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Some fundamental groups in arithmetic geometry

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Abstract. We report on Deligne's finiteness theorem for ℓ-adic representations on smooth varieties defined over a finite field, on its crystalline version, and on how the geometric étale fundamental group of a smooth projective variety defined over a field of positive characteristic controls crystals on the infinitesimal site and should control those on the crystalline site.

1. Acknowledgments

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2. Deligne's conjectures: ℓ-adic theory

The classical Hermite-Minkowski theorem asserts that there are finitely many numbers fields with bounded discriminant. In this section we present Deligne's program aiming at showing an analog finiteness theorem on complex varieties for variations of Hodge structures and on varieties over finite fields for ℓ-adic sheaves.

Theorem 2.1 (Deligne, [**Del84**], Thm. 0.5.). *Let X be a complex smooth connected variety, let r, w be natural numbers with $r \neq 0$. Then there are finitely many rank r $\mathbb{Q}$-local systems which are definable over $\mathbb{Z}$ and are direct factors of a $\mathbb{Q}$-variation of polarizable pure Hodge structure of weight w.*

The inspiration for this finiteness theorem in Hodge theory comes from Faltings's finiteness theorem [**Fal83**, Cor. p.344] for abelian schemes, that is in weight $w = 1$. We refer e.g. to [**Cat14**, Section 2] for the notion of a variation of Hodge structure. In particular, as it is regular singular at infinity, the analog of the discriminant appearing in Hermite-Minkowski's theorem is just the reduced divisor at

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infinity in a good normal crossings compactification $X \hookrightarrow \bar{X}$. This explains why it is enough to fix X and no multiplicities along the components of $\bar{X} \setminus X$.

On the other hand, for varieties defined over finite fields $\mathbb{F}_q$, in ℓ-adic theory, ℓ prime to q, one has the following finiteness theorem.

THEOREM 2.2 (Deligne, [**EK12**], Thm. 2.1, [**Esn17**], Thm. 3.1). *Let X be a normal separated scheme of finite type defined over a finite field $\mathbb{F}_q$, let $0 \neq r$ be a natural number. Let $D \subset \bar{X}$ be an effective Cartier divisor of a normal compactification $\bar{X}$ with support $\bar{X} \setminus X$. Then there are finitely many isomorphism classes of irreducible Weil (resp. étale) rank r lisse $\bar{\mathbb{Q}}_\ell$-sheaves with ramification bounded by D, up to twist with Weil (resp. étale) characters of $\mathbb{F}_q$. The number does not depend on the choice of ℓ.*

We refer to [**Del80**, Section 1.1] for ℓ-adic sheaves and to [**Del80**, 1.1.7] for Weil sheaves, also to [**EK11**, Notations]. Here 'ramification bounded by D' means that the Swan conductor on the pull-back of the sheaf on any smooth curve mapping non-trivially to X is bounded by the pull-back of D (see [**EK12**, Definition 3.6]). One could formulate the finiteness theorem by replacing this notion by the one used by Drinfeld in [**Dri12**, Thm. 2.5 (ii)], counting the isomorphism classes of the sheaves which become tame on a given finite étale cover $X' \to X$. That this assumption is stronger is proved in [**EK12**, Proof of Prop. 3.9].

Deligne's proof relies on Lafforgue's main theorem which in particular implies Theorem 2.2 over a smooth curve ([**Laf02**, Thm. VII.6]). The whole question, how to reduce to curves, is geometric. Some of the key ideas of the proof go back to Wiesend ([**Wie06**] and [**Wie07**]).

The chronology is a bit intricate. First Deligne gave a direct proof of 2.3. This can be understood as a corollary of Theorem 2.2 which itself was proved later (see [**EK12**, Thm. 8.2] for the deduction).

COROLLARY 2.3 (Deligne, [**Del12**], Thm. 3.1, Deligne's conjecture (ii) in [**Del80**], 1.2.10). *Given a lisse étale $\bar{\mathbb{Q}}_\ell$-sheaf V with determinant of finite order, the subfield of $\bar{\mathbb{Q}}_\ell$ spanned by the coefficients of the minimal polynomials of the Frobenii F_x at closed points $x \in |X|$ acting on $V_{\bar{x}}$ is a number field.*

Using this in an essential way, Drinfeld proved the existence of ℓ-adic companions on smooth quasi-projective varieties defined over a finite field [**Dri12**, Thm. 1.1], which is part (v) of the conjecture [**Del80**, 1.2.10]. Then, using Drinfeld's theorem in an essential way, Deligne proved Theorem 2.2 under the additional assumption that X is smooth. Finally, a simple reduction of the problem to the smooth locus of X enables one to extend the finiteness theorem to the case where X is normal ([**Esn17**, Thm. 3.1]).

Fixing a good compactification $X \hookrightarrow \bar{X}$, with a strict normal crossings divisor at infinity, then a curve $\bar{C}$, complete intersection of ample divisors in $\bar{X}$ in good position, fulfils the Lefschetz theorem on topological fundamental groups, that is the homomorphism

$$\pi_1^{\mathrm{top}}(C := X \cap \bar{C}) \to \pi_1^{\mathrm{top}}(X)$$

is surjective. In particular, this reduces Theorem 2.1 to the case where X is of dimension 1. However, for X of dimension ≥ 2 in characteristic $p > 0$, there is no Lefschetz theorem. Thus Theorem 2.2 does not obviously reduce to dimension 1.

All one has at disposal are two kinds of Lefschetz theorems, one for reducing all tame coverings of X to one well chosen curve, one for reducing one specific object (ℓ-adic representation or given Galois cover) to one curve adapted to this object.

THEOREM 2.4 (Drinfeld, [**Dri12**], Prop. C.2, [**EK15**], Section 6). *Let $\bar{X} \supset X$ be a projective normal geometrically connected compactification of a smooth scheme of finite type X defined over a field k, let $\Sigma \subset \bar{X}$ be a closed subset of codimension ≥ 2 such that $(\bar{X} \setminus \Sigma)$ and $(\bar{X} \setminus \Sigma) \cap (\bar{X} \setminus X)$ are smooth. Let $\bar{C} \subset \bar{X} \setminus \Sigma$ be a smooth projective curve, complete intersection of ample divisors, meeting $\bar{X} \setminus X$ transversally. Then the restriction to $C = \bar{C} \cap X$ of any finite étale connected cover of $X \setminus \Sigma$, which is tame along $(\bar{X} \setminus \Sigma) \cap (\bar{X} \setminus X)$, is connected. In particular, the homomorphism on the tame fundamental groups $\pi_1^t(C) \to \pi_1^t(X)$ is surjective.*

An important point is that one does not need a good compactification for Drinfeld's theorem. If one has one, one can enhance the theorem to a complete version of the Lefschetz theorems under the Lefschetz conditions $Lef(X, Y)$ (formal sections along Y of vector bundles lift to an open neighbourhood of Y) and under the effective Lefschetz conditions $Leff(X, Y)$ (formal bundles along Y lift to an open neighbourhood of Y) (see [**EK15**, Thm 2.5], and [**SGA2**, X.2,p.90] for Grothendieck's Lefschetz and effective Lefschetz conditions). Using Theorem 2.4, together with the existence alterations [**dJ97**], one can prove Theorem 2.2 with the stronger assumption on the ramification being killed by one fixed finite étale cover $X' \to X$ purely geometrically, without using the existence of ℓ-adic companions (see [**Esn17**, Thm. 1.4]).

For non-tame ℓ-adic sheaves or covers, only a much weaker version of the Lefschetz theorems is available.

THEOREM 2.5 (see e.g. [**EK12**], Prop. B.1, Lem. B.2). *Let X be a smooth quasi-projective variety defined over $\mathbb{F}_q$, let $S \subset |X|$ be a finite set of closed points.*

1) *Let V be an irreducible $\bar{\mathbb{Q}}_\ell$-Weil or -étale lisse sheaf, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that $V|_C$ is irreducible.*
2) *Let $H \subset \pi_1(X)$ be an open normal subgroup, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that the homomorphism $\pi_1(C) \to \pi_1(X)/H$ is surjective.*

Nonetheless, it has the important following consequences.

COROLLARY 2.6. (Drinfeld [**Dri12**], Thm. 1.1, Deligne's conjecture (v) in [**Del80**] 1.2.10)

1) *If V is an irreducible Weil sheaf, such that* $\det(V)$ *is of finite order, then V has weight* 0.
2) *If V is an irreducible Weil lisse $\bar{\mathbb{Q}}_\ell$-sheaf with determinant of finite order, and $\sigma : \bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_{\ell'}$ is an isomorphism for ℓ' a prime number different from p, there is an irreducible Weil lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaf V_σ, called the* σ-companion *of V, with determinant of finite order, such that the characteristic polynomials $f_V \in \bar{\mathbb{Q}}_\ell[t], f_{V_\sigma} \in \bar{\mathbb{Q}}_{\ell'}[t]$ of the local Frobenii F_x satisfy $f_{V_\sigma} = \sigma(f_V)$.*

In 1) and 2), V and V_σ are in fact étale by [**Del80**, Thm. 1.3.1]. Deligne's finiteness Theorem 2.2 for rank 1 sheaves can be proven directly.

THEOREM 2.7 (Kerz-Saito, [**KS14**], Thm. 1.1). *Let X be a smooth quasi-projective variety over a perfect field k, let $X \subset \bar{X}$ be a projective compactification*

with simple normal crossings at infinity, let D be an effective divisor with support in $\bar{X} \setminus X$. Define $\pi_1^{\mathrm{ab}}(X, D)$ by the condition that a character $\chi : \pi_1(X) \to \mathbb{Q}/\mathbb{Z}$ factors through $\pi_1^{\mathrm{ab}}(X, D)$ if and only if the Artin conductor of χ pulled-back to any curve $C \to X$ is bounded by the pull-back of D via $\bar{C} \to \bar{X}$, where $\bar{C}$ is a compactification of C, smooth along $\bar{C} \setminus C$. Then the full package of the Lefschetz theorems holds:

For a sufficiently ample divisor $i : \bar{Y} \subset \bar{X}$ in good position with respect to $\bar{X} \setminus X$, the homomorphism

$$i_* : \pi_1^{\mathrm{ab}}(Y, \bar{Y} \cap D) \to \pi_1^{\mathrm{ab}}(X, D)$$

is an isomorphism if $\dim Y \geq 2$, and is surjective if $\dim Y = 1$, where $Y = \bar{Y} \cap X$. In particular, if $k = \mathbb{F}_q$, then

$$\mathrm{Ker}\big(\pi_1^{\mathrm{ab}}(X, D) \to \pi_1^{\mathrm{ab}}(k)\big)$$

is finite.

Theorem 2.7 implies the rank 1 case of Deligne's finiteness Theorem 2.2 in case one has a good compactification. In fact, one does not need the full package, only that if $\bar{C}$ is a complete intersection curve of such hypersurfaces $\bar{Y}$ as in the theorem, then

$$\pi_1^{\mathrm{ab}}(C, \bar{C} \cap D) \to \pi_1^{\mathrm{ab}}(X, D)$$

is surjective. So far, one does not have tools to understand a version of this for the whole fundamental group, which would explain Theorem 2.2 in general.

3. Deligne's conjectures: crystalline theory

Let X be a smooth geometrically connected scheme of finite type over a perfect field k of characteristic $p > 0$, $W := W(k)$ be the ring of Witt vectors, $K = \mathrm{Frac}(W)$ be its field of fractions. We refer to [**ES15**, Section 1] for the following presentation.

One defines the *crystalline sites* X/W_n as PD-thickenings $(U \hookrightarrow T/W_n, \delta)$, where the coverings come from $U \subset X$ Zariski open. The crystalline site X/W is then the 2-inductive limit of the X/W_n, see [**BO78**, Ch. 7,p. 7-22]. The category of *crystals* $\mathrm{Crys}(X/W)$ is the category of sheaves of $\mathcal{O}_{X/W}$-modules of finite presentation, with transition maps being isomorphisms. It is W-linear. The category of *isocrystals* $\mathrm{Crys}(X/W)_{\mathbb{Q}}$ is its $\mathbb{Q}$-linearisation. It is K-linear, tannakian.

The absolute Frobenius F acts on $\mathrm{Crys}(X/W)_{\mathbb{Q}}$. The largest full subcategory $\mathrm{Conv}(X/K) \subset \mathrm{Crys}(X/W)_{\mathbb{Q}}$ on which every object is F^{∞}-divisible is the K-tannakian subcategory of *convergent* isocrystals (Berthelot-Ogus) (Ogus defines the site of enlargements from X/W, then convergent isocrystals are crystals on it of $\mathcal{O}_{X/K}$-modules of finite presentation).

We introduce various categories of *F-isocrystals*. One defines the category F-$\mathrm{Conv}(X/K)$ of *convergent F-isocrystals* as pairs $(\mathcal{E}, \Phi)$ where $\mathcal{E} \in \mathrm{Conv}(X/K)$ and $\Phi : F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ is a K-linear isomorphism. Its is a $\mathbb{Q}_p$-linear tannakian category. The category F-$\mathrm{Overconv}(X/K)$ of *overconvergent F-isocrystals*, defined analytically by Berthelot in ([**Ber96**, 2.3.6, 2.3.7] (see also [**LeS07**, p. 288]), has a more algebraic description due to Kedlaya. It consists of those convergent F-isocrystals $\mathcal{E}$ which have unipotent local monodromy after alteration in the sense of Kedlaya ([**Ked04**, Introduction], [**Ked07**, Introduction and Section 3.2]). It is a $\mathbb{Q}_p$-linear category, fully embedded in F-$\mathrm{Conv}(X/K)$ ([**Ked04**, Thm. 1.1]). We shall just need that if

X is proper, then Kedlaya's full embedding is an equivalence, and that the group of extensions of the trivial object by itself in this category is $H^1_{\mathrm{rig}}(X/K)$, the first rigid cohomology group.

When k is a finite field $\mathbb{F}_q$, $q = p^s$, one defines $F_{\mathbb{F}_q} = F^s$-Overconv(X/K) by the same formulae as before, but now F^s acts instead of F. It is a K-linear category. Fixing an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, one defines F-Overconv$(X/K)_{\bar{\mathbb{Q}}_p}$ as its $\bar{\mathbb{Q}}_p$-linearization. It is called is the category of *overconvergent F-isocrystals over $\bar{\mathbb{Q}}_p$*. See [**Abe13**, 1.4.11, 4.1.2].

With those notations at hand, one can formulate the general hope. For smooth geometrically irreducible schemes of finite type X defined over a finite field $\mathbb{F}_q$, there should be an analogy between

i) irreducible objects in F-Overconv$(X/K)_{\bar{\mathbb{Q}}_p}$ with determinant of finite order;
ii) irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves with determinant of finite order.

Upon bounding ramification at infinity in i) and ii), the analogy should extend to irreducible $\mathbb{Q}$-variations of polarisable pure Hodge structures definable over $\mathbb{Z}$ over complex varieties. Of course, since there are many categories of isocrystals, one may wonder why those particular ones are the right analogs. The best demonstration is clearly Abe's Theorem 3.1. But already before the theorem was known, one knew that isocrystals have slopes (a topic not discussed here) and that those isocrystals which come from the variation of crystalline cohomology of the fibers of a smooth projective family have pure slope parts which are convergent subisocrystals. However, the lisse $\bar{\mathbb{Q}}_\ell$-sheaves computing the variation of the ℓ-adic cohomology of the fibers tend to be irreducible if the geometric variation of the family is big, e.g. if the family is the universal family on a moduli space.

THEOREM 3.1 (Abe, [**Abe13**], Thm. 4.3.1). *Let X be a smooth curve defined over a finite field $\mathbb{F}_q$. Then*

1) *an irreducible object in F-Overconv$(X/K)_{\bar{\mathbb{Q}}_p}$ with determinant of finite order is ι-pure of weight 0;*
2) *an irreducible lisse $\bar{\mathbb{Q}}_\ell$- étale sheaf with determinant of finite order has a companion which is an irreducible overconvergent F-isocrystal over $\bar{\mathbb{Q}}_p$ with determinant of finite order; vice-versa, an irreducible overconvergent F-isocrystal over $\bar{\mathbb{Q}}_p$ with determinant of finite order has a companion which is an irreducible lisse $\bar{\mathbb{Q}}_\ell$- étale sheaf with determinant of finite order.*

Here ι is a fixed isomorphism $\iota : \bar{\mathbb{Q}}_p \to \mathbb{C}$, and ι-pure means that the Frobenii at all closed points x of X act on the fiber E_x in

$$F-\mathrm{Overconv}(x/\mathrm{Frac}(W(k(x)))_{\bar{\mathbb{Q}}_p} = F-\mathrm{Conv}(x/\mathrm{Frac}(W(k(x)))_{\bar{\mathbb{Q}}_p}$$

of the object $\mathcal{E}$ in F-Overconv$(X/K)_{\bar{\mathbb{Q}}_p}$ with eigenvalues of complex absolute value $q^{w/2}$ for a fixed real number w called the weight. (Similarly, ι-mixed means that $\mathcal{E}$ is filtered in F-Overconv$(X/K)_{\bar{\mathbb{Q}}_p}$ such that the associated graded $gr\mathcal{E}$ in a sum of ι-pure objects. See [**AC13**, Defn. 2.1.3].)

Deligne's program [**Del80**, 1.2.10] in higher dimension on the crystalline side is not yet achieved. However, a Lefschetz theorem such as Theorem 2.5 for overconvergent F-isocrystals over $\bar{\mathbb{Q}}_p$ has been proven.

THEOREM 3.2 (Abe-Esnault, [**AE16**], Thm. 0.1). *Let X be a smooth connected quasi-projective variety defined over $\mathbb{F}_q$. Let M be an irreducible overconvergent F-isocrystal. Then there is a open dense subscheme $U \hookrightarrow X$ such that for any finite set $S \subset U$ of closed points, there is a smooth irreducible curve $C \to X$ such that $S \subset C$ and such that $M|_C$ is irreducible.*

This implies the following.

1) M is ι-pure of weight 0 ([**Abe13**, Thm. 4.2.2]);
2) Corollary 2.3 remains true with V replaced by M, that is there is a number field containing all the coefficients of all local eigenpolynomials ([**AE16**, 1.5]) at closed points ([**AE16**, Lem. 4.1]);
3) M has ℓ-adic companions which are irreducible $\bar{\mathbb{Q}}_\ell$-lisse sheaves ([**AE16**, Thm. 4.3]);
4) There is a crystalline version Theorem 2.2: Let X be a smooth connected quasi-projective variety defined over $\mathbb{F}_q$, (r, D) be as in Theorem 2.2, σ be a field isomorphism from $\bar{\mathbb{Q}}_p$ to $\bar{\mathbb{Q}}_\ell$ for some prime number ℓ different from p. Then there are finitely many isomorphism classes of irreducible overconvergent F-isocrystals, up to twist with rank 1 isocrystals on $\mathbb{F}_q$, such that the σ-companion (which by 3) is an irreducible $\bar{\mathbb{Q}}_\ell$-lisse sheaf) has ramification bounded by D.

We note that in an 'unstable preprint' posted on his webpage, unstable in the author's terminology, Kedlaya uses weights to deduce 1) and 2) as well as the part of 3) concerning the existence of ℓ-adic companions. The properties of being lisse and irreducible seems to be inaccessible without the Lefschetz theorem 3.2.

We also mention [**Kos15**, Thm. 1.2] in which a weak analog to Theorem 2.2 is proven: if X is a smooth geometrically connected variety defined over a finite field, then an absolutely semi-simple unit-root overconvergent F-isocrystal in F-$\mathrm{Overconv}(X/K)_{\bar{\mathbb{Q}}_p}$ is isotrivial. The point is that such an object necessarily is locally isotrivial at infinity, which reduces the problem to the case of X smooth projective, thus by the standard Lefschetz theorem to the curve case. One then applies Abe's theorem [**Abe13**, Thm. 4.1] which reduces the statement to Lafforgue's theorem [**Laf02**, Thm. VII.6]).

4. Malčev-Grothendieck's theorem, Gieseker's conjecture, de Jong's conjecture

Let X be a smooth geometrically irreducible scheme of finite type over field k of characteristic 0. Grothendieck defined the *infinitesimal site* X_∞ ([**Gro68**]) with objects $U \hookrightarrow T$ where T is an infinitesimal thickening of a Zariski open subscheme U, and where coverings come from the Us. *Crystals* are finitely presented crystals on X_∞. The category is equivalent to the category of bundles on X with an integrable connection (E, ∇) or equivalently to the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. It is a k-linear category, which is tannakian.

THEOREM 4.1 (Malčev [**Mal40**], Grothendieck [**Gro70**], Thm. 4.2). *Let X be a complex smooth variety. If its étale fundamental group is trivial, then there are no non-trivial crystals in the infinitesimal site (with regular singularities at infinity in case X is not projective).*

Here one uses the Riemann-Hilbert correspondence to translate the assertion on representations of groups which are finitely generated, applied to the topological fundamental group, to the assertion on crystals in the infinitesimal site. The proof then just uses that a $GL(r, A)$-representation, where A is $\mathbb{Z}$-algebra of finite type, is trivial if and only if it is by restriction to the closed points of $\mathrm{Spec}(A)$.

Gieseker [**Gie75**, p. 8] conjectured that the analog theorem remains true in characteristic $p > 0$. Let X be a smooth projective geometrically irreducible variety over a field k of characteristic $p > 0$. One defines X_∞ and crystals as in characteristic 0. By Katz' theorem [**Gie75**, Thm. 1.3], which relies on Cartier descent, it is also equivalent to the category of Frobenius divisible $\mathcal{O}_X$-coherent sheaves, that is infinite sequences $(E_0, E_1, \cdots, \sigma_0, \sigma_1, \cdots)$ of bundles E_n on the n-th Frobenius twist $X^{(n)}$ of X, together with isomorphisms σ_n between E_n and the Frobenius pull-back of E_{n+1}. Then Gieseker's conjecture predicts that Theorem 4.1 holds in characteristic $p > 0$. It has been proved in 2010.

THEOREM 4.2 (Esnault-Mehta, [**EM10**], Thm. 1.1). *Let X be a smooth projective geometrically irreducible variety over a field k of characteristic $p > 0$. If its geometric étale fundamental group is trivial, then there are no non-trivial crystals in the infinitesimal site.*

What in the proof replaces the finite generation of the topological fundamental group is the existence of quasi-projective moduli for stable bundles with vanishing Chern classes (Langer, [**Lan04**, Thm. 4.1]). What then replaces the criterion for triviality is Hrushovski's theorem on the existence of preperiodic points on dominant correspondences over finite fields [**Hru04**]. Varshavsky in [**Var18**] gave a proof of it in the framework of arithmetic geometry, without using model theory.

One can formulate variants of Gieseker's conjecture. If X is not proper, then the theory of *regular singular* crystals in the infinitesimal site has been developed by Kindler [**Kin15**], in such a way that for those objects with a finite Tannaka group, it coincides with the notion of *tame* quotient of the étale fundamental group. There is no good higher ramification theory so far, nor does one have an analog of Theorem 4.2, except for the tame abelian quotient of the geometric fundamental group (Kindler, [**Kin13**, Thm. 1.4]), and in the case where X is the smooth locus of a normal projective variety defined over $k = \bar{\mathbb{F}}_q$ (Esnault-Srinivas [**ESB15**, Thm 1.1]; the proof uses Bost's improvement of Grothendieck's LEF theorem, see [**ESB15**, Appendix]).

In 2010, de Jong formulated the corresponding conjecture in the category of isocrystals. Let X be a smooth projective geometrically irreducible variety over a perfect field k of characteristic $p > 0$. If its geometric étale fundamental group is trivial, then the conjecture predicts that there are no non-trivial isocrystals. As of today, there is no complete understanding of the conjecture. We now list the known results concerning it.

One defines $N(1) = \infty, N(2) = 2, N(3) = 1, N(r) = 1/M(r)$ where for any natural number $r \geq 4$, $M(r)$ is the maximum of the lower common multiples of a and b for all choices of natural numbers $a, b \geq 1$ with $a + b \leq r$. For any torsion-free coherent sheaf $\mathcal{F}$, one denotes by $\mu_{\max}(\mathcal{F})$ its maximal slope.

THEOREM 4.3 (Esnault-Shiho). *Let X be a smooth projective geometrically irreducible variety over a perfect field k of characteristic $p > 0$.*

1) *If the abelian quotient of the geometric étale fundamental group of X is trivial, there are no non-trivial isocrystals which are successive extensions of rank* 1 *isocrystals.* ([**ES15**, Prop. 2.9, Prop. 2.10]).
2) *If the geometric étale fundamental group of X is trivial, $\mu_{\max}(\Omega^1_X) < N(r)$, the irreducible constituents of the Jordan-Hölder filtration of $\mathcal{E}$ have rank $\leq r$, and $\mathcal{E}$ itself is either in* $\mathrm{Conv}(X/K)$ *or else each of its irreducible constituents has a locally free lattice and has rank $\geq r$, then $\mathcal{E}$ is trivial.* ([**ES15**, Thm. 1.1] *and* [**ES15b**, Thm. 1.2]).

Let $f : Y \to X$ be a smooth proper morphism between smooth proper schemes of finite type.

3) *If the geometric étale fundamental group of X is trivial, then all the Gauß-Manin convergent isocrystals $R^n f_* \mathcal{O}_{Y/K}$ are trivial. If k is finite, f is projective and $p \geq 3$, one can drop the properness assumption on X* ([**ES15b**, Thm. 1.3, Rmk. 1.4]).

We first discuss 3). Le us assume that k is a finite field. Then the statement relies on

THEOREM 4.4 (Abe's Čebotarev's density theorem, [**Abe13**], A.3). *Let X be a smooth scheme of finite type defined over a finite field k. If $\mathcal{E}$ and $\mathcal{E}'$ are ι-mixed overconvergent F-isocrystals over $\bar{\mathbb{Q}}_p$ with the same set of Frobenius eigenvalues on closed points of X, then the semi-simplifications of $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.*

When X is proper or f is projective, the convergent F-isocrystal $R^n f_* \mathcal{O}_{Y/K}$ is an overconvergent F-isocrystal over $\bar{\mathbb{Q}}_p$ via the faithful embedding ([**Laz15**, Cor. 5.4]), thus obeys Theorem 4.4. The Weil conjectures [**KM74**, Thm. 1.1], [**CLS98**, Cor. 1.3] in the proper case, enable one to conclude that the semi-simplification of $R^n f_* \mathcal{O}_{Y/K}$ is constant. Forgetting the F-structure, it is thus a successive extension of the trivial overconvergent isocrystal by itself, thus is trivial, as the first rigid cohomology of X is controlled by the first ℓ-adic cohomology of $X \otimes_k \bar{k}$ when X is proper or $p \geq 3$. The latter is trivial if the geometric fundamental group is trivial. One can alternatively use the existence of ℓ-adic companions ([**AE16**, Thm.4.2]). Over a general field k, the properness assumption on X allows to compare the statement to the one over finite fields by base change.

We discuss 2). Since the conjecture concerns isocrystals, it is not natural to try to argue with lattices, that is p-torsion free crystals E in a given isocrystal class $\mathcal{E}$. Unfortunately, there is at present no other way to do, and indeed, basically 2) is proven by showing that under the given assumptions, the value E_X on X of a well chosen crystal E in the isocrystal class $\mathcal{E}$ is trivial. Then one studies the possible liftings modulo p-powers. In order to show triviality of E_X, one applies Theorem 4.2. To do so, one has to show the existence of such an E_X which is semi-stable with vanishing numerical Chern classes, so as to be able to define its moduli point. If E_X was F^∞-divisible, then one could apply Theorem 4.2 directly. This is not the case, even if $\mathcal{E} \in \mathrm{Conv}(X/K)$, that is even if $\mathcal{E}$ is F^∞-divisible. Instead, one shows that F^a-divisibility is enough to trivialize a moduli point, for a large enough depending only on X and the rank of $\mathcal{E}$ ([**ES15**, Prop. 3.2]). One applies this to the Frobenius pull-backs of E_X. For this one needs that they are semistable as well, and this is the reason for the assumption on $\mu_{\max}(\Omega^1_X)$. On the other hand, a Langton type argument guarantees that one finds a crystal E such

that E_X is semi-stable. The issue is then to show vanishing of the Chern classes of E_X. It is easy to show that all lattices E_X of a given isocrystal $\mathcal{E}$ have the same crystalline Chern classes $c_n^{\mathrm{crys}}(\mathcal{E})$ in $H^{2n}(X/W), n \geq 1$, and thus $c_n^{\mathrm{crys}}(\mathcal{E}) = 0$ if $\mathcal{E} \in \mathrm{Conv}(X/K)$ ([**ES15**, Prop. 3.1]). If E_X is locally free as a coherent sheaf, it is true, but by no means trivial, that $c_n^{\mathrm{crys}}(\mathcal{E}) = 0$, where $\mathcal{E}$ is the isocrystal class of E (see [**ES15b**, Section 2/3]). However, we do not know whether or not any isocrystal $\mathcal{E}$ admits a lattice E which is locally free (see [**ES15c**]). This explains the restriction on the type of isocrystals considered in the theorem.

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