Limit shapes in the Schur process

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Outline

- Pyramid partitions
- Interlude into partitions and the Schur process
- Asymptotics of pyramid partitions
Let them eat cake!

“S’ils n’ont pas de pain, qu’ils mangent de la brioche!”
–Marie Antoinette d’Autriche (1755–1793)
Pyramid partitions

Figure: Piles of $2 \times 2 \times 1$ boxes, each viewed as a pair of dominoes in the 2D projection looking downwards. On the left, the empty pyramid partition.
Flips and the volume

- pyramid partition = what’s left after a finite number of box removals from the empty configuration (introduced by Kenyon and Szendrői)
- removal = flip (adjacent vertical dominoes ↔ adjacent horizontal dominoes)
- Volume = Number of flips

Theorem (Young 2010)

\[ \sum_{\Lambda} q^{\text{Volume}(\Lambda)} = \prod_{n \geq 1} \frac{(1 + q^{2n-1})^{2n-1}}{(1 - q^{2n})^{2n}}. \]

This is a consequence of RSK for supersymmetric Schur functions (BCC 2014).
How do large pyramid partitions look like?
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Partitions

Figure: Partition \((2, 2, 2, 1, 1)\) in English, French and Russian notation, with associated Maya diagram (particle-hole representation).
Horizontal and vertical strips

Given partitions $\mu \subseteq \lambda$, we can form skew diagram $\lambda/\mu$, which we call a

- horizontal strip, and write $\mu \prec \lambda$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \ldots$$

- vertical strip, and write $\mu \prec' \lambda$, if $\lambda' \prec \mu'$ (‘ = conjugate) or

$$\lambda_i - \mu_i \in \{0, 1\}$$
The Schur process

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \{\prec, \succ, \prec', \succ'\}^n$ be a word. We say a sequence of partitions $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$ is $\omega$-interlaced if $\lambda(i-1) \omega_i \lambda(i)$, for $i = 1, \ldots, n$.

The Schur process of word $\omega$ with parameters $Z = (z_1, \ldots, z_n)$ is the measure on the set of $\omega$-interlaced sequences of partitions

$$\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$$

given by

$$Prob(\Lambda) \propto \prod_{i=1}^{n} z_i^{||\lambda(i)|-|\lambda(i-1)||}.$$ 

Remark

For a more general definition, see the original work of Okounkov–Reshetikhin 2003, or Borodin–Rains 2006.
The Schur process is a determinantal point process.

**Theorem (OR 2003; BR 2006)**

\[
\text{Prob}(\lambda(i_s) \text{ contains a particle at position } k_s, 1 \leq s \leq n) = \det_{1 \leq u, v \leq n} K(i_u, k_u; i_v, k_v)
\]

where

\[
K(i, k; i', k') = \begin{cases} 
\left[ \frac{z^k}{w^{k'}} \right] \frac{\Phi(z; Z, \omega; i)}{\Phi(w; Z, \omega; i')} \sqrt{zw}, & i \leq i', \\
-\left[ \frac{z^k}{w^{k'}} \right] \frac{\Phi(z; Z, \omega; i')}{\Phi(w; Z, \omega; i)} \sqrt{zw}, & i > i'
\end{cases}
\]

with

\[
\Phi(z; Z, \omega; i) = \prod_{j: \ j \leq i, \ \omega_j \in \{\prec, \prec'\}, \ \epsilon_j = 1} (1 + \epsilon_j z_j z)^{\epsilon_j} \prod_{j: \ j > i, \ \omega_j \in \{\succ, \succ'\}, \ \epsilon_j = -1} (1 + \epsilon_j \frac{z_j}{z})^{-\epsilon_j} \prod_{j: \ j > i, \ \omega_j \in \{\succ, \succ'\}, \ \epsilon_j = 1} (1 + \epsilon_j \frac{z_j}{z})^{-\epsilon_j}
\]
Pyramid partitions as Schur processes, pictorially

Figure: A pyramid partition of width 5 corresponding to the sequence

\( \emptyset \prec (1) \prec' (2) \prec (2, 2) \prec' (3, 3) \prec (3, 3, 2) \succ'(2, 2, 1) \succ (2, 1) \succ'(1, 1) \succ (1) \succ' \emptyset. \)
Let $n = 2n_0$ be an even integer. A pyramid partition is (bijectively) a sequence of $2n + 1$ partitions

$$\Lambda = (\emptyset = \lambda(-n) \prec \lambda(-n+1) \prec' \lambda(-n+2) \prec \cdots \prec' \lambda(0) \succ \lambda(1) \succ' \lambda(2) \succ \cdots \succ' \lambda(n) = \emptyset).$$

It is this a Schur process for the word $\omega_{\text{pyr}} = (\prec, \prec')^{n_0}(\succ, \succ')^{n_0}$ and parameters $Z = (z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n)$.

**Remark**

For volume weighting, $z_{-i} = z_i = q^{i-\frac{1}{2}}, \; 1 \leq i \leq n$. 
Everything we’d like to know about asymptotics of large pyramid partitions can be translated into asymptotics of large particle–hole systems associated to the corresponding Schur process.
How to compute the limit shape

Let $t = 2t_0 < n$, $k \in \mathbb{Z} + \frac{1}{2}$. A weak Wick lemma shows that:

Lemma (db–Boutillier–Vuletić 2015)

$$\text{Prob}(\lambda(-t) \text{ contains a particle at position } k) =$$

$$= \left[ \frac{z^k}{w^k} \right] \frac{J(z; t_0)}{J(w; t_0)} \frac{\sqrt{zw}}{z - w}$$

$$= \int \int \frac{J(z; t_0)}{J(w; t_0)} \frac{1}{z^{k - \frac{1}{2}} w^{k - \frac{1}{2}}} \frac{1}{z - w} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w}$$

where (with $(u; q)_m = \prod_{i=0}^{m-1} (1 - q^i u)$)

$$J(z; t_0) = \frac{(-q^{2t_0 + \frac{1}{2}} z; q^2)_{n_0 - t_0} (\frac{q^{\frac{1}{2}}}{z}; q^2)_{n_0}}{(q^{2t_0 + \frac{3}{2}} z; q^2)_{n_0 - t_0} (-\frac{q^{3/2}}{z}; q^2)_{n_0}}.$$
Asymptotics regime

We let the size of the partition grow with $q \to 1$ as $\epsilon \to 0$ like so:

\[
q(\epsilon) = \exp(-\gamma \epsilon),
\]

\[
n_0(\epsilon) = a_0 / \epsilon,
\]

\[
t_0(\epsilon) = x_0 / \epsilon,
\]

\[
k(\epsilon) = y / \epsilon.
\]
A few limit formulas

If $q = \exp(-r)$ and $r \to 0+$, we have

$$\log(z; q)_{\infty} \sim -\frac{Li_2(z)}{r}$$

and furthermore,

$$\log(z; q)_{\frac{A}{r}} \sim \frac{1}{r} (Li_2(e^{-A}z) - Li_2(z))$$

where

$$Li_2(z) = \sum_{n \geq 1} \frac{z^2}{n^2}, \quad |z| < 1$$

with analytic continuation given by

$$Li_2(z) = -\int_0^z \frac{\log(1-u)}{u} du, \quad z \in \mathbb{C}\backslash[1, \infty).$$
Asymptotics of the kernel

Lemma (db–Boutillier–Vuletić 2015)

In the limit ($x = 2x_0$ is rescaled $t$, $y$ is rescaled $k$),

$$\text{Prob}(\lambda(-t) \text{ contains a particle at position } k) \sim \int \int e^{S(z;x,y) - S(w;x,y)} \frac{dT}{z - w}$$

where

$$S(z; x, y) = \frac{1}{2\gamma} \left( Li_2(-Az) - Li_2(-Xz) + Li_2\left(\frac{A}{z}\right) - Li_2\left(\frac{1}{z}\right) + Li_2(Xz) - Li_2(Az) + Li_2\left(-\frac{1}{z}\right) - Li_2\left(-\frac{A}{z}\right) \right) - y \log z$$

and $X = \exp(-\gamma x)$, $A = \exp(-2\gamma a_0)$.
The arctic curve

To compute the arctic curve, one solves for \((x, y)\) (or \(X = \exp(-\gamma x), Y = \exp(2\gamma y)\)) corresponding to double critical points of \(S(z; x, y)\). That is,

**Theorem (db–Boutillier–Vuletić 2015)**

The arctic curve is the locus \((x, y)\) satisfying:

\[
\begin{align*}
  f(z; X) &= Y, \\
  f'(z; X) &= 0
\end{align*}
\]

where \(f(z; X) = \frac{(z+1)(z-A)(z-1/A)(z+1/X)}{(z-1)(z+A)(z+1/A)(z-1/X)}\).

**Remark**

Alternatively, it can be seen as given by the algebraic equation

\[
\Delta [(z + 1)(z - A)(z - 1/A)(z + 1/X) - Y(z - 1)(z + A)(z + 1/A)(z - 1/X)] = 0
\]

where \(\Delta\) represents taking the discriminant.
The arctic curve, pictorially

Notice the cusps (which correspond to the *double* critical point of $S$ at $z = 0$). A similar cusp phenomenon has appeared in the case of (skew) plane partitions with two different $q$'s, Mkrtchyan 2013.
Something similar, but not quite the same: Mkrtchyan 2013
Arctic curve in the infinite regime

What happens when $a_0 \to \infty$, or equivalently, $A \to 0$?

The cusps move to $\infty$ and the arctic curve becomes

$$(1 + Z + W - ZW)(1 + Z - W + ZW)(1 - Z + W + ZW)(1 - Z - W - ZW) = 0$$

where $(Z, W) = (\sqrt{X}, \sqrt{Y})$ which is the boundary of the amoeba of the (square lattice determined) polynomial

$$P(Z, W) = 1 + Z + W - ZW.$$
Arctic curve in the infinite regime, pictorially
A large sample in the infinite regime, up to affine transformations
A word on what happens on the arctic curve

Everywhere but at the cusps and tangency points, fluctuations are of Airy type (cf., for example, Okounkov–Reshetikhin 2006). At the turning points, one (probably) has two correlated GUE minors processes. At the cusps, one would conjecture and expect the Pearcey process fluctuations. Alas, in the absence of a triple critical point and due to additional constraints, what (probably) actually happens is one gets the cusp Airy process of Duse–Johansson–Metcalfe (work in progress, 2015) with kernel:

\[
K(x, k; x', k') = \int \int w^{k'} z^{k} \frac{e^{\frac{x^3}{3} + xz}}{e^{\frac{w^3}{3} + x'w}} \frac{1}{z - w} \, d\mathbb{T}
\]

where \(x\) is the continuous time direction, and \(k\) the discrete space direction.
Other stuff: “skew pyramid partitions”

**Figure:** Skew pyramid partitions: word \((\prec, \prec')^{50}(\succ, \succ')^{50}(\prec, \prec')^{50}(\succ, \succ')^{50}\), \(q = 0.99\). The analogue in pyramid partition land of OR 2006’s skew plane partitions.
Other stuff: symmetric “pyramid partitions”
Symmetric “pyramid partitions” as plane overpartitions

This limit shape seems to be the same that Vuletić 2009 analyzed in the context of strict plane partitions and Pfaffian processes.
Thank you!