Patterns in the primes

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Primes are the ‘atoms’ of the integers from the point of view of multiplication.

Theorem (Fundamental Theorem of Arithmetic)

Every integer \( n > 1 \) can be written as a product of primes \( n = p_1 \times p_2 \times \cdots \times p_k \). Moreover, this is unique apart from rearranging the product.

Example

\[ 6 = 2 \times 3 = 3 \times 2. \]

\[ 1024 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2. \]
Primes: Why care?

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Mathematicians are *lazy*. This means we can simplify a problem about integers to one about primes.
Fermat’s last theorem

Theorem (Example: Fermat’s last theorem)

There are no integer solutions to $x^n + y^n = z^n$ with $n > 2$ apart from the obvious ones when $xyz = 0$. 

Claim

It is enough to prove FLT with $n$ prime

Proof.

1. Imagine $n = ab$ is composite, and there is a solution to $x^n + y^n = z^n$ for $n$.

2. Then $(x^b)^a + (y^b)^a = (z^b)^a$, so $(x^b, y^b, z^b)$ is a solution for $a$.

3. So there is a solution for every prime factor of $n$.

4. Contradiction to FLT for primes! Unless $n = 2k$.

5. Fermat: There are no solutions for $n = 4$.

\[ \square \]
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Counting primes

Question

How many primes are there?
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Theorem

There are infinitely many primes.
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*How many primes are there?*

Theorem

*There are infinitely many primes.*

Gauss: "Around $x$ the primes should occur with density $1 / \ln x$".

Theorem (Prime Number Theorem)

$$\pi(x) = \#\{\text{primes} \leq x\} \approx \int_2^x \frac{dt}{\ln t}.$$
**Counting primes**

**Question**

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**Theorem (Prime Number Theorem)**

$$\pi(x) = \#\{\text{primes } \leq x\} \approx \int_2^x \frac{dt}{\ln t}.$$  

$$\pi(10^{10}) = 455,052,511, \quad \int_2^{10^{10}} \frac{dt}{\ln t} = 455,055,613.8...$$

Difference 3102.8... (< 0.0007%).
Instead of counting primes with weight 1, it is easier to compensate for the density by counting with weight $\ln p$. 

**Theorem (Riemann’s explicit formula)**

If $x$ is not an integer, then

$$
\sum_{p \leq x} \frac{1}{p} = \log x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2} \log \left( \frac{2\pi}{x^2} \right)
$$

Here $\zeta(s)$ is the ‘Riemann zeta function’, and the sum is over zeros of the zeta function.
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**Theorem (Riemann’s explicit formula)**

*If* $x$ *is not an integer, then*

$$\sum_{n,p} \ln p = x - \sum_{\rho} \frac{x^\rho}{\rho} - \ln(2\pi).$$

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Instead of counting primes with weight 1, it is easier to compensate for the density by counting with weight \( \ln p \).

**Theorem (Riemann’s explicit formula)**

*If \( x \) is not an integer, then*

\[
\sum_{n,p} \ln p = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \ln(2\pi).
\]

Here \( \zeta(s) \) is the ‘Riemann zeta function’, and the sum is over zeros of the zeta function.
Therefore the zeros tell us exactly where the primes are!

Our step function is a sum of ‘waves’: ‘The music of the primes’
Riemann’s Hypothesis

The size of $x^\rho$ is $x^{\Re(\rho)}$.

**Conjecture (Riemann’s Hypothesis, $1,000,000$)**

*All the non-trivial zeros of $\zeta(s)$ have real part $1/2$.***
Riemann’s Hypothesis

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**Conjecture (Riemann’s Hypothesis, $1,000,000$)**

*All the non-trivial zeros of $\zeta(s)$ have real part $1/2$.***

This means all the terms $x^\rho$ have size $\sqrt{x}$, which is much smaller than $x$.

**Corollary**

*Assume RH. Then for all $x > 2$*

$$\left| \pi(x) - \int_2^x \frac{dt}{\ln t} \right| < 4 \sqrt{x} \ln x$$

This would completely explain why $\int_2^x \frac{dt}{\ln t}$ is such a good approximation! This explains the large-scale structure!
"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"

"So if you could be the Devil and offer a mathematician to sell his soul for the proof of one theorem - what theorem would most mathematicians ask for? I think it would be the Riemann Hypothesis."
Small-scale distribution

What about primes on a small scale - the gaps between them?

Theorem (Prime Number Theorem)

\[ \text{primes} \leq x \approx \int x^2 \, dt \ln t \approx x \ln x. \]

Corollary

The average size gap \( p_{n+1} - p_n \) amongst primes \( p_n \leq x \) is \( \approx \ln x \).

Proof.

Average gap \( = \sum_{p_n < x} (p_{n+1} - p_n) \approx p_N - 2\pi(x) \approx \frac{x}{\ln x} \approx \ln x. \) \( \square \)
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The average size gap \( p_{n+1} - p_n \) amongst primes \( p_n \leq x \) is \( \approx \ln x \)

**Proof.**

Average gap \[ = \frac{\sum_{p_n < x} (p_{n+1} - p_n)}{\#\{p_n \leq x\}} \]

\[ = \frac{p_N - 2}{\pi(x)} \approx \frac{x}{x/\ln x} \]

\[ \approx \ln x. \]
Question

Are prime gaps always this big?

- (2, 3) is the only pair of primes which differ by 1.
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- (2, 3) is the only pair of primes which differ by 1. (One of \( n \) and \( n + 1 \) is a multiple of 2 for every integer \( n \)).
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There are lots of pairs of primes which differ by 2: (3, 5), (5, 7), (11, 13), \ldots, (1031, 1033), \ldots, (1000037, 1000039), \ldots, (1000000007, 1000000009), \ldots
Question

Are prime gaps always this big?

(2, 3) is the only pair of primes which differ by 1. (One of $n$ and $n + 1$ is a multiple of 2 for every integer $n$).

There are lots of pairs of primes which differ by 2:
(3, 5), (5, 7), (11, 13), …, (1031, 1033), …, (1000037, 1000039), …, (1000000007, 1000000009), …

Conjecture (Twin prime conjecture)

There are infinitely many pairs of primes $(p, p')$ which differ by 2.

This is one of the oldest problems in mathematics, and is very much open!
How many Twin primes are there?

- If we randomly picked a number $n$ of size $x$, then the probability $n$ is prime is about $1/\ln x$. 

\[ \int_{10^8}^{10^{10}} \left( \frac{1}{\ln t} \right)^2 \, dt = 333530. \]

The difference is 106781 (about 24.2%).
How many Twin primes are there?

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- If we randomly picked a number $n$ of size $x$, then the probability $n + 2$ is prime is about $1/\ln x$.
- If these were independent events, then the probability $n$ and $n + 2$ are both prime would be $1/\ln^2 x$. 

Guess $\{\text{twin primes } \leq x\} \approx \int x^2 \, dt (\ln t)^2$ 

But this can't be right, as $n$ and $n + 1$ can't both be prime!

$\{\text{twin primes } \leq 10^8\} = 440312$

$\int 10^8 \, dt (\ln t)^2 = 333530$.

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Guess

$$\#\{\text{twin primes } \leq x\} \approx \int_2^x \frac{dt}{(\ln t)^2}$$

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$$\#\{\text{twin primes} \leq x\} \approx \int_{2}^{x} \frac{dt}{(\ln t)^2}$$

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$$\#\{\text{twin primes} \leq 10^8\} = 440312, \quad \int_{2}^{10^8} \frac{dt}{(\ln t)^2} = 333530.2...$$

Difference 106781.8... (about 24.2%)
Second Attempt

Let's use the fact primes > 2 are odd.

- If we randomly picked an odd number $n$ of size $x$, then the probability $n$ is prime is about $\frac{2}{\ln x}$.

Guess (Second attempt)

$\{\text{twin primes} \leq x\} \approx \int x^2 \, dt \left( \ln t \right)^2$

Worse!

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- If we randomly picked an odd number $n$ of size $x$, then the probability $n + 2$ is prime is about $2/\ln x$.
- If these were independent events, then the probability $n$ and $n + 2$ are both prime would be $4/(\ln x)^2$. 

Guess (Second attempt)

$\#\{\text{twin primes} \leq x\} \approx 2 \int_x^x \left(\frac{1}{\ln t}\right)^2 dt$
Second Attempt

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Guess (Second attempt)

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Worse!
If $n$ and $n + 2$ are prime, $n$ must be 2 more than a multiple of 3, and so 1 less than a multiple of 6.

- If we randomly picked a number $n$ of the form $6k − 1$ of size $x$, then the probability $n$ is prime is about $\frac{3}{\ln x}$. 

Error $\approx 13\%$. Better!
If $n$ and $n + 2$ are prime, $n$ must be 2 more than a multiple of 3, and so 1 less than a multiple of 6.

- If we randomly picked a number $n$ of the form $6k - 1$ of size $x$, then the probability $n$ is prime is about $\frac{3}{\ln x}$.
- The probability $n + 2$ is prime is also about $\frac{3}{\ln x}$.
If $n$ and $n + 2$ are prime, $n$ must be 2 more than a multiple of 3, and so 1 less than a multiple of 6.

- If we randomly picked a number $n$ of the form $6k - 1$ of size $x$, then the probability $n$ is prime is about $3/\ln x$.
- The probability $n + 2$ is prime is also about $3/\ln x$.
- If these were independent events, then the probability $n$ and $n + 2$ are both prime would be $9/(\ln x)^2$. 

Guess (Third attempt)

\[
\{\text{twin primes} \leq x\} \approx \frac{3}{2} \int_1^x \left(\frac{1}{\ln t}\right)^2 \, dt.
\]

Error $\approx 13\%$. Better!
If $n$ and $n + 2$ are prime, $n$ must be 2 more than a multiple of 3, and so 1 less than a multiple of 6.

- If we randomly picked a number $n$ of the form $6k - 1$ of size $x$, then the probability $n$ is prime is about $3/\ln x$.
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**Guess (Third attempt)**

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Error $\approx 13\%$. Better!
Let's try to correct for all primes $p > 2$.

- There are $p - 2$ possible remainders for $n$ after dividing by $p$ if $n$ and $n + 2$ are prime.
Let's try to correct for all primes $p > 2$.

- There are $p - 2$ possible remainders for $n$ after dividing by $p$ if $n$ and $n + 2$ are prime.
- So the probability than neither $n$ nor $n + 2$ are a multiple of $p$ is $(p - 2)/p$. 

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- So the probability than neither $n$ nor $n+2$ are a multiple of $p$ is $(p-2)/p$.
- If $n$ and $n+2$ were ‘independent’, then the probability neither were a multiple of $p$ is $(p-1)/p \times (p-1)/p$. 

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- So the probability than neither $n$ nor $n + 2$ are a multiple of $p$ is $(p - 2)/p$.
- If $n$ and $n + 2$ were ‘independent’, then the probability neither were a multiple of $p$ is $(p - 1)/p \times (p - 1)/p$.
- So we were off by a factor $\frac{p(p-2)}{(p-1)^2}$.

**Guess (Fourth attempt)**

$$\#\{\text{twin primes } \leq x\} \approx 2C_2 \int_{2}^{x} \frac{dt}{(\ln t)^2}$$

with $C_2 = \prod_{p > 2} \frac{p(p - 2)}{(p - 1)^2}$.

$$\#\{\text{twin primes } \leq 10^8\} = 440312, \quad 2C_2 \int_{2}^{10^8} \frac{dt}{(\ln t)^2} = 440367.8...$$

Difference 55.8... (this is $< 0.2\%$). Success!
We can look at more than just gaps of size 2.

**Conjecture (De Polignac)**

*For every positive integer $h$, there are infinitely many pairs of primes which differ by $2h$.***

Again, we guess the number less than $x$ is roughly $C_h x / (\ln x)^2$ for some constant $C_h$. 
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**Theorem**

*This is true for at least one $h$ in $\{1, \ldots, 123\}$.***

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In particular:

**Theorem**

*There are infinitely many pairs* $(p_1, p_2)$ *of primes such that* $|p_1 - p_2| \leq 246$. 
We can also look for **triples** of primes \( n, n + h_1, n + h_2 \) for some fixed shifts \( h_1, h_2 \)

1. If \( h_1 = 2, h_2 = 4 \) then \((3, 5, 7)\) is the only triple.
   (one of \( n, n + 2, n + 4 \) must be a multiple of 3)
2. If \( h_1 = 2, h_2 = 6 \) then there are many such triples.

**Conjecture**

*There are infinitely many \( n \) such that \( n, n + h_1, \ldots, n + h_k \) are prime if there isn’t an obvious reason why they can’t be.*

‘Obvious reason’ means one is always a multiple of some prime for all \( n \).

**Theorem**

*There exists \( h_1, \ldots, h_k \) such that \( n, n + h_1, \ldots, n + h_k \) are all primes for infinitely many \( n \).*
If we assume a well-believed technical conjecture about primes in arithmetic progressions, then we can get close to the twin prime conjecture!

**Theorem**

Assume ‘GEH’. Then there are infinitely many pairs \((p_1, p_2)\) of primes with \(|p_1 - p_2| \leq 6\).
Goldbach’s conjecture

This conjecture also allows us to say something about another old conjecture

Conjecture (Goldbach’s conjecture)

Every even number can be written as the sum of at most two primes.

Theorem

Assume ‘GEH’. Then at least one of the following is true:

1. There are infinitely many twin primes
2. For every large even integer $N$, one of $N$, $N + 2$ or $N - 2$ is the sum of two primes.

Of course we expect both to be true!
Alice wants to send Bob a Facebook message containing sensitive gossip.
1. Alice wants to send Bob a Facebook message containing sensitive gossip.

2. This can be done securely if her laptop can find $N = pq$ which is hard to factor into primes.
Alice wants to send Bob a Facebook message containing sensitive gossip.

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If \( p - 1 \) has only small prime factors, then there is a way to factor \( N \) easily (Bad).
Real-World Example

1. Alice wants to send Bob a Facebook message containing sensitive gossip.
2. This can be done securely if her laptop can find \( N = pq \) which is hard to factor into primes.
3. If \( p - 1 \) has only small prime factors, then there is a way to factor \( N \) easily (Bad).
4. On Wikipedia it had been suggested that one could choose \( p, q \) such that \( (p - 1)/2 \) and \( (q - 1)/2 \) are prime.
Alice wants to send Bob a Facebook message containing sensitive gossip.

This can be done securely if her laptop can find $N = pq$ which is hard to factor into primes.

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On Wikipedia it had been suggested that one could choose $p, q$ such that $(p - 1)/2$ and $(q - 1)/2$ are prime.

If there are only 10 (say) 1024-digit primes $p$ such that $(p - 1)/2$ is prime, then this is a VERY bad idea! Bob would die before Alice finds one!

A slight generalization of our model predicts there are many such primes.
It is an exciting time for prime number theory!

Any questions?