Codes and designs in Johnson graphs with high symmetry

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Abstract

The Johnson graph $J(v,k)$ has, as vertices, all k-subsets of a v-set $\mathcal{V}$, with two k-subsets adjacent if and only if they share $k-1$ common elements of $\mathcal{V}$. Subsets of vertices of $J(v,k)$ can be interpreted as the blocks of an incidence structure, or as the codewords of a code, and automorphisms of $J(v,k)$ leaving the subset invariant are then automorphisms of the corresponding incidence structure or code. This approach leads to interesting new designs and codes. For example, numerous actions of the Mathieu sporadic simple groups give rise to examples of Delandtsheer designs (which are both flag-transitive and anti-flag transitive), and codes with large minimum distance (and hence strong error-correcting properties). The paper surveys recent progress, explores links between designs and codes in Johnson graphs which have a high degree of symmetry, and discusses several open questions.

Key-words: designs, codes in graphs, Johnson graph, 2-transitive permutation group, neighbour-transitive, Delandtsheer design, flag-transitive, antiflag-transitive.

Mathematics Subject Classification (2010): 05C25, 20B25, 94B60.

1 Introduction

The Johnson graphs are ubiquitous in mathematics, perhaps because of their many useful properties. They are distance transitive, and indeed geodesic-transitive, they underpin the Johnson association schemes – and 'everyone's favourite graph', the Petersen Graph, occurs as the complement of one of them[1]. Also, the class of Johnson graphs played a key role in Babai's recent breakthrough [2] to a quasipolynomial bound on the complexity of graph isomorphism testing[2].

Our focus in this paper is studying Johnson graphs as 'carrier spaces' of error-correcting codes and combinatorial designs, and in particular using group theory to find surprisingly rich families of examples of codes and designs with 'high symmetry'.

The Johnson graph $J(\mathcal{V},k)$ (or $J(v,k)$) is based on a set $\mathcal{V}$ of v elements, called *points*. Its vertices are the k-subsets of $\mathcal{V}$, and distinct k-subsets are adjacent precisely when their intersection has size $k-1$. For distinct k-subsets γ,γ', their distance $d(\gamma,\gamma')$ (length of shortest path from γ to γ') in $J(\mathcal{V},k)$ is therefore $k-|\gamma\cap\gamma'|$. The *complementing map* τ which maps each k-subset of $\mathcal{V}$ to its complement induces a graph isomorphism $\tau: J(\mathcal{V},k)\to J(\mathcal{V},v-k)$. For this reason we may, and sometimes we do, replace $J(\mathcal{V},k)$ by $J(\mathcal{V},v-k)$ in our analysis and thereby assume that $k\leqslant v/2$. The symmetric group $\mathrm{Sym}(\mathcal{V})$ acts as automorphisms on $J(v,k)$, and if $k\neq v/2$ then $\mathrm{Sym}(\mathcal{V})$ is the full automorphism group, while if $k=v/2$ then the

[1] The Petersen graph is the complement of the Johnson graph $J(5,2)$; a whole book has been written about it [17].

[2] Babai, whose work relies on both group-theoretic and combinatorial techniques, found that "in a well-defined sense, the Johnson graphs are the only obstructions to effective canonical partitioning". See also Helfgott's lecture [15].

automorphism group is $\mathrm{Sym}(\mathcal{V}) \times \langle\tau\rangle$, (see, for example, [28] for $k \neq v/2$ and [12] for $k = v/2$). We will work with symmetry provided by $\mathrm{Sym}(\mathcal{V})$.

The link between $J(\mathcal{V}, k)$ and combinatorial designs is fairly clear: points of the design are elements of $\mathcal{V}$, and if each block of the design is incident with exactly k points, then the blocks are vertices of $J(\mathcal{V}, k)$. Similarly, a code in $J(\mathcal{V}, k)$ is a subset of vertices. We next make some further brief comments on each of these links in turn.

1.1 Codes in Johnson graphs

In 1973, Philippe Delsarte [7] introduced the notion of a code in a distance-regular graph: a vertex subset $\mathcal{C}$ is considered to be a code, its elements are the codewords, and distance between codewords is the natural distance in the graph. In particular the *minimum distance* $\delta(\mathcal{C})$ of $\mathcal{C}$ is the minimum length of a path between distinct codewords, and the *automorphism group* $\mathrm{Aut}(\mathcal{C})$ is the setwise stabiliser of $\mathcal{C}$ in the automorphism group of the graph.

Delsarte defined a special type of code, now called a completely-regular code, 'which enjoys combinatorial (and often algebraic) symmetry akin to that observed for perfect codes' (see [24, page 1], and also Section 2), and he posed explicitly the question of existence of completely-regular codes in Johnson graphs. Such codes in Johnson graphs were studied by Meyerowitz [25, 26] and Martin [22, 23], but disappointingly, not many were found with good error-correcting properties (large distance between distinct codewords). In joint work with Liebler [21], the stringent regularity conditions imposed for complete regularity, were replaced by a 'local transitivity' property: a *neighbour-transitive code* in $J(\mathcal{V}, k)$ was defined as a vertex-subset $\mathcal{C}$ such that $\mathrm{Aut}(\mathcal{C})$ (*which we will take here as the setwise stabiliser of* $\mathcal{C}$ *in* $\mathrm{Sym}(\mathcal{V})$*, even if* $k = v/2$) is transitive both on $\mathcal{C}$ and on the set $\mathcal{C}_1$ of 'code-neighbours' (the non-codewords which are adjacent in $J(\mathcal{V}, k)$ to some codeword). If $\delta(\mathcal{C}) \geqslant 3$, for a neighbour-transitive code $\mathcal{C}$, it turns out (see [21, Theorem 1.2]) that $\mathrm{Aut}(\mathcal{C})$ has an even stronger property, namely, $\mathrm{Aut}(\mathcal{C})$ is transitive on the set of triples

$$(u, u', \gamma) \text{ where } \gamma \in \mathcal{C},\ u \in \gamma, \text{ and } u' \in \mathcal{V} \setminus \gamma. \tag{1.1}$$

A code $\mathcal{C}$ with this property is called *strongly-incidence-transitive*, and we will say that $\mathcal{C}$ *has the SIT-property* and is an *SIT-code*. We will describe surprisingly rich classes of SIT-codes in Johnson graphs arising from both combinatorial and geometric constructions, and we will mention some open problems.

We observe that a code consisting of a single codeword γ trivially has the SIT-property, since its automorphism group is $\mathrm{Sym}(\gamma) \times \mathrm{Sym}(\mathcal{V} \setminus \gamma)$. To avoid such trivial cases, we will always therefore assume that *SIT-codes* $\mathcal{C}$ *have size at least* 2. We call such codes *nontrivial*. 'Nontriviality' still allows the following examples:

- $k = v/2$ and $\mathcal{C}$ consists of a single k-subset γ together with its complement. Here the automorphism group is $\mathrm{Sym}(\gamma) \wr \mathrm{Sym}(2)$, it has the SIT-property and is transitive on $\mathcal{V}$.
- $\mathcal{C} = \binom{\mathcal{V}}{k}$ is the *complete code* containing all k-subsets of $\mathcal{V}$. Here the automorphism group $\mathrm{Sym}(\mathcal{V})$ has the SIT-property and of course is transitive on $\mathcal{V}$.

Let $\mathcal{C}$ be a nontrivial SIT-code in $J(v,k)$. If $k = v/2$ and $\mathcal{C}$ consists of a single k-subset and its complement, then as we noted above its automorphism group is transitive on $\mathcal{V}$. In all other cases there exist distinct codewords $\gamma, \gamma' \in \mathcal{C}$ with $\gamma' \neq \mathcal{V} \setminus \gamma$. Then for $G = \mathrm{Aut}(\mathcal{C})$, the stabiliser G_γ has two orbits in $\mathcal{V}$, namely γ and $\mathcal{V} \setminus \gamma$, and similarly $G_{\gamma'}$ is transitive on γ' and $\mathcal{V} \setminus \gamma'$. Also, at least one of γ', $\mathcal{V} \setminus \gamma'$ meets both γ and $\mathcal{V} \setminus \gamma$, and hence $\langle G_\gamma, G_{\gamma'} \rangle$ is transitive on $\mathcal{V}$. It follows from this discussion that every nontrivial SIT-code has automorphism group transitive on $\mathcal{V}$. We record this fact, and also a property on the minimum distance proved in [21].

Lemma 1.1 *[21, Lemma 2.1] Let $\mathcal{C}$ be an SIT- code in $J(\mathcal{V},k)$. If $|\mathcal{C}| \geqslant 2$, then* $\mathrm{Aut}(\mathcal{C})$ *is transitive on $\mathcal{V}$. If $\mathcal{C}$ is not the complete code $\binom{\mathcal{V}}{k}$, then the minimum distance $\delta(\mathcal{C}) \geqslant 2$.*

1.2 Designs in Johnson graphs: Delandtsheer designs

As mentioned above a vertex-subset $\mathcal{C}$ of $J(\mathcal{V},k)$ can be interpreted as the block-set of a design. We will usually assume some additional regularity properties on $\mathcal{C}$. For $1 \leqslant t < k < v$, a $t-(v,k,\lambda)$ *design* $\mathcal{D} = (\mathcal{V},\mathcal{C})$ consists of a point-set $\mathcal{V}$ of size v, and a subset $\mathcal{C}$ of k-subsets of $\mathcal{V}$ (called blocks) with the property that each t-subset of $\mathcal{V}$ is contained in exactly λ blocks. A point-block pair (u,γ) is called a *flag*, or an *antiflag* according as $u \in \gamma$ or $u \notin \gamma$, respectively. If $t = 2$ and $\lambda = 1$, then $\mathcal{D}$ is called a *linear space*. The automorphism group $\mathrm{Aut}(\mathcal{D})$ is the setwise stabiliser of $\mathcal{C}$ in $\mathrm{Sym}(\mathcal{V})$ – *hence the same group occurs whether we regard $\mathcal{C}$ as a code or the block set of a design.*

In 1984, Delandtsheer [5] classified all antiflag-transitive linear spaces $\mathcal{D} = (\mathcal{V},\mathcal{C})$, proving that the linear spaces with this property are the projective and affine spaces, the Hermitian unitals, and two exceptional examples. In her paper she noticed that all, apart from the two exceptional examples, possessed a stronger property, namely the block set $\mathcal{C}$ has the SIT-property when viewed as a code in $J(\mathcal{V},k)$. Requiring $\mathcal{C}$ to have the SIT-property implies both flag-transitivity and antiflag-transitivity of $\mathcal{D}$, and it is even more restrictive, as illustrated by the two exceptions in [5]. Here is a simpler example: the unique 1-$(4,2,2)$ design, which is the edge set of a 4-cycle in the complete graph $K_4 = J(4,2)$, is flag-transitive and antiflag-transitive, but does not have the SIT property. It is only very recently that we noticed that the SIT-property for codes coincides with this property observed by Delandtsheer for antiflag-transitive linear spaces. We want to consider designs, not necessarily linear spaces, with the SIT-property. Trivially, if $\mathcal{C}$ contains only one block then, as we remarked above, $\mathcal{C}$ has the SIT-property. We avoid this case and consider only designs with more than one block. We therefore (appropriately) refer to any design $\mathcal{D} = (\mathcal{V},\mathcal{C})$ in $J(\mathcal{V},k)$ as a *Delandtsheer design* if $\mathcal{C}$ has the SIT-property and $|\mathcal{C}| \geqslant 2$.

Thus each new Delandtsheer design corresponds to a new SIT code, and conversely.

1.3 Imprimitive examples and a dichotomy

For each nontrivial strongly incidence-transitive (SIT) code in $J(\mathcal{V}, k)$, the automorphism group is transitive on $\mathcal{V}$ and apart from the complete code, the minimum distance is at least 2, see Lemma 1.1. In Section 2 we show how to produce an infinite family of SIT-codes via a *blow-up* procedure: Construction 2.1 provides a general method for building SIT-codes and Delandtsheer designs $\mathcal{C}$ from smaller codes/designs $\mathcal{C}_0$ possessing the SIT-property. As long as the input code $\mathcal{C}_0$ has size at least two, the output 'blow-up' code $\mathcal{C}$ admits as automorphism group a full wreath product in its transitive imprimitive action. Moreover $\mathcal{C}$ is the block set of a Delandtsheer 1-design, but never of a 2-design, see Lemmas 2.2 and 2.3.

Importantly, Proposition 2.4 shows that the imprimitive SIT-codes are precisely those arising from Construction 2.1, and by Theorem 2.5, each imprimitive SIT-code can be obtained by applying Construction 2.1 to a primitive SIT-code.

Despite this structural link between the imprimitive and primitive strongly incidence-transitive codes, the behaviour of these two sub-families of examples is vastly different, especially from the viewpoint of designs. To begin with, the primitive examples have very restricted automorphism groups.

Theorem 1.2 *[21, Theorem 1.2] Let $\mathcal{C}$ be a strongly incidence-transitive code in $J(\mathcal{V}, k)$ such that $|\mathcal{C}| \geqslant 2$, and suppose that* $\mathrm{Aut}(\mathcal{C})$ *is primitive on $\mathcal{V}$. Then* $\mathrm{Aut}(\mathcal{C})$ *is 2-transitive on $\mathcal{V}$.*

A subgroup $G \leqslant \mathrm{Sym}(\mathcal{V})$ is called 2-*transitive* if it is transitive on the ordered pairs of distinct points from $\mathcal{V}$. We note that Theorem 1.2 is valid even if the condition $2 \leqslant k \leqslant v-2$ in [21, Theorem 1.2] is removed, since $k = 1$ or $k = v - 1$ implies that $\mathcal{C} = \binom{\mathcal{V}}{k}$ is a complete code with 2-transitive automorphism group $\mathrm{Aut}(\mathcal{C}) = \mathrm{Sym}(\mathcal{V})$.

If a design $(\mathcal{V}, \mathcal{C})$ in $J(\mathcal{V}, k)$ has automorphism group 2-transitive on $\mathcal{V}$ then there is a constant λ such that each point-pair is contained in exactly λ blocks, that is, $(\mathcal{V}, \mathcal{C})$ is a $2-(v, k, \lambda)$ design. Thus Theorem 1.2 leads to a striking dichotomy between the imprimitive and primitive Delandtsheer designs:

Theorem 1.3 *Let $\mathcal{D} = (\mathcal{V}, \mathcal{C})$ be a Delandtsheer design in $J(\mathcal{V}, k)$ with automorphism group G. If G is primitive on $\mathcal{V}$ then $\mathcal{D}$ is a $2-(v, k, \lambda)$ design, for some λ. On the other hand if G is not primitive on $\mathcal{V}$, then $\mathcal{D}$ is a $1-(v, k, \lambda)$ design, for some λ, but is never a 2-design.*

1.4 Primitive examples: towards a classification

Thus understanding the primitive SIT-codes and Delandtsheer designs, is of central importance. Numerous actions of the sporadic Mathieu groups [27] have been shown to give rise to examples of Delandtsheer designs which, when viewed as SIT-codes in Johnson graphs, have large minimum distances and hence strong error-correcting properties. Other natural geometrical examples of Delandtsheer designs come from subspaces and classical unitals [21] (not all of them linear spaces), binary quadratic forms [18], and more exotic geometrical examples linked to cones, cylinders and maximal arcs have been constructed by Durante [11]. Yet more examples

	Group	Degree
Mathieu	M_n	$n \in \{11, 12, 22, 23, 24\}$, or
	M_{11}	12
alternating	A_7	15
projective	$\mathrm{PSL}(2, 11)$	11
sporadic	HS	176, or
	Co_3	276

Table 1: 'Sporadic' 2-transitive permutation groups

projective	$\mathrm{PSL}(n, q) \leqslant G \leqslant \mathrm{P\Gamma L}(n, q)$ on $\mathrm{PG}(n-1, q)$
rank 1	the Suzuki, Ree and Unitary groups
affine	$G \leqslant \mathrm{A\Gamma L}(\mathcal{V})$ acting on $\mathcal{V} = \mathbb{F}_q^n$
symplectic	$\mathrm{Sp}(2n, 2)$ on $Q^{\varepsilon}(2)$, $\varepsilon = \pm$

Table 2: Infinite families of 2-transitive permutation groups

arise from symmetric 2-designs [18]. We discuss some of these examples and give references for locating other known examples.

The fact that all finite 2-transitive permutation groups are known explicitly, as a consequence of the finite simple group classification (see for example [10, Chapter 7.7]), suggests that an obvious strategy for finding all SIT-codes and Delandtsheer designs is to analyse all these groups, or families of groups, one by one. Significant progress has been made, which we will outline. We also point out some major open cases.

We may subdivide the finite 2-transitive permutation groups according to whether or not they lie in an infinite family of 2-transitive groups. Those which do not lie in an infinite family we call *sporadic*; these are listed in Table 1. Note that the Mathieu group M_{11} occurs in both line 1 and line 2 of Table 1, corresponding to its two 2-transitive actions of degrees 11 and 12 (with point stabilisers M_{10} and $\mathrm{PSL}(2, 11)$), respectively. The infinite families of finite 2-transitive groups G, not containing the alternating group, are listed in Table 2.

When determining examples,

we will from now on assume that $3 \leqslant k \leqslant v/2$.

We may do this because, if $k = 1, 2, v-2$ or $v-1$, then the 2-transitive group G is transitive on k-subsets of $\mathcal{V}$, and hence the only G-SIT-code in $J(v, k)$ is the complete code consisting of all k-subsets of $\mathcal{V}$. Also if $v/2 < k \leqslant v-3$ and $\mathcal{C}$ is an SIT-code in $J(v, k)$, then $\{\mathcal{V} \setminus \gamma \mid \gamma \in \mathcal{C}\}$ is an SIT-code in $J(v, v-k)$ with the same automorphism group as $\mathcal{C}$, and we have $3 \leqslant v-k < v/2$. In these cases for each G-SIT-code $\mathcal{C}$ in $J(v, k)$, the pair $(\mathcal{V}, \mathcal{C})$ is a Delandtsheer $2-(v, k, \lambda)$ design, for some λ. To determine the value of λ we first note that the number of blocks is $b = |\mathcal{C}| = |G : G_\gamma|$ for $\gamma \in \mathcal{C}$. Then, counting the number of triples (u, u', γ) with γ a block and u, u' distinct points of γ, gives

$$\lambda = \frac{bk(k-1)}{v(v-1)}. \tag{1.2}$$

Analysis of the sporadic cases was completed in [27], yielding 27 strongly-incidence-transitive (code, group) pairs in $J(v,k)$ with $3 \leqslant k \leqslant v/2$, which we list in Table 3 in Section 3. Table 3 also gives details of the corresponding Delandtsheer designs, many but not all of which are well-known. They include the Steiner systems from the Mathieu groups and the 11-point biplane.

This leaves the infinite families of 2-transitive groups G to be considered. We discuss the 'projective' and 'rank 1' lines of Table 2 in Section 4. These cases are completely resolved using work in [11, 21]. The examples are well known, and comprise the classical unital, designs of projective subspaces of given dimension, designs of Baer sublines of a projective line, and one exceptional example, see Table 4.

The 'affine' line of Table 2 deserves special mention because the classification here is still open. As in the projective linear case, we obtain examples by taking the set of affine subspaces of given dimension. Also, in [11, Section 3.2], Durante constructed additional infinite families of examples geometrically in affine spaces over fields of order $q \in \{4, 16\}$: from cylinders in $\mathrm{AG}(n,4)$ with base the hyperoval in $\mathrm{PG}(2,4)$ or its complement, and from unions of two or four parallel hyperplanes, for $q = 4$ or $q = 16$, respectively. Moreover, Durante [11, Theorem 27] showed that when $q \in \{4, 16\}$ these are the only examples with $3 \leqslant k \leqslant v/2$. By the results in [21, Section 6] and [11] it was believed that the classification of the affine SIT-codes was complete. However Mark Ioppolo discovered an error in the proof of [21, Proposition 6.6] in the case $q = 2$. He showed [18, Lemma A.1] that the analysis in [11, 21] completed the classification for affine spaces over all fields of size at least 3, and produced [18, Example 7.11, Theorem 7.13], two additional infinite families of affine SIT-codes and Delandtsheer designs with $q = 2$. They are related to the symplectic symmetric 2-designs arising in Kantor's classification of 2-transitive symmetric designs [20]. Ioppolo [18, Theorem 7.9] also showed how to use the blowup Construction 2.1 to produce larger examples admitting an affine group (not 2-transitive). The known affine examples are summarised in Table 5 in Section 5, and in particular we discuss there what is required to resolve the following problem.

Problem 1 *Complete the classification of SIT-codes and Delandtsheer designs admitting a 2-transitive affine group over a field of order 2.*

The last infinite family of 2-transitive groups, in the 'symplectic' line of Table 2, corresponds to the Jordan–Steiner actions of the symplectic groups $G = \mathrm{Sp}(2n,2)$ on nondegenerate quadratic forms. Investigating this was the major topic of Ioppolo's thesis [18], and we discuss his findings in Section 6. He identifies two distinct infinite families of examples arising from subspace actions of the symplectic groups [18, Chapter 4], and shows that, for any further examples of G-SIT codes, the stabiliser of a codeword G_γ is an almost simple group acting absolutely irreducibly on the underlying space $\mathbb{F}_2^{2n}$, see [18, Theorem 8.2] or [3, Theorem 1.4]. In the case where G_γ is almost simple, there is at least one further example: one in $J(136,10)$ with automorphism group S_{10} constructed in [18, Section 5.4]. On the other hand G_γ is not a sporadic group or an exceptional group of Lie type, [18, Theorem 8.5]. It would be very nice to see a full classification.

Problem 2 *Complete the classification of SIT-codes and Delandtsheer designs admitting a 2-transitive symplectic group in one of its Jordan–Steiner actions.*

In the final Section 7 we discuss several related families of codes and designs. These include pairwise transitive designs and binary linear codes.

Remark I began the study of SIT-codes in 2005 in collaboration with R. A. (Bob) Liebler. Sadly, Bob died in July 2009 while hiking in California, some years before our joint paper [21] was completed. When proof-reading the paper [21], I failed to identify several misprints introduced during the typesetting process: namely, around 5% of expressions of the form $\binom{\mathcal{V}}{k}$ had been changed to $\frac{\mathcal{V}}{k}$. This affects the following statements in [21]: Lemma 3.2 parts (a) and (c), Proposition 6.1 (ii), and Example 8.1. There are also a few instances of these misprints in the proofs in [21].

2 Completely regular codes and a blow-up Construction

A code $\mathcal{C}$ in $J(\mathcal{V},k)$ determines a distance partition $\{\mathcal{C}_0,\ldots,\mathcal{C}_{r-1}\}$ of the vertex set of $J(\mathcal{V},k)$, where $\mathcal{C}_0=\mathcal{C}$ is the code itself, $\mathcal{C}_1$ is the set of code-neighbours, and in general $\mathcal{C}_i$ is the set of k-subsets which are at distance i from at least one codeword in $\mathcal{C}$, and at distance at least i from every codeword. For the last non-empty set $\mathcal{C}_{r-1}$, the parameter r is called the covering radius of $\mathcal{C}$. The code is *completely-regular* if, for any $i,j\in\{0,\ldots,r-1\}$ and $\gamma\in\mathcal{C}_i$, the number of vertices of $\mathcal{C}_j$ which are adjacent to γ in $J(\mathcal{V},k)$ is independent of the choice of γ in $\mathcal{C}_i$, and depends only on i and j. Further, $\mathcal{C}$ is called *completely-transitive* if $\mathrm{Aut}(\mathcal{C})$ (which fixes each $\mathcal{C}_i$ setwise) is transitive on each $\mathcal{C}_i$.

For our blow-up construction we begin with a code $\mathcal{C}_0$ in $J(\mathcal{U},k_0)$ and essentially replace each element of $\mathcal{U}$ with a set of constant size $a>1$ to obtain a code in $J(\mathcal{V},ak_0)$ with base set $\mathcal{V}$ of size $a|\mathcal{U}|$. These blow-up codes are a generalisation of the *groupwise complete designs* introduced by Martin in [22] which correspond to the special case where $\mathcal{C}_0$ is the complete code $\binom{\mathcal{U}}{k_0}$. Martin [22, Theorem 2.1] determined all groupwise complete designs that are completely-regular codes, and as noted in [21, Remark 4.5], most of the codes we introduce below are *not* completely regular. We note that complete codes provide somewhat trivial examples of strongly incidence-transitive codes and Delandtsheer designs.

Construction 2.1 *Let $\mathcal{U}=\{U_1|U_2|\ldots|U_{v_0}\}$ be a partition of the v-set $\mathcal{V}$ with v_0 parts of size a, where $v=av_0, a>1, v_0\geqslant 2$, and let $k=ak_0$ where $1\leqslant k_0\leqslant v_0-1$. For a code $\mathcal{C}_0$ in $J(\mathcal{U},k_0)$ with $|\mathcal{C}_0|\geqslant 2$, the blow-up code $\mathcal{C}(a,\mathcal{C}_0)$ in $J(\mathcal{V},k)$ is the set of all k-subsets of $\mathcal{V}$ of the form $\cup_{U\in\gamma_0}U$, for some $\gamma_0\in\mathcal{C}_0$.*

This construction, in the special case where $\mathcal{C}_0$ is the complete code $\binom{\mathcal{U}}{k_0}$, was introduced by Bill Martin [22] in 1994. He called such codes $\mathcal{C}(a,\mathcal{C}_0)$ *groupwise complete designs*, and in particular he determined the groupwise complete designs which are completely-regular codes in $J(\mathcal{V},k)$. The general Construction 2.1 above is [21, Example 4.4] except that we allow v_0 to take any value at least 2. In [21, Example 4.4], $v_0\geqslant 4$ is assumed, but we note that [21, Lemma 4.6] is valid for all $v_0\geqslant 2$. In particular Construction 2.1 allows the possibility $k_0=1$, in which case the smaller Johnson graph $J(\mathcal{U},1)$ is the complete graph with vertex set $\mathcal{U}$. In all cases the construction preserves the property of strong incidence-transitivity. It is inherently an 'imprimitive construction' in the sense that the natural automorphism group preserving the code leaves the nontrivial partition $\mathcal{U}$ invariant. Recall that we define $\mathrm{Aut}(\mathcal{C}_0)$ as the stabiliser of $\mathcal{C}_0$ in $\mathrm{Sym}(\mathcal{U})$.

Lemma 2.2 *[21, Lemma 4.6] Let $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ as in Construction 2.1, and let $A = \mathrm{Aut}(\mathcal{C}_0)$.*

(a) Then $\delta(\mathcal{C}) = a\,\delta(\mathcal{C}_0)$ and $\mathrm{Aut}(\mathcal{C})$ contains $\mathrm{Sym}(a) \wr A$ in its imprimitive action on $\mathcal{V}$;

(b) if $a \geqslant 3$, or if $a = 2 \leqslant \delta(\mathcal{C}_0)$, then $\mathcal{C}$ is $(\mathrm{Sym}(a) \wr A)$-strongly incidence-transitive if and only if $\mathcal{C}_0$ is A-strongly incidence-transitive;

(c) in particular if $\mathcal{C}_0$ is strongly incidence-transitive, then $\mathcal{C}$ is also strongly incidence-transitive.

Thus Construction 2.1 produces many strongly incidence-transitive codes, and hence also many Delandtsheer designs. It turns out that essentially all are 1-designs and none are 2-designs, even if the input $(\mathcal{U}, \mathcal{C}_0)$ is a 2-design.

Lemma 2.3 *Let $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ be a code in $J(\mathcal{V}, k)$ as in Construction 2.1, for some strongly incidence-transitive code $\mathcal{C}_0$ in $J(\mathcal{U}, k_0)$, where $|\mathcal{C}_0| \geqslant 2$, $v = |\mathcal{V}| = a|\mathcal{U}| = av_0$, and $k = ak_0$. Then $(\mathcal{U}, \mathcal{C}_0)$ is a Delandtsheer $1-(v_0, k_0, r)$ design, for some r, and $(\mathcal{V}, \mathcal{C})$ is a Delandtsheer $1-(v, k, r)$ design but is not a 2-design.*

Proof The fact that $\mathcal{C}$ is strongly incidence-transitive follows from Lemma 2.2(c). We note that $|\mathcal{C}| = |\mathcal{C}_0| \geqslant 2$, from Construction 2.1, and hence, by definition, both $(\mathcal{U}, \mathcal{C}_0)$ and $(\mathcal{V}, \mathcal{C})$ are Delandtsheer designs. Also, by Lemma 1.1, $A = \mathrm{Aut}(\mathcal{C}_0)$ is transitive on $\mathcal{U}$ and $\mathrm{Sym}(a) \wr A$ is transitive on $\mathcal{V}$, and hence both are 1-designs, say, $(\mathcal{U}, \mathcal{C}_0)$ is a $1-(v_0, k_0, r)$ design. Then it is easily seen from the construction that $(\mathcal{V}, \mathcal{C})$ is a $1-(v, k, r)$ design with the same parameter r. It remains to show that $(\mathcal{V}, \mathcal{C})$ is not a 2-design.

Consider distinct points u, u' in the same class U_i of $\mathcal{U}$. Then as U_i lies in exactly r blocks of $\mathcal{C}_0$, the pair $\{u, u'\}$ lies in exactly r blocks of $\mathcal{C}$. Suppose that $(\mathcal{V}, \mathcal{C})$ is a $2-(v, k, \lambda)$ design. Let u, u' lie in distinct classes U, U' of $\mathcal{U}$. Then by the definition of $\mathcal{C}$, the set of blocks of $\mathcal{C}$ containing $\{u, u'\}$ is in one-to-one correspondence with the set of blocks of $\mathcal{C}_0$ containing $\{U, U'\}$. Since this set has size $\lambda > 0$, independent of the choices of u, u' it follows that $(\mathcal{U}, \mathcal{C}_0)$ is also a $2-(v_0, k_0, \lambda)$ design (with the same λ). In particular $k_0 > 1$, since otherwise there would be no blocks of $\mathcal{C}_0$ containing $\{U, U'\}$. Moreover λ must be equal to the parameter r (the number of blocks of $\mathcal{C}$ containing a point pair within a single class). Counting triples (U, U', γ_0), with distinct $U, U' \in \mathcal{U}$ and $\gamma_0 \in \mathcal{C}_0$ such that $U, U' \in \gamma_0$, we have $v_0(v_0 - 1)\lambda = |\mathcal{C}_0|\, k_0(k_0 - 1) = v_0 r(k_0 - 1)$, and since $\lambda = r$, this implies that $v_0 = k_0$. This means, however, that $|\mathcal{C}_0| = 1$, which is a contradiction. $\square$

It turns out that the family of examples from Construction 2.1 essentially exhausts all possibilities for G-SIT-codes, and Delandtsheer designs, in $J(\mathcal{V}, k)$ for which the automorphism group G is not primitive on the base set $\mathcal{V}$. This assertion is proved in [19] and draws together a number of results from [21] concerning the larger class of neighbour-transitive codes. It is instructive to repeat the argument here.

Proposition 2.4 [19, Theorem 2] *Suppose that $\mathcal{C} \subseteq \binom{\mathcal{V}}{k}$ is a code in $J(\mathcal{V}, k)$, where $2 \leqslant k \leqslant v-2$ and $|\mathcal{C}| \geqslant 2$, and suppose that $G \leqslant \mathrm{Aut}(\mathcal{C})$ is not primitive on $\mathcal{V}$. Then the following are equivalent:*

(a) $\mathcal{C}$ is a G-SIT code;

(b) G is transitive on $\mathcal{V}$ and, for some divisor a of $\gcd(v,k)$ with $a \geqslant 2$, G leaves invariant a partition $\mathcal{U}$ of $\mathcal{V}$ with $|\mathcal{U}| = v/a$ parts of size a, and $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ as in Construction 2.1, for some G_0-SIT-code $\mathcal{C}_0$ in $J(\mathcal{U}, k_0)$, where $k = ak_0$ and G_0 is the group induced by G on $\mathcal{U}$.

Proof Let $\mathcal{C}, G$ be as in the statement. Then $|\mathcal{C}| \geqslant 2$. Also, since G is not primitive on $\mathcal{V}$, it follows in particular that $G \neq \mathrm{Sym}(\mathcal{V})$ or $\mathrm{Alt}(\mathcal{V})$. Suppose first that $\mathcal{C}$ is G-strongly incidence-transitive. If $\mathcal{C}$ were the complete code $\binom{\mathcal{V}}{k}$, then G would be transitive on the k-subsets of $\mathcal{V}$. By [10, Theorem 9.4B], such a group G is either (i) 2-transitive on $\mathcal{V}$, or (ii) $k = 2$, $v \equiv 3 \pmod 4$, a prime power, and a point stabiliser in G has orbits in $\mathcal{V}$ of lengths $1, (v-1)/2, (v-1)/2$. Now each 2-transitive permutation group is primitive, and also each group in case (ii) is primitive on $\mathcal{V}$. This contradicts our assumption that G is imprimitive on $\mathcal{V}$. Therefore $\mathcal{C}$ is not the complete code and so G is transitive on $\mathcal{V}$ and $\delta(\mathcal{C}) \geqslant 2$, by Lemma 1.1. Hence by assumption, G is imprimitive on $\mathcal{V}$, and so by [21, Proposition 4.7], $\mathcal{C}$ is as in Example 4.1 or Example 4.4 of [21] relative to some G-invariant nontrivial partition $\mathcal{U}$ of $\mathcal{V}$ with $|\mathcal{U}| = v/a$ parts of size a. If $\mathcal{C}$ comes from [21, Example 4.4], then since this set of examples arises from our Construction 2.1 we have $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ for some code $\mathcal{C}_0$ in $J(\mathcal{U}, k/a)$. Now G leaves $\mathcal{U}$ invariant and the group G_0 induced by G on $\mathcal{U}$ must preserve $\mathcal{C}_0$, so $G \leqslant \mathrm{Sym}(a) \wr G_0$ and $G_0 \leqslant \mathrm{Aut}(\mathcal{C}_0)$. Moreover, since $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ is a G-SIT-code, the small code $\mathcal{C}_0$ must also be a G_0-SIT-code. Thus part (b) holds. Suppose now that $\mathcal{C}$ comes from [21, Example 4.1]. Then since $\delta(\mathcal{C}) \geqslant 2$, it follows from [21, Lemma 4.3] that $\mathcal{C}$ is as in 'Line 1 of Table 3 for [21, Example 4.1] with $k = a$'. This means that $\mathcal{U}$ has parts of size $a = k$ and $\mathcal{C}$ is equal to the set of parts of $\mathcal{U}$. Thus $\mathcal{C}$ is a blow-up code $\mathcal{C}(a, \mathcal{C}_0)$ as in Construction 2.1, where $k_0 = 1$ and the small code $\mathcal{C}_0$ consists of all the singletons from $\mathcal{U}$. Since $\mathcal{C}$ is a G-SIT-code, the stabiliser G_γ of any part γ of $\mathcal{U}$ is transitive on $\mathcal{V} \setminus \gamma$ and hence the group G_0 induced by G on $\mathcal{U}$ is 2-transitive, and $\mathcal{C}_0$ is a G_0-SIT-code. Thus again part (b) holds.

Conversely, assume that part (b) holds. Then $\mathcal{C}$ is strongly incidence transitive by Lemma 2.2. □

We deduce that an imprimitive strongly incidence-transitive code can be obtained directly by applying Construction 2.1 to a primitive SIT-code.

Theorem 2.5 *Suppose that $\mathcal{C} \subseteq \binom{\mathcal{V}}{k}$ is a G-SIT-code in $J(\mathcal{V}, k)$, where $|\mathcal{V}| = v$, $2 \leqslant k \leqslant v-2$, $|\mathcal{C}| \geqslant 2$, and $G \leqslant \mathrm{Aut}(\mathcal{C})$, such that G is imprimitive on $\mathcal{V}$.*

(a) *Then* $v = av_0$ *and* $k = ak_0$ *for some* $a \geqslant 2$*, there exists a strongly incidence-transitive code* $\mathcal{C}_0$ *in* $J(\mathcal{U}, k_0)$*, where* $|\mathcal{U}| = v_0$*,* $1 \leqslant k_0 \leqslant v_0 - 1$*, such that* $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ *and* $\mathrm{Aut}(\mathcal{C}_0)$ *is primitive on* $\mathcal{U}$*.*

(b) *Moreover,* $\mathrm{Aut}(\mathcal{C}_0)$ *is* 2*-transitive on* $\mathcal{U}$*,* $\mathrm{Aut}(\mathcal{C}) = \mathrm{Sym}(a) \wr \mathrm{Aut}(\mathcal{C}_0)$*, and either*

(i) $k_0 = 1$*,* $\mathcal{C}_0 = \binom{\mathcal{U}}{1}$ *is the complete code,* $\mathcal{C} = \mathcal{U}$*, and* $(\mathcal{V}, \mathcal{C})$ *is a Delandtsheer* $1-(v, k, 1)$*-design; or*

(ii) $k_0 \geqslant 2$*,* $(\mathcal{U}, \mathcal{C}_0)$ *is a Delandtsheer* $2-(v_0, k_0, \lambda_0)$ *design, for some* λ_0*, and* $(\mathcal{V}, \mathcal{C})$ *is a Delandtsheer* $1-(v, k, r)$ *design where* $r = (v_0 - 1)\lambda_0/(k_0 - 1)$*.*

Proof (a) By Proposition 2.4, $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ for some strongly incidence-transitive code $\mathcal{C}_0$ in $J(\mathcal{U}, k_0)$ where $v = av_0, k = ak_0$. Suppose that $\mathrm{Aut}(\mathcal{C}_0)$ is imprimitive on $\mathcal{U}$. Then a further application of Proposition 2.4 shows that $\mathcal{C}_0 = \mathcal{C}(a', \mathcal{C}_0')$ for some strongly incidence-transitive code $\mathcal{C}_0'$ in $J(\mathcal{U}', k_0')$ where $v_0 = a'|\mathcal{U}'|, k_0 = a'k_0'$. It follows from Construction 2.1 that $\mathcal{C} = \mathcal{C}(aa', \mathcal{C}_0')$. Thus we see recursively that, if a is maximal such that $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ for some strongly incidence-transitive code $\mathcal{C}_0$ in $J(\mathcal{U}, k_0)$ with $v = av_0$ and $k = ak_0$, then $\mathrm{Aut}(\mathcal{C}_0)$ is primitive on $\mathcal{U}$.

(b) Suppose that $\mathcal{C} = \mathcal{C}(a, \mathcal{C}_0)$ with $\mathcal{C}_0$ as in part (a). Since $\mathcal{C}_0$ is strongly incidence-transitive and $|\mathcal{C}_0| = |\mathcal{C}| \geqslant 2$ with $A = \mathrm{Aut}(\mathcal{C}_0)$ primitive on $\mathcal{V}$, the group A is 2-transitive on $\mathcal{V}$ by Theorem 1.2. If $k_0 \geqslant 2$ this implies that $(\mathcal{U}, \mathcal{C}_0)$ is a Delandtsheer $2-(v_0, k_0, \lambda_0)$ design, for some λ_0. This means that each $U_i \in \mathcal{U}$ lies in $r = |\mathcal{C}_0|k_0/v_0 = (v_0 - 1)\lambda_0/(k_0 - 1)$ blocks in $\mathcal{C}_0$, and hence each point of $\mathcal{V}$ lies in r blocks of $\mathcal{C}$. On the other hand if $k_0 = 1$ then $\mathcal{C}_0 = \binom{\mathcal{U}}{1}$ is the complete code with $v_0 = |\mathcal{U}|$ blocks, and $\mathcal{C} = \mathcal{U}$ with each point lying in a unique block.

Finally we show that $\mathrm{Aut}(\mathcal{C})$ is equal to $X := \mathrm{Sym}(a) \wr A$ where $A = \mathrm{Aut}(\mathcal{C}_0)$. Now $\mathrm{Aut}(\mathcal{C})$ contains X by Lemma 2.2. Note that X contains a transposition so its only primitive overgroup is $\mathrm{Sym}(\mathcal{V})$, which does not leave $\mathcal{C}$ invariant. Therefore $\mathrm{Aut}(\mathcal{C})$ is imprimitive. If $\mathcal{U}'$ is a nontrivial $\mathrm{Aut}(\mathcal{C})$-invariant partition of $\mathcal{V}$, then considering the action of X we see that each part of $\mathcal{U}$ (on which X induces the full group $\mathrm{Sym}(a)$) must be contained in a part of $\mathcal{U}'$, that is, $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Then the set of parts of $\mathcal{U}$ contained in a fixed part of $\mathcal{U}'$ forms a block of imprimitivity for the induced action of X on $\mathcal{U}$, and since X induces the primitive group A on $\mathcal{U}$, it follows that this block of imprimitivity has size 1 and $\mathcal{U}' = \mathcal{U}$. Thus $\mathrm{Aut}(\mathcal{C})$ leaves $\mathcal{U}$ invariant. This implies that $\mathrm{Aut}(\mathcal{C})$ induces a subgroup of $\mathrm{Sym}(\mathcal{U})$ leaving $\mathcal{C}_0$ invariant. It follows that $\mathrm{Aut}(\mathcal{C}) = X$. □

3 Sporadic SIT-codes and Delandtsheer 2-designs

Although the codes arising from Construction 2.1 can have arbitrarily large minimum distance (since $\delta(\mathcal{C}(a, \mathcal{C}_0) = a\delta(\mathcal{C}_0) \geqslant a)$ by Lemma 2.2, the codes are not very large since $|\mathcal{C}(a, \mathcal{C}_0)| = |\mathcal{C}_0|$ remains fixed as a grows. Thus the interesting strongly incidence-transitive codes in $J(\mathcal{V}, k)$ have automorphism groups primitive on $\mathcal{V}$, as we discussed in Subsection 1.4. In this section we examine the SIT-codes and Delandtsheer 2-designs admitting a sporadic 2-transitive group. It turns out that each of the sporadic 2-transitive groups, as listed in Table 1, acts on at least one SIT-code and Delandtsheer design. The approach to this classification in [27] was computational, using the computer system GAP [13]. Note that, if $\mathcal{C}$ is a G-SIT-code and

$\gamma \in \mathcal{C}$, then G_γ has two orbits in $\mathcal{V}$, namely γ and its complement. For each of the sporadic 2-transitive groups G in Table 1,

- we started with a list $\mathcal{L}$ of representatives from the conjugacy classes of maximal subgroups of G (apart from the stabilisers of points of $\mathcal{V}$). For each subgroup $H \in \mathcal{L}$,
- we computed the orbits of H on $\mathcal{V}$, and discarded any subgroups with more than two orbits on $\mathcal{V}$;
- if H had two orbits γ and $\mathcal{V} \setminus \gamma$, with say $2 \leqslant |\gamma| \leqslant v/2$, then we checked whether or not H is transitive on $\gamma \times (\mathcal{V} \setminus \gamma)$, and if not then H was discarded;
- if H was transitive on $\gamma \times (\mathcal{V} \setminus \gamma)$, then we enumerated the corresponding code $\mathcal{C} := \{\gamma^g \mid g \in G\}$ in $J(v, |\gamma|)$, determined the minimal distance $\delta(\mathcal{C})$, and checked if H was the full stabiliser of $\mathcal{C}$ in G (since later in this process the subgroup H might not be maximal in G); in this case we would have found an interesting SIT-code listed in Table 3;
- finally, if H was transitive on $\mathcal{V}$, then we appended to $\mathcal{L}$ a representative of each of the conjugacy classes of maximal subgroups of H, and continued with these steps on the next listed subgroup in $\mathcal{L}$.

Since the groups in Table 1 were explicitly available in GAP as permutation groups of reasonable degree v, the initial list $\mathcal{L}$ could either be constructed by explicit computation, or by inspecting the Atlas of Finite Group Representations [31]. The procedure terminates because the groups are finite, and the fact that it yields all examples is justified by [27, Lemma 1]. To aid with an understanding of the beautiful geometric and group-theoretic structures underpinning the examples in Table 3, mathematical arguments were given in [27, Section 2] in most cases (with occasionally reference to computations to finish off the case).

Some of the Delandtsheer designs corresponding to these SIT-codes in Table 3 are very familiar, such as the Steiner systems in lines 4, 8, 12, 15, 24, and 26 associated with the Mathieu groups. Others might not be so easily identifiable: the design in line 21 for the Conway group Co_3 is the $2-(276, 100, 1458)$ design found by Haemers et al [14] in 1993. This and other 2-designs for Co_3 were constructed and the parameters found using the DESIGN package for GAP [29].

4 Classical SIT-codes and Delandtsheer designs

In this section we describe the SIT-codes and Delandtsheer designs admitting either a 'projective' or a 'rank 1' group from Table 2. There are three infinite families of examples and one exceptional case. We list them in Table 4 and describe how they were classified. In the table, $\binom{m}{r}_q$ denotes the number of r-dimensional subspaces of $\mathbb{F}_q^m$.

Line	G	v	k	$\delta(\mathcal{C})$	$\|\mathcal{C}\|$	Delandtsheer design	G_γ
1	$L_2(11)$	11	5	3	11	2-$(11,5,2)$ biplane	A_5
2	A_7	15	7	4	15	planes of PG$(3,2)$	$L_2(7)$
3	M_{11}	12	6	3	22	totals	A_6
4	M_{22}	22	6	4	77	3-$(22,6,1)$ design	$2^4 : A_6$
5			7	4	176	heptads	A_7
6			8	4	330	octads	$2^3 : L_3(2)$
7			10	4	616	decads	$\mathrm{M}_{10} \cong A_6 \cdot 2_3$
8	$\mathrm{M}_{22}.2$	22	6	4	77	3-$(22,6,1)$ design	$2^4 : S_6$
9			7	3	352	heptads	A_7
10			8	4	330	octads	$2 \times 2^3 : L_3(2)$
11			10	4	616	decads	$A_6 \cdot (2^2)$
12	M_{23}	23	7	4	253	4-$(23,7,1)$ design	$2^4 : A_7$
13			8	4	506	octads	A_8
14			11	4	1288	endecads	M_{11}
15	M_{24}	24	8	4	759	5-$(24,8,1)$ design	$2^4 : A_8$
16			12	4	2576	duum	M_{12}
17	HS	176	50	36	176	2-$(176,50,14)$	$\mathrm{U}_3(5) : 2$
18			56	32	1100	2-$(176,56,110)$	$L_3(4).2$
19	Co_3	276	6	3	708400	2-$(276,6,280)$	$3_+^{1+4} : 4S_6$
20			36	24	170775	2-$(276,36,2835)$	$2{\cdot}\mathrm{Sp}_6(2)$
21			100	50	11178	2-$(276,100,1458)$	HS
22			126	36	655776	2-$(276,126,136080)$	$\mathrm{U}_3(5) : S_3$
23	A_7	15	3	2	35	lines of PG$(3,2)$	$(A_3 \times A_4).2$
24	M_{11}	11	5	2	66	4-$(11,5,1)$ design	S_5
25	M_{11}	12	6	2	110	halves of quadrisect.	$3^2 : Q_8$
26	M_{12}	12	6	2	132	5-$(12,6,1)$ design	$A_6.2$
27	M_{24}	24	12	2	35420	5-$(24,12,660)$	$2^6 : 3.(S_3 \times S_3)$

Table 3: Sporadic 2-transitive SIT-codes $\mathcal{C}$ with group G and $3 \leqslant k \leqslant v/2$.

Line	G	v	k	$\delta(\mathcal{C})$	λ	Blocks of $\mathcal{D}(\mathcal{C})$
1	$\mathrm{P\Gamma U}(3,q)$	q^3+1	$q+1$	q	1	Classical unital
2	$\mathrm{P\Gamma U}(3,3)$	28	12	6	11	'Bases'
3	$\mathrm{P\Gamma L}(2,q)$	$q+1$	q_0+1	q_0-1	$\frac{q_0-1}{2}$	Baer sublines, $q=q_0^2$
4	$\mathrm{P\Gamma L}(n,q)$	$\frac{q^n-1}{q-1}$	$\frac{q^s-1}{q-1}$	q^{s-1}	$\binom{n-2}{s-2}_q$	Subspaces, $2 \leqslant s < n$

Table 4: Classical 2-transitive SIT-codes $\mathcal{C}$ and Delandtsheer $2-(v,k,\lambda)$ designs $\mathcal{D}(\mathcal{C})$ with group G and $3 \leqslant k \leqslant v/2$

Constructions for the SIT-codes $\mathcal{C}$ in Table 4 giving G, v, k and $\delta(\mathcal{C})$ are given in [21, Examples 7.1, 7.3, 8.1, and Theorem 1.3(b)]. It follows from Theorem 1.2 that in each case $\mathcal{D}(\mathcal{C}) = (\mathcal{V}, \mathcal{C})$ is a Delandtsheer $2-(v, k, \lambda)$ design, and the value of λ is determined using (1.2). More details on the classical unital may be found in [4, 30]. The fact that there are no other examples from groups in the rank 1 and projective lines of Table 2 needs some comment.

This classification is given for the 2-transitive Suzuki, Ree and unitary groups in [21, Theorem 1.3], showing that only the unitary groups yield any examples, and that these examples are the ones in Lines 1 and 2 of Table 4. The case of 2-transitive subgroups of $\mathrm{P\Gamma L}(2, q)$ acting on the projective line $\mathrm{PG}(1, q)$ was dealt with in [21, Proposition 7.2], which showed that examples arise only if $q = q_0^2$ and the blocks are the Baer sublines as in Line 3 of Table 4.

This leaves the case where $\mathrm{PSL}(n, q) \leqslant G \leqslant \mathrm{P\Gamma L}(n, q)$ with $n \geqslant 3$, acting on the point set $\mathcal{V}$ of the projective space $\mathrm{PG}(n-1, q)$. Here the restrictions required for a subset $\gamma \subseteq \mathcal{V}$ to be a codeword of a G-SIT-code, or equivalently, a block of a Delandtsheer design admitting G, may be interpreted as a condition on the lines of the projective geometry $\mathrm{PG}(n-1, q)$. For a pair of points $u \in \gamma$ and $u' \in \mathcal{V} \setminus \gamma$, there is a unique line ℓ of $\mathrm{PG}(n-1, q)$ containing u and u', so transitivity of G on the triples in (1.1) implies that G is transitive on the projective lines which meet both γ and its complement. Hence such 'shared lines' meet γ in a constant number of points, say x, where $0 < x < |\ell| = q + 1$. Thus each projective line ℓ meets γ in 0, or x or $q + 1$ points. A subset γ of $\mathcal{V}$ with these properties is called a *subset of class* $[0, x, q + 1]_1$. If $x = 1$ then each line containing two distinct points of γ must be contained in γ, and so γ is a projective subspace of $\mathrm{PG}(n-1, q)$ – and the set of subspaces of a given dimension is an example, as in Line 4 of Table 4. Similarly if $x = q$ then each line containing two distinct points of $\mathcal{V} \setminus \gamma$ lies in $\mathcal{V} \setminus \gamma$, and so γ is a subspace complement: however in this case $k = |\gamma| > v/2$ so we do not list these examples separately.

In the case where $1 < x < q$ we restrict analysis to the case where $2 \leqslant x \leqslant (q + 1)/2$, since $J(v, k) \cong J(v, v - k)$ and if γ is a subset of class $[0, x, q + 1]_1$, then $\mathcal{V} \setminus \gamma$ is a subset of class $[0, q + 1 - x, q + 1]_1$. Here, by [21, Proposition 7.4], $x \in \{2, \sqrt{q} + 1\}$. Using this information it was shown by Durante [11, Theorem 15] that there are no further examples of SIT-codes admitting projective groups. In fact Durante [11, Theorem 14] described all subsets γ of $\mathrm{PG}(n-1, q)$ of class $[0, x, q+1]_1$, showing that either (i) γ or its complement is a subspace, or (ii) $n = 3$, q is even, and γ is a proper x-maximal arc or the complement of a proper $(q+1-x)$-maximal arc. Applying a result of Delandtsheer and Doyen [6, Theorem], among the examples in case (ii), the only ones for which G_γ is transitive on 'shared lines' are the hyperoval in $\mathrm{PG}(2, 4)$, for which $(k, x) = (6, 2)$, and the dual of a regular hyperoval, for which $(k, x) = (q(q-1)/2, q/2)$. Since $x \in \{2, \sqrt{q}\}$, this leaves only $q = 4$ with γ a hyperoval in $\mathrm{PG}(2, 4)$. However, it was noted in [21, Remark 7.5] that, although the stabiliser in $\mathrm{PGL}(3, 4)$ of a hyperoval γ is transitive on 'shared lines', the more restrictive SIT-property fails and this is not an additional example.

5 Affine SIT-codes and Delandtsheer designs

Line	q	n	k	$\delta(\mathcal{C})$	λ	Blocks of $\mathcal{D}(\mathcal{C})$
1	any	$n \geqslant 2$	q^s	$q^{s-1}(q-1)$	$\binom{n-1}{s-1}_q$	Subspaces, $1 \leqslant s < n$
2	16	1	4	3	1	Baer sublines
3	16	$n \geqslant 2$	4.16^{n-1}	?	?	Four parallel hyperplanes
4	4	$n \geqslant 2$	2.4^{n-1}	?	?	Two parallel hyperplanes
5	4	$n = 2$	6	3	6	Hyperovals in $\mathrm{AG}(2, 4)$
6	4	$n \geqslant 3$	?	?	?	Cylinders in $\mathrm{AG}(n, 4)$ with base as in Line 5
7	2	$n \geqslant 4$ n even	$2^{n-1}+$ $\varepsilon 2^{n/2-1}$	?	?	Blocks of symplectic design $\mathcal{S}^\varepsilon$, where $\varepsilon = \pm$
8	2	$n \geqslant 4$ n even	$2^{n-1}+$ $\varepsilon 2^{n/2-1}$	?	?	Union of all symplectic designs on $\mathcal{V}$ of type ε

Table 5: Affine 2-transitive SIT-codes $\mathcal{C}$ and Delandtsheer $2-(q^n, k, \lambda)$ designs $\mathcal{D}(\mathcal{C})$ with 2-transitive group $G \leqslant \mathrm{A\Gamma L}(n, q)$, $q \geqslant 4$, and $k \leqslant q^n/2$.

As we discussed in Subsection 1.4, there are many interesting examples of affine SIT-codes and Delandtsheer designs. Let $G \leqslant \mathrm{A\Gamma L}(n, q)$ be 2-transitive on the point-set $\mathcal{V}$ of $\mathrm{AG}(n, q)$, and suppose that $\mathcal{C}$ it a G-SIT-code in $J(q^n, k)$ with $3 \leqslant k \leqslant q^n/2$. Combining [21, Propositions 6.1 and 6.6] and [18, Lemma A.1], either $\mathcal{C}$ is the set of all affine subspaces of fixed dimension, as in Line 1 of Table 5, or the field size $q \in \{2, 4, 16\}$. If $n = 1$ then the only non-subspace example is the set of Baer sublines of $\mathrm{AG}(1, 16)$, by [21, Proposition 6.1], as in Line 2 of Table 5. Suppose then that $n \geqslant 2$, and that γ is not a subspace, so $q \in \{2, 4, 16\}$.

For $\gamma \in \mathcal{C}$ and points $u \in \gamma$ and $u' \in \mathcal{V} \setminus \gamma$, there is a unique affine line ℓ of $\mathrm{AG}(n, q)$ containing u and u', and transitivity of G on the triples in (1.1) implies that G is transitive on the affine lines which meet both γ and its complement. Hence such 'shared lines' meet γ in a constant number x of points, where $0 < x < |\ell| = q$, and therefore each line meets γ in 0, or x or q points, that is to say, γ is a *subset of class* $[0, x, q]_1$ in $\mathrm{AG}(n, q)$. For $q \geqslant 4$, Durante [11] determined all subsets of class $[0, x, q]_1$: in Theorem 22 of [11] if $n \geqslant 3, q > 4$, Theorem 24 if $n \geqslant 3, q = 4$, and Proposition 18 if $n = 2$. He then examined all these examples and determined which of them have the SIT-property in [11, Theorem 27]. The examples give all possibilities for $q = 4, 16$, and are as in Lines 3–6 of Table 5. Unfortunately he does not give the parameters $\delta(\mathcal{C})$ and λ for $\mathcal{D}(\mathcal{C})$, but we have these for Line 5 deduced from [21, Example 6.7] (with the help of John Bamberg).

Problem 3 *Determine the minimum distance $\delta(\mathcal{C})$ and the parameter λ for the affine SIT-codes and Delandtsheer designs in Lines* $3, 4, 6, 7, 8$ *of Table* 5*.*

There remains the exceptional case of $q = 2$, studied by Ioppolo in [18, 19]. The 2-transitive affine group $G \leqslant \mathrm{AGL}(n, 2)$ contains the translation group $T \cong C_2^n$ as its unique minimal normal subgroup, and Ioppolo showed that the structure of a G-SIT-code $\mathcal{C}$ in $J(2^n, k)$ depends on the intersections $M(\gamma) = G_\gamma \cap T$, for $\gamma \in \mathcal{C}$. Note that $M(\gamma) \neq T$ since $\gamma \neq \mathcal{V}$. Since G is transitive on $\mathcal{C}$, these subgroups $M(\gamma)$ form a conjugacy class $\mathcal{M}$ of G, and the code $\mathcal{C}$ is partitioned into pairwise disjoint subcodes

$\mathcal{C}(M) = \{\gamma \in \mathcal{C} | M(\gamma) = M\}$, called *components*, for $M \in \mathcal{M}$. Each component $\mathcal{C}(M)$ is itself an $N_G(M)$-SIT-code, by [18, L:emma 7.2]. If $2^a = |M| > 1$, then the group $N_G(M)$ acts imprimitively on $\mathcal{V}$ preserving the partition $\mathcal{U}$ into M-orbits, and hence, by Proposition 2.4, $\mathcal{C}(M) = \mathcal{C}(2^a, \mathcal{C}_0)$ as in Construction 2.1, with $\mathcal{C}_0$ a G_0-SIT-code in $J(2^{n-a}, k_0)$, where $k = 2^a k_0$ and G_0 is the group induced by $N_G(M)$ on the set $\mathcal{U}$ of M-orbits in $\mathcal{V}$. Since $M \neq T$, the group G_0 is an affine group on $\mathcal{U}$, and by the definition of $\mathcal{C}(M)$, each codeword $\gamma_0 \in \mathcal{C}_0$ intersects the translation group for $\mathcal{C}_0$ trivially: such a code $\mathcal{C}_0$ is said to be *translation-free.* Since G permutes the components $\mathcal{C}(M)$ transitively, it follows that each affine SIT-code $\mathcal{C}$ in $J(2^n, k)$ corresponds, up to isomorphism, to a unique affine translation-free SIT-code $\mathcal{C}_0$ in $J(2^{n-a}, k/2^a)$, for some $a \geqslant 0$.

The affine translation-free SIT-codes are therefore central to understanding the affine SIT-codes over $\mathbb{F}_2$, and they have been completely classified by Ioppolo [18, Chapter 7] (see also [19]). The examples are as in Lines 7–8 of Table 5. For the examples in Line 7, which are constructed in [18, Example 7.11], the Delandtsheer design $\mathcal{D}(\mathcal{C})$ is the symplectic symmetric design $\mathcal{S}^\varepsilon(\mathcal{V})$, where n is even and $\varepsilon \in \{\pm\}$. The automorphism group of $\mathcal{S}^\varepsilon(\mathcal{V})$, and of the corresponding SIT-code, is the symplectic affine group $A\mathrm{Sp}(n, 2) = C_2^n \rtimes \mathrm{Sp}(n, 2)$ (or $C_2^4 \rtimes S_6$ if $n = 4$). It is shown in [18, Theorem 7.13] that every affine translation-free SIT-code in $J(2^n, k)$ has n even and $k = 2^{n-1} + \varepsilon 2^{n/2-1}$, for some $\varepsilon \in \{+, -\}$, and is a disjoint union of several copies of $\mathcal{S}^\varepsilon(\mathcal{V})$ permuted transitively by some overgroup of $A\mathrm{Sp}(n, 2)$ in $\mathrm{AGL}(n, 2)$. Since $\mathrm{Sp}(n, 2)$ is a maximal subgroup of $\mathrm{GL}(n, 2)$ (or S_6 maximal in $\mathrm{GL}(4, 2) \cong A_8$ if $n = 4$), by [1], it follows that the only other examples are the union of all images of a code in Line 7 of Table 5 under elements of $\mathrm{AGL}(n, 2)$, or equivalently, under elements of $\mathrm{GL}(n, 2)$ (since the translation subgroup leaves the codes $\mathcal{S}^\varepsilon(\mathcal{V})$ invariant). These are the examples in Line 8 of Table 5. Note that, for each ε, there is one copy of $\mathcal{S}^\varepsilon(\mathcal{V})$ for each nondegenerate alternating form on $\mathcal{V}$.

It is not clear whether it is possible to use a translation-free code $\mathcal{C}_0$ to build a component code $\mathcal{C}(M) = \mathcal{C}(2^a, \mathcal{C}_0)$, with $M \neq 1$, in such a way that $\mathrm{Aut}(\mathcal{C}(M)) \cap \mathrm{AGL}(n, 2)$ acts strongly incidence transitively, noting that $\mathrm{Aut}(\mathcal{C}(M)) = \mathrm{Sym}(2^a) \wr \mathrm{Aut}(\mathcal{C}_0)$. *The task remaining to solve Problem 1 is essentially to resolve this issue.* The reason is that each example $\mathcal{C}(M) = \mathcal{C}(2^a, \mathcal{C}_0)$ of an H-SIT code in $J(2^{a+n}, k)$, with $a > 0$, $\mathcal{C}_0$ as in Table 5 line 7 or 8, and $C_2^{a+n} \trianglelefteq H \leqslant \mathrm{AGL}(a + n, 2)$, can be used to construct an affine G-SIT-code $\mathcal{C}$ for any 2-transitive group G satisfying $H < G \leqslant \mathrm{AGL}(a + n, 2)$ by taking $\mathcal{C} = \cup_{g \in G} \mathcal{C}(M)^g$. And each such code would arise in this way.

6 Symplectic SIT-codes and Delandtsheer designs

The last infinite family of 2-transitive permutation groups to consider are the symplectic groups $G = \mathrm{Sp}(2n, 2)$, with $n \geqslant 2$, (line 'symplectic' of Table 2). The group G is the isometry group of a nondegenerate alternating form B on $V = \mathbb{F}_2^{2n}$, and G acts on the associated set $\mathcal{Q}$ of all nondegenerate quadratic forms $\phi : V \to \mathbb{F}_2$ which polarise to B, that is to say:

$$B(x, y) = \phi(x + y) - \phi(x) - \phi(y), \text{ for all } x, y \in V.$$

For $g \in G$ and $\phi \in \mathcal{Q}$, we write ϕ^g for the function $\phi^g(x) = \phi(xg^{-1})$, for $x \in V$, and note that $\phi^g \in \mathcal{Q}$. We let $\mathrm{sing}(\phi) = \{x \in V \mid \phi(x) = 0\}$, the set of *$\phi$-singular vectors*. The group G has exactly two orbits in $\mathcal{Q}$, which we denote $\mathcal{Q}^+$ and $\mathcal{Q}^-$: a form $\phi \in \mathcal{Q}^+$ if the maximum dimension of a subspace of V contained in $\mathrm{sing}(\phi)$ is n, while for $\phi \in \mathcal{Q}^-$ this dimension is $n-1$. We say that ϕ has *type* $+$ or $-$, respectively. For $\varepsilon \in \{+,-\}$, this G-action on $\mathcal{Q}^\varepsilon$ is 2-transitive, and $|\mathcal{Q}^\varepsilon| = 2^{n-1}(2^n + \epsilon.1)$. These actions are known as the *Jordan-Steiner actions* of G. We sometimes write ε instead of $\varepsilon.1$.

For a given $\varepsilon \in \{+,-\}$, Ioppolo constructed two families of G-SIT codes in $J(\mathcal{Q}^\varepsilon, k)$, for certain k, as follows. The first family is based on nondegenerate subspaces of V of fixed dimension.

Construction 6.1 *[3, Construction 1.2] Let $\varepsilon' \in \{+,-\}$, $d \in \mathbb{Z}$ with $1 \leqslant d < n$, and $k = 2^n(2^d + \varepsilon')(2^{n-d} + \varepsilon\varepsilon')$. Define the code*

$$\Gamma(n, d, \varepsilon, \varepsilon') = \{\gamma(U) \mid U < V, U \text{ nondegenerate}, \dim(U) = 2d\}$$

in $J(\mathcal{Q}^\varepsilon, k)$, where

$$\gamma(U) = \{\phi \in \mathcal{Q}^\varepsilon \mid \varphi|_U \textit{ has type } \varepsilon' \textit{ and } \varphi|_{U^\perp} \textit{ has type } \varepsilon\varepsilon'\}.$$

The second family uses totally isotropic subspaces of fixed dimension.

Construction 6.2 *[3, Construction 1.3] Let $c, d \in \mathbb{Z}$ with $c = 0$ or 1 and $1 \leqslant d \leqslant n$ such that $(d, \varepsilon) \neq (n, -)$, and let $k = 2^{n-1}(2^{n-d} + \varepsilon)$. Define the code*

$$\Gamma(n, d, \varepsilon, c) = \{\gamma(U) \mid U < V, U \text{ totally isotropic}, \dim(U) = d\}$$

in $J(\mathcal{Q}^\varepsilon, k)$, where

$$\gamma(U) = \{\phi \in \mathcal{Q}^\epsilon \mid \dim(\mathrm{sing}(\phi) \cap U) = d - c\}.$$

Ioppolo showed that these are the only examples which are 'geometrically based' in the following sense. The SIT-property requires a codeword stabiliser G_γ to have two orbits in $\mathcal{Q}^\varepsilon$, namely γ and its complement, and analysis of the various cases considers the possible maximal subgroups M of G containing G_γ. The family of maximal subgroups of $G = \mathrm{Sp}(2n, 2)$ have been classified into various families by Aschbacher [1]. One of these families consists of the subspace stabilisers appearing in Constructions 6.1 and 6.2. The union of several of the families of maximal subgroups, including these subspace stabilisers, is called the family of *geometric subgroups*, and Aschbacher proved that each maximal subgroup M of G that is not geometric, in this sense, is an almost simple group, that is, $T \trianglelefteq M \leqslant \mathrm{Aut}(T)$ where T is a nonabelian simple group, called the socle of M, and T is absolutely irreducible on $\mathbb{F}_2^{2n}$. The major result of Ioppolo's thesis ([18, Theorem 4.3], or see [3, Theorem 1.4]) is that the only SIT-codes for which a codeword stabiliser G_γ is contained in a maximal geometric subgroup are those from Constructions 6.1 and 6.2. It is not known (to us) whether the corresponding Delandtsheer designs are previously known 2-designs.

Questions 6.3 *Are the Delandtsheer designs corresponding to the SIT-codes in Constructions 6.1 and 6.2 previously known 2-designs?*

Ioppolo's work shows that for any other examples the codeword stabiliser itself must be almost simple. There is at least one additional example with $3 \leqslant k \leqslant v/2$ where G_γ is a maximal almost simple group, namely an $\mathrm{Sp}(8,2)$-SIT-code in $J(136,10)$ based on $\mathcal{V} = \mathcal{Q}^+$, with $G_\gamma = \mathrm{Sym}(10)$. Here the underlying space $\mathbb{F}_2^8$ is the deleted permutation module for the natural action of $\mathrm{Sym}(10)$. This example is the only one where $\mathbb{F}_2^{2n}$ is the deleted permutation module for $\mathrm{Sym}(2n+2)$, [18, Theorem 8.3], and it is the only one for dimensions $2n \leqslant 12$, [18, Section 6.2]. Moreover, there are no examples if T is a sporadic simple group or an exceptional group of Lie type [18, Theorem 6.25 and Appendix C.2], and if T is a classical simple group then existence of a corresponding SIT-code places very strong restrictions on T [18, Chapter 6]. Nevertheless, there is much work to be done yet to deal completely with Problem 2.

7 Related codes and designs

In this final section we discuss several families of designs and codes with links to Delandtsheer designs and SIT-codes.

7.1 Transitivity properties on flags and antiflags of designs

Flag-transitive 2-designs have been studied extensively, particularly following the seminal result of D. G. Higman and J. E. McLaughlin [16] that a flag-transitive $2-(v,k,1)$ design (linear space) is point-primitive. It is however Delandtsheer's work [5] on antiflag-transitive linear spaces which is most relevant to the theme of this exposition, especially her observation that most examples in her classification possessed the stronger SIT-property: hence our definition of a Delandtsheer design $(\mathcal{V},\mathcal{C})$ with block size k as one in which the block set $\mathcal{C}$ is an SIT-code in $J(\mathcal{V},k)$. Proposition 2.4 together with Theorem 1.2 essentially reduce the problem of classifying the Delandtsheer designs to the case of (point) 2-transitive Delandtsheer 2-designs. And the discussion in Sections 3, 4, 5 and 6 outlines the current status of this classification problem.

It would be interesting to know if progress could be made in understanding the intermediate family where the stringent symmetry conditions for Delandtsheer designs are relaxed somewhat, but not as far as simply antiflag-transitivity.

Problem 4 *Investigate designs that are both flag-transitive and antiflag-transitive.*

Motivated by a problem in graph theory, investigations have begun [8, 9] of the family of pairwise transitive designs which are in particular both flag-transitive and antiflag-transitive. This family of designs is defined by its symmetry: a design $\mathcal{D} = (\mathcal{V},\mathcal{B})$ is said to be *pairwise transitive* if its automorphism group is transitive on the following six (possibly empty) sets of ordered pairs from $\mathcal{V}\cup\mathcal{B}$: collinear point-pairs (that is, distinct points contained in some block of $\mathcal{B}$), non-collinear point-pairs, incident point-block pairs, non-incident point-block pairs, intersecting block-pairs, and non-intersecting block-pairs. To compare pairwise transitive designs with Delandtsheer designs we make the following observations.

(a) The families of pairwise transitive designs and Delandtsheer designs have *significant overlap*, for example, both contain the designs of points and hyperplanes of projective and affine geometries (see Tables 4 and 5, and [9, Tables 1 and 2]), as well as the symplectic symmetric designs $\mathcal{S}^{\varepsilon}(\mathcal{V})$ described in Section 5 and [9, Table 1], and other 2-transitive symmetric designs (Lines 1 and 17 of Table 3 and [9, Table 1]).

(b) *Neither family is contained in the other.* For example most Delandtsheer designs arising from small dimensional subspaces of projective and affine geometries are not pairwise transitive, since there is more than one kind of nontrivial block-intersection and hence the group is not transitive on pairs of intersecting blocks. On the other hand the exceptional design consisting of the 21 points of $\mathrm{PG}(2,4)$ and one of the three $\mathrm{PSL}(3,4)$-orbits of 56 hyperovals as blocks, forms a pairwise transitive design [9, Theorem 4.6] which, as discussed in Section 4, is not a Delandtsheer design (see [21, Remark 7.5]).

(c) The two exceptional antiflag-transitive linear spaces in Delandtsheer's classification [5], which are not Delandtsheer designs, are both 2-transitive nondesarguesian affine planes, namely the nearfield plane of order 9 and Hering's plane of order 27. Neither of these is pairwise transitive, by [9, Theorem 1.1]. Thus there are examples of antiflag-transitive 2-designs which are *neither Delandtsheer designs not pairwise transitive designs.*

(d) By Lemma 2.3 each code $\mathcal{C}(a, \mathcal{C}_0)$, arising from Construction 2.1 with $\mathcal{C}_0$ an SIT-code in $J(\mathcal{U}, k_0)$, is a Delandtsheer 1-*design, but not a* 2-*design.* However the group $G = \mathrm{Sym}(a) \wr \mathrm{Aut}(\mathcal{C}_0)$ is not transitive on pairs of collinear points unless $k_0 = 1$ in which case $\mathcal{C}$ is the set of parts of the partition $\mathcal{U}$. Thus these designs are not G-pairwise transitive if $k_0 > 1$, but are indeed pairwise transitive if $k_0 = 1$, see [8, Example 2.1].

(e) Let $\mathcal{V} = \mathbb{F}_q^n$ and $0 \neq u \in \mathcal{V}$, and consider the subset $\mathcal{C}$ of $J(\mathcal{V}, q)$ consisting of all hyperplanes of $\mathrm{AG}(n, q)$ which do not contain a line in the direction $\langle u \rangle$. Then for $G = [q^n].U < \mathrm{AGL}(n, 1)$, where U is the stabiliser in $\mathrm{GL}(n, q)$ of $\langle u \rangle$, the design $(\mathcal{V}, \mathcal{C})$ is a G-pairwise transitive $1 - (q^n, q^{n-1}, q^{n-1})$ design, by [8, Lemma 3.4], but it is not a Delandtsheer design (the SIT-property is easily seen to fail for γ an $(n-1)$-subspace complementing $\langle u \rangle$).

The pairwise transitive 2-designs have been classified by Devillers and the author in [9, Theorem 1.1]. Since all pairs of distinct points are collinear, the automorphism group G is 2-transitive on points, and since G is transitive on intersecting block-pairs and on non-intersecting block-pairs, there are at most two possible values for the intersection size of a block-pair. Thus a pairwise transitive 2-design is either symmetric (if all block-pairs intersect nontrivially) or quasisymmetric if there exist non-intersecting block-pairs, [9, Lemma 2.4]. The 2-transitive symmetric designs were classified by Kantor [20], while the quasisymmetric pairwise transitive 2-designs were classified in [9].

The examples given in (d) and (e) show that Delandtsheer 1-designs and pairwise transitive 1-designs exist, and that neither family contains the other; while the examples in (c) illustrate that these two families do not cover all the designs addressed in Problem 4. Nevertheless, a better understanding of both families would be helpful.

Problem 5 *[8, Problem 1.11] Classify pairwise transitive* 1*-designs.*

In particular, a classification of the sub-family of pairwise transitive 1-designs for which the block set can be partitioned into parallel classes would have significant graph theoretic application. Such a design would be an affine design (see [8, Section 2.2]). A bipartite graph Γ with ordered bipartition $(B|B')$ corresponds to a design $\mathcal{D}$ with point set B, block set B' and incidence given by adjacency; also Γ is said to be *locally* (G, s)*-distance transitive* with automorphism group G if, for each vertex x, the stabiliser G_x is transitive on the set of vertices at distance i from x, for $i = 1, \ldots, s$. If the design $\mathcal{D}$ is affine, then the graph Γ is locally $(G, 4)$-distance transitive if and only if $\mathcal{D}$ is G-pairwise transitive, [8, Proposition 2.7(ii)].

7.2 SIT-codes and codes in binary Hamming graphs

Some of the examples of sporadic SIT-codes and Delandtsheer designs in Table 3 have surprisingly large minimum distance, and this prompted us to think in [27] about links between codes in Johnson graphs and codes in binary Hamming graphs - the traditional block codes. The *binary Hamming graph* $H(v, 2)$ has as vertices the ordered v-tuples with entries from $\{0, 1\}$, and edges those pairs of v-tuples which agree in all but one entry. If we write $\mathcal{V} = \{1, 2, \ldots, v\}$, then each vertex γ of the Johnson graph $J(v, k)$ can be identified with the binary v-tuple $h(\gamma)$ with i-entry 1 if and only if $i \in \gamma$, (see [27, Section 1.3]). This allows $J(v, k)$ to be identified with the set of all weight k vertices of $H(v, 2)$, and each code $\mathcal{C}$ in $J(v, k)$ to be identified with a constant weight code $H(\mathcal{C})$ in $H(v, 2)$. Two vertices at distance d in $J(v, k)$ correspond to two vertices in $H(v, 2)$ at distance $2d$ so the minimum distance of $H(\mathcal{C})$ is equal to $2\delta(\mathcal{C})$. Moreover $\mathrm{Aut}(\mathcal{C})$, in its action on entries, acts as automorphisms of the code $H(\mathcal{C})$ in $H(v, 2)$, so for an SIT code $\mathcal{C}$ in $J(v, k)$, $\mathrm{Aut}(\mathcal{C})$ is transitive on the codewords of $H(\mathcal{C})$. The neighbours of $H(\mathcal{C})$ in $H(v, 2)$ have weights $k \pm 1$, yielding two $\mathrm{Aut}(\mathcal{C})$-orbits on code neighbours for an SIT code $\mathcal{C}$.

Lemma 7.1 *Let $\mathcal{C}$ be an SIT-code in $J(v, k)$. Then* $\mathrm{Aut}(\mathcal{C})$ *has two orbits on the code-neighbours of $H(\mathcal{C})$, namely those of weight $k-1$ and those of weight $k+1$.*

Proof Extend the map h to act on each subset $\nu \subseteq \mathcal{V}$, so that $h(\nu)$ is the v-tuple with i-entry 1 if and only if $i \in \nu$. Since $\mathcal{C}$ is an SIT-code, $\mathrm{Aut}(\mathcal{C})$ is transitive on the set of triples (u, u', γ) such that $\gamma \in \mathcal{C}$, $u \in \gamma$ and $u' \in \mathcal{V} \setminus \gamma$. In particular $\mathrm{Aut}(\mathcal{C})$ is transitive on pairs (u, γ) with $\gamma \in \mathcal{C}$, $u \in \gamma$ (the flags of the corresponding Delandtsheer design $\mathcal{D}(\mathcal{C})$). Since each code-neighbour of weight $k-1$ is of the form $h(\gamma \setminus \{u\})$ for some flag (u, γ) of $\mathcal{D}(\mathcal{C})$, it follows that $\mathrm{Aut}(\mathcal{C})$ is transitive on the set of all code-neighbours of weight $k-1$. An analogous proof, using anti-flags of $\mathcal{D}(\mathcal{C})$, shows that $\mathrm{Aut}(\mathcal{C})$ is transitive on the set of all code-neighbours of weight $k+1$. $\square$

In fact, the conclusion of Lemma 7.1 holds if the assumption on $\mathcal{C}$ is weakened to simply requiring $\mathrm{Aut}(\mathcal{C})$ to be transitive on flags and on antiflags, as in Problem 4.

Acknowledgements

The Clay Plenary Lecture at the British Combinatorial Conference BCC21 was based on this paper. The author thanks the Clay Mathematics Institute (CMI) for support as the Clay Lecturer for the BCC21. The author is grateful to Mark Ioppolo, John Bamberg, and an anonymous referee for their careful reading of the manuscript. The paper forms part of Australian Research Council grant DP200100080.

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