

# Tropical Mathematics

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These are the notes for the Clay Mathematics Institute Senior Scholar Lecture which was delivered by Bernd Sturmfels in Park City, Utah, on July 22, 2004. The topic of this lecture is the “tropical approach” in mathematics, which has gotten a lot of attention recently in combinatorics, algebraic geometry and related fields. It offers an elementary introduction to this subject, touching upon Arithmetic, Polynomials, Curves, Phylogenetics and Linear Spaces. Each section ends with a suggestion for further research. The bibliography contains numerous references for further reading in this field.

The adjective “tropical” was coined by French mathematicians, including Jean-Eric Pin [19], in the honor of their Brazilian colleague Imre Simon [23], who was one of the pioneers in min-plus algebra. There is no deeper meaning in the adjective “tropical”. It simply stands for the French view of Brazil.

## 1 Arithmetic

Our basic object of study is the *tropical semiring*  $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ . As a set this is just the real numbers  $\mathbb{R}$ , together with an extra element  $\infty$  which represents infinity. However, we redefine the basic arithmetic operations of addition and multiplication of real numbers as follows:

$$x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y.$$

In words, the *tropical sum* of two numbers is their minimum, and the *tropical product* of two numbers is their sum. Here are some examples of how to do

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arithmetic in this strange number system. The tropical sum of 3 and 7 is 3. The tropical product of 3 and 7 equals 10. We write this as follows:

$$3 \oplus 7 = 3 \quad \text{and} \quad 3 \odot 7 = 10.$$

Many of the familiar axioms of arithmetic remain valid in tropical mathematics. For instance, both addition and multiplication are *commutative*:

$$x \oplus y = y \oplus x \quad \text{and} \quad x \odot y = y \odot x.$$

The *distributive law* holds for tropical addition and tropical multiplication:

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z.$$

Here is a numerical example to show distributivity:

$$\begin{aligned} 3 \odot (7 \oplus 11) &= 3 \odot 7 = 10, \\ 3 \odot 7 \oplus 3 \odot 11 &= 10 \oplus 14 = 10. \end{aligned}$$

Both arithmetic operations have a neutral element. Infinity is the *neutral element* for addition and zero is the *neutral element* for multiplication:

$$x \oplus \infty = x \quad \text{and} \quad x \odot 0 = 0.$$

Elementary school students tend to prefer tropical arithmetic because the multiplication table is easier to memorize, and even long division becomes easy. Here are the tropical *addition table* and the tropical *multiplication table*:

$\oplus$	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	$\odot$	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>
<b>1</b>	1	1	1	1	1	1	1	<b>1</b>	2	3	4	5	6	7	8
<b>2</b>	1	2	2	2	2	2	2	<b>2</b>	3	4	5	6	7	8	9
<b>3</b>	1	2	3	3	3	3	3	<b>3</b>	4	5	6	7	8	9	10
<b>4</b>	1	2	3	4	4	4	4	<b>4</b>	5	6	7	8	9	10	11
<b>5</b>	1	2	3	4	5	5	5	<b>5</b>	6	7	8	9	10	11	12
<b>6</b>	1	2	3	4	5	6	6	<b>6</b>	7	8	9	10	11	12	13
<b>7</b>	1	2	3	4	5	6	7	<b>7</b>	8	9	10	11	12	13	14

But watch out: tropical arithmetic is tricky when it comes to subtraction. There is no  $x$  which we can call “10 minus 3” because the equation  $3 \oplus x = 10$  has no solution  $x$  at all. To stay on safe ground, in this lecture, we shall content ourselves with using addition  $\oplus$  and multiplication  $\odot$  only.

It is extremely important to remember that “0” is the multiplicatively neutral element. For instance, the tropical *Pascal’s triangle* looks like this:

$$\begin{array}{cccccc}
 & & & & & 0 \\
 & & & & 0 & 0 \\
 & & & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots
 \end{array}$$

The rows of Pascal’s triangle are the coefficients appearing in the *Binomial Theorem*. For instance, the third row in the triangle represents the identity

$$\begin{aligned}
 (x \oplus y)^3 &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\
 &= 0 \odot x^3 \oplus 0 \odot x^2y \oplus 0 \odot xy^2 \oplus 0 \odot y^3.
 \end{aligned}$$

Of course, the zero coefficients can be dropped in this identity:

$$(x \oplus y)^3 = x^3 \oplus x^2y \oplus xy^2 \oplus y^3.$$

Moreover, the *Freshman’s Dream* holds for all powers in tropical arithmetic:

$$(x \oplus y)^3 = x^3 \oplus y^3.$$

The validity of the three displayed identities is easily verified by noting that the following equations hold in classical arithmetic for all  $x, y \in \mathbb{R}$ :

$$3 \cdot \min\{x, y\} = \min\{3x, 2x + y, x + 2y, 3y\} = \min\{3x, 3y\}.$$

**Research problem:** The set of convex polyhedra in  $\mathbb{R}^n$  can be made into a semiring by taking  $\odot$  as “Minkowski sum” and  $\oplus$  as “convex hull of the union”. A natural subalgebra is the set of all polyhedra which have a fixed *recession cone*  $C$ . If  $n = 1$  and  $C = \mathbb{R}_{\geq 0}$  then we get the tropical semiring. Develop linear algebra and algebraic geometry over these semirings, and implement efficient software for doing arithmetic with polyhedra when  $n \leq 4$ .

## 2 Polynomials

Let  $x_1, x_2, \dots, x_n$  be variables which represent elements in the tropical semiring  $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ . A *monomial* is any product of these variables, where

repetition is allowed. (Technical note: we allow negative integer exponents.) By commutativity, we can sort the product and write monomials in the usual notation, with the variables raised to exponents:

$$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 = x_1^2 x_2^3 x_3^2 x_4.$$

A monomial represents a function from  $\mathbb{R}^n$  to  $\mathbb{R}$ . When evaluating this function in classical arithmetic, what we get is a linear function:

$$x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 = 2x_1 + 3x_2 + 2x_3 + x_4.$$

Every linear function with integer coefficients arises in this manner.

**Fact 1.** *Tropical monomials are the linear functions with integer coefficients.*

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1, \dots, x_n) = a \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus b \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots$$

Here the coefficients  $a, b, \dots$  are real numbers and the exponents  $i_1, j_1, \dots$  are integers. Every tropical polynomial represents a function  $\mathbb{R}^n \rightarrow \mathbb{R}$ . When evaluating this function in classical arithmetic, what we get is the minimum of a finite collection of linear functions, namely,

$$p(x_1, \dots, x_n) = \min(a + i_1 x_1 + \cdots + i_n x_n, b + j_1 x_1 + \cdots + j_n x_n, \dots)$$

This function  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  has the following three important properties:

- $p$  is continuous,
- $p$  is piecewise-linear, where the number of pieces is finite, and
- $p$  is concave, i.e.,  $p(\frac{x+y}{2}) \geq \frac{1}{2}(p(x) + p(y))$  for all  $x, y \in \mathbb{R}^n$ .

It is known that every function which satisfies these three properties can be represented as the minimum of a finite set of linear functions. We conclude:

**Fact 2.** *The tropical polynomials in  $n$  variables  $x_1, \dots, x_n$  are precisely the piecewise-linear concave functions on  $\mathbb{R}^n$  with integer coefficients.*

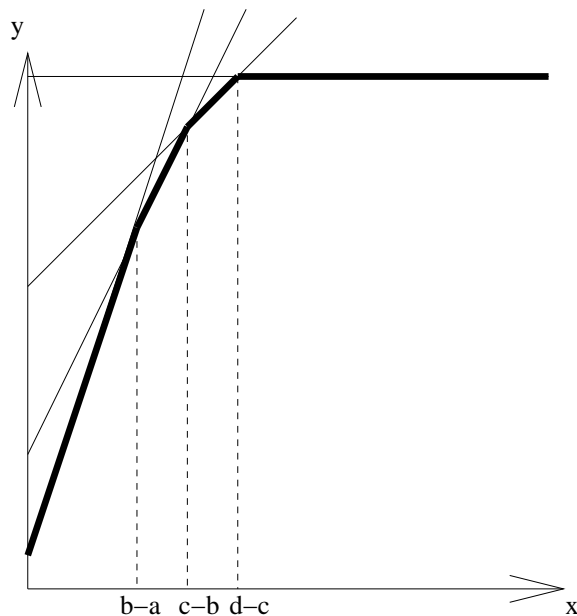


Figure 1: The graph of a cubic polynomial and its roots

As a first example consider the general cubic polynomial in one variable  $x$ ,

$$p(x) = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d. \quad (1)$$

To graph this function we draw four lines in the  $(x, y)$  plane:  $y = 3x + a$ ,  $y = 2x + b$ ,  $y = x + c$  and the horizontal line  $y = d$ . The value of  $p(x)$  is the smallest  $y$ -value such that  $(x, y)$  is on one of these four lines, i.e., the graph of  $p(x)$  is the lower envelope of the lines. All four lines actually contribute if

$$b - a \leq c - b \leq d - c. \quad (2)$$

These three values of  $x$  are the breakpoints where  $p(x)$  fails to be linear, and the cubic has a corresponding factorization into three linear factors:

$$p(x) = a \odot (x \oplus (b - a)) \odot (x \oplus (c - b)) \odot (x \oplus (d - c)). \quad (3)$$

See Figure 1 for the graph and the roots of the cubic polynomial  $p(x)$ .

Every tropical polynomial function can be written uniquely as a tropical product of tropical linear functions (i.e., the *Fundamental Theorem of Algebra* holds tropically). In this statement we must underline the word “function”.

Distinct polynomials can represent the same function. We are not claiming that every polynomial factors into linear functions. What we are claiming is that every polynomial can be replaced by an equivalent polynomial, representing the same function, that can be factored into linear factors, e.g.,

$$x^2 \oplus 17 \odot x \oplus 2 = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2.$$

Unique factorization of polynomials no longer holds in two or more variables. Here the situation is more interesting. Understanding it is our next problem.

**Research problem:** The factorization of multivariate tropical polynomials into irreducible tropical polynomials is not unique. Here is a simple example:

$$\begin{aligned} & (0 \odot x \oplus 0) \odot (0 \odot y \oplus 0) \odot (0 \odot x \odot y \oplus 0) \\ = & (0 \odot x \odot y \oplus 0 \odot x \oplus 0) \odot (0 \odot x \odot y \oplus 0 \odot y \oplus 0). \end{aligned}$$

Develop an algorithm (with implementation and complexity analysis) for computing all the irreducible factorizations of a given tropical polynomial. Gao and Lauder [12] have shown the importance of tropical factorization for the problem of factoring multivariate polynomials in the classical sense.

### 3 Curves

A tropical polynomial function  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  is given as the minimum of a finite set of linear functions. We define the *hypersurface*  $\mathcal{H}(p)$  to be the set of all points  $x \in \mathbb{R}^n$  at which this minimum is attained at least twice. Equivalently, a point  $x \in \mathbb{R}^n$  lies in  $\mathcal{H}(p)$  if and only if  $p$  is not linear at  $x$ . For example, if  $n = 1$  and  $p$  is the cubic in (1) with the assumption (2), then

$$\mathcal{H}(p) = \{b - a, c - b, d - c\}.$$

Thus the hypersurface  $\mathcal{H}(p)$  is the set of “roots” of the polynomial  $p(x)$ .

In this section we consider the case of a polynomial in two variables:

$$p(x, y) = \bigoplus_{(i,j)} c_{ij} \odot x^i \odot y^j.$$

**Fact 3.** *The tropical curve  $\mathcal{H}(p)$  is a finite graph which is embedded in the plane  $\mathbb{R}^2$ . It has both bounded and unbounded edges, all edge directions are rational, and this graph satisfies a zero tension condition around each node.*

The zero tension condition is the following geometric condition. Consider any node  $(x, y)$  of the graph and suppose it is the origin, i.e.,  $(x, y) = (0, 0)$ . Then the edges adjacent to this node lie on lines with rational slopes. On each such ray emanating from the origin consider the first non-zero lattice vector. *Zero tension* at  $(x, y)$  means that the sum of these vectors is zero.

Our first example is a *line* in the plane. It is defined by a polynomial:

$$p(x, y) = a \odot x \oplus b \odot y \oplus c \quad \text{where } a, b, c \in \mathbb{R}.$$

The curve  $\mathcal{H}(p)$  consists of all points  $(x, y)$  where the function

$$p : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto \min(a + x, b + y, c)$$

is not linear. It consists of three half-rays emanating from the point  $(x, y) = (c - a, c - b)$  into northern, eastern and southwestern direction.

Here is a general method for drawing a tropical curve  $\mathcal{H}(p)$  in the plane. Consider any term  $\gamma \odot x^i \odot y^j$  appearing in the polynomial  $p$ . We represent this term by the point  $(\gamma, i, j)$  in  $\mathbb{R}^3$ , and we compute the convex hull of these points in  $\mathbb{R}^3$ . Now project the lower envelope of that convex hull into the plane under the map  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $(\gamma, i, j) \mapsto (i, j)$ . The image is a planar convex polygon together with a distinguished subdivision  $\Delta$  into smaller polygons. The tropical curve  $\mathcal{H}(p)$  is the dual graph to this subdivision.

As an example we consider the general quadratic polynomial

$$p(x, y) = a \odot x^2 \oplus b \odot xy \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f.$$

Then  $\Delta$  is a subdivision of the triangle with vertices  $(0, 0)$ ,  $(0, 2)$  and  $(2, 0)$ . The lattice points  $(0, 1)$ ,  $(1, 0)$ ,  $(1, 1)$  are allowed to be used as vertices in these subdivisions. Assuming that  $a, b, c, d, e, f \in \mathbb{R}$  are general solutions of

$$2b \leq a + c, \quad 2d \leq a + f, \quad 2e \leq c + f,$$

the subdivision  $\Delta$  consists of four triangles, three interior edges and six boundary edges. The curve  $\mathcal{H}(p)$  has four vertices, three bounded edges and six half-rays (two northern, two eastern and two southwestern). In Figure 2,  $\mathcal{H}(p)$  is shown in bold, and the subdivision  $\Delta$  is shown in thin lines.

**Fact 4.** *Tropical curves intersect and interpolate like algebraic curves do.*

1. *Two general lines meet in one point, a line and a quadric meet in two points, two quadrics meet in four points, etc....*

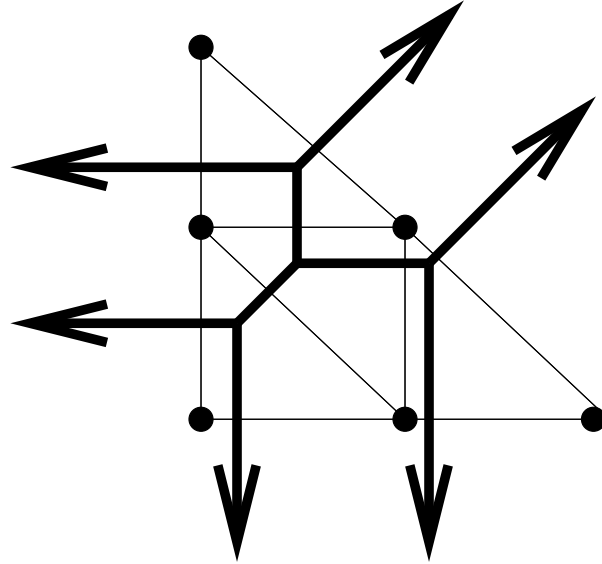


Figure 2: The subdivision  $\Delta$  and the tropical curve

2. *Two general points lie on a unique line, five general points lie on a unique quadric, etc...*

For a general discussion of *Bézout's Theorem* in tropical algebraic geometry, and for many pictures illustrating Fact 4, we refer to the article [20].

**Research problem:** Classify all combinatorial types of *tropical curves in 3-space* of degree  $d$ . Such a curve is a finite embedded graph of the form

$$C = \mathcal{H}(p_1) \cap \mathcal{H}(p_2) \cap \cdots \cap \mathcal{H}(p_r) \subset \mathbb{R}^3,$$

where the  $p_i$  are tropical polynomials,  $C$  has  $d$  unbounded parallel halfrays in each of the four coordinate directions, and all other edges of  $C$  are bounded.

## 4 Phylogenetics

An important problem in computational biology is to construct a *phylogenetic tree* from distance data involving  $n$  taxa. These taxa might be organisms or genes, each represented by a DNA sequence. For an introduction to phylogenetics we recommend [11] and [21]. Here is an example, for  $n = 4$ , to illustrate how such data might arise. Consider an alignment of four genomes:



Human: *ACAATGTCATTAGCGAT...*  
 Mouse: *ACGTTGTCAATAGAGAT...*  
 Rat: *ACGTAGTCATTACACAT...*  
 Chicken: *GCACAGTCAGTAGAGCT...*

From such sequence data, computational biologists infer the distance between any two taxa. There are various algorithms for carrying out this inference. They are based on statistical models of evolution. For our discussion, we may think of the distance between any two strings as a refined version of the Hamming distance (= the proportion of characters where they differ). In our (Human, Mouse, Rat, Chicken) example, the inferred distance matrix might be the following symmetric  $4 \times 4$ -matrix:

	<i>H</i>	<i>M</i>	<i>R</i>	<i>C</i>
<i>H</i>	0	1.1	1.0	1.4
<i>M</i>	1.1	0	0.3	1.3
<i>R</i>	1.0	0.2	0	1.2
<i>C</i>	1.4	1.3	1.2	0

The problem of phylogenetics is to construct a tree with edge lengths which represent this distance matrix, provided such a tree exists. In our example, a tree does exist. It is depicted in Figure 3. The number next to the each edge is its length. The distance between two leaves in the tree is the sum of the lengths of the edges on the unique path between the two leaves. For instance, the distance in this tree between “Human” and “Mouse” equals  $0.6 + 0.3 + 0.2 = 1.1$ , which is the corresponding entry in the  $4 \times 4$ -matrix.

In general, considering  $n$  taxa, the *distance* between taxon  $i$  and taxon  $j$  is a positive real number  $d_{ij}$  which has been determined by some bio-statistical method. So, what we are given is a real symmetric  $n \times n$ -matrix

$$D = \begin{pmatrix} 0 & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{12} & 0 & d_{23} & \cdots & d_{2n} \\ d_{13} & d_{23} & 0 & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & d_{3n} & \cdots & 0 \end{pmatrix}.$$

We may assume that  $D$  is a *metric*, i.e., the triangle inequalities  $d_{ik} \leq d_{ij} + d_{jk}$  hold for all  $i, j, k$ . This can be expressed by matrix multiplication:

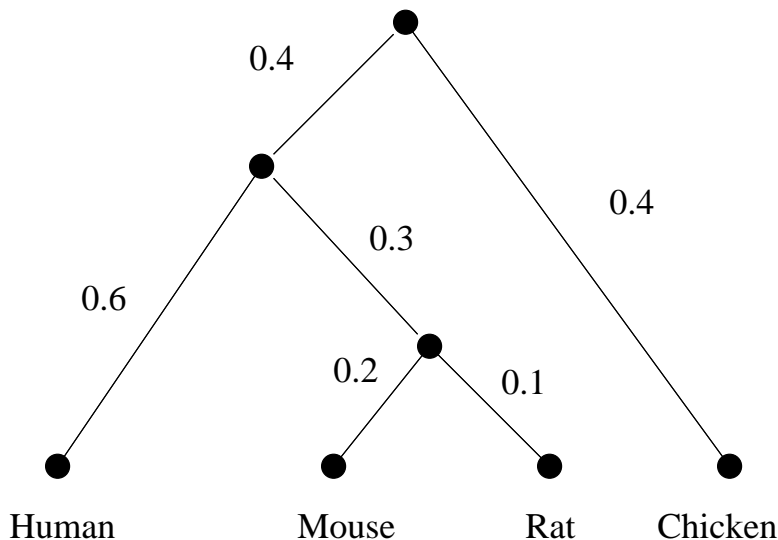


Figure 3: A Phylogenetic Tree

**Fact 5.** *The matrix  $D$  represents a metric if and only if  $D \odot D = D$ .*

We say that  $D$  is a *tree metric* if there exists a tree  $T$  with  $n$  leaves, labeled  $1, 2, \dots, n$ , and a positive length for each edge of  $T$ , such that the distance from leaf  $i$  to leaf  $j$  equals  $d_{ij}$  for all  $i, j$ . Tree metrics occur naturally in biology because they model an evolutionary process that led to the  $n$  taxa.

Most metrics  $D$  are not tree metrics. If we are given a metric  $D$  that arises from some biological data then it is reasonable to assume that there exists a tree metric  $D_T$  which is close to  $D$ . Biologists use a variety of algorithms (e.g. “neighbor joining”) to construct such a nearby tree  $T$  from the given data  $D$ . In what follows we state a tropical characterization of tree metrics.

Let  $X = (X_{ij})$  be a symmetric matrix with zeros on the diagonal whose  $\binom{n}{2}$  distinct off-diagonal entries are unknowns. For each quadruple  $\{i, j, k, l\} \subset \{1, 2, \dots, n\}$  we consider the following tropical polynomial of degree two:

$$p_{ijkl} = X_{ij} \odot X_{kl} \oplus X_{ik} \odot X_{jl} \oplus X_{il} \odot X_{jk}. \quad (4)$$

This polynomial is the *tropical Grassmann-Plücker relation*. It defines a hypersurface  $\mathcal{H}(p_{ijkl})$  in the space  $\mathbb{R}^{\binom{n}{2}}$ . The *tropical Grassmannian* is the intersection of these  $\binom{n}{4}$  hypersurfaces. This is a polyhedral fan, denoted

$$Gr_{2,n} = \bigcap_{1 \leq i < j < k < l \leq n} \mathcal{H}(p_{ijkl}) \subset \mathbb{R}^{\binom{n}{2}}.$$

**Fact 6.** A metric  $D$  on  $\{1, 2, \dots, n\}$  is a tree metric if and only if its negative  $X = -D$  is a point in the tropical Grassmannian  $Gr_{2,n}$ .

The statement is a reformulation of the *Four Point Condition* in phylogenetics, which states that  $D$  is a tree metric if and only if, for all  $1 \leq i < j < k < l \leq n$ , the *maximum* of the three numbers  $D_{ij} + D_{kl}$ ,  $D_{ik} + D_{jl}$  and  $D_{il} + D_{jk}$  is attained at least twice. For  $X = -D$ , this means that the *minimum* of the three numbers  $X_{ij} + X_{kl}$ ,  $X_{ik} + X_{jl}$  and  $X_{il} + X_{jk}$  is attained at least twice, or, equivalently,  $X \in \mathcal{H}(p_{ijkl})$ . The tropical Grassmannian  $Gr_{2,n}$  is also known as the *space of phylogenetic trees* [5]. The combinatorial structure of this beautiful space is well-studied and well-understood.

Our research suggestion for this section concerns a certain reembedding of the tropical Grassmannian  $Gr_{2,n}$  into a higher-dimensional space.

**Research problem:** Let  $n \geq 5$  and consider a metric  $D$  on  $\{1, 2, \dots, n\}$ . The *triple weights* of the metric  $D$  are defined as follows:

$$D_{ijk} := D_{ij} + D_{ik} + D_{jk} \quad (1 \leq i < j < k \leq n).$$

This formula specifies a linear map  $\psi : \mathbb{R}^{\binom{n}{2}} \rightarrow \mathbb{R}^{\binom{n}{3}}$ . Our problem is to characterize the image  $\psi(Gr_{2,n})$  of tree space  $Gr_{2,n}$  under this linear map.

This problem arises from phylogenies of sequence alignments in genomics. For such taxa, it can be more reliable statistically to estimate the triple weights  $D_{ijk}$  rather than the pairwise distances  $D_{ij}$ . Pachter and Speyer [18] showed that tree metrics are uniquely determined by their triple weights, i.e., the map from  $Gr_{2,n}$  onto  $\psi(Gr_{2,n})$  is a bijection. Find a natural system of tropical polynomials which define  $\psi(Gr_{2,n})$  as a tropical subvariety of  $\mathbb{R}^{\binom{n}{3}}$ .

## 5 Linear Spaces

This section is concerned with tropical linear spaces. Generalizing the notion of a line from Section 3, we define a *tropical hyperplane* to be a subset of  $\mathbb{R}^n$  which has the form  $\mathcal{H}(\ell)$ , where  $\ell$  is a tropical linear form in  $n$  unknowns:

$$\ell(x) = a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus \dots \oplus a_n \odot x_n.$$

Here  $a_1, \dots, a_n$  are arbitrary real constants. Solving linear equations in tropical mathematics means computing the intersection of finitely many hyperplanes  $\mathcal{H}(\ell)$ . It is tempting to define tropical linear spaces simply as intersections of tropical hyperplanes. However, this would not be a good definition

because such arbitrary intersections are not always pure dimensional, and they do not behave the way linear spaces do in classical geometry.

A better notion of tropical linear space is derived by allowing only those intersections of hyperplanes which are “sufficiently complete”. In what follows we offer a definition which is a direct generalization of our discussion in Section 4. The idea is that phylogenetic trees are lines in tropical projective space, whose Plücker coordinates  $X_{ij}$  are the negated pairwise distances  $d_{ij}$ .

We consider the  $\binom{n}{d}$ -dimensional space  $\mathbb{R}^{\binom{n}{d}}$  whose coordinates  $X_{i_1 \dots i_d}$  are indexed by  $d$ -element subsets  $\{i_1, \dots, i_d\}$  of  $\{1, 2, \dots, n\}$ . Let  $S$  be any  $(d-2)$ -element subset of  $\{1, 2, \dots, n\}$  and let  $i, j, k$  and  $l$  be any four distinct indices in  $\{1, \dots, n\} \setminus S$ . The corresponding *three-term Grassmann Plücker relation*  $p_{S,ijkl}$  is the following tropical polynomial of degree two:

$$p_{S,ijkl} = X_{Sij} \odot X_{Sk l} \oplus X_{Sik} \odot X_{Sj l} \oplus X_{Sil} \odot X_{Sj k}. \quad (5)$$

We define the *three-term tropical Grassmannian* to be the intersection

$$Gr_{d,n} = \bigcap_{S,i,j,k,l} \mathcal{H}(p_{S,ijkl}) \subset \mathbb{R}^{\binom{n}{d}},$$

where the intersection is over all  $S, i, j, k, l$  as above. Note that in the special case  $d = 2$  we have  $S = \emptyset$ , the polynomial (5) is the four point condition in (4), and  $Gr_{d,n}$  is the space of trees which was discussed in Section 4.

We now fix an arbitrary point  $X = (X_{i_1 \dots i_d})$  in the three-term tropical Grassmannian  $Gr_{d,n}$ . For any  $(d+1)$ -subset  $\{j_0, j_1, \dots, j_d\}$  of  $\{1, 2, \dots, n\}$  we consider the following tropical linear form in the variables  $x_1, \dots, x_n$ :

$$\ell_{j_0 j_1 \dots j_d}^X = \bigoplus_{r=0}^d X_{j_0 \dots \widehat{j_r} \dots j_d} \odot x_r. \quad (6)$$

The  $\widehat{\phantom{x}}$  means to omit  $j_r$ . The *tropical linear space* associated with the point  $X$  is the following set:

$$L_X = \bigcap \mathcal{H}(\ell_{j_0 j_1 \dots j_n}^X) \subset \mathbb{R}^n.$$

Here the intersection is over all  $(d+1)$ -subsets  $\{j_0, j_1, \dots, j_d\}$  of  $\{1, 2, \dots, n\}$ .

The tropical linear spaces are precisely the sets  $L_X$  where  $X$  is any point in  $Gr_{d,n} \subset \mathbb{R}^{\binom{n}{d}}$ . The “sufficient completeness” referred to in the first paragraph of this section means that we need to solve linear equations using

*Cramer's rule*, in all possible ways, in order for an intersection of hyperplanes to actually be a linear space. The definition of linear space given here is more inclusive than the one used in [10, 20, 24], where  $L_X$  was required to come from ordinary algebraic geometry over a field with a suitable valuation.

For example, a 3-dimensional tropical linear subspace of  $\mathbb{R}^n$  (a.k.a. two-dimensional plane in tropical projective  $(n - 1)$ -space) is the intersection of  $\binom{n}{4}$  tropical hyperplanes, each of whose defining linear form has four terms:

$$\ell_{j_0 j_1 j_2 j_3}^X = X_{j_0 j_1 j_2} \odot x_{j_3} \oplus X_{j_0 j_1 j_3} \odot x_{j_2} \oplus X_{j_0 j_2 j_3} \odot x_{j_1} \oplus X_{j_1 j_2 j_3} \odot x_{j_0}.$$

We note that even the very special case when each coordinate of  $X$  is either 0 (the multiplicative unit) or  $\infty$  (the additive unit) is really interesting. Here  $L_X$  is a polyhedral fan known as the *Bergman fan* of a matroid [2, 25].

Tropical linear spaces have many of the properties of ordinary linear spaces. First, they are pure polyhedral complexes of the correct dimension:

**Fact 7.** *Each maximal cell of the tropical linear space  $L_X$  is  $d$ -dimensional.*

Every tropical linear space  $L_X$  determines its vector of tropical Plücker coordinates  $X$  uniquely up to tropical multiplication (= classical addition) by a common scalar. If  $L$  and  $L'$  are tropical linear spaces of dimensions  $d$  and  $d'$  with  $d + d' \geq n$ , then  $L$  and  $L'$  meet. It is not quite true that two tropical linear spaces intersect in a tropical linear space but it is almost true. If  $L$  and  $L'$  are tropical linear spaces of dimensions  $d$  and  $d'$  with  $d + d' \geq n$  and  $v$  is a generic small vector then  $L \cap (L' + v)$  is a tropical linear space of dimension  $d + d' - n$ . Following [20], it makes sense to define the *stable intersection* of  $L$  and  $L'$  by taking the limit of  $L \cap (L' + v)$  as  $v$  goes to zero, and this limit will again be a tropical linear space of dimension  $d + d' - n$ .

**Research Problem:** It is not true that a  $d$ -dimensional tropical linear space can always be written as the intersection of  $n - d$  tropical hyperplanes. The definition shows that  $\binom{n}{d+1}$  hyperplanes are always enough. What is the minimum number of tropical hyperplanes needed to cut out any tropical linear space of dimension  $d$  in  $n$ -space? Are  $n$  hyperplanes always enough?

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