \emptyset$ for all places ν of K (including the infinite ones). This principle is usually referred to as the "Hasse principle". A natural question which arises from this theorem is whether or not this principle holds for an arbitrary variety. This turns out not to be the case [**24**], but there is often an explanation for the failure of this principle arising from a cohomological obstruction discovered by Manin [**21**]. To describe this obstruction, a few minor technical remarks are in order.

Since A is a K-algebra, we can consider the adelic points $X(A)$. Restriction gives a map $X(A) \to \prod_\nu X(K_\nu)$. In fact, this map is a bijection. (To prove this, one can use a regular proper model of X over the scheme of integers of K to reduce to the case in which X is affine, where this follows from the universal property of the product.) From this point of view, the Hasse principle says that $X(A) \neq \emptyset$ if and only if $X(K) \neq \emptyset$. Moreover, the K-algebra structure on A gives a map $X(K) \to X(A)$. Manin's idea is to produce a pairing whose "kernel" contains the image of $X(K)$ in $X(A)$. The pairing arises as follows: restriction gives a map

$$\mathrm{Br}(X) \times X(A) \to \mathrm{Br}(A) \to \mathbf{Q}/\mathbf{Z},$$

where the last map comes from the usual local invariants of class field theory. The standard reciprocity law implies that the Brauer group of K is in the left kernel of this pairing, yielding an invariant map

$$X(A) \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(K), \mathbf{Q}/\mathbf{Z}).$$

Write $X(A)^{\mathrm{Br}(X)}$ for the "kernel" of this map (i.e., the elements sent to the map $0 : \mathrm{Br}(X)/\mathrm{Br}(K) \to \mathbf{Q}/\mathbf{Z}$). The same reciprocity law shows that $X(K)$ is contained in $X(A)^{\mathrm{Br}(X)}$. In particular, if $X(A)^{\mathrm{Br}(X)} = \emptyset$ then $X(K) = \emptyset$. The obvious question concerning this pairing is the following.

QUESTION 2.1. If $X(A)^{\mathrm{Br}(X)} \neq \emptyset$ then is $K(K) \neq \emptyset$?

As suspected from the beginning, the answer turns out to be "no," but there was no counterexample until Skorobogatov discovered a bielliptic surface with no rational points and vanishing Brauer-Manin obstruction [**25**]. There is a refinement of this question due to Colliot-Thélène that is still the subject of much current research. (Cf. Conjecture 1.5(a) of [**4**] and Conjecture 2.4 of [**5**].)

CONJECTURE 2.2. *If $X(A)^{\mathrm{Br}(X)} \neq \emptyset$ then there is a 0-cycle of degree 1 (over K) on X.*

We will call this property "the Hasse principle for 0-cycles". A famous theorem of Saito affirms Conjecture 2.2 when X is a curve, under the assumption that the Tate-Shafarevich group of the curve is finite (the original is [**23**], with another account of this result in [**5**]); the general case is still wide open. According to Colliot-Thélène, it is not known if Skorobogatov's negative answer to Question 2.1 has a 0-cycle of degree 1. One of our primary goals in this paper will be to link certain cases of Conjecture 2.2 to the period-index problem for function fields of arithmetic surfaces.

There is one case of Conjecture 2.2 which will come up below.

CONJECTURE 2.3. *If X is smooth and geometrically rational and $\mathrm{Pic}(X \otimes \overline{K})$ is isomorphic to $\mathbf{Z}$ then the Hasse principle holds (for 0-cycles) for X.*

PROOF THAT CONJECTURE 2.3 FOLLOWS FROM CONJECTURE 2.2. By [**11**], we know $\mathrm{Br}(X \otimes \overline{K}) = 0$, since X is smooth and geometrically rational. The Leray spectral sequence for $\mathbf{G}_m$ then shows that $\mathrm{Br}(X)/\mathrm{Br}(K) = \mathrm{H}^1(K, \mathbf{Z})$, and the latter group is trivial (since the first cohomology of a finite group acting trivially on $\mathbf{Z}$ is trivial, and the Galois cohomology is a colimit of such). Thus, $X(A) = X(A)^{\mathrm{Br}(X)}$. □

The statement of Conjecture 2.3 is meant to include both the strong form (classical Hasse principle) and weak form (Hasse principle for 0-cycles). We will discuss two different relationships between Conjecture 2.3 and the period-index problem; one will relate to the strong form, while one will relate to the weak form.

3. Moduli spaces of stable twisted sheaves on curves and period-index theorems

In this section, we start to explain the connections among the various problems described in the preceding section. In particular, we will use classical theorems about rational points on various kinds of varieties over various C_1-fields to solve moduli problems encoding period-index problems. Then we will prove the Theorem from the Introduction.

To begin, we give another result which is a transparent translation of a simple spectral sequence argument.

THEOREM 3.1. *If C is a proper curve over a finite field $\mathbf{F}_q$ then* $\mathrm{Br}(C) = 0$.

SKETCH OF PROOF. An exercise in deformation theory (see Section A.2) reduces the theorem to the case in which C is smooth. Let $\mathscr{C} \to C$ be a $\boldsymbol{\mu}_n$-gerbe. Consider the stack $\mathscr{M}_{\mathscr{C}/\mathbf{F}_q}(1,0)$ parameterizing invertible $\mathscr{C}$-twisted sheaves of degree 0. Just as in the classical case, $\mathscr{M}_{\mathscr{C}/\mathbf{F}_q}(1,0)$ is a $\mathbf{G}_m$-gerbe over a $\mathrm{Jac}(C)$-torsor T. By Lang's theorem, T has a rational point p. By Wedderburn's theorem, p lifts to an object of the stack, giving an invertible $\mathscr{C}$-twisted sheaf. As described in the appendix, this invertible twisted sheaf trivializes the Brauer class associated to $[\mathscr{C}]$. Since $\mathscr{C}$ was an arbitrary $\boldsymbol{\mu}_n$-gerbe, this shows that the Brauer group of C is trivial. □

Next, we will sketch a proof of the following theorem.

THEOREM 3.2 (de Jong). *Let X be a surface over an algebraically closed field k. For all $\alpha \in \mathrm{Br}(k(X))$, we have* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.

Unlike Theorem 3.1, this is not merely a geometric realization of a standard cohomological argument. There are various proofs of this result – de Jong's original proof [**7**], a proof due to de Jong and Starr [**9**], and the one we present (found with details in [**20**]). They each ultimately rest on deformation theory and the definition of a suitable moduli problem. The latter two both reduce the result to the existence of a section for a rationally connected fibration over a curve and use the Graber-Harris-Starr theorem.

SKETCH OF PROOF. We assume that $\mathrm{char}\, k = 0$ for the sake of simplicity; the reader will find a reduction to this case in [**20**]. We proceed in steps. We will write n for the period of α.

(1) Blowing up in the base locus of a very ample pencil of divisors (one of which contains the ramification divisor of α) on X, we may assume that there is a fibration $X \to \mathbf{P}^1$ with a section such that the ramification of α is entirely contained in a fiber and the generic fiber is smooth of genus $g \geq 2$. Let $C/k(t)$ be the generic fiber; this is a proper smooth curve of genus $g \geq 2$ with a rational point p, and α lies in $\mathrm{Br}(C)$.

(2) Choose a $\boldsymbol{\mu}_n$-gerbe $\mathscr{C} \to C$ such that $[\mathscr{C}] = \alpha \in \mathrm{Br}(C)$ and $[\mathscr{C} \otimes \overline{k(t)}] = 0 \in \mathrm{H}^2(C, \boldsymbol{\mu}_n)$. (We can do this because C has a rational point.)

(3) Consider the modular $\boldsymbol{\mu}_n$-gerbe $\mathscr{M}_{\mathscr{C}/k(t)}(n, \mathscr{O}(p)) \to M_{\mathscr{C}/k(t)}(n, \mathscr{O}(p))$. As discussed in Section 1, there is an isomorphism

$$M_{\mathscr{C}/k(t)}(n, \mathscr{O}(p)) \otimes \overline{k(t)} \xrightarrow{\sim} M_{C \otimes \overline{k(t)}/\overline{k(t)}}(n, \mathscr{O}(p)),$$

and we know that the latter (hence, the former) is unirational (in fact, rational), and thus rationally connected.

(4) The Graber-Harris-Starr theorem implies that there is a rational point $q \in M_{\mathscr{C}/k(t)}(n, \mathscr{O}(p))(k(t))$.

(5) Tsen's theorem implies that q lifts to an object $\mathscr{V}$: a locally free $\mathscr{C}$-twisted sheaf of rank n.

(6) The algebra $\mathscr{E}nd(\mathscr{V})|_\eta$ is a central division algebra of degree n with Brauer class α, thus proving that the index of α divides n. Since we already know the converse divisibility relation, we are done. □

REMARK 3.3. The reader will note that we use the Graber-Harris-Starr theorem in two places in the proof: once to find a rational point on the coarse moduli space, and once (in the guise of Tsen's theorem) to lift that rational point to an object of the stack. While classical algebraic geometry only "sees" the former, the difference between fine and coarse moduli problems necessitates the latter.

If we start with an arithmetic surface instead of a surface over an algebraically closed field, things get more complicated. (For example, it is no longer true that the period and index are always equal if the class is allowed to ramify.) For the rest of this section, we will discuss unramified classes. In Section 5 below, we will discuss certain ramified classes. In both cases, we will tie the period-index problem to Conjecture 2.3.

Let K be a global field, S the scheme integers of K, and $C \to S$ a proper relative curve with a section and smooth generic fiber. (When $\operatorname{char}(K) > 0$, the scheme of integers is assumed to be proper over the prime field; this ensures that it is unique.)

THEOREM 3.4. *If Conjecture 2.3 is true, then any* $\alpha \in \operatorname{Br}_\infty(C)$ *satisfies* $\operatorname{per}(\alpha) = \operatorname{ind}(\alpha)$.

PROOF. Write $n = \operatorname{per}(\alpha)$. Since $\mathcal{C} \to S$ has a section, we can choose a $\boldsymbol{\mu}_n$-gerbe $\mathscr{C} \to C$ such that $[\mathscr{C} \otimes \overline{K}] = 0 \in \mathrm{H}^2(C \otimes \overline{K}, \boldsymbol{\mu}_n)$. By class field theory, the restriction of α to the point $p \in C(K)$ is trivial.

Consider the stack $\chi : \mathscr{M}_{\mathscr{C}_K/K}(n, \mathscr{O}(p)) \to M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$. We claim that to prove the theorem it suffices to show that for every place ν of K, the category $\mathscr{M}_{\mathscr{C}_K/K}(n, \mathscr{O}(p))_{K_\nu}$ is nonempty. Indeed, the map χ is a $\boldsymbol{\mu}_n$-gerbe; let $\beta \in \operatorname{Br}(M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$ be the associated Brauer class. Since $M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$ is geometrically rational with Picard group **Z** [**12**] and has a point over every completion of K, we know that the pullback map $\operatorname{Br}(K) \to \operatorname{Br}(M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$ is an isomorphism. Thus, β is the pullback of a class over K. The fact that each $\mathscr{M}_{\mathscr{C}_K/K}(n, \mathscr{O}(p))_{K_\nu}$ is non-empty implies that $\beta_{K_\nu} = 0$; class field theory again shows that $\beta = 0$. But then any rational point of $M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$ lifts to an object of $\mathscr{M}_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$.

Let us assume we have found a collection of local objects as in the previous paragraph. The assumption that Conjecture 2.3 holds yields a 0-cycle of degree 1 on the coarse space $M_{\mathscr{C}_K/K}(n, \mathscr{O}(p))$, which lifts to the stack, producing a complex $P^\bullet$ of locally free $\mathscr{C}_K$-twisted sheaves such that $\operatorname{rk} P^\bullet = n$. Indeed, if there is a $\mathscr{C} \otimes L$-twisted sheaf of rank n for some finite algebra L/K then pushing forward along $\mathscr{C} \otimes L \to \mathscr{C}$ gives a $\mathscr{C} \otimes K$-twisted sheaf of rank $[L : K]n$. A 0-cycle of degree 1 yields two algebras L_1/K and L_2/K such that $[L_1 : K] -$

$[L_2 : K] = 1$ and $\mathscr{M}_{\mathscr{C}\otimes L_i/L_i}(n, \mathscr{O}(p))_{L_i} \neq \emptyset$. There result two $\mathscr{C}\otimes K$-twisted sheaves V_1 and V_2 such that $\operatorname{rk} V_1 - \operatorname{rk} V_2 = n$, whence we can let $P^\bullet$ be the complex with V_1 in degree 0 and V_2 in degree 1 (and the trivial differential). Since the category of coherent twisted sheaves over the generic point of C is semisimple (it is just the category of finite modules over a division ring), it follows that there is a $\mathscr{C}_\eta$-twisted sheaf of rank n, whence $\operatorname{per}(\alpha) = \operatorname{ind}(\alpha)$, as desired.

So it remains to produce local objects of $\mathscr{M}_{\mathscr{C}_K/K}(n, \mathscr{O}(p))_{K_\nu}$ for all places ν. If ν is Archimedean, then $[\mathscr{C}\otimes K_\nu] = 0 \in \mathrm{H}^2(C\otimes K_\nu, \boldsymbol{\mu}_n)$ by assumption, so that stable $\mathscr{C}\otimes K_\nu$-twisted sheaves are equivalent (upon twisting down by an invertible $\mathscr{C}\otimes K_\nu$-twisted sheaf) to stable sheaves on $C\otimes K_\nu$. Since moduli spaces of stable vector bundles with fixed invariants have rational points over every infinite field, we find a local object.

Now assume that ν is finite, and let R be the valuation ring of K_ν, with finite residue field F. By Theorem 3.1 and the assumption that $[\mathscr{C}\otimes \overline{F}] = 0 \in \mathrm{H}^2(C\otimes \overline{F}, \boldsymbol{\mu}_n)$ we know that $[\mathscr{C}\otimes F] = 0 \in \mathrm{H}^2(C\otimes F, \boldsymbol{\mu}_n)$. It follows as in the previous paragraph that it suffices to show the existence of a stable sheaf on $C\otimes F$ of determinant $\mathscr{O}(p)$ and rank n. Consider the stack $\mathscr{M}_{C\otimes F/F}(n, \mathscr{O}(p)) \to M_{C\otimes F/F}(n, \mathscr{O}(p))$. Just as above, the space $M_{C\otimes F/F}(n, \mathscr{O}(p))$ is a smooth projective rationally connected variety. By Esnault's theorem [**6**], it has a rational point. Since $\mathscr{M}_{C\otimes F/F}(n, \mathscr{O}(p)) \to M_{C\otimes F/F}(n, \mathscr{O}(p))$ is a $\boldsymbol{\mu}_n$-gerbe and F is finite, the moduli point lifts to an object, giving rise to a stable $\mathscr{C}\otimes F$-twisted sheaf V of rank n and determinant $\mathscr{O}(p)$). Since $\mathscr{M}_{\mathscr{C}/S}(n, \mathscr{O}(p))$ is smooth over S, the sheaf V deforms to a family over R, whose generic fiber gives the desired local object. □

4. Parabolic bundles on $\mathbf{P}^1$

In this section, we review some basic elements of the moduli theory of parabolic bundles of rank 2 on the projective line. We will focus on the case of interest to us and describe it in stack-theoretic language. We refer the reader to [**2**] for a general comparison between the classical and stacky descriptions of parabolic bundles.

Let $D = p_1 + \cdots + p_r \subset \mathbf{P}^1$ be a reduced divisor, and let $\pi : \mathscr{P} \to \mathbf{P}^1$ be the stack given by extracting square roots of the points of D as in [**3**]. The stack has r non-trivial residual gerbes $\xi_1, \ldots, \xi_r$, each isomorphic to $\mathsf{B}\boldsymbol{\mu}_2$ over its field of moduli (i.e., the residue field of p_i). Recall that the category of quasi-coherent sheaves on $\mathsf{B}\boldsymbol{\mu}_2$ is naturally equivalent to the category of representations of $\boldsymbol{\mu}_2$. Given a sheaf $\mathscr{F}$ on $\mathsf{B}\boldsymbol{\mu}_2$, we will call the representation arising by this equivalence the *associated representation* of $\mathscr{F}$.

DEFINITION 4.1. A locally free sheaf V on $\mathscr{P}$ is *regular* if, for each $i = 1, \ldots, r$, the associated representation of the restriction $V|_{\xi_i}$ is a direct sum of copies of the regular representation.

DEFINITION 4.2. Let $\{a_i \leq b_i\}_{i=1}^r$ be elements of $\{0, 1/2\}$. A *parabolic bundle* V_* of rank N with parabolic weights $\{a_i \leq b_i\}$ is a pair (W, F), where W is a locally free sheaf of rank 2 and $F \subset W_D$ is a subbundle. The parabolic degree of V_* is

$$\operatorname{pardeg}(V_*) = \deg W + \sum_i a_i(\operatorname{rk} W - \operatorname{rk} F_{p_i}) + b_i(\operatorname{rk} F_{p_i})$$

We will only consider parabolic bundles with weights in $\{0, 1/2\}$ in this paper. More general weights in $[0, 1)$ are often useful. The stack-theoretic interpretation of this more general situation is slightly more complicated; it is explained clearly in [**2**].

In [**26**], Vistoli defined a Chow theory for Deligne-Mumford stacks and showed that pushforward defines an isomorphism $A(\mathscr{P}) \otimes \mathbf{Q} \xrightarrow{\sim} A(\mathbf{P}^1) \otimes \mathbf{Q}$ of Chow rings. In particular, any invertible sheaf L on $\mathscr{P}$ has a degree, $\deg L \in \mathbf{Q}$. One can make a more ad hoc definition of the degree of an invertible sheaf L on $\mathscr{P}$ in the following way. The sheaf $L^{\otimes 2}$ is the pullback of a unique invertible sheaf $\mathscr{M}$ on $\mathbf{P}^1$, and we can define $\deg_{\mathscr{P}} L = \frac{1}{2} \deg_{\mathbf{P}^1} \mathscr{M}$. Thus, for example, $\deg \mathscr{O}(\xi_i) = [\kappa_i : k]/2$, where κ_i is the field of moduli of ξ_i (the residue field of p_i).

The following is a special case of a much more general result. The reader is referred to (e.g.) [**2**] for the generalities.

PROPOSITION 4.3. *There is an equivalence of categories between locally free sheaves V on $\mathscr{P}$ and parabolic sheaves V_* on $\mathbf{P}^1$ with parabolic divisor D and parabolic weights contained in* $\{0, 1/2\}$. *Moreover, we have* $\deg_{\mathscr{P}} V = \operatorname{pardeg}(V_*)$.

PROOF. Given V, define V_* as follows: the underlying sheaf W of V_* is $\pi_* V$. To define the subbundle $F \subset W_D$, consider the inclusion $V(-\sum \xi_i) \subset V$. Pushing forward by π yields a subsheaf $W' \subset W$, and we let F be the image of the induced map $W'_D \to W_D$.

We leave it to the reader as an amusing exercise to check that 1) this defines an equivalence of categories, and 2) this equivalence respects degrees, as claimed. □

DEFINITION 4.4. Given a non-zero locally free sheaf V on $\mathscr{P}$, the *slope* of V is

$$\mu(V) = \frac{\deg V}{\operatorname{rk} V}.$$

The sheaf V on $\mathscr{P}$ is *stable* if for all locally split proper subsheaves $W \subset V$ we have $\mu(W) < \mu(V)$.

The stability condition of Definition 4.4 is identical to the classical notion for sheaves on a proper smooth curve. The reader familiar with the classical definition of stability for parabolic bundles can easily check (using Proposition 4.3) that a sheaf V on $\mathscr{P}$ is stable if and only if the associated parabolic bundle V_* is stable in the parabolic sense. One can check (by the standard methods) that stable parabolic bundles form an Artin stack of finite type over k. The stack of stable parabolic bundles of rank n with fixed determinant is a $\boldsymbol{\mu}_n$-gerbe over an algebraic space.

NOTATION 4.5. Given $L \in \operatorname{Pic}(\mathscr{P})$, let $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ denote the stack of regular locally free sheaves on $\mathscr{P}$ of rank r and determinant L, and let $M^*_{\mathscr{P}/k}(n, L)$ denote the coarse space.

PROPOSITION 4.6. *If $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ is non-empty then it is geometrically unirational and geometrically integral. Moreover, the stack $\mathscr{M}^*_{\mathscr{P}/k}(2, \mathscr{O}(\sum \xi_i))$ is non-empty if $r > 3$.*

PROOF. Basic deformation theory shows that $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ is smooth. Thus, to show that it is integral, it suffices to show that it is connected. We will do this by showing that it is unirational (i.e., finding a surjection onto $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ from an open subset of a projective space. This is a standard trick using a space of extensions. (There are in fact two versions, one using finite cokernels and one using invertible cokernels. We show the reader the former, as it is useful in situations where the latter does not apply and the latter seems to be more easily available in the literature.)

Let V be a parabolic sheaf of rank n and determinant L. Since $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ is of finite type, there exists a positive integer N such that for every parabolic sheaf of rank n and determinant L, a general map $V \to W(N)$ is injective with cokernel Q supported on a reduced divisor E in $\mathscr{P} \setminus \{\xi_1, \ldots, \xi_r\}$ such that $E \in |\mathscr{O}(nN)|$. Let $U \subset |\mathscr{O}(nN)|$ be the open subset parametrizing divisors supported in $\mathscr{P} \setminus \{\xi_1, \ldots, \xi_r\}$, and let $\mathscr{E} \subset \mathscr{P} \times U$ be the universal divisor. The sheaf $\mathbf{R}^1(\mathrm{pr}_2)_* \mathbf{R}\mathscr{H}om(\mathscr{O}_{\mathscr{E}}, \mathrm{pr}_1^* V)$ is locally free on U, and gives rise to a geometric vector bundle $B \to U$ and an extension

$$0 \to V_{\mathscr{P} \times B} \to \mathscr{W} \to \mathscr{Q}_{\mathscr{P} \times B} \to 0$$

such that every extension $0 \to V \to W(N) \to Q \to 0$ as above arises is a fiber over B. Passing to the open subscheme B° over which the extension $\mathscr{W}$ is locally free with stable fibers yields a surjective map $B^\circ \to \mathscr{M}^*_{\mathscr{P}/k}(n, L)$ from an open subset of a projective space, proving that $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ is geometrically integral and unirational.

To prove that $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ is nonempty is significantly more subtle. The proof is similar to a result of Biswas [**1**] for parabolic bundles with parabolic degree 0, but including it here would take us too far afield. ☐

5. Forms of parabolic moduli via split ramification

5.1. Some generalities. In this section we show how to produce forms of the stack of parabolic bundles on $\mathbf{P}^1$ from Brauer classes over $k(t)$. For the sake of simplicity, we restrict our attention to classes in $\mathrm{Br}(k(t))[2]$ and parabolic bundles of rank 2. A generalization to higher period/rank should be relatively straightforward.

Let $\alpha \in \mathrm{Br}(k(t))[2]$ be a Brauer class. Suppose $D = p_1 + \cdots + p_r$ is the ramification divisor of α, and let $\mathscr{P} \to \mathbf{P}^1$ be the stacky branched cover as in Section 4. By Corollary C.2, α extends to a class $\alpha' \in \mathrm{Br}(\mathscr{P})[2]$. Suppose $\mathscr{C} \to \mathscr{P}$ is a $\boldsymbol{\mu}_2$-gerbe representing α' such that $[\mathscr{C} \otimes \overline{k}] = 0 \in \mathrm{H}^2(\mathscr{P} \otimes \overline{k}, \boldsymbol{\mu}_2)$. (For a proof that $\mathscr{C}$ is itself an algebraic stack, the reader is referred to [**19**]. We cannot always ensure that the cohomology class of $[\mathscr{C} \otimes \overline{k}]$ is trivial; we make that as a simplifying assumption. More general cases can be analyzed by similar methods.)

DEFINITION 5.1.1. A *regular $\mathscr{C}$-twisted sheaf* is a locally free $\mathscr{C}$-twisted sheaf V such that for each $i = 1, \ldots, r$, the restriction $V_{\mathscr{C} \times_{\mathbf{P}^1} \mathrm{Spec}\, \overline{\kappa}_i}$ has the form

$\mathscr{L} \otimes \rho^{\oplus m}$ for some integer $m > 0$, where $\mathscr{L}$ is an invertible $\mathscr{C} \otimes \overline{\kappa}_i$-twisted sheaf and ρ is the sheaf on $\mathsf{B}\boldsymbol{\mu}_2$ associated to the regular representation of $\boldsymbol{\mu}_2$.

Just as in Definition 4.4 and Definition A.1.2, we can define stable regular $\mathscr{C}$-twisted sheaves.

NOTATION 5.1.2. Let $\mathscr{M}^*_{\mathscr{C}/k}(n, L)$ denote the stack of stable regular $\mathscr{C}$-twisted sheaves of rank n and determinant L, and $M^*_{\mathscr{C}/k}(n, L)$ its coarse moduli space (sheafification).

PROPOSITION 5.1.3. *For any section σ of $\mathscr{C} \otimes k \to \mathscr{P} \otimes k$ there is an isomorphism $\mathscr{M}^*_{\mathscr{C}/k}(n, L) \otimes \overline{k} \xrightarrow{\sim} \mathscr{M}^*_{\mathscr{P}/k}(n, L)$.*

PROOF. We may assume that $k = \overline{k}$. The section σ corresponds to an invertible $\mathscr{C}$-twisted sheaf $\mathscr{L}$ such that $\mathscr{L}^{\otimes 2} = \mathscr{O}_{\mathscr{C}}$. Twisting by $\mathscr{L}$ defines the isomorphism. (Note that the regularity condition implies that n must be even for either space to be nonempty.) □

In other words, the stack $\mathscr{M}^*_{\mathscr{C}/k}(n, L)$ (resp. the quasi-projective coarse moduli space $M^*_{\mathscr{C}/k}(n, L)$) is a form of $\mathscr{M}^*_{\mathscr{P}/k}(n, L)$ (resp. $M^*_{\mathscr{P}/k}(n, L)$).

COROLLARY 5.1.4. *The space $M^*_{\mathscr{C}/k}(n, L)$ is geometrically (separably) unirational when it is nonempty.*

One can use a generalization of Corollary 5.1.4 to higher genus curves and arbitrary period to give another proof of Theorem 3.2 without having to push the ramification into a fiber, by simply taking any pencil and using the generic points of the ramification divisor (and a point of the base locus) to define the parabolic divisor. (This is not substantively different from the proof we give here.) The main interest for us, however, will be for arithmetic surfaces of mixed characteristic.

5.2. An extended example. Let $\alpha \in \mathrm{Br}(\mathbf{Q}(t))[2]$ be a class whose ramification divisor $D \subset \mathbf{P}^1_{\mathbf{Z}}$ is non-empty with simple normal crossings. Let $\mathscr{P} \to \mathbf{P}^1_{\mathbf{Q}}$ be the stacky cover branched over D to order 2 as in the first paragraph of Section 4 above. Let $\mathscr{C} \to \mathscr{P}$ be a $\boldsymbol{\mu}_2$-gerbe with Brauer class α; if the degree $[D \otimes \mathbf{Q} : \mathbf{Q}]$ is odd, one can ensure that $\mathscr{C}$ such that $[\mathscr{C} \otimes \overline{\mathbf{Q}}] \in \mathrm{H}^2(\mathscr{P} \otimes \overline{Q}, \boldsymbol{\mu}_2)$ has the form $[\Lambda]^{1/2}$ for some invertible sheaf $\Lambda \in \mathrm{Pic}(\mathscr{P} \otimes \overline{\mathbf{Q}})$ of half-integral degree.

DEFINITION 5.2.1. Given a field extension L/K and a Brauer class $\alpha \in \mathrm{Br}(L)$, the *Faddeev index of* α is $\min_{\beta \in \mathrm{Br}(K)} \mathrm{ind}(\alpha + \beta_L)$.

PROPOSITION 5.2.2. *The class α has Faddeev index 2 if and only if the space $M^*_{\mathscr{C}/\mathbf{Q}}(2, L)^{ss}$ has a $\mathbf{Q}$-rational point for some invertible sheaf L. If the degree $[D \otimes \mathbf{Q} : \mathbf{Q}]$ is odd, we need only quantify over L of half-integral degree and look for points in the stable locus $M^*_{\mathscr{C}/\mathbf{Q}}(2, L)$.*

The point of Proposition 5.2.2 is that the computation of the Faddeev index is reduced to the existence of a rational point on one of a sequence of geometrically rational smooth (projective if $[D \otimes \mathbf{Q} : \mathbf{Q}]$ is odd) geometrically connected varieties over $\mathbf{Q}$.

PROOF. First, suppose that $P : \operatorname{Spec} \mathbf{Q} \to M^*_{\mathscr{C}/\mathbf{Q}}(2, L)$ is a rational point. Pulling back $\mathscr{M}_{\mathscr{C}/\mathbf{Q}}(2, L) \to M_{\mathscr{C}/\mathbf{Q}}(2, L)$ along P yields a class $\beta = -[\mathscr{S}] \in \operatorname{Br}(\mathbf{Q})[2]$. Just as in [**14**], we see that

$$\mathscr{M}_{\mathscr{C} \wedge \mathscr{S}_{\mathscr{P}}/\mathbf{Q}}(2, L) \to M_{\mathscr{C} \wedge \mathscr{S}/\mathbf{Q}}(2, L) = M_{\mathscr{C}/\mathbf{Q}}(2, L)$$

is split over P, whence there is a $\mathscr{C} \wedge \mathscr{S}$-twisted sheaf of rank 2. This shows that $\alpha - \beta$ has index 2 and thus that α has Faddeev index dividing 2. Since α is ramified, it cannot have Faddeev index 1.

Now suppose that α has Faddeev index 2, so that there is some $\mathscr{S} \to \operatorname{Spec} \mathbf{Q}$ such that there is locally free $\mathscr{C} \wedge \mathscr{S}$-twisted sheaf of rank 2. Since there is a canonical isomorphism $M_{\mathscr{C} \wedge \mathscr{S}/\mathbf{Q}}(2, L) = M_{\mathscr{C}/\mathbf{Q}}(2, L)$, upon replacing $\mathscr{C}$ by $\mathscr{C} \wedge \mathscr{S}$ we may assume that α has period 2, and is thus represented by a quaternion algebra $[(a, b)] \in \operatorname{Br}(\mathbf{Q}(t))$. To prove the result, it suffices to show that in this case there is a regular $\mathscr{C}$-twisted sheaf of rank 2.

By Proposition A.2.3(1), it suffices to prove this for $\mathscr{C} \otimes \widehat{\mathscr{O}}_{\mathbf{P}^1, D}$. Thus, we are reduced to the following: let R be a complete discrete valuation ring with residual characteristic 0 and (a, b) a quaternion algebra over the fraction field $K(R)$. Suppose (a, b) is ramified, and let $\mathscr{C} \to \mathscr{R}_2$ be a $\boldsymbol{\mu}_2$-gerbe with Brauer class $[(a, b)]$. Then there is a regular $\mathscr{C}$-twisted sheaf of rank 2. To prove this, note that the bilinearity and skew-symmetry of the symbol allows us to assume that b is a uniformizer for R and a has valuation at most 1. We will define a $\boldsymbol{\mu}_2$-equivariant Azumaya algebra A in the restriction of (a, b) to $R' = R[\sqrt{b}]$ such that the induced representation of $\boldsymbol{\mu}_2$ on the fiber $\overline{A}$ over the closed point of R' is $\rho^{\oplus 2}$; it then follows that any twisted sheaf V on $\mathscr{R}_2$ such that $\mathscr{E}nd(V) = A$ is regular by a simple geometric computation over the residue field.

Let x and y be the standard generators for (a, b), so that $x^2 = a$ and $y^2 = b$, and write $a = ub^\varepsilon$ with $u \in R^\times$ and $\varepsilon \in \{0, 1\}$. Let $\tilde{x} = x/\sqrt{b}^\varepsilon$ and $\tilde{y} = y/\sqrt{b}$; we have $\tilde{x}^2 = u$ and $\tilde{y}^2 = 1$, which means that $\tilde{x}$ and $\tilde{y}$ generate an Azumaya algebra with generic fiber $(a, b) \otimes K(R)(\sqrt{b})$. A basis for A as a free R' module is given by $1, \tilde{x}, \tilde{y}, \tilde{x}\tilde{y}$; this also happens to be an eigenbasis for the action of $\boldsymbol{\mu}_2$. Let χ^1 denote the non-trivial character of $\boldsymbol{\mu}_2$ and χ^0 the trivial character. The eigensheaf decomposition of A corresponding to the basis can be written as $\chi^0 \oplus \chi^1 \oplus \chi^\varepsilon \oplus \chi^{1+\varepsilon}$, where the last sum is taken modulo 2. For either value of ε this is isomorphic to $\rho^{\oplus 2}$, as desired.

Gluing the local models (as in Proposition A.2.3(1)) produces a regular $\mathscr{C}$-twisted sheaf V of rank 2. Since α is non-trivial, we see that this sheaf must be stable, hence geometrically semistable (in fact, geometrically polystable [**14**]). If $[D \otimes \mathbf{Q} : \mathbf{Q}]$ is odd, then V is semistable with coprime rank and degree, hence geometrically stable. □

Proposition 5.2.2 has an amusing consequence. The starting point is the following (somewhat useful) lemma. Fix a place ν of $\mathbf{Q}$.

LEMMA 5.2.3. *If there is an object of* $\mathscr{M}^*_{\mathscr{C}/\mathbf{Q}}(2, L)^{ss}_{\mathbf{Q}_\nu}$ *then there is an object of* $\mathscr{M}^*_{\mathscr{C}/\mathbf{Q}}(2, L)_{\mathbf{Q}_\nu}$.

PROOF. Let V be a regular semistable $\mathscr{C} \otimes \mathbf{Q}_\nu$-twisted sheaf of rank 2 and determinant L, and let Y be an algebraization of a versal deformation space of V. Since deformations of vector bundles on curves are unobstructed, we know that Y is smooth over $\mathbf{Q}_\nu$. On the other hand, we also know that the field $\mathbf{Q}_\nu$

has the property that any smooth variety with a rational point has a Zariski-dense set of rational points. Finally, we know that the locus of geometrically stable $\mathscr{C}\otimes\mathbf{Q}_\nu$-twisted sheaves is open and dense in Y. The result follows. □

In [**15**], the authors produce examples of biquaternion algebras A over $\mathbf{Q}(t)$ of Faddeev index 4 such that for all places ν of $\mathbf{Q}$ the algebra $A\otimes\mathbf{Q}_\nu(t)$ has index 2. They describe this as the failure of a sort of "Hasse principle". However, a careful examination of their examples shows that in fact there is a local obstruction: for each class of such algebras they write down, there is always a place ν over which there is no regular $\mathscr{C}\otimes\mathbf{Q}_\nu$-twisted sheaf! (Moreover, the proofs that their examples work use this local failure in an essential way, although the authors do not phrase things this way.) Why doesn't this contradict Proposition 5.2.2? Because for the place ν where things fail, one of the original ramification sections ends up in the locus where the algebra $A\otimes\mathbf{Q}_\nu$ is unramified, and now the condition that the sheaf be regular around the stacky point over that section is non-trivial!

Let us give an explicit example. In Proposition 4.3 of [**15**], the authors show that the biquaternion algebra $A=(17,t)\otimes(13,(t-1)(t-11))$ has Faddeev index 4 while for all places ν of $\mathbf{Q}$ the algebra $A\otimes\mathbf{Q}_\nu$ has index 2. Consider $A_{17}=A\otimes\mathbf{Q}_{17}$; since 13 is a square in $\mathbf{Q}_{17}$, we have that $A_{17}=(17,t)$ as $\mathbf{Q}_{17}$-algebras. Elementary calculations show that $(17,1)=0\in\operatorname{Br}(\mathbf{Q}_{17})$ and $(17,11)\neq 0\in\operatorname{Br}(\mathbf{Q}_{17})$. It follows that any $\mathbf{Q}_{17}(t)$-algebra Faddeev-equivalent to A_{17} has the property that precisely one of its specializations at 1 and 11 will be nontrivial.

LEMMA 5.2.4. *Given a field F and a nontrivial Brauer class $\gamma\in\operatorname{Br}(F)[2]$, let $\mathscr{G}\to\mathsf{B}\boldsymbol{\mu}_2\times\operatorname{Spec}F$ be a $\boldsymbol{\mu}_2$-gerbe representing the pullback of γ in $\operatorname{Br}(\mathsf{B}\boldsymbol{\mu}_2\times\operatorname{Spec}F)$. There is no regular $\mathscr{G}$-twisted sheaf of rank 2.*

PROOF. The inertia stack of $\mathscr{G}$ is isomorphic to $\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2$, where the first factor comes from the gerbe structure and the second factor comes from the inertia of $\mathsf{B}\boldsymbol{\mu}_2$. Given a $\mathscr{G}$-twisted sheaf F, the eigendecomposition of F with respect to the action of the second factor gives rise to $\mathscr{G}$-twisted subsheaves of F. Since $\mathscr{G}$ has period 2, if $\operatorname{rk}F=2$ there can be no proper twisted subsheaves. □

COROLLARY 5.2.5. *In the example above, all of the sets $M^*_{\mathscr{C}/\mathbf{Q}}(2,L)(\mathbf{Q}_{17})$ are empty.*

PROOF. Suppose $Q\in M^*_{\mathscr{C}/\mathbf{Q}}(2,L)(\mathbf{Q}_{17})$. As above, after replacing A_{17} by a Faddeev-equivalent algebra, we can lift Q to an object of $\mathscr{M}^*_{\mathscr{C}/\mathbf{Q}}(2,L)_{\mathbf{Q}_{17}}$. In other words, there would be a regular (stable!) $\mathscr{C}\otimes\mathbf{Q}_{17}$-twisted sheaf of rank 2. But this contradicts Lemma 5.2.4 and the remarks immediately preceding it. □

Combining Proposition 5.2.2 and Corollary 5.2.5, we see that the failure of the "Hasse principle" to which the authors of [**15**] refer in relation to the algebra A is in fact a failure of the existence of a *local* point in the associated moduli problem! This is in great contrast to the unramified case, which we saw above was directly related to the Hasse principle. It is somewhat disappointing that the examples we actually have of classes over arithmetic surfaces whose period and index are distinct cannot be directly related to the Hasse principle (except insofar as both sets under consideration are empty!).

There is one mildly interesting question which arises out of this failure.

PROPOSITION 5.2.6. *If Conjecture 2.3 is true then any element* $\alpha \in \mathrm{Br}(\mathbf{Q}(t))[2]$ *such that*

(1) *the ramification of* α *is a simple normal crossings divisor* $D = D_1 + \cdots + D_r$ *in* $\mathbf{P}^1_{\mathbf{Z}}$ *which is a union of fibers and an odd number of sections of* $\mathbf{P}^1_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$*,*
(2) *for every crossing point* $p \in D_i \cap D_j$*, both ramification extensions are either split, non-split, or ramified at* p*, and*
(3) *all points* d *of* $D \times_{\mathbf{P}^1_{\mathbf{Z}}} \mathbf{P}^1_{\mathbf{R}}$ *which are not ramification divisors of* $\alpha_{\mathbf{R}(t)}$ *give rise to the same element* $(\alpha_{\mathbf{R}(t)})_d \in \mathrm{Br}(\mathbf{R})$

satisfies $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.

The second condition of the proposition is almost equivalent to the statement that the restriction of α to $\mathbf{Q}_\nu(t)$ has no hot points (in the sense of Saltman) on $\mathbf{P}^1_{\mathbf{Z}_\nu}$; it is not quite equivalent because Saltman's hot points are all required to lie on intersections of ramification divisors, while some of the ramification divisors of α may no longer be in the ramification divisor of $\alpha_{\mathbf{Q}_\nu(t)}$. The proof of Proposition 5.2.6 uses Proposition A.2.3 (2) as a starting point for a deformation problem. It is similar in spirit to the proof of Theorem 3.4 and will be omitted.

6. A list of questions

We record several questions arising from the preceding discussion. Let C be a curve over a field k.

(1) Are there biquaternion algebras in $\mathrm{Br}_\infty(\mathbf{Q}(t))$ of Faddeev index 4 with non-hot secondary ramification (in the sense of Proposition 5.2.6)? For example, let p be a prime congruent to 1 modulo $3 \cdot 4 \cdot 7 \cdot 13 \cdot 17$ and congruent to 2 modulo 5. What is the Faddeev index of the algebra $(p, t) \otimes (13, 15(t-1)(t+13))$?
(2) What is the Brauer-Manin obstruction for $M^*_{\mathscr{P}/k}(2, L)$? If algebras as in the first question exist, is the resulting failure of the Hasse principle explained by the Brauer-Manin obstruction?
(3) Let $\mathscr{C} \to C$ be a $\boldsymbol{\mu}_n$-gerbe. What is the index of the Brauer class $\mathscr{M}_{\mathscr{C}/k}(n, \mathscr{O}) \to M_{\mathscr{C}/k}(n, \mathscr{O})$? If C has a rational point, this index must divide n. More generally, it must divide $n\,\mathrm{ind}(C)$. Is this sharp?
(4) Does every fiber of $M_{C/k}(n, L) \to \mathrm{Pic}^d_{C/k}$ over a rational point contain a rational point? This is (indirectly) related to the index-reduction problem. It is not too hard to see that if the rational point of $\mathrm{Pic}^d_{C/k}$ comes from an invertible sheaf then there is always a rational point in the fiber.
(5) Suppose k is a global field. Let $\mathcal{C} \to S$ be a regular proper model of C. Is there a class $\alpha \in \mathrm{Br}_\infty(\mathcal{C})$ such that $\mathrm{per}(\alpha) \neq \mathrm{ind}(\alpha)$? When k has positive characteristic and the period of α is invertible in k, this is impossible [**20**]. The existence of such a class would disprove Conjecture 2.3 (even the strong form).

Appendix A. Gerbes, twisted sheaves, and their moduli

In this appendix, we remind the reader of the basic facts about twisted sheaves, their moduli, and their applications to the Brauer group. For more comprehensive references, the reader can consult [**18, 20**]. The basic setup will be the following: let $\mathscr{X} \to X \to S$ be a $\boldsymbol{\mu}_n$-gerbe on a proper flat morphism of finite presentation. We assume n is invertible on S. At various points in this appendix, we will impose conditions on the morphism.

We first recall what it means for $\mathscr{X} \to X$ to be a $\boldsymbol{\mu}_n$-gerbe.

Definition A.1. A $\boldsymbol{\mu}_n$-gerbe is an S-stack $\mathscr{Y}$ along with an isomorphism $\boldsymbol{\mu}_{n,\mathscr{Y}} \to \mathscr{I}(\mathscr{Y})$. We say that $\mathscr{Y}$ is a $\boldsymbol{\mu}_n$-gerbe on Y if there is a morphism $\mathscr{Y} \to Y$ such that the natural map $\mathrm{Sh}(\mathscr{Y}) \to Y$ is an isomorphism, where $\mathrm{Sh}(\mathscr{Y})$ denotes the sheafification of $\mathscr{Y}$ on the big étale site of S.

Because $\mathscr{X}$ is a $\boldsymbol{\mu}_n$-gerbe, any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ admits a decomposition $\mathscr{F} = \mathscr{F}_0 \oplus \cdots \oplus \mathscr{F}_{n-1}$ into eigensheaves, where the natural (left) action of the stabilizer on $\mathscr{F}_i$ is via the ith power map.

Definition A.2. An $\mathscr{X}$-twisted sheaf is a sheaf $\mathscr{F}$ of $\mathscr{O}_{\mathscr{X}}$-modules such that the natural left action (induced by inertia) $\boldsymbol{\mu}_n \times \mathscr{F} \to \mathscr{F}$ is equal to scalar multiplication.

A.1. Moduli. It is a standard fact (see, for example, [**17**]) that the stack of flat families of quasi-coherent $\mathscr{X}$-twisted sheaves of finite presentation is an algebraic stack locally of finite presentation over S. For the purposes of this paper, we will focus on only a single case, where we will consider stability of twisted sheaves. *From now on, we assume that $S = \operatorname{Spec} k$ with k a field, and X is a proper smooth curve over k.*

To define stability, we need a notion of degree for invertible $\mathscr{X}$-twisted sheaves. For the sake of simplicity, we give an ad hoc definition. Given an invertible $\mathscr{X}$-twisted sheaf L, we note that $L^{\otimes n}$ is the pullback of a unique invertible sheaf L' on X. We can thus define $\deg L$ to be $\frac{1}{n}\deg L'$. With this definition, we can define the degree of a locally free $\mathscr{X}$-twisted sheaf V as $\deg V = \deg \det V$. With this notion of degree, we can define stability.

Definition A.1.1. The *slope* of a non-zero locally free $\mathscr{X}$-twisted sheaf V is

$$\mu(V) = \frac{\deg V}{\operatorname{rk} V}.$$

Definition A.1.2. A locally free $\mathscr{X}$-twisted sheaf V is *stable* if for all proper locally split subsheaves $W \subset V$ we have

$$\mu(W) < \mu(V).$$

The sheaf is *semistable* of for all proper locally split subsheaves $W \subset V$ we have

$$\mu(W) \leq \mu(V).$$

By "locally split" we mean that there is a faithfully flat map $Z \to \mathscr{X}$ such that there is a retraction of the inclusion $W_Z \subset V_Z$. (I.e., W is locally a direct summand of V.) As in the classical case, the stack of stable sheaves is a $\mathbf{G}_m$-gerbe over an algebraic space. If in addition we fix a determinant, then the resulting stack is a $\boldsymbol{\mu}_r$-gerbe over an algebraic space, where r is the rank of the sheaves in question.

NOTATION A.1.3. Given an invertible sheaf L on X, the stack of stable (resp. semistable) $\mathscr{X}$-twisted sheaves of rank n and determinant L will be denoted by $\mathscr{M}_{\mathscr{X}/k}(n, L)$ (resp. $\mathscr{M}_{\mathscr{X}/k}(n, L)^{ss}$). The coarse moduli space (which in the stable case is also the sheafification) will be denoted by $M_{\mathscr{X}/k}(n, L)$ (resp. $M_{\mathscr{X}/k}(n, L)^{ss}$).

Given an integer d, the stack of stable $\mathscr{X}$-twisted sheaves of rank n and degree d will be denoted by $\mathscr{M}_{\mathscr{X}/k}(n, d)$, and its coarse moduli space will be denoted by $M_{\mathscr{X}/k}(n, d)$.

As mentioned above, $\mathscr{M}_{\mathscr{X}/k}(n, d) \to M_{\mathscr{X}/k}(n, d)$ is a $\mathbf{G}_m$-gerbe; similarly, $\mathscr{M}_{\mathscr{X}/k}(n, L) \to M_{\mathscr{X}/k}(n, L)$ is a $\boldsymbol{\mu}_n$-gerbe. In fact, the stack $\mathscr{M}_{\mathscr{X}/k}(n, L)$ is the stack theoretic fiber of the determinant morphism $\mathscr{M}_{\mathscr{X}/k}(n, d) \to \mathscr{P}ic^d_{X/k}$ over the morphism $\operatorname{Spec} k \to \mathscr{P}ic^d_{X/k}$ corresponding to L. The comparison results described in Section 1, combined with classical results on the stack of stable sheaves on a curve, show that $\mathscr{M}_{\mathscr{X}/k}(n, d)$ is of finite type over k; if we assume that the class $\mathscr{X}$ is zero in $\mathrm{H}^2(X \otimes \overline{k}, \boldsymbol{\mu}_n)$ we know that $\mathscr{M}_{\mathscr{X}/k}(n, d)$ is quasi-proper (i.e., satisfies the existence part of the valuative criterion of properness) whenever n and d are relatively prime.

A.2. Deformation theory. As $\mathscr{O}$-modules on a ringed topos (the étale topos of $\mathscr{X}$), $\mathscr{X}$-twisted sheaves are susceptible to the usual deformation theory of Illusie. The following theorem summarizes the consequences of this fact. Suppose $S = \operatorname{Spec} A$ is affine, $I \to \widetilde{A} \to A$ is a small extension (so that the kernel I is an A-module), $X/\widetilde{A}$ is flat, $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe with n invertible in A, and $\mathscr{F}$ is an A-flat family of quasi-coherent $\mathscr{X} \otimes_{\widetilde{A}} A$-twisted sheaves of finite presentation.

THEOREM A.2.1. *There is an element* $\mathfrak{o} \subset \operatorname{Ext}^2(\mathscr{F}, \mathscr{F}_A \otimes I)$ *such that*

1. $\mathfrak{o} = 0$ *if and only if there is an* $\widetilde{A}$*-flat quasi-coherent* $\mathscr{X}$*-twisted sheaf* $\widetilde{\mathscr{F}}$ *and an isomorphism* $\widetilde{\mathscr{F}} \otimes_{\widetilde{A}} A \xrightarrow{\sim} \mathscr{F}$*;*
2. *if such an extension exists, the set of such extensions is a torsor under* $\operatorname{Ext}^1(\mathscr{F}, \mathscr{F} \otimes I)$*;*
3. *given one such extension, the group of automorphisms of* $\widetilde{F}$ *which reduce to the identity on* $\mathscr{F}$ *is identified with* $\operatorname{Hom}(\mathscr{F}, \mathscr{F} \otimes I)$.

COROLLARY A.2.2. *If* X/S *is a relative curve, then the stack of locally free* $\mathscr{X}$*-twisted sheaves is smooth over* S.

Deformation theory can also be used to construct global objects from local data. The key technical tool is the following; see [**19**] for a more detailed proof and further references.

PROPOSITION A.2.3. *Let* $\mathscr{P}$ *be a separated tame Artin stack of finite type over* k *which is pure of dimension* 1*. Let* P *be the coarse moduli space of* $\mathscr{P}$ *and let* $\mathscr{C} \to \mathscr{P}$ *be a* $\boldsymbol{\mu}_n$*-gerbe. Suppose* $\mathscr{P}$ *is regular away from the (finitely many) closed residual gerbes* $\xi_1, \ldots, \xi_r$*. Suppose the map* $\operatorname{Pic}(\mathscr{P}) \to \prod \operatorname{Pic}(\xi_i)$ *has kernel generated by the image of* $\operatorname{Pic}(P)$ *under pullback.*

(1) *Given locally free $\mathscr{C}_{\xi_i}$-twisted sheaves V_i of rank m, $i = 1, \dots, r$, and locally free $\mathscr{C}_\eta$-twisted sheaf V_η of rank m, there is a locally free $\mathscr{C}$-twisted sheaf W of rank m such that $W_\eta \cong V_\eta$ and $W_{\xi_i} \cong V_i$.*

(2) *Suppose that k is finite, $P \cong \mathbf{P}^1$, $\mathscr{P} \to P$ is generically a $\boldsymbol{\mu}_n$-gerbe, and the ramification extension $R \to P$ of $\mathscr{C}$ (see Proposition C.4 below) is geometrically connected. Given an invertible sheaf $L \in \operatorname{Pic}(\mathscr{P})$, if there are $\mathscr{C}_{\xi_i}$-twisted sheaves V_i of rank m such that $\det V_i \cong L_{\xi_i}$ and a $\mathscr{C}_\eta$-twisted sheaf V_η of rank m, then there is a locally free $\mathscr{C}$-twisted sheaf V of rank m such that $\det V \cong L$.*

PROOF. Let P be the coarse moduli space of $\mathscr{P}$, and let $p_i \in P$ be the image of ξ_i. Let $\widehat{\eta}_i$ denote the localization of $\operatorname{Spec} \widehat{\mathscr{O}}_{p_i}$ at the set of maximal (i.e., generic) points. Let $\mathscr{O}(1)$ be an ample invertible sheaf on P.

To prove the first item, note that the stack $\mathscr{C}_{\xi_i}$ is tame, as its inertia group is an extension of a reductive group by a $\boldsymbol{\mu}_n$. Thus, the infinitesimal deformations of each V_i are unobstructed. Since $\mathscr{C}$ is proper over P, the Grothendieck existence theorem implies that for each i there is a $\mathscr{C} \times \operatorname{Spec} \widehat{\mathscr{O}}_{P,p_i}$-twisted sheaf $\widetilde{V}_i$ of rank m whose restriction to $\mathscr{C}_{\xi_i}$ is V_i. On the other hand, since the scheme of generic points of P is 0-dimensional, we know that for each i there is an isomorphism $(\widetilde{V}_i)_{\widehat{\eta}_i} \cong (V_\eta)_{\widehat{\eta}_i}$. (See [**19**] for a similar situation, with more details.) The basic descent result of [**22**] shows that we can glue the $\widetilde{V}_i$ to V_η to produce W, as desired.

To prove the second item, choose any W as in the first part, and let $L' = \det W$. This is an invertible sheaf which is isomorphic to L in a neighborhood of each ξ_i, and therefore (by hypothesis) there is an invertible sheaf M on P such that $L \otimes (L')^{-1} \cong M_{\mathscr{P}}$. Twisting W by a suitable (negative!) power of $\mathscr{O}(1)$, we may assume that M is ample of arbitrarily large degree. By the Lang-Weil estimates and hypothesis that $P \cong \mathbf{P}^1$, there is a point $q \in R$ whose image p in P is an element of $|M|$. Making an elementary transformation of W along p (over which the Brauer class associated to $\mathscr{C}$ is trivial) produces a locally free $\mathscr{C}$-twisted sheaf with the desired properties. □

Appendix B. Basic facts on the Brauer group

In this appendix, we review a few basic facts about the Brauer group of a scheme. We freely use the technology of twisted sheaves, as introduced in the previous appendix. Let Z be a quasi-compact separated scheme and $\mathscr{Z} \to Z$ a $\mathbf{G}_m$-gerbe. We start with a (somewhat idiosyncratic) definition of the Brauer group of Z.

DEFINITION B.1. The cohomology class $[\mathscr{Z}] \in \mathrm{H}^2(Z, \mathbf{G}_m)$ is said to *belong to the Brauer group of Z* if there is a non-zero locally free $\mathscr{Z}$-twisted sheaf of finite rank.

LEMMA B.2. *The Brauer group of Z is a group.*

PROOF. Given two $\mathbf{G}_m$-gerbes $\mathscr{Z}_1 \to Z$ and $\mathscr{Z}_2 \to Z$ which belong to the Brauer group of Z, let V_i be a locally free $\mathscr{Z}_i$-twisted sheaf. Then $V_1 \otimes V_2$ is a locally free $\mathscr{Z}_1 \wedge \mathscr{Z}_2$-twisted sheaf, where $\mathscr{Z}_1 \wedge \mathscr{Z}_2$ is the $\mathbf{G}_m$-gerbe considered in [**10, 14**], which represents the cohomology class $[\mathscr{Z}_1] + [\mathscr{Z}_2]$. The neutral element in the group is represented by the trivial gerbe $\mathsf{B}\mathbf{G}_m \times Z \to Z$. □

We thus find a distinguished subgroup $\mathrm{Br}(Z) \subset \mathrm{H}^2(Z, \mathbf{G}_m)$ containing those classes which belong to the Brauer group of Z. What are the properties of this group?

PROPOSITION B.3. *The group* $\mathrm{Br}(Z)$ *has the following properties.*

(1) *An element* $[\mathscr{X}]$ *is trivial in* $\mathrm{Br}(Z)$ *if and only if there is an invertible* $\mathscr{X}$*-twisted sheaf.*
(2) $\mathrm{Br}(Z)$ *is a torsion abelian group.*
(3) *(Gabber) If* Z *admits an ample invertible sheaf, then the inclusion* $\mathrm{Br}(Z) \subset \mathrm{H}^2(Z, \mathbf{G}_m)_{tors}$ *is an isomorphism.*

SKETCH OF PROOF. If $\mathscr{L}$ is an invertible $\mathscr{X}$-twisted sheaf, then we can define a morphism of $\mathbf{G}_m$-gerbes $\mathrm{B}\mathbf{G}_m \to \mathscr{X}$ by sending an invertible sheaf L to the invertible $\mathscr{X}$-twisted sheaf $L \otimes \mathscr{L}$. Since any $\mathbf{G}_m$-gerbe admitting a morphism of $\mathbf{G}_m$-gerbes from $\mathrm{B}\mathbf{G}_m$ is trivial, we see that $\mathscr{X}$ is trivial [**10**].

Now suppose that $\mathscr{X}$ is an arbitrary element of $\mathrm{Br}(Z)$, and let V be a locally free $\mathscr{X}$-twisted sheaf of rank r. Writing $\mathscr{X}_r$ for the gerbe corresponding to $r[\mathscr{X}]$, one can show that the sheaf $\det V$ is an invertible $\mathscr{X}_r$-twisted sheaf, thus showing that $r[\mathscr{X}] = 0$, as desired.

The last part of the proposition is due to Gabber; a different proof has been written down by de Jong [**8**]. □

Given a $\mathbf{G}_m$-gerbe $\mathscr{X} \to Z$ and a locally free $\mathscr{X}$-twisted sheaf V, the $\mathscr{O}_{\mathscr{X}}$-algebra $\mathscr{E}nd(V)$ is acted upon trivially by the inertia stack of $\mathscr{X}$, and thus is the pullback of a unique sheaf of algebras $\mathscr{A}$ on Z. The algebras $\mathscr{A}$ which arise in this manner are precisely the Azumaya algebras: étale forms of $\mathrm{M}_r(\mathscr{O}_Z)$ (for positive integers r). Moreover, starting with an Azumaya algebra, we can produce a $\mathbf{G}_m$-gerbe by solving the moduli problem of trivializing the algebra (i.e., making it isomorphic to a matrix algebra). Further details about this correspondence may be found in [**10**, §V.4].

When $Z = \operatorname{Spec} K$ for some field K, an Azumaya algebra is precisely a central simple algebra over K, and thus we recover the classical Brauer group of the field. Note that in this case if $\mathscr{X} \to Z$ is a $\mathbf{G}_m$-gerbe, any nonzero coherent $\mathscr{X}$-twisted sheaf is locally free. Moreover, we know that $\mathscr{X} \otimes \overline{K}$ is isomorphic to $\mathrm{B}\mathbf{G}_m \otimes \overline{K}$, and therefore that there is an invertible $\mathscr{X} \otimes \overline{K}$-twisted sheaf. Pushing forward to $\mathscr{X}$, we see that there is a nontrivial quasi-coherent $\mathscr{X}$-twisted sheaf $\mathscr{Q}$. Since $\mathscr{X}$ is Noetherian, $\mathscr{Q}$ is a colimit of coherent $\mathscr{X}$-twisted subsheaves. This shows that $\mathscr{X}$ belongs to the Brauer group of Z. We have just given a geometric proof of the classical Galois cohomological result $\mathrm{Br}(K) = \mathrm{H}^2(\operatorname{Spec} K, \mathbf{G}_m)$.

Now assume that Z is integral and Noetherian with generic point η, and let $\mathscr{X} \to Z$ in arbitrary $\mathbf{G}_m$-gerbe. By the preceding, there is a coherent $\mathscr{X}$-twisted sheaf of positive rank.

DEFINITION B.4. The *index* of $[\mathscr{X}]$, written $\operatorname{ind}([\mathscr{X}])$, is the minimal nonzero rank of a coherent $\mathscr{X}$-twisted sheaf, and the *period* of $[\mathscr{X}]$, written $\operatorname{per}([\mathscr{X}])$, is the order of $[\mathscr{X}]_\eta$ in $\mathrm{H}^2(\eta, \mathbf{G}_m)$.

We have written Definition B.4 so that it only pertains to generic properties of $\mathscr{X}$ and Z. One can imagine more general definitions (e.g., using locally free $\mathscr{X}$-twisted sheaves; when Z is regular of dimension 1 or 2, the basic properties of reflexive sheaves tell us that the natural global definition will actually equal the generic definition. When $Z = \operatorname{Spec} K$, it is easy to say what the index of a Brauer class α is: α parameterizes a unique central division algebra over K, whose dimension over K is n^2 for some positive integer n. The index of α is then n.

The basic fact governing the period and index is the following.

PROPOSITION B.5. *For any* $\alpha \in \operatorname{Br}(K)$*, there is a positive integer* h *such that* $\operatorname{per}(\alpha) \mid \operatorname{ind}(\alpha)$ *and* $\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^h$.

The proof of Proposition B.5 is an exercise in Galois cohomology; the reader is referred to (for example) [**20**, Lemma 2.1.1.3]. One immediate consequence of Proposition B.5 is the question: how can we understand h? For example, does h depend only on K (in the sense that there is a value of h which works for all $\alpha \in \operatorname{Br}(K)$)? If so, what properties of K are being measured by h? And so on. Much work has gone into this problem; for a summary of our current expectation the reader is referred to [**20**].

Appendix C. Ramification of Brauer classes

In this section, we recall the basic facts about the ramification theory of Brauer classes. We also describe how to split ramification by a stack. To begin with, we consider the case $X = \operatorname{Spec} R$ with R a complete discrete valuation ring with valuation v. Fix a uniformizer t of R; let K and κ denote the fraction field and residue field of R, respectively. Write $j : \operatorname{Spec} K \to \operatorname{Spec} R$ for the natural inclusion. Throughout, we only consider Brauer classes $\alpha \in \operatorname{Br}(K)$ of period relatively prime to $\operatorname{char}(\kappa)$; we write $\operatorname{Br}(K)'$ for the subgroup of classes satisfying this condition.

The theory of divisors yields an exact sequence of étale sheaves on $\operatorname{Spec} R$,

$$0 \to \mathbf{G}_m \to j_*\mathbf{G}_m \to \mathbf{Z}_{(t)} \to 0,$$

which yields a map

$$\mathrm{H}^2(\operatorname{Spec} R, j_*\mathbf{G}_m) \to \mathrm{H}^2(\operatorname{Spec} R, \mathbf{Z}_{(t)}) = \mathrm{H}^2(\kappa, \mathbf{Z}) = \mathrm{H}^1(\kappa, \mathbf{Q}/\mathbf{Z}).$$

Since any $a \in \operatorname{Br}(K)'$ has an unramified splitting field, the Leray spectral sequence for $\mathbf{G}_m$ on j shows that $\mathrm{H}^2(\operatorname{Spec} R, j_*\mathbf{G}_m)' = \operatorname{Br}(K)'$ (where the $'$ denotes classes with orders invertible in R). Putting this together yields the *ramification sequence*

$$0 \to \operatorname{Br}(R)' \to \operatorname{Br}(K)' \to \mathrm{H}^1(\kappa, \mathbf{Q}/\mathbf{Z}).$$

The last group in the sequence parameterizes cyclic extensions of the residue field κ. Suppose for the sake of simplicity that κ contains a primitive nth root of unity. The ramification of a cyclic algebra (a, b) is given by the extension of κ generated by the nth root of $(-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)}$. In particular, given any element $\overline{u} \in \kappa^*$, the algebra (u, t) has ramification extension $\kappa(\overline{u}^{1/n})$, where u is any lift of $\overline{u}$ in R^*. With our assumption about roots of unity, any cyclic extension of degree n is given by extracting roots of some $\overline{u}$. This has the following two useful consequences. Given a positive integer n, let $\mathscr{R}_n \to \operatorname{Spec} R$ denote the stack of nth roots of the closed point of $\operatorname{Spec} R$, as in [**3**].

PROPOSITION C.1. *Assume that* κ *contains a primitive* n*th root of unity* ζ. *Fix an element* $\alpha \in \operatorname{Br}(K)[n]$.

(1) *There exists* $u \in R^*$ *and* $\alpha' \in \mathrm{Br}(R)$ *such that* $\alpha = \alpha' + (u,t) \in \mathrm{Br}(K)$.
(2) *There exists* $\beta \in \mathrm{Br}(\mathscr{R}_n)$ *whose image in* $\mathrm{Br}(K)$ *under the restriction map is* α.

PROOF. The first item follows immediately from the paragraph preceding this proposition: we can find u such that (u,t) has the same ramification as α, and subtracting this class yields an element of $\mathrm{Br}(R)$. To prove the second item, it follows in the first that it suffices to prove it for the class (u,t). Recall that $\mathscr{R}_n$ is the stacky quotient of $\operatorname{Spec} R[t^{1/n}]$ by the natural action of $\boldsymbol{\mu}_n$; to extend (u,t) to $\mathscr{R}_n$, it suffices to find a $\boldsymbol{\mu}_n$-equivariant Azumaya algebra in $(u,t)_{K(t^{1/n})}$. Recall that (u,t) is generated by x and y such that $x^n = u$, $y^n = t$, and $xy = \zeta yx$. Letting $\tilde{y} = y/t^{1/n}$, the natural action of $\boldsymbol{\mu}_n$ on $t^{1/n}$ yields an equivariant Azumaya algebra in $(u,t)_{K(t^{1/u})}$, as desired. □

The following corollary is an example of how one applies Proposition C.1 in a global setting. More general results are true, using various purity theorems.

COROLLARY C.2. *Let* C *be a proper regular curve over a field* k *which contains a primitive* n*th root of unity for some* n *invertible in* k. *Suppose* $\alpha \in \mathrm{Br}(k(C))[n]$ *is ramified at* $p_1, \ldots, p_r$, *and let* $\mathscr{C} \to C$ *be a stack of* n*th roots of* $p_1 + \cdots + p_r$. *There is an element* $\alpha' \in \mathrm{Br}(\mathscr{C})[n]$ *whose restriction to the generic point is* α.

PROOF. Let A be a central simple algebra over $k(C)$ with Brauer class α. For any point $q \in C$, we know (at the very least) that there is a positive integer m and an Azumaya algebra over the localization $\mathscr{C}_q$ which is contained in $M_m(A)$. Suppose $U \subset \mathscr{C}$ is an open substack over which there is an Azumaya algebra B with generic Brauer class α. Given a closed point $q \in C \setminus U$, we can choose some m such that there is an Azumaya algebra B' over $\widehat{\mathscr{C}_q}$ whose restriction to the generic point $\widehat{\eta}$ of $\widehat{C}_q$ is isomorphic to $M_m(B)_{\widehat{\eta}}$. Since we know that $U \times_C \widehat{\mathscr{C}_q} = \widehat{\eta}$, we can glue B' to $M_m(B)$ as in the proof of Proposition A.2.3 (using [**22**]) to produce an Azumaya algebra over $U \cup \{q\}$. Since $C \setminus U$ is finite, we can find some m such that $M_m(A)$ extends to an Azumaya algebra over $\mathscr{C}$, as desired. □

The following is a more complicated corollary of Proposition C.1, using purity of the Brauer group on a surface. We omit the proof.

COROLLARY C.3. *Let* X *be a connected regular Noetherian scheme pure of dimension* 2 *with function field* K. *Suppose* $U \subset X$ *is the complement of a simple normal crossings divisor* $D = D_1 + \cdots + D_r$ *and* $\alpha \in \mathrm{Br}(U)[n]$ *is a Brauer class such that* n *is invertible in* $\kappa(D_i)$ *for* $i = 1, \ldots, r$. *If* $\mathscr{X} \to X$ *is the root construction of order* n *over each* D_i *then there is a class* $\widetilde{\alpha} \in \mathrm{Br}(\mathscr{X})$ *such that* $\widetilde{\alpha}_U = \alpha$.

We end this discussion with an intrinsic characterization of the ramification extension as a moduli space.

PROPOSITION C.4. *Let* X *be a scheme on which* n *is invertible such that* $\Gamma(X, \mathscr{O})$ *contains a primitive* n*th root of unity. Suppose* $\pi : \mathscr{X} \to X$ *is a* $\boldsymbol{\mu}_n$-*gerbe. Given a further* $\boldsymbol{\mu}_n$-*gerbe* $\mathscr{Y} \to \mathscr{X}$, *the relative Picard stack* ${}_{\tau}\mathscr{P}ic_{\mathscr{Y}/X}$ *of invertible* $\mathscr{Y}$-*twisted sheaves is a* $\mathbf{G}_m$-*gerbe over a* $\mathbf{Z}/n\mathbf{Z}$-*torsor* T. *Moreover, the*

Brauer class of $\mathscr{Y}$ in $\mathrm{H}^2(\mathscr{X}, \mathbf{G}_m)$ is the pullback of a class from X if and only if T is trivial.

PROOF. Standard methods show that ${}_{\tau}\mathscr{P}ic_{\mathscr{Y}/X}$ is a $\mathbf{G}_m$-gerbe over a flat algebraic space of finite presentation $P \to X$. Since the relative Picard space $\mathrm{Pic}_{\mathscr{X}/X}$ is isomorphic to the constant group scheme $\mathbf{Z}/n\mathbf{Z}$, tensoring with invertible sheaves on $\mathscr{X}$ gives an action

$$\mathbf{Z}/n\mathbf{Z} \times P \to P.$$

To check that this makes B a torsor, it suffices (by the obvious functoriality of the construction) to treat the case in which X is the spectrum of an algebraically closed field k. In this case, $\mathrm{Br}(\mathscr{X}) = 0$; choosing an invertible $\mathscr{Y}$-twisted sheaf L, we see that all invertible $\mathscr{Y}$-twisted sheaves M of the form $L \otimes \Lambda$, where Λ is invertible sheaf on $\mathscr{X}$. This gives the desired result. □

Starting with a complete discrete valuation ring R with fraction field K, an element $\alpha \in \mathrm{Br}(K)$ gives rise to two cyclic extensions of the residue field κ of R: the classical ramification extension and the moduli space produced in Proposition C.4. In fact, these two extensions are isomorphic. We will use this fact, but we omit the details.

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Existence of α-Stable Coherent Systems on Algebraic Curves

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Contents

This survey covers the theory of moduli spaces of α-stable coherent systems on a smooth projective curve C of genus $g \geq 2$. (The cases $g = 0$ and $g = 1$ are special cases, for which we refer to [**30, 31, 32, 33**].) I have concentrated on the issues of emptiness and non-emptiness of the moduli spaces. Other questions concerning irreducibility, smoothness, singularities, etc., and the relationship with

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This survey is a much extended version of a lecture given at the Clay workshop held in Cambridge, Massachusetts, Oct 6–10, 2006. The survey began life earlier than that as a set of notes on a series of three lectures given in Liverpool in August 2005 at a meeting of the research group VBAC (Vector Bundles on Algebraic Curves) which formed part of the activity of the Marie Curie Training Site LIMITS. My thanks are due to the Clay Mathematics Institute for supporting me as a Senior Scholar and to the European Commission for its support of LIMITS (Contract No. HPMT-CT-2001-00277). My thanks are due also to Cristian González-Martínez, who wrote the first draft of this survey shortly after the VBAC meeting.

Brill-Noether theory are mentioned where relevant but play a subsidiary rôle. In the bibliography, I have tried to include a full list of all relevant papers which contain results on coherent systems on algebraic curves. In general, I have not included papers concerned solely with Brill-Noether theory except where they are needed for reference, but I have included some which have clear relevance for coherent systems but have not been fully developed in this context. For another survey of coherent systems, concentrating on structural results and including an! appendix on the cases $g = 0$ and $g = 1$, see [**13**]; for a survey of higher rank Brill-Noether theory, see [**27**].

We work throughout over an algebraically closed field of characteristic 0. Many results are undoubtedly valid in finite characteristic, but we omit this possibility here since most of the basic papers on coherent systems assume characteristic 0. As stated above, we assume that C is a smooth projective curve of genus $g \geq 2$. We denote by K the canonical bundle on C.

1. Definitions

Coherent systems are simply the analogues for higher rank of classical linear systems.

Definition 1.1. A *coherent system* on C of type (n, d, k) is a pair (E, V) where E is a vector bundle of rank n and degree d over C and V is a linear subspace of dimension k of $H^0(E)$. A *coherent subsystem* of (E, V) is a pair (F, W), where F is a subbundle of E and $W \subseteq V \cap H^0(F)$. We say that (E, V) is *generated* (*generically generated*) if the canonical homomorphism $V \otimes \mathcal{O} \to E$ is surjective (has torsion cokernel).

Stability of coherent systems is defined with respect to a real parameter α. We define the α-slope $\mu_\alpha(E, V)$ of a coherent system (E, V) by

$$\mu_\alpha(E, V) := \frac{d}{n} + \alpha\frac{k}{n}.$$

Definition 1.2. (E, V) is *α-stable* (*α-semistable*) if, for every proper coherent subsystem (F, W) of (E, V),

$$\mu_\alpha(F, W) < (\leq)\ \mu_\alpha(E, V).$$

Every α-semistable coherent system (E, V) admits a Jordan-Hölder filtration

$$0 = (E_0, V_0) \subset (E_1, V_1) \subset \cdots \subset (E_m, V_m) = (E, V)$$

such that all (E_j, V_j) have the same α-slope and all the quotients $(E_j, V_j)/(E_{j-1}, V_{j-1})$ are defined and α-stable. The associated graded object

$$\bigoplus_1^m (E_j, V_j)/(E_{j-1}, V_{j-1})$$

is determined by (E, V) and denoted by $\mathrm{gr}(E, V)$. Two α-semistable coherent systems (E, V), (E', V') are said to be *S-equivalent* if

$$\mathrm{gr}(E, V) \cong \mathrm{gr}(E', V').$$

There exists a moduli space $G(\alpha; n, d, k)$ for α- stable coherent systems of type (n, d, k) [**29, 36, 43, 17**]; this is a quasi-projective scheme and has a natural completion to a projective scheme $\widetilde{G}(\alpha; n, d, k)$, whose points correspond to S-equivalence classes of α-semistable coherent systems of type (n, d, k). In particular, we have $G(\alpha; n, d, 0) = M(n, d)$ and $\widetilde{G}(\alpha; n, d, 0) = \widetilde{M}(n, d)$, where $M(n, d)$ ($\widetilde{M}(n, d)$) is the usual moduli space of stable bundles (S-equivalence classes of semistable bundles).

We define also the *Brill-Noether loci*

$$B(n, d, k) := \{E \in M(n, d) : h^0(E) \geq k\}$$

and

$$\widetilde{B}(n, d, k) := \{[E] \in \widetilde{M}(n, d) : h^0(\operatorname{gr} E) \geq k\},$$

where, for any semistable bundle E, $[E]$ denotes the S-equivalence class of E and $\operatorname{gr} E$ is the graded object associated to E.

For $k \geq 1$, we must have $\alpha > 0$ for the existence of α-stable coherent systems (for $\alpha \leq 0$, $(E, 0)$ contradicts the α-stability of (E, V)). Given this, the moduli space $G(\alpha; 1, d, k)$ is independent of α and we denote it by $G(1, d, k)$; it coincides with the classical variety of linear series G^{k-1}_d.

So suppose $n \geq 2$ and $k \geq 1$.

Lemma 1.3. *If* $G(\alpha; n, d, k) \neq \emptyset$ *then*

$$\alpha > 0,\ d > 0,\ \alpha(n-k) < d. \tag{1}$$

Proof. Let $(E, V) \in G(\alpha; n, d, k)$ with $n \geq 2$, $k \geq 1$. As already remarked, $(E, 0)$ contradicts α-stability for $\alpha \leq 0$. If (E, V) is generically generated, then certainly $d \geq 0$ with equality if and only if $(E, V) \cong (\mathcal{O}^n, H^0(\mathcal{O}^n))$, which is not α-stable for $n \geq 2$. Otherwise, the subsystem (F, V), where F is the subbundle of E generically generated by V, contradicts stability for $d \leq 0$. Finally, let W be a 1-dimensional subspace of V and let L be the line subbundle of E generically generated by W. Then (L, W) contradicts α-stability if $k < n$ and $\alpha(n-k) \geq d$. □

The range of permissible values of α is divided up by a finite set of critical values

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L < \begin{cases} \frac{d}{n-k} & \text{where } k < n \\ \infty & \text{where } k \geq n, \end{cases}$$

and α-stability of (E, V) cannot change within an interval (α_i, α_{i+1}) (see [**17**]). For $\alpha \in (\alpha_i, \alpha_{i+1})$, we write $G_i(n, d, k) := G(\alpha; n, d, k)$. For $k \geq n$ and $\alpha > \alpha_L$ we write $G_L(n, d, k) := G(\alpha; n, d, k)$, with a similar definition for $k < n$ and $\alpha_L < \alpha < \frac{d}{n-k}$. The semistable moduli spaces $\widetilde{G}_i(n, d, k)$ are defined similarly.

Lemma 1.4. *We have a morphism* $\widetilde{G}_0(n, d, k) \to \widetilde{B}(n, d, k)$*, whose image contains* $B(n, d, k)$*. In fact, for any coherent system* (E, V)*,*

- E *stable* $\Rightarrow (E, V) \in G_0(n, d, k)$
- $(E, V) \in G_0(n, d, k) \Rightarrow E$ *semistable.*

Proof. This follows easily from the definition of α-stability. □

Definition 1.5. The *Brill-Noether number* $\beta(n, d, k)$ is defined by

$$\beta(n, d, k) := n^2(g-1) + 1 - k(k - d + n(g-1)).$$

This is completely analogous to the classical case ($n = 1$) and is significant for two reasons. In the first place, we have

Lemma 1.6. ([**28**, Corollaire 3.14]; [**17**, Corollary 3.6]) *Every irreducible component of* $G(\alpha; n, d, k)$ *has dimension* $\geq \beta(n, d, k)$*.*

The second reason concerns the local structure of the moduli spaces.

Definition 1.7. Let (E,V) be a coherent system. The *Petri map* of (E,V) is the linear map

$$V \otimes H^0(E^* \otimes K) \longrightarrow H^0(\mathrm{End}E \otimes K)$$

given by multiplication of sections.

Lemma 1.8. ([**17**, Proposition 3.10]) *Let $(E,V) \in G(\alpha; n, d, k)$. The following conditions are equivalent:*

- *$G(\alpha; n, d, k)$ is smooth of dimension $\beta(n,d,k)$ at (E,V)*
- *the Petri map of (E,V) is injective.*

Definition 1.9. The curve C is called a *Petri curve* if the Petri map of $(L, H^0(L))$ is injective for every line bundle L.

It is a standard result of classical Brill-Noether theory that the general curve of any genus is a Petri curve [**25, 35**].

2. The Existence Problem

Among the possible questions one can ask about the existence of α-stable coherent systems are the following.

(i) Given n, d, k, α satisfying (1), is $G(\alpha; n, d, k) \neq \emptyset$?
(ii) Given n, d, k, does there exist (E,V) which is α-stable for all α satisfying (1)?
(iii) Given n, d, k, does there exist (E,V) with E stable which is α-stable for all α satisfying (1)?

We can restate (ii) and (iii) in a nice way by defining

$$U(n,d,k) := \{(E,V) \in G_L(n,d,k) \mid E \text{ is stable}\}$$

and

$$U^s(n,d,k) := \{(E,V) \mid (E,V) \text{ is } \alpha\text{-stable for } \alpha > 0, \alpha(n-k) < d\}.$$

Then (ii) and (iii) become

(ii)′ Given n, d, k with $d > 0$, is $U^s(n,d,k) \neq \emptyset$?
(iii)′ Given n, d, k with $d > 0$, is $U(n,d,k) \neq \emptyset$?

The "obvious" conjecture is that the answers are affirmative if and only if $\beta(n,d,k) \geq 0$. However, this is false in both directions; see, for example, Remarks 4.2 and 8.4. A basic result, for which a complete proof has only been given very recently, is

Theorem 2.1. *For any fixed n, k, $U(n,d,k) \neq \emptyset$ for all sufficiently large d.*

Proof. For $k \leq n$, see [**16, 18**]. For $k > n$, see [**3**] and [**49**]. □

Now write

$$d_0 = d_0(n,k) := \min\{d : G(\alpha; n, d, k) \neq \emptyset \text{ for some } \alpha \text{ satisfying (1)}\}.$$

The following would be an "ideal" result.

Model Theorem. *For fixed n, k with $n \geq 2$, $k \geq 1$,*

(a) *$G(\alpha; n, d, k) \neq \emptyset$ if and only if $\alpha > 0$, $(n-k)\alpha < d$ and $d \geq d_0$*
(b) *$B(n,d,k) \neq \emptyset$ if and only if $d \geq d_0$*

(c) *if* $d \geq d_0$, $U(n,d,k) \neq \emptyset$.

Note that (c) is stronger than (a) and (b) combined.

One may also ask whether, for given n, d, there is an upper bound on k for which $G(\alpha; n, d, k) \neq \emptyset$. In fact the existence of an upper bound on $h^0(E)$ for any $(E, V) \in G(\alpha; n, d, k)$ (possibly depending on α) is used in the proof of existence of the moduli spaces (see, for example [**29**]). An explicit such bound is given in [**17**, Lemmas 10.2 and 10.3]. The more restricted question of an upper bound on k has now been solved by the following generalization of Clifford's Theorem.

Theorem 2.2. ([**34**]) *Let* (E, V) *be a coherent system of type* (n, d, k) *with* $k > 0$ *which is* α*-semistable for some* $\alpha > 0$*. Then*

$$k \leq \begin{cases} \frac{d}{2} + n & \text{if } 0 \leq d \leq 2gn \\ d + n(1-g) & \text{if } d \geq 2gn. \end{cases}$$

3. Methods

We now describe some of the methods which can be used for tackling the existence problem.

- Degeneration Methods. These have been used very successfully by M. Teixidor i Bigas [**46, 47, 49**] (see section 8).
- Extensions $0 \to E_1 \to E \to E_2 \to 0$. For $k < n$, a special case, used in [**16, 18**], is
$$0 \to \mathcal{O}^k \to E \to E_2 \to 0.$$
The method is not so useful when $V \not\subseteq H^0(E_1)$ because there is a problem of lifting sections of E_2 to E. A more promising approach is to use extensions of coherent systems
$$0 \to (E_1, V_1) \to (E, V) \to (E_2, V_2) \to 0,$$
which are classified by $\mathrm{Ext}^1((E_2, V_2), (E_1, V_1))$. This approach was pioneered in [**17**] and has been used successfully in several papers [**19, 12, 32**].
- Syzygies and projective embeddings. Given a generated coherent system (E, V) of type (n, d, k) with $k > n$, one can construct a morphism $C \to \mathrm{Grass}(k-n, k)$. One can also use the sections of $\det E$ to get a morphism from C to a projective space. The relationship between these morphisms can be used to settle existence problems (see section 7 and [**26**]).
- Elementary transformations. Consider an exact sequence
$$0 \to E \to E' \to T \to 0,$$
where T is a torsion sheaf. If (E, V) is a coherent system, then so is (E', V). This method has been used to construct α-stable coherent systems for low (even negative) values of $\beta(n, d, k)$ (see sections 6, 7 and 8).
- Flips. Flips were pioneered by M. Thaddeus [**50**] and introduced in this context in [**17**]. They are given by extensions of coherent systems and can allow results for one value of α to be transmitted to another value.

- Dual Span. This was originally introduced by D. C. Butler [**24**] and has been used very successfully in the case $k = n + 1$ (see section 6). The idea is as follows. Let (E, V) be a coherent system. We define a new coherent system $D(E, V) = (D_V(E), V')$, where $D_V(E)^*$ is the kernel of the evaluation map $V \otimes \mathcal{O} \to E$ and V' is the image of V^* in $H^0(D_V(E))$. In the case where (E, V) is generated and $H^0(E^*) = 0$, we have dual exact sequences
$$0 \to D_V(E)^* \to V \otimes \mathcal{O} \to E \to 0$$
and
$$0 \to E^* \to V^* \otimes \mathcal{O} \to D_V(E) \to 0$$
and $D(D(E, V)) = (E, V)$, so this is a true duality operation. The main point here is that the stability properties of $D(E, V)$ are closely related to those of (E, V) [**17**, section 5.4].
- Covering Spaces. Suppose that $f : Y \to C$ is a covering (maybe ramified). If (E, V) is a coherent system on Y, then (f_*E, V) is a coherent system on C. One can for example take E to be a line bundle. The rank of f_*E is then equal to the degree of the covering and $\deg E$ is easy to compute. The problem is to prove α-stability. Preliminary work suggests that this may be interesting, but to get really good results one needs to take into account the fact that Y is not a general curve and may possess line bundles with more than the expected number of independent sections.
- Homological methods. In classical Brill-Noether theory, homological methods have been very successful in proving non-emptiness of Brill-Noether loci; essentially one views the loci as degeneration loci and uses the Porteous formula [**1**, II (4.2) and VII (4.4)]. It is not trivial to generalize this to higher rank Brill-Noether loci because of non-compactness of the moduli space of stable bundles (when $\gcd(n, d) \neq 1$) but more particularly because the cohomology is much more complicated. There is some unpublished work in a special case by Seongsuk Park [**42**] (see also section 9). Once one knows that $B(n, d, k) \neq \emptyset$, one gets also $G_0(n, d, k) \neq \emptyset$ by Lemma 1.4.
- Gauge theory. This has been used for constructing the moduli spaces of coherent systems [**14, 15**], but not (so far as I know) for proving they are non-empty.

4. $0 < k \leq n$

In this case the existence problem is completely solved and there are some results on irreducibility and smoothness.

Theorem 4.1. ([**18**, Theorem 3.3]) *Suppose* $0 < k \leq n$. *Then* $G(\alpha; n, d, k) \neq \emptyset$ *if and only if (1) holds and in addition*

$$k \leq n + \frac{1}{g}(d - n), \quad (n, d, k) \neq (n, n, n). \tag{2}$$

Moreover, if (2) holds and $d > 0$, *then* $U(n, d, k) \neq \emptyset$ *and is smooth and irreducible.*

Indication of proof. It is known [**17**, Theorems 5.4 and 5.6] that $G_L(n, d, k) \neq \emptyset$ under the stated conditions. Moreover $G_L(n, d, k)$ is smooth and irreducible and every element has the form

$$0 \to \mathcal{O}^k \to E \to F \to 0 \quad (k < n),$$
$$0 \to \mathcal{O}^n \to E \to T \to 0 \quad (k = n),$$

with F semistable, T torsion. Necessity of (2) is proved in [**18**, Corollary 2.5 and Lemma 2.10] and [**17**, Remark 5.7]. In both cases, given (2) and $d > 0$, the general E of this form is stable; this is a consequence of [**38**, Théorème A.5] (see also [**21, 22, 37**]). □

Remark 4.2. Conditions (2) are strictly stronger than $\beta(n, d, k) \geq 0$.

Corollary 4.3. *Model Theorem holds with*

$$d_0 = \begin{cases} \max\{1, n - g(n-k)\} & (k < n) \\ n+1 & (k = n). \end{cases}$$

5. $0 < d \leq 2n$

In this case the existence problem is completely solved over any curve and all non-empty moduli spaces are irreducible.

Theorem 5.1. ([**19**, Theorem 5.4]) *Suppose $g \geq 3$ and C is not hyperelliptic. If $0 < d \leq 2n$, then $G(\alpha; n, d, k) \neq \emptyset$ if and only if either (1) and (2) hold or $(n, d, k) = (g-1, 2g-2, g)$. Whenever it is non-empty, $G(\alpha; n, d, k)$ is irreducible. Moreover $U(n, d, k) \neq \emptyset$ and is smooth.*

Indication of proof. The necessity of the conditions for $0 < d < 2n$ follows from [**37**]. For $d = 2n$, further calculations are necessary. For $k \leq n$, the sufficiency of (1) and (2) has already been proved in Theorem 4.1. For $k > n$, one requires the results of [**21, 37, 39**] and a lemma stating that under these conditions $B(n, d, k) \neq \emptyset \Rightarrow U(n, d, k) \neq \emptyset$. For $(n, d, k) = (g-1, 2g-2, g)$ we take $(E, V) = D(K, H^0(K))$ (see section 6). □

Remark 5.2. For C hyperelliptic, the theorem remains true for $0 < d < 2n$, but some modification is needed for $d = 2n$ [**19**, Theorem 5.5].

The reference [**19**] includes an example to show that these nice results do not extend beyond $d = 2n$.

Example 5.3. ([**19**, section 7]) Let (E, V) be a coherent system of type (n, d, k) with

(3) $$n + \frac{1}{g}(d - n) < k < \frac{ng}{g-1}.$$

Then (E, V) is not α-semistable for large α. Moreover, if C is non-hyperelliptic and $3 \leq r \leq g-1$, there exists (E, V) of type $(rg - r + 1, 2rg - 2r + 3, rg + 1)$ with E stable, and (3) holds in this case.

6. $k = n + 1$

In this case the existence problem is almost completely solved when C is a Petri curve, which we assume until further notice. Full details are contained in [**12**].

We use the dual span construction for (L, V), where L is a line bundle of degree $d > 0$, (L, V) is generated and $\dim V = n + 1$.

Lemma 6.1. ([**17**, Corollary 5.10]) *$D(L,V)$ is α-stable for large α.*

Proof. Let (F,W) be a coherent subsystem of $D(L,V)$ with $\operatorname{rk} F < n$. Since $D(L,V)$ is generated and $H^0(D_V(L)^*) = 0$, it follows that $D_V(L)/F$ is non-trivial and that it is generated by the image of V^* in $H^0(D_V(L)/F)$. This implies that this image has dimension $\geq n - \operatorname{rk} F + 1$. So $\frac{\dim W}{\operatorname{rk} F} \leq 1 < \frac{n+1}{n}$ which implies the result. □

Theorem 6.2. *$G_L(n,d,n+1)$ is birational to $G(1,d,n+1) = G^n_d$.*

Proof. Note that $\beta(n,d,n+1) = \beta(1,d,n+1) = \dim(G(1,d,n+1))$. Now, do a parameter count to show that non-generated (L,V) and (E,V^*) contribute $< \beta(n,d,n{+}1)$ to the dimension (see the proof of [**17**, Theorem 5.11] for details). □

Corollary 6.3. *$G_L(n,d,n+1) \neq \emptyset$ if and only if $\beta(n,d,n+1) \geq 0$, i.e. if and only if*

$$d \geq g + n - \left[\frac{g}{n+1}\right]. \tag{4}$$

Proof. This follows from classical Brill-Noether theory, since C is Petri. □

Theorem 6.4. ([**20**, Theorem 2]) *If (4) holds and $d \leq g+n$, then*

$$U^s(n,d,n+1) \neq \emptyset.$$

Moreover, except for $g = n = 2$, $d = 4$, $U(n,d,n+1) \neq \emptyset$

Proof. If (4) holds and $d \leq g+n$, there exists a line bundle L of degree d with $h^0(L) = n+1$, so we can take $V = H^0(L)$ for such L, and suppose that V generates L. By Lemma 6.1, $D(L,V)$ is α-stable for large α, while by [**24**, Theorem 2] (see also [**20**, Proposition 4.1]), $D_L(V)$ is stable for general L. The result for $g = n = 2$, $d = 4$ is contained in [**20**, Proposition 4.1]. □

Theorem 6.5. ([**20**, Theorem 2]) *If $G(\alpha;n,d,n+1) \neq \emptyset$ for some α, then (4) holds.*

Indication of proof. One can show that, for $d \leq g+n$,

$$G(\alpha;n,d,n+1) = G_L(n,d,n+1)$$

for all $\alpha > 0$. Now use Corollary 6.3. □

Corollary 6.6. $d_0(n,n+1) = g + n - \left[\frac{g}{n+1}\right]$.

Theorem 6.7. ([**20**, Theorem 3]) *If $g \geq n^2 - 1$, then Model Theorem holds.*

Theorem 6.8. ([**12**, Theorems 7.1, 7.2, 7.3]) *Model Theorem holds for $n = 2,3,4$ and $g \geq 3$.*

For $g = 2$, Model Theorem does not quite hold; the result is as follows.

Theorem 6.9. ([**12**, Theorem 8.2]) *Let X be a curve of genus 2. Then $d_0 = n+2$ and*

- *$U^s(n,d,n+1) \neq \emptyset$ if and only if $d \geq d_0$*
- *$U(n,d,n+1) \neq \emptyset$ if and only if $d \geq d_0$, $d \neq 2n$.*

The proofs of these theorems depend on combining several techniques including those of [**20**], extensions of coherent systems and the following result.

Proposition 6.10. *Suppose that $d \geq d_1$, where*

$$d_1 := \begin{cases} \frac{n(g+3)}{2} + 1 & \text{if } g \text{ is odd} \\ \frac{n(g+4)}{2} + 1 & \text{if } g \text{ is even and } n > \frac{g!}{(\frac{g}{2})!(\frac{g}{2}+1)!} \\ \frac{n(g+2)}{2} + 1 & \text{if } g \text{ is even and } n \leq \frac{g!}{(\frac{g}{2})!(\frac{g}{2}+1)!}. \end{cases}$$

Then $U(n, d, n+1) \neq \emptyset$.

Proof. This is proved using elementary transformations (for details see [**12**, Proposition 6.6]). The restriction on n in the third formula is required because we need to have n non-isomorphic line bundles L_i of degree $\frac{d_1-1}{n}$ with $h^0(L_i) = 2$. For the number of such line bundles in this case, see [**1**, V (1.2)]. □

Remark 6.11. Although Model Theorem is not established in all cases, one can say that $U(n, d, n+1)$ is always smooth and is irreducible of dimension $\beta(n, d, n+1)$ whenever it is non-empty and $\beta(n, d, n+1) > 0$ [**12**, Remark 6.2].

Now let us replace the Petri condition by the condition that C be general (in some unspecified sense). Teixidor's result (see Theorem 8.1) takes the following form when $k = n+1$.

Theorem 6.12. *Suppose that C is a general curve of genus g and that $d \geq d_1'$, where*

$$d_1' = \begin{cases} \frac{n(g+1)}{2} + 1 & \text{if } g \text{ is odd} \\ \frac{n(g+2)}{2} + 1 & \text{if } g \text{ is even.} \end{cases}$$

Then $U(n, d, n+1) \neq \emptyset$.

Using this, we can prove

Theorem 6.13. *Suppose that C is a general curve of genus 3. Then Model Theorem holds.*

Proof. The methods of [**12**] are sufficient to prove (for any Petri curve of genus 3) that $U(n, d, n+1) \neq \emptyset$ if $d \geq d_0$ and $d \neq 2n+2$ (see [**12**, Theorem 8.3]). The exceptional case is covered by Theorem 6.12. □

Remark 6.14. Suppose now that C is any smooth curve of genus g. Since $\gcd(n, d, n+1) = 1$, a specialization argument shows that, if $G(\alpha; n, d, n+1) \neq \emptyset$ on a general curve and α is not a critical value, then $G(\alpha; n, d, n+1) \neq \emptyset$ on C. A priori, this does not imply that $U(n, d, n+1) \neq \emptyset$, but Ballico [**6**] has used Teixidor's result to show that, when $d \geq d_1'$, we have $U^s(n, d, n+1) \neq \emptyset$. If $\gcd(n, d) = 1$, this gives $U(n, d, n+1) \neq \emptyset$.

7. $n = 2$, $k = 4$

This is the first case in which we do not know the value of d_0 even on a general curve. Let us define

$$d_2 := \begin{cases} g+3 & \text{if } g \text{ is even} \\ g+4 & \text{if } g \text{ is odd.} \end{cases} \tag{5}$$

Note that $\beta(2, d, 4) = 4d - 4g - 11$, so $\beta(2, g+3, 4) = 1$ and $g+3$ is the smallest value of d for which $\beta(2, d, 4) \geq 0$. Moreover, on a Petri curve, $\frac{d_2-1}{2}$ is the smallest degree for which there exists a line bundle L with $h^0(L) \geq 2$.

Teixidor has proved the following result by degeneration methods.

Theorem 7.1. [**46**] *Let C be a general curve of genus $g \geq 3$. If $d \geq d_2$, then $U(2,d,4) \neq \emptyset$.*

By completely different methods, we can prove the following stronger result (further details will appear in [**26**]).

Theorem 7.2. *Let C be a Petri curve of genus $g \geq 3$. Then*
(i) *$U(2,d,4) \neq \emptyset$ for $d \geq d_2$,*
(ii) *if $d < d_2 - 1$ and (E,V) is α-semistable of type $(2,d,4)$ for some α, then $(E,V) \in U(2,d,4)$. Moreover, $U(2,d',4) \neq \emptyset$ for $d \leq d' \leq d_2 - 1$,*
(iii) *$G(\alpha;2,d,4) = \emptyset$ for all α for $d < min\{\frac{4g}{5} + 4, d_2 - 1\}$.*

Proof. (i) Consider

$$0 \to L_1 \oplus L_2 \to E \to T \to 0,$$

where the L_i are line bundles with $\deg(L_i) = \frac{d_2-1}{2}$, $h^0(L_i) = 2$, $L_1 \not\cong L_2$ and T is a torsion sheaf of length $d - d_2 + 1$. By classical Brill-Noether theory, such line bundles always exist on a Petri curve. By [**38**, Théorème A.5], the general such E is stable. Now, it is easy to show that $(E,V) \in U(2,d,4)$ where $V = H^0(L_1) \oplus H^0(L_2)$.

(ii) Let (F,W) be a coherent subsystem of (E,V) with $\operatorname{rk} F = 1$, where $W = V \cap H^0(F)$. Suppose E is not stable and choose F with $\deg F \geq \frac{d}{2}$. Then $\deg(E/F) \leq \frac{d}{2} < \frac{d_2-1}{2}$. Hence, by classical Brill-Noether theory, $h^0(E/F) \leq 1$, so $\dim W \geq 3$. This contradicts α-semistability for all α.

So E is stable. Now, for any (F,W), $\deg F < \frac{d}{2}$, so $h^0(F) \leq 1$, hence $\dim W \leq 1$. So

$$\mu_\alpha(F,W) < \frac{d}{2} + \alpha < \frac{d + 4\alpha}{2},$$

for all $\alpha > 0$, so $(E,V) \in U(2,d,4)$.

For the last part, we proceed by induction; we need to prove that, if $d < d_2 - 1$, then

$$U(2,d,4) \neq \emptyset \Rightarrow U(2,d+1,4) \neq \emptyset.$$

So suppose $(E,V) \in U(2,d,4)$ and consider an elementary transformation

$$0 \to E \to E' \to T \to 0,$$

where T is a torsion sheaf of length 1. Then $(E',V) \in U(2,d+1,4)$ if and only if E' is stable. It is easy to see that E' is semistable. If E' is strictly semistable, then it possesses a line subbundle L of degree $\frac{d+1}{2}$. Since E is stable, $L \cap E$ must have degree $\frac{d-1}{2}$ and so, by classical Brill-Noether theory, $\dim(H^0(L) \cap V) \leq 1$. Hence $h^0(E/L) \geq 3$, which is a contradiction since $\deg(E/L) = \frac{d+1}{2} \leq \frac{d_2-1}{2}$.

(iii) If $(E,V) \in G(\alpha;2,d,4)$ with $d < min\{\frac{4g}{5} + 4, d_2 - 1\}$, we have $(E,V) \in U(2,d,4)$ by (ii). It is easy to check that (E,V) is generically generated and the proof of (ii) shows that the subsheaf E' of E generated by V is stable. We have an exact sequence

$$0 \to D_V(E')^* \to V \otimes \mathcal{O} \to E' \to 0,$$

which induces two further exact sequences

$$0 \to N \to \wedge^2 V \otimes \mathcal{O} \to \det E' \to 0 \tag{6}$$

and

$$0 \to (\det E')^* \to N \to E' \otimes D_V(E')^* \to 0. \tag{7}$$

Now $D_V(E')$ is stable of the same slope as E', so, by (7),

$$h^0(N) \le h^0(E' \otimes D_V(E')^*) \le 1.$$

So $h^0(\det E') \ge 5$ by (6). Hence, by classical Brill-Noether theory,

$$\deg E \ge \deg E' \ge \frac{4g}{5} + 4.$$

□

Corollary 7.3. *If $\frac{4g}{5} + 4 > d_2 - 1$, then $d_0 = d_2$. If $\frac{4g}{5} + 4 \le d_2 - 1$, then $\frac{4g}{5} + 4 \le d_0 \le d_2$. Moreover, in all cases, Model Theorem holds.*

Proof. The second part follows immediately from the theorem. If $\frac{4g}{5} + 4 > d_2 - 1$, then certainly $d_2 - 1 \le d_0 \le d_2$. In fact, if $(E, V) \in G(\alpha; 2, d_2 - 1, 4)$, the argument used to prove part (iii) of the theorem shows that $\deg E \ge \frac{4g}{5} + 4$, which is a contradiction.

Model Theorem is now clear except possibly when $d_0 = d_2 - 1$. But in this case, there exists $(E, V) \in G(\alpha; 2, d_2 - 1, 4)$ for some α. If E is not stable, then, in the proof of part (ii) of the theorem, we obtain $h^0(E/F) \le 2$, so $\dim W \ge 2$, contradicting α-stability. The rest of the proof works to show that $(E, V) \in U(2, d_2 - 1, 4)$. □

Remark 7.4. Theorems 7.1 and 7.2(i) and the first statement of Corollary 7.3 fail for $g = 2$. In this case $d_2 = 5$ and the proof of Theorem 7.2(i) fails because there is only one line bundle L of degree 2 with $h^0(L) = 1$. Moreover it follows from Riemann-Roch that any stable bundle E of rank 2 and degree 5 has $h^0(E) = 3$ and one can easily deduce that $G(\alpha; 2, 5, 4) = \emptyset$ for all $\alpha > 0$. In fact $d_0 = 6$ and Model Theorem holds (see statement (3) preceding Lemma 6.6 in [**20**]).

We finish this section with an example

Example 7.5. If $4 \le g \le 8$ and g is even, Corollary 7.3 gives $d_0 = d_2 = g + 3$. Suppose now that $g = 10$. Then $\frac{4g}{5} + 4 = 12 = g + 2 = d_2 - 1$, so $d_0 = 12$ or 13.

Suppose $(E, V) \in G(\alpha; 2, 12, 4)$. The argument in the proof of Theorem 7.2 (ii)/(iii) shows that E' and $D(E')$ are stable and $h^0(\det E') \ge 5$. By classical Brill-Noether theory, $\deg E' \ge 12$, so $E' = E$. Also $h^0(\det E) = 5$.

Write $L = \det E$. We can choose a 3-dimensional subspace W of $H^0(L)$ such that there is an exact sequence

$$0 \to E^* \to W \otimes \mathcal{O} \to L \to 0,$$

which induces

$$0 \to H^0(L \otimes E^*) \to W \otimes H^0(L) \xrightarrow{\psi} H^0(L^{\otimes 2}) \to 0.$$

Now $L \otimes E^* \cong E$, so $h^0(L \otimes E^*) \ge 4$. Hence $\dim \operatorname{Ker} \psi \ge 4$, from which it follows that the linear map

$$S^2(H^0(L)) \to H^0(L^{\otimes 2}) \tag{8}$$

is not injective. Both of the spaces in (8) have dimension 15, so (8) is not surjective. On the other hand, Voisin has proved that, for general C, (8) is surjective [**51**]. So, for general C, $G(\alpha; 2, 12, 4) = \emptyset$ and $d_0 = 13$.

If (8) is not surjective, the sections of L determine an embedding $C \hookrightarrow \mathbb{P}^4$ whose image lies on the quadric q whose equation generates the kernel of (8).

If q has rank 5, it is easy to check that (E,V) exists. This situation does in fact arise for certain Petri curves lying on K3 surfaces (see [**51**]), so there exist Petri curves for which $d_0 = 12$. If q has rank 4, it is possible that (E,V) is strictly α-semistable for all $\alpha > 0$, so we can make no deduction about d_0. The quadric q cannot have rank < 4.

Further details of this example will appear in [**26**].

8. $k \geq n+2$

The situation in section 7 is typical of that for $k \geq n+2$, except that in general we know even less. Until recently, no reasonable bound for d_0 had been established. This has now been rectified by Teixidor, who has obtained a bound for general C by degeneration methods. We state her result in a slightly adapted version.

Theorem 8.1. ([**49**]) *Let C be a general curve of genus $g \geq 2$. If $k > n$, then $U(n,d,k) \neq \emptyset$ for $d \geq d_3$, where*

$$d_3 := \begin{cases} k + n(g-1) - n\left[\frac{g-1}{[k/n]}\right] + 1 & \text{if } n \mid k \\ k + n(g-1) - n\left[\frac{g-1}{[k/n]+1}\right] & \text{if } n \nmid k. \end{cases} \tag{9}$$

Moreover $U(n,d,k)$ has a component of dimension $\beta(n,d,k)$.

It should be noted that this is not best possible; in particular, when $n = 2$, $k = 4$, it is weaker than Theorem 7.1. Also, Teixidor states her result in terms of non-emptiness of all $G(\alpha; n,d,k)$, but her proof gives the stronger statement that $U(n,d,k) \neq \emptyset$. For other formulations of (9), see [**44, 38**], where the corresponding result for Brill-Noether loci is proved.

For an arbitrary smooth curve, we have the following theorem of Ballico [**6**], obtained from Theorem 8.1 by a specialization argument.

Theorem 8.2. *Let C be a smooth curve of genus g and $k > n$. If* $\gcd(n,d,k) =$ 1 *and $d \geq d_3$, then $U^s(n,d,k) \neq \emptyset$. If moreover* $\gcd(n,d) = 1$*, then $U(n,d,k) \neq \emptyset$.*

When $n \mid k$, we can prove the following theorem without any coprimality assumptions by using the methods of [**38**].

Theorem 8.3. *Let C be a smooth curve of genus $g \geq 2$. If $k > n$, $n \mid k$, then $U(n,d,k) \neq \emptyset$ for $d \geq d_3$. If in addition C is Petri and $k' := \frac{k}{n} \mid g$, then $U(n,d,k) \neq \emptyset$ for $d \geq d_3 - n$, provided that*

$$n \leq n' := g! \prod_{i=0}^{k'-1} \frac{i!}{(i+\frac{g}{k'})!}.$$

Proof. Note that $d_3 - 1$ is divisible by n and $\beta\left(1, \frac{d_3-1}{n}, \frac{k}{n}\right) > 0$. Hence, by classical Brill-Noether theory, we can find n non-isomorphic line bundles L_i with $\deg L_i = \frac{d_3-1}{n}$ and $h^0(L_i) \geq \frac{k}{n}$. Choose subspaces V_i of $H^0(L_i)$ of dimension $\frac{k}{n}$ and consider an exact sequence

$$0 \to L_1 \oplus \cdots \oplus L_n \to E \to T \to 0, \tag{10}$$

where T is a torsion sheaf of length $d - d_3 + 1 > 0$. Let $V := \bigoplus H^0(L_i) \subseteq H^0(E)$. Then $(L_1 \oplus \cdots \oplus L_n, V)$ is α-semistable for all $\alpha > 0$. If E is stable, it follows that $(E,V) \in U(n,d,k)$. On the other hand, the general E given by (10) is stable by [**38**, Théorème A.5]. The first result follows.

Now suppose $k' \mid g$. Then $\left[\frac{g-1}{k'}\right] = \frac{ng}{k} - 1$. Let

$$d' := \frac{d_3 - 1}{n} - 1 = k' + g - 1 - \frac{g}{k'}$$

and $\beta(1, d', k')$=0. So, again by classical Brill-Noether theory, there exist line bundles L_i with $\deg L_i = d'$ and $h^0(L_i) \geq k'$, and, if C is Petri, there are precisely n' such line bundles up to isomorphism [**1**, V (1.2)]. The proof now goes through as before. □

Remark 8.4. It is worth noting that, when $n \mid k$ and $k' \mid g$,

$$\beta(n, d_3 - n, k) = k - n^2 + 1,$$

so one can have $U(n, d, k) \neq \emptyset$, even on a Petri curve, with negative Brill-Noether number.

For a lower bound on d_0, we have the following result of Brambila-Paz.

Theorem 8.5. ([**20**, Theorem 1]) *Let C be a Petri curve of genus g. Suppose $k > n$ and $\beta(n, d, n+1) < 0$. Then $G(\alpha; n, d, k) = \emptyset$ for all $\alpha > 0$.*

Combining Theorems 8.1 and 8.5, we obtain that, on a general curve,

$$n + g - \left[\frac{g}{n+1}\right] \leq d_0(n, k) \leq d_3.$$

9. $n = 2$, $\det E = K$

In this section we consider coherent systems (E, V) with $\operatorname{rk} E = 2$, $\det E \cong K$. These have not to my knowledge been studied in their own right, but the corresponding problem for Brill-Noether loci has attracted quite a lot of attention and this has some implications for coherent systems. In what follows, we consistently denote the corresponding spaces by $B(2, K, k)$, $G(\alpha; 2, K, k)$ etc.

The first point to note is that some definitions have to be changed. There is a canonical skew-symmetric isomorphism between E and $E^* \otimes K$. As a result of this, the Petri map has to be replaced by the *symmetrized Petri map*

$$S^2 H^0(E) \to H^0(S^2 E). \tag{11}$$

This map governs the infinitesimal behaviour of $B(2, K, k)$. Moreover, the Brill-Noether number giving the expected dimension of $B(2, K, k)$ and $G(\alpha; 2, K, k)$ must be replaced by

$$\beta(2, K, k) := 3g - 3 - \frac{k(k+1)}{2}.$$

All components of $B(2, K, k)$ have dimension $\geq \beta(2, K, k)$. The intriguing thing is that often $\beta(2, K, k) > \beta(2, 2g-2, k)$, so that the expected dimension of $B(2, K, k)$ is greater than that of $B(2, 2g-2, k)$, although the latter contains the former!

These Brill-Noether loci were studied some time ago by Bertram and Feinberg [**9**] and by Mukai [**41**], who independently proposed the following conjecture.

Conjecture. *Let C be a general curve of genus g. Then $B(2, K, k)$ is non-empty if and only if $\beta(2, K, k) \geq 0$.*

This was verified by Bertram and Feinberg and Mukai for low values of g. Of course, if $B(2,K,k) \neq \emptyset$, then $G_0(2,K,k) \neq \emptyset$, but we cannot make deductions about $U(2,K,k)$ without further information.

Some of these results have been extended and there are also some general existence results due to Teixidor.

Theorem 9.1. ([**45**]) *Let C be a general curve of genus g. If either $k = 2k_1$ and $g \geq k_1^2$ $(k_1 > 2)$, $g \geq 5$ $(k_1 = 2)$, $g \geq 3$ $(k_1 = 1)$ or $k = 2k_1 + 1$ and $g \geq k_1^2 + k_1 + 1$, then $B(2,K,k) \neq \emptyset$ and has a component of dimension $\beta(2,K,k)$.*

Corollary 9.2. *Under the conditions of Theorem 9.1, $G_0(2,K,k) \neq \emptyset$.*

The most significant result to date is also due to Teixidor.

Theorem 9.3. ([**48**]) *Let C be a general curve of genus g. Then the symmetrized Petri map (11) is injective for every semistable bundle E.*

Corollary 9.4. *Let C be a general curve of genus g. If $\beta(2,K,k) < 0$, then $G_0(2,K,k) = \emptyset$.*

To prove the conjecture, it is therefore sufficient to extend Theorem 9.1 to all cases where $\beta(2,K,k) \geq 0$. Progress on this has been made recently by Seongsuk Park using homological methods [**42**].

Remark 9.5. Theorem 9.3 and Corollary 9.4 fail for certain Petri curves lying on K3 surfaces. In fact, based on work of Mukai, Voisin [**51**, Théorème 0.1] observed that on these curves $B\left(2,K,\frac{g}{2}+2\right) \neq \emptyset$ for any even g. However, for g even, $g \geq 10$, the Brill-Noether number $\beta\left(2,K,\frac{g}{2}+2\right) < 0$.

10. Special curves

The results of sections 3 and 4 are valid for arbitrary smooth curves. On the other hand, while some of the results of the remaining sections (dealing with the case $k > n$) are valid on arbitrary smooth curves, most of them hold on general curves. Indeed, if $k > n$, the value of d_0 certainly depends on the geometry of C and not just on g. This is of course no different from what happens for $n = 1$, but it is already clear that the distinctions in higher rank are more subtle than those for the classical case (see, for instance, Example 7.5 and Remark 9.5).

There is as yet little work in the literature concerning coherent systems on special curves; here by *special* we mean a curve for which the moduli spaces of coherent systems exhibit behaviour different from that on a general curve. So far as I am aware, the only papers relating specifically to special curves are those of Ballico on bielliptic curves [**2**] and Brambila-Paz and Ortega on curves with specified Clifford index [**23**]. In the latter paper, the authors define positive numbers

$$d_u = d_u(n,g,\gamma) := \begin{cases} n+g-1+\frac{g-1}{n-1} & \text{if } \gamma \geq g-n \\ 2n+\gamma+\frac{\gamma}{n-1} & \text{if } \gamma < g-n \end{cases}$$

and

$$d_\ell = d_\ell(n,g,\gamma) := \begin{cases} n+g-1 & \text{if } \gamma \geq g-n \\ 2n+\gamma & \text{if } \gamma < g-n \end{cases}$$

and prove, among other things, the following three results.

Theorem 10.1. ([**23**, Theorem 1.2]) *Let C be a curve of genus g and Clifford index γ and suppose $k > n$. Then $d_0(n,k) \geq d_\ell$.*

Theorem 10.2. ([**23**, Theorem 4.1]) *Let C be a curve of genus g and Clifford index γ and suppose that there exists a generated coherent system (G, W) of type $(m, d, n+m)$ with $H^0(G^*) = 0$ and $d_\ell \leq d < d_u$. Then $U(n, d, n+m) \neq \emptyset$.*

Corollary 10.3. ([**23**, Corollary 5.1]) *Let C be a smooth plane curve of degree $d \geq 5$. Then $d_0(2, 3) = d_\ell(2, g, \gamma) = d$ and $U(2, d, 3) \neq \emptyset$.*

In the corollary, one may note that the value of d_0 here is much smaller than the value for a general curve of the same genus given by Corollary 6.6. The paper [**23**] includes also examples where $d_0(n, n+1) = g - 1$.

11. Singular curves

The construction of moduli spaces for coherent systems can be generalized to arbitrary curves (see [**29**]). Of course, Teixidor's work depends on constructing coherent systems on reducible curves and in particular the concept of limit linear series (see [**44, 45, 46, 47, 48, 49**] and Teixidor's article in this volume), but apart from this there has been little work on coherent systems on singular curves. There is some work of Ballico proving the existence of α-stable coherent systems for large d [**4, 5**] and a paper of Ballico and Prantil for C a singular curve of genus 1 [**8**].

The situation for nodal curves has now changed due to work of Bhosle, who has generalized results of [**21**] and [**17**]. We refer to [**10, 11**] for details.

12. Open Problems

There are many open problems in the theory of moduli spaces of coherent systems on algebraic curves. As in the main part of this article, we restrict attention here to problems of emptiness and non-emptiness. The basic outstanding problems can be formulated as follows.

Problem 12.1. Given C and (n, k) with $n \geq 2$, $k \geq 1$, determine the value of $d_0(n, k)$ (the minimum value of d for which $G(\alpha; n, d, k) \neq \emptyset$ for some $\alpha > 0$).

Problem 12.2. Given C and (n, k), prove or disprove Model Theorem. If the theorem fails to hold, Theorem 2.1 implies that there are finitely many values of $d \geq d_0(n, k)$ for which $U(n, d, k) = \emptyset$. For each such d, find the range of α (possibly empty) for which $G(\alpha; n, d, k) \neq \emptyset$. Note that we already have a solution for this problem when $g = 2$ and $k = n + 1$; in this case (a) is true but not (b) or (c) (Theorem 6.9). In Example 5.3, (a) fails but (b) may possibly be true.

Problem 12.3. For fixed genus g and fixed (n, d), we know that there exists a bound, independent of α, on $h^0(E)$ for E to be an α-semistable coherent system. The bound given by [**17**, Lemmas 10.2 and 10.3] looks weak. Find a better (even best possible) bound. An answer to this would be useful in estimating codimensions of flip loci.

We now turn to some more specific problems, which we state for a general curve, although versions for special curves could also be produced.

Problem 12.4. A conjecture of D. C. Butler [**24**, Conjecture 2] can be stated in the following form.

Conjecture. (Butler's Conjecture) *Let C be a general curve of genus $g \geq 3$ and let (E, V) be a general generated coherent system in $G_0(m, d, n+m)$. Then the dual span $D(E, V)$ belongs to $G_0(n, d, n+m)$.*

(Since $G_0(m, d, n+m)$ can be reducible, one needs to be careful over the meaning of the word "general" here.) There are many possible versions of this conjecture. For example, if one replaces G_0 by G_L, then the conjecture holds for all $(E, V) \in G_L(m, d, n+m)$ if $\gcd(m, n) = 1$ and, even in the non-coprime case, a slightly weaker version holds [**17**, section 5.4]. We seem, however, to be some distance from proving the general conjecture.

Problem 12.5. A slightly stronger version of Butler's conjecture is

Conjecture. (Butler's Conjecture (strong form)) *Let C be a general curve of genus $g \geq 3$ and let (E, V) be a general generated coherent system in $G_0(m, d, n+m)$. Then $D_V(E)$ is stable.*

When $m = 1$ and C is Petri, this conjecture holds for $V = H^0(E)$ (see [**24**, Theorem 2] and [**20**, Proposition 4.1]). For general V, it is equivalent in this case to

Conjecture. *Let C be a Petri curve of genus $g \geq 3$. If $\beta(n, d, n+1) \geq 0$, then $U(n, d, n+1) \neq \emptyset$.*

(For the proof of this equivalence, see [**12**, Conjecture 9.5 and Proposition 9.6].) This result narrowly fails for $g = 2$ and $d = 2n$ (see Theorem 6.9), but is proved in many cases in section 6. For some further results on the strong form of Butler's Conjecture when $m = 1$, see [**40**].

Problem 12.6. Let C be a general curve of genus $g \geq 3$. Calculate $d_0(2, 4)$. For the current state of information on this problem, see section 7, especially Corollary 7.3.

Problem 12.7. Let C be a general curve of genus g. Extend the results of section 9 to $G(\alpha; 2, K, k)$ for arbitrary $\alpha > 0$ and hence to $U(2, K, k)$.

Problem 12.8. Related to Problem 12.7, we can state an extended version of the conjecture of Bertram/Feinberg/Mukai.

Conjecture. *Let C be a general curve of genus g. Then*

- $\beta(2, K, k) < 0 \Rightarrow G(\alpha; 2, K, k) = \emptyset$ *for all* $\alpha > 0$
- $\beta(2, K, k) \geq 0 \Rightarrow U(2, K, k) \neq \emptyset$ *for all* $\alpha > 0$.

For the current state of information on this problem, see section 9.

Turning now to special curves, we give two problems.

Problem 12.9. Solve the basic problems for hyperelliptic curves. For Brill-Noether loci, a strong result is known [**22**, section 6]. One could try to generalize this to coherent systems.

Problem 12.10. There are several possible notions of higher rank Clifford indices. Determine what values these indices can take and what effect they have on the geometry of the moduli spaces of coherent systems. What other invariants can be introduced (generalizing gonality, Clifford dimension, ...)?

We finish with two problems on a topic not touched on in this survey.

Problem 12.11. The moduli spaces can be constructed in finite characteristic (see [**29**]). Solve the basic problems in this case and compare (or contrast) them with those for characteristic 0. Much of the basic theory should be unchanged, but detailed structure of the moduli spaces might well be different.

Problem 12.12. Obtain results for coherent systems on curves defined over finite fields. This would allow the possibility of computing zeta-functions and hence obtaining cohomological information in the characteristic 0 case.

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Regularity on Abelian Varieties III: Relationship with Generic Vanishing and Applications

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Contents

1. Introduction

In previous work we have introduced the notion of *M-regularity* for coherent sheaves on abelian varieties ([**PP1**], [**PP2**]). This is useful because M-regular sheaves enjoy strong generation properties, in such a way that M-regularity on abelian varieties presents close analogies with the classical notion of Castelnuovo-Mumford regularity on projective spaces. Later we studied objects in the derived category of a smooth projective variety subject to Generic Vanishing conditions (*GV-objects* for short, [**PP4**]). The main ingredients are Fourier-Mukai transforms and the systematic use of homological and commutative algebra techniques. It turns out that, from the general perspective, M-regularity is a natural strenghtening of a Generic Vanishing condition. In this paper we describe in detail the relationship between the two notions in the case of abelian varieties, and deduce new basic properties of both M-regular and GV-sheaves. We also collect a few extra applications of the generation properties of M-regular sheaves, mostly announced

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but not contained in [**PP1**] and [**PP2**]. This second part of the paper is based on our earlier preprint [**PP6**].

We start in §2 by recalling some basic definitions and results from [**PP4**] on GV-conditions, restricted to the context of the present paper (coherent sheaves on abelian varieties). The rest of the section is devoted to the relationship between GV-sheaves and M-regular sheaves. More precisely, we prove a criterion, Proposition 2.8, characterizing the latter among the former: M-regular sheaves are those GV-sheaves $\mathcal{F}$ for which the Fourier-Mukai transform of the Grothendieck-dual object $\mathbf{R}\Delta\mathcal{F}$ is a torsion-free sheaf. (This will be extended to higher regularity conditions, or strong Generic Vanishing conditions, in our upcoming work [**PP5**].)

We apply this relationship in §3 to the basic problem of the behavior of cohomological support loci under tensor products. We first prove that tensor products of GV-sheaves are again GV when one of the factors is locally free, and then use this and the torsion-freeness characterization to deduce a similar result for M-regular sheaves. The question of the behavior of M-regularity under tensor products had been posed to us by A. Beauville as well. It is worth mentioning that Theorem 3.2 does not seem to follow by any more standard methods.

In the other direction, in §4 we prove a result on GV-sheaves based on results on M-regularity. Specifically, we show that GV-sheaves on abelian varieties are nef. We deduce this from a theorem of Debarre [**De2**], stating that M-regular sheaves are ample, and the results in §2. This is especially interesting for the well-known problem of semipositivity: higher direct images of dualizing sheaves via maps to abelian varieties are known to be GV (cf. [**Hac**], [**PP3**]).

In §5 we survey generation properties of M-regular sheaves. This section is mostly expository, but the presentation of some known results, as Theorem 5.1$(a) \Rightarrow (b)$ (which was proved in [**PP1**]), is new and more natural with respect to the Generic Vanishing perspective, providing also the new implication $(b) \Rightarrow (a)$. In combination with well-known results of Green-Lazarsfeld and Ein-Lazarsfeld, we deduce some basic generation properties of the canonical bundle on a variety of maximal Albanese dimension, used in the following section.

The second part of the paper contains miscellaneous applications of the generation properties enjoyed by M-regular sheaves on abelian varieties, extracted or reworked from our older preprint [**PP6**]. In §6 we give effective results for pluricanonical maps on irregular varieties of general type and maximal Albanese dimension via M-regularity for direct images of canonical bundles, extending work in [**PP1**] §5. In particular we show, with a rather quick argument, that on a smooth projective variety Y of general type, maximal Albanese dimension, and whose Albanese image is not ruled by subtori, the pluricanonical series $|3K_Y|$ is very ample outside the exceptional locus of the Albanese map (Theorem 6.1). This is a slight strengthening, but also under a slightly stronger hypothesis, of a result of Chen and Hacon ([**CH**], Theorem 4.4), both statements being generalizations of the fact that the tricanonical bundle is very ample for curves of genus at least 2.

In §7.1 we look at bounding the Seshadri constant measuring the local positivity of an ample line bundle. There is already extensive literature on this in the case of abelian varieties (cf. [**La1**], [**Nak**], [**Ba1**], [**Ba2**], [**De1**] and also [**La2**] for further references). Here we explain how the Seshadri constant of a polarization L on an abelian variety is bounded below by an asymptotic version – and in particular by the usual – M-regularity index of the line bundle L, defined in [**PP2**] (cf. Theorem

7.4). Combining this with various bounds for Seshadri constants proved in [**La1**], we obtain bounds for M-regularity indices which are not apparent otherwise.

In §7.2 we shift our attention towards a cohomological study of Picard bundles, vector bundles on Jacobians of curves closely related to Brill-Noether theory (cf. [**La2**] 6.3.C and 7.2.C for a general introduction). We combine Fourier-Mukai techniques with the use of the Eagon-Northcott resolution for special determinantal varieties in order to compute their regularity, as well as that of their relatively small tensor powers (cf. Theorem 7.15). This vanishing theorem has practical applications. In particular we recover in a more direct fashion the main results of [**PP1**] §4 on the equations of the W_d's in Jacobians, and on vanishing for pull-backs of pluritheta line bundles to symmetric products.

By work of Mukai and others ([**Muk3**], [**Muk4**], [**Muk1**], [**Um**] and [**Or**]) it has emerged that on abelian varieties the class of vector bundles most closely resembling semistable vector bundles on curves and line bundles on abelian varieties is that of *semihomogeneous* vector bundles. In §7.3 we show that there exist numerical criteria for their geometric properties like global or normal generation, based on their Theta regularity. More generally, we give a result on the surjectivity of the multiplication map on global sections for two such vector bundles (cf. Theorem 7.30). Basic examples are the projective normality of ample line bundles on any abelian variety, and the normal generation of the Verlinde bundles on the Jacobian of a curve, coming from moduli spaces of vector bundles on that curve.

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2. GV-sheaves and M-regular sheaves on abelian varieties

GV-sheaves. We recall definitions and results from [**PP4**] on *Generic Vanishing* conditions (GV for short). In relationship to the treatment of [**PP4**] we confine ourselves to a more limited setting, with respect to the following three aspects: (a) we consider only coherent sheaves (rather than complexes) subject to generic vanishing conditions; (b) we consider only the simplest such condition, i.e. GV_0, henceforth denoted GV; (c) we work only on abelian varieties, with the classical Fourier-Mukai functor associated to the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$ (rather than arbitrary integral transforms).

Let X be an abelian variety of dimension g over an algebraically closed field, $\widehat{X} = \mathrm{Pic}^0(X)$, P a normalized Poincaré bundle on $X \times \widehat{X}$, and $\mathbf{R}\widehat{\mathcal{S}} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$ the standard Fourier-Mukai functor given by $\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}) = \mathbf{R}p_{\widehat{A}*}(p_A^*\mathcal{F} \otimes P)$. We denote by $\mathbf{R}\mathcal{S} : \mathbf{D}(\widehat{X}) \to \mathbf{D}(X)$ the functor in the other direction defined analogously. For a coherent sheaf $\mathcal{F}$ on X, we will consider for each $i \geq 0$ its *i-th cohomological support locus*

$$V^i(\mathcal{F}) := \{\alpha \in \widehat{X} \mid h^i(X, \mathcal{F} \otimes \alpha) > 0\}.$$

By base-change, the support of $R^i\widehat{\mathcal{S}}\mathcal{F}$ is contained in $V^i(\mathcal{F})$.

Proposition/Definition 2.1 (GV-sheaf, [**PP4**]). Given a coherent sheaf $\mathcal{F}$ on X, the following conditions are equivalent:
(a) codim $\mathrm{Supp}(R^i\widehat{\mathcal{S}}\mathcal{F}) \geq i$ for all $i > 0$.
(b) codim $V^i(\mathcal{F}) \geq i$ for all $i > 0$.
If one of the above conditions is satisfied, $\mathcal{F}$ is called a *GV-sheaf*. (The proof of the equivalence is a standard base-change argument – cf. [**PP4**] Lemma 3.6.)

Notation/Terminology 2.2. (a) (IT_0-sheaf). The simplest examples of GV-sheaves are those such that $V^i(\mathcal{F}) = \emptyset$ for every $i > 0$. In this case $\mathcal{F}$ is said to *satisfy the Index Theorem with index 0* (IT_0 for short). If $\mathcal{F}$ is IT_0 then $\mathbf{R}\widehat{\mathcal{S}}\mathcal{F} = R^0\widehat{\mathcal{S}}\mathcal{F}$, which is a locally free sheaf.
(b) (Weak Index Theorem). Let $\mathcal{G}$ be an object in $\mathbf{D}(X)$ and $k \in \mathbf{Z}$. $\mathcal{G}$ is said *to satisfy the Weak Index Theorem with index k* (WIT_k for short), if $R^i\widehat{\mathcal{S}}\mathcal{G} = 0$ for $i \neq k$. In this case we denote $\widehat{\mathcal{G}} = R^k\widehat{\mathcal{S}}\mathcal{G}$. Hence $\mathbf{R}\widehat{\mathcal{S}}\mathcal{G} = \widehat{\mathcal{G}}[-k]$.
(c) The same terminology and notation holds for sheaves on $\widehat{X}$, or more generally objects in $\mathbf{D}(\widehat{X})$, by considering the functor $\mathbf{R}\mathcal{S}$.

We now state a basic result from [**PP4**] only in the special case of abelian varieties considered in this paper. In this case, with the exception of the implications from (1) to the other parts, it was in fact proved earlier by Hacon [**Hac**]. We denote $\mathbf{R}\Delta\mathcal{F} := \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Theorem 2.3. *Let X be an abelian variety of dimension g and $\mathcal{F}$ a coherent sheaf on X. Then the following are equivalent:*
(1) $\mathcal{F}$ is a GV-sheaf.
(2) For any sufficiently positive ample line bundle A on $\widehat{X}$,

$$H^i(\mathcal{F} \otimes \widehat{A^{-1}}) = 0, \text{ for all } i > 0.$$

(3) $\mathbf{R}\Delta\mathcal{F}$ satisfies WIT_g.

Proof. This is Corollary 3.10 of [**PP4**], with the slight difference that conditions (1), (2) and (3) are all stated with respect to the Poincaré line bundle P, while condition (3) of Corollary 3.10 of *loc. cit.* holds with respect to $P^\vee$. This can be done since, on abelian varieties, the Poincaré bundle satisfies the symmetry relation $P^\vee \cong ((-1_X) \times 1_{\widehat{X}})^*P$. Therefore Grothendieck duality (cf. Lemma 2.5 below) gives that the Fourier-Mukai functor defined by $P^\vee$ on $X \times \widehat{X}$ is the same as $(-1_X)^* \circ \mathbf{R}\widehat{\mathcal{S}}$. We can also assume without loss of generality that the ample line bundle A on $\widehat{X}$ considered below is symmetric. □

Remark 2.4. The above Theorem holds in much greater generality ([**PP4**], Corollary 3.10). Moreover, in [**PP5**] we will show that the equivalence between (1) and (3) holds in a local setting as well. Condition (2) is a Kodaira-Kawamata-Viehweg-type vanishing criterion. This is because, up to an étale cover of X, the vector bundle $\widehat{A^{-1}}$ is a direct sum of copies of an ample line bundle (cf. [**Hac**], and also [**PP4**] and the proof of Theorem 4.1 in the sequel).

Lemma 2.5 ([**Muk1**] 3.8). *The Fourier-Mukai and duality functors satisfy the exchange formula:*

$$\mathbf{R}\Delta \circ \mathbf{R}\widehat{\mathcal{S}} \cong (-1_{\widehat{X}})^* \circ \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\Delta[g].$$

A useful immediate consequence of the equivalence of (a) and (c) of Theorem 2.3, together with Lemma 2.5, is the following (cf. [**PP4**], Remark 3.11.):

Corollary 2.6. *If $\mathcal{F}$ is a GV-sheaf on X then*

$$R^i\widehat{\mathcal{S}}\mathcal{F} \cong \mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{X}}).$$

M-regular sheaves and their characterization. We now recall the M-regularity condition, which is simply a stronger (by one) generic vanishing condition, and relate it to the notion of GV-sheaf. The reason for the different terminology is that the notion of M-regularity was discovered – in connection with many geometric applications – before fully appreciating its relationship with generic vanishing theorems (see [**PP1**], [**PP2**], [**PP3**]).

Proposition/Definition 2.7. Let $\mathcal{F}$ be a coherent sheaf on an abelian variety X. The following conditions are equivalent:
(a) codim $\mathrm{Supp}(R^i\widehat{\mathcal{S}}\mathcal{F}) > i$ for all $i > 0$.
(b) codim $V^i(\mathcal{F})i$ for all $i > 0$.
If one of the above conditions is satisfied, $\mathcal{F}$ is called an *M-regular sheaf.*

The proof is identical to that of Proposition/Definition 2.1. By definition, every M-regular sheaf is a GV-sheaf. Non-regular GV-sheaves are those whose support loci have dimension as big as possible. As shown by the next result, as a consequence of the Auslander-Buchsbaum theorem, this is equivalent to the presence of torsion in the Fourier transform of the Grothendieck dual object.

Proposition 2.8. *Let X be an abelian variety of dimension g, and let $\mathcal{F}$ be a GV-sheaf on X. The following conditions are equivalent:*
(1) $\mathcal{F}$ is M-regular.
(2) $\widehat{\mathbf{R}\Delta\mathcal{F}} = \mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F})[g]$ is a torsion-free sheaf.[1]

Proof. By Corollary 2.6, $\mathcal{F}$ is M-regular if and only if for each $i > 0$

$$\text{codim Supp}(\mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{X}})) > i.$$

The theorem is then a consequence of the following commutative algebra fact, which is surely known to the experts. □

Lemma 2.9. *Let $\mathcal{G}$ be a coherent sheaf on a smooth variety X. Then $\mathcal{G}$ is torsion-free if and only if* codim $\mathrm{Supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_X)) > i$ *for all* $i > 0$.

Proof. If $\mathcal{G}$ is torsion-free then it is a subsheaf of a locally free sheaf $\mathcal{E}$. From the exact sequence

$$0 \longrightarrow \mathcal{G} \to \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{G} \longrightarrow 0$$

it follows that, for $i > 0$, $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{X}}) \cong \mathcal{E}xt^{i+1}(\mathcal{E}/\mathcal{G}, \mathcal{O}_{\widehat{X}})$. But then a well-known consequence of the Auslander-Buchsbaum Theorem applied to $\mathcal{E}/\mathcal{G}$ implies that

$$\text{codim Supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{X}})) > i, \text{ for all } i > 0.$$

Conversely, since X is smooth, the functor $\mathbf{R}\mathcal{H}om(\,\cdot\,, \mathcal{O}_X)$ is an involution on $\mathbf{D}(X)$. Thus there is a spectral sequence

$$E_2^{ij} := \mathcal{E}xt^i\Big((\mathcal{E}xt^j(\mathcal{G}, \mathcal{O}_X), \mathcal{O}_X\Big) \Rightarrow H^{i-j} = \mathcal{H}^{i-j}\mathcal{G} = \begin{cases} \mathcal{G} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

[1] Note that it is a sheaf by Theorem 2.3.

If codim $\mathrm{Supp}(\mathcal{E}xt^i(\mathcal{G},\mathcal{O}_X)) > i$ for all $i > 0$, then $\mathcal{E}xt^i\Big(\mathcal{E}xt^j(\mathcal{G},\mathcal{O}_X),\mathcal{O}_X\Big) = 0$ for all i,j such that $j > 0$ and $i - j \leq 0$, so the only E_∞^{ii} term which might be non-zero is E_∞^{00}. But the differentials coming into E_p^{00} are always zero, so we get a sequence of inclusions

$$\mathcal{F} = H^0 = E_\infty^{00} \subset \cdots \subset E_3^{00} \subset E_2^{00}.$$

The extremes give precisely the injectivity of the natural map $\mathcal{G} \to \mathcal{G}^{**}$. Hence $\mathcal{G}$ is torsion-free. □

Remark 2.10. It is worth noting that in the previous proof, the fact that we are working on an abelian variety is of no importance. In fact, an extension of Proposition 2.8 holds in the generality of [**PP4**], and even in a local setting, as it will be shown in [**PP5**].

3. Tensor products of GV and M-regular sheaves

We now address the issue of preservation of bounds on the codimension of support loci under tensor products. Our main result in this direction is (2) of Theorem 3.2 below, namely that the tensor product of two M-regular sheaves on an abelian variety is M-regular, provided that one of them is locally free. Note that the same result holds for Castelnuovo-Mumford regularity on projective spaces ([**La2**], Proposition 1.8.9). We do not know whether the same holds if one removes the local freeness condition on $\mathcal{E}$ (in the case of Castelnuovo-Mumford regularity it does not).

Unlike the previous section, the proof of the result is quite specific to abelian varieties. One of the essential ingredients is Mukai's main inversion result (cf. [**Muk1**], Theorem 2.2), which states that the functor $\mathbf{R}\widehat{\mathcal{S}}$ is an equivalence of derived categories and, more precisely,

$$\mathbf{R}\mathcal{S} \circ \mathbf{R}\widehat{\mathcal{S}} \cong (-1_A)^*[-g] \ \text{ and } \ \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\mathcal{S} \cong (-1_{\widehat{A}})^*[-g]. \tag{1}$$

Besides this, the argument uses the characterization of M-regularity among GV-sheaves given by Proposition 2.8.

Proposition 3.1. *Let $\mathcal{F}$ be a GV-sheaf and H a locally free sheaf satisfying IT_0 on an abelian variety X. Then $\mathcal{F} \otimes H$ satisfies IT_0.*

Proof. Consider any $\alpha \in \mathrm{Pic}^0(X)$. Note that $H \otimes \alpha$ also satisfies IT_0, so $\mathbf{R}\widehat{\mathcal{S}}(H \otimes \alpha) = R^0\widehat{\mathcal{S}}(H \otimes \alpha)$ is a vector bundle N_α on $\widehat{X}$. By Mukai's inversion theorem (1) N_α satisfies WIT_g with respect to $\mathbf{R}\mathcal{S}$ and $H \otimes \alpha \cong \mathbf{R}\mathcal{S}((-1_X)^*N_\alpha)[g]$. Consequently for all i we have

$$H^i(X, \mathcal{F} \otimes H \otimes \alpha) \cong H^i(X, \mathcal{F} \otimes \mathbf{R}\mathcal{S}((-1_{\widehat{X}})^*N_\alpha)[g]). \tag{2}$$

But a basic exchange formula for integral transforms ([**PP4**], Lemma 2.1) states, in the present context, that

$$H^i(X, \mathcal{F} \otimes \mathbf{R}\mathcal{S}((-1_{\widehat{X}})^*N_\alpha)[g]) \cong H^i(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \underline{\otimes} (-1_{\widehat{X}})^*N_\alpha[g]). \tag{3}$$

Putting (2) and (3) together, we get that

$$H^i(X, \mathcal{F} \otimes H \otimes \alpha) \cong H^i(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \underline{\otimes} (-1_{\widehat{X}})^*N_\alpha[g]) = H^{g+i}(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \underline{\otimes} (-1_{\widehat{X}})^*N_\alpha). \tag{4}$$

The hypercohomology groups on the right hand side are computed by the spectral sequence

$$E_2^{jk} := H^j(Y, R^k\mathcal{S}\mathcal{F} \otimes (-1_{\widehat{X}})^* N_\alpha[g]) \Rightarrow H^{j+k}(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F}\underline{\otimes}(-1_{\widehat{X}})^* N_\alpha[g]).$$

Since $\mathcal{F}$ is GV, we have the vanishing of $H^j(Y, R^k\mathcal{S}\mathcal{F}\otimes(-1_{\widehat{X}})^*N_\alpha[g])$ for $j+k > g$, and from this it follows that the hypercohomology groups in (4) are zero for $i > 0$. □

Theorem 3.2. *Let X be an abelian variety, and $\mathcal{F}$ and $\mathcal{E}$ two coherent sheaves on X, with $\mathcal{E}$ locally free.*
(1) If $\mathcal{F}$ and $\mathcal{E}$ are GV-sheaves, then $\mathcal{F}\otimes\mathcal{E}$ is a GV-sheaf.
(2) If $\mathcal{F}$ and $\mathcal{E}$ are M-regular, then $\mathcal{F}\otimes\mathcal{E}$ is M-regular.

Proof. (1) Let A be a sufficiently ample line bundle on $\widehat{X}$. By Proposition 3.1 $\mathcal{E}\otimes\widehat{A^{-1}}$ satisfies IT_0, and then $(\mathcal{F}\otimes\mathcal{E})\otimes\widehat{A^{-1}}$ satisfies IT_0 as well. Applying Theorem 2.3(2), we deduce that $\mathcal{F}\otimes\mathcal{E}$ is GV.
(2) Both $\mathcal{F}$ and $\mathcal{E}$ are GV, so (1) implies that $\mathcal{F}\otimes\mathcal{E}$ is also a GV. We use Proposition 2.8. This implies to begin with that $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F})$ and $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{E}) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathcal{E}^\vee)$ are torsion-free sheaves (we harmlessly forget about what degree they live in). Going backwards, it also implies that we are done if we show that $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta(\mathcal{F}\otimes\mathcal{E}))$ is torsion-free. But note that

$$\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta(\mathcal{F}\otimes\mathcal{E})) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F}\otimes\mathcal{E}^\vee) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F})\underline{*}\mathbf{R}\widehat{\mathcal{S}}(\mathcal{E}^\vee)$$

where $\underline{*}$ denotes the (derived) Pontrjagin product of sheaves on abelian varieties, and the last isomorphism is the exchange of Pontrjagin and tensor products under the Fourier-Mukai functor (cf. [**Muk1**] (3.7)). Note that this derived Pontrjagin product is in fact an honest Pontrjagin product, as we know that all the objects above are sheaves. Recall that by definition the Pontrjagin product of two sheaves $\mathcal{G}$ and $\mathcal{H}$ is simply $\mathcal{G}*\mathcal{H} := m_*(p_1^*\mathcal{G}\otimes p_2^*\mathcal{H})$, where $m : \widehat{X}\times\widehat{X}\to\widehat{X}$ is the group law on $\widehat{X}$. Since m is a surjective morphism, if $\mathcal{G}$ and $\mathcal{H}$ are torsion-free, then so is $p_1^*\mathcal{G}\otimes p_2^*\mathcal{H}$ and its push-forward $\mathcal{G}*\mathcal{H}$. □

Remark 3.3. As mentioned in §2, Generic Vanishing conditions can be naturally defined for objects in the derived category, rather than sheaves (see [**PP4**]). In this more general setting, (1) of Theorem 3.2 holds for $\mathcal{F}\underline{\otimes}\mathcal{G}$, where $\mathcal{F}$ is any GV-*object* and $\mathcal{E}$ any GV-*sheaf*, while (2) holds for $\mathcal{F}$ any M-regular *object* and $\mathcal{E}$ any M-regular *locally free sheaf*. The proof is the same.

4. Nefness of GV-sheaves

Debarre has shown in [**De2**] that every M-regular sheaf on an abelian variety is ample. We deduce from this and Theorem 2.3 that GV-sheaves satisfy the analogous weak positivity.

Theorem 4.1. *Every GV-sheaf on an abelian variety is nef.*

Proof. *Step 1.* We first reduce to the case when the abelian variety X is principally polarized. For this, consider A any ample line bundle on $\widehat{X}$. By Theorem 2.3 we know that the GV-condition is equivalent to the vanishing

$$H^i(\mathcal{F}\otimes\widehat{A^{-m}}) = 0, \text{ for all } i > 0, \text{ and all } m \gg 0.$$

But A is the pullback $\hat{\psi}^*L$ of a principal polarization L via an isogeny $\hat{\psi} : \widehat{X} \to \widehat{Y}$ (cf. [**LB**] Proposition 4.1.2). We then have

$$0 = H^i(\mathcal{F} \otimes \widehat{A^{-m}}) \cong H^i(\mathcal{F} \otimes (\widehat{\hat{\psi}^*(L^{-m})})) \cong H^i(\mathcal{F} \otimes \psi_* \widehat{L^{-m}}) \cong H^i(\psi^*\mathcal{F} \otimes \widehat{L^{-m}}).$$

Here ψ denotes the dual isogeny. (The only thing that needs an explanation is the next to last isomorphism, which is the commutation of the Fourier-Mukai functor with isogenies, [**Muk1**] 3.4.) But this implies that $\psi^*\mathcal{F}$ is also GV, and since nefness is preserved by isogenies this completes the reduction step.

Step 2. Assume now that X is principally polarized by Θ (and we use the same notation for a principal polarization on $\widehat{X}$, via the standard identification). For any $m \geq 1$ we have that $\mathcal{O}(-m\Theta)$ satisfies IT_g on $\widehat{X}$, with $g = \dim X$. Hence $\widehat{\mathcal{O}(-m\Theta)}$ is locally free and satisfies IT_0 on X, so the same is in fact true for $\widehat{\mathcal{O}(-m\Theta)} \otimes \alpha$ for any $\alpha \in \mathrm{Pic}^0(X)$. By Proposition 3.1 we get

$$H^i(\mathcal{F} \otimes \widehat{\mathcal{O}(-m\Theta)} \otimes \alpha) = 0, \text{ for all } i > 0, \text{ all } \alpha \in \mathrm{Pic}^0(X) \text{ and all } m \gg 0.$$

If we denote by $\phi_m : X \to X$ multiplication by m, i.e. the isogeny induced by $m\Theta$, then this implies that

$$H^i(\phi_m^*\mathcal{F} \otimes \mathcal{O}(m\Theta) \otimes \beta) = 0, \text{ for all } i > 0 \text{ and all } \beta \in \mathrm{Pic}^0(X)$$

as $\phi_m^*\widehat{\mathcal{O}(-m\Theta)} \cong \bigoplus \mathcal{O}(m\Theta)$ by [**Muk1**] Proposition 3.11(1). This means that the sheaf $\phi_m^*\mathcal{F} \otimes \mathcal{O}(m\Theta)$ satisfies IT_0 on X, so in particular it is M-regular. By Debarre's result [**De2**] Corollary 3.2, it is then ample.

But ϕ_m is a finite cover, and $\phi_m^*\Theta \equiv m^2\Theta$. The statement above is then same as saying that, in the terminology of [**La2**] §6.2, the $\mathbb{Q}$-twisted[2] sheaf $\mathcal{F} < \frac{1}{m} \cdot \Theta >$ on X is ample, since $\phi_m^*(\mathcal{F} < \frac{1}{m} \cdot \Theta >)$ is an honest ample sheaf. As m goes to ∞, we see that $\mathcal{F}$ is a limit of ample $\mathbb{Q}$-twisted sheaves, and so it is nef by [**La2**] Proposition 6.2.11. □

Combining the result above with the fact that higher direct images of canonical bundles are GV (cf. [**PP3**] Theorem 5.9), we obtain the following result, one well-known instance of which is that the canonical bundle of any smooth subvariety of an abelian variety is nef.

Corollary 4.2. *Let X be a smooth projective variety and $a : Y \to X$ a (not necessarily surjective) morphism to an abelian variety. Then $R^j a_*\omega_Y$ is a nef sheaf on X for all j.*

One example of an immediate application of Corollary is to integrate a result of Peternell-Sommese in the general picture. For a finite surjective morphism $f : Y \to X$, we denote by E_f the dual of the kernel of the (split) trace map $f_*\mathcal{O}_Y \to \mathcal{O}_X$, so that

$$f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus E_f^\vee.$$

(Cf. the Introduction to [**De2**] for a discussion of the significance of this vector bundle in the topological study of coverings.)

Corollary 4.3 ([**PS**], Theorem 1.17). *Let $a : Y \to X$ be a finite surjective morphism of smooth projective varieties, with X an abelian variety. Then the vector bundle E_a is nef.*

[2]Note that the twist is indeed only up to numerical equivalence.

Proof. By duality we have $a_*\omega_Y \cong \mathcal{O}_X \oplus E_a$. Thus E_a is a quotient of $a_*\omega_Y$, so by Corollary 4.2 it is nef. $\square$

5. Generation properties of M-regular sheaves on abelian varieties

The interest in the notion of M-regularity comes from the fact that M-regular sheaves on abelian varieties have strong generation properties. In this respect, M-regularity on abelian varieties parallels the notion of Castelnuovo-Mumford regularity on projective spaces (cf. the survey [**PP3**]). In this section we survey the basic results about generation properties of M-regular sheaves. The presentation is somewhat new, since the proof of the basic result (the implication $(a) \Rightarrow (b)$ of Theorem 5.1 below) makes use of the relationship between M-regularity and GV-sheaves (Proposition 2.8). The argument in this setting turns out to be more natural, and provides as a byproduct the reverse implication $(b) \Rightarrow (a)$, which is new.

Another characterization of M-regularity. M-regular sheaves on abelian varieties are characterized as follows:

Theorem 5.1. *([**PP1**], Theorem 2.5) Let $\mathcal{F}$ be a GV-sheaf on an abelian variety X of dimension g. Then the following conditions are equivalent:*
(a) $\mathcal{F}$ is M-regular.
(b) For every locally free sheaf H on X satisfying IT_0, and for every non-empty Zariski open set $U \subset \widehat{X}$, the sum of multiplication maps of global sections

$$\mathcal{M}_U : \bigoplus_{\alpha \in U} H^0(X, \mathcal{F} \otimes \alpha^{-1}) \otimes H^0(X, H \otimes \alpha) \xrightarrow{\oplus m_\alpha} H^0(X, \mathcal{F} \otimes H)$$

is surjective.

Proof. Since $\mathcal{F}$ is a GV-sheaf, by Theorem 2.3 the transform of $\mathbf{R}\Delta\mathcal{F}$ is a sheaf in degree g, i.e. $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F}) = \widehat{\mathbf{R}\Delta\mathcal{F}}[-g]$. If H is a coherent sheaf satisfying IT_0 then $\mathbf{R}\widehat{\mathcal{S}}H = \widehat{H}$, a locally free sheaf in degree 0. It turns out that the following natural map is an isomorphism

$$\mathrm{Ext}^g(H, \mathbf{R}\Delta\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}). \tag{5}$$

This simply follows from Mukai's Theorem (1), which yields that

$$\mathrm{Ext}^g(H, \mathbf{R}\Delta F) = \mathrm{Hom}_{\mathbf{D}(X)}(H, \mathbf{R}\Delta\mathcal{F}[g]) \cong \mathrm{Hom}_{\mathbf{D}(\widehat{X})}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) = \mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}).$$

Proof of (a) $\Rightarrow$ (b). Since $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is torsion-free by Proposition 2.8, the evaluation map at the fibres

$$\mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \to \prod_{\alpha \in U} \mathcal{H}om(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes_{\mathcal{O}_{\widehat{X},\alpha}} k(\alpha) \tag{6}$$

is injective for all open sets $U \subset \mathrm{Pic}^0(X)$. Therefore, composing with the isomorphism (5), we get an injection

$$\mathrm{Ext}^g(H, \mathbf{R}\Delta F) \to \prod_{\alpha \in U} \mathcal{H}om(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes_{\mathcal{O}_{\widehat{X},\alpha}} k(\alpha). \tag{7}$$

By base-change for the top direct image of the complex $\mathbf{R}\Delta\mathcal{F}$ and Serre duality, this is the dual map of the map in (b), which is therefore surjective.

Proof of (b) $\Rightarrow$ (a). Let A be an ample symmetric line bundle on $\widehat{X}$. From Mukai's Theorem (1), it follows that $A^{-1} = \widehat{H_A}$, where H_A is a locally free sheaf on X

satisfying IT_0 and such that $\widehat{H_A} = A^{-1}$. We have that (b) is equivalent to the injectivity of (7). We now take $H = H_A$ in both (5) and (7). The facts that (5) is an isomorphism and that (7) is injective yield the injectivity, *for all open sets* $U \subset \mathrm{Pic}^0(X)$, of the evaluation map at fibers

$$H^0(\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A) \xrightarrow{ev_U} \prod_{\alpha \in U} (\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A) \otimes_{\mathcal{O}_{\widehat{X},\alpha}} k(\alpha).$$

Letting A be sufficiently positive so that $\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A$ is globally generated, this is equivalent to the torsion-freeness of $\widehat{\mathbf{R}\Delta\mathcal{F}}$[3] and hence, by Proposition 2.8, to the M-regularity of $\mathcal{F}$. □

Continuous global generation and global generation. Recall first the following:

Definition 5.2 ([**PP1**], Definition 2.10). Let Y be a variety equipped with a morphism $a : Y \to X$ to an abelian variety X.
(a) A sheaf $\mathcal{F}$ on Y is *continuously globally generated with respect to* a if the sum of evaluation maps

$$\mathcal{E}v_U : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes a^*\alpha) \otimes a^*\alpha^{-1} \longrightarrow \mathcal{F}$$

is surjective for every non-empty open subset $U \subset \mathrm{Pic}^0(X)$.
(b) More generally, let T be a proper subvariety of Y. The sheaf $\mathcal{F}$ is said to be *continuously globally generated with respect to a away from* T if $\mathrm{Supp}(\mathrm{Coker}\,\mathcal{E}v_U) \subset T$ for every non-empty open subset $U \subset \mathrm{Pic}^0(X)$.
(c) When a is the Albanese morphism, we will suppress a from the terminology, speaking of *continuously globally generated* (resp. *continuously globally generated away from* T) sheaves.

In Theorem 5.1, taking H to be a sufficiently positive line bundle on X easily yields (cf [**PP1**], Proposition 2.13):

Corollary 5.3. *An M-regular sheaf on X is continuously globally generated.*

The relationship between continuous global generation and global generation comes from:

Proposition 5.4 ([**PP1**], Proposition 2.12). *(a) In the setting of Definition* 5*, let $\mathcal{F}$ (resp. A) be a coherent sheaf on Y (resp. a line bundle, possibly supported on a subvariety Z of Y), both continuously globally generated. Then $\mathcal{F} \otimes A \otimes a^*\alpha$ is globally generated for all $\alpha \in \mathrm{Pic}^0(X)$.*
(b) More generally, let $\mathcal{F}$ and A as above. Assume that $\mathcal{F}$ is continuously globally generated away from T and that A is continuously globally generated away from W. Then $F \otimes A \otimes a^*\alpha$ is globally generated away from $T \cup W$ for all $\alpha \in \mathrm{Pic}^0(X)$.

The proposition is proved via the classical method of *reducible sections*, i.e. those sections of the form $s_\alpha \cdot t_{-\alpha}$, where s_α (resp. $t_{-\alpha}$) belongs to $H^0(F \otimes a^*\alpha)$ (resp. $H^0(A \otimes a^*\alpha^{-1})$).

Generation properties on varieties of maximal Albanese dimension via Generic Vanishing. The above results give effective generation criteria once

[3]Note that the kernel of ev_U generates a torsion subsheaf of $\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A$ whose support is contained in the complement of U.

one has effective *Generic Vanishing* criteria ensuring that the dimension of each cohomological support locus is not too big. The main example of such a criterion is the Green-Lazarsfeld Generic Vanishing Theorem for the canonical line bundle of an irregular variety, proved in [**GL1**] and further refined in [**GL2**] using the deformation theory of cohomology groups.[4] For the purposes of this paper, it is enough to state the Generic Vanishing Theorem in the case of varieties Y of *maximal Albanese dimension*, i.e. such that the Albanese map $a : Y \to \mathrm{Alb}(Y)$ is generically finite onto its image. More generally, we consider a morphism $a : Y \to X$ to an abelian variety X. Then, as in §1 one can consider the *cohomological support loci* $V^i_a(\omega_Y) = \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(\omega_Y \otimes a^*\alpha) > 0\}$. (In case a is the Albanese map we will suppress a from the notation.)

The result of Green-Lazarsfeld (see also [**EL**] Remark 1.6) states that, *if the morphism a is generically finite, then*

$$\mathrm{codim}\ V^i_a(\omega_Y) \geq i \quad \text{for all } i > 0.$$

Moreover, in [**GL2**] it is proved that the $V^i_a(\omega_Y)$ *are unions of translates of subtori.* Finally, an argument of Ein-Lazarsfeld [**EL**] yields that, *if there exists an $i > 0$ such that* codim $V^i_a(\omega_Y) = i$, *then the image of a is ruled by subtori of X.* All of this implies the following typical application of the concept of M-regularity.

Proposition 5.5. *Assume that* $\dim Y = \dim a(Y)$ *and that $a(Y)$ is not ruled by tori. Let Z be the exceptional locus of a, i.e. the inverse image via a of the locus of points in $a(Y)$ having non-finite fiber. Then:*
(i) $a_*\omega_Y$ is an M-regular sheaf on X.
(ii) $a_*\omega_Y$ is continuously globally generated.
(iii) ω_Y is continuously globally generated away from Z.
(iv) For all $k \geq 2$, $\omega_Y^{\otimes k} \otimes a^*\alpha$ is globally generated away from Z for any $\alpha \in \mathrm{Pic}^0(X)$.

Proof. By Grauert-Riemenschneider vanishing, $R^i a_*\omega_Y = 0$ for all $i \neq 0$. By the Projection Formula we get $V^i_a(\omega_Y) = V^i(a_*\omega_Y)$. Combined with the Ein-Lazarsfeld result, (i) follows. Part (ii) follows from Corollary 5.3. For (iii) note that, as with global generation (and by a similar argument), continuous global generation is preserved by finite maps: if a is finite and $a_*\mathcal{F}$ is continuously globally generated, then $\mathcal{F}$ is continuously globally generated. (iv) for $k = 2$ follows from (iii) and Proposition 5.4. For arbitrary $k \geq 2$ it follows in the same way by induction (note that if a sheaf $\mathcal{F}$ is such that $\mathcal{F} \otimes a^*\alpha$ is globally generated away from Z for every $\alpha \in \mathrm{Pic}^0(X)$, then it is continuously globally generated away from Z). □

6. Pluricanonical maps of irregular varieties of maximal Albanese dimension

One of the most elementary results about projective embeddings is that every curve of general type can be embedded in projective space by the tricanonical line bundle. This is sharp for curves of genus two. It turns out that this result can be generalized to arbitrary dimension, namely to varieties of maximal Albanese dimension. In fact, using Vanishing and Generic Vanishing Theorems and the Fourier-Mukai transform, Chen and Hacon proved that *for every smooth complex*

[4]More recently, Hacon [**Hac**] has given a different proof, based on the Fourier-Mukai transform and Kodaira Vanishing. Building in part on Hacon's ideas, several extensions of this result are given in [**PP4**].

variety of general type and maximal Albanese dimension Y such that $\chi(\omega_Y) > 0$, the tricanonical line bundle $\omega_Y^{\otimes 3}$ gives a birational map (cf. [**CH**], Theorem 4.4). The main point of this section is that the concept of M-regularity (combined of course with vanishing results) provides a quick and conceptually simple proof of on one hand a slightly more explicit version of the Chen-Hacon Theorem, but on the other hand under a slightly more restrictive hypothesis. We show the following:

THEOREM 6.1. *Let Y be a smooth projective complex variety of general type and maximal Albanese dimension. If the Albanese image of Y is not ruled by tori, then $\omega_Y^{\otimes 3}$ is very ample away from the exceptional locus of the Albanese map.*

Here the *exceptional locus* of the Albanese map $a : Y \to \mathrm{Alb}(Y)$ is $Z = a^{-1}(T)$, where T is the locus of points in $\mathrm{Alb}(Y)$ over which the fiber of a has positive dimension.

REMARK 6.2. A word about the hypothesis of the Chen-Hacon Theorem and of Theorem 6.1 is in order. As a consequence of the Green-Lazarsfeld Generic Vanishing Theorem (end of §5), it follows that $\chi(\omega_Y) \geq 0$ for every variety Y of maximal Albanese dimension. Moreover, Ein-Lazarsfeld [**EL**] prove that for Y of maximal Albanese dimension, *if $\chi(\omega_Y) = 0$, then $a(Y)$ is ruled by subtori of* $\mathrm{Alb}(Y)$. In dimension ≥ 3 there exist examples of varieties of general type and maximal Albanese dimension with $\chi(\omega_Y) = 0$ (cf. *loc. cit.*).

In the course of the proof we will invoke $\mathcal{J}(Y, \| L \|)$, the *asymptotic multiplier ideal sheaf* associated to a complete linear series $|L|$ (cf. [**La2**] §11). One knows that, given a line bundle L of non-negative Iitaka dimension,

$$H^0(Y, L \otimes \mathcal{J}(\| L \|)) = H^0(Y, L), \tag{8}$$

i.e. the zero locus of $\mathcal{J}(\| L \|)$ is contained in the base locus of $|L|$ ([**La2**], Proposition 11.2.10). Another basic property we will use is that, for every k,

$$\mathcal{J}(\| L^{\otimes(k+1)} \|) \subseteq \mathcal{J}(\| L^{\otimes k} \|). \tag{9}$$

(Cf. [**La2**], Theorem 11.1.8.) A first standard result is

LEMMA 6.3. *Let Y be a smooth projective complex variety of general type. Then:*
(a) $h^0(\omega_Y^{\otimes m} \otimes \alpha)$ is constant for all $\alpha \in \mathrm{Pic}^0(Y)$ and for all $m > 1$.
(b) The zero locus of $\mathcal{J}(\| \omega_Y^{\otimes(m-1)} \|)$ is contained in the base locus of $\omega_Y^{\otimes m} \otimes \alpha$, for all $\alpha \in \mathrm{Pic}^0(Y)$.

PROOF. Since bigness is a numerical property, all line bundles $\omega_Y \otimes \alpha$ are big, for $\alpha \in \mathrm{Pic}^0(Y)$. By Nadel Vanishing for asymptotic multiplier ideals ([**La2**], Theorem 11.2.12)

$$H^i(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) = 0$$

for all $i > 0$ and all $\alpha, \beta \in \mathrm{Pic}^0(X)$. Therefore, by the invariance of the Euler characteristic,

$$h^0(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) = \text{constant} = \lambda_\alpha$$

for all $\beta \in \mathrm{Pic}^0(Y)$. Now

$$h^0(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) \leq h^0(Y, \omega_Y^m \otimes \beta)$$

for all $\beta \in \mathrm{Pic}^0(X)$ and, because of (8) and (9), equality holds for $\beta = \alpha^m$. By semicontinuity it follows that $h^0(Y, \omega_Y^{\otimes m} \otimes \beta) = \lambda_\alpha$ for all β contained in a Zariski open set U_α of $\mathrm{Pic}^0(X)$ which contains α^m. Since this is true for all α, the statement follows. Part (b) follows from the previous argument. □

LEMMA 6.4. *Let Y be a smooth projective complex variety of general type and maximal Albanese dimension, such that its Albanese image is not ruled by tori. Let Z be the exceptional locus of its Albanese map. Then, for every $\alpha \in \mathrm{Pic}^0(Y)$:*
(a) the zero-locus of $\mathcal{J}(\| \omega_Y \otimes \alpha \|)$ is contained (set-theoretically) in Z.
(b) $\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)$ is globally generated away from Z.

PROOF. (a) By (8) and (9) the zero locus of $\mathcal{J}(\| \omega \otimes \alpha \|)$ is contained in the base locus of $\omega^{\otimes 2} \otimes \alpha^2$. By Proposition 5.5, the base locus of $\omega^{\otimes 2} \otimes \alpha^2$ is contained Z. (b) Again by Proposition 5.5, the base locus of $\omega^{\otimes 2} \otimes \alpha$ is contained in Z. By Lemma 6.3(b), the zero locus of $\mathcal{J}(\| \omega_Y \|)$ is contained in Z. □

PROOF. *(of Theorem* 6.1*)* As above, let $a : Y \to \mathrm{Alb}(Y)$ be the Albanese map and let Z be the exceptional locus of a. As in the proof of Prop. 5.5, the Ein-Lazarsfeld result at the end of §3 (see also Remark 6.2), the hypothesis implies that $a_*\omega_Y$ is M-regular, so ω_Y is continuously globally generated away from Z. We make the following:
Claim. *For every $y \in Y - Z$, the sheaf $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is M-regular.*
We first see how the Claim implies Theorem 6.1. The statement of the Theorem is equivalent to the fact that, *for any $y \in Y - Z$, the sheaf $I_y \otimes \omega_Y^{\otimes 3}$ is globally generated away from Z.* By Corollary 5.3, the Claim yields that $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is continuously globally generated. Therefore $I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|)$ is continuously globally generated away from Z. Hence, by Proposition 5.4, $I_y \otimes \omega_Y^{\otimes 3} \otimes \mathcal{J}(\| \omega_Y \|)$ is globally generated away from Z. Since the zero locus of $\mathcal{J}(\| \omega_Y \|)$ is contained in Z (by Lemma 6.4)(a)), the Theorem follows from the Claim.
Proof of the Claim. We consider the standard exact sequence
(10)
$$0 \to I_y \otimes \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|) \to \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|) \to (\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|))_{|y} \to 0.$$

(Note that y does not lie in the zero locus of $\mathcal{J}(\| \omega_Y \|)$.) By Nadel Vanishing for asymptotic multiplier ideals, $H^i(Y, \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)) = 0$ for all $i > 0$ and $\alpha \in \mathrm{Pic}^0(Y)$. Since, by Lemma 6.4, y is not in the base locus of $\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)$, taking cohomology in (10) it follows that

$$H^i(Y, I_y \otimes \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)) = 0 \tag{11}$$

for all $i > 0$ and $\alpha \in \mathrm{Pic}^0(X)$ as well. Since y does not belong to the exceptional locus of a, the map $a_*(\omega_Y^2 \otimes \mathcal{J}(\| \omega_Y \|)) \to a_*((\omega_Y^2 \otimes \mathcal{J}(\| \omega_Y \|))_{|y})$ is surjective. On the other hand, since a is generically finite, by a well-known extension of Grauert-Riemenschneider vanishing, $R^i a_*(\omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ vanishes for all $i > 0$.[5] Therefore (10) implies also that for all $i > 0$

$$R^i a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|)) = 0. \tag{12}$$

Combining (11) and (12) one gets, by the projection formula, that the sheaf $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is IT_0 on X, hence M-regular. □

[5]The proof of this is identical to that of the usual Grauert-Riemenschneider vanishing theorem in [**La2**] §4.3.B, replacing Kawamata-Viehweg vanishing with Nadel vanishing.

Remark 6.5. It follows from the proof that $\omega_Y^{\otimes 3} \otimes \alpha$ is very ample away from Z for all $\alpha \in \mathrm{Pic}^0(Y)$ as well.

Remark 6.6 (**The Chen-Hacon Theorem**). The reader might wonder why, according to the above quoted theorem of Chen-Hacon, the tricanonical bundle of varieties of general type and maximal Albanese dimension is birational (but not necessarily very ample outside the Albanese exceptional locus) even under the weaker assumption that $\chi(\omega_Y)$ is positive, which does not ensure the continuous global generation of $a_*\omega_Y$. The point is that, according to Generic Vanishing, if the Albanese dimension is maximal, then $\chi(\omega_Y) > 0$ implies $h^0(\omega_Y \otimes \alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0(Y)$. Hence, even if ω_Y is not necessarily continuously globally generated away from the exceptional locus of the Albanese map, the following condition holds: for *general* $y \in Y$, there is a Zariski open set $U_y \subset \mathrm{Pic}^0(Y)$ such that y is not a base point of $\omega_Y \otimes \alpha$ for all $\alpha \in U_y$. Using the same argument as in Proposition 5.4 – based on reducible sections – it follows that such y is not a base point of $\omega_Y^{\otimes 2} \otimes \alpha$ for *all* $\alpha \in \mathrm{Pic}^0(Y)$. Then the Chen-Hacon Theorem follows by an argument analogous to that of Theorem 6.1.

To complete the picture, it remains to analyze the case of varieties Y of maximal Albanese dimension and $\chi(\omega_Y) = 0$. Chen and Hacon prove that if the Albanese dimension is maximal, then $\omega_Y^{\otimes 6}$ is always birational (and $\omega_Y^{\otimes 6} \otimes \alpha$ as well). The same result can be made slightly more precise as follows, extending also results in [**PP1**] §5:

Theorem 6.7. *If Y is a smooth projective complex variety of maximal Albanese dimension then, for all $\alpha \in \mathrm{Pic}^0(Y)$, $\omega_Y^{\otimes 6} \otimes \alpha$ is very ample away from the exceptional locus of the Albanese map. Moreover, if L a big line bundle on Y, then $(\omega_Y \otimes L)^{\otimes 3} \otimes \alpha$ gives a birational map.*

The proof is similar to that of Theorem 6.1, and left to the interested reader. For example, for the first part the point is that, by Nadel Vanishing for asymptotic multiplier ideals, $H^i(Y, \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)) = 0$ for all $\alpha \in \mathrm{Pic}^0(Y)$. Hence, by the same argument using Grauert-Riemenschneider vanishing, $a_*(\omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is M-regular.

Finally, we remark that in [**CH**], Chen-Hacon also prove effective birationality results for pluricanonical maps of irregular varieties of arbitrary Albanese dimension (as a function of the minimal power for which the corresponding pluricanonical map on the general Albanese fiber is birational). It is likely that the methods above apply to this context as well.

7. Further applications of M-regularity

7.1. M-regularity indices and Seshadri constants. Here we express a natural relationship between Seshadri constants of ample line bundles on abelian varieties and the M-regularity indices of those line bundles as defined in [**PP2**]. This result is a theoretical improvement of the lower bound for Seshadri constants proved in [**Nak**]. In the opposite direction, combined with the results of [**La1**], it provides bounds for controlling M-regularity. For a general overview of Seshadri constants, in particular the statements used below, one can consult [**La2**] Ch.I §5. Note only that one way to interpret the Seshadri constant of a line bundle L at a point x is

$$\epsilon(L,x) = \sup_n \frac{s(L^n,x)}{n}, \tag{13}$$

where for a line bundle A we denote by $s(A,x)$ the largest number $s \geq 0$ such that A separates s-jets at x. On abelian varieties this is independent of the chosen point, and we denote $\epsilon(L) = \epsilon(L,x)$ for any $x \in X$.

We start by recalling the basic definition from [**PP2**] and by also looking at a slight variation. We will denote by X an abelian variety of dimension g over an algebraically closed field and by L an ample line bundle on X. Given $x \in X$, we denote by $m_x \subset \mathcal{O}_X$ its ideal sheaf.

Definition 7.1. The *M-regularity index* of L is defined as

$$m(L) := \max\{l \mid L \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_p}^{k_p} \text{ is } M\text{−regular for all distinct } x_1, \ldots, x_p \in X \text{ with } \Sigma k_i = l\}.$$

Definition 7.2. We also define a related invariant, associated to just one given point $x \in X$:

$$p(L,x) := \max\{l \mid L \otimes m_x^l \text{ is } M\text{−regular}\}.$$

The definition does not depend on x because of the homogeneity of X, so we will denote this invariant simply by $p(L)$.

Our main interest will be in the asymptotic versions of these indices, which turn out to be related to the Seshadri constant associated to L.

Definition 7.3. The *asymptotic M-regularity index* of L and its punctual counterpart are defined as

$$\rho(L) := \sup_n \frac{m(L^n)}{n} \text{ and } \rho'(L) := \sup_n \frac{p(L^n)}{n}.$$

The main result of this section is:

Theorem 7.4. *We have the following inequalities:*

$$\epsilon(L) = \rho'(L) \geq \rho(L) \geq 1.$$

In particular $\epsilon(L) \geq \max\{m(L), 1\}$.

This improves a result of Nakamaye (cf. [**Nak**] and the references therein). Nakamaye also shows that $\epsilon(L) = 1$ for some line bundle L if and only if X is the product of an elliptic curve with another abelian variety. As explained in [**PP2**] §3, the value of $m(L)$ is reflected in the geometry of the map to projective space given by L. Here is a basic example:

Example 7.5. If L is very ample – or more generally gives a birational morphism outside a codimension 2 subset – then $m(L) \geq 2$, so by the Theorem above $\epsilon(L) \geq 2$. Note that on an arbitrary smooth projective variety very ampleness implies in general only that $\epsilon(L,x) \geq 1$ at each point.

The proof of Theorem 7.4 is a simple application Corollary 5.3 and Proposition 5.4, via the results of [**PP2**] §3. We use the relationship with the notions of k-jet ampleness and separation of jets. Recall the following:

DEFINITION 7.6. A line bundle L is called *k-jet ample*, $k \geq 0$, if the restriction map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_X/m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_p}^{k_p})$$

is surjective for any distinct points $x_1, \ldots, x_p$ on X such that $\sum k_i = k+1$. Note that if L is k-jet ample, then it separates k-jets at every point.

PROPOSITION 7.7 ([**PP2**] Theorem 3.8 and Proposition 3.5). *(i) L^n is $(n+m(L)-2)$-jet ample, so in particular $s(L^n, x) \geq n + m(L) - 2$.*
(ii) If L is k-jet ample, then $m(L) \geq k+1$.

Given (13), this points in the direction of local positivity. To establish the connection with the asymptotic invariants above we also need the following:

LEMMA 7.8. *For any $n \geq 1$ and any $x \in X$ we have $s(L^{n+1}, x) \geq m(L^n)$.*

PROOF. This follows immediately from Corollary 5.3 and Proposition 5.4: if $L^n \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_p}^{k_p}$ is M-regular, then $L^{n+1} \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_p}^{k_p}$ is globally generated, and so by [**PP2**] Lemma 3.3, L^{n+1} is $m(L)$-jet ample. □

PROOF. (*of Theorem* 7.4.) Note first that for every $p \geq 1$ we have

$$m(L^n) \geq m(L) + n - 1, \tag{14}$$

which follows immediately from the two parts of Proposition 7.7. In particular $m(L^n)$ is always at least $n-1$, and so $\rho(L) \geq 1$. Putting together the definitions, (14) and Lemma 7.8, we obtain the main inequality $\epsilon(L) \geq \rho(L)$. Finally, the asymptotic punctual index computes precisely the Seshadri constant. Indeed, by completely similar arguments as above, we have that for any ample line bundle L and any $p \geq 1$ one has

$$p(L^n) \geq s(L^n, x) \text{ and } s(L^{n+1}, x) \geq p(L^n, x).$$

The statement then follows from the definition. □

REMARK 7.9. What the proof above shows is that one can give an interpretation for $\rho(L)$ similar to that for $\epsilon(L)$ in terms of separation of jets. In fact $\rho(L)$ is precisely the "asymptotic jet ampleness" of L, namely:

$$\rho(L) = \sup_n \frac{a(L^n)}{n},$$

where $a(M)$ is the largest integer k for which a line bundle M is k-jet ample.

QUESTION 7.10. Do we always have $\epsilon(L) = \rho(L)$? Can one give independent lower bounds for $\rho(L)$ or $\rho'(L)$ (which would then bound Seshadri constants from below)?

In the other direction, there are numerous bounds on Seshadri constants, which in turn give bounds for the M-regularity indices that (at least to us) are not obvious from the definition. All of the results in [**La2**] Ch.I §5 give some sort of bound. Let's just give a couple of examples:

COROLLARY 7.11. *If $(J(C), \Theta)$ is a principally polarized Jacobian, then $m(n\Theta) \leq \sqrt{g} \cdot n$. On an arbitrary abelian variety, for any principal polarization Θ we have $m(n\Theta) \leq (g!)^{\frac{1}{g}} \cdot n$.*

Proof. It is shown in [**La1**] that $\epsilon(\Theta) \leq \sqrt{g}$. We then apply Theorem 7.4. For the other bound we use the usual elementary upper bound for Seshadri constants, namely $\epsilon(\Theta) \leq (g!)^{\frac{1}{g}}$. □

Corollary 7.12. *If (A, Θ) is a very general PPAV, then there exists at least one n such that $p(n\Theta) \geq \frac{2^{\frac{1}{g}}}{4}(g!)^{\frac{1}{g}} \cdot n$.*

Proof. Here we use the lower bound given in [**La1**] via a result of Buser-Sarnak. □

There exist more specific results on $\epsilon(\Theta)$ for Jacobians (cf. [**De1**], Theorem 7), each giving a corresponding result for $m(n\Theta)$. We can ask however:

Question 7.13. Can we calculate $m(n\Theta)$ individually on Jacobians, at least for small n, in terms of the geometry of the curve?

Example 7.14 (Elliptic curves). As a simple example, the above question has a clear answer for elliptic curves. We know that on an elliptic curve E a line bundle L is M-regular if and only if $\deg(L) \geq 1$, i.e. if and only if L is ample. From the definition of M-regularity we see then that if $\deg(L) = d > 0$, then $m(L) = d - 1$. This implies that on an elliptic curve $m(n\Theta) = n - 1$ for all $n \geq 1$. This is misleading in higher genus however; in the simplest case we have the following general statement: *If (X, Θ) is an irreducible principally polarized abelian variety of dimension at least 2, then $m(2\Theta) \geq 2$.* This is an immediate consequence of the properties of the Kummer map. The linear series $|2\Theta|$ induces a $2:1$ map of X onto its image in $\mathbf{P}^{2^g-1}$, with injective differential. Thus the cohomological support locus for $\mathcal{O}(2\Theta) \otimes m_x \otimes m_y$ consists of a finite number of points, while the one for $\mathcal{O}(2\Theta) \otimes m_x^2$ is empty.

7.2. Regularity of Picard bundles and vanishing on symmetric products. In this subsection we study the regularity of Picard bundles over the Jacobian of a curve, twisted by positive multiples of the theta divisor. Some applications to the degrees of equations cutting out special subvarieties of Jacobians are drawn in the second part. Let C be a smooth curve of genus $g \geq 2$, and denote by $J(C)$ the Jacobian of C. The objects we are interested in are the Picard bundles on $J(C)$: a line bundle L on C of degree $n \geq 2g - 1$ – seen as a sheaf on $J(C)$ via an Abel-Jacobi embedding of C into $J(C)$ – satisfies IT_0, and the Fourier-Mukai transform $E_L = \widehat{L}$ is called an *n-th Picard bundle* . When possible, we omit the dependence on L and write simply E. Note that any other such n-th Picard bundle E_M, with $M \in \mathrm{Pic}^n(C)$, is a translate of E_L. The line bundle L induces an identification between $J(C)$ and $\mathrm{Pic}^n(C)$, so that the projectivization of E – seen as a vector bundle over $\mathrm{Pic}^n(C)$ – is the symmetric product C_n (cf. [**ACGH**] Ch.VII §2).

The following Proposition is the main cohomological result needed for the proof of Theorem 7.18. It is worth noting that Picard bundles are known to be negative (i.e. with ample dual bundle), so vanishing theorems are not automatic. For simplicity we prove only a non-effective result, as the value of n does not affect the applications, but cf. Remark 7.17.

Proposition 7.15. *If $n \gg 0$, for every $1 \leq k \leq g - 1$ the vector bundle $\otimes^k E \otimes \mathcal{O}(\Theta)$ satisfies IT_0.*

Proof. [6] The first step in the proof is to record the following description of tensor powers of Picard bundles as Fourier-Mukai transforms.

Lemma 7.16. *For any $k \geq 1$, let $\pi_k : C^k \to J(C)$ be a desymmetrized Abel-Jacobi mapping and let L be a line bundle on C of degree $n \gg 0$ on C. Then $\pi_{k*}(L \boxtimes \cdots \boxtimes L)$ satisfies IT_0, and*

$$(\pi_{k*}(L \boxtimes \cdots \boxtimes L))\widehat{\ } = \otimes^k E,$$

where E is the n-th Picard bundle of C.

Proof. The first assertion is clear. Concerning the second assertion note that, by definition, $\pi_{k*}(L \boxtimes \cdots \boxtimes L)$ is the Pontrjagin product $L * \cdots * L$. Moreover, this is the same as the entire derived Pontrjagin product, as

$$R^i \pi_{k*}(L \boxtimes \cdots \boxtimes L) = 0 \text{ for all } i > 0$$

by relative Serre vanishing. But then by the exchange of derived Pontrjagin and tensor product under the Fourier-Mukai transform ([**Muk1**] (3.7)), it follows that $(L * \cdots * L)\widehat{\ } \cong \widehat{L} \otimes \cdots \otimes \widehat{L} = \otimes^k E$. □

Continuing with the proof of the Proposition, we will loosely use the notation Θ for any translate of the canonical theta divisor. The statement then becomes equivalent to the vanishing

$$h^i(\otimes^k E \otimes \mathcal{O}(\Theta)) = 0, \ \forall\, i > 0, \ \forall\, 1 \leq k \leq g-1.$$

To prove this we use the Fourier-Mukai transform. The first point is that Lemma 7.16, combined with Grothendieck duality (Lemma 2.5 above), tells us precisely that $\otimes^k E$ satisfies WIT_g and, by Mukai's inversion theorem (1) its Fourier transform is

$$\widehat{\otimes^k E} = (-1_J)^* \pi_{k*}(L \boxtimes \cdots \boxtimes L).$$

Using once again the fact that the Fourier-Mukai transform is an equivalence, we have the following sequence of isomorphisms:

$$H^i(\otimes^k E \otimes \mathcal{O}(\Theta)) \cong \mathrm{Ext}^i(\mathcal{O}(-\Theta), \otimes^k E) \cong \mathrm{Ext}^i(\widehat{\mathcal{O}(-\Theta)}, \widehat{\otimes^k E})$$

$$\cong \mathrm{Ext}^i(\mathcal{O}(\Theta), (-1_J)^* \pi_{k*}(L \boxtimes \cdots \boxtimes L)) \cong H^i((-1_J)^* \pi_{k*}(L \boxtimes \cdots \boxtimes L) \otimes \mathcal{O}(-\Theta)).$$

(Here we are using the fact that both $\mathcal{O}(-\Theta)$ and $\otimes^k E$ satisfy WIT_g and that $\widehat{\mathcal{O}(-\Theta)} = \mathcal{O}(\Theta)$.)

As we are loosely writing Θ for any translate, multiplication by -1 does not influence the vanishing, so the result follows if we show:

$$H^i(\pi_{k*}(L \boxtimes \cdots \boxtimes L) \otimes \mathcal{O}(-\Theta)) = 0, \ \forall\, i > 0.$$

But we have seen in the proof of Lemma 7.16 that $R^i \pi_{k*}(L \boxtimes \cdots \boxtimes L) = 0$ for $i > 0$, so from Leray and the Projection Formula we get that it is enough to have

$$H^i(C^k, (L \boxtimes \cdots \boxtimes L) \otimes \pi_k^* \mathcal{O}(-\Theta)) = 0, \ \forall\, i > 0.$$

This follows again from Serre vanishing. □

[6] We are grateful to Olivier Debarre for pointing out a numerical mistake in the statement, in a previous version of this paper.

REMARK 7.17. Note that the proof above works identically if one replaces $\mathcal{O}(\Theta)$ by any vector bundle satisfying IT_0. On the other hand, in the particular case of $\mathcal{O}(\Theta)$, with more work one can show that the effective bound $n > 4g-4$ suffices for Proposition 7.15 to hold. For this one first needs to apply the theorem on formal functions to show the effective vanishing of $R^i\pi_{k*}(L \boxtimes \cdots \boxtimes L) = 0$ (under very mild positivity for L) similarly to [**Ei**] Theorem 2.7 or [**Pol**] Lemma 3.3(b). Then one uses the well-known fact that

$$\pi_k^*\mathcal{O}(\Theta) \cong ((\omega_C \otimes A^{-1}) \boxtimes \cdots \boxtimes (\omega_C \otimes A^{-1})) \otimes \mathcal{O}(-\Delta),$$

where Δ is the union of all the diagonal divisors in C^k (themselves symmetric products) and A is a line bundle of degree $g-k-1$, from which the effective bound for the vanishing of the groups at the end of the proof of the Proposition follows by chasing cohomology via inductive restriction to diagonals.

An interesting consequence of the vanishing result for Picard bundles proved above is a new – and in some sense more classical – way to deduce Theorem 4.1 of [**PP1**] on the M-regularity of twists of ideal sheaves $\mathcal{I}_{W_d}$ on the Jacobian $J(C)$. This theorem has a number of applications to the equations of the W_d's inside $J(C)$, and also to vanishing results for pull-backs of theta divisors to symmetric products. For this circle of ideas we refer the reader to [**PP1**] §4. For any $1 \le d \le g-1$, $g \ge 3$, consider $u_d : C_d \longrightarrow J(C)$ to be an Abel-Jacobi mapping of the symmetric product (depending on the choice of a line bundle of degree d on C), and denote by W_d the image of u_d in $J(C)$.

THEOREM 7.18. *For every* $1 \le d \le g-1$, $\mathcal{I}_{W_d}(2\Theta)$ *satisfies* IT_0.

PROOF. We have to prove that

$$h^i(\mathcal{I}_{W_d} \otimes \mathcal{O}(2\Theta) \otimes \alpha) = 0, \ \forall\ i > 0, \ \forall\ \alpha \in \mathrm{Pic}^0(J(C)).$$

In the rest of the proof, by Θ we will understand generically any translate of the canonical theta divisor, and so α will disappear from the notation.

It is well known that W_d has a natural determinantal structure, and its ideal is resolved by an Eagon-Northcott complex. We will chase the vanishing along this complex. This setup is precisely the one used by Fulton and Lazarsfeld in order to prove for example the existence theorem in Brill-Noether theory – for explicit details on this cf. [**ACGH**] Ch.VII §2. Concretely, W_d is the "highest" degeneracy locus of a map of vector bundles

$$\gamma : E \longrightarrow F,$$

where $\mathrm{rk}\, F = m$ and $\mathrm{rk}\, E = n = m + d - g + 1$, with $m >> 0$ arbitrary. The bundles E and F are well understood: E is the n-th Picard bundle of C, discussed above, and F is a direct sum of topologically trivial line bundles. (For simplicity we are again moving the whole construction on $J(C)$ via the choice of a line bundle of degree n.) In other words, W_d is scheme theoretically the locus where the dual map

$$\gamma^* : F^* \longrightarrow E^*$$

fails to be surjective. This locus is resolved by an Eagon-Northcott complex (cf. [**Ke1**]) of the form

$$0 \to \wedge^m F^* \otimes S^{m-n}E \otimes \det E \to \cdots \to \wedge^{n+1}F^* \otimes E \otimes \det E \to \wedge^n F^* \to \mathcal{I}_{W_d} \to 0.$$

As it is known that the determinant of E is nothing but $\mathcal{O}(-\Theta)$, and since F is a direct sum of topologically trivial line bundles, the statement of the theorem follows by chopping this into short exact sequences, as long as we prove

$$h^i(S^kE\otimes\mathcal{O}(\Theta))=0,\ \forall\ i>0,\ \forall\ 1\leq k\leq m-n=g-d-1.$$

Since we are in characteristic zero, S^kE is naturally a direct summand in $\otimes^kE$, and so it is sufficient to prove that

$$h^i(\otimes^kE\otimes\mathcal{O}(\Theta))=0,\ \forall\ i>0,\ \forall\ 1\leq k\leq g-d-1.$$

But this follows from Proposition 7.15. □

Remark 7.19. Using [**PP1**] Proposition 2.9, it follows that $\mathcal{I}_{W_d}(k\Theta)$ satisfies IT_0 for all $k\geq 2$.

Remark 7.20. It is conjectured, based on a connection with minimal cohomology classes (cf. [**PP7**] for a discussion), that the only nondegenerate subvarieties Y of a principally polarized abelian variety (A,Θ) such that $\mathcal{I}_Y(2\Theta)$ satisfies IT_0 are precisely the W_d's above, in Jacobians, and the Fano surface of lines in the intermediate Jacobian of the cubic threefold.

Question 7.21. What is the minimal k such that $\mathcal{I}_{W_d^r}(k\Theta)$ is M-regular, for r and d arbitrary?

We describe below one case in which the answer can already be given, namely that of the singular locus of the Riemann theta divisor on a non-hyperelliptic Jacobian. It should be noted that in this case we do not have that $\mathcal{I}_{W^1_{g-1}}(2\Theta)$ satisfies IT_0 any longer (but rather $\mathcal{I}_{W^1_{g-1}}(3\Theta)$ does, by the same [**PP1**] Proposition 2.9).

Proposition 7.22. *$\mathcal{I}_{W^1_{g-1}}(2\Theta)$ is M-regular.*

Proof. It follows from the results of [**vGI**] that

$$h^i(\mathcal{I}_{W^1_{g-1}}\otimes\mathcal{O}(2\Theta)\otimes\alpha)=\begin{cases}0 & \text{for } i\geq g-2,\ \forall\alpha\in\mathrm{Pic}^0(J(C))\\ 0 & \text{for } 0<i<g-2,\ \forall\alpha\in\mathrm{Pic}^0(J(C)) \text{ such that } \alpha\neq\mathcal{O}_{J(C)}.\end{cases}$$

For the reader's convenience, let us briefly recall the relevant points from Section 7 of [**vGI**]. We denote for simplicity, via translation, $\Theta=W_{g-1}$ (so that $W^1_{g-1}=\mathrm{Sing}(\Theta)$). In the first place, from the exact sequence

$$0\to\mathcal{O}(2\Theta)\otimes\alpha\otimes\mathcal{O}(-\Theta)\to\mathcal{I}_{W^1_{g-1}}(2\Theta)\otimes\alpha\to\mathcal{I}_{W^1_{g-1}/\Theta}(2\Theta)\otimes\alpha\to 0$$

it follows that

$$h^i(J(C),\mathcal{I}_{W^1_{g-1}}(2\Theta)\otimes\alpha)=h^i(\Theta,\mathcal{I}_{W^1_{g-1}/\Theta}(2\Theta)\otimes\alpha) \text{ for } i>0.$$

Hence one is reduced to a computation on Θ. It is a standard fact (see e.g. [**vGI**], 7.2) that, via the Abel-Jacobi map $u=u_{g-1}:C_{g-1}\to\Theta\subset J(C)$,

$$h^i(\Theta,\mathcal{I}_{W^1_{g-1}/\Theta}(2\Theta)\otimes\alpha)=h^i(C_{g-1},L^{\otimes 2}\otimes\beta\otimes\mathcal{I}_Z),$$

where $Z=u^{-1}(W^1_{g-1})$, $L=u^*\mathcal{O}_X(\Theta)$ and $\beta=u^*\alpha$. We now use the standard exact sequence ([**ACGH**], p.258):

$$0\to T_{C_{g-1}}\stackrel{du}{\to}H^1(C,\mathcal{O}_C)\otimes\mathcal{O}_{C_{g-1}}\to L\otimes\mathcal{I}_Z\to 0.$$

Tensoring with $L\otimes\beta$, we see that it is sufficient to prove that

$$H^i(C_{g-1},T_{C_{g-1}}\otimes L\otimes\beta)=0,\ \forall i\geq 2,\ \forall\beta\neq\mathcal{O}_{C_{g-1}}.$$

To this end we use the well known fact (cf. *loc. cit.*) that

$$T_{C_{g-1}} \cong p_* \mathcal{O}_D(D)$$

where $D \subset C_{g-1} \times C$ is the universal divisor and p is the projection onto the first factor. As $p_{|D}$ is finite, the degeneration of the Leray spectral sequence and the projection formula ensure that

$$h^i(C_{g-1}, T_{C_{g-1}} \otimes L \otimes \beta) = h^i(D, \mathcal{O}_D(D) \otimes p^*(L \otimes \beta)),$$

which are zero for $i \geq 2$ and β non-trivial by [**vGI**], Lemma 7.24. □

7.3. Numerical study of semihomogeneous vector bundles. An idea that originated in work of Mukai is that on abelian varieties the class of vector bundles to which the theory of line bundles should generalize naturally is that of *semihomogeneous* bundles (cf. [**Muk1**], [**Muk3**], [**Muk4**]). These vector bundles are semistable, behave nicely under isogenies and Fourier transforms, and have a Mumford type theta group theory as in the case of line bundles (cf. [**Um**]). The purpose of this section is to show that this analogy can be extended to include effective global generation and normal generation statements dictated by specific numerical invariants measuring positivity. Recall that normal generation is Mumford's terminology for the surjectivity of the multiplication map $H^0(E) \otimes H^0(E) \to H^0(E^{\otimes 2})$.

In order to set up a criterion for normal generation, it is useful to introduce the following notion, which parallels the notion of Castelnuovo-Mumford regularity.

Definition 7.23. A coherent sheaf $\mathcal{F}$ on a polarized abelian variety (X, Θ) is called *m-Θ-regular* if $\mathcal{F}((m-1)\Theta)$ is M-regular.

The relationship with normal generation comes from (3) of the following "abelian" Castelnuovo-Mumford Lemma. Note that (1) is Corollary 5.3 plus Proposition 5.4.

Theorem 7.24 ([**PP1**], Theorem 6.3). *Let $\mathcal{F}$ be a 0-Θ-regular coherent sheaf on X. Then:*
(1) $\mathcal{F}$ is globally generated.
(2) $\mathcal{F}$ is m-Θ-regular for any $m \geq 1$.
(3) The multiplication map

$$H^0(\mathcal{F}(\Theta)) \otimes H^0(\mathcal{O}(k\Theta)) \longrightarrow H^0(\mathcal{F}((k+1)\Theta))$$

is surjective for any $k \geq 2$.

Basics on semihomogeneous bundles. Let X be an abelian variety of dimension g over an algebraically closed field. As a general convention, for a numerical class α we will use the notation $\alpha > 0$ to express the fact that α is ample. If the class is represented by an effective divisor, then the condition of being ample is equivalent to $\alpha^g > 0$. For a line bundle L on X, we denote by ϕ_L the isogeny defined by L:

$$\begin{array}{cccc} \phi_L : & X & \longrightarrow & \mathrm{Pic}^0(X) \cong \widehat{X} \\ & x & \rightsquigarrow & t_x^* L \otimes L^{-1}. \end{array}$$

Definition 7.25. ([**Muk3**]) A vector bundle E on X is called *semihomogeneous* if for every $x \in X$, $t_x^* E \cong E \otimes \alpha$, for some $\alpha \in \mathrm{Pic}^0(X)$.

Mukai shows in [**Muk3**] §6 that the semihomogeneous bundles are Gieseker semistable (while the simple ones – i.e. with no nontrivial automorphisms – are in fact stable). Moreover, any semihomogeneous bundle has a Jordan-Hölder filtration in a strong sense.

Proposition 7.26. *([**Muk3**] Proposition 6.18) Let E be a semihomogeneous bundle on X, and let δ be the equivalence class of $\frac{\det(E)}{\mathrm{rk}(E)}$ in $NS(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. Then there exist simple semihomogeneous bundles $F_1,\cdots,F_n$ whose corresponding class is the same δ, and semihomogeneous bundles $E_1,\cdots,E_n$, satisfying:*

- *$E\cong\bigoplus_{i=1}^n E_i$.*
- *Each E_i has a filtration whose factors are all isomorphic to F_i.*

Since the positivity of E is carried through to the factors of a Jordan-Hölder filtration as in the Proposition above, standard inductive arguments allow us to immediately reduce the study below to the case of simple semihomogeneous bundles, which we do freely in what follows.

Lemma 7.27. *Let E be a simple semihomogeneous bundle of rank r on X.*
*(1) ([**Muk3**], Proposition 7.3) There exists an isogeny $\pi: Y\to X$ and a line bundle M on Y such that $\pi^*E\cong\bigoplus_r M$.*
*(2) ([**Muk3**], Theorem 5.8(iv)) There exists an isogeny $\phi: Z\to X$ and a line bundle L on Z such that $\phi_*L=E$.*

Lemma 7.28. *Let E be a nondegenerate (i.e. $\chi(E)\neq 0$) simple semihomogeneous bundle on X. Then exactly one cohomology group $H^i(E)$ is nonzero, i.e. E satisfies the Index Theorem.*

Proof. This follows immediately from the similar property of the line bundle L in Lemma 7.27(2). □

Lemma 7.29. *A semihomogeneous bundle E is m-Θ-regular if and only if $E((m-1)\Theta)$ satisfies IT_0.*

Proof. The more general fact that an M-regular semihomogeneous bundle satisfies IT_0 follows quickly from Lemma 7.27(1) above. More precisely, the line bundle M in its statement is forced to be ample since it has a twist with global sections and positive Euler characteristic. □

A numerical criterion for normal generation. The main result of this section is that the normal generation of a semihomogeneous vector bundle is dictated by an explicit numerical criterion. We assume throughout that all the semihomogeneous vector bundles involved satisfy the minimal positivity condition, namely that they are 0-Θ-regular, which in particular is a criterion for global generation by Theorem 7.24. We will in fact prove a criterion which guarantees the surjectivity of multiplication maps for two arbitrary semihomogeneous bundles. This could be seen as an analogue of Butler's theorem [**Bu**] for semistable bundles on curves.

Theorem 7.30. *Let E and F be semihomogeneous bundles on (X,Θ), both 0-Θ-regular. Then the multiplication maps*

$$H^0(E)\otimes H^0(t_x^*F)\longrightarrow H^0(E\otimes t_x^*F)$$

are surjective for all $x\in X$ if the following holds:

$$\frac{1}{r_F}\cdot c_1(F(-\Theta))+\frac{1}{r'_E}\cdot\phi_\Theta^*c_1(\widehat{E(-\Theta)})>0,$$

where $r_F:=\mathrm{rk}(F)$ and $r'_E:=\mathrm{rk}(\widehat{E(-\Theta)})$. (Recall that ϕ_Θ is the isogeny induced by Θ.)

REMARK 7.31. Although most conveniently written in terms of the Fourier-Mukai transform, the statement of the theorem is indeed a numerical condition intrinsic to E (and F), since by [**Muk2**] Corollary 1.18 one has:

$$c_1(\widehat{E(-\Theta)}) = -PD_{2g-2}(\mathrm{ch}_{g-1}(E(-\Theta))),$$

where PD denotes the Poincaré duality map

$$PD_{2g-2} : H^{2g-2}(J(X), \mathbb{Z}) \to H^2(J(X), \mathbb{Z}),$$

and ch_{g-1} the $(g-1)$-st component of the Chern character. Note also that

$$\mathrm{rk}(\widehat{E(-\Theta)}) = h^0(E(-\Theta)) = \frac{1}{r^{g-1}} \cdot \frac{c_1(E(-\Theta))^g}{g!}$$

by Lemma 7.29 and [**Muk1**] Corollary 2.8.

We can assume E and F to be simple by the considerations in §2, and we will do so in what follows. We begin with a few technical results. In the first place, it is useful to consider the *skew Pontrjagin product*, a slight variation of the usual Pontrjagin product (see [**Pa**] §1). Namely, given two sheaves $\mathcal{E}$ and $\mathcal{G}$ on X, one defines

$$\mathcal{E} \hat{*} \mathcal{G} := d_*(p_1^*(\mathcal{E}) \otimes p_2^*(\mathcal{G})),$$

where p_1 and p_2 are the projections from $X \times X$ to the two factors and $d : X \times X \to X$ is the difference map.

LEMMA 7.32. *For all $i \geq 0$ we have:*

$$h^i((E \hat{*} F) \otimes \mathcal{O}_X(-\Theta)) = h^i((E \hat{*} \mathcal{O}_X(-\Theta)) \otimes F).$$

PROOF. This follows from Lemma 3.2 in [**Pa**] if we prove the following vanishings:

(1) $h^i(t_x^* E \otimes F) = 0, \ \forall i > 0, \ \forall x \in X.$
(2) $h^i(t_x^* E \otimes \mathcal{O}_X(-\Theta)) = 0, \ \forall i > 0, \ \forall x \in X.$

We treat them separately:

(1) By Lemma 7.27(1) we know that there exist isogenies $\pi_E : Y_E \to X$ and $\pi_F : Y_F \to X$, and line bundles M on Y_E and N on Y_F, such that $\pi_E^* E \cong \underset{r_E}{\oplus} M$ and $\pi_F^* F \cong \underset{r_F}{\oplus} N$. Now on the fiber product $Y_E \times_X Y_F$, the pull-back of $t_x^* E \otimes F$ is a direct sum of line bundles numerically equivalent to $p_1^* M \otimes p_2^* N$. This line bundle is ample and has sections, and so no higher cohomology by the Index Theorem. Consequently the same must be true for $t_x^* E \otimes F$.
(2) Since E is semihomogeneous, we have $t_x^* E \cong E \otimes \alpha$ for some $\alpha \in \mathrm{Pic}^0(X)$, and so

$$h^i(t_x^* E \otimes \mathcal{O}_X(-\Theta)) = h^i(E \otimes \mathcal{O}_X(-\Theta) \otimes \alpha) = 0,$$

since $E(-\Theta)$ satisfies IT_0.

□

Let us assume from now on for simplicity that the polarization Θ is symmetric. This makes the proofs less technical, but the general case is completely similar since everything depends (via suitable isogenies) only on numerical equivalence classes.

Proposition 7.33. *Under the hypotheses above, the multiplication maps*

$$H^0(E)\otimes H^0(t_x^*F)\longrightarrow H^0(E\otimes t_x^*F)$$

are surjective for all $x\in X$ if we have the following vanishing:

$$h^i(\phi_\Theta^*((-1_X)^*E\otimes\mathcal{O}_X(-\Theta))\widehat{\ }\otimes F(-\Theta))=0,\ \forall i>0.$$

Proof. By [**Pa**] Theorem 3.1, all the multiplication maps in the statement are surjective if the skew-Pontrjagin product $E\hat{*}F$ is globally generated, so in particular if $(E\hat{*}F)$ is 0-Θ-regular. On the other hand, by Lemma 7.32, we can check this 0-regularity by checking the vanishing of $h^i((E\hat{*}\mathcal{O}_X(-\Theta))\otimes F)$. To this end, we use Mukai's general Lemma 3.10 in [**Muk1**] to see that

$$E\hat{*}\mathcal{O}_X(-\Theta)\cong\phi_\Theta^*((-1_X)^*E\otimes\mathcal{O}_X(-\Theta))\widehat{\ }\otimes\mathcal{O}(-\Theta).$$

This implies the statement. □

We are now in a position to prove Theorem 7.30: we only need to understand the numerical assumptions under which the cohomological requirement in Proposition 7.33 is satisfied.

Proof. (*of Theorem* 7.30.) We first apply Lemma 7.27(1) to $G:=\phi_\Theta^*\widehat{(-1_X)^*E(-\Theta)}$ and $H:=F(-\Theta)$: there exist isogenies $\pi_G:Y_G\to X$ and $\pi_H:Y_H\to X$, and line bundles M on Y_G and N on Y_H, such that $\pi_G^*G\cong\underset{r_G}{\oplus}M$ and $\pi_H^*H\cong\underset{r_H}{\oplus}N$. Consider the fiber product $Z:=Y_G\times_X Y_H$, with projections p_G and p_H. Denote by $p:Z\to X$ the natural composition. By pulling everything back to Z, we see that

$$p^*(G\otimes H)\cong\bigoplus_{r_G\cdot r_F}(p_1^*M\otimes p_2^*N).$$

This implies that our desired vanishing $H^i(G\otimes H)=0$ (cf. Proposition 7.33) holds as long as

$$H^i(p_G^*M\otimes p_H^*N)=0,\ \forall i>0.$$

Now $c_1(p_G^*M)=p_G^*c_1(M)=\frac{1}{r_G}p^*c_1(G)$ and similarly $c_1(p_H^*N)=p_H^*c_1(N)=\frac{1}{r_H}p^*c_1(G)$. Finally we get

$$c_1(p_G^*M\otimes p_H^*N)=p^*(\frac{1}{r_G}\cdot c_1(G)+\frac{1}{r_H}\cdot c_1(H)).$$

Thus all we need to have is that the class

$$\frac{1}{r_G}\cdot c_1(G)+\frac{1}{r_H}\cdot c_1(H)$$

be ample, and this is clearly equivalent to the statement of the theorem. □

(−1)-Θ-regular vector bundles. It can be easily seen that Theorem 7.30 implies that a (−1)-Θ-regular semihomogeneous bundle is normally generated. Under this regularity hypothesis we have however a much more general statement, which works for every vector bundle on a polarized abelian variety.

Theorem 7.34. *For (−1)-Θ-regular vector bundles E and F on X, the multiplication map*

$$H^0(E)\otimes H^0(F)\to H^0(E\otimes F)$$

is surjective.

Proof. We use an argument exploited in [**PP1**], inspired by techniques introduced by Kempf. Let us consider the diagram

$$\begin{array}{ccc} \bigoplus_{\xi\in U} H^0(E(-2\Theta)\otimes P_\xi)\otimes H^0(2\Theta\otimes P_\xi^\vee)\otimes H^0(F) & \longrightarrow & H^0(E)\otimes H^0(F) \\ \downarrow & & \downarrow \\ \bigoplus_{\xi\in U} H^0(E(-2\Theta)\otimes P_\xi)\otimes H^0(F(2\Theta)\otimes P_\xi^\vee) & \longrightarrow & H^0(E\otimes F) \end{array}$$

Under the given hypotheses, the bottom horizontal arrow is onto by the general Theorem 5.1. On the other hand, the abelian Castelnuovo-Mumford Lemma Theorem 7.24 ensures that each one of the components of the vertical map on the left is surjective. Thus the composition is surjective, which gives the surjectivity of the vertical map on the right. □

Corollary 7.35. *Every (-1)-Θ-regular vector bundle is normally generated.*

Examples. There are two basic classes of examples of (-1)-Θ-regular bundles, and both turn out to be semihomogeneous. They correspond to the properties of linear series on abelian varieties and on moduli spaces of vector bundles on curves, respectively.

Example 7.36. (**Projective normality of line bundles.**) For every ample divisor Θ on X, the line bundle $L = \mathcal{O}_X(m\Theta)$ is (-1)-Θ-regular for $m \geq 3$. Thus we recover the classical fact that $\mathcal{O}_X(m\Theta)$ is projectively normal for $m \geq 3$.

Example 7.37. (**Verlinde bundles.**) Let $U_C(r,0)$ be the moduli space of rank r and degree 0 semistable vector bundles on a a smooth projective curve C of genus $g \geq 2$. This comes with a natural determinant map $\det : U_C(r,0) \to J(C)$, where $J(C)$ is the Jacobian of C. To a generalized theta divisor Θ_N on $U_C(r,0)$ (depending on the choice of a line bundle $N \in \mathrm{Pic}^{g-1}(C)$) one associates for any $k \geq 1$ the (r,k)-Verlinde bundle on $J(C)$, defined by $E_{r,k} := \det_*\mathcal{O}(k\Theta_N)$ (cf. [**Po**]). It is shown in *loc. cit.* that the numerical properties of $E_{r,k}$ are essential in understanding the linear series $|k\Theta_N|$ on $U_C(r,0)$. It is noted there that $E_{r,k}$ are polystable and semihomogeneous.

A basic property of these vector bundles is the fact that $r_J^*E_{r,k} \cong \oplus\mathcal{O}_J(kr\Theta_N)$, where r_J denotes multiplication by r on $J(C)$ (cf. [**Po**] Lemma 2.3). Noting that the pull-back $r_J^*\mathcal{O}_J(\Theta_N)$ is numerically equivalent to $\mathcal{O}(r^2\Theta_N)$, we obtain that $E_{r,k}$ is 0-Θ-regular iff $k \geq r+1$, and (-1)-Θ-regular iff $k \geq 2r+1$. This implies by the statements above that $E_{r,k}$ is globally generated for $k \geq r+1$ and normally generated for $k \geq 2r+1$. These are precisely the results [**Po**] Proposition 5.2 and Theorem 5.9(a), the second obtained there by ad hoc (though related) methods.

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Vector Bundles on Reducible Curves and Applications

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1. Introduction

Let C be a projective, irreducible, non-singular curve of given genus $g \geq 2$ defined over an algebraically closed field. Let E be a vector bundle on C. The slope of E is defined as

$$\mu(E) = \frac{\deg E}{\mathrm{rank} E}$$

A vector bundle E is said to be semistable (resp. stable) if and only if for every subbundle F of E, $\mu(F) \leq$ (resp. $<$)$\mu(E)$. The set of all stable bundles of a given rank r and degree d on C form a moduli space $U(r,d)$. If r and d are coprime, the condition for semistability is the same as the condition for stability and $U(r,d)$ is projective. If r and d are not coprime, this is no longer the case. A projective moduli space can be obtained, though, by considering semistable vector bundles modulo a suitable equivalence relation.

If C is a singular curve, the results above are not true. The moduli space of stable bundles $U(r,d)$ is no longer projective even when r and d are relatively prime. Newstead (see [**N**]) gave a compactification for a fixed irreducible curve by considering torsion-free sheaves instead of just vector bundles on the curve. This was later generalized by Seshadri ([**Se**]) to include reducible curves and by Pandariphande ([**P**]) when the curve moves in the moduli space of a fixed genus.

A different approach was taken by Gieseker in the case of an irreducible nodal curve for $r = 2$ by considering vector bundles on various semistable models of the given curve. His methods were extended in [**X**] to the case of a reducible curve (and $r = 2$) and then by Schmitt [**Sc2**] to any rank and curves moving in the moduli space.

In the case of rank one, many results about non-singular curves and their linear series have been proved by degenerating to singular curves. This requires knowing what is the equivalent in the singular case of a linear series. For a reducible curve of compact type, the theory of limit linear series from [**EH1**] gave an excellent solution. In order to do something similar in higher rank, one needs to extend the concept of limit linear series to rank greater than one. At the same time, one must deal with the stability condition.

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In this paper, we present some scattered results about the moduli spaces of (semi) stable vector bundles on reducible nodal curves, the concept of generalized limit linear series and some of their applications. On neither of these themes does the paper try to be comprehensive. Its main purpose is to present a few examples of some techniques that have proved useful in the resolution of problems related to vector bundles with the hope that they will become more widely used.

2. Moduli spaces of slope-stable torsion-free sheaves

Seshadri in [**Se**] gave the following compactification(s) of a moduli space of vector bundles on a reducible curve. While his construction is more general, we shall concentrate on the nodal case.

Let C be a nodal curve with components C_i. When the curve is reducible, the notion of torsion-free sheaf mentioned in the introduction for irreducible curves can be generalized to the concept of depth one sheaves. A sheaf E on C is said to be of depth one if any torsion section vanishes identically on some component of C.

For an irreducible curve, the concept of stability could have been formulated equivalently by the condition that for every subbundle F of E,

$$\frac{\chi(F)}{\text{rank}(F)} \leq (<) \frac{\chi(E)}{\text{rank}(E)}$$

where χ denotes the Euler-Poincaré characteristic. On a reducible curve, even if E has constant rank, there will be subsheaves of E with different ranks on the various components. Therefore, in order to generalize the definition above, one needs to make clear how to count the relative rank of every component. This is done in the following way:

A polarization of C is the choice of rational weights

$$w_i, \quad 0 < w_i < 1, \quad \sum w_i = 1.$$

A depth one sheaf E of rank n on C is said to be (semi)stable for the given polarization if for every subsheaf F of E with rank r_i on the component C_i,

$$\frac{\chi(G)}{\sum w_i r_i} (\leq) < \frac{\chi(E)}{r}.$$

We will refer to the number $\sum w_i r_i$ as the w-rank of F. There is then a moduli space parameterizing (equivalence classes of semi)stable torsion-free sheaves on C.

The moduli space of torsion-free sheaves on a nodal reducible curve is itself reducible. The description of its components was given in [**T2**] for curves of compact type (that is, curves whose Jacobian is compact or whose dual graph has no non-trivial cycles) and in [**T3**] for arbitrary nodal curves. If each component of the curve has genus at least one, the number of components of the moduli space depends only on the rank and the dual graph of the curve rather than the genus of the various components of C. For example, if the curve is of compact type with M components all of genus at least one, then the moduli space of vector bundles of rank r on C has r^{M-1} components.

More generally, consider any nodal curve C. Assume that C is the fiber of a one-dimensional family $\pi : \mathcal{C} \to S$. Fix a vector bundle $\mathcal{E}$ on $\mathcal{C}$ so that the restriction to every fiber of π is of degree d. The multidegree (d_i) is the collection of degrees of the restriction of this vector bundle to the components C_i of C. If we tensor $\mathcal{E}$ by a line bundle $\mathcal{E} = \mathcal{O}_{\mathcal{C}}(\sum a_i C_i)$, the numbers d_i will be modified by multiples of r and the total degree d will remain invariant. For example, if C has only two components glued at one point, the degree on one of the components can be increased by an arbitrary multiple of r while the degree on the other component will decrease by subtracting the same multiple of r. Let us say that two multidegrees are equivalent if they can be obtained from one another by such a transformation. Then, the set of components of the moduli space corresponds one to one with the set of equivalence classes of multidegrees by this equivalence relation.

We provide a sketch of proof of this fact in the case of a curve C with two components C_1, C_2 and a single node obtained by identifying $Q \in C_1$ with $P \in C_2$ (for the more general statements and proofs see [**T3**]):

2.1. **Proposition** *Let C be the nodal curve of genus g obtained by identifying $Q \in C_1$ with $P \in C_2$, where C_1, C_2 are two non-singular curves of genus at least one. Then the moduli space of torsion-free sheaves on C semistable with respect to a generic polarization is connected and has r irreducible components each of dimension $r^2(g-1)+1$.*

Proof. Assume that E is a vector bundle stable by a generic polarization w_1, w_2. Consider the subsheaf F of E consisting of the sections of E that vanish identically on the second component C_2. Then, $\chi(F) = \chi(E_{|C_1}) - r$ while the w-rank of F is $w_1 r$. The stability condition gives

$$\frac{\chi(E_{|C_1}) - r}{w_1 r} = \frac{\chi(F)}{w_1 r} \leq \frac{\chi(E)}{r}$$

Hence, $\chi(E_{|C_1}) \leq w_1\chi(E) + r$. Reversing the role of C_1, C_2 and using that $\chi(E) = \chi(E_{|C_1}) + \chi(E_{|C_2}) - r, w_1 + w_2 = 1$, one obtains

$$(*) \qquad w_1\chi(E) \leq \chi(E_{|C_1}) \leq w_1\chi(E) + r.$$

If w_1 is generic, $w_1\chi(E)$ is not an integer and there are therefore r possible values of $\chi(E_{|C_1})$ that satisfy $(*)$. Moreover (see (cf. [**T2**], [**T3**]) if $(*)$ is satisfied and $E_{|C_1}, E_{|C_2}$ are semistable then the vector bundle E is also semistable. If in addition, at least one of the vector bundles $E_{|C_1}, E_{|C_2}$ is stable, then so is E. In fact it suffices if they are both semistable and no destabilizing subsheaf of $E_{|C_1}$ glues with a destabilizing subsheaf of $E_{|C_2}$ for E to be stable.

The moduli space of semistable vector bundles of a given rank and degree on each of the curves C_1, C_2 is irreducible of dimension $r^2(g_1-1)+1, r^2(g_2-1)+1$ with $g_1 \geq 1,\ g_2 \geq 1$ the genera of the corresponding curves. Once $E_{|C_1}, E_{|C_2}$ have been fixed, a vector bundle on C is determined by giving a projective isomorphism of the fibers $(E_{|C_1})_Q \cong (E_{|C_2})_P$. Fix a degree for E_1 satisfying (*). As the degree of E is fixed, this determines the degree for $E_{|C_2}$. One gets a variety of dimension

$$r^2(g_1-1)+1+r^2(g_2-1)+1+r^2-1 = r^2(g-1)+1.$$

This is the dimension of the moduli space of vector bundles on C. We obtain in this way an open set of one of the components of the moduli space of vector bundles on C. The closure of this component may contain some points corresponding to vector bundles that are stable on C but whose restriction to each of the components is not necessarily semistable. Moreover, the different components of the moduli space are not disjoint, the points of intersection correspond to torsion-free sheaves that are not locally free at the nodes. For example, in the situation above, the components corresponding to $\deg(E_{|C_1}) = d_1, \deg(E_{|C_2}) = d - d_1$ and $\deg(E'_{|C_1}) =$

$d_1 - 1, \deg(E'_{|C_2}) = d - d_1 + 1$ have an intersection whose generic point is a torsion-free sheaf F such that $\deg(F_{|C_1}) = d_1 - 1, \deg(F_{|C_2}) = d - d_1$ and the fiber at the node N is of the form $(\mathcal{O}_N)^{r-1} \oplus \mathcal{M}_N$ with $\mathcal{M}_N$ the maximal ideal at the node.

On an elliptic curve, there are no stable vector bundles of rank r and degree d when r, d are not coprime. Assume that C_1 is an elliptic curve and that $d_1 = \deg(E_{|C_1})$ gives a solution to $(*)$ (or a suitable set of relations deduced from the graph of the nodal curve). Write h for the greatest common divisor of r, d_1. Then, the generic point in the component corresponding to this value d_1 is a vector bundle whose restriction to C_1 is a direct sum of h vector bundles of rank $\frac{r}{h}$ and degree $\frac{d_1}{h}$. Note that this still gives the right dimension to the component of the moduli space of vector bundles on C: while there is an h-dimensional space of these restrictions of vector bundles to C_1 there is also an h-dimensional set of automorphisms of these restrictions acting on the set that identify isomorphic bundles. □

As it was mentioned before, the stability of a vector bundle for a given polarization on a reducible curve does not imply that the restriction of this bundle to each component is stable. But these restrictions cannot be too unstable either. As an example we present the following:

2.2. **Proposition** *Let C be a nodal curve with an irreducible component $\bar{C}$ intersecting the rest of the curve in α nodes. Let E be a vector bundle on C that is stable by a generic polarization. Assume that the restriction of E to $\bar{C}$ is the direct sum of r line bundles $L_1 \oplus \cdots \oplus L_r$. Then, $|\deg L_i - \deg L_j| \leq \alpha - 1$.*

Proof. Consider the subsheaf F of E that vanishes on all components of $C - \bar{C}$ and that restricts to $\bar{C}$ to the sheaf of sections of L_1 that vanish at the nodes. Then $\chi(F) = \chi(L_1) - k$ while the weighted rank $\operatorname{rank}(F) = \bar{w}$. The stability condition then gives

$$\chi(L_1) - \alpha \leq \bar{w}\frac{\chi(E)}{r}.$$

Consider the subsheaf F'_i of E that restricts to $\bar{C}$ to the sheaf of sections of $L_2 \oplus \cdots \oplus L_r$ and on all components of $C - \bar{C}$ restricts to the sections of E that glue with the above on $\bar{C}$. Then $\chi(F') = \chi(E) - \chi(L_1)$ while the weighted rank $\operatorname{rank}(F') = (r-1)\bar{w} + r(1-\bar{w}) = r - \bar{w}$. The stability condition then gives

$$\frac{\chi(E) - \chi(L_1)}{r - \bar{w}} \leq \frac{\chi(E)}{r}.$$

One obtains

$$\bar{w}\frac{\chi(E)}{r} \leq \chi(L_1) \leq \bar{w}\frac{\chi(E)}{r} + \alpha$$

As $\bar{w}$ is generic, $\bar{w}\frac{\chi(E)}{r}$ is not an integer. As the above equation is valid for all L_i, not just L_1, the result follows. □

While the choice of a polarization can be arbitrary, there is a natural one that can be defined as follows: For a given semistable curve C, define d_i as the degree of the canonical sheaf of C restricted to the component C_i. The canonical polarization has weights

$$w_i = \frac{d_i}{2g-2}$$

Pandharipande showed in [**P**] that there is a moduli space of torsion-free sheaves over the moduli space of stable curves if one considers the canonical polarization.

3. Hilbert compactification of the moduli space and its relationship with the slope-stable moduli space

Gieseker took the following approach towards compactifying the moduli space of vector bundles on a nodal curve. There is an isomorphism between moduli spaces of vector bundles $U(r,d) \cong U(r,d+kr)$ given by tensoring with a line bundle of degree k. Hence, one can assume that d is sufficiently large. In this situation, on a non-singular curve, a stable vector bundle is globally generated, hence it induces a map from the curve to the Grassmannian.

Start now the other way around. Consider the Hilbert scheme of curves in $\mathrm{Gr}(H^0(E),r)$ of suitable Hilbert polynomial. Take the connected component of points corresponding to immersions of a given nodal curve and consider its closure. The restriction to the curve of the universal bundle on the Grassmannian gives a vector bundle on the curve. In order to obtain a moduli space for vector bundles, one should first restrict to the points that are stable under the action of the linear group and then take quotient by this action.

This provides a natural compactification of the locus of vector bundles. Its main advantage is that all the points of this space correspond to vector bundles on a curve, namely the restriction of the universal bundle on the Grassmannian. Its main drawback is that the curve is not fixed. Other curves may appear that differ from the given nodal curve in adding a few rational components between the nodes of the original nodal curve. As this is precisely what happens with limit liner series (see next section), this seems a suitable model to use in this case.

Several extensions of Gieseker's results to higher rank were given by Nagaraj-Seshadri [**NS1**], [**NS2**] and Kausz [**K**] using different methods. Caporaso ([**C**]) gave a compactification of the Hilbert stable set of curves over $\bar{\mathcal{M}}_g$ when $r=1$. Finally, Schmitt (see [**Sc2**]) extended Gieseker's construction not only to higher rank but also to the whole moduli space of semistable curves in $\bar{\mathcal{M}}_g$.

The two compactifications of Seshadri and Gieseker-Schmitt are in fact closely related: for a non-singular curve, slope stability and Hilbert stability are equivalent. This was first proved for rank 2 by Gieseker and Morrison in [**GM**] (see also [**T4**] for a different proof). For general rank, one of the implications was proved in [**T5**] and the equivalence in [**Sc1**].

On reducible curves, Hilbert stability implies that the distribution of degrees on the various components is the same as that allowed by slope stability in the case of the canonical weights (that is conditions similar to those in $(*)$ in the proof of 2.1 are satisfied). Similarly, the restriction to the various components cannot be too unstable, so conditions as those stated in 2.2 are also satisfied. For instance, rational components on Hilbert stable curves must have at least two nodes and in this case the restriction of the vector bundles to the components is of the form $\mathcal{O}^{\oplus j} \oplus \mathcal{O}(1)^{\oplus r-j}$.

4. Limit linear series

Assume that we have a family of curves $\pi : \mathcal{C} \to S$ such that $\mathcal{C}_t$ is non-singular for $t \neq t_0$ while the special fiber $\mathcal{C}_{t_0}$ is a curve of compact type (that is, a curve whose Jacobian is compact or whose dual graph has no cycles).

We want to consider a linear series for vector bundles on $\mathcal{C} - \mathcal{C}_{t_0}$ and we would like to describe the limit of this object on $\mathcal{C}_{t_0}$.

A linear series for vector bundles on $\mathcal{C} - \mathcal{C}_{t_0}$ will be given by a vector bundle $\mathcal{E}$ on $\mathcal{C} - \mathcal{C}_{t_0}$ such that the restriction to each fiber has a preassigned rank r and degree d together with a family of spaces of sections given by a locally free subsheaf $\mathcal{V}$ of rank k of $\pi_*(\mathcal{E})$.

Up to making a few base changes and blow-ups (that may add a few rational components to the central fiber), we can assume that the vector bundle and locally-free subsheaf can be extended to the whole family.

On the central fiber, one obtains then a limit linear series in the sense of [**T1**] section 2, that can be defined as follows:

4.1. **Definition. Limit linear series** *A limit linear series of rank r, degree d and dimension k on a curve of compact type with M components consists of data (I), (II) below for which data (III), (IV) exist satisfying conditions (a)-(c).*

(I) For every component C_i, a vector bundle E_i of rank r and degree d_i and a k-dimensional space of sections V_i of E_i.

(II) For every node obtained by gluing $Q \in C_i$ with $P \in C_j$ an isomorphism of the projectivisation of the fibers $(E_i)_Q$ and $(E_j)_P$

(III) A positive integer a

(IV) For every node obtained by gluing $Q \in C_i$ and $P \in C_j$, bases $s^t_{Q,i}, s^t_{P,j}, t = 1...k$ of the vector spaces V_i and V_j

Subject to the conditions

(a) $(\sum_{i=1}^M d_i) - r(M-1)a = d$

(b) At the nodes, the sections glue with each other through the isomorphism in II and the orders of vanishing at P_j, Q_i of corresponding sections of the chosen basis satisfy $ord_P s^t_{P,j} + ord_Q s^t_{Q,i} \geq a$

(c) Sections of the vector bundles $E_i(-aQ)$ are completely determined by their value at the nodes.

This is a generalization of the concept of limit linear series for line bundles. In the line bundle case, $a = d = \deg E_i$ for all i. Hence, (III) is irrelevant and so is the projective isomorphism in (II) as the fibers are one-dimensional. Conditions (a), (c) and the first part of (b) are automatically satisfied and one only needs to impose the second part of (b).

Let us check that the limit of a linear series on a non-singular curve is actually a limit linear series. Given a family of curves $\pi : \mathcal{C} \to S$ as above, fix a vector bundle on the generic fiber and extend it (after some base change and normalizations) to a vector bundle on the central fiber. This limit vector bundle on the central fiber is not unique. We could modify it by, for example, tensoring the bundle on the whole curve $\mathcal{C}$ with a line bundle with support on the central fiber. This would leave the vector bundle on the generic curve unchanged but would modify the vector bundle on the reducible curve (by adding to the restriction to each component a linear combination of the nodes). In this way, for each component C_i, one can choose (in many different ways) a version $\mathcal{E}_i$ of the limit vector bundle such that the sections of $\mathcal{E}_{i|C_j}$, $i \neq j$ are trivial. In particular, this implies that $\mathcal{V}_i$ restricts to a space of sections of dimension k on the component C_i. Taking $E_i = \mathcal{E}_{i|C_i}$, $V_i = \mathcal{V}_{i|C_i}$, one obtains data in (I) satisfying condition (c).

We want to see how to choose the data above in order to obtain the integer a in (III).

Order the components of C so that C_1 has only one node and $C_1 \cup \cdots \cup C_i$ is connected. From the connectivity, C_i intersects some $C_{j(i)}$, $j(i) < i$ and from the fact that the curve is of compact type, this $j(i)$ is uniquely determined. Let C'_i be the connected component of $C - C_i$ containing $C_{j(i)}$. The restriction of $\mathcal{E}_{j(i)}(-bC'_i)$ to any component of C other than $C_i, C_{j(i)}$ is identical to the restriction of $\mathcal{E}_{j(i)}$ to that component. The restriction to C_i (resp $C_{j(i)}$) is changed by tensoring with $\mathcal{O}_{C_i}(bP_i)$ (resp $\mathcal{O}_{C_{j(i)}}(-bQ_{j(i)})$) if $P_i, Q_{j(i)}$ are the two points that get identified to form the node. Hence, in order to satisfy the conditions in the previous paragraph, we can take $\mathcal{E}_i = \mathcal{E}_{j(i)}(-b_iC'_i)$ for some integer b_i. As the curve has only a finite number of components, we may assume the b_i to be all identical. We denote this number by a.

It is easy to check then that condition (a) is satisfied.

As $\mathcal{E}_i = \mathcal{E}_j(-aC'_i)$ the gluing defining $\mathcal{E}_{i|C}$ determine those of $\mathcal{E}_{j|C}$. Hence, the isomorphisms in (II) are well determined and the first part of b) is satisfied.

Let us check the second part of (b). If $C_i, C_j,\ j < i$ intersect, then $j = j(i)$ (in the notations above). Hence $\mathcal{E}_i = \mathcal{E}_j(-aC'_i)$. Let s be a section of $\mathcal{V}_{j|C}$. Assume that s vanishes on C'_i with multiplicity α (and therefore vanishes at the point P of intersection with C'_i with order at least α). Let t_0 be a local equation for C. Then $(t_0^{a-\alpha})s$ is a non-trivial section of $\mathcal{E}_j(-aC'_i) = \mathcal{E}_i$. Hence it gives rise to a section on $\mathcal{V}_i$ that vanishes to order at least $a - \alpha$ on P.

Given a vector bundle $\mathcal{E}$ stable by some polarization, the vector bundles $\mathcal{E}_i$ that give rise to the limit linear series are no longer stable by this polarisation, as the distribution of degrees among the components gets changed. But conditions of "quasistability" are preserved, that is, conditions that say that the restriciton of the vector bundle to each component is not far from being stable (analogous to 2.2). This fact is very important in applications as it restricts the possibilities for these vector bundles.

5. Subbundles of a vector bundle

Let E be a vector bundle of rank r and degree d. Let E' be a subsheaf of E of fixed rank r'. Up to increasing the degree of E', we can assume that the quotient is again a vector bundle. Hence, we have an exact sequence

$$(**) \quad 0 \to E' \to E \to E'' \to 0$$

Define

$$s_{r'}(E) = r'd - r\max\{\deg(E')\}$$

where E' varies among all subsheaves of E of rank r'. If E is stable, then $s_{r'}(E) \geq 0$.

If $E',\ E''$ are fixed, the set of vector bundles that fit in an exact sequence as $(**)$ is parameterized by $H^1(C, (E'')^* \otimes E')$. If there exists a stable E fitting in one of these exact sequences, then $H^0(C, (E'')^* \otimes E') = 0$, otherwise there would be a non-trivial map $E'' \to E'$ and therefore a non-trivial endomorphism of E contradicting stability. This implies that $h^1(C, (E'')^* \otimes E')$ is constant.

Let $E',\ E''$ vary

$$E' \in U(r', d'),\ E'' \in U(r - r', d - d').$$

Then, the set of stable E varies in a space of dimension

$$(r')^2(g-1)+1+(r-r')^2(g-1)+1+r'(d-d')-(r-r')d'+r'(r-r')(g-1)-1 =$$

$$= r^2(g-1) + 1 + r'd - rd' - r'(r-r')(g-1)$$

The following result was conjectured by Lange in [**L**]

5.1. **Theorem** *If $r'd - rd' - r'(r-r')(g-1) \geq 0$, the generic vector bundle in $U(r,d)$ has subbundles of rank r' and degree d'. For $0 \leq r'd - rd' \leq r'(r-r')(g-1)$, the set of vector bundles with such a subbundle is an irreducible set of the moduli space of codimension $r'(r-r')(g-1) - (r'd - rd')$.*

The rank two case was proved by Lange in [**L**] . Several special cases were obtained (see[**BL**] and the references there).

The result was proved for the generic curve in [**T7**] and then in its full generality in [**RT**].

The main point of any proof of the result is to show that there exist extensions as in $(*)$ with a stable E and for $r'd - rd' < r'(r-r')(g-1)$ that such an E has only a finite number of subbundles of the given rank and degree.

Reducible curves were used in [**T6**], [**T7**], [**T8**]. We will sketch below the main points of the proof when $r = 2,\ r' = 1, d = 1$. In order to prove the result for the generic curve, it suffices to show it for a special curve. Take g elliptic curves $C_1, ..., C_g$ with marked points P_i, Q_i on them and identify Q_i with P_{i+1} to form a nodal curve of arithmetic genus g. Consider the component of the moduli space of vector bundles on this curve with distribution of degrees one on the first component and zero on the remaining ones. For a generic point on this component, E_1 is an indecomposable vector bundle of degree one while the remaining E_i are direct sums of two line bundles.

The largest degree of a line subbundle of E_1 is zero. In fact, every line subbundle of degree zero on C_1 can be immersed in E_1. For a particular choice of such a subbundle, we can assume that it glues at Q_1 with a fixed direction. On the remaining components, the degree of the largest subbundle is zero and there are precisely two subbundles of this degree inside each E_i. On the other hand, every line bundle of degree -1 on C_i can be immersed in C_i by a two-dimensional family of maps. Given a fixed direction at both P_i and Q_i, one can find a subbundle of E_i of degree -1 that glues with these two directions at the nodes.

In order to obtain a line subbundle on the total curve, we must choose line subbundles of each of the components that glue with each other. If all the gluing are generic, the choice that gives the largest degree will be as follows: Take a subbundle of degree zero of E_1 that glues with one of the two subbundles of degree zero of E_2. Choose a subbundle of degree zero on each of the components C_i for even i. Choose a subbundle of degree -1 on the components C_i for odd i that glues with the chosen subbundles in adjacent components. The degree of the subbundle obtained in this way is $d' = -\frac{g-2}{2}$ if this number is an integer and $d' = -\frac{g-1}{2}$ otherwise.

In order to obtain subbundles of higher degree, take the gluing in the last k components so that line subbundles of degree zero glue with each other and glue with the chosen subbundle in the previous $g-k$ components. One then obtains a subbundle of degree $d' = -\frac{g-k-2}{2}$ (again if this number is an integer). In this case, the gluing depends on three rather than four parameters at the last k nodes. Hence, the codimension of the set is $k = g + 2d' - 2$ as expected.

6. Some applications to Brill-Noether Theory

A Brill-Noether subvariety $B^k_{r,d}$ of the stable set in $U(r,d)$ is a subset of $U(r,d)$ whose points correspond to stable bundles having at least k independent sections (often denoted by $W^{k-1}_{r,d}$). Brill-Noether varieties can be locally represented as determinantal varieties. This implies that, when non-empty, the dimension of $B^k_{r,d}$ at any point is at least the so called Brill-Noether number

$$\rho^k_{r,d} = dimU(r,d) - h^0(E)h^1(E) = r^2(g-1) + 1 - k(k-d+r(g-1))$$

with expected equality. Moreover, $B^{k+1}_{r,d}$ is contained in the singular locus, again with expected equality. It is not true that the above expectations actually occur for all meaningful values of r, d, k. For an account of the state of the art in Brill-Noether Theory see [**GT**].

Using limit linear series, one can prove the following:

6.1. **Theorem** *(see* [**T1**], [**T9**]*) Let C be a generic non-singular curve of genus $g \geq 2$. Let d, r, k be positive integers with $k > r$. Write*

$$d = rd_1 + d_2,\ k = rk_1 + k_2,\ d_2 < r,\ k_2 < r$$

and all $d_i,\ k_i$ non-negative integers. Assume that one of the following conditions is satisfied

$$(1) g - (k_1+1)(g - d_1 + k_1 - 1) \geq 1,\ d_2 \geq k_2 \neq 0$$
$$(2) g - k_1(g - d_1 + k_1 - 1) > 1,\ k_2 = 0$$
$$(3) g - (k_1+1)(g - d_1 + k_1) \geq 1,\ d_2 < k_2.$$

Then the set $B^k_{r,d}$ of rank r degree d and with k sections on C is non-empty and has (at least) one component of the expected dimension ρ.

Given a family of curves

$$\mathcal{C} \to T$$

construct the Brill-Noether locus for the family

$$\mathcal{B}^k_{r,d} = \{(t,E) | t \in T,\ E \in B^k_{r,d}(\mathcal{C}_t)\}.$$

The dimension of this scheme at any point is at least $\dim T + \rho^k_{r,d}$. Assume that we can find a point (t_0, E_0) such that the dimension at the point (t_0, E_0) of the fiber of $\mathcal{B}^k_{r,d}$ over t_0 $\dim(B^k_{r,d}(\mathcal{C}_{t_0}))_{E_0} = \rho^k_{n,d}$. Then $\dim \mathcal{B}^k_{r,d} \leq \rho^k_{r,d} + \dim T$ and therefore we have equality. The dimension of the generic fiber of the projection $\mathcal{B}^k_{r,d} \to T$ is at most the dimension of the fiber over t_0, namely $\rho^k_{r,d}$. But it cannot be any smaller, as the fiber over a point $t \in T$ is $B^k_{r,d}(\mathcal{C}_t)$ which has dimension at least $\rho^k_{n,d}$ at any point. Hence, there is equality which is the result we are looking for.

As the particular curve $\mathcal{C}_{t_0}$, we take g elliptic curves $C_1, ..., C_g$ with marked points P_i, Q_i on them and identify Q_i with P_{i+1} to form a node. Then, one needs to prove that there is a component of the set of limit linear series on the curve of dimension exactly ρ with the generic point corresponding to a stable bundle.

Let us sketch how one could proceed in the particular case $g = 5, k = 4, d = 9$ (and hence $\rho = 5$).

On the curve C_1, take the vector bundle to be a generic indecomposable vector bundle of degree nine. On the curve C_2 take $\mathcal{O}(P_2 + 3Q_2) \oplus \mathcal{O}(2P_2 + 2Q_2)$. On the curve C_3, take $\mathcal{O}(P_3 + 3Q_3)^{\oplus 2}$. On the curve C_4, take $\mathcal{O}(3P_4 + Q_4)^{\oplus 2}$. On the curve C_5, take the direct sum of two generic line bundles of degree four.

The sections and gluing at the nodes are taken as follows: there is a unique section s_1 of E_1 that vanishes at Q_1 with order four. Take as space of sections the space that contains s_1, two sections that vanish at Q_1 to order three and one section that vanishes to order two.

On C_2, glue $\mathcal{O}(2P_2+2Q_2)$ with the direction of s_1. Take a section that vanishes at P_2 to order three, the section that vanishes at P_2 to order one and at Q_2 to order three, one section that vanishes at P_2 to order one and at Q_2 to order two and the section that vanishes at P_2 to order two and at Q_2 to order two.

On C_3, take generic gluing. Take two sections that vanish at P_3 with order one and at Q_3 to order three and two sections that vanish at P_3 to order two and at Q_3 to order one.

On C_4, take generic gluing. Take two sections that vanish at P_4 to order one and at Q_4 to order two and two sections that vanish at P_4 to order three and at Q_4 to order one.

On C_5, take generic gluing. Take two sections that vanish at P_5 to order two and two sections that vanish at P_5 to order three.

In this way, each section on C_i glues at Q_i with a section on C_{i+1} so that the sum of the orders of vanishing is four. Note also that we have been using sections of our vector bundles that vanish as much as possible between the two nodes. So it is not possible to obtain a limit linear series of dimension larger than four with these vector bundles.

Let us count the dimension of the family so obtained. The vector bundle E_1 varies in a one-dimensional family and has a one-dimensional family of endomorphisms. The vector bundles E_2, E_3, E_4 are completely determined and have a family of endomorphisms of dimensions two, four and four respectively while E_5 varies in a two-dimensional family and has a two-dimensional family of endomorphisms. The resulting vector bundle is stable as the restriction to each component is semistable and the first one is actually stable (see section 2). Therefore it has a one-dimensional family of automorphisms. The gluings at each of the nodes vary in a four-dimensional family except for the first one that varies in a three-dimensional family. Therefore the total dimension of the family is

$$1+0+0+0+2-(1+2+4+4+4+2)+1+(3+4+4+4)=5=\rho.$$

If we try to deform the vector bundle by either taking more general restrictions to some component curves or more general gluing at some nodes, the limit linear series does not extend to this deformation. Hence, the point we describe is a general point in the set of limit linear series. Therefore, the result is proved.

One of the main questions for classical Brill-Noether Theory (that is in the case $r=1$) comes from the infinitesimal study of $B^k_{r,d}$. The tangent space to $U(r,d)$ can be identified with the set of infinitesimal deformations of the vector bundle E which is parameterized by $H^1(C,E\otimes E^*)\cong H^0(C,K\otimes E^*\otimes E)$. The tangent space to $B^k_{r,d}$ inside the tangent space to $U(r,d)$ can be identified with the orthogonal to the image of the Petri map

$$P_V : H^0(C,E)\otimes H^0(C,K\otimes E^*)\to H^0(C,K\otimes E\otimes E^*).$$

If this map is injective for a given E, then $B^k_{r,d}$ is non-singular of the right dimension at E. Hence, in order to prove that the expected results hold, it would be sufficient to prove the injectivity of this map for all possible E on say, a generic curve. This is true in rank one ([**G1**]). Unfortunately, for rank greater than one, the map is not injective in general. There is one case though in which one has an analogous result.

Assume that E is a vector bundle of rank two and canonical determinant. Let $U(2,K)$ be the moduli space of stable rank two vector bundles with determinant the canonical sheaf. Consider the set

$$B^k_{2,K} = \{E \in U(2,K) | h^0(C,E) \geq k\}$$

Then $B^k_{2,K}$ can be given a natural scheme structure. Its expected dimension is (see [**GT**], [**T10**])

$$\dim U(2,K) - \binom{k+1}{2}$$

The tangent space to $B^k_{2,K}$ at a point E is naturally identified with the orthogonal to the image of the symmetric Petri map

$$S^2(H^0(C,E)) \to H^0(C, S^2(E))$$

6.2. **Theorem** *(cf.* [**T10**]*) Let C be a generic curve of genus g defined over an algebraically closed field of characteristic different from two. Let E be a semistable vector bundle on C of rank two with canonical determinant. Then, the canonical Petri map*

$$S^2H^0(C,E) \to H^0(S^2E)$$

is injective.

The proof of this fact is in many ways similar to the proof of the injectivity of the classical Petri map in [**EH2**] with the (considerable) added complication brought in by higher rank.

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This collection of cutting-edge articles on vector bundles and related topics originated from a CMI workshop, held in October 2006, that brought together a community indebted to the pioneering work of P. E. Newstead, visiting the United States for the first time since the 1960s. Moduli spaces of vector bundles were then in their infancy, but are now, as demonstrated by this volume, a powerful tool in symplectic geometry, number theory, mathematical physics, and algebraic geometry. In fact, the impetus for this volume was to offer a sample of the vital convergence of techniques and fundamental progress, taking place in moduli spaces at the outset of the twenty-first century.

This volume contains contributions by J. E. Andersen and N. L. Gammelgaard (Hitchin's projectively flat connection and Toeplitz operators), M. Aprodu and G. Farkas (moduli spaces), D. Arcara and A. Bertram (stability in higher dimension), L. Jeffrey (intersection cohomology), J. Kamnitzer (Langlands program), M. Lieblich (arithmetic aspects), P. E. Newstead (coherent systems), G. Pareschi and M. Popa (linear series on Abelian varieties), and M. Teixidor i Bigas (bundles over reducible curves).

These articles do require a working knowledge of algebraic geometry, symplectic geometry and functional analysis, but should appeal to practitioners in a diversity of fields. No specialization should be necessary to appreciate the contributions, or possibly to be stimulated to work in the various directions opened by these path-blazing ideas; to mention a few, the Langlands program, stability criteria for vector bundles over surfaces and threefolds, linear series over abelian varieties and Brauer groups in relation to arithmetic properties of moduli spaces.

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