The two-time distribution in random growth

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Consider the last-passage times

\[ G(m, n) = \max_{\pi:(1,1) \rightarrow (m,n)} \sum_{(i,j) \in \pi} w(i,j), \]

where \( w(i,j) \) are i.i.d. geometric random variables

\[ \mathbb{P}[w(i,j) = k] = (1 - q)q^k, \quad k \geq 0. \]
Directed last-passage times

We have the limit theorem

$$\mathbb{P}\left[ \frac{G(n, \lfloor \lambda n \rfloor) - an}{bn^{1/3}} \leq \xi \right] \rightarrow F_2(\xi)$$

as $n \rightarrow \infty$, where

$$F_2(\xi) = \det \left( I - K_{Ai} \right)_{L^2(\xi, \infty)}$$

is the *Tracy-Widom distribution*, $K_{Ai}$ the *Airy kernel*. 
Directed last-passage times

We have the limit theorem

\[ \mathbb{P} \left[ \frac{G(n, \lfloor \lambda n \rfloor) - an}{bn^{1/3}} \leq \xi \right] \to F_2(\xi) \]

as \( n \to \infty \), where

\[ F_2(\xi) = \det(I - K_{\text{Ai}})_{L^2(\xi, \infty)} \]

is the \textit{Tracy-Widom distribution}, \( K_{\text{Ai}} \) the \textit{Airy kernel}.

Fluctuation exponent 1/3.
Two-point distribution, space direction

Joint distribution of $G(m_1, n_1)$ and $G(m_2, n_2)$ when $m_1 < m_2$ and $n_1 > n_2$.

\[
\mathbb{P} \left[ \frac{G(n + \nu_1 n^{2/3}, n - \nu_1 n^{2/3}) - an}{bn^{1/3}} \leq \xi_1, \frac{G(n + \nu_2 n^{2/3}, n - \nu_2 n^{2/3}) - an}{bn^{1/3}} \leq \xi_2 \right]
\]

converges to a Fredholm determinant involving the extended Airy kernel.

Fluctuation exponent 2/3.
Two-point distribution, time direction

Two-time joint distribution

\[ \mathbb{P} [ G(m_1, m_1) \leq v_1, G(m_2, m_2) \leq v_2 ] \]

We expect non-trivial fluctuations if

\[ \frac{m_1}{m_2 - m_1} \sim c, \quad c > 0. \]

Exponent 1

*Slow de-correlation phenomenon*

(P. L. Ferrari and I. Corwin, P.L. Ferrari, S. Péché)
Simulations

Figure: Clusters at different times in an Eden model
Simulations


Slow de-correlation
Zero temperature Brownian semi-discrete directed polymer

Last-passage time

\[ H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}). \]

Distributed as the largest eigenvalue of a GUE-matrix

\[
\mathbb{P}[H(\mu, n) \leq \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{2\mu}} \, d^n x.
\]
Zero temperature Brownian semi-discrete directed polymer

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\[ H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}). \]

Distributed as the largest eigenvalue of a GUE-matrix

\[ \mathbb{P}[H(\mu, n) \leq \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{2\mu}} \, d^n x. \]

Limit of \( G(m, n) \)

\[ \frac{G([\mu T], n) - \frac{q}{1-q} [\mu T]}{\frac{\sqrt{q}}{1-q} \sqrt{T}} \to H(\mu, n) \]

in distribution as \( T \to \infty. \)
Main theorem

Theorem
Let $0 < t_1 < t_2$, $\eta_1, \eta_2, \nu_1, \nu_2 \in \mathbb{R}$ be given. Set

$$\alpha = \left(\frac{t_1}{t_2 - t_1}\right)^{1/3}.$$ 

Introduce the scaling, $i = 1, 2,$

$$\mu_i = t_i M - \nu_i (t_i M)^{2/3}, \quad n_i = t_i M + \nu_i (t_i M)^{2/3}, \quad \xi_i = 2t_i M + (\eta_i - \nu_i^2)(t_i M)^{1/3}. \quad i = 1, 2.$$

With this scaling,

$$\lim_{M \to \infty} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = F_{tt}(\eta_1, \eta_2; \alpha, \nu_1, \nu_2).$$

There is an explicit, but rather complicated, formula for the two-time distribution $F_{tt}$.

A different formula was derived non-rigorously by V. Dotsenko using the replica method.
Let

\[ \mathbf{G}(m) = (G(m, 1), \ldots, G(m, n)) \]

and

\[ w_m(x) = (1 - q)^m \binom{x + m - 1}{x} q^x 1(x \geq 0). \]

For \( x, y \in \{ x \in \mathbb{Z}^n ; x_1 \leq x_2 \leq \cdots \leq x_n \} \) and \( m > \ell \geq 0 \),

\[ \mathbb{P}[\mathbf{G}(m) = y | \mathbf{G}(\ell) = x] = \det \left( \Delta^{j-i} w_{m-\ell} (y_j - x_i) \right)_{1 \leq i, j \leq n}. \]

In particular

\[ \mathbb{P}[\mathbf{G}(m) = x] = \det \left( \Delta^{j-i} w_{m} (x_j) \right)_{1 \leq i, j \leq n}. \]

(Inspired by similar results by J. Warren.)
Joint distribution

We get the formula

\[
\mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2] = \sum_{u=-\infty}^{v_1} \sum_{x \in W_{n_2}} \sum_{y \in W_{n_2}} \det (\Delta^{j-i} w_{m_1}(x_j))^{1 \leq i, j \leq n_2} \det (\Delta^{j-i} w_{m_2-m_1}(y_j - x_i))^{1 \leq i, j \leq n_2},
\]
We get the formula

\[ \mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2] \]

\[ = \sum_{u=-\infty}^{v_1} \sum_{x \in W_{n_2}} \sum_{y \in \mathbb{N}_{n_2}} \det \left( \Delta^{j-i} w_{m_1}(x) \right)_{1 \leq i, j \leq n_2} \left( \Delta^{j-i} w_{m_2-m_1} (y - x) \right)_{1 \leq i, j \leq n_2}, \]

Into this formula we can insert

\[ \Delta^k w_m(x) = \frac{(1 - q)^m}{2\pi i} \int_{\gamma_r} \frac{(1 - z)^k dz}{(1 - qz)^m z^{x+k+1}}. \]
After a non-trivial computation we obtain

\[ \mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2] = \sum_{u=-\infty}^{v_1} \frac{(1-q)^{m_2 n_2} (-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2 n_1} (\Delta n)!^2} \int_{\gamma_1^{n_1}} d^n z \int_{\gamma_1^{\Delta n}} d^n z \int_{\gamma_2^{n_1}} d^n w \int_{\gamma_2^{\Delta n}} d^n w \times \det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} \det \left( w_j^{i-1} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{w_j - z_i} \right)_{1 \leq i, j \leq n_1} \det \left( \frac{1}{z_j - w_i} \right)_{n_1 < i, j \leq n_2} \times \prod_{j=n_1+1}^{n_2} \frac{1-z_j}{1-w_j} \left( 1 - \prod_{j=1}^{n_1} \frac{z_j}{w_j} \right) \prod_{j=1}^{n_2} \frac{w_j^{u-v_2-\Delta n}}{z_j^{u+n_1} (1-z_j)^{\Delta n} (1-qz_j)^{m_1} (1-w_j)^{n_1} (1-qw_j)^{\Delta m}}, \]

Here \( \Delta n = n_2 - n_1, \Delta m = m_2 - m_1, 0 < s_1 < r_1 < 1, 0 < r_2 < s_2 < 1. \)
Two identities

An important ingredient in the above derivation are two symmetrization identities:

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \left( \frac{1 - w_{\sigma(j)}}{w_{\sigma(j)}} \right)^j \frac{1}{(1 - w_{\sigma(1)})(1 - w_{\sigma(1)}w_{\sigma(2)}) \cdots (1 - w_{\sigma(1)} \cdots w_{\sigma(n)})}
\]

\[= (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \frac{1}{w_j^{n-1}} \det \left( w_{j-1}^{i-1} \right)_{1 \leq i,j \leq n}, \]

(Tracy-Widom ASEP identity) and

\[
\sum_{\sigma_1, \sigma_2 \in S_n} \text{sgn}(\sigma_1 \sigma_2) \prod_{j=1}^{n} \left( \frac{w_{\sigma_2(j)}(1 - z_{\sigma_1(j)})}{z_{\sigma_1(j)}(1 - w_{\sigma_2(j)})} \right)^j \\
\times \frac{1}{\left( 1 - \frac{z_{\sigma_1(1)}}{w_{\sigma_2(1)}} \right) \left( 1 - \frac{z_{\sigma_1(1)}z_{\sigma_1(2)}}{w_{\sigma_2(1)}w_{\sigma_2(2)}} \right) \cdots \left( 1 - \frac{z_{\sigma_1(1)} \cdots z_{\sigma_1(n)}}{w_{\sigma_2(1)} \cdots w_{\sigma_2(n)}} \right)}
\]

\[= \prod_{j=1}^{n} \frac{w_j^{n+1}(1 - z_j)^n}{z_j^n(1 - w_j)^n} \det \left( \frac{1}{w_j - z_i} \right)_{1 \leq i,j \leq n}. \]
Taking the appropriate limit to the Brownian directed polymer we obtain

\[ \frac{\partial}{\partial \xi_1} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = \frac{\partial}{\partial h} \bigg|_{h=0} Q(h), \]

where we recall that

\[ H(\mu, n) = \sup_{0=\tau_0 < \tau_1 < \cdots < \tau_n = \mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}). \]
Limit to the Brownian directed polymer

Taking the appropriate limit to the Brownian directed polymer we obtain

\[
\frac{\partial}{\partial \xi_1} P \left[ H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2 \right] = \frac{\partial}{\partial h} \bigg|_{h=0} Q(h),
\]

where

\[
Q(h) = \frac{(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2} n_1! \Delta n!} \times \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \det \left( z_j^{i-1} \right)_{1 \leq i,j \leq n_2} \det \left( w_j^{i-1} \right)_{1 \leq i,j \leq n_2} \\
\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w^2_j - \Delta \xi w_j}}{z_j^{\Delta n} w_j^{n_1}} \left( \frac{1}{z_j - w_j} - h \right) \prod_{j=n_1+1}^{n_2} \frac{e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w^2_j - \Delta \xi w_j}}{z_j^{\Delta n-1} w_j^{n_1+1} (w_j - z_j)}.
\]

\(d_1 < d_3 < 0\), \(d_4 < d_2 < 0\) and \(\Delta \xi = \xi_2 - \xi_1\), \(\Delta \mu = \mu_2 - \mu_1\).
Limit to the Brownian directed polymer

\[
Q(h) = \frac{(-1)^{n_2(n_2-1)}}{(2\pi i)^{2n_2} n_1! \Delta n!} \times \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \det (z_j^{i-1})_{1 \leq i, j \leq n_2} \det (w_j^{i-1})_{1 \leq i, j \leq n_2} \\
\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j}}{z_j^{\Delta n} w_j^{n_1}} \left( \frac{1}{z_j - w_j} - h \right) \prod_{j=n_1+1}^{n_2} \frac{e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j}}{z_j^{\Delta n-1} w_j^{n_1+1}} (w_j - z_j).
\]

\[d_1 < d_3 < 0, \quad d_4 < d_2 < 0\] and \(\Delta \xi = \xi_2 - \xi_1\), \(\Delta \mu = \mu_2 - \mu_1\).

In order to understand the asymptotics of \(Q(h)\) we need to rewrite it further so that we can do an appropriate expansion.
Limit to the Brownian directed polymer

\[ Q(h) = \frac{1}{(2\pi i)^{4n_2} n_1!(\Delta n)!} \int_{\Gamma_{d_1}} d^{n_1} z \int_{\Gamma_{d_2}} d^{\Delta n} z \int_{\Gamma_{d_3}} d^{n_1} w \int_{\Gamma_{d_4}} d^{\Delta n} w \int_{\gamma_{\tau_1}} d^{n_2} \zeta \int_{\gamma_{\tau_2}} d^{n_2} \omega \]

\times \det \left( \frac{1}{\zeta_j^i} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{\omega^{n_2+1-i}} \right)_{1 \leq i, j \leq n_2}

\times \prod_{j=1}^{n_1} \frac{z_j^{n_1} w_j^{\Delta n} e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j} \left( \frac{1}{z_j - w_j} - h \right)}{e^{\frac{1}{2} \mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2} \Delta \mu \omega_j^2 - \Delta \xi \omega_j} (\zeta_j - z_j)(\omega_j - w_j)}

\times \prod_{j=n_1+1}^{n_2} \frac{z_j^{n_1+1} w_j^{\Delta n-1} e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j} (w_j - z_j)(\zeta_j - z_j)(\omega_j - w_j)}{e^{\frac{1}{2} \mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2} \Delta \mu \omega_j^2 - \Delta \xi \omega_j} (w_j - z_j)(\zeta_j - z_j)(\omega_j - w_j)},

where

\[ d_1 < d_3 < -\max(\tau_1, \tau_2) < 0 , \quad d_4 < d_2 < -\max(\tau_1, \tau_2) < 0. \]
Expansion – Analogous simpler computation

Largest eigenvalue $\lambda_{\text{max}}$ of a GUE matrix

$$\mathbb{P}_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{Z_n} \int_{(-\infty,a]^n} \det \left( x_j^{i-1} \right)^2 \prod_{j=1}^{n} e^{-x_j^2} d^n x$$
Largest eigenvalue $\lambda_{\text{max}}$ of a GUE matrix

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\mathbb{P}_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{Z_n} \int_{(-\infty,a]^n} \det \left( x_j^{i-1} \right)^2 \prod_{j=1}^{n} e^{-x_j^2} \, d^n x
\]

\[
= \frac{1}{n!} \int_{(-\infty,a]^n} \det (h_{i-1}(x_j)) \det (h_{i-1}(x_j)e^{-x_j^2}) \, d^n x,
\]

where $h_m(x)$ is the $m$:th normalized Hermite polynomial.
Largest eigenvalue $\lambda_{\text{max}}$ of a GUE matrix

\[
\mathbb{P}_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{Z_n} \int_{(-\infty,a]^n} \det (x_j^{i-1})^2 \prod_{j=1}^{n} e^{-x_j^2} d^n x
\]

\[
= \frac{1}{n!} \int_{(-\infty,a]^n} \det (h_{i-1}(x_j)) \det (h_{i-1}(x_j)e^{-x_j^2}) d^n x,
\]

where $h_m(x)$ is the $m$:th normalized Hermite polynomial.

\[
= \det \left( \delta_{ij} - \int_{a}^{\infty} h_{i-1}(x) h_{j-1}(x) e^{-x^2} dx \right).
\]
Largest eigenvalue $\lambda_{\text{max}}$ of a GUE matrix

$$
P_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{Z_n} \int_{(-\infty,a]^n} \det \left( x_j^{i-1} \right)^2 \prod_{j=1}^n e^{-x_j^2} \, d^n x$$

$$= \frac{1}{n!} \int_{(-\infty,a]^n} \det \left( h_{i-1}(x) \right) \det \left( h_{i-1}(x) e^{-x_j^2} \right) \, d^n x,$$

where $h_m(x)$ is the $m$:th normalized Hermite polynomial.

$$= \det \left( \delta_{ij} - \int_a^\infty h_{i-1}(x) h_{j-1}(x) e^{-x^2} \, dx \right).$$

Can be expanded in a Fredholm expansion.
Expansion – Analogous simpler computation

Contour integral formulas

\[ h_n(x) = \frac{\sqrt{n!}}{2^{n/2} \pi^{1/4} 2\pi i} \int_{|w|=1} \frac{e^{-w^2+2xw}}{w^{n+1}} \, dw \]

\[ e^{-x^2} h_n(x) = \frac{2^{n/2}}{\sqrt{n!} \pi^{3/4} i} \int_{\text{Re } z = -2} z^n e^{z^2-2xz} \, dz. \]
Contour integral formulas

\[ h_n(x) = \frac{\sqrt{n!}}{2^{n/2} \pi^{1/4} 2\pi i} \int_{|w|=1} e^{-w^2+2xw} \frac{dw}{w^{n+1}} \]

\[ e^{-x^2} h_n(x) = \frac{2^{n/2}}{\sqrt{n!} \pi^{3/4} i} \int_{\text{Re } z = -2} z^n e^{z^2-2xz} \, dz. \]

Insert into

\[ \mathbb{P}_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{n!} \int_{(-\infty,a]^n} \det (h_{i-1}(x_j)) \det (h_{i-1}(x_j) e^{-x_j^2}) \, d^n x. \]
Expansion – Analogous simpler computation

We get

\[ P_{\text{GUE}}(n)[\lambda_{\max} \leq a] = \frac{2^n}{n!(2\pi i)^2n} \int_{(-\infty,a]^n} \int_{\Re z_j = -2} d^n x \int_{|w_j| = 1} d^n z \int d^n w \]

\[ \det \left( z_j^{-1} \right) \det \left( \frac{1}{w_j^i} \right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2x_j z}}{e^{w_j^2 - 2x_j w_j}}. \]
We get

\[
\mathbb{P}_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{2^n}{n!(2\pi i)^{2n}} \int_{(-\infty,a]^n} d^n x \int_{\text{Re } z_j = -2} d^n z \int_{|w_j| = 1} d^n w \\
\det \left( z_j^{i-1} \right) \det \left( \frac{1}{w_j^i} \right) \prod_{j=1}^n \frac{e^{z_j^2 - 2x_j z}}{e^{w_j^2 - 2x_j w}}.
\]

Now,

\[
\int_{-\infty}^a e^{2x_j(w_j - z_j)} \, dx_j = \frac{e^{2a(w_j - z_j)}}{2(w_j - z_j)}
\]

since \(\text{Re } (w_j - z_j) > 0\).
Expansion – Analogous simpler computation

We obtain

\[ P_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] = \frac{1}{n!(2\pi i)^{2n}} \int_{\text{Re} z_j = -2} d^n z \int_{|w_j| = 1} d^n w \det \left( z_j^{i-1} \right) \det \left( \frac{1}{w_j^i} \right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{(w_j - z_j)e^{w_j^2 - 2aw_j}}. \]
Expanding the $w$-determinant and using symmetry yields

$$\mathbb{P}_{\text{GUE}(n)}[\lambda_{\max} \leq a] = \frac{1}{(2\pi i)^{2n}} \int_{\text{Re } z_j = -2} d^n z \int_{|w_j| = 1} d^n w \det (z_j^{i-1}) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{w_j^i (w_j - z_j) e^{w_j^2 - 2aw_j}}.$$
Expansion – Analogous simpler computation

\[ p_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \frac{1}{(2\pi i)^{2n}} \int_{\text{Re} z_j = -2} d^n z \int_{|w_j| = 1} d^n w \det (z_j^{-1}) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{w_j^j (w_j - z_j) e^{w_j^2 - 2aw_j}}. \]

We can now expand the z-determinant.
Expansion – Analogous simpler computation

\[ \mathbb{P}_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] = \frac{1}{(2\pi i)^{2n}} \int_{\text{Re } z = -2} d^n z \int_{|w| = 1} d^n w \det \left( z_j^{-1} \right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{w_j^i (w_j - z_j) e^{w_j^2 - 2aw_j}}. \]

We can now expand the \( z \)-determinant.

\[ \mathbb{P}_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \frac{1}{(2\pi i)^2} \int_{\text{Re } z = -2} dz \int_{|w| = 1} dw \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^i (w - z) e^{w^2 - 2aw}}. \]
Expansion – Analogous simpler computation

\[ P_{\text{GUE}(n)}[\lambda_{\text{max}} \leq a] = \sum_{\sigma \in S_n} \text{sgn} (\sigma) \prod_{j=1}^{n} \frac{1}{(2\pi i)^2} \int_{\text{Re} z = -2} dz \int_{|w|=1} dw \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^j (w - z) e^{w^2 - 2aw}}. \]

An asymptotic analysis at the edge requires the \( z \)-contour to be on the right side of the \( w \)-contour.
Expansion – Analogous simpler computation

\[ \mathbb{P}_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] \]
\[ = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \frac{1}{(2\pi i)^2} \int_{\text{Re } z = -2} dz \int_{|w| = 1} dw \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^j(w - z) e^{w^2 - 2aw}}. \]

An asymptotic analysis at the edge requires the \( z \)-contour to be on the right side of the \( w \)-contour.
Expansion – Analogous simpler computation

This gives

$$P_{\mathrm{GUE}(n)}[\lambda_{\text{max}} \leq a]$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \left( \delta_{j,\sigma(j)} - \frac{1}{(2\pi i)^2} \int_{\text{Re } z=2} dz \int_{|w|=1} dw \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^{j} (w - z) e^{w^2 - 2aw}} \right).$$

Expand the product.
Expansion – Analogous simpler computation

This gives

\[ P_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \left( \delta_{j,\sigma(j)} - \frac{1}{(2\pi i)^2} \int_{\text{Re } z = 2} \int_{|w|=1} dz \int dw \frac{z^{\sigma(j)-1} e^{z^2-2az}}{w^j (w - z) e^{w^2-2aw}} \right). \]

Expand the product.

This is nothing but

\[ P_{\text{GUE}}(n)[\lambda_{\text{max}} \leq a] = \det \left( \delta_{ij} - \int_{a}^{\infty} h_{i-1}(x) h_{j-1}(x) e^{-x^2} dx \right). \]

in a new form.
The two-time distribution

Recall

$$\lim_{M \to \infty} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = F_{tt}(\eta_1, \eta_2; \alpha).$$

where

$$\mu_i = t_i M = n_i = t_i M, \quad \xi_i = 2t_i M + \eta_i (t_i M)^{1/3},$$

$$i = 1, 2,$$ and

$$\alpha = \left(\frac{t_1}{(t_2 - t_1)}\right)^{1/3}.$$
The two-time distribution

We have the formula

\[
F_{tt}(\eta_1^*, \eta_2; \alpha) = F_2(\eta_2) - \sum_{r,s,t=0}^{\infty} \frac{1}{(r!)^2 s! t!}
\times \int_0^{\infty} d\eta_1^* \int_0^{r} d^r x \int_0^{s} d^s x' \int_0^{t} d^t y \int_0^{t} \eta_1^* d\eta_1 \int_0^{r} d^r x \int_0^{s} d^s x' \int_0^{r-1} d^r y \int_0^{t} d^t y' W^{(1)}_{r,s,r,t}(x, x', y, y')
\]

\[
- \sum_{r=1}^{\infty} \sum_{s,t=0}^{\infty} \frac{1}{r!(r-1)! s! t!}
\times \int_0^{\infty} d\eta_1^* \int_0^{r} d^r x \int_0^{s} d^s x' \int_0^{r-1} d^r y \int_0^{t} d^t y' W^{(2)}_{r,s,r-1,t}(x, x', y, y'),
\]
The two-time distribution

Block determinant

\[
W^{(1)}_{r_1, s, r_2, t}(x, x', y, y') = \begin{vmatrix}
\psi(x, x) & \psi(x, x') & \psi(x, 0) & \psi(x, y) & \psi(x, y') \\
\phi(x', x) & \phi(x', x') & \phi(x', 0) & \phi(x', y) & \phi(x', y') \\
\psi(0, x) & \psi(0, x') & \psi(0, 0) & \psi(0, y) & \psi(0, y') \\
\phi(y, x) & \phi(y, x') & \phi(y, 0) & \phi(y, y) & \phi(y, y') \\
\psi(y', x) & \psi(y', x') & \psi(y', 0) & \psi(y', y) & \psi(y', y')
\end{vmatrix}
\]

\(x \in \mathbb{R}^{r_1}, x' \in \mathbb{R}^s, y \in \mathbb{R}^{r_2}, y' \in \mathbb{R}^t\) and \(0 \in \mathbb{R}\).

\(W^{(2)}_{r_1, s, r_2, t}\) is very similar (\(\psi\) in middle row replaced by \(\phi\)).
The two-time distribution

The functions \( \phi \) and \( \psi \) are defined by

\[
\phi(x, y) = \phi_1(x, y) + 1(y \geq 0)\phi_2(x, y) - 1(x < 0)\phi_3(x, y),
\]

and

\[
\psi(x, y) = -\psi_1(x, y) - 1(y > 0)\phi_2(x, y) + 1(x \leq 0)\phi_3(x, y),
\]

where

\[
\phi_1(x, y) = -\alpha \int_0^\infty K_{Ai}(\eta_1 - \tau, \eta_1 - y)K_{Ai}(\Delta \eta + \alpha \tau, \Delta \eta + \alpha x) \, d\tau,
\]

\[
\psi_1(x, y) = \alpha \int_0^\infty K_{Ai}(\eta_1 + \tau, \eta_1 - y)K_{Ai}(\Delta \eta - \alpha \tau, \Delta \eta + \alpha x) \, d\tau,
\]

\[
\phi_2(x, y) = \alpha K_{Ai}(\Delta \eta + \alpha x, \Delta \eta + \alpha y),
\]

\[
\phi_3(x, y) = K_{Ai}(\eta_1 - x, \eta_1 - y).
\]

and

\[
\Delta \eta = \eta_2 \left( \frac{t_2}{\Delta t} \right)^{1/3} - \eta_1 \left( \frac{t_1}{\Delta t} \right)^{1/3}.
\]
Thank you!