Brownian surfaces

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based on ongoing joint work with Jérémie Bettinelli

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Maps on surfaces

Definition

A map on a surface $M$ (orientable, compact) is an embedding of a finite graph on $M$, that dissects the latter into topological polygons, and considered up to direct homeomorphisms of $M$.

A rooted map: distinguish one oriented edge.

- $V(m)$ Vertices
- $E(m)$ Edges
- $F(m)$ Faces
- $d_m(u,v)$ combinatorial graph distance
- The degree of a face is the number of incident corners
- A $p$-angulation has only faces of degree $p$. 
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On the torus

On a disk
Motivation

- Maps are seen as discretized 2D Riemannian manifolds.
- This comes from 2D quantum gravity, in which a basic object is the partition function

\[ \int_{\mathcal{R}(M)/\text{Diff}^+(M)} [\mathcal{D}g] \exp(-\alpha \text{Area}_g(M)) \]

- \( M \) is a 2-dimensional orientable manifold,
- \( \mathcal{R}(M) \) is the space of Riemannian metrics on \( M \),
- \( \text{Diff}^+(M) \) the set of orientation-preserving diffeomorphisms,
- \( \mathcal{D}g \) is a “Lebesgue” measure on \( \mathcal{R}(M) \) invariant under the action of \( \text{Diff}^+(M) \). This, and the induced measure \([\mathcal{D}g]\), are the problematic objects.
How to deal with $[\mathcal{D}g]$?

One can replace

$$
\int_{\mathcal{R}(M)/\text{Diff}^+(M)} [\mathcal{D}g] \rightarrow \sum_{T \in \text{Tr}(M)} \delta_T
$$

where $\text{Tr}(M)$ is the set of triangulations of $M$.

- Then one tries to take a **scaling limit** of the right-hand side, in which triangulations approximate a continuum surface, the Brownian map.

- Analog to **path integrals**, in which random walks can be used to approximate Brownian motion.

- The success of this approach comes from the rich literature on enumerative theory of maps, after Tutte’s work or the literature on matrix integrals.

- However, metric aspects of maps could only be dealt with recently, using **bijective approaches**.
The quantum Liouville theory approach

- Another approach is **quantum Liouville theory** (Polyakov, David, ...). The challenge is to make sense of the “metric”

\[ g = e^{2u} g_0 \]

where \( g_0 \) is the usual metric on the sphere, and \( u \) is a **Gaussian free field** (a random distribution).

- In 2008, Duplantier-Sheffield have made sense of the measure associated with this ill-defined metric, which can be constructed using Kahane’s **Gaussian multiplicative chaos theory**.

- Much more recently (2013), Miller-Sheffield have introduced the **QLE(8/3,0)**, which is a candidate for the growth process of the metric balls of Liouville quantum gravity.
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QLE growth and peeling process

- The way in which QLE grows is inspired by the *Eden model* on random triangulations.
- Note that the peeling process was first introduced by Ambjørn-Watabiki (1995) to derive (semi-rigorously, see Budd 2014) the two-point function of random triangulations (law of the distance between two uniformly chosen points).
- The Miller-Sheffield QLE(8/3,0) construction asks whether one can recover the Brownian map from the scaling limit of the peeling process (described in terms of branching and Lévy processes, see Curien-Le Gall 2014).
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Some simulations

Figure: A random quadrangulation with 30000 vertices, by J.-F. Marckert
Some simulations

Figure: A random bipartite quadrangulation of the torus by G. Chapuy
The plane case

- $Q_n$ uniform random variable in the set $Q_n$, of rooted plane quadrangulations with $n$ faces.
- The set $V(Q_n)$ of its vertices is endowed with the graph distance $d_{Q_n}$.
- Typically $d_{Q_n}(u, v)$ scales like $n^{1/4}$ (Chassaing-Schaeffer (2004)).
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Convergence to the Brownian map

**Theorem**

There exists a random metric space \((S, D)\), called the Brownian map, such that the following convergence in distribution holds

\[
(V(Q_n), (8n/9)^{-1/4}d_{Q_n}) \xrightarrow{(d)} (S, D)
\]

as \(n \to \infty\), for the Gromov-Hausdorff topology.

- \(X_n \to X\) for the Gromov-Hausdorff topology if \(X'_n \to X'\) in the Hausdorff sense for suitable isometric embeddings of \(X_n, X\) in a common space.
- This result has been proved independently by Le Gall (2011) and Miermont (2011), via different approaches.
- Before this work, convergence was only known up to extraction of subsequences — the uniqueness of the limiting law was open.
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Some (of the many) pre-2011 results in this topic

- Chassaing-Schaeffer (2004)
  - identify $n^{1/4}$ as the proper scaling and
  - compute limiting functionals for random quadrangulations, including the two-point function.


- Le Gall (2007)
  - Gromov-Hausdorff tightness for rescaled $2p$-angulations
  - the limiting topology is the same as that of the Brownian map.
  - all subsequential limits have Hausdorff dimension 4

- Le Gall-Paulin (2008), and later M. (2008) show that the limiting topology is that of the 2-sphere.

- Bouttier-Guitter (2008) identify the limiting three-point function: joint law of distances between three uniformly chosen vertices.
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- **Marckert-Mokkadem (2006)** introduce the Brownian map.

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Universality

• Le Gall (2011) gives an efficient method to extend the result to many other classes of maps, and applies it to $p$-angulations, $p \in \{3, 4, 6, 8, 10, \ldots \}$: if $M_n^{(p)}$ is a uniform $p$-angulation with $n$ faces then

$$\left( V(M_n^{(p)}), c_p n^{-1/4} d_{M_n^{(p)}} \right) \xrightarrow{d} (S, D) \quad n \to \infty$$

for some scaling constant $c_p \in (0, \infty)$.

• The general method has been applied successfully to other families of maps:
  ▶ quadrangulations with no pendant edges (Beltran and Le Gall 2012)
  ▶ simple triangulations and simple quadrangulations (Addario-Berry and Albenque 2013)
  ▶ general maps with fixed number of edges (Bettinelli, Jacob and Miermont)
  ▶ bipartite maps with fixed number of edges (C. Abraham)
  ▶ and probably more to come ($p$-angulations for odd $p \geq 5$, planar graphs?)
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Other topologies

- Let $Q_n^{(g)}$ be the set of rooted, bipartite quadrangulations of the $g$-torus $\mathbb{T}_g$.
- Let $Q_{n,m}$ be the set of rooted plane maps, with $n + 1$ faces, all of degree 4 except the root face, that has degree $2m$.

In ongoing joint work with Jérémie Bettinelli, we show that

- if $Q_n$ is uniform in $Q_n^{(g)}$, then

  $$(V(Q_n), n^{-1/4} d_{Q_n}) \xrightarrow{(d)} M^{(g)},$$

- if $Q_n$ is uniform in $Q_{n,m}^{\partial}$ and $m = m(n)$ satisfies $m/\sqrt{n} \to \lambda \in (0, \infty)$, then

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- Let $Q^{(g)}_n$ be the set of rooted, bipartite quadrangulations of the $g$-torus $T_g$.
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Topologies are preserved

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Previous results by Bettinelli (2011, 2012) show that a.s.,

- \(M^{(g)}\) is homeomorphic to \(\mathbb{T}_g\) and \(M_{\lambda}^\partial\) is homeomorphic to the unit disk.
- Moreover, \(\dim(M^{(g)}) = \dim(M_{\lambda}^\partial) = 4\) and \(\dim(\partial M_{\lambda}^\partial) = 2\).

For the case \(\lambda \in \{0, \infty\}\):

**Theorem (Bettinelli, 2012)**

- If \(m \ll \sqrt{n}\), the space \(n^{-1/4} Q_{n,m}^\partial\) converges to the Brownian map.
- If \(\sqrt{n} \ll m\), the space \(m^{-1/2} Q_{n,m}^\partial\) converges to the Brownian continuum random tree.
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Let $T_n$ be the set of rooted plane trees with $n$ edges,

$\mathbb{T}_n$ be the set of labeled trees $(t, \ell)$ where $\ell : V(t) \to \mathbb{Z}$ satisfies $\ell(\text{root}) = 0$ and

$$|\ell(u) - \ell(v)| \leq 1, \quad u, v \text{ neighbors}.$$
The Cori-Vauquelin-Schaeffer bijection: coding maps with trees

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  \[ |\ell(u) - \ell(v)| \leq 1, \quad u, v \text{ neighbors}. \]

**Theorem (Cori-Vauquelin 1981, Schaeffer)**

The construction to follow yields a bijection between $\mathbb{T}_n \times \{0, 1\}$ and $\mathbb{Q}_n^*$, the set of rooted, pointed plane quadrangulations with $n$ faces.
The Cori-Vauquelin-Schaeffer bijection

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\[
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The Cori-Vauquelin-Schaeffer bijection
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Note that the labels are geodesic distances in the map. Key formula:

$$d_q(v_*, v) = \ell(v) - \inf \ell + 1$$
The modified CVS bijection

• We saw that a pointed, rooted map is a labelled tree. We can view it alternatively by forbidding arcs that “wrap around” the root corner of the tree.

• Instead, we connect those arcs to a “shuttle”, giving a quadrangulation with a geodesic boundary.

• These maps play an important part in Le Gall’s 2011 proof of uniqueness of the Brownian map.
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The Brownian CRT with Brownian labels

Let $(T_n, \ell_n)$ be uniform in $\mathbb{T}_n$.

It is well-known that in some natural sense (encoding by contour functions)

$$\left( \frac{T_n}{\sqrt{n}}, \frac{\ell_n}{n^{1/4}} \right) \longrightarrow (\mathcal{T}, Z)$$

$\mathcal{T}$ is the Brownian continuum random tree (Aldous)

$Z$ is a process of Brownian labels indexed by $\mathcal{T}$. 
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Brownian map with geodesic boundaries

- The scaling limit $(\mathcal{T}, Z)$ is the building block for the Brownian map $(S, D)$: perform a continuum analog of the CVS bijection. Distinguished points are the root $\rho$ of $\mathcal{T}$, and the point $x_*$ where $Z$ attains its overall minimum.

- Cutting along the (a.s. unique) geodesic from $\rho$ to $x_*$, one gets a Brownian map with geodesic boundary.

- The latter is the same as the scaling limit for the modified CVS bijection.
Quadrangulations with a boundary

- One can code a quadrangulation with \( n \) faces and a boundary of length \( 2m \) using another variant of the CVS bijection (special case of the Bouttier-Di Francesco-Guitter bijection).

- Let \((x_1, x_2, \ldots, x_m)\) be an integer walk of length \( m \), with \( x_1 = 0 \), and \( x_{i+1} - x_i \geq -1 \) for every \( i \in \{1, \ldots, m\} \) (with the convention that \( x_{m+1} = 0 \)), chosen uniformly at random.

- Consider a forest of well-labeled trees \((T_1, \ell_1), \ldots, (T_m, \ell_m)\) with a total of \( n \) edges, chosen uniformly at random with the constraint that the root of \( T_i \) has label \( x_i \).

- Arrange these trees in cyclic order and perform the CVS bijection rules. This gives a pointed quadrangulation with a boundary. Modulo some rooting convention that we omit, this is a bijection.
Quadrangulations with a boundary

- One can code a quadrangulation with $n$ faces and a boundary of length $2m$ using another variant of the CVS bijection (special case of the Bouttier-Di Francesco-Guitter bijection).
- Let $(x_1, x_2, \ldots, x_m)$ be an integer walk of length $m$, with $x_1 = 0$, and $x_{i+1} - x_i \geq -1$ for every $i \in \{1, \ldots, m\}$ (with the convention that $x_{m+1} = 0$), chosen uniformly at random.
- Consider a forest of well-labeled trees $(T_1, \ell_1), \ldots, (T_m, \ell_m)$ with a total of $n$ edges, chosen uniformly at random with the constraint that the root of $T_i$ has label $x_i$.
- Arrange these trees in cyclic order and perform the CVS bijection rules. This gives a pointed quadrangulation with a boundary. Modulo some rooting convention that we omit, this is a bijection.
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Scaling limit

In the scaling limit $m \sim \lambda n^{1/2}$, labels renormalized by $n^{1/4}$,

- The random walk $x_1, \ldots, x_m$ becomes a Brownian bridge.
- The labeled trees become a Brownian forest (trees encoded by the excursions of a reflected Brownian motion until it has accumulated local time $\lambda$ at 0) plus Brownian labels.
- The bridge and labels have different diffusion constants!
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- Get a gluing of “shards” (or “slices”, Bouttier-Guitter 2012).

A difficulty when performing this metric gluing is that we glue infinitely many shards. Fortunately, these only accumulate near the boundary, where we can show that the geodesics do not typically go.
Closed surfaces

- For bipartite quadrangulations of the $g$-torus $\mathbb{T}_g$, we use similar ideas, relying on a generalization of the CVS bijection by Chapuy-Marcus-Schaeffer (2008).

- In this context, pointed (rooted) bipartite quadrangulations are in correspondence with well-labeled $g$-trees, that are maps on $\mathbb{T}_g$ with only one face, carrying integer labels with variations of $\{-1, 0, 1\}$ along edges.
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Labeled $g$-trees can be decomposed canonically into maximal doubly-pointed labeled tree components, each of which encodes a quadrilateral with geodesic boundaries. In the scaling limit, these can be obtained from the Brownian map by cutting along two geodesics from two uniform points $x_1, x_2$ to the root $x_*$. 
Brownian quadrilaterals

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Gluing these quadrilaterals in the scaling limit is easier than for shards, because there are only finitely many of these. On the other hand, their exact definition requires singular conditionings of the Brownian map (precisely, one conditions on the value of $D(x_1, x_*) - D(x_2, x_*)$).
Some future directions and open questions

We expect to get answers to the following questions reasonably soon:

- Extend this construction to more general surfaces (genus $g$ with $p$ holes): should come from Bettinelli (2014)’s study of geodesics on Brownian surfaces
- Prove universality for these maps (more face degrees allowed, topological constraints e.g. simple boundaries, etc.)
- Show the metric gluing of two Brownian discs along their boundaries is well-defined (motivated by the self-avoiding walk on random maps)

Much more challenging problems for the future:

- Conformal structure of Brownian surfaces? For instance, what is the asymptotic law of the moduli of a random uniform bipartite quadrangulation of $\mathbb{T}_g$? Can it be recovered from $M(g)$?
- Can one make sense of the double scaling limit, where $n$ and $g$ go to $\infty$ together?
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Thanks!