Clay Mathematics Proceedings
Volume 18

A Celebration of Algebraic Geometry

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A Conference in Honor of Joe Harris' 60th Birthday
Harvard University
Cambridge, MA
August 25–28, 2011

Brendan Hassett
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2010 *Mathematics Subject Classification.* Primary 14C30, 14D22, 14H45, 14J28, 14K10, 14N15, 14N20.

Library of Congress Cataloging-in-Publication Data

A celebration of algebraic geometry : in honor of Joe Harris' 60th birthday, August 25–28, 2011, Harvard University, Cambridge, Massachusetts / Brendan Hassett, James M^{c}Kernan, Jason Starr, Ravi Vakil, editors.

pages cm. — (Clay mathematics proceedings : volume 18)

"Clay Mathematics Institute."

Includes bibliographical references.

ISBN 978-0-8218-8983-1 (alk. paper)

1. Geometry, Algebraic. I. Harris, Joe, 1951– II. Hassett, Brendan. III. M^{c}Kernan, James, 1964– IV. Starr, Jason M. (Jason Michael) V. Vakil, Ravi. VI. American Mathematical Society. VII. Clay Mathematics Institute.

QA564.C425 2013
516.3′5—dc23

2013014118

10 9 8 7 6 5 4 3 2 1 18 17 16 15 14 13

To Joe Harris, for all he has taught us over the years

Contents

Introduction

Joseph Daniel Harris, universally known as "Joe" in the algebraic geometry community, celebrated his 60th birthday in 2011. On 25–28 August 2011, an algebraic geometry conference, "A Celebration of Algebraic Geometry", was held at Harvard University with over 300 mathematicians participating.

This milestone was cause for celebration not only for Joe, but for all of us who have been fortunate to know and work with Joe over the years, and for the many more who have been touched by Joe's research, his books, and his vivacious lectures.

This is not the moment to catalogue Joe's numerous accomplishments and accolades. However, we would like to recall some of Joe's most lasting impacts on our subject. Joe Harris was both an undergraduate and graduate student at Harvard University, completing his Ph.D thesis with Phillip Griffiths in 1978. During those student years, Griffiths and Harris wrote together their textbook in complex geometry, "Principles of Algebraic Geometry". This has been *the* textbook for both complex algebraic geometry and complex differential geometry ever since, as well as the source of many "Joe legends".

Joe has over 100 published research articles. From the beginning, Joe was a superstar in the subject of algebraic curves, particularly the moduli spaces of curves. Signal achievements were Joe's completion with Griffiths of the Brill-Noether theorem; the proof with Mumford, and then Eisenbud, that moduli spaces of curves are of general type for $g > 23$; the many consequences of the theory of "limit linear series" of Eisenbud-Harris; and the proof of the Severi conjecture on irreducibility of Severi varieties of plane curves. But this captures only a small part of Joe's work: he has also worked on local differential geometry of algebraic varieties, variations of Hodge structure, extension of Thom-Porteous theory to the symmetric and skew-symmetric setting, varieties of minimal degree, Noether-Lefschetz theory, the Harris-Morrison slope conjecture, Cayley-Bacharach theory, the uniformity of rational points of curves over number fields, the enumerative theory of plane curves with specified contact orders (i.e., relative Gromov-Witten invariants of the projective plane, in modern parlance), unirationality of low degree varieties, rationally connected varieties, and moduli spaces of rational curves on Fano manifolds, among many others.

For many of us, equally remarkable is Joe's unflagging support of younger mathematicians, reflected in his teaching and his mentoring. Joe served as the graduate program director of the Harvard mathematics department for many years, shepherding generations of graduate students. Joe was the mentor for many Benjamin Peirce instructors and other postdoctoral fellows. Joe has been the thesis advisor for about fifty Ph.Ds, and has about 120 total "mathematical descendants" according to the Mathematics Genealogy Project. Joe's graduate textbooks have become instant classics: "Principles of Algebraic Geometry" with Griffiths, "Geometry of Algebraic Curves" with Arbarello, Cornalba and Griffiths, "Representation Theory" with William Fulton, "The Geometry of Schemes" with David Eisenbud, "Moduli of Curves" with Ian Morrison, and "Algebraic Geometry, A First Course". Recently, Joe has turned his attention to an even broader audience of young people: with long-time colleague Benedict Gross, Joe pioneered a quantitative reasoning course for non-technically oriented undergraduates culminating in their textbook, "The Magic of Numbers."

The articles gathered in this volume are contributed from many of Joe's colleagues, postdoctoral mentees and students. The topics reflect the breadth of Joe's own interests, indeed the breadth of "classical" algebraic geometry in our post-classical era. From all of us whom Joe has led and inspired, we dedicate this volume to Joe as a token of our gratitude.

The conference was funded by the National Science Foundation (DMS-1045217), the Clay Mathematics Institute, and Harvard University. Its success is due in large part to the tireless efforts of Irene Minder and the rest of the staff of the Harvard Mathematics Department. We are grateful to Izzet Coskun for his contributions to the planning of the conference, as well as the other members of the organizing committee (David Ellwood, Benedict Gross, and David Smyth) and the scientific committee (Dan Abramovich, Lucia Caporaso, Kieran O'Grady, and Rahul Pandharipande).

Brendan Hassett, James McKernan, Jason Starr, Ravi Vakil

Clay Mathematics Proceedings
Volume **18**, 2013

Classification of 2-Fano manifolds with high index

Carolina Araujo and Ana-Maria Castravet

Dedicated to Joe Harris.

Abstract. In this paper we classify n-dimensional Fano manifolds with index $\geq n-2$ and positive second Chern character.

Contents

1. Introduction

A Fano manifold is a smooth complex projective variety X having ample anticanonical class, $-K_X > 0$. This simple condition has far reaching geometric implications. For instance, any Fano manifold X is *rationally connected*, i.e., there are rational curves connecting any two points of X ([**Cam92**] and [**KMM92a**]).

The Fano condition $-K_X > 0$ also plays a distinguished role in arithmetic geometry. In the landmark paper [**GHS03**], Graber, Harris and Starr showed that proper families of rationally connected varieties over curves always admit sections. This generalizes Tsen's theorem in the case of function fields of curves.

Theorem (Tsen's Teorem). *Let K be a field of transcendence degree r over an algebraically closed field k. Let $\mathcal{X} \subset \mathbf{P}^n_K$ be a hypersurface of degree d. If $d^r \leq n$, then $\mathcal{X}$ has a K-point.*

2010 *Mathematics Subject Classification.* Primary 14J45; Secondary 14M20.

For hypersurfaces of degree d in $\mathbf{P}^n$, being Fano or rationally connected is equivalent to the numerical condition $d \leq n$. So, for $r = 1$, [**GHS03**] replaces the condition of $\mathcal{X}$ being a hypersurface of degree $d \leq n$ with the condition of $\mathcal{X}$ being rationally connected. It turns out that rationally connected varieties form the largest class of varieties for which such statement holds true when $r = 1$ (see [**GHMS02**] for the precise statement).

Since then, there has been quite some effort towards finding suitable geometric conditions on $\mathcal{X}$ that generalize Tsen's theorem for function fields of higher dimensional varieties. In [**deJHS08**], de Jong and Starr considered a possible notion of *rationally simply connectedness*. They established a version of Tsen's theorem for function fields of surfaces, replacing the condition of $\mathcal{X}$ being a hypersurface of degree d, $d^2 \leq n$, with the condition of $\mathcal{X}$ being rationally simply connected (see [**deJHS08**, Corollary 1.1] for a precise statement). Several attempts have been made to define the appropriate notion of rationally simply connectedness. Roughly speaking, one would like to ask that a suitable irreducible component of the space of rational curves through two general points of X is itself rationally connected. However, in order to make the definition applicable, one is led to introduce some technical hypothesis, which makes this condition difficult to verify in practice. It is then desirable to have natural geometric conditions that imply rationally simply connectedness. In this context, *2-Fano manifolds* were introduced by de Jong and Starr in [**deJS06**] and [**deJS07**]. In order to define these, we introduce some notation. Given a smooth projective variety X and a positive integer k, we denote by $N_k(X)$ the $\mathbf{R}$-vector space of k-cycles on X modulo numerical equivalence, and by $\overline{\mathrm{NE}}_k(X)$ the closed convex cone in $N_k(X)$ generated by classes of effective k-cycles. Recall that the second Chern character of X is

$$\mathrm{ch}_2(X) = \frac{c_1(X)^2}{2} - c_2(X),$$

where $c_i(X) = c_i(T_X)$. We say that a manifold X is *2-Fano* (respectively *weakly 2-Fano*) if it is Fano and $\mathrm{ch}_2(X) \cdot \alpha > 0$ (respectively $\mathrm{ch}_2(X) \cdot \alpha \geq 0$) for every $\alpha \in \overline{\mathrm{NE}}_2(X) \setminus \{0\}$.

QUESTIONS 1. Do 2-Fano manifolds satisfy some version of rationally simply connectedness? Is this a good condition to impose on the general member of fibrations over surfaces in order to prove existence of rational sections (modulo the vanishing of Brauer obstruction)?

Motivated by these questions, in [**AC12**], we investigated and classified certain spaces of rational curves on 2-Fano manifolds, and gave evidence for a positive answer to Questions 1. In that work, we announced the following threefold classification.

THEOREM 2. *The only 2-Fano threefolds are $\mathbf{P}^3$ and the smooth quadric hypersurface $Q^3 \subset \mathbf{P}^4$.*

In this paper we write down a complete proof of Theorem 2. In fact, Theorem 2 will follow from a more general classification. Recall that the index i_X of a Fano manifold X is the largest integer dividing $-K_X$ in $\mathrm{Pic}(X)$. Our main result is the following.

THEOREM 3. *Let X be a 2-Fano manifold of dimension $n \geq 3$ and index $i_X \geq n-2$. Then X is isomorphic to one of the following.*

- $\mathbf{P}^n$.
- *Complete intersections in projective spaces:*
  - *Quadric hypersurfaces* $Q^n \subset \mathbf{P}^{n+1}$ *with* $n > 2$*;*
  - *Complete intersections of quadrics* $X_{2\cdot 2} \subset \mathbf{P}^{n+2}$ *with* $n > 5$*;*
  - *Cubic hypersurfaces* $X_3 \subset \mathbf{P}^{n+1}$ *with* $n > 7$*;*
  - *Quartic hypersurfaces in* $X_4 \subset \mathbf{P}^{n+1}$ *with* $n > 15$*;*
  - *Complete intersections* $X_{2\cdot 3} \subset \mathbf{P}^{n+2}$ *with* $n > 11$*;*
  - *Complete intersections* $X_{2\cdot 2\cdot 2} \subset \mathbf{P}^{n+3}$ *with* $n > 9$*.*
- *Complete intersections in weighted projective spaces:*
  - *Degree* 4 *hypersurfaces in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n > 11$*;*
  - *Degree* 6 *hypersurfaces in* $\mathbf{P}(3, 2, 1, \ldots, 1)$ *with* $n > 23$*;*
  - *Degree* 6 *hypersurfaces in* $\mathbf{P}(3, 1, \ldots, 1)$ *with* $n > 26$*;*
  - *Complete intersections of two quadrics in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n > 14$*.*
- $G(2, 5)$.
- $OG_+(5, 10)$ *and its linear sections of codimension* $c < 4$*.*
- $SG(3, 6)$.
- G_2/P_2.

Here $OG_+(5, 10)$ denotes a connected component of the 10-dimensional orthogonal Grassmannian $OG(5, 10)$ in the half-spinor embedding (see Section 6.2), $SG(3, 6)$ is a 6-dimensional symplectic Grassmannian (see Section 6.3), and G_2/P_2 is a 5-dimensional homogeneous variety for a group of type G_2 (see Section 6.4).

In order to prove Theorem 3, we will go through the classification of Fano manifolds of dimension $n \geq 3$ and index $i_X \geq n - 2$, and check positivity of the second Chern character for each of them. In the course of the proof, we also determine (with two exceptions) which of these manifolds are weakly 2-Fano. We summarize the results in the following Theorem.

THEOREM 4. *Let* X *be a weakly 2-Fano, but not 2-Fano, manifold of dimension* $n \geq 3$ *and index* $i_X \geq n - 2$*.*

If $\rho(X) = 1$*, then* X *is isomorphic to one of the following:*

- *Complete intersections in projective spaces:*
  - *Complete intersections of quadrics* $X_{2\cdot 2} \subset \mathbf{P}^7$*;*
  - *Cubic hypersurfaces* $X_3 \subset \mathbf{P}^8$*;*
  - *Quartic hypersurfaces in* $X_4 \subset \mathbf{P}^{16}$*;*
  - *Complete intersections* $X_{2\cdot 3} \subset \mathbf{P}^{13}$*;*
  - *Complete intersections* $X_{2\cdot 2\cdot 2} \subset \mathbf{P}^{12}$*.*
- *Complete intersections in weighted projective spaces:*
  - *Degree* 4 *hypersurfaces in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n = 11$*;*
  - *Degree* 6 *hypersurfaces in* $\mathbf{P}(3, 2, 1, \ldots, 1)$ *with* $n = 23$*;*
  - *Degree* 6 *hypersurfaces in* $\mathbf{P}(3, 1, \ldots, 1)$ *with* $n = 26$*;*
  - *Complete intersections of two quadrics in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n = 14$*.*
- *Linear sections of codimension* 1 *in* $G(2, 5)$ *and possibly codimension* 2 *(see Question* 39*).*
- *Linear sections of codimension* 4 *in* $OG_+(5, 10)$*.*
- $G(2, 6)$ *and possibly linear sections of codimension* 2 *in* $G(2, 6)$ *(see Question* 41*).*
- *Linear sections of codimension* 1 *in* $SG(3, 6)$*.*
- *Linear sections of codimension* 1 *in* G_2/P_2 *.*

If $\rho(X) > 1$, then X is isomorphic to one of the following:

- *Dimension $n = 3$:*
  - $\mathbf{P}^1 \times \mathbf{P}^2$;
  - $\mathbf{P}(T_{\mathbf{P}^2})$;
  - $\mathbf{P}_{\mathbf{P}^2}\big(\mathcal{O}(1) \oplus \mathcal{O}\big) \cong V_7$ *(V_7 is the blow-up of $\mathbf{P}^3$ at a point);*
  - $\mathbf{P}_{\mathbf{P}^2}\big(\mathcal{O}(2) \oplus \mathcal{O}\big)$;
  - $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$;
  - $\mathbf{P}^1 \times \mathbf{F}_1$;
  - $\mathbf{P}_{\mathbf{P}^1 \times \mathbf{P}^1}\big(\mathcal{O}(1,1) \oplus \mathcal{O}\big)$;
  - *The blow-up of V_7 along the proper transform of a line l passing through the center of the blow-up $V_7 \to \mathbf{P}^3$.*
- *Dimension $n = 4$:*
  - $\mathbf{P}^2 \times \mathbf{P}^2$;
  - $\mathbf{P}^1 \times \mathbf{P}^3$;
  - $\mathbf{P}_{\mathbf{P}^3}\big(\mathcal{O}(1) \oplus \mathcal{O}(-1)\big)$;
  - $\mathbf{P}_{Q^3}\big(\mathcal{O} \oplus \mathcal{O}(-1)\big)$;
  - $\mathbf{P}_{\mathbf{P}^3}\big(\mathcal{E}\big)$, *where $\mathcal{E}$ is the null-correlation bundle (see Section 9, case (11));*
  - $\mathbf{P}^1 \times \mathbf{P}(T_{\mathbf{P}^2})$;
  - $\mathbf{P}^1 \times V_7$;
  - $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.
- *Dimension $n > 4$:*
  - $\mathbf{P}^2 \times Q^3$;
  - $\mathbf{P}^3 \times \mathbf{P}^3$.

The paper is organized as follows. In Section 2 we revise the classification of Fano manifolds of high index. In Section 3, we check the 2-Fano condition for the simplest ones: (weighted) projective spaces and complete intersections on them, and Grassmannians. Most of the others can be described as double covers, blow-ups or projective bundles over simpler ones. So in Section 4 we compute Chern characters for these constructions. In Section 5, we revise some results from [**AC12**], which describe certain families of rational curves on 2-Fano manifolds. These results are then used in Section 6 to check the 2-Fano condition for certain Fano manifolds described as complete intersections on homogeneous spaces. After all these computations, we are ready to prove Theorems 3 and 4. The proof occupies Sections 7, 8 and 9, with Section 10 being a summary. In Section 7, we address n-dimensional Fano manifolds with index $i_X \geq n-2$, except Fano threefolds and fourfolds with Picard number ≥ 2. These are treated in Sections 8 and 9 respectively.

We remark that toric 2-Fano manifolds have been addressed in [**Nob11**], [**Sat11**] and [**Nob12**]. At present, the only known examples are projective spaces.

Notation. Given a vector bundle $\mathcal{E}$ on a variety X, we denote by $\mathbf{P}_X(\mathcal{E})$, or simply $\mathbf{P}(\mathcal{E})$, the projective bundle of one-dimensional quotients of the fibers of $\mathcal{E}$, i.e., $\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\mathcal{E})$.

We denote by $G(k, n)$ the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V, and we always assume that $2 \leq k \leq \frac{n}{2}$. We write

$$0 \to \mathcal{S} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0$$

for the universal sequence on $G(k,n)$. For subvarieties X of $G(k,n)$, we denote by the same symbols $\sigma_{a_1,\dots,a_k}$ the restrictions to X of the corresponding Schubert cycles.

Aknowledgements. We thank the referee for very useful suggestions. The first named author was partially supported by CNPq and Faperj Research Fellowships. The second named author was partially supported by the NSF grants DMS-1001157, DMS-1160626 and a travel grant from the Association for Women in Mathematics.

2. Classification of Fano manifolds

In this section we discuss the classification of Fano manifolds. A modern survey on this subject can be found in [**IP99**].

Notation. When X is an n-dimensional Fano manifold with $\rho(X)=1$, we denote by L the ample generator of $\mathrm{Pic}(X)$, and define the degree of X as $d_X := c_1(L)^n$.

For a fixed positive integer n, Fano n-folds form a bounded family ([**KMM92b**]). For $n \le 3$, Fano n-folds are completely classified. The classification of Fano surfaces, also known as *del Pezzo surfaces*, is a classical result. They are $\mathbf{P}^2$, $\mathbf{P}^1\times\mathbf{P}^1$, and the blow-up S_{9-n} of $\mathbf{P}^2$ at n points in general position, $1\le n\le 8$. It is easy to check that among those only $\mathbf{P}^2$ is 2-Fano, and among the others only $S_8=\mathbf{F}_1$ and $\mathbf{P}^1\times\mathbf{P}^1$ are weakly 2-Fano (see 4.3.1).

The classification of Fano threefolds of Picard number $\rho=1$ was established by Iskovskikh in [**Isk77**] and [**Isk78**]. There are 17 deformation types of these. The classification of Fano threefolds of Picard number $\rho\ge 2$ was established by Mori and Mukai in [**MM81**] and [**MM03**]. There are 88 deformation types of those. We will revise this list in Section 8.

In higher dimensions, there is no complete classification. On the other hand, one can get results in this direction if one fixes some invariants of the Fano manifold. For instance, we have the following result by Wiśniewski.

Theorem 5 ([**Wiś91**]). *Let X be an n-dimensional Fano manifold with index $i_X \ge \frac{n+1}{2}$. Then X satisfies one of the following conditions:*

- $\rho(X)=1$*;*
- $X\cong \mathbf{P}^{\frac{n}{2}}\times\mathbf{P}^{\frac{n}{2}}$ *(n even);*
- $X\cong \mathbf{P}^{\frac{n-1}{2}}\times Q^{\frac{n+1}{2}}$ *(n odd);*
- $X\cong \mathbf{P}(T_{\mathbf{P}^{\frac{n+1}{2}}})$ *(n odd); or*
- $X\cong \mathbf{P}_{\mathbf{P}^{\frac{n+1}{2}}}(\mathcal{O}(1)\oplus\mathcal{O}^{\frac{n-1}{2}})$ *(n odd).*

Fano manifolds of dimension n and index $i_X\ge n-2$ have been classified. A classical result of Kobayachi-Ochiai's asserts that $i_X\le n+1$, and equality holds if and only if $X\simeq\mathbf{P}^n$. Moreover, $i_X=n$ if and only if X is a quadric hypersurface $Q^n\subset\mathbf{P}^{n+1}$ ([**KO73**]). Fano manifolds with index $i_X=n-1$ are called *del Pezzo* manifolds. They were classified by Fujita in [**Fuj82a**] and [**Fuj82b**]:

Theorem 6. *Let X be an n-dimensional Fano manifold with index $i_X=n-1$, $n\ge 3$.*

(1) *Suppose that $\rho(X)=1$. Then $1\le d_X\le 5$. Moreover, for each $1\le d\le 4$ and $n\ge 3$, and for $d=5$ and $3\le n\le 6$, there exists a unique deformation*

class of n-dimensional Fano manifolds Y_d with $\rho(X)=1$, $i_X=n-1$ and $d_X=d$. They have the following description:

(i) *Y_5 is a linear section of the Grassmannian $G(2,5)\subset \mathbf{P}^9$ (embedded via the Plücker embedding).*

(ii) *$Y_4=Q\cap Q'\subset \mathbf{P}^{n+2}$ is an intersection of two quadrics in $\mathbf{P}^{n+2}$.*

(iii) *$Y_3\subset \mathbf{P}^{n+1}$ is a cubic hypersurface.*

(iv) *$Y_2\to \mathbf{P}^n$ is a double cover branched along a quartic $B\subset \mathbf{P}^n$ (alternatively, Y_2 is a hypersurface of degree 4 in the weighted projective space $\mathbf{P}(2,1,\dots,1)$).*

(v) *Y_1 is a hypersurface of degree 6 in the weighted projective space $\mathbf{P}(3,2,1,\dots,1)$.*

(2) *Suppose that $\rho(X)>1$. Then X is isomorphic to one of the following:*
- $\mathbf{P}^2\times\mathbf{P}^2$ *($n=4$);*
- $\mathbf{P}(T_{\mathbf{P}^2})$ *($n=3$);*
- $\mathbf{P}\big(\mathcal{O}_{\mathbf{P}^2}(1)\oplus\mathcal{O}_{\mathbf{P}^2}\big)$ *($n=3$); or*
- $\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$ *($n=3$).*

An n-dimensional Fano manifold X with index $i_X=n-2$ is called a *Mukai* manifold. The classification of such manifolds was first announced in [**Muk89**] (see also [**IP99**] and references therein). First we note that, by Theorem 5, if $n\geq 5$, then n-dimensional Mukai manifolds have Picard number $\rho=1$, except in the cases of $\mathbf{P}^3\times\mathbf{P}^3$, $\mathbf{P}^2\times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1)\oplus\mathcal{O}^2)$.

So we start by considering n-dimensional Mukai manifolds X with $\rho(X)=1$. In this case there is an integer $g=g_X$, called the *genus* of X, such that $d_X=c_1(L)^n=2g-2$. The linear system $|L|$ determines a morphism

$$\phi_{|L|}:X\to\mathbf{P}^{g+n-2},$$

which is an embedding if $g\geq 4$ (see [**IP99**, Theorem 5.2.1]).

Theorem 7. *Let X be an n-dimensional Mukai manifold with $\rho(X)=1$. Then X has genus $g\leq 12$ ($g\neq 11$) and we have the following descriptions.*

(1) *If $g=12$, then $n=3$ and X is the zero locus of a global section of the vector bundle $\wedge^2\mathcal{S}^*\oplus\wedge^2\mathcal{S}^*\oplus\wedge^2\mathcal{S}^*$ on the Grassmannian $G(3,7)$.*

(2) *If $6\leq g\leq 10$, then X is a linear section of a variety*

$$\Sigma^{n(g)}_{2g-2}\subset\mathbf{P}^{g+n(g)-2}$$

of dimension $n(g)$ and degree $2g-2$, which can be described as follows:

(g=6) *$\Sigma^6_{10}\subset\mathbf{P}^{10}$ is a quadric section of the cone over the Grassmannian $G(2,5)\subset\mathbf{P}^9$ in the Plücker embedding.*

(g=7) *$\Sigma^{10}_{12}=OG_+(5,10)\subset\mathbf{P}^{15}$ is a connected component of the orthogonal Grassmannian $OG(5,10)$ in the half-spinor embedding.*

(g=8) *$\Sigma^8_{14}=G(2,6)\subset\mathbf{P}^{14}$ is the Grassmannian $G(2,6)$ in the Plücker embedding.*

(g=9) *$\Sigma^6_{16}=SG(3,6)\subset\mathbf{P}^{13}$ is the symplectic Grassmannian $SG(3,6)$ in the Plücker embedding.*

(g=10) *$\Sigma^5_{18}=(G_2/P_2)\subset\mathbf{P}^{13}$ is a 5-dimensional homogeneous variety for a group of type G_2.*

(3) *If $g \leq 5$, and the map $\phi_{|L|}$ is an embedding, then X is a complete intersection as follows:*

(g=3) $X_4 \subset \mathbf{P}^{n+1}$ *a quartic hypersurface;*
(g=4) $X_{2\cdot 3} \subset \mathbf{P}^{n+2}$ *a complete intersection of a quadric and a cubic;*
(g=5) $X_{2\cdot 2\cdot 2} \subset \mathbf{P}^{n+3}$ *a complete intersection of three quadrics.*

(4) *If $g \leq 3$, and the map $\phi_{|L|}$ is not an embedding, then:*

(g=2) $\phi_{|L|} : X \to \mathbf{P}^n$ *is a double cover branched along a sextic (alternatively, X is a degree 6 hypersurface in the weighted projective space $\mathbf{P}(3,1,\ldots,1)$);*
(g=3) $\phi_{|L|} : X \to Q^n \subset \mathbf{P}^{n+1}$ *is a double cover branched along the intersection of Q with a quartic hypersurface (alternatively, X is a complete intersection of two quadric hypersurfaces in the weighted projective space $\mathbf{P}(2,1,\ldots,1)$).*

We will go through the classification of Mukai manifolds with Picard number $\rho \geq 2$ and dimension $n \in \{3,4\}$ in Sections 8 and 9.

3. First Examples

In this section we compute the first Chern characters for the simplest examples of Fano manifolds with high index: (weighted) projective spaces and complete intersection on them, and Grassmannians.

3.1. Projective spaces. Set $h := c_1(\mathcal{O}_{\mathbf{P}^n}(1))$. Then

$$\mathrm{ch}(\mathbf{P}^n) = n + \sum_{k=1}^{n} \frac{n+1}{k!} h^k.$$

In particular, $\mathbf{P}^n$ is 2-Fano.

3.2. Weighted projective spaces. Let $\mathbf{P} = \mathbf{P}(a_0,\ldots,a_n)$ be a weighted projective space. We always assume that $\gcd(a_0,\ldots,a_n) = 1$, and, for every $i \in \{0,\ldots,n\}$, $\gcd(a_0,\ldots,\hat{a}_i,\ldots a_n) = 1$. Denote by H the effective generator of the class group $\mathrm{Cl}(\mathbf{P}) \cong \mathbf{Z}$. Recall that H is an ample $\mathbf{Q}$-Cartier divisor. From the Euler sequence, on the smooth locus of $\mathbf{P}$, we have:

$$\mathrm{ch}(\mathbf{P}) = n + \sum_{k=1}^{n} \frac{a_0^k + \ldots + a_n^k}{k!} c_1(H)^k.$$

3.3. Zero loci of sections of vector bundles. Several Fano manifolds with $\rho(X) = 1$ and high index are described as the zero locus $X = Z(s) \subseteq Y$ of a global section s of a vector bundle $\mathcal{E}$ on a simpler variety Y. So we investigate the 2-Fano condition in this situation.

LEMMA 8. *Let Y be a smooth projective variety, and $\mathcal{E}$ a vector bundle on Y. Let s be a global section of $\mathcal{E}$, and X its zero locus $Z(s)$. Assume that X is smooth of dimension $dim(Y) - rk(\mathcal{E})$. Then*

$$ch_i(X) = \big(ch_i(Y) - ch_i(\mathcal{E})\big)_{|X}.$$

PROOF. Since the normal bundle $N_{X|Y}$ is $\mathcal{E}_{|X}$, the lemma follows from the normal bundle sequence. $\square$

Special cases of these are complete intersections. If Y is a smooth projective variety, and X is a smooth complete intersection of divisors $D_1, \ldots, D_c$ in X, then Lemma 8 becomes:

$$\mathrm{ch}_k(X) = \big(ch_k(Y) - \frac{1}{k!}\sum D_i^k\big)_{|X}. \tag{3.1}$$

3.3.1. **Complete intersections in $\mathbf{P}^n$.** Let X be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$ in $\mathbf{P}^n$. Then by (3.1):

$$\mathrm{ch}_k(X) = \frac{1}{k!}\big((n+1) - \sum d_i^k\big)h^k_{|X}.$$

It follows that

(i) X is 2-Fano if and only if $\sum d_i^2 \le n$.
(ii) X is weakly 2-Fano if and only if $\sum d_i^2 \le n+1$.

3.3.2. **Complete intersections in weighted projective spaces.** We use the same notation and assumptions as in 3.2. Let X be a smooth complete intersection of hypersurfaces with classes $d_1H, \ldots, d_cH$ in $\mathbf{P}$, and assume that X is contained in the smooth locus of $\mathbf{P}$. Then the Chern character of X is given by

$$\mathrm{ch}(X) = (n-c) + \sum_{k=1}^{n} \frac{a_0^k + \ldots + a_n^k - \sum d_i^k}{k!} c_1(H_{|X})^k.$$

It follows that

(i) X is 2-Fano if and only if $\sum d_i^2 < \sum a_i^2$.
(ii) X is weakly 2-Fano if and only if $\sum d_i^2 \le \sum a_i^2$.

3.4. Grassmannians. Consider the Grassmannian $G(k,n)$ of k-dimensional subspaces of an n-dimensional vector space V, and recall our convention that $2 \le k \le \frac{n}{2}$. Let $\mathcal{S}^*$ denote the dual of the universal rank k vector bundle $\mathcal{S}$ on $G(k,n)$. The Chern classes of $\mathcal{S}^*$ are given by:

$$c_i(\mathcal{S}^*) = \sigma_{1,\ldots,1}, \quad (i \ge 1)$$

where $\sigma_{a_1,\ldots,a_k}$ denotes the usual Schubert cycle on $G(k,n)$. Recall that σ_1 is the class of a hyperplane via the Plücker embedding and generates $\mathrm{Pic}(G(k,n))$. Since the tangent bundle of $G(k,n)$ is given by

$$T_{G(k,n)} = \mathcal{S}^* \otimes \mathcal{Q},$$

the Chern character of $G(k,n)$ can be calculated from

$$\mathrm{ch}(G(k,n)) = \mathrm{ch}(\mathcal{S}^*)\mathrm{ch}(\mathcal{Q}) = \mathrm{ch}(\mathcal{S}^*)\big(n - \mathrm{ch}(\mathcal{S})\big).$$

The Chern character of $\mathcal{S}^*$ is given by

$$\mathrm{ch}(\mathcal{S}^*) = k + \sigma_1 + \frac{1}{2}(\sigma_2 - \sigma_{1,1}) + \frac{1}{6}(\sigma_3 - \sigma_{2,1} + \sigma_{1,1,1}) + \ldots.$$

As computed in [**deJS06**, 2.4]),

$$\mathrm{ch}(G(k,n)) = k(n-k) + n\sigma_1 + \big(\frac{n+2-2k}{2}\sigma_2 - \frac{n-2-2k}{2}\sigma_{1,1}\big) + \\ + \frac{n-2k}{6}\big(\sigma_3 - \sigma_{2,1} + \sigma_{1,1,1}\big) + \ldots.$$

The cone $\overline{\mathrm{NE}}_2(G(k,n))$ is generated by the dual Schubert cycles σ_2^* and $\sigma_{1,1}^*$. It follows that $G(k,n)$ is 2-Fano if and only if $n = 2k$ or $2k+1$. Moreover, $G(k,n)$ is weakly 2-Fano if and only if $n = 2k$, $2k+1$ or $2k+2$.

Remark 9. Complete intersections and, more generally, zero loci of vector bundles on Grassmannians will be addressed in Section 6. We will need the following formulas, obtained by standard Chern class computations:

$$\mathrm{ch}(\wedge^2(\mathcal{S}^*)) = \binom{k}{2} + (k-1)\sigma_1 + \big(\frac{k-1}{2}\sigma_2 - \frac{k-3}{2}\sigma_{1,1}\big) +$$

$$+\big(\frac{k-1}{6}\sigma_3 - \frac{k-4}{6}\sigma_{2,1} + \frac{k-7}{6}\sigma_{1,1,1}\big) + \dots \quad ,$$

$$\mathrm{ch}(\mathrm{Sym}^2(\mathcal{S}^*)) = \binom{k+1}{2} + (k+1)\sigma_1 + \big(\frac{k+3}{2}\sigma_2 - \frac{k+1}{2}\sigma_{1,1}\big) +$$

$$+\big(\frac{k+7}{6}\sigma_3 - \frac{k+4}{6}\sigma_{2,1} + \frac{k+1}{6}\sigma_{1,1,1}\big) + \dots \quad .$$

4. Chern class computations

From the classification of Fano manifolds with high index, we see that many of those with $\rho = 1$ are described as double covers, while most of the ones with $\rho > 1$ are obtained from simpler ones by blow-ups and taking projective bundles. So in this section we compute Chern characters for these constructions.

4.1. Double covers.

Lemma 10. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective varieties X and Y. Let $R \subset X$ denote the ramification divisor, and $B = f(R) \subset Y$ the branch divisor. Then $f^*(B) = 2R$ and there is an exact sequence:*

$$0 \to T_X \to f^*T_Y \to \mathcal{O}(2R)_{|R} \to 0.$$

The first and second Chern characters are related by

$$c_1(X) = f^*\big(c_1(Y) - \frac{1}{2}B\big),$$

$$ch_2(X) = f^*\big(ch_2(Y) - \frac{3}{8}B^2\big).$$

Proof. This follows from the exact sequence:

$$0 \to f^*\Omega_Y \to \Omega_X \to \mathcal{O}(-R)_{|R} \to 0.$$

□

As an immediate consequence of Lemma 10 we have the following.

Corollary 11. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective varieties X and Y. Let $B \subset Y$ be the branch divisor. Then:*

(i) *X is Fano if and only if $-K_Y - \frac{1}{2}B$ is an ample divisor. In particular, if X is Fano and B is nef, then Y is Fano.*

(ii) *X is 2-Fano (respectively weakly 2-Fano) if and only if X is Fano and*

$$ch_2(Y) - \frac{3}{8}B^2 > 0 \quad (\textit{respectively } \geq 0).$$

4.2. Projective Bundles. The following two lemmas appear in [**deJS06**]. (Note that in [**deJS06**] the notation $\mathbf{P}(E)$ stands for $\mathrm{Proj}(\mathrm{Sym}E^*)$.)

Lemma 12. [**deJS06**, Lemma 4.1] *Let X be a smooth projective variety and let $\mathcal{E}$ be a rank r vector bundle on X. Denote by $\pi : \mathbf{P}(\mathcal{E}) \to X$ the natural projection and set $\xi = c_1(\mathcal{O}_\pi(1))$. Then*

$$c_1(\mathbf{P}(\mathcal{E})) = \pi^*\big(c_1(X) + c_1(\mathcal{E}^*)\big) + r\xi,$$

$$ch_2(\mathbf{P}(\mathcal{E})) = \pi^*\big(ch_2(X) + ch_2(\mathcal{E}^*)\big) + \pi^* c_1(\mathcal{E}^*) \cdot \xi + \frac{r}{2}\xi^2.$$

Lemma 13. [**deJS06**, Proposition 4.3] *Let X be a smooth projective variety and let $\mathcal{E}$ be a rank 2 vector bundle on X. Denote by $\pi : \mathbf{P}(\mathcal{E}) \to X$ the natural projection and set $\xi = c_1(\mathcal{O}_\pi(1))$. Then*

$$c_1(\mathbf{P}(\mathcal{E})) = 2\xi + \pi^*\big(c_1(X) - c_1(\mathcal{E})\big),$$

$$ch_2(\mathbf{P}(\mathcal{E})) = \pi^*\Big(ch_2(X) + \frac{1}{2}(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}))\Big).$$

Therefore, $ch_2(\mathbf{P}(\mathcal{E})) \geq 0$ if and only if

$$ch_2(X) + \frac{1}{2}(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})) \geq 0. \tag{4.1}$$

If $\dim(X) > 0$, then $\mathbf{P}(\mathcal{E})$ is not 2-Fano. $\mathbf{P}(\mathcal{E})$ is weakly 2-Fano if it is Fano and condition (4.1) holds.

As an immediate consequence of Lemma 13, we have the following criterion.

Corollary 14. *Let X be a smooth projective variety and let L be a line bundle on X. The projective bundle $\mathbf{P}_X(\mathcal{O} \oplus L)$ is not 2-Fano and it is weakly 2-Fano if and only if it is Fano and we have:*

$$ch_2(X) + \frac{1}{2}c_1(L)^2 \geq 0. \tag{4.2}$$

In particular, (4.2) holds if X is weakly 2-Fano and L is nef. For example:

(i) *$\mathbf{P}_{\mathbf{P}^n}(\mathcal{O} \oplus \mathcal{O}(a))$ is weakly 2-Fano if and only if $|a| \leq n$.*
(ii) *$\mathbf{P}_{\mathbf{P}^n \times \mathbf{P}^m}(\mathcal{O} \oplus \mathcal{O}(a,b))$ is weakly 2-Fano if and only if $|a| \leq n$, $|b| \leq m$, and $ab \geq 0$.*

Example 15. Consider the Fano manifold $X = \mathbf{P}(T_{\mathbf{P}^n})$, $n \geq 2$.
If $n = 2$, then X is not 2-Fano, but weakly 2-Fano by Lemma 13 since

$$\mathrm{ch}_2(\mathbf{P}^2) + \frac{1}{2}\big(c_1(\mathbf{P}^2)^2 - 4c_2(\mathbf{P}^2)\big) = c_1(\mathbf{P}^2)^2 - 3c_2(\mathbf{P}^2) = 0.$$

Suppose $n \geq 3$. Denote by $\pi : X \to \mathbf{P}^n$ the natural morphism, and let $\ell \subset \mathbf{P}^n$ be a line. Consider the surface S in $\pi^{-1}(\ell)$, ruled over ℓ, corresponding to the surjection

$$T_{\mathbf{P}^n}|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1} \twoheadrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Using the formula for ch_2 from Lemma 12, one gets that $\mathrm{ch}_2(X) \cdot S = -1$. Hence X is not weakly 2-Fano.

Example 16. The exact same calculation as in Example 15 shows that

$$\mathbf{P}_{\mathbf{P}^n}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1})$$

is not weakly 2-Fano for $n \geq 3$.

Lemma 17. *A product $X \times Y$ of smooth projective varieties is not 2-Fano. It is weakly 2-Fano if and only if both X and Y are weakly 2-Fano.*

Proof. This follows from the projection formula and the formula

$$\mathrm{ch}_2(X \times Y) = \pi_1^* \mathrm{ch}_2(X) + \pi_2^* \mathrm{ch}_2(Y),$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections. Given two curves $B \subset X$ and $C \subset Y$, set $S = B \times C$. Then $\mathrm{ch}_2(X \times Y) \cdot S = 0$, and thus $X \times Y$ is not 2-Fano. □

4.2.1. **Complete intersections in products of projective spaces.** Let Y be a smooth divisor of type $(a_1, \dots, a_r)$ in $\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r}$, and set $h_i := c_1(\pi_i^* \mathcal{O}(1))$. By a direct computation using the normal bundle sequence, we have:

$$\mathrm{ch}_2(Y) = \frac{1}{2} \sum_{i=1}^{r} (n_i + 1 - a_i^2) \big(h_i^2\big)_{|Y} - \sum_{i<j} (a_i a_j) \big(h_i \cdot h_j\big)_{|Y}.$$

We compute some examples of intersection numbers $\mathrm{ch}_2(Y) \cdot S$, where

$$S = {h_1}_{|Y}^{c_1} \cdot \ldots \cdot {h_r}_{|Y}^{c_r}, \quad \sum c_i = \sum n_i - 3 \quad (c_i \geq 0). \tag{4.3}$$

Example 18. Let Y be a divisor of type (a, b) on $\mathbf{P}^n \times \mathbf{P}^m$ $(a, b > 0)$. It follows from (4.3) that

$$\mathrm{ch}_2(Y) \cdot {h_1}_{|Y}^{n-2} \cdot {h_2}_{|Y}^{m-1} = \frac{b}{2}(n + 1 - 3a^2).$$

In particular, Y is not weakly 2-Fano if either $3a^2 > n + 1$ or $3b^2 > m + 1$.

Example 19. Let Y be a divisor of type (a, b, c) on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. It follows from (4.3) that

$$\mathrm{ch}_2(Y) \cdot h_{3|Y} = -3abc.$$

In particular, if $a, b, c > 0$ then Y is not weakly 2-Fano.

Example 20. Let Y be a divisor of type $(a_1, \dots, a_r)$ on $(\mathbf{P}^1)^r$. It follows from (4.3) that

$$\mathrm{ch}_2(Y) \cdot h_{1|Y} \cdot h_{2|Y} \cdot \ldots \cdot h_{r-3|Y} = -3 a_{r-2} a_{r-1} a_r.$$

In particular, if $a_i > 0$ for all $i = 1, \dots r$, then Y is not weakly 2-Fano.

Example 21. Let Y be a complete intersection in $\mathbf{P}^n \times \mathbf{P}^m$ $(m, n \geq 2)$ of a divisor D_1 of type (a_1, b_1) and a divisor D_2 of type (a_2, b_2). Then

$$\mathrm{ch}_2(Y) = \frac{1}{2}(n + 1 - {a_1}^2 - {a_2}^2) h_{1|Y}^2 + \frac{1}{2}(m + 1 - {b_1}^2 - {b_2}^2) h_{2|Y}^2 -$$

$$-(a_1 b_1 + a_2 b_2)(h_1 \cdot h_2)_{|Y}.$$

It follows that

$$\mathrm{ch}_2(Y) \cdot {h_1}_{|Y}^{n-2} \cdot {h_2}_{|Y}^{m-2} = 1 + \frac{n + m}{2} - \frac{3}{2}(a_1 b_1 + a_2 b_2)(a_1 b_2 + a_2 b_1).$$

In particular, Y is not weakly 2-Fano if $(a_1 b_1 + a_2 b_2)(a_1 b_2 + a_2 b_1) > \frac{n+m+2}{3}$.

4.3. Blow-ups. The following Lemma appeared first in [**deJS06**]. See also [**Nob12**] for a detailed computation.

LEMMA 22. [**deJS06**, Lemma 5.1] *Let X be a smooth projective variety and let $i : Y \hookrightarrow X$ be a smooth subvariety of codimension $c \geq 2$. Let $f : \tilde{X} \to X$ be the blow-up of X along Y and let E be the exceptional divisor. Denote by $j : E \hookrightarrow \tilde{X}$ the natural inclusion map and set $\pi = f_{|E} : E \to Y$. Let N be the normal bundle of Y in X. The Chern characters of $\tilde{X}$ are given by the following formulas:*

$$c_1(\tilde{X}) = f^* c_1(X) - (c-1)[E],$$

$$ch_2(\tilde{X}) = f^* ch_2(X) + \frac{c+1}{2}[E]^2 - j_* \pi^* c_1(N).$$

4.3.1. **Del Pezzo surfaces.** Let S_d $(1 \leq d \leq 9)$ denote a del Pezzo surface of degree d, i.e., S_d is the blow-up of $\mathbf{P}^2$ at $9-d$ points in general position. By Lemma 22, we have

$$\mathrm{ch}_2(S_d) = \mathrm{ch}_2(\mathbf{P}^2) - \frac{3}{2}(9-d) = \frac{3}{2}(d-8). \tag{4.4}$$

It follows that the only 2-Fano del Pezzo surface is $\mathbf{P}^2$, while $S_8 = \mathbf{F}_1$ and $\mathbf{P}^1 \times \mathbf{P}^1$ (Lemma 17) are the only other weakly 2-Fano surfaces.

4.3.2. **The case of threefolds.** We compute several intersection numbers of $\mathrm{ch}_2(\tilde{X})$ with surfaces in the case when $\tilde{X}$ is a blow-up of a threefold, first along a smooth curve (Lemma 23), and then along points (Lemma 24).

LEMMA 23. *Let X be a smooth projective variety of dimension* 3 *and let C be a smooth irreducible curve in X. Let $\tilde{X}$ be the blow-up of X along C, E the exceptional divisor, and N the normal bundle of C in X. Then $E^3 = -\deg(N)$ and*

$$ch_2(\tilde{X}) \cdot E = -\frac{1}{2}\deg(N).$$

Let T be a smooth surface in X and let $\tilde{T}$ be its proper transform in $\tilde{X}$.

(i) *If $T \cap C$ is a* 0*-dimensional reduced scheme of length r, then*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - \frac{3r}{2}.$$

(ii) *If $C \subset T$ and $(C^2)_T$ denotes the self-intersection of C on T, then*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T + \frac{3}{2}(C^2)_T - \deg(N).$$

PROOF. By Lemma 22,

$$\mathrm{ch}_2(\tilde{X}) \cdot E = \frac{3}{2}E^3 - \big(j_* \pi^* c_1(N)\big) \cdot E.$$

Set $\xi = c_1(\mathcal{O}_E(1))$. Since $E \cong \mathbf{P}_C(N^*)$, by [**Ful98**, Rmk. 3.2.4]) we have

$$\xi^2 + \pi^* c_1(N)\xi + \pi^* c_2(N) = 0.$$

(Note that in [**Ful98**], the notation $\mathbf{P}(E)$ stands for $\mathrm{Proj}(\mathrm{Sym}E^*)$.)

It follows that $\xi^2 = -\deg(N)$ and hence,

$$E^3 = \xi^2 = -\deg(N).$$

If α is a cycle on E and D is a divisor on X, then

$$j_* \alpha \cdot D = (j^* D \cdot \alpha)_E, \tag{4.5}$$

where $(\,,)_E$ denotes the intersection on E. Applying (4.5) for $D = E$ and $\alpha = \pi^* c_1(N)$, it follows that

$$\big(j_* \pi^* c_1(N)\big) \cdot E = -\big(\xi \cdot \pi^* c_1(N)\big)_E = -\deg(N),$$
$$\mathrm{ch}_2(T_{\tilde{X}}) \cdot E = -\frac{3}{2}\deg(N) + \deg(N) = -\frac{1}{2}\deg(N).$$

For Cases (i) and (ii), by Lemma 22, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(X) \cdot T + \frac{3}{2}E^2 \cdot \tilde{T} - \big(j_* \pi^* c_1(N)\big) \cdot \tilde{T}.$$

Consider now Case (i). Then $\tilde{T}$ is the blow-up of T along the r points in $C \cap T$, and $E \cap \tilde{T}$ is the union of the r exceptional divisors of the blow-up $\tilde{T} \to T$. Since $\tilde{T}_{|E}$ consists of fibers of $\pi : E \to C$, it follows using (4.5) that

$$\big(j_* \pi^* c_1(N)\big) \cdot \tilde{T} = \tilde{T}_{|E} \cdot \pi^* c_1(N) = 0.$$

As $E^2 \cdot \tilde{T} = \big(E^2_{|\tilde{T}}\big)_{\tilde{T}} = -r$, the result follows.

Consider now Case (ii). Then $\tilde{T} \cong T$ and $E \cap \tilde{T}$ is a section of $\pi : E \to C$. By (4.5) it follows that

$$\big(j_* \pi^* c_1(N)\big) \cdot \tilde{T} = \tilde{T}_{|E} \cdot \pi^* c_1(N) = \deg(N).$$

Since $E^2 \cdot \tilde{T} = (C^2)_T$, the result follows. $\square$

Lemma 24. *Let X be a smooth projective variety of dimension 3, $q \in X$ a point, and $\tilde{X}$ the blow-up of X at q, with exceptional divisor E. Then $E^3 = 1$ and*

$$ch_2(\tilde{X}) \cdot E = 2.$$

Let T be a surface in X and let $\tilde{T}$ be its proper transform in $\tilde{X}$. If $m \geq 0$ is the multiplicity of T at q, then we have:

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - 2m.$$

Proof. Since $E_{|E}$ is the tautological line bundle $\mathcal{O}_E(-1)$ on $E \cong \mathbf{P}^2$, it follows that $E^3 = \big(\mathcal{O}_E(-1)^2\big)_E = 1$. By Lemma 22, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot E = 2E^3 = 2.$$

If T is a surface that contains q with multiplicity m, then

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(\tilde{X}) \cdot (f^*T - mE) = \mathrm{ch}_2(X) \cdot T - 2mE^3 = \mathrm{ch}_2(X) \cdot T - 2m.$$

$\square$

As an immediate consequence of Lemmas 23 and 24, we have the following criterion.

Corollary 25. *Let X be a smooth projective variety of dimension 3 and $\tilde{X}$ be the blow-up of X along disjoint smooth curves $C_1, \ldots, C_k$ and l distinct points. If T is a smooth surface in X containing $0 \leq s \leq l$ of the blown-up points, and intersecting $\cup_i C_i$ along a zero-dimensional reduced scheme of length r, then*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - \frac{3r}{2} - 2s.$$

In particular, $\tilde{X}$ is not weakly 2-Fano if

$$ch_2(X) \cdot T < \frac{3r}{2} + 2s.$$

The results in 4.3.2 (for example Corollary 25) give ways to check that some blow-ups of threefolds are not weakly 2-Fano. Here we list a few more.

COROLLARY 26. *Let the assumptions and notation be as in Lemma 23. Suppose that either $g(C) > 0$ and $-K_X \cdot C > 0$, or $C \cong \mathbf{P}^1$ and $-K_X \cdot C > 2$. Then*

$$ch_2(\tilde{X}) \cdot E < 0.$$

In particular, if X is Fano and either $g(C) > 0$, or X has index $i_X \geq 3$, then $ch_2(\tilde{X})$ is not nef.

If $C \cong \mathbf{P}^1$ and $-K_X \cdot C = 2$, then

$$ch_2(\tilde{X}) \cdot E = 0.$$

PROOF. The result follows immediately from Lemma 23 since

$$\deg(N) = \deg(T_{X|C}) - \deg(T_C) = -K_X \cdot C + 2g(C) - 2.$$

□

LEMMA 27. *Let X be a smooth projective threefold. Assume X has a semiample divisor T such that*

$$ch_2(X) \cdot T < 0.$$

Then any blow-up of X along points and smooth curves is not weakly 2-Fano.

PROOF. By replacing T with a multiple, we may assume that $|T|$ is a base-point free linear system. In this case we can find a surface T that avoids any of the blown-up points and intersects each of the blown-up curves in a reduced 0-dimensional scheme of length $r \geq 0$. By Lemma 25, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(X) \cdot T - \frac{3r}{2} < 0.$$

In particular, $\tilde{X}$ is not weakly 2-Fano. □

As a consequence of Lemma 27, we have the following criterion.

COROLLARY 28. *Let X be a smooth projective threefold with $\rho = 1$. If X is not weakly 2-Fano, then any blow-up of X along points and smooth curves is not weakly 2-Fano.*

COROLLARY 29. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective threefolds with ample branch divisor B. Moreover, assume Y has a semiample divisor T such that*

$$ch_2(Y) \cdot T \leq 0.$$

Then any blow-up of X along points and smooth curves is not weakly 2-Fano.

PROOF. By replacing T with a multiple, we may assume that $|T|$ is base-point free. Note that $|f^*T|$ is also base-point free. By Lemma 10, we have:

$$\mathrm{ch}_2(X) \cdot f^*(T) = \big(\mathrm{ch}_2(Y) - \frac{3}{8}B^2\big) \cdot T < 0.$$

The result now follows from Lemma 27. □

5. Families of rational curves on 2-Fano manifolds

In this section we revise some results from [**AC12**], to which we refer for details and further references.

Let X be a smooth complex projective uniruled variety, and $x \in X$ a general point. There is a scheme $\mathrm{RatCurves}^n(X,x)$ parametrizing rational curves on X passing through x, and it always contains a smooth and proper irreducible component H_x. For instance, one can take H_x to be an irreducible component of $\mathrm{RatCurves}^n(X,x)$ parametrizing rational curves through x having minimal degree with respect to some fixed ample line bundle on X. We denote by $\pi_x : U_x \to H_x$ and $\mathrm{ev}_x : U_x \to X$ the usual universal family morphisms, and set $d := \dim(H_x)$. Since ev_x is proper and π_x is a $\mathbf{P}^1$-bundle, we have a linear map

$$T_1 = \mathrm{ev}_{x*}\pi_x^* : N_1(H_x) \to N_2(X),$$

which maps $\overline{\mathrm{NE}}_1(H_x) \setminus \{0\}$ into $\overline{\mathrm{NE}}_2(X) \setminus \{0\}$.

The variety H_x comes with a natural polarization L_x, which can be defined as follows. By [**Keb02**, Theorems 3.3 and 3.4], there is a finite morphism $\tau_x : H_x \to \mathbf{P}(T_xX^*)$ that sends a point parametrizing a curve smooth at x to its tangent direction at x. We then set $L_x := \tau_x^*\mathcal{O}(1)$.

The pair (H_x, L_x) is called a *polarized minimal family of rational curves through* x, and reflects much of the geometry of X. It is well understood for homogeneous spaces and complete intersections on them (see [**Hwa01**]). In [**AC12**], we computed all the Chern classes of the variety H_x in terms of the Chern classes of X and $c_1(L_x)$. For instance,

$$c_1(H_x) = \pi_{x*}\mathrm{ev}_x^*\big(\mathrm{ch}_2(X)\big) + \frac{d}{2}c_1(L_x). \tag{5.1}$$

In particular, if X is 2-Fano (respectively weakly 2-Fano), then $-2K_{H_x} - dL_x$ is ample (respectively nef). This necessary condition is also sufficient provided that $T_1\big(\overline{\mathrm{NE}}_1(H_x)\big) = \overline{\mathrm{NE}}_2(X)$.

Example 30. We consider the special case of the Grassmannian $G(k,n)$ ($2 \le k \le \frac{n}{2}$). The variety H_x of lines in $G(k,n)$ that pass through a general point $x = [W]$ can be identified with $\mathbf{P}(W) \times \mathbf{P}(V/W)^* \cong \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}$, and the map $\tau_x : \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1} \to \mathbf{P}(T_xX^*)$ is the Segre embedding. So the polarization L_x corresponds to a divisor of type $(1,1)$. We denote by π_1 and π_2 the projections from $\mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}$. The map

$$T_1 : \overline{\mathrm{NE}}_1(H_x) \to \overline{\mathrm{NE}}_2(G(k,n))$$

sends classes of lines in the fibers of π_1 and π_2, to the dual cycles σ_2^* and $\sigma_{1,1}^*$, respectively.

Now let $X = H_1 \cap \ldots \cap H_c \subseteq G(k,n)$ be a smooth complete intersection of hyperplane sections $H_1, \ldots, H_c$ under the Plücker embedding, with $c \le n-2$. We may assume that $x \in X$ is a general point, and consider the variety of lines in X, $Z_x \subset H_x$. Notice that Z_x is a complete intersection of c divisors D_i of type $(1,1)$ in H_x:

$$Z_x = D_1 \cap \ldots \cap D_c \subset H_x \cong \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}.$$

Claim.

(i) If $c \le k-1$ and $n > 2k$, then Z_x contains a line from a fiber of π_2. In particular, X contains a surface with class $\sigma_{1,1}^*$.

(ii) If $c < n-k-1$, then Z_x contains a line from a fiber of π_1. In particular, X contains a surface with class σ_2^*.

In particular, if $c \leq k-1$ and $n > 2k$, then the natural map

$$u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(G(k,n))$$

is surjective.

Proof. Let $x_0, \ldots, x_{k-1}$ (respectively $y_0, \ldots, y_{n-k-1}$) denote the coordinates on $\mathbf{P}^{k-1}$ (respectively on $\mathbf{P}^{n-k-1}$). Each divisor D_i has an equation of type:

$$x_0 F_0^{(i)} + \ldots + x_{k-1} F_{k-1}^{(i)} = 0,$$

where $F_j^{(i)}$ are linear forms in (y_i). Clearly, if $c < k-1$, then Z_x contains a line from any fiber of π_2. By the same argument, if $c < n-k-1$, then Z_x contains a line from any fiber of π_1, and this proves (ii).

Note that if $c = k-1$ and $k \leq n-k-1$, then the locus in $\mathbf{P}^{n-k-1}$ where the k minors of size $(k-1)\times(k-1)$ of the matrix of linear forms $\big(F_j^{(i)}\big)$ vanish is non-empty. This proves (i) and the claim follows. □

Note that inequalities (i) and (ii) are optimal. Indeed, consider the case when $X = H_1 \cap H_2 \subset G(2,5)$. Let x_0, x_1 (respectively y_0, y_1, y_2) denote the coordinates on $\mathbf{P}^1$ (respectively on $\mathbf{P}^2$). The variety Z_x is a complete intersection in $\mathbf{P}^1 \times \mathbf{P}^2$ of two divisors of type $(1,1)$:

$$D_1 : x_0 F_0 + x_1 F_1 = 0,$$
$$D_2 : x_0 G_0 + x_1 G_1 = 0,$$

where F_i, G_i are linear forms in (y_i). Thus Z_x is isomorphic to the smooth conic $F_0 G_1 - F_1 G_0 = 0$ in $\mathbf{P}^2$, and $Z_x \subset \mathbf{P}^1 \times \mathbf{P}^2$ is a curve of type $(2,2)$. It follows that $T_1 : \overline{\mathrm{NE}}_1(Z_x) \to \overline{\mathrm{NE}}_2(X)$ maps the fundamental class of Z_x to $\sigma_2^* + \sigma_{1,1}^*$.

6. Complete intersections in homogeneous spaces

6.1. Complete intersections in Grassmannians. We apply the results from Section 3.3 to the case when the ambient space is a Grassmannian.

Proposition 31. *Consider a smooth complete intersection*

$$X = (d_1 H) \cap \ldots \cap (d_c H) \subseteq G(k,n) \quad (2 \leq k \leq \frac{n}{2}, 1 \leq c),$$

where $H = \sigma_1$ is the class of a hyperplane class via the Plücker embedding.

The Chern character of X is given by:

$$ch(X) = \big(k(n-k) - c\big) + (n - \sum d_i)\sigma_1 +$$
$$\big(\frac{n+2-2k-\sum d_i^2}{2}\sigma_2 - \frac{n-2-2k+\sum d_i^2}{2}\sigma_{1,1}\big) +$$
$$+\big(\frac{n-2k-\sum d_i^3}{6}\big(\sigma_3 + \sigma_{1,1,1}\big) - \frac{n-2k+\sum d_i^3}{6}\sigma_{2,1}\big) + \ldots.$$

Then X is Fano if and only if $\sum d_i < n$. Moreover, X is not weakly 2-Fano if

$$\sum d_i^2 \geq n - 2k + 2,$$

with the exception of the case when $n = 2k, c = 2, d_1 = d_2 = 1$, in which case X is weakly 2-Fano (see also Proposition 32).

Proof of Proposition 31. The formula for $\mathrm{ch}(X)$ follows from the formula for $\mathrm{ch}(G(k,n))$ computed in 3.4 and the formula for complete intersections from 3.3. The criterion for X to be Fano follows immediately.

To prove the last statement, we may assume X is Fano. If $\sum d_i^2 \geq n-2k+2$ then $\mathrm{ch}_2(X) = a\sigma_2 + b\sigma_{1,1}$, with $a, b \leq 0$. Note that $b = 0$ if and only if $n = 2k$ and $\sum d_i^2 = 2$, i.e., $c = 2$ and $d_1 = d_2 = 1$. In this case, $\mathrm{ch}_2(X) = 0$. But if $b < 0$, then either $a < 0$, in which case $\mathrm{ch}_2(X) \cdot S < 0$ for any surface $S \subset X$, or $a = 0$ and we have:

$$\sum d_i^2 = n-2k+2, \quad \mathrm{ch}_2(X) = -(n-2k)\sigma_{1,1|X} \quad (n > 2k).$$

Since $\sigma_{1,1} \cdot \sigma_1^{\dim G(k,n)-2} > 0$ and $\sigma_{1|X}$ is ample, in the latter case X is not weakly 2-Fano for

$$\mathrm{ch}_2(X) \cdot \sigma_1^{\dim(X)-2} < 0. \tag{6.1}$$

□

Proposition 32. *Consider a smooth complete intersection*

$$X = H_1 \cap \ldots \cap H_c \subseteq G(k,n) \quad (2 \leq k \leq \frac{n}{2}, 1 \leq c < n),$$

of hyperplane sections $H_1, \ldots, H_c$ under the Plücker embedding. Then

$$ch_2(X) = \frac{n+2-2k-c}{2}\sigma_2 - \frac{n-2-2k+c}{2}\sigma_{1,1}.$$

Moreover:

(i) *If $c \geq n-2k+2$ (with $c \neq 2$ if $n = 2k$) then X is not weakly 2-Fano.*
(ii) *If $c \leq k-1$ and $n \geq 2k+2$, then X is not weakly 2-Fano.*
(iii) *If $n = 2k$ then X is (weakly) 2-Fano if and only if $c = 1$ ($c = 1, 2$).*
(iv) *If $n = 2k+1$, then X is not 2-Fano; X is weakly 2-Fano if and only if $c = 1$, and possibly when $X = H_1 \cap H_2 \subset G(2,5)$.*

Proof. By Proposition 31, if $c \geq n-2k+2$, and if we are not in the case when $n = 2k$ and $c = 2$, then X is not weakly 2-Fano. This gives (i). For (ii), note that if $c \leq k-1$ then, by Example 30, the natural map $u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(G(k,n))$ is surjective. If $n-2k-2+c > 0$ and $n > 2k$, then part (ii) follows, since

$$\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = -\frac{n-2k-2+c}{2} < 0.$$

Part (iii) follows immediately, since if $n = 2k$ then

$$\mathrm{ch}_2(X) = \frac{2-c}{2}\big(\sigma_2 + \sigma_{1,1}\big) = \frac{2-c}{2}\sigma_1^2.$$

We now prove (iv). Assume that $n = 2k+1$. We have:

$$\mathrm{ch}_2(X) = \frac{3-c}{2}\sigma_2 + \frac{1-c}{2}\sigma_{1,1}.$$

By (i), if $c \geq 3$ then X is not weakly 2-Fano. Assume that $c \leq 2$. If $k \geq 3$, then $c \leq 2 \leq k-1$. By Example 30, the natural map $u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(G(k,n))$ is surjective. It follows that, in this case, X is weakly 2-Fano if and only if the coefficients of σ_2 and $\sigma_{1,1}$ in the formula for $\mathrm{ch}_2(X)$ are non-negative, i.e., $c = 1$. Note that X is not 2-Fano in this case, as $\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = 0$.

We are left to analyze what happens in the case when $k < 3$, i.e., the case of $G(2,5)$. If $c = 1$ then $\mathrm{ch}_2(X) = \sigma_2 \geq 0$, and X is weakly 2-Fano. By Example 30, the natural map $u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(G(2,5))$ is surjective, and X is not 2-Fano since $\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = 0$. Now assume $c = 2$. Then we have:

$$\mathrm{ch}_2(X) = \frac{1}{2}\sigma_2 - \frac{1}{2}\sigma_{1,1}.$$

By Example 30, X contains a surface S with class $\sigma_2^* + \sigma_{1,1}^*$. Clearly, X is not 2-Fano, since $\mathrm{ch}_2(X) \cdot S = 0$. □

6.2. Orthogonal Grassmannians. We fix Q a nondegenerate symmetric bilinear form on the n-dimensional vector space V. Let $OG(k,n)$ be the subvariety of the Grassmannian $G(k,n)$ parametrizing linear subspaces that are isotropic with respect to Q.

If $n \neq 2k$ then $OG(k,n)$ is a Fano manifold of dimension $\frac{k(2n-3k-1)}{2}$ and $\rho = 1$. On the other hand, $OG(k,2k)$ has two connected components [**GH**, p. 737]: If $\Sigma \subset V$ is a fixed isotropic subspace of dimension k in V, then one component $OG_+(k,2k)$, corresponds to $[W] \in OG(k,2k)$ such that $\dim(W \cap \Sigma) \equiv k \pmod 2$, while the other component $OG_-(k,2k)$ corresponds to those $[W] \in OG(k,2k)$ such that $\dim(W \cap \Sigma) \not\equiv k \pmod 2$. The two components are disjoint and isomorphic. Note also that

$$OG(k-1, 2k-1) \cong OG_+(k,2k).$$

The orthogonal Grassmannian $OG(k,n)$ is the zero locus in $G(k,n)$ of a global section of the vector bundle $\mathrm{Sym}^2(\mathcal{S}^*)$. Using this description and the formula for $\mathrm{ch}\big(G(k,n)\big)$ described in 3.4, standard Chern class computations show that for any component X of $OG(k,n)$ we have:

$$\begin{aligned}\mathrm{ch}(X) = \frac{k(2n-3k-1)}{2} + (n-k-1)\sigma_1 + \\ +\big(\frac{n-3k-1}{2}\sigma_2 - \frac{n-3k-3}{2}\sigma_{1,1}\big) + \\ +\big(\frac{n-3k-7}{6}\sigma_3 - \frac{n-3k-4}{6}\sigma_{2,1} + \frac{n-3k-1}{6}\sigma_{1,1,1}\big) + \dots .\end{aligned}$$

6.2.1. **Complete intersections in $OG_+(k,2k)$.** Our main reference in what follows is [**Cos09**]. We consider now one component $OG_+(k,2k)$ of the orthogonal Grassmannian $OG(k,2k)$. For the reader's convenience, we recall the description of Schubert varieties in $OG_+(k,2k)$. Let

$$F_1 \subset F_2 \subset \ldots \subset F_k$$

be an isotropic flag in V, with $[F_k] \in OG_+(k,2k)$. This induces a second flag

$$F_{k-1} \subset F_{k-1}^{\perp} \subset F_{k-2}^{\perp} \subset \ldots \subset F_1^{\perp} \subset V.$$

Here, by abuse of notation, we denote by $F_{k-1}^{\perp}$ an isotropic subspace of dimension k parametrized by $OG_-(k,2k)$ and such that $F_{k-1} \subset F_{k-1}^{\perp}$.

For each decreasing sequence

$$\lambda : k-1 \geq \lambda_1 > \lambda_2 > \ldots > \lambda_s \geq 0 \quad (s \leq k),$$

(where we assume $k-s$ is even) we denote by

$$\mu : k-1 \geq \mu_{s+1} > \mu_{s+2} > \ldots > \mu_k \geq 0$$

the sequence obtained by removing $k-1-\lambda_i$ from $k-1,\dots,0$. For each sequence λ as above, we have a Schubert variety of codimension $\sum \lambda_i$:

$$\Omega^0_\lambda = \{[W] \in OG(k,2k) \mid \dim(W \cap F_{k-\lambda_i}) = i,\ \dim(W \cap F^\perp_{\mu_j}) = j\}.$$

Let Ω_λ be the closure of Ω^0_λ and denote by τ_λ its cohomology class. The cohomology of $OG_+(k,2k)$ is generated by the classes τ_λ. In particular, $b_4(OG_+(k,2k)) = 1$.

Claim 33. On $OG_+(k,2k)$ we have $\sigma_2 = \sigma_{1,1} = \frac{1}{2}\sigma_1^2$.

Proof. Since $b_4 = 1$, it is enough to find a surface S in $OG_+(k,2k)$ such that $\sigma_2 \cdot S = \sigma_{1,1} \cdot S$. Let $S = \Omega_{k-1,k-2,\dots,3,1}$ (the unique Schubert variety of dimension 2). One can show that $\sigma_2 \cdot S = \sigma_{1,1} \cdot S = 2$. We leave this fun computation to the reader. □

Proposition 34. *$OG_+(k,2k)$ is a 2-Fano manifold.*
Consider a smooth complete intersection

$$X = (d_1H) \cap \ldots \cap (d_cH) \subseteq OG_+(k,2k) \quad (k \geq 3),$$

where $H = \frac{1}{2}\sigma_1$ denotes a hyperplane section of the half-spinor embedding of $OG_+(k,2k)$. The Chern character of X is given by:

$$ch(X) = \frac{k(k-1)}{2} + \big(2k-2-\sum d_i\big)H + \big(\frac{4-\sum d_i^2}{2}\big)H^2 + \dots.$$

Then X is Fano if and only if $\sum d_i < 2k-2$. Moreover, X is 2-Fano if and only if all $d_i = 1$ and $c \leq 3$. The only other cases when X is weakly 2-Fano are when $c = 4$, $d_1 = \ldots = d_4 = 1$ and $c = 2$, $d_1 = d_2 = 2$.

Proof. Since $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$, by Claim 33, we obtain

$$\mathrm{ch}_2(OG_+(k,2k)) = \frac{1}{2}\sigma_1^2.$$

In particular, $OG_+(k,2k)$ is 2-Fano. Recall that a hyperplane section of $OG_+(k,2k)$ via the Plücker embedding is linearly equivalent to $2H$, where H is a hyperplane section of the spinor embedding [**Muk95**, Proposition 1.7]. It follows that $2H = \sigma_1$. The result now follows from the formula for the Chern character of $OG(k,n)$. □

6.3. Symplectic Grassmannians. We fix ω a non-degenerate antisymmetric bilinear form on the n-dimensional vector space V, n even. Let $SG(k,n)$ be the subvariety of the Grassmannian $G(k,n)$ parametrizing linear subspaces that are isotropic with respect to ω. Then $SG(k,n)$ is a Fano manifold of dimension $\frac{k(2n-3k+1)}{2}$ and $\rho(X) = 1$. Notice that X is the zero locus in $G(k,n)$ of a global section of the vector bundle $\wedge^2(\mathcal{S}^*)$. Using this description and the formula for $\mathrm{ch}\big(G(k,n)\big)$ described in 3.4, standard Chern class computations show that

$$\mathrm{ch}(SG(k,n)) = \frac{k(2n-3k+1)}{2} + (n-k+1)\sigma_1 +$$
$$+\big(\frac{n-3k+3}{2}\sigma_2 - \frac{n-3k+1}{2}\sigma_{1,1}\big) +$$
$$+\big(\frac{n-3k+1}{6}\sigma_3 - \frac{n-3k+4}{6}\sigma_{2,1} + \frac{n-3k+7}{6}\sigma_{1,1,1}\big) + \dots.$$

6.3.1. **Complete intersections in $SG(k, 2k)$.** The symplectic Grassmannian $SG(k, 2k)$ is a Fano manifold with $b_4 = 1$. For example, note that $b_4\big(SG(k, 2k)\big) = b_4\big(OG(k, 2k+1)\big)$ (see for instance [**BS02**, Section 3.1]), and $b_4\big(OG(k, 2k+1)\big) = 1$ (see Section 6.2.1).

Claim 35. On $SG(k, 2k)$ we have $\sigma_2 = \sigma_{1,1} = \frac{1}{2}\sigma_1^2$.

Proof. Since $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$, it is enough to prove that on $SG(k, 2k)$ we have $\sigma_2 = \sigma_{1,1}$. Since $b_4(SG(k, 2k)) = 1$, we are done if we find a surface S in $SG(k, 2k)$ such that $S \cdot \sigma_2 = S \cdot \sigma_{1,1}$. Let x be a general point on $SG(k, 2k)$ and let H_x denote the space of lines on $SG(k, 2k)$ that pass through x. Recall from [**AC12**, 5.5]) that

$$H_x \cong \mathbf{P}^{k-1} \subset \mathbf{P}^{k-1} \times \mathbf{P}^{k-1}$$

is the diagonal embedding. Let S be the surface in $SG(k, 2k)$ corresponding to a line in $H_x \cong \mathbf{P}^{k-1}$ via the map $T_1 : \overline{\mathrm{NE}}_1(H_x) \to \overline{\mathrm{NE}}_2(SG(k, 2k))$. It follows that the class of S is $\sigma_2^* + \sigma_{1,1}^*$. Clearly, $S \cdot \sigma_2 = S \cdot \sigma_{1,1} = 1$. □

It follows from 6.3 that

$$\mathrm{ch}(SG(k, 2k)) = \big(\frac{k(k+1)}{2}\big) + (k+1)\sigma_1 + \frac{1}{2}\sigma_1^2 + \dots$$

In particular, $SG(k, 2k)$ is 2-Fano (as proved also in [**AC12**, 5.5]) and we have the following consequence:

Proposition 36. *Consider a smooth complete intersection*

$$X = (d_1 H) \cap \ldots \cap (d_c H) \subseteq SG(k, 2k) \quad (k \geq 2, c \geq 1),$$

where $H = \sigma_1$ is a hyperplane section under the Plücker embedding.

The Chern character of X is given by:

$$ch(X) = \big(\frac{k(k+1)}{2} - r\big) + (k + 1 - \sum d_i)\sigma_1 + \frac{1 - \sum d_i^2}{2}\sigma_1^2 + \dots$$

Then X is Fano if and only if $\sum d_i < k$. Moreover, X is weakly 2-Fano if and only if $c = d_1 = 1$. In this case X is not 2-Fano.

6.4. Complete intersections in homogeneous spaces G_2/P_2. If G is a group of type G_2, there exist two maximal parabolic subgroups P_1 and P_2 in G. The quotient variety G/P_1 is isomorphic to a 5-dimensional quadric $Q \subset \mathbf{P}^6$, and G/P_2 is a Mukai variety of genus $g = 10$ (see Theorem 7):

$$G_2/P_2 \subset \mathbf{P}^{13}.$$

One has $b_4(G_2/P_2) = 1$ (see for instance [**And11**, Proposition 4.5 and Appendix A.3]), and G_2/P_2 is 2-Fano by [**AC12**, 5.7.].

Recall from [**Hwa01**, 1.4.5] the polarized minimal family of rational curves through a general point $y \in G_2/P_2$ is

$$(H_y, L_y) \cong (\mathbf{P}^1, \mathcal{O}(3)).$$

Let H denote the hyperplane class (the generator of the Picard group). We claim that

$$\mathrm{ch}_2(G_2/P_2) = \frac{1}{2}H^2.$$

This will follow from the following more general remark, applied to $Y = G_2/P_2$.

Remark 37 (Complete intersections in varieties with $\rho(Y) = b_4(Y) = 1$). Let Y be a Fano manifold with $\rho(X) = 1$, and H an ample generator of $\mathrm{Pic}(Y)$. Let $y \in Y$ be a general point, and (H_y, L_y) a polarized minimal family of rational curves through y, as defined in Section 5. Suppose that $\dim(H_y) \geq 1$. If $b_4(Y) = 1$, then the map

$$T_1 : \overline{\mathrm{NE}}_1(H_y) \to \overline{\mathrm{NE}}_2(Y)$$

is clearly surjective. Let $C \subset H_y$ be a complete curve, and $S = T_1([C])$ the corresponding surface class on Y. By (5.1), the second Chern character of Y is given by:

$$\mathrm{ch}_2(Y) = aH^2, \quad a \in \frac{1}{2}\mathbf{Z}, \quad a(H^2 \cdot S) = -(K_{H_y} - \frac{d}{2}L_y) \cdot C.$$

In particular, $a \leq -(K_{H_y} - \frac{d}{2}L_y) \cdot C$.

Now consider a complete intersection:

$$X = (d_1H) \cap \ldots \cap (d_cH) \subset Y.$$

The natural map $u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(Y)$ is surjective. Thus, by (3.1), X is 2-Fano (respectively weakly 2-Fano) if and only if it is Fano and $\sum d_i^2 < 2a$ (respectively $\leq 2a$).

We have the following consequence:

Proposition 38. *A linear section $H_1 \subset G_2/P_2$ is weakly 2-Fano, but not 2-Fano. A linear section $H_1 \cap H_2 \subset G_2/P_2$ is not 2-Fano.*

7. Fano manifolds with high index and $\rho = 1$

In this section we address n-dimensional Fano manifolds X with index $i_X \geq n-2$ and $\rho(X) = 1$. We also treat those with bigger Picard number for $n > 4$.

Recall from Section 3 that $\mathbf{P}^n$ and $Q^n \subset \mathbf{P}^{n+1}$ are 2-Fano for $n \geq 3$.

7.1. Del Pezzo manifolds. We go through the classification in Theorem 6. We first consider manifolds with $\rho = 1$.

7.1.1. **Degree $d = 5$.** We saw in Section 3.4 that the Grassmannian $G(2,5)$ is 2-Fano. Consider now a linear section

$$X = H_1 \cap \ldots \cap H_c \subset G(2,5) \quad (c \geq 1).$$

By Proposition 32(iv), if $c = 3$, the threefold X is not weakly 2-Fano. If $c = 1$, then X is weakly 2-Fano, but not 2-Fano. If $c = 2$, then X is not 2-Fano. We could not decide if in this case X is weakly 2-Fano. We raise the following:

Question 39. *Is a linear section $\mathbf{P}^7 \cap G(2,5) \subset \mathbf{P}^9$ weakly 2-Fano?*

7.1.2. **Degree $d = 4$.** By 3.3.1, a del Pezzo variety of type Y_4 is 2-Fano if and only if $n \geq 6$ and weakly 2-Fano if and only if $n \geq 5$.

7.1.3. **Degree $d = 3$.** By 3.3.1, a del Pezzo variety of type Y_3 is 2-Fano if and only if $n \geq 8$ and weakly 2-Fano if and only if $n \geq 7$.

7.1.4. **Degree $d = 2$.** By Corollary 11(ii) or 3.3.2, del Pezzo varieties of type Y_2 are 2-Fano (respectively weakly 2-Fano) if and only if $n > 11$ (respectively $n \geq 11$).

7.1.5. **Degree $d = 1$.** By Corollary 3.3.2, del Pezzo varieties of type Y_1 are (weakly) 2-Fano if and only if $n > 23$ ($n \geq 23$).

7.1.6. **Del Pezzo manifolds with $\rho > 1$.** All the del Pezzo manifolds with $\rho > 1$ are weakly 2-Fano but not 2-Fano. For $\mathbf{P}^2 \times \mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ this follows from Lemma 17, for $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$ from Corollary 14, and for $\mathbf{P}(T_{\mathbf{P}^2})$ from Example 15.

REMARK 40. We get the following classification of weakly 2-Fano del Pezzo manifolds:

- All the del Pezzo manifolds with $\rho > 1$ are weakly 2-Fano.
- The only del Pezzo manifolds with $\rho = 1$ that are weakly 2-Fano are:

($d = 5$) $G(2,5)$ and its linear sections of codimension 1 (and possibly codimension 2, see Question 39);
($d = 4$) Complete intersections of quadrics $Q \cap Q' \subset \mathbf{P}^{n+2}$ if $n \geq 5$;
($d = 3$) Cubic hypersurfaces $Y_3 \subset \mathbf{P}^{n+1}$ if $n \geq 7$;
($d = 2$) Degree 4 hypersurfaces in $\mathbf{P}(2, 1, \ldots, 1)$ if $n \geq 11$;
($d = 1$) Degree 6 hypersurfaces in $\mathbf{P}(3, 2, 1, \ldots, 1)$ if $n \geq 23$.

7.2. Mukai manifolds.

7.2.1. **Mukai manifolds of dimension > 4 and $\rho > 1$.** Recall that the only Mukai manifolds of dimension > 4 and $\rho > 1$ are $\mathbf{P}^3 \times \mathbf{P}^3$, $\mathbf{P}^2 \times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}^2)$. The manifolds $\mathbf{P}^3 \times \mathbf{P}^3$ and $\mathbf{P}^2 \times Q^3$ are weakly 2-Fano but not 2-Fano by Lemma 17, while $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}^2)$ are not weakly Fano by Example 15 and Example 16, respectively.

Next we go through the classification of Mukai manifolds with $\rho = 1$ in Theorem 7.

7.2.2. **Genus $g \leq 5$.** Consider the case of complete intersections. By 3.3.1:

- $X_4 \subset \mathbf{P}^{n+1}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 15$ (respectively $n \geq 14$).
- $X_{2\cdot 3} \subset \mathbf{P}^{n+2}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 11$ (respectively $n \geq 10$)
- $X_{2\cdot 2\cdot 2} \subset \mathbf{P}^{n+3}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 9$ (respectively $n \geq 8$).

Consider the case of double covers. Using Corollary 11, we have:

(i) A double cover $X \to \mathbf{P}^n$ branched a long a sextic is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 27$ (respectively $n \geq 26$).
(ii) A double cover $X \to Q \subset \mathbf{P}^{n+1}$ branched a long the intersection of the quadric Q with a quartic hypersurafce, is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 15$ (respectively $n \geq 14$).

7.2.3. **Genus 6.** A linear section X of Σ^6_{10} is isomorphic to one of the following ([**IP99**, Proposition 5.2.7]):

(i) A complete intersection in $G(2,5)$ of a linear subspace and a quadric.
(ii) A double cover of a smooth linear section Y of $G(2,5)$, branched along a quadric section B of Y.

In Case (i), it follows from Proposition 31 that X is not weakly 2-Fano. Consider now Case (ii). Set $c := \operatorname{codim}(Y)$. Then X is not weakly 2-Fano by Corollary

11, since we have:

$$\mathrm{ch}_2(Y) - \frac{3}{8}B^2 = \frac{3-c}{2}\sigma_2 + \frac{1-c}{2}\sigma_{1,1} - \frac{3}{8}(2\sigma_1)^2 = -\frac{c}{2}\sigma_2 - \frac{c+2}{2}\sigma_{1,1},$$
$$\big(\mathrm{ch}_2(Y) - \frac{3}{8}B^2\big) \cdot {\sigma_1}_{|Y}^{\dim(Y)-2} = \big(-\frac{c}{2}\sigma_2 - \frac{c+2}{2}\sigma_{1,1}\big) \cdot \sigma_1^4 < 0.$$

7.2.4. **Genus** 7**.** By 6.2.1, the manifold $OG_+(5,10)$ is 2-Fano and a linear section of codimension c is 2-Fano (respectively weakly 2-Fano) if and only if $c < 4$ (respectively $c \leq 4$).

7.2.5. **Genus** 8**.** We saw in Section 3.4 that the Grassmannian $G(2,6)$ is weakly 2-Fano. Let $X \subset G(2,6)$ be a linear section of codimension c. By Proposition 32, if $c \geq 4$ or $c = 1$, then X is not weakly 2-Fano. Assume that $c = 3$. Then

$$\mathrm{ch}_2(X) = \frac{1}{2}(\sigma_2 - 3\sigma_{1,1}).$$

By a straightforward calculation, $\sigma_1^6 = 9\sigma_2^* + 5\sigma_{1,1}^*$. It follows that

$$\mathrm{ch}_2(X) \cdot {\sigma_1}_{|X}^3 = \mathrm{ch}_2(X) \cdot \sigma_1^6 = -3 < 0.$$

In particular, X is not weakly 2-Fano.

Assume now that $c = 2$. Then $\mathrm{ch}_2(X) = \sigma_2 - \sigma_{1,1}$. By Example 30, the variety $H_x \subset \mathbf{P}^1 \times \mathbf{P}^3$ defined in Section 5 is isomorphic to a smooth quadric surface in $\mathbf{P}^3$ (via the projection π_2). The map $T_1 : \overline{\mathrm{NE}}_1(H_x) \to \overline{\mathrm{NE}}_2(G(k,n))$ sends the classes of the lines in the two rulings of the quadric surface H_x to the classes σ_2^* and $\sigma_2^* + \sigma_{1,1}^*$. In particular, X is not 2-Fano, as $\mathrm{ch}_2(X) \cdot (\sigma_2^* + \sigma_{1,1}^*) = 0$. We could not decide if in this case X is weakly 2-Fano. We raise the following:

Question 41. *Is a linear section $\mathbf{P}^{12} \cap G(2,6) \subset \mathbf{P}^{14}$ weakly 2-Fano?*

7.2.6. **Genus** 9**.** We saw in Section 6.3 that the symplectic Grassmannian $SG(k,2k)$ is 2-Fano. By 6.3.1, a codimension $c \geq 1$ linear section X of $SG(k,2k)$ is not 2-Fano. The only case when X is weakly 2-Fano is for $c = 1$.

7.2.7. **Genus** 10**.** We saw in Section 6.4 that the variety G_2/P_2 is 2-Fano. By Proposition 38, a codimension $c \geq 1$ linear section in G_2/P_2 is not 2-Fano and it is weakly 2-Fano if and only if $c = 1$.

7.2.8. **Genus** 12**.** By Remark 9,

$$c_1(\wedge^2(\mathcal{S}^*)) = 2\sigma_1, \quad \mathrm{ch}_2(\wedge^2(\mathcal{S}^*)) = \sigma_2.$$

It follows from Lemma 8 and the computation of $\mathrm{ch}(G(3,7))$ made in Section 3.4 that the Chern characters of the threefold $X = X_{22}$ are given by:

$$c_1(X) = \big(c_1\big(G(3,7)\big) - 3c_1\big(\wedge^2(\mathcal{S}^*)\big)\big)_{|X} = {\sigma_1}_{|X},$$
$$\mathrm{ch}_2(X) = \big(\mathrm{ch}_2\big(G(3,7)\big) - 3\mathrm{ch}_2\big(\wedge^2(\mathcal{S}^*)\big)\big)_{|X} = -\frac{3}{2}{\sigma_2}_{|X} + \frac{1}{2}{\sigma_{1,1}}_{|X}.$$

Since $\rho = 1$ and $\dim(X) = 3$, $b_4(X) = 1$. In particular, the restrictions ${\sigma_2}_{|X}$ and ${\sigma_{1,1}}_{|X}$ are multiples of the positive codimension 2-cycle

$$A := (\sigma_1^2)_{|X}.$$

We claim that

$${\sigma_{1,1}}_{|X} = \frac{5}{11}A, \quad {\sigma_2}_{|X} = \frac{6}{11}A.$$

To see this, it is enough to prove that

$$(\sigma_2 \cdot \sigma_1)_X = 12, \quad (\sigma_{1,1} \cdot \sigma_1)_X = 10,$$

where $(,)_X$ denotes the intersection on X. Since X is the zero locus of a global section of the rank 9 vector bundle $\mathcal{E} = (\wedge^2(\mathcal{S}^*))^{\oplus 3}$, it follows that

$$\big(\sigma_2 \cdot \sigma_1\big)_X = \sigma_2 \cdot \sigma_1 \cdot c_9(\mathcal{E}).$$

By a standard computation with Chern classes,

$$c_9(\mathcal{E}) = c_3(\wedge^2(\mathcal{S}^*))^3 = \big(c_1(\mathcal{S}^*)c_2(\mathcal{S}^*) - c_3(\mathcal{S}^*)\big)^3 = \sigma_{2,1}^3.$$

It is a straightforward exercise in Schubert calculus to check that

$$\sigma_{2,1}^3 = 4\sigma_{4,4,1} + 8\sigma_{4,3,2} + 2\sigma_{3,3,3}.$$

It follows that

$$\sigma_1 \cdot \sigma_{2,1}^3 = 12\sigma_2^* + 10\sigma_{1,1}^*.$$

Then $\mathrm{ch}_2(X) \cdot A = -13 < 0$. Hence, X_{22} is not weakly 2-Fano.

Remark 42. We get the following classification of weakly 2-Fano Mukai manifolds with $\rho = 1$:

(1) Complete intersection in projective spaces:
($g = 3$) Degree 4 hypersurfaces in $\mathbf{P}^{n+1}$ if $n \geq 15$;
($g = 4$) Complete intersections $X_{2\cdot 3} \subset \mathbf{P}^{n+2}$ if $n \geq 11$;
($g = 5$) Complete intersections $X_{2\cdot 2\cdot 2} \subset \mathbf{P}^{n+3}$ if $n \geq 9$.

(2) Complete intersection in weighted projective spaces:
($g = 2$) Degree 6 hypersurfaces in $\mathbf{P}(3, 1, \ldots, 1)$ if $n \geq 26$;
($g = 3$) Complete intersections of two quadrics in $\mathbf{P}(2, 1, \ldots, 1)$, $n \geq 14$.

(3) With genus $g \geq 6$:
($g = 7$) $OG_+(5, 10)$ and linear sections of codimension $c \leq 4$;
($g = 8$) $G(2, 6)$ and possibly a linear section of codimension 2 in $G(2, 6)$ (see Question 41);
($g = 9$) $SG(3, 6)$ and linear sections of codimension 1;
($g = 10$) G_2/P_2 and linear sections of codimension 1.

8. Fano threefolds with Picard number $\rho \geq 2$

By the results of Mori-Mukai [**MM81**] (see also [**MM03**]) there are 88 types of Fano threefolds with Picard number $\rho(X) \geq 2$, up to deformation. We will go through the list in [**MM81**] and check that none of them is 2-Fano. We point out those that are weakly 2-Fano. We recall the terminology and notation from [**MM81**]:

(i) V_d $(1 \leq d \leq 5)$ denotes a Fano 3-fold of index 2, with $\rho(X) = 1$ and degree d (See Theorem 6).

(ii) W is a smooth divisor of $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 1)$. It is isomorphic to the $\mathbf{P}^1$-bundle $\mathbf{P}(T_{\mathbf{P}^2})$ over $\mathbf{P}^2$, and appears as (32) in the following list.

(iii) The blow-up of $\mathbf{P}^3$ at a point is denoted by V_7. It appears as (35) in the following list. The smooth quadric in $\mathbf{P}^4$ is denoted by Q.

(iv) S_d $(1 \le d \le 7)$ is a del Pezzo surface of degree d. $\mathbf{F}_1$ is the blow-up of $\mathbf{P}^2$ at a point.

(v) All curves are understood to be smooth and irreducible, and all intersections are understood to be scheme theoretic.

(vi) A divisor D (respectively a curve C) on the product variety

$$M = \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_m}$$

is of multi-degree $(a_1, \ldots, a_m)$ if $\mathcal{O}_M(D) \cong \otimes_{i=1}^m \pi_i^* \mathcal{O}_{\mathbf{P}^{n_i}}(a_i)$ (respectively if $C \cdot \pi_i^* \mathcal{O}_{\mathbf{P}^{n_i}}(a_i) = a_i$ for all $i = 1, \ldots, m$), where π_i is the projection of M onto the i-th factor.

8.1. Fano 3-folds with $\rho = 2$. We go through the list in [**MM81**, Table 2] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) The blow-up of V_1 with center an elliptic curve which is an intersection of two members of $\left| -\frac{1}{2}K_{V_1} \right|$. This is not weakly 2-Fano by Corollary 26.

(**2**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2, 4)$. Since $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano, this is not weakly 2-Fano by Corollary 11(ii).

(**3**) The blow-up of V_2 with center an elliptic curve which is an intersection of two members of $\left| -\frac{1}{2}K_{V_2} \right|$. This is not weakly 2-Fano by Corollary 26.

(**4**) The blow-up of $\mathbf{P}^3$ with center an intersection of two cubics. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**5**) The blow-up of $V_3 \subset \mathbf{P}^4$ with center a plane cubic on it. This is not weakly 2-Fano by Corollary 26.

(**6a**) A divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(2, 2)$ is not weakly 2-Fano by Example 18.

(**6b**) A double cover of W whose branch locus is a member of $|-K_W|$. Since $W = \mathbf{P}(T_{\mathbf{P}^2})$ is not 2-Fano by Lemma 13, its double cover is not weakly 2-Fano by Corollary 11(ii).

(**7**) The blow-up of $Q \subset \mathbf{P}^4$ with center an intersection of two members of $\left|\mathcal{O}_Q(2)\right|$. Since Q is a Fano threefold with index 3, this is not weakly 2-Fano by Corollary 26.

(**8**) A double cover of V_7 whose branch locus is a member B of $|-K_{V_7}|$. Since $V_7 = \mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ is not 2-Fano by Lemma 13, this 3-fold is not weakly 2-Fano by Corollary 11(ii).

(**9**) The blow-up of $\mathbf{P}^3$ with center a curve of degree 7 and genus 5 which is an intersection of cubics. This is not weakly 2-Fano by Corollary 26.

(**10**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center an elliptic curve which is an intersection of two hyperplane sections. This is not weakly 2-Fano by Corollary 26.

(**11**) The blow-up of $V_3 \subset \mathbf{P}^4$ with center a line on it. Since V_3 has Picard number 1 and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**12**) The blow-up of $\mathbf{P}^3$ with center a curve of degree 6 and genus 3 which is an intersection of cubics. This is not weakly 2-Fano by Corollary 26.

(**13**) The blow-up of $Q \subset \mathbf{P}^4$ with center a curve of degree 6 and genus 2. This is not weakly 2-Fano by Corollary 26.

(**14**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center an elliptic curve which is an intersection of two hyperplane sections. This is not weakly 2-Fano by Corollary 26.

(**15**) The blow-up of $\mathbf{P}^3$ with center an intersection of a quadric A and a cubic B. This is not weakly 2-Fano by Corollary 26.

(**16**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center a conic on it. Since V_4 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**17**) The blow-up of $Q \subset \mathbf{P}^4$ with center an elliptic curve of degree 5 on it. This is not weakly 2-Fano by Corollary 26.

(**18**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$. Since $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano by Lemma 17, this is not weakly 2-Fano by Corollary 11(ii).

(**19**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center a line on it. Since V_4 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**20**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a twisted cubic on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**21**) The blow-up of $Q \subset \mathbf{P}^4$ with center a twisted quartic (i.e., a smooth rational curve of degree 4 which spans $\mathbf{P}^4$) on it. Since Q is a Fano threefold with index 3, this is not weakly 2-Fano by Corollary 26.

(**22**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a twisted conic on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**23**) The blow-up of $Q \subset \mathbf{P}^4$ with center an intersection of $A \in |\mathcal{O}_Q(1)|$ and $B \in |\mathcal{O}_Q(2)|$. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**24**) A divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 2)$ is not weakly 2-Fano by Example 18.

(**25**) The blow-up of $\mathbf{P}^3$ with center an elliptic curve which is an intersection of two quadrics. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**26**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a line on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 28.

(**27**) The blow-up of $\mathbf{P}^3$ with center a twisted cubic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**28**) The blow-up of $\mathbf{P}^3$ with center a plane cubic. This is not weakly 2-Fano by Corollary 26.

(**29**) The blow-up of $Q \subset \mathbf{P}^4$ with center a conic on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**30**) The blow-up of $\mathbf{P}^3$ with center a conic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**31**) The blow-up of $Q \subset \mathbf{P}^4$ with center a line on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**32**) $W \cong \mathbf{P}(T_{\mathbf{P}^2})$. This is not 2-Fano, but weakly 2-Fano by Example 15.

(**33**) The blow-up of $\mathbf{P}^3$ with center a line. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**34**) The product $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano, but weakly 2-Fano by Lemma 17.

(**35**) $V_7 \cong \mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$. This is not 2-Fano, but weakly 2-Fano by Corollary 14.

(**36**) The blow-up of the Veronese cone $W_4 \subseteq \mathbf{P}^6$ with center the vertex, that is the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbf{P}^2$. This is not 2-Fano, but weakly 2-Fano by Corollary 14.

8.2. Fano 3-folds with $\rho = 3$. We go through the list in [**MM81**, Table 3] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus is a divisor of tridegree $(2, 2, 2)$. Since $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is not 2-Fano by Lemma 17, this is not weakly 2-Fano by Corollary 11(ii).

(**2**) A member X of the linear system $|\mathcal{O}_\pi(1)^{\otimes 2} \otimes \mathcal{O}(2,3)|$ on the $\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2})$ over $\mathbf{P}^1 \times \mathbf{P}^1$ such that $X \cap Y$ is irreducible, where Y is a member of $|\mathcal{O}_\pi(1)|$.

We prove that X is not weakly 2-Fano by a direct computation. Set $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}$ and let $\pi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1 \times \mathbf{P}^1$ be the natural projection. If π_1, π_2 are the two projections from $\mathbf{P}^1 \times \mathbf{P}^1$, we set $H_i = \pi_i^*\mathcal{O}(1)$ $(i = 1, 2)$. Set $\xi = c_1(\mathcal{O}_\pi(1))$. By Lemma 12 and formula (3.1),

$$\mathrm{ch}_2(\mathbf{P}(\mathcal{E})) = 6\pi^*(H_1 \times H_2) + 2\pi^*(H_1 + H_2) \cdot \xi + \frac{3}{2}\xi^2,$$

$$\mathrm{ch}_2(X) = \big(2\pi^*(-H_1 - 2H_2) \cdot \xi - \frac{1}{2}\xi^2\big)_{|X}.$$

We claim that $\mathrm{ch}_2(X) \cdot (\pi^* H_1)_{|X} < 0$. This is a direct computation:

$$\mathrm{ch}_2(X) \cdot (\pi^* H_1)_X = \mathrm{ch}_2(X) \cdot \pi^* H_1 \cdot \big(\pi^*(2H_1 + 3H_2) + 2\xi\big) = -\frac{15}{2}.$$

(**3**) A divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of tridegree $(1, 1, 2)$ is not weakly 2-Fano by Example 19.

(**4**) The blow-up of Y (No. (18) in the list for $\rho = 2$) with center a smooth fiber of $Y \to \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$. Recall that $Y \to \mathbf{P}^1 \times \mathbf{P}^2$ is a double cover branched along a divisor of bidegree $(2, 2)$. Apply Corollary 29 to deduce that this is not weakly 2-Fano.

(**5**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a curve C of bidegree $(5, 2)$ such that the composition $C \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$ is an embedding. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C = 16 > 2$, this is not weakly 2-Fano by Corollary 26.

(**6**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and an elliptic curve of degree 4. This is not weakly 2-Fano by Corollary 26.

(**7**) The blow-up of W with center an elliptic curve of degree 4. This is not weakly 2-Fano by Corollary 26.

(**8**) A member X of the linear system $|\pi_1^* g^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(2)|$ on $\mathbf{F}_1 \times \mathbf{P}^2$, where π_i $(i = 1, 2)$ is the projection onto the i-th factor and $g : \mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up map. We prove that X is not weakly 2-Fano by a direct computation. Set $h_1 := c_1\big(\pi_1^* g^* \mathcal{O}(1)\big)$ and $h_2 := c_1\big(\pi_2^* \mathcal{O}(1)\big)$. By (4.3.1), $\mathrm{ch}_2(\mathbf{F}_1) = 0$. By (3.1), we have:

$$\mathrm{ch}_2(X) = \big(\mathrm{ch}_2(\mathbf{F}_1 \times \mathbf{P}^2) - \frac{1}{2}X^2\big)_{|X} = \big(\frac{3}{2}h_2^2 - \frac{1}{2}(h_1 + 2h_2)^2\big)_{|X} =$$

$$= \big(-\frac{1}{2}h_1^2 - \frac{1}{2}h_2^2 - 2h_1 h_2\big)_{|X}.$$

We claim that $\mathrm{ch}_2(X) \cdot h_{2|X} < 0$. This is a direct computation:

$$\mathrm{ch}_2(X) \cdot h_{2|X} = \mathrm{ch}_2(X) \cdot h_2 \cdot X = \mathrm{ch}_2(X) \cdot h_2 \cdot (h_1 + 2h_2) = -3.$$

(**9**) The blow-up of the cone $W_4 \subset \mathbf{P}^6$ over the Veronese surface $R_4 \subset \mathbf{P}^5$ with center a disjoint union of the vertex and a quartic curve in $R_4 \cong \mathbf{P}^2$. Since the center of the blow-up is a curve of genus 3, this is not weakly 2-Fano by Corollary 26.

(**10**) The blow-up of $Q \subset \mathbf{P}^4$ with center a disjoint union of two conics on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**11**) The blow-up of V_7 with center an elliptic curve which is an intersection of two members of of $|-\frac{1}{2}K_{V_7}|$. This is not weakly 2-Fano by Corollary 26.

(**12**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and a twisted cubic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**13**) The blow-up of $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ with center a curve C of bidegree $(2,2)$ on it such that the composition of $C \hookrightarrow W \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ with the projection π_i : $\mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ is an embedding for both $i = 1, 2$. Since $-K_W \cdot C = 8$, this is not weakly 2-Fano by Corollary 26.

(**14**) The blow-up of $\mathbf{P}^3$ with center a union of a cubic in a plane S and a point not in S. This is not weakly 2-Fano by Corollary 26.

(**15**) The blow-up of $Q \subset \mathbf{P}^4$ with center a disjoint union of a line and a conic on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**16**) The blow-up of V_7 with center the strict transform of a twisted cubic passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. Since $-K_{V_7} \cdot C = 10$, this is not weakly 2-Fano by Corollary 26.

(**17**) A smooth divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of tridegree $(1,1,1)$ is not weakly 2-Fano by Example 19.

(**18**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and a conic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**19**) The blow-up X of $Q \subset \mathbf{P}^4$ with center two points p and q on it which are not collinear. By 3.3.1,

$$\mathrm{ch}_2(Q) = \frac{1}{2} h^2_{|Q}.$$

It $\tilde{T}$ is the proper transform of a general hyperplane section T of Q that passes through p, by Lemma 25, we have

$$\mathrm{ch}_2(X) \cdot \tilde{T} = \mathrm{ch}_2(Q) \cdot T - 2 = \frac{1}{2} h^3 - 2 = -1.$$

In particular, X is not weakly 2-Fano.

Remark 43. Moreover, note that $\tilde{T}$ is a base-point free divisor on X. It follows from Corollary 27 that no blow-up of X along points and disjoint smooth curves is weakly 2-Fano.

(**20**) The blow-up of $Q \subset \mathbf{P}^4$ with center two disjoint lines on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 26.

(**21**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a curve C of bidegree $(2,1)$. Since $-K_{\mathbf{P}^1\times\mathbf{P}^2} \cdot C = 7$, this is not weakly 2-Fano by Corollary 26.

(**22**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a conic C in $\{t\} \times \mathbf{P}^2$ ($t \in \mathbf{P}^1$). Since $-K_{\mathbf{P}^1\times\mathbf{P}^2} \cdot C = 6$, this is not weakly 2-Fano by Corollary 26.

(**23**) The blow-up of V_7 with center a conic C passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. Recall that V_7 is the blow-up of $\mathbf{P}^3$ at a point. Since $-K_{V_7} \cdot C = 6$, this is not weakly 2-Fano by Corollary 26.

(**24**) The fiber product $X = W \times_{\mathbf{P}^2} \mathbf{F}_1$ where $W \to \mathbf{P}^2$ is the $\mathbf{P}^1$-bundle $\mathbf{P}(T_{\mathbf{P}^2})$ and $\pi : \mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up map. This is not weakly 2-Fano: Since $X = \mathbf{P}_{\mathbf{F}_1}(\pi^* T_{\mathbf{P}^2})$, by Lemma 13, $\mathrm{ch}_2(X) \geq 0$ if and only if

$$\mathrm{ch}_2(\mathbf{F}_1) + \frac{1}{2}\pi^*\big(c_1(\mathbf{P}^2)^2 - 4c_2(\mathbf{P}^2)\big) \geq 0.$$

By Lemma 22, $\mathrm{ch}_2(\mathbf{F}_1) = \pi^*\mathrm{ch}_2(\mathbf{P}^2) + \frac{3}{2}E^2$, where E is the exceptional divisor of $\mathbf{F}_1$. Hence, X is not weakly 2-Fano, since

$$\pi^*\big(c_1(\mathbf{P}^2)^2 - 3c_2(\mathbf{P}^2)\big) + \frac{3}{2}E^2 = -\frac{3}{2} < 0.$$

(**25**) The blow-up of $\mathbf{P}^3$ with center two disjoint lines, that is, $\mathbf{P}(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ over $\mathbf{P}^1 \times \mathbf{P}^1$. This is not weakly 2-Fano by Corollary 14.

(**26**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a point and a line. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**27**) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is weakly 2-Fano and not 2-Fano by Lemma 17.

(**28**) $\mathbf{P}^1 \times \mathbf{F}_1$ is weakly 2-Fano and not 2-Fano by Lemma 17.

(**29**) The blow-up X of V_7 with center a line L on the exceptional divisor $E \cong \mathbf{P}^2$ of the blow-up $\pi : V_7 \to \mathbf{P}^3$ at a point p. The line L corresponds to a plane $\Lambda \subset \mathbf{P}^3$ passing through p. By Lemma 22, we have:

$$\mathrm{ch}_2(V_7) = 2(\pi^*h)^2 + 2E^2.$$

Let T be a plane through the point p, different from Λ. The proper transform $\tilde{T}$ of T in X intersects L in a point. By Lemma 25,

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(V_7) \cdot T - \frac{3}{2} = (2(\pi^*h)^2 + 2E^2) \cdot (\pi^*h - E) - \frac{3}{2} = -\frac{3}{2}.$$

In particular, $\tilde{X}$ is not weakly 2-Fano.

(**30**) The blow-up X of V_7 along the proper transform of a line l passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. We denote by $\pi : X \to \mathbf{P}^3$ the composition of the two blowups. By Lemma 23, we have $\mathrm{ch}_2(X) \cdot E = 0$, where E is the exceptional divisor over l. In particular, X is not 2-Fano.

We now prove that X is weakly 2-Fano. Since X is a toric variety, the cone $\overline{\mathrm{NE}}_2(X)$ is generated by the exceptional divisor E, the proper transform E' of the exceptional divisor of the blow-up $V_7 \to \mathbf{P}^3$ and the proper transform T of a plane in $\mathbf{P}^3$ that contains l. To prove that X is weakly 2-Fano, it is therefore enough to prove that $\mathrm{ch}_2(X) \cdot E' \geq 0$ and $\mathrm{ch}_2(X) \cdot T \geq 0$. Using Lemma 23 and Lemma 24, one easily computes that $\mathrm{ch}_2(X) \cdot E' = \frac{1}{2}$ and $\mathrm{ch}_2(X) \cdot T = 0$. Hence, X is weakly 2-Fano.

(**31**) The blow-up of the cone over a smooth quadric surface in $\mathbf{P}^3$ with center the vertex, that is, the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1,1))$ over $\mathbf{P}^1 \times \mathbf{P}^1$. This is weakly 2-Fano and not 2-Fano by Corollary 14.

8.3. Fano 3-folds with $\rho = 4$. We go through the list in [**MM81**, Table 4], [**MM03**] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) A smooth divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ of tridegree $(1,1,1,1)$ is not weakly 2-Fano by Example 20.

(**2**) The blow-up of the cone over a smooth quadric surface $S \subset \mathbf{P}^3$ with center a disjoint union of the vertex and an elliptic curve on S. This is not weakly 2-Fano by Corollary 26.

(**3**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve of tridegree $(1,1,2)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 8$, this is not weakly 2-Fano by Corollary 26.

(**4**) The blow-up of X (No. (19) in the list for $\rho = 3$) with center the strict transform of a conic on Q passing through p and q. This is not weakly 2-Fano by Remark 43.

(**5**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center two disjoint curves C_1 and C_2 of bidegree $(2,1)$ and $(1,0)$ respectively. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C_1 = 7$, this is not weakly 2-Fano by Corollary 26.

(**6**) The blow-up of $\mathbf{P}^3$ with center three disjoint lines, that is, the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center the tridiagonal curve. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 26.

(**7**) The blow-up of $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ with center two disjoint curves C_1 and C_2 of bidegree $(0,1)$ and $(1,0)$. Since $-K_W \cdot C_i = 3$, this is not weakly 2-Fano by Corollary 26.

(**8**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve C of tridegree $(0,1,1)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 4$, this is not weakly 2-Fano by Corollary 26.

(**9**) The blow-up X of Y (No. (25) in the list for $\rho = 3$) with center an exceptional line of the blowing up $Y \to \mathbf{P}^3$. Recall that Y is the blow-up of $\mathbf{P}^3$ along two disjoint lines. If T is the proper transform in Y of a plane in $\mathbf{P}^3$

intersecting the two lines at general points, then by Corollary 25 we have:

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H - 3 = -1.$$

Since T is disjoint from the exceptional line blown-up,

$$\mathrm{ch}_2(X) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -1.$$

In particular, X is not weakly 2-Fano.

(**10**) $\mathbf{P}^1 \times S_7$ is not weakly 2-Fano by 4.3.1 and Lemma 17.

(**11**) The blow-up X of $\mathbf{P}^1 \times \mathbf{F}_1$ with center $\{t\} \times C$, where $t \in \mathbf{P}^1$ and C is the exceptional curve of the first kind on $\mathbf{F}_1$. If $\mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up of a point $p \in \mathbf{P}^2$, let T be the surface $\mathbf{P}^1 \times L$, where L is the proper transform of a general line through the point p. Since T intersects $\{t\} \times C$ in one point, it follows from Corollary 25, that

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(\mathbf{P}^1 \times \mathbf{F}_1) \cdot T - \frac{3}{2} = -\frac{3}{2}.$$

In particular, X is not weakly 2-Fano.

(**12**) The blow-up X of Y (No. (33) in the list for $\rho = 2$) with center two exceptional lines of the blowing-up $Y \to \mathbf{P}^3$ along a line L. Let T be the proper transform on Y of a plane in $\mathbf{P}^3$ that contains L. It follows from Lemma 22 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H + \frac{3}{2}\left(L^2\right)_T - \deg(N_{L|\mathbf{P}^3}) = \frac{3}{2}.$$

Let $\tilde{T}$ be the proper transform of T in X. Since $\tilde{T}$ intersects the blown-up curves in two points, it follows by Corollary 25 that

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T - 3 = -\frac{3}{2}.$$

In particular, X is not weakly 2-Fano.

(**13**) (See [**MM03**].) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve of tridegree $(1, 1, 3)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 10$, this is not weakly 2-Fano by Corollary 26.

8.4. Fano 3-folds with $\rho \geq 5$. We go through the list in [**MM81**, Table 3] and check that each Fano 3-fold in the list is not weakly 2-Fano.

(**1**) The blow-up X of Y (No. (29) in the list for $\rho = 2$) with center three exceptional lines of the blowing-up $Y \to Q$ along a conic C.

Let T be the proper transform on Y of a general hyperplane section of Q. Note that T will intersect C in two points. It follows from Corollary 25 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(Q) \cdot H - 3 = -2.$$

Since T is disjoint from the three exceptional lines of the blow-up, it follows that $\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -2$. In particular, X is not weakly 2-Fano.

(**2**) The blow-up X of Y (No. (25) in the list for $\rho = 3$) with center two exceptional lines l, l' of the blow-up

$$\phi : Y \to \mathbf{P}^3$$

such that l, l' lie on the same irreducible component of the exceptional set of ϕ. Recall that ϕ is the blow-up of two disjoint lines L_1 and L_2 in $\mathbf{P}^3$.

Let T be the proper transform on Y of a general plane in $\mathbf{P}^3$ that intersects both L_1 and L_2. It follows from Corollary 25 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H - 3 = -1.$$

Since T is disjoint from l and l', $\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -1$. In particular, X is not weakly 2-Fano.

(**3**), (**4**) $\mathbf{P}^1 \times S_d$ is not weakly 2-Fano if $d \leq 6$, by 4.3.1 and Lemma 17.

9. Fano fourfolds with index $i \geq 2$ and Picard number $\rho \geq 2$

By Theorem 6, the only Fano fourfold with index 3 and $\rho > 1$ is $\mathbf{P}^2 \times \mathbf{P}^2$, which is weakly 2-Fano, but not 2-Fano by Lemma 17. The classification of Fano fourfolds of index 2 and $\rho > 1$ can be found in [**IP99**, Table 12.7]. We go through this list, check that none of them is 2-Fano, and point out the cases in which the 4-fold is weakly 2-Fano. We use the same notation as in the previous section.

(**1**) $\mathbf{P}^1 \times V_1$. This is not weakly 2-Fano by Lemma 17.

(**2**) $\mathbf{P}^1 \times V_2$. This is not weakly 2-Fano by Lemma 17.

(**3**) $\mathbf{P}^1 \times V_3$. This is not weakly 2-Fano by Lemma 17.

(**4**) A double cover of $\mathbf{P}^2 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$. Since $\mathbf{P}^2 \times \mathbf{P}^2$ is not 2-Fano by Lemma 17, this is not weakly 2-Fano by Corollary 11(ii).

(**5**) A divisor of $\mathbf{P}^2 \times \mathbf{P}^3$ of bidegree $(1, 2)$. This is not weakly 2-Fano by Example 18.

(**6**) $\mathbf{P}^1 \times V_4$. This is not weakly 2-Fano by Lemma 17.

(**7**) An intersection Y of two divisors of bidegree $(1, 1)$ on $\mathbf{P}^3 \times \mathbf{P}^3$. This is not weakly 2-Fano by Example 21. With the same notation as in the example:

$$\mathrm{ch}_2(Y) \cdot (h_1 \cdot h_2)_{|Y} = -2.$$

(**8**) A divisor of $\mathbf{P}^2 \times Q^3$ of bidegree $(1, 1)$. By making a computation similar to those in 4.2.1, one can check that this is not weakly 2-Fano.

(**9**) $\mathbf{P}^1 \times V_5$. This is not weakly 2-Fano by Lemma 17.

(**10**) The blow-up of Q^4 along a conic C which is not contained in a plane lying on Q^4. We claim that this is not weakly 2-Fano.

The normal bundle of C in Q^4 is $N \cong \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. Let $\pi : X \to Q^4$ denote the blow-up, and $E \cong \mathbf{P}(N^*)$ the exceptional divisor. Consider the surface S in E, ruled over C, corresponding to a surjection

$$N^* \cong \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2) \twoheadrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-2).$$

Using the formula for ch_2 from Lemma 22, one gets that $\mathrm{ch}_2(X) \cdot S = -2$.

(**11**) $\mathbf{P}_{\mathbf{P}^3}(\mathcal{E})$, where $\mathcal{E}$ is the null-correlation bundle on $\mathbf{P}^3$. Recall that $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = h^2$. Therefore this is weakly 2-Fano but not 2-Fano by Lemma 13.

(**12**) The blow-up of $Q^4 \subset \mathbf{P}^5$ along a line ℓ. We claim that this is not weakly 2-Fano.

The normal bundle of ℓ in Q^4 is $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}$. Let $\pi : X \to Q^4$ denote the blow-up, and $E \cong \mathbf{P}(N^*)$ the exceptional divisor. Consider the surface S in E, ruled over ℓ, corresponding to the surjection

$$N^* \cong \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \twoheadrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Using the formula for ch_2 from Lemma 22, one gets that $\mathrm{ch}_2(X) \cdot S = -2$.

(**13**) $\mathbf{P}_{Q^3}(\mathcal{O}(-1) \oplus \mathcal{O})$, where $Q^3 \subset \mathbf{P}^4$ is a smooth quadric. This is weakly 2-Fano but not 2-Fano by Lemma 13.

(**14**) $\mathbf{P}^1 \times \mathbf{P}^3$. This is weakly 2-Fano but not 2-Fano by Lemma 17.

(**15**) $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$. This is weakly 2-Fano but not 2-Fano by Corollary 14.

(**16**) $\mathbf{P}^1 \times W$. This is weakly 2-Fano but not 2-Fano by Lemma 17.

(**17**) $\mathbf{P}^1 \times V_7$. This is weakly 2-Fano but not 2-Fano by Lemma 17.

(**18**) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. This is weakly 2-Fano but not 2-Fano by Lemma 17.

10. Proof of the main theorem

This section contains a guideline to the proof of Theorems 3 and 4.

The case of n-dimensional Fano manifolds with index $i_X \geq n-2$, except Fano threefolds and fourfolds with Picard number ≥ 2, is treated in Section 7. There are two cases to consider: del Pezzo manifolds (Section 7.1) and Mukai manifolds (Section 7.2).

We refer to Theorem 6 for a classification of del Pezzo manifolds. Del Pezzo manifolds with $\rho > 1$ are analyzed in 7.1.6. The rest of Section 7.1 analyzes del Pezzo manifolds with $\rho = 1$. A complete list of weakly 2-Fano del Pezzo manifolds can be found in Remark 40.

We refer to Theorem 7 for a classification of Mukai manifolds with $\rho = 1$. If $n \geq 5$, then n-dimensional Mukai manifolds have Picard number $\rho = 1$, except in the cases of $\mathbf{P}^3 \times \mathbf{P}^3$, $\mathbf{P}^2 \times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}^2)$ (see Section 2 and the remarks preceding Theorem 7). The latter cases are treated in 7.2.1. The rest of

Section 7.2 analyzes Mukai manifolds with $\rho = 1$. A complete list of weakly 2-Fano Mukai manifolds with $\rho = 1$ can be found in Remark 42.

We analyze separately Fano threefolds with $\rho > 1$ in Section 8 and Fano fourfolds with $\rho > 1$ in Section 9.

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Clay Mathematics Proceedings
Volume **18**, 2013

Abelian varieties associated to Gaussian lattices

Arnaud Beauville

To Joe Harris on his 60th birthday

Abstract. We associate to a unimodular lattice Γ, endowed with an automorphism i of square -1, a principally polarized abelian variety $A_\Gamma = \Gamma_\mathbb{R}/\Gamma$. We show that the configuration of i-invariant theta divisors of A_Γ follows a pattern very similar to the classical theory of theta characteristics; as a consequence we find that A_Γ has a high number of vanishing thetanulls. When $\Gamma = E_8$ we recover the 10 vanishing thetanulls of the abelian fourfold discovered by R. Varley.

Introduction

A *Gaussian lattice* is a free, finitely generated $\mathbb{Z}[i]$-module Γ with a positive hermitian form $\Gamma \times \Gamma \to \mathbb{Z}[i]$. Equivalently, we can view Γ as a lattice over $\mathbb{Z}$ endowed with an automorphism i of square -1_Γ. This gives a complex structure on the vector space $\Gamma_\mathbb{R} := \Gamma \otimes_\mathbb{Z} \mathbb{R}$; we associate to Γ the complex torus $A_\Gamma := \Gamma_\mathbb{R}/\Gamma$.

As a complex torus A_Γ is isomorphic to E^g, where E is the complex elliptic curve $\mathbb{C}/\mathbb{Z}[i]$ and $g = \frac{1}{2}\,\mathrm{rk}_\mathbb{Z}\,\Gamma$. More interestingly, the hermitian form provides a *polarization* on A_Γ (see (1.3) below); in particular, if Γ is unimodular, A is a principally polarized abelian variety (p.p.a.v. for short), which is indecomposable if Γ is indecomposable.

The first non-trivial case is $g = 4$, with Γ the root lattice of type E_8 (Example 1.2.1). The resulting p.p.a.v. is the abelian fourfold discovered by Varley [**V**] with a different (and more geometric) description; it has 10 "vanishing thetanulls" (even theta functions vanishing at 0), the maximum possible for a 4-dimensional indecomposable p.p.a.v. In fact this property characterizes the Varley fourfold outside the hyperelliptic Jacobian locus [**D**].

Our aim is to explain this property from the lattice point of view, and to extend it to all unimodular lattices. It turns out that we can mimic the classical theory of theta characteristics, replacing the automorphism (-1) by i. We will show:

• The group A_i of i-invariant points of A_Γ is a vector space of dimension g over $\mathbb{Z}/2$; it admits a natural non-degenerate bilinear symmetric form b.

• The set of i-invariant theta divisors of A_Γ is an affine space over A_i, isomorphic to the space of quadratic forms on A_i associated to b (see (2.1)).

• Let Θ be an i-invariant theta divisor, and Q the corresponding quadratic form. The multiplicity $m_0(\Theta)$ of Θ at 0 satisfies

$$2m_0(\Theta) \equiv \sigma(Q) + g \pmod{8},$$

where σ is the *Brown invariant* of the form Q (2.1).

As a consequence, we obtain a high number of i-invariant divisors Θ with $m_0(\Theta) \equiv 2 \pmod{4}$; each of them corresponds to a vanishing thetanull. When Γ is even, this number is $2^{\frac{g}{2}-1}(2^{\frac{g}{2}} - (-1)^{\frac{g}{4}})$; for $g = 4$ we recover the 10 vanishing thetanulls of the Varley fourfold.

1. Gaussian lattices

1.1. Lattices. As recalled in the Introduction, a Gaussian lattice is a free finitely generated $\mathbb{Z}[i]$-module Γ endowed with a positive hermitian form[1] $H : \Gamma\times\Gamma \to \mathbb{Z}[i]$. We write $H(x,y) = S(x,y) + iE(x,y)$; S and E are $\mathbb{Z}$-bilinear forms on Γ, S is symmetric, E is skew-symmetric, and we have

$$S(ix,iy) = S(x,y) \quad , \quad E(ix,iy) = E(x,y) \quad , \quad E(x,y) = S(ix,y)\ .$$

We will rather view a Gaussian lattice as an ordinary lattice (over $\mathbb{Z}$) with an automorphism i such that $i^2 = -1_\Gamma$: the last formula above defines E, and we have $H = S + iE$.

We have $\det S = \det E = (\det H)^2$; the lattice is *unimodular* when these numbers are equal to 1. It is *even* if $S(x,x)$ is even for all $x \in \Gamma$. We say that Γ is *indecomposable over* $\mathbb{Z}[i]$ if it cannot be written as the orthogonal sum of two nonzero Gaussian lattices; this is of course the case if Γ is indecomposable over $\mathbb{Z}$, but the converse is false (Example 3 below).

1.2. Examples. 1) For g even, the lattice Γ_{2g} is

$$\Gamma_{2g} := \{(x_j) \in \mathbb{R}^{2g} \mid x_j \in \frac{1}{2}\mathbb{Z}\ ,\ x_j - x_k \in \mathbb{Z}\ ,\ \sum x_j \in 2\mathbb{Z}\}\ .$$

The inner product is inherited from the euclidean structure of $\mathbb{R}^{2g}$, and the automorphism i is given in the standard basis (e_j) by

$$ie_{2j-1} = e_{2j} \qquad ie_{2j} = -e_{2j-1} \quad \text{for} \ \ 1 \le j \le g\ .$$

The lattice Γ_{2g} is unimodular, indecomposable when $g > 2$, and even if g is divisible by 4. The first case $g = 4$ gives the root lattice E_8.

The automorphism i is *unique* up to conjugacy: for $g = 4$ this is classical [**C**], and for $g \ge 6$ this follows easily from the fact that $\mathrm{Aut}(\Gamma_{2g})$ is the semi-direct product $(\mathbb{Z}/2)^{2g-1} \rtimes \mathfrak{S}_{2g}$, acting by permutation and even changes of sign of the basis vectors (e_j).

[1] Our convention is that $H(x,y)$ is $\mathbb{C}$-linear in y.

2) The Leech lattice Λ_{24} admits an automorphism of square -1 [**C-S**], also unique up to conjugacy.

3) Let Γ_0 be a lattice, and $\Gamma := \Gamma_0 \otimes_{\mathbb{Z}} \mathbb{Z}[i]$. The inner product of Γ_0 extends to an hermitian inner product on Γ, which is then a gaussian lattice. If Γ_0 is unimodular, resp. even, resp. indecomposable, Γ is unimodular, resp. even, resp. indecomposable over $\mathbb{Z}[i]$.

1.3. The abelian variety A_Γ. Let Γ be a Gaussian lattice, of rank $2g$ over $\mathbb{Z}$. We put $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $A_\Gamma := \Gamma_{\mathbb{R}}/\Gamma$. The automorphism i defines a complex structure on $\Gamma_{\mathbb{R}}$, so that A_Γ is a complex torus. Since Γ is a free $\mathbb{Z}[i]$-module, A_Γ is isomorphic to E^g, where E is the complex elliptic curve $\mathbb{C}/\mathbb{Z}[i]$.

The positive hermitian form H extends to $\Gamma_{\mathbb{R}}$, and its imaginary part E takes integral values on Γ: this is by definition a *polarization* on A_Γ. The polarization is principal if and only if Γ is unimodular; the p.p.a.v. A_Γ is indecomposable (i.e. is not a product of two nontrivial p.p.a.v.) if and only if Γ is indecomposable over $\mathbb{Z}[i]$.

The multiplication by i on $\Gamma_{\mathbb{R}}$ induces an automorphism of A_Γ, that we simply denote i. Conversely, let $A = V/\Gamma$ be a complex torus, of dimension g, with an automorphism inducing on $T_0(A) = V$ the multiplication by i. Then Γ is a $\mathbb{Z}[i]$-module, thus isomorphic to $\mathbb{Z}[i]^g$, so that A is isomorphic to E^g; polarizations of A correspond bijectively to positive hermitian forms on Γ.

2. Linear algebra over $(\mathbb{Z}/2)[i]$

2.1. Linear algebra over $\mathbb{Z}/2$. We consider a vector space V over $\mathbb{Z}/2$, of dimension g, with a non-degenerate symmetric bilinear form b on V. The form $x \mapsto b(x,x)$ is linear. Two different situations may occur:

- $b(x,x) = 0$ for all $x \in V$; in that case b is a symplectic form.
- $b(x,x)$ is not identically zero; it is then easy (using induction on g) to prove that V admits an orthonormal basis with respect to b.

A *quadratic form associated to* b is a function $q : V \to \mathbb{Z}/4$ such that

$$q(x+y) = q(x) + q(y) + 2b(x,y) \qquad \text{for } x, y \in V ,$$

where multiplication by 2 stands for the isomorphism $\mathbb{Z}/2 \xrightarrow{\sim} 2\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$.

Observe that this implies $q(0) = 0$ and $q(x) \equiv b(x,x) \pmod{2}$. We denote by $\mathcal{Q}_b$ the set of quadratic forms associated to b; $\mathcal{Q}_b$ is an affine space over V, the action of V being given by $(\alpha + q)(x) = q(x) + 2b(\alpha, x)$ for $q \in \mathcal{Q}_b$, $\alpha, x \in V$.

When b is symplectic, q takes it values in $2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2$; the corresponding form $q' : V \to \mathbb{Z}/2$ is a quadratic form associated to b in the usual sense, that is satisfies $q'(x+y) = q'(x) + q'(y) + b(x,y)$ for $x, y \in V$.

The *Brown invariant* $\sigma(q) \in \mathbb{Z}/8$ of a form $q \in \mathcal{Q}_b$ has been introduced in [**B**] as a generalization of the Arf invariant; it can be defined as follows. If b is symplectic, we put $\sigma(q) := 4\,\mathrm{Arf}(q')$, where $q' : V \to \mathbb{Z}/2$ is the form defined above. Otherwise b admits an orthonormal basis $(e_1,\ldots,e_g)$; we have $q(e_i)=\pm 1$, and we let g^+ (resp. g^-) be the number of basis vectors e_i such that $q(e_i)=1$ (resp. -1). Then $\sigma(q)=g^+-g^-$ (mod. 8).

2.2. Linear algebra over $(\mathbf{Z}/2)[i]$. Let Γ be a unimodular Gaussian lattice of rank $2g$ over $\mathbb{Z}$. We put $A_2:=\Gamma/2\Gamma$; this is naturally identified with the 2-torsion subgroup of A_Γ. We have the following structures on A_2:

a) A_2 is a free $(\mathbb{Z}/2)[i]$-module of rank g. We put $\varepsilon:=1+i$ in $(\mathbb{Z}/2)[i]$; then $(\mathbb{Z}/2)[i]=(\mathbb{Z}/2)[\varepsilon]$, with $\varepsilon^2=0$. The subgroup A_i of i-invariant elements is $\operatorname{Ker}\varepsilon=\varepsilon A_2$; it is a vector space of dimension g over $\mathbb{Z}/2$.

b) The form E induces on A_2 a symplectic form e (the Weil pairing for A_Γ). Since $E(x,iy)=-E(ix,y)$, we have, for $\alpha,\beta\in A_2$,

$$e(\alpha,\varepsilon\beta)=e(\varepsilon\alpha,\beta)\quad\text{hence}\quad e(\varepsilon\alpha,\varepsilon\beta)=0\ ;$$

thus A_i is a Lagrangian subspace of A_2.

c) The form $x\mapsto S(x,x)$ induces a quadratic form $Q:A_2\to\mathbb{Z}/4$ associated with the bilinear symmetric form $(\alpha,\beta)\mapsto e(\alpha,i\beta)$ (2.1). In particular we have $Q(\alpha)\equiv e(\alpha,i\alpha)$ (mod. 2).

Since $S((1+i)x,(1+i)x)=2S(x,x)$, we have $Q(\varepsilon\alpha)=2Q(\alpha)=2e(\alpha,i\alpha)$.

LEMMA 1. *Let $q:A_2\to\mathbb{Z}/4$ be an i-invariant quadratic form associated to e. The formulas*

$$b(\varepsilon\alpha,\varepsilon\beta)=e(\alpha,\varepsilon\beta)\quad,\quad Q_q(\varepsilon\alpha)=q(\alpha)-Q(\alpha)\qquad\textit{for}\ \ \alpha,\beta\in A_2\,,$$

define on $A_i=\varepsilon A_2$ a non-degenerate symmetric form b and a quadratic form $Q_q:A_i\to\mathbb{Z}/4$ associated with b.

Proof : Since $A_i=\operatorname{Ker}\varepsilon$ is isotropic for e, the expression $e(\alpha,\varepsilon\beta)$ is a bilinear function b of $\varepsilon\alpha$ and $\varepsilon\beta$; it is symmetric by *b*). If $e(\alpha,\varepsilon\beta)=0$ for all β in A_2 we have $\alpha\in A_i$ because A_i is Lagrangian, hence $\varepsilon\alpha=0$, so b is non-degenerate.

Put $\tilde{Q}_q(\alpha)=q(\alpha)-Q(\alpha)\in\mathbb{Z}/4$ for $\alpha\in A_2$. We have

$$\tilde{Q}_q(\alpha+\beta)=\tilde{Q}_q(\alpha)+\tilde{Q}_q(\beta)+2e(\alpha,\varepsilon\beta)\ .$$

Take $\beta=\varepsilon\gamma$. Since q is i-invariant we have $q(\varepsilon\gamma)=2e(\gamma,i\gamma)=Q(\varepsilon\gamma)$ by *c*), hence $\tilde{Q}_q(\varepsilon\gamma)=0$ and $\tilde{Q}_q(\alpha+\varepsilon\gamma)=\tilde{Q}_q(\alpha)$. Thus $\tilde{Q}_q$ defines a quadratic form Q_q on A_i associated to b. ■

Let $\mathcal{Q}_e^{(i)}$ be the set of i-invariants quadratic forms on A_2 associated to e. If $q\in\mathcal{Q}_e^{(i)}$ and $\alpha\in A_2$, we have $\alpha+q\in\mathcal{Q}_e^{(i)}$ if and only if α belongs to $A_i^\perp=A_i$; in other words, $\mathcal{Q}_e^{(i)}$ is an affine subspace of $\mathbb{Q}_e$, with direction A_i.

LEMMA 2. *The map $q\mapsto Q_q$ is an affine isomorphism of $\mathcal{Q}_e^{(i)}$ onto $\mathcal{Q}_b$.*

Proof : We just have to prove the equality $Q_{\alpha+q}=\alpha+Q_q$ for $q\in\mathcal{Q}_e^{(i)}$, $\alpha\in A_i$. Let $\beta\in A_i$; we write $\beta=\varepsilon\beta'$ for some $\beta'\in A_2$. Then

$$Q_{\alpha+q}(\beta)=2e(\alpha,\beta')+q(\beta')-Q(\beta')=2b(\alpha,\beta)+Q_q(\beta)\ .\quad ■$$

REMARK 1. Let $\alpha \in A_2$; we have $b(\varepsilon\alpha,\varepsilon\alpha) = e(\alpha,\varepsilon\alpha) = e(\alpha,i\alpha) \equiv Q(\alpha)$ (mod. 2), hence *the form b is symplectic if and only if Γ is even.* In this case we have $e(\alpha,i\alpha)=0$ for all $\alpha \in A_2$; it follows that $\mathcal{Q}_e^{(i)}$ is the set of forms vanishing on A_i. Since A_i is Lagrangian for e, this implies that these forms, viewed as quadratic forms $A_2 \to \mathbb{Z}/2$, are all even (that is, their Arf invariant is 0).

3. i-invariant theta divisors

3.1. Reminder on theta characteristics. We first recall the classical theory of theta characteristics on an arbitrary p.p.a.v. $A = V/\Gamma$. Let $A_2 \cong \Gamma/2\Gamma$ be the 2-torsion subgroup of A, $\mathcal{T}$ the set of symmetric theta divisors on A, and $\mathcal{Q}_e$ the set of quadratic forms on A_2 associated to the Weyl pairing e. The $\mathbb{Z}/2$-vector space A_2 acts on $\mathcal{T}$ by translation, and on $\mathcal{Q}_e$ by the action defined in (2.1); both sets are affine spaces over A_2, and there is a canonical affine isomorphism $q \mapsto \Theta_q$ of $\mathcal{Q}_e$ onto $\mathcal{T}$. It can be defined as follows ([**M**], §2). Let $\gamma \in \Gamma$, and let $\bar\gamma$ be its class in A_2. For $z \in V$, we put

$$e_\gamma(z) = i^{q(\bar\gamma)} e^{\pi H(\gamma, z+\frac{\gamma}{2})} .$$

We define an action of Γ on the trivial bundle $V \times \mathbb{C}$ by $\gamma.(z,t) = (z+\gamma, e_\gamma(z)t)$; then the quotient of $V \times \mathbb{C}$ by this action is the line bundle $\mathcal{O}_A(\Theta_q)$ on A.

3.2. The main results. We go back to the abelian variety A_Γ associated to a Gaussian lattice Γ. We assume that Γ is unimodular. We use the notation of (2.2). The isomorphism $\mathcal{Q}_e \xrightarrow{\sim} \mathcal{T}$ is compatible with the action of i, so i-invariant theta divisors correspond to forms $q \in \mathcal{Q}_e^{(i)}$.

Let $q \in \mathcal{Q}_e^{(i)}$, and let L be the line bundle $\mathcal{O}_{A_\Gamma}(\Theta_q)$. We have $i^*L \cong L$; we denote by $\iota : i^*L \to L$ the unique isomorphism inducing the identity of L_0. For each $\alpha \in A_i$, ι induces an isomorphism $\iota(\alpha) : L_\alpha \to L_\alpha$.

PROPOSITION 1. *$\iota(\alpha)$ is the homothety of ratio $i^{Q_q(\alpha)}$.*

Proof : The isomorphism $\iota^{-1} : L \xrightarrow{\sim} i^*L$ corresponds to a linear automorphism j of L above i:

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d] & L \arrow[d] \\ A_\Gamma \arrow[r, "i"] & A_\Gamma \end{tikzcd} .$$

Consider the automorphism $\tilde{j} : (z,t) \mapsto (iz,t)$ of $\Gamma_{\mathbb{R}} \times \mathbb{C}$. Since $e_{i\gamma}(iz) = e_\gamma(z)$, we have $\tilde{j}(\gamma.(z,t)) = (i\gamma).\tilde{j}(z,t)$. Thus $\tilde{j}$ factors through an isomorphism $L \to L$

above i which is the identity on L_0, hence equal to j; that is, we have a commutative diagram:

$$\begin{tikzcd} \Gamma_{\mathbb{R}} \times \mathbb{C} \arrow[r, "\tilde{j}"] \arrow[d, "\pi"'] & \Gamma_{\mathbb{R}} \times \mathbb{C} \arrow[d, "\pi"] \\ L \arrow[r, "j"] & L \end{tikzcd}$$

where π is the quotient map.

Let $\alpha \in A_i$, and let γ be an element of Γ whose class (mod. 2Γ) is α. Then $\delta := \frac{i\gamma}{2} - \frac{\gamma}{2}$ belongs to Γ. We have

$$j(\pi(\frac{\gamma}{2}, t)) = \pi(\frac{i\gamma}{2}, t) = \pi(\frac{\gamma}{2}, e_\delta(\frac{\gamma}{2})^{-1} t) \ ,$$

hence $\iota(\alpha) = j(\alpha)^{-1}$ is the homothety of ratio $e_\delta(\frac{\gamma}{2})$. Let β be the class of δ in A_2. Since $\gamma = -(1+i)\delta$, we have $\alpha = \varepsilon\beta$, hence

$$e_\delta(\frac{\gamma}{2}) = i^{q(\beta)} e^{\frac{\pi}{2} H(\delta, \gamma+\delta)} = i^{q(\beta) - H(\delta,\delta)} = i^{Q_q(\alpha)} \ . \quad \blacksquare$$

From $\iota : i^*L \to L$ we deduce an isomorphism $\iota^\flat : L \xrightarrow{\sim} i_*L$, inducing on global sections an automorphism of $H^0(A_\Gamma, L)$.

PROPOSITION 2. *$\iota^\flat$ acts on $H^0(A_\Gamma, L)$ by multiplication by $e^{\frac{i\pi}{4}(\sigma(Q_q)+g)}$.*

Note that $\sigma(Q_q) \equiv g$ (mod. 2) ([**B**], Thm. 1.20, (vi)), so this number is a power of i.

Proof : Since $\dim H^0(A_\Gamma, L) = 1$ it suffices to compute $\operatorname{Tr} \iota^\flat$. This is given by the holomorphic Lefschetz formula [**A-B**] applied to (i, ι). Since $H^i(A_\Gamma, L) = 0$ for $i > 0$, we find

$$\operatorname{Tr} \iota^\flat = \sum_{\alpha \in A_i} \frac{\operatorname{Tr} \iota(\alpha)}{(1-i)^g} = (1-i)^{-g} \sum_{\alpha \in A_i} i^{Q_q(\alpha)} \ .$$

We have $(1-i)^{-g} = 2^{-\frac{g}{2}} e^{\frac{i\pi g}{4}}$ and $\sum_{\alpha \in A_i} i^{Q_q(\alpha)} = 2^{\frac{g}{2}} e^{\frac{i\pi}{4}\sigma(Q_q)}$ ([**B**], Thm. 1.20, (xi)), hence the result. $\blacksquare$

PROPOSITION 3. *Let $\alpha \in A_i$, and let $m_\alpha(\Theta_q)$ be the multiplicity of Θ_q at α. We have*

$$2m_\alpha(\Theta_q) \equiv \sigma(Q_q) + g - 2Q_q(\alpha) \quad (\text{mod. } 8) \, .$$

Proof : Let θ be a nonzero section of $H^0(A_\Gamma, L)$. Choose a local non-vanishing section s of L around α. We can write $\theta = fs$ in a neighborhood of α, with $f \in \mathcal{O}_{A_\Gamma, \alpha}$. We have $\iota^\flat(\theta) = i^k\theta$ with $2k \equiv \sigma(Q_q) + g$ (mod. 8) (Proposition 2), hence

$$(i^*f)\iota^\flat(s) = i^k fs \ .$$

We look at this equality in $\mathfrak{m}_\alpha^m L / \mathfrak{m}_\alpha^{m+1} L$, where $\mathfrak{m}_\alpha$ is the maximal ideal of $\mathcal{O}_{A_\Gamma, \alpha}$ and $m := m_\alpha(\Theta)$. We have $i^*f = i^m f$ (mod. $\mathfrak{m}_\alpha^{m+1}$), and $\iota^\flat(s) = \iota(\alpha)s$ (mod. $\mathfrak{m}_\alpha L$). We obtain $i^m \iota(\alpha) = i^k$, hence the result in view of Proposition 1. $\blacksquare$

Corollary. *The number of i-invariant theta divisors Θ with $m_0(\Theta) \equiv 2 \pmod 4$ is*

$$2^{\frac{g}{2}-1}(2^{\frac{g}{2}} - (-1)^{\frac{g}{4}}) \quad \textit{if } \Gamma \textit{ is even, and} \quad 2^{g-2} - 2^{\frac{g}{2}-1}\cos\frac{\pi g}{4} \quad \textit{if } \Gamma \textit{ is odd;}$$

each of these divisors corresponds to a vanishing thetanull.

Proof : According to the Proposition, we have $m_0(\Theta_q) \equiv 2 \pmod 4$ if and only if $\sigma(Q_q) \equiv 4 - g \pmod 8$. When q runs over $\mathcal{Q}_e^{(i)}$, Q_q runs over $\mathcal{Q}_b$ (Lemma 2.2), so we must find how many elements Q of $\mathcal{Q}_b$ satisfy $\sigma(Q) \equiv 4 - g \pmod 8$.

If Γ is even (so that g is divisible by 4), we identify $\mathcal{Q}_b$ with the set of quadratic forms $Q : A_2 \to \mathbb{Z}/2$ associated with the symplectic form b; the previous congruence becomes $\mathrm{Arf}(Q) \equiv 1 + \frac{g}{4} \pmod 2$. There are $2^{\frac{g}{2}-1}(2^{\frac{g}{2}}+1)$ such forms with Arf invariant 0 and $2^{\frac{g}{2}-1}(2^{\frac{g}{2}}-1)$ with Arf invariant 1, hence the result.

Assume that Γ is odd; we choose an orthonormal basis $(e_1, \ldots, e_g)$ for b. The forms $Q \in \mathcal{Q}_b$ are determined by their values $Q(e_i) = \pm 1$; the condition is that the number g^+ of $+1$ values satisfies

$$2g^+ - g \equiv 4 - g \pmod 8 \ , \ \text{hence} \ \ g^+ \equiv 2 \pmod 4 \ .$$

The number of forms with the required property is thus the number of subsets $E \subset \{1, \ldots, g\}$ with $\mathrm{Card}(E) \equiv 2 \pmod 4$, that is

$$\binom{g}{2} + \binom{g}{6} + \ldots = \frac{1}{4}\big[(1+1)^g + (1-1)^g - (1+i)^g - (1-i)^g\big] = 2^{g-2} - 2^{\frac{g}{2}-1}\cos\frac{\pi g}{4} \ . \quad \blacksquare$$

Thus we find a number of vanishing thetanulls asymptotically equivalent to 2^{g-1} when Γ is even, and 2^{g-2} when Γ is odd. These numbers are rather modest, at least by comparison with the number of vanishing thetanulls of a hyperelliptic Jacobian, which is asymptotically equivalent to 2^{2g-1}. However, when Γ is even, the vanishing thetanulls of A_Γ have the particular property of being "syzygetic" in the classical terminology, which just means that the corresponding quadratic forms (3.1) lie in an affine subspace of $\mathcal{Q}_e$ which consists of even forms (Remark 1). Such a subspace has dimension $\leq g$, and it might be that the number given by the Corollary in the even case is the maximum possible for a syzygetic subset of vanishing thetanulls.

4. Complements

4.1. Automorphisms. The automorphism group of A_Γ is the centralizer of i in $\mathrm{Aut}(\Gamma)$. This group can be rather large: it has order 46080 for $\Gamma = E_8$ and 2012774400 for $\Gamma = \Lambda_{24}$ [**C-S**]. For the lattice Γ_{2g} (Example 1.2.1) with $g > 4$, it has order $2^{2g-1}g!$.

For the lattice $\Gamma = \Gamma_0 \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ of Example 1.2.3, $\mathrm{Aut}(A_\Gamma)$ is generated by i and the group $\mathrm{Aut}(\Gamma_0)$. Note that there are examples of unimodular lattices (even or odd) Γ_0 with $\mathrm{Aut}(\Gamma_0) = \{\pm 1\}$ [**Ba**], so that $\mathrm{Aut}(A_\Gamma)$ is reduced to $\{\pm 1, \pm i\}$.

4.2. Jacobians. We observe that for $g>1$ the p.p.a.v. A_Γ can *not* be a Jacobian. Indeed, let C be a curve of genus g; if $JC\cong A_\Gamma$, Torelli theorem provides an automorphism u of C inducing either i or $-i$ on JC, hence also on $T_0(JC)=H^0(C,K_C)^*$. Then u acts trivially on the image of the canonical map $C\rightarrow\mathbb{P}(H^0(C,K_C)^*)$; this implies that u is the identity or that C is hyperelliptic and u is the hyperelliptic involution. But in these cases u acts on $H^0(C,K_C)$ by multiplication by ± 1, a contradiction. ∎

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Clay Mathematics Proceedings
Volume **18**, 2013

Rank of divisors on graphs: an algebro-geometric analysis

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Dedicated to Joe Harris, for his sixtieth birthday

Abstract. The divisor theory for graphs is compared to the theory of linear series on curves through the correspondence associating a curve to its dual graph. An algebro-geometric interpretation of the combinatorial rank is proposed, and proved in some cases.

Contents

The goal of this paper is to apply the divisor theory for graphs to the theory of linear series on singular algebraic curves, and to propose an algebro-geometric interpretation for the rank of divisors on graphs. Let us begin with a simple question.

What is the maximum dimension of a linear series of degree $d \geq 0$ on a smooth projective curve of genus g?

We know what the answer is. If $d \geq 2g-1$ by Riemann's theorem every complete linear series of degree d on every smooth curve of genus g has dimension $d-g$. If $d \leq 2g-2$ the situation is more interesting: Clifford's theorem states that the answer is $\lfloor d/2 \rfloor$, and the bound is achieved only by certain linear series on hyperelliptic curves; see [**3**].

Now let us look at the combinatorial side of the problem. The dual graph of any smooth curve of genus g is the (weighted) graph with one vertex of weight equal to g and no edges, let us denote it by G_g. This graph admits a unique divisor of degree d, whose rank, as we shall see, is equal to $d-g$ if $d \geq 2g-1$, and to $\lfloor d/2 \rfloor$ otherwise.

We draw the following conclusion: the maximum dimension of a linear series of degree d on a smooth curve of genus g equals the rank of the degree d divisor on the dual graph of the curve. In symbols, denoting by $\underline{d}$ the unique divisor of degree d on G_g and by $r_{G_g}(\underline{d})$ its rank (see below),

$$r_{G_g}(\underline{d}) = \max\{r(X,D), \ \ \forall X \in M_g, \ \forall D \in \mathrm{Pic}^d(X)\} \tag{0.1}$$

2010 *Mathematics Subject Classification.* Primary 14Hxx, 05Cxx .

where M_g is the moduli space of smooth projective curves of genus g. This is quite pleasing for at least two reasons. First, the graph is fixed, whereas the curve varies (in a moduli space of dimension $3g-3$ if $g \geq 2$); also the divisor on G_g is fixed, whereas $\mathrm{Pic}^d(X)$ has dimension g. Second: computing the rank of a divisor on a graph is simpler than computing the dimension of a linear series on a curve; a computer can do that.

Therefore we shall now ask how this phenomenon generalizes to singular curves. For every graph G we have a family, $M^{\mathrm{alg}}(G)$, of curves having dual graph equal to G. We want to give an interpretation of the rank of a divisor on G in terms of linear series on curves in $M^{\mathrm{alg}}(G)$.

This is quite a delicate issue, as for such curves we do not have a good control on the dimension of a linear series; in fact, as we shall see, both Riemann's theorem and Clifford's theorem fail. Furthermore, asking for the maximal dimension of a linear series of degree d is not so interesting, as the answer easily turns out to be $+\infty$. By contrast, the rank of a divisor of degree $d \geq 0$ on a graph is always at most equal to d. In fact, to set-up the problem precisely we need a few more details. Let us assume some of them for now, and continue with this overview.

For any curve X having G as dual graph, we have an identification of the set of irreducible components of X with the set of vertices, $V(G)$, of G, and we write

$$X = \cup_{v \in V(G)} C_v. \tag{0.2}$$

The group of divisors of G is the free abelian group, $\mathrm{Div}\, G$, generated by $V(G)$. Hence there is a natural map sending a Cartier divisor D on X to a divisor on G:

$$\mathrm{Div}\, X \longrightarrow \mathrm{Div}(G); \quad D \mapsto \sum_{v \in V(G)} (\deg D_{|C_v})v,$$

so that the divisor of G associated to D is the multidegree of D; the above map descends to $\mathrm{Pic}(X) \to \mathrm{Div}(G)$, as linearly equivalent divisors have the same multidegree. Therefore we can write

$$\mathrm{Pic}(X) = \bigsqcup_{\underline{d} \in \mathrm{Div}(G)} \mathrm{Pic}^{\underline{d}}(X). \tag{0.3}$$

On the other hand, linearly equivalent divisors on G have the same rank, so the combinatorial rank is really a function on divisor classes. Let $\delta \in \mathrm{Pic}(G)$ be a divisor class on G and write $r_G(\delta) := r_G(\underline{d})$ for any representative $\underline{d} \in \delta$.

How does $r_G(\delta)$ relate to $r(X, L)$ as X varies among curves having G as dual graph, and $L \in \mathrm{Pic}(X)$ varies by keeping its multidegree class equal to δ? We conjecture that the following identity holds:

$$r_G(\delta) = \max_{X \in M^{\mathrm{alg}}(G)} \Big\{ \min_{\underline{d} \in \delta} \big\{ \max_{L \in \mathrm{Pic}^{\underline{d}}(X)} \{ r(X, L) \} \big\} \Big\}. \tag{0.4}$$

An accurate discussion of this conjecture is at the beginning of Section 2. In Section 1, after some combinatorial preliminaries, a comparative analysis of the graph-theoretic and algebraic situation is carried out highlighting differences and analogies; this also serves as motivation. In Section 2 we prove the above identity in a series of cases, summarized at the end of the paper.

The techniques we use are mostly algebro-geometric, while the combinatorial aspects are kept at a minimum. The hope is, of course, that using more sophisticated combinatorial arguments the validity range of above identity could be completely determined.

I am grateful to Margarida Melo and to the referee for some very useful remarks.

1. Combinatorial and algebraic rank

We apply the following conventions throughout the paper.
X is a projective algebraic curve over some algebraically closed field.
X is connected, reduced and has at most nodes as singularities.
G is a finite, connected, vertex weighted graph.
Capital letters $D, E, \dots$ are Cartier divisors on curves.
Underlined lowercase letters $\underline{d}, \underline{e}, \dots$ are divisors on graphs.
$r(X, D) := h^0(X, D) - 1$ is the (algebraic) rank of D on X.
$r_G(\underline{d})$ is the (combinatorial) rank of $\underline{d}$ on G.
$\mathrm{Div}(*)$ is the set of divisors on $*$, $\mathrm{Div}_+(*)$ the set of effective divisors.
$\mathrm{Div}^d(*)$ is the set of divisors of degree d, for $d \in \mathbb{Z}$.
$\sim$ is the linear equivalence on $\mathrm{Div}^d(*)$.
$\mathrm{Pic}(*) := \mathrm{Div}(*)/\sim$ and $\mathrm{Pic}^d(*) := \mathrm{Div}^d(*)/\sim$.

1.1. Basic divisor theory on graphs. We begin by reviewing the combinatorial setting following [**6**] and [**2**]. The basic reference is [**6**], which deals with loopless weightless graphs, we use the extension to general weighted graphs given in [**2**]; see [**1**] for a different approach.

Let G be a (finite, connected, weighted) graph; we allow loops. We write $V(G)$ and $E(G)$ for its vertex set and edge set; G is given a weight function $\omega : V(G) \to \mathbb{Z}_{\geq 0}$. If $\omega = 0$ we say that G is *weightless*. The genus of G is $b_1(G) + \sum_{v \in V(G)} \omega(v)$.

We always fix an ordering $V(G) = \{v_1, \dots, v_\gamma\}$. The group of divisors of G is the free abelian group on $V(G)$:

$$\mathrm{Div}(G) := \{\sum_{i=1}^{\gamma} d_i v_i,\ d_i \in \mathbb{Z}\} \cong \mathbb{Z}^\gamma.$$

Throughout the paper we identify $\mathrm{Div}(G)$ with $\mathbb{Z}^\gamma$, so that divisors on graphs are usually represented by ordered sequences of integers, $\underline{d} = (d_1, \dots, d_\gamma)$; we write $\underline{d} \geq 0$ if $d_i \geq 0$ for every $i = 1, \dots, \gamma$.

We set $|\underline{d}| = \sum_{i=1}^{\gamma} d_i$, so that $\mathrm{Div}^d(G) = \{\underline{d} \in \mathrm{Div}(G) :\ |\underline{d}| = d\}$; also $\mathrm{Div}_+(G) := \{\underline{d} \in \mathrm{Div}(G) :\ \underline{d} \geq 0\}$.

For $v \in V(G)$ we denote by $\underline{d}(v)$ the coefficient of v in $\underline{d}$, so that $\underline{d}(v_i) = d_i$.

If $Z \subset V(G)$ we write $\underline{d}(Z) = \sum_{v \in Z} \underline{d}(v)$ and $\underline{d}_Z = (\underline{d}(v),\ \forall v \in Z) \in \mathbb{Z}^{|Z|}$. We set $Z^c = V(G) \smallsetminus Z$.

The local geometry of G can be described by its so-called intersection product, which we are going to define. Fix two vertices v and w of G; we want to think of v and w as "close" in G if they are joined by some edges. To start with we set, if $v \neq w$,

$$(v \cdot w) := \text{number of edges joining } v \text{ and } w.$$

So, the greater $(v \cdot w)$ the closer v and w. Next we set

$$(v \cdot v) = -\sum_{w \neq v} (v \cdot w) \tag{1.1}$$

and the *intersection* product, $\mathrm{Div}(G) \times \mathrm{Div}(G) \to \mathbb{Z}$, is defined as the $\mathbb{Z}$-linear extension of $(v, w) \mapsto (v \cdot w)$.

Given $Z, W \subset V(G)$, we shall frequently abuse notation by writing $(W \cdot Z) = \sum_{w\in W, z\in Z}(w \cdot z)$. Notice that if $v \notin W$ the quantity $(v \cdot W)$ is the number of edges joining v with a vertex of W, whereas if $v \in W$ we have $(v \cdot W) \leq 0$.

We are going to study functions on G, and their divisors. A rational function f on G is a map $f : V(G) \to \mathbb{Z}$. To define the associated divisor, $\operatorname{div}(f)$, we proceed in analogy with classical geometry. We begin by requiring that if f is constant its divisor be equal to 0. The set of rational functions on G is a group under addition; so we require that if $c : V(G) \to \mathbb{Z}$ is constant then $\operatorname{div}(f + c) = \operatorname{div}(f)$. Now we need to study the analogue of zeroes and poles, i.e. the local behaviour of a function near each $v \in V(G)$. We write

$$\operatorname{div}(f) := \sum_{v\in V(G)} \operatorname{ord}_v(f)v$$

where $\operatorname{ord}_v(f) \in \mathbb{Z}$ needs to be defined so as to depend on the behaviour of f near v, that is on the value of f at each w close to v, and on how close v and w are. We are also requiring that $\operatorname{ord}_v(f)$ be invariant under adding a constant to f, this suggests that $\operatorname{ord}_v(f)$ be a function of the difference $f(v) - f(w)$, proportional to $(v \cdot w)$. That was an intuitive motivation for the following definition

$$\operatorname{ord}_v(f) := \sum_{w\neq v}(f(v) - f(w))(v \cdot w). \tag{1.2}$$

Loosely speaking, $\operatorname{ord}_v(f) = 0$ means f is locally constant at v, and $\operatorname{ord}_v(f) > 0$ (resp. $\operatorname{ord}_v(f) < 0$), means v is a local maximum for f (resp. a local minimum).

Notice the following useful simple fact.

Remark 1.1. Let $Z \subset V(G)$ be the set of vertices where the function f takes its minimum value. Then $\operatorname{div}(f)(Z) \leq -(Z \cdot Z^c)$ and for every $v \in Z$ we have $\operatorname{div}(f)(v) \leq 0$.

Note that $\operatorname{ord}_v(f) = -\operatorname{ord}_{-f}(v)$ and $\operatorname{ord}_v(f) + \operatorname{ord}_v(g) = \operatorname{ord}_v(f + g)$. The divisors of the form $\operatorname{div}(f)$ are called *principal*, and are easily seen to have degree zero. Thus they form a subgroup of $\operatorname{Div}^0(G)$, denoted by $\operatorname{Prin}(G)$.

Two divisors $\underline{d}, \underline{d}' \in \operatorname{Div}(G)$ are linearly equivalent, written $\underline{d} \sim \underline{d}'$, if $\underline{d} - \underline{d}' \in \operatorname{Prin}(G)$. We write $\operatorname{Pic}(G) = \operatorname{Div}(G)/\sim$; we usually denote an element of $\operatorname{Pic}(G)$ by δ and write $\underline{d} \in \delta$ for a representative; we also write $\delta = [\underline{d}]$. Now, $\underline{d} \sim \underline{d}'$ implies $|\underline{d}| = |\underline{d}'|$ hence we set

$$\operatorname{Pic}^d(G) = \operatorname{Div}^d(G)/\sim$$

(often in the graph-theory literature the notation $\operatorname{Jac}(G)$ is used for what we here denote by $\operatorname{Pic}(G)$ to stress the analogy with algebraic geometry).

The group $\operatorname{Pic}^0(G)$ appears in several different places of the mathematical literature, with various names and notations; see for example [**5**], [**15**], [**16**].

It is well known that $\operatorname{Pic}^d(G)$ is a finite set whose cardinality equals the complexity, i.e. the number of spanning trees, of the graph G.

Remark 1.2. The intersection product does not depend on the loops or the weights of G, hence the same holds for $\operatorname{Prin}(G)$ and $\operatorname{Pic}(G)$.

To define the combinatorial rank we proceed in two steps, treating loopless, weightless graphs first.

Let G be a loopless, weightless graph, and $\underline{d} \in \mathrm{Div}(G)$. Following [**6**], we define the *(combinatorial) rank* of $\underline{d}$ as follows

$$r_G(\underline{d}) = \max\{k : \forall \underline{e} \in \mathrm{Div}_+^k(G) \;\; \exists \underline{d}' \sim \underline{d} \text{ such that } \underline{d}' - \underline{e} \geq 0\} \tag{1.3}$$

with $r_G(\underline{d}) = -1$ if the set on the right is empty.

The combinatorial rank defined in (1.3) satisfies a Riemann-Roch formula (see below) if the graph is free from loops and weights, but not in general. This is why a different definition is needed for weighted graphs admitting loops. To do that we introduce the weightless, loopless graph $G^\bullet$ obtained from G by first attaching $\omega(v)$ loops based at v for every $v \in V(G)$, and then by inserting a vertex in every loop edge. This graph $G^\bullet$ (obviously free from loops) is assigned the zero weight function. Now G and $G^\bullet$ have the same genus.

As $V(G) \subset V(G^\bullet)$ we have a natural injection $\iota : \mathrm{Div}(G) \hookrightarrow \mathrm{Div}(G^\bullet)$. It is easy to see that $\iota(\mathrm{Prin}(G)) \subset \mathrm{Prin}(G^\bullet)$, hence we have

$$\mathrm{Pic}(G) \hookrightarrow \mathrm{Pic}(G^\bullet). \tag{1.4}$$

We define the rank for a divisor $\underline{d}$ on any graph G as follows:

$$r_G(\underline{d}) := r_{G^\bullet}(\iota(\underline{d})) \tag{1.5}$$

where the right-hand-side is defined in (1.3).

REMARK 1.3. If $\underline{d} \sim \underline{d}'$ we have $r_G(\underline{d}) = r_G(\underline{d}')$.

EXAMPLE 1.4. The picture below represents $G^\bullet$ for a graph having one vertex of weight 1 and one loop based at a vertex of weight zero. We have $\mathrm{Pic}^0(G) = 0$ and it is easy to check that $\mathrm{Pic}^0(G^\bullet) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Consider the divisor $v \in \mathrm{Div}(G)$; then $r_G(v) = 0$.

FIGURE 1. Weightless loopless model of G

In our figures, weight-zero vertices are represented by a "$\circ$".

It is clear that different graphs may have the same $G^\bullet$, see for example the picture in the proof of 2.7. Other examples will be given in the sequel, also during some proofs.

1.2. Simple comparisons. As is well known, the combinatorial rank is the analogue of the rank for a divisor on a smooth curve, in the following sense. If X is smooth and D is a divisor on it we have

$$r(X, D) = h^0(X, D) - 1 =$$
$$= \max\{k : \forall p_1, \dots, p_k \in X \;\; \exists D' \sim D : \; D' - p_i \geq 0 \;\; \forall i = 1 \dots k\}.$$

Now, if X is singular the above identity may fail, as the next example shows. First, recall that two Cartier divisors, D and D', on X are defined to be linearly equivalent, in symbols $D \sim D'$, if the corresponding line bundles, or invertible sheaves, $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$, are isomorphic.

Example 1.5. Let $X = C_1 \cup C_2$ be the union of two smooth rational curves meeting at a point (a node of X). Let $q \in C_2$ be a smooth point of X; then $r(X, q) = 1$ (see the next remark). Now, for any smooth point p of X lying on C_1 we have $q \not\sim p$ (these two divisors have different multidegree).

We will use the following simple facts.

Remark 1.6. Let $X = Z \cup Y$ with Z and Y connected subcurves with no common components, set $k := |Z \cap Y|$. Pick $L \in \operatorname{Pic} X$, then:

(1) $r(Z, L_Z) + r(Y, L_Y) - k + 1 \leq r(X, L) \leq r(Z, L_Z) + r(Y, L_Y) + 1$.
(2) If $k = 1$ we have $r(X, L) = r(Z, L_Z) + r(Y, L_Y) + 1$ if and only if L_Z and L_Y have a base point at the branch over $Z \cap Y$.
(3) If $\underline{\deg} L_Z < 0$ we have $r(X, L) = r(Y, L_Y(-Y \cdot Z))$, where $Y \cdot Z$ denotes the degree-k divisor cut by Z on Y.

Let X be a nodal connected curve and G its *dual graph*. Recall that G is defined so that the set of its vertices is identified with the set of irreducible components of X (we always use notation (0.2)), the set of its edges is identified with the set of nodes of X, and for $v, w \in V(G)$ we have $(v \cdot w) = |C_v \cap C_w|$. The weight function on G assigns to the vertex v the genus of the desingularization of the corresponding component, C_v. The arithmetic genus of X is equal to the genus of its dual graph.

The divisor theory of G is best connected to the divisor theory of X by adding to the picture variational elements, i.e. by considering one-parameter families of curves specializing to X, as follows.

Let $\phi : \mathcal{X} \to B$ be a regular one-parameter smoothing of a curve X. That is, B is a smooth connected one-dimensional variety with a marked point $b_0 \in B$, $\mathcal{X}$ is a regular surface, and $\phi^{-1}(b_0) \cong X$ while $\phi^{-1}(b)$ is a smooth curve for every $b \neq b_0$. Such a ϕ determines a discrete subgroup $\operatorname{Tw}_\phi X$ of $\operatorname{Pic}^0(X)$:

$$\operatorname{Tw}_\phi X := \{\mathcal{O}_{\mathcal{X}}(D)_{|X},\ \forall D \in \operatorname{Div}(\mathcal{X}) : \operatorname{Supp} D \subset X\}/\cong . \tag{1.6}$$

Elements of $\operatorname{Tw}_\phi X$ are called twisters. The multidegree map

$$\underline{\deg} : \operatorname{Tw}_\phi X \longrightarrow \mathbb{Z}^\gamma = \operatorname{Div}(G)$$

has image, independent of ϕ, written

$$\Lambda_X = \underline{\deg}\ (\operatorname{Tw}_\phi X) \subset \operatorname{Div}^0(G).$$

We now connect with the divisor theory of G. Write $X = \cup_{v_i \in V(G)} C_{v_i}$; it is obvious that Λ_X is generated by $\underline{\deg}\ \mathcal{O}(C_{v_i})$ for $i = 1, \ldots, \gamma$. On the other hand we clearly have

$$\underline{\deg}\ \mathcal{O}(C_{v_i}) = ((v_1 \cdot v_i), \ldots, (v_\gamma \cdot v_i)) = -\operatorname{div} f_i$$

where $f_i : V(G) \to \mathbb{Z}$ is the function taking value $+1$ at v_i and zero elsewhere. Therefore $\underline{\deg}\, \mathcal{O}(C_{v_i}) \in \operatorname{Prin}(G)$. Finally, as the set $\{\operatorname{div}(f_i),\ \ i = 1, \ldots \gamma\}$ generates $\operatorname{Prin}(G)$, we obtain

$$\Lambda_X = \operatorname{Prin}(G).$$

For $v \in V(G)$ we shall denote

$$\underline{t}_v := \underline{\deg}\ \mathcal{O}(C_v) = ((v_1 \cdot v), \ldots, (v_\gamma \cdot v)) \in \operatorname{Prin}(G). \tag{1.7}$$

By (1.1) any $\gamma - 1$ elements of type $\underline{t}_v$ generate $\operatorname{Prin}(G)$.

We denote by $q_\phi : \mathrm{Pic}(X) \to \mathrm{Pic}(X)/\operatorname{Tw}_\phi X$ the quotient map. Summarizing, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "\underline{\deg}"] \arrow[d, "q_\phi"'] & \mathrm{Div}(G) \arrow[d, "q_G"] \\ & \mathrm{Pic}(X)/\operatorname{Tw}_\phi X \arrow[r] & \mathrm{Div}(G)/\operatorname{Prin} G = \mathrm{Pic}(G). \end{tikzcd} \tag{1.8}$$

We are going to use the diagram to compare the combinatorial rank $r_G(\underline{d})$ to the algebraic rank $r(X, L)$, where L is a line bundle on X. The next statement summarizes a series of well known facts by highlighting opposite behaviours.

Proposition 1.7 (Differences in combinatorial and algebraic setting). *Let X be a reducible curve and G its dual graph.*

(1) (a) *For every $d \in \mathbb{Z}$ and $\underline{d} \in \mathrm{Div}^d(G)$ we have $r_G(\underline{d}) \leq \max\{-1, d\}$.*
(b) *For every $d, n \in \mathbb{Z}$ there exist infinitely many $\underline{d}$ with $|\underline{d}| = d$ such that $r(X, L) > n$ for every $L \in \mathrm{Pic}^{\underline{d}}(X)$.*

(2) (a) *For any $\underline{d}, \underline{d}' \in \mathrm{Div}(G)$ with $\underline{d} \sim \underline{d}'$ (i.e. $q_G(\underline{d}) = q_G(\underline{d}')$) we have $r_G(\underline{d}) = r_G(\underline{d}')$.*
(b) *For every regular one-parameter smoothing ϕ of X there exist infinitely many $L, L' \in \mathrm{Pic}(X)$ with $q_\phi(L) = q_\phi(L')$ and $r(X, L) \neq r(X, L')$.*

(3) (a) [**6**, Lemma 2.1] *For any $\underline{d}, \underline{d}' \in \mathrm{Div}(G)$ with $r_G(\underline{d}) \geq 0$ and $r_G(\underline{d}') \geq 0$ we have*
$$r_G(\underline{d}) + r_G(\underline{d}') \leq r_G(\underline{d} + \underline{d}').$$
(b) *There exist infinitely many $L, L' \in \mathrm{Pic}(X)$ with $r(X, L) \geq 0$ and $r(X, L') \geq 0$ such that*
$$r(X, L) + r(X, L') > r(X, L \otimes L').$$

(4) (a) [**6**, Cor. 3.5]*(Clifford for graphs) For any $0 \leq d \leq 2g - 2$ and any $\underline{d} \in \mathrm{Div}^d(G)$ we have*
$$r_G(\underline{d}) \leq d/2.$$
(b) *For any $0 \leq d \leq 2g - 2$ there exist infinitely many $\underline{d}$ with $|\underline{d}| = d$ such that for any $L \in \mathrm{Pic}^{\underline{d}}(X)$*
$$r(X, L) > d/2.$$

Remark 1.8. In [**6**] the authors work with loopless, weightless graphs, but it is clear that the two above results extend, using definition (1.5).

Proof. Part (1). The assertion concerning r_G follows immediately from the definition. The second part follows from the next observation.

Let $\underline{d} = (d_1, \ldots, d_\gamma)$ be any multidegree on X. For any integer m we pick $\underline{d}' = (d'_1, \ldots, d'_\gamma) \sim \underline{d}$ such that $d'_1 \geq m$ (for example $\underline{d}' = \underline{d} - \underline{\deg}\, \mathcal{O}_X((m + d_1)C_1))$. It is clear that for any $n \in \mathbb{N}$ we can choose m large enough so that for every $L' \in \mathrm{Pic}^{\underline{d}'}(X)$ we have $r(X, L') \geq n$. In particular, for every $L \in \mathrm{Pic}^{\underline{d}}(X)$, any regular smoothing ϕ of X, there exists $L' \in \mathrm{Pic}(X)$ such that $q_\phi(L) = q_\phi(L')$ and $r(X, L') \geq n$. From this argument we derive item (b) for parts (1), (2) and (4).

It remains to prove item (b) of part (3). Fix an irreducible component C of X and set $Z = \overline{X \smallsetminus C}$. Pick any effective Cartier divisor E on X with $\operatorname{Supp} E \subset Z$ and such that, setting $L' = \mathcal{O}_X(E)$, we have

$$r(X, L') \geq 1. \tag{1.9}$$

Now pick $m \geq 2g_C + k$ where g_C is the arithmetic genus of C and $k = |C \cap Z|$. Let $\underline{d}$ be a multidegree with $d_C = m$ and such that

$$\underline{d}_Z + \underline{\deg}_Z \mathcal{O}_X(E) < 0.$$

In particular $\underline{d}_Z < 0$, hence for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have

$$r(X, L) = r(C, L(-C \cdot Z)) = m - k - g_C \geq g_C \geq 0$$

(writing $C \cdot Z$ for the divisor cut on C by Z; see Remark 1.6). Now consider $L \otimes L' = L(E)$. We have $\underline{\deg}_Z L(E) = \underline{d}_Z + \underline{\deg}_Z \mathcal{O}_X(E) < 0$ hence

$$r(X, L \otimes L') = r(C, L(E - C \cdot Z)) = r(C, L(-C \cdot Z)) = r(X, L).$$

By (1.9), we have $r(X, L \otimes L') < r(X, L) + r(X, L')$ and are done. ■

We now mention, parenthetically but using the same set-up, a different type of result on the interplay between algebraic geometry and graph theory, when families of curves are involved. This is the Specialization Lemma of [**8**], concerning a regular one-parameter smoothing $\phi : \mathcal{X} \to B$ of a curve X as before (so that X is the fiber over $b_0 \in B$). This lemma states that if $\mathcal{L}$ is a line bundle on the total space $\mathcal{X}$ then, up to shrinking B near b_0, for every $b \in B \smallsetminus \{b_0\}$ the algebro-geometric rank of the restriction of $\mathcal{L}$ to the fiber over b is at most equal to the combinatorial rank of the multidegree of the restriction of $\mathcal{L}$ to X. In symbols, for all $b \neq b_0$, we have $r(\phi^{-1}(b), \mathcal{L}_{|\phi^{-1}(b)}) \leq r_G(\underline{\deg}\ \mathcal{L}_{|X})$. (This form is actually a generalization of the one proved in [**8**]; see [**1**] and [**2**].) Apart from being interesting in its own right, the Specialization Lemma has some remarkable applications, like a new proof of the classical Brill-Noether theorem (see [**3**]) given in [**11**]. We view this as yet another motivation to study the algebro-geometric meaning of the combinatorial rank.

A fundamental analogy between the algebraic and combinatorial setting is the Riemann-Roch formula, which holds for every nodal curve X and every graph G. The algebraic case is classical: let $K_X \in \operatorname{Pic}(X)$ be the dualizing line bundle (equal to the canonical bundle if X is smooth), then for any Cartier divisor D on X we have

$$r(X, D) - r(X, K_X(-D)) = \deg D - g + 1$$

where g is the arithmetic genus of X.

The same formula holds for graphs. To state it, we introduce the canonical divisor, $\underline{k}_G$, of a graph G:

$$\underline{k}_G := \sum_{v \in V(G)} \big(2\omega(v) - 2 + \operatorname{val}(v)\big) v \tag{1.10}$$

where $\operatorname{val}(v)$ is the valency of v. If G is the dual graph of X we have

$$\underline{k}_G = \underline{\deg}\ K_X. \tag{1.11}$$

THEOREM 1.9 (Riemann-Roch formula for graphs). *Let G be a graph of genus g; for every $\underline{d} \in \operatorname{Div}^d(G)$ we have*

$$r_G(\underline{d}) - r_G(\underline{k}_G - \underline{d}) = d - g + 1.$$

This is [**6**, Thm 1.12] for loopless, weightless graphs; the extension to general graphs can be found in [**2**].

From Riemann-Roch we immediatly derive the following facts.

REMARK 1.10. Let $\underline{d} \in \mathrm{Div}^0(G)$. Then $r_G(\underline{d}) \leq 0$ and equality holds if and only if $\underline{d} \sim \underline{0}$.

Let $\underline{d} \in \mathrm{Div}^{2g-2}(G)$. Then $r_G(\underline{d}) \leq g-1$ and equality holds if and only if $\underline{d} \sim \underline{k}_G$.

1.3. Edge contractions and smoothings of nodes. Let $S \subset E(G)$ be a set of edges. By G/S we denote the graph obtained by contracting to a point (i.e. a vertex of G/S) every edge in S; the associated map will be denoted by

$$\sigma : G \to G/S.$$

There is an obvious identification $E(G/S) = E(G) \smallsetminus S$. The map σ induces a surjection

$$\sigma_V : V(G) \longrightarrow V(G/S); \quad v \mapsto \sigma(v).$$

For $\overline{v} \in V(G/S)$ we set $\overline{\omega}(\overline{v}) = \sum_{v \in \sigma_V^{-1}(\overline{v})} \omega(v) + b_1(\sigma^{-1}(\overline{v}))$ for its weight, so that $\overline{\omega}(\overline{v})$ is the genus of the (weighted) graph $\sigma^{-1}(\overline{v})$. We refer to G/S as a *contraction* of G; notice that G and G/S have the same genus. A picture can be found in Example 1.13.

REMARK 1.11. Contractions are particularly interesting for us, as they correspond to "smoothings" of algebraic curves. More precisely, let $\phi : \mathcal{X} \to B$ be a one-parameter family of curves having X as special fiber, and let $n \in X$ be a node; we say that ϕ is a smoothing of n if n is not the specialization of a node of the generic fiber (i.e. if there is an open neighborhood $U \subset \mathcal{X}$ of n such that the restriction of ϕ to $U \smallsetminus n$ has smooth fibers). Let G be the dual graph of X and let $S \subset E(G)$ be the set of edges corresponding to nodes n such that ϕ is a smoothing of n. Then, the contraction G/S is the dual graph of the fibers of ϕ near X. The converse also holds, i.e. for any contraction $G \to G/S$ there exists a deformation of X smoothing precisely the nodes corresponding to S.

Observe now that associated to $\sigma : G \to G/S$ there is a map

$$\sigma_* : \mathrm{Div}(G) \longrightarrow \mathrm{Div}(G/S); \qquad \sum_{v \in V(G)} n_v v \mapsto \sum_{\overline{v} \in V(G/S)} \Big(\sum_{v \in \sigma_V^{-1}(\overline{v})} n_v \Big) \overline{v}.$$

We need the following fact (essentially due to Baker-Norine, [**7**]).

PROPOSITION 1.12. *Let G be a graph, $e \in E(G)$, and let $\sigma : G \to G/e$ be the contraction of e. Then*

(1) $\sigma_* : \mathrm{Div}(G) \to \mathrm{Div}(G/e)$ *is a surjective group homomorphism such that* $\sigma_*(\mathrm{Prin}(G)) \supset \mathrm{Prin}(G/e)$.

(2) $\mathrm{Pic}(G) \cong \mathrm{Pic}(G/e)$ *if and only if e is a bridge (i.e. a separating edge). In this case the above isomorphism is induced by σ_*, and σ_* preserves the rank.*

PROOF. It is clear that σ_* is a surjective homomorphism. Let $v_0, v_1 \in V(G)$ be the endpoints of e. Set $\overline{G} := G/e$, now write $V(G) = \{v_0, v_1, \ldots, v_n\}$ and $V(\overline{G}) = \{\overline{v_1}, \ldots, \overline{v_n}\}$ with $\sigma_V(v_i) = \overline{v_i}$ for $i \geq 1$.

Denote by $\underline{t}_i = ((v_0 \cdot v_i), (v_1 \cdot v_i), \ldots, (v_n \cdot v_i)) \in \mathrm{Prin}(G)$ the principal divisor corresponding to v_i, defined in (1.7), and by $\overline{\underline{t}_i}$ the principal divisor of $\overline{G}$ corresponding to $\overline{v_i}$. As we mentioned earlier, it suffices to show that $\overline{\underline{t}_i} \in \sigma_*(\Lambda_G)$ for $i = 2, \ldots, n$. This follows from the identity

$$\sigma_*(\underline{t}_i) = \overline{\underline{t}_i}, \quad \forall i = 2, \ldots, n. \tag{1.12}$$

Let us prove it for $i = 2$ (which is obviously enough). We have

$$\sigma_*(\underline{t}_2) = ((v_0 \cdot v_2) + (v_1, \cdot v_2), (v_2 \cdot v_2), \ldots, (v_n \cdot v_2)),$$

now $(v_0 \cdot v_2) + (v_1, \cdot v_2) = (\overline{v_1} \cdot \overline{v_2})$ and $(v_i \cdot v_2) = (\overline{v_i} \cdot \overline{v_2})$ for every $i \geq 2$ hence (1.12) is proved.

Part (2). Suppose e is a bridge; then by [**7**, Lm. 5.7, Cor. 5.10] there is a rank-preserving isomorphism $\mathrm{Pic}(G^\bullet) \cong \mathrm{Pic}(G^\bullet/e)$. Of course, $G^\bullet/e = (G/e)^\bullet$, hence by (1.4), we obtain a rank preserving isomorphism $\mathrm{Pic}(G) \cong \mathrm{Pic}(G/e)$.

Assume e is not a bridge. Recall that for any d and any G the set $\mathrm{Pic}^d G$ has cardinality equal to the complexity, $c(G)$, of G. Therefore it is enough to prove that G and $\overline{G}$ have different complexity. Now, it is easy to see that the contraction map $\sigma : G \to \overline{G}$ induces a bijection between the spanning trees of $\overline{G}$ and the spanning trees of G containing e. On the other hand, since e is not a bridge, G admits a spanning tree not containing e (just pick a spanning tree of the connected graph $G - e$). We thus proved that $c(G) > c(\overline{G})$, and we are done. ∎

We observed in Remark 1.11 that one-parameter families of curves correspond to edge contractions of graphs. Now, in algebraic geometry the rank of a divisor is an upper-semicontinuous function: given a family of curves X_t specializing to a curve X, with a family of divisors $D_t \in \mathrm{Div}(X_t)$ specializing to $D \in \mathrm{Div}(X)$, we have $r(X_t, D_t) \leq r(X, D)$.

Do we have a corresponding semicontinuity for the combinatorial rank? The answer in general is no. By Proposition 1.12, contraction of bridges preserves the rank. But the following example illustrates that the rank can both decrease or increase if a non-bridge is contracted.

Example 1.13. *Failure of semicontinuity under edge contractions.* Consider the contraction of the edge $e_4 \in E(G)$ for the graph G in the picture below.

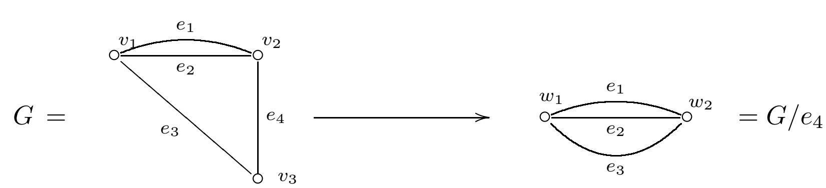


Figure 2. Contraction of e_4

Let us first show that the combinatorial rank may decrease. Pick

$$\underline{d} = (-2, 3, -1) \in \mathrm{Div}(G);$$

then $r_G(\underline{d}) = 0$ as

$$\underline{d} = -\underline{t}_{v_2} \sim (0, 0, 0).$$

Now $\sigma_*(\underline{d}) = (-2, 2)$ and hence

$$r_{G/e_4}(\sigma_*(\underline{d})) = -1 < r_G(\underline{d}).$$

Now let us show that the combinatorial rank may go up. Consider $\underline{d} = (1, -1, 1) \in \mathrm{Div}(G)$; then one checks easily (or by Lemma 1.14) that $r_G(\underline{d}) = -1$. Now $\sigma_*(\underline{d}) = (1, 0)$ hence $r_{G/e_4}(1, 0) = 0 > r_G(\underline{d})$.

Let us give also an example with $r_G \geq 0$. Pick $\underline{e} = (1, -1, 2)$ so that

$$r_{G/e_4}(\sigma_*(\underline{e})) = r_{G/e_4}(1,1) = 1.$$

Now $\underline{e} + \underline{t}_{v_3} = (1,-1,2) + (1,1,-2) = (2,0,0)$, hence $r_G(\underline{e}) \geq 0$. To show that $r_G(\underline{e}) \leq 0$ we note that if we subtract $(0,0,1)$ from $\underline{e}$ we get $(1,-1,1)$, which has rank -1, as observed above.

A convenient computational tool is provided by the following Lemma, of which we had originally a slightly less general version; the following version was suggested by the referee.

LEMMA 1.14. *Fix an integer $r \geq 0$ and let $\underline{d} \in \mathrm{Div}(G)$ be such that for some $v \in V(G)$ we have $\underline{d}(v) < r$. Assume that for every subset of vertices $Z \subset V(G) \smallsetminus \{v\}$ we have $\underline{d}(Z) < (Z \cdot Z^c)$. Then $r_G(\underline{d}) \leq r - 1$.*

PROOF. Since both hypotheses remain valid in $G^\bullet$, and $r_G(\underline{d})$ is defined as the rank of $\underline{d}$ on $G^\bullet$, we can assume G weightless and loopless.

For notational consistency, write $\underline{e} \in \mathrm{Div}^1_+(G)$ for the (effective) divisor corresponding to v. By contradiction, suppose $r_G(\underline{d}) \geq r$; hence $r_G(\underline{d} - r\underline{e}) \geq 0$, but $\underline{d} - r\underline{e}$ is not effective by hypothesis. Therefore for some nontrivial principal divisor $\underline{t} = \mathrm{div}(f) \in \mathrm{Prin}(G)$ we have

$$0 \leq \underline{d} - r\underline{e} + \underline{t}.$$

We use Remark 1.1; let $Z \subset V(G)$ be the set of vertices where f assumes its minimum; then $\underline{t}(Z) \leq -(Z \cdot Z^c)$. We have $v \notin Z$, for otherwise $\underline{t}(v) \leq 0$ hence $(\underline{d} - r\underline{e} + \underline{t})(v) < r - r = 0$ which is impossible. Therefore, by hypothesis, $\underline{d}(Z) < (Z \cdot Z^c)$, which yields (as $\underline{e}(Z) = 0$)

$$0 \leq (\underline{d} - r\underline{e} + \underline{t})(Z) = \underline{d}(Z) - r\underline{e}(Z) + \underline{t}(Z) \leq \underline{d}(Z) - (Z \cdot Z^c) < 0,$$

a contradiction. ∎

2. Algebraic interpretation of the combinatorial rank

Let G be a graph of genus least 2. We say G is *semistable* if every vertex of weight zero has valency at least 2, and we say G is *stable* if every vertex of weight zero has valency at least 3. This terminology is motivated by the fact that a curve X of arithmetic genus at least 2 is semistable, or stable, if and only if so is its dual graph.

2.1. A conjecture. If G is a stable graph, the locus of isomorphism classes of curves whose dual graph is G is an interesting subset of the moduli space of stable curves, denoted $M^{\mathrm{alg}}(G) \subset \overline{M_g}$; it is well known that $M^{\mathrm{alg}}(G)$ is irreducible, quasiprojective of dimension $3g - 3 - |E(G)|$. More generally, i.e. for any graph, we denote by $M^{\mathrm{alg}}(G)$ the set of isomorphism classes of curves having G as dual graph.

Let $X \in M^{\mathrm{alg}}(G)$ and $\underline{d} \in \mathrm{Div}(G)$, we denote

$$r^{\max}(X, \underline{d}) := \max\{r(X, L), \ \ \forall L \in \mathrm{Pic}^{\underline{d}}(X)\}.$$

By Riemann-Roch we have

$$r^{\max}(X, \underline{d}) \geq \max\{-1, |\underline{d}| - g\}. \tag{2.1}$$

We want to study the relation between $r_G(\underline{d})$ and $r^{\max}(X,\underline{d})$. Now, the combinatorial rank r_G is constant in an equivalence class, hence we set, for any $\delta \in \operatorname{Pic}(G)$ and $\underline{d} \in \delta$

$$r_G(\delta) := r_G(\underline{d}).$$

On the other hand, we saw in Proposition 1.7 that the algebraic rank behaves badly with respect to linear equivalence of multidegrees, indeed, it is unbounded on the fibers of q_ϕ. Therefore we set

$$r(X,\delta) := \min\{r^{\max}(X,\underline{d}), \quad \forall \underline{d} \in \delta\}.$$

Now, having the analogy with (0.1) in mind, we state

Conjecture 1. *Let G be a graph and $\delta \in \operatorname{Pic}^d(G)$. Then*

$$r_G(\delta) = \max\{r(X,\delta), \quad \forall X \in M^{\mathrm{alg}}(G)\}.$$

We set

$$r^{\mathrm{alg}}(G,\delta) := \max\{r(X,\delta), \quad \forall X \in M^{\mathrm{alg}}(G)\},$$

so that the above conjecture becomes

$$r^{\mathrm{alg}}(G,\delta) = r_G(\delta). \tag{2.2}$$

We think of $r^{\mathrm{alg}}(G,\delta)$ as the "algebro-geometric" rank of the combinatorial class δ. We shall prove that (2.2) holds in low genus and for $d \geq 2g-2$.

Remark 2.1. Stable and semistable curves are of fundamental importance in algebraic geometry; see [**4**], [**12**], [**14**]. We shall see, as a consequence of Lemma 2.4, that if Identity (2.2) holds for semistable graphs, it holds for any graph.

The following is a simple evidence for the conjecture.

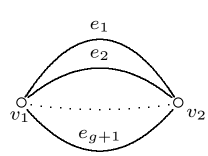
Lemma 2.2. *Conjecture 1 holds for $\delta = 0$. More precisely for every G and $X \in M^{\mathrm{alg}}(G)$ we have $r^{\max}(X,\underline{d}) = r_G(\delta) = 0$.*

Proof. We have $r_G(\delta) = 0$, of course. Now, as we explained in Subsection 1.2, every $\underline{d} \in \delta$ is the multidegree of some twister of X; pick one of them, T, so that $T \in \operatorname{Pic}^{\underline{d}}(X) \cap \operatorname{Tw}_\phi(X)$ for some regular one-parameter smoothing ϕ. By upper-semicontinuity of the algebraic rank, the twister T, being the specialization of the trivial line bundle, satisfies $r(X,T) \geq 0$. On the other hand $r(X,\mathcal{O}_X) = 0$ and it is easy to check that any other $L \in \operatorname{Pic}^{\underline{0}}(X)$ has rank -1; so we are done. ■

Here is an example where Conjecture 1 holds, and the equality $r(X,\delta) = r_G(\delta)$ does not hold for every $X \in M^{\mathrm{alg}}(G)$.

Example 2.3. Let G be a *binary graph* of genus $g \geq 2$, i.e. G is the graph with two vertices of weight zero joined by $g+1$ edges. (This graph is sometimes named "banana" graph; we prefer the word binary for consistency with the terminology used in other papers, such as [**10**].)

G =



Let $\underline{d} = (1,1) \in \mathrm{Div}(G)$. It is clear that $r_G(\underline{d}) = 1$.

Let now X be a curve whose dual graph is G, so X has two smooth rational components intersecting in $g+1$ points; we say X is a binary curve. It is easy to check that Clifford's theorem holds in this case (i.e. for this multidegree), hence $r(X,L) \leq 1$ for every $L \in \mathrm{Pic}^{(1,1)}(X)$.

Suppose first that $g = 2$. Then we claim that for every such X we have $r^{\max}(X,\underline{d}) = 1$ and there exists a unique $L \in \mathrm{Pic}^{(1,1)}(X)$ for which $r(X,L) = 1$. Indeed, to prove the existence it suffices to pick $L = K_X$. The fact that there are no other line bundles with this multidegree and rank follows from Riemann-Roch.

Now let $g \geq 2$. We say that a binary curve $X = C_1 \cup C_2$ is *special* if there is an isomorphism of pointed curves

$$(C_1; p_1, \dots p_{g+1}) \cong (C_2; q_1, \dots q_{g+1})$$

where p_i, q_i are the branches of the i-th node of X, for $i = 1, \dots g+1$ (if $g = 2$ every binary curve is special).

We claim that $r^{\max}(X,\underline{d}) = 1$ if and only if X is special, and in this case there exists a unique $L \in \mathrm{Pic}^{(1,1)}(X)$ for which $r(X,L) = 1$. We use induction on g; the base case $g = 2$ has already been done . Set $g \geq 3$ and observe that the desingularization of a special binary curve at a node is again special.

Let $\nu_1 : X_1 \to X$ be the desingularization of X at one node, so that X_1 has genus $g-1$. Let $p, q \in X_1$ be the branches of the desingularized node. By induction X_1 admits a line bundle L_1 of bidegree $(1,1)$ and rank 1 if and only if X_1 is special, and in this case L_1 is unique. Next, there exists $L \in \mathrm{Pic}^{(1,1)}(X)$ having rank 1 if and only if X_1 is special, $\nu_1^* L = L_1$ and,

$$r(X_1, L_1(-p)) = r(X_1, L_1(-q)) = r(X_1, L_1(-p-q)) = 0;$$

moreover such L is unique if it exists (see [**10**, Lm. 1.4]). Therefore $L_1 = \mathcal{O}(p+q)$, hence X is a special curve. The claim is proved.

Let us now consider $\underline{d}' \sim \underline{d}$ with $\underline{d}' \neq \underline{d}$:

$$\underline{d}' = (1 + n(g+1), 1 - n(g+1)).$$

By symmetry we can assume $n \geq 1$. Then for any $L \in \mathrm{Pic}^{\underline{d}'} X$ we have

$$r(X,L) = r(C_1, L_{C_1}(-C_1 \cdot C_2)) = r(\mathbb{P}^1, \mathcal{O}((n-1)g+n)) = (n-1)g+n \geq 1.$$

Therefore, denoting by $\delta \in \mathrm{Pic}(G)$ the class of $\underline{d} = (1,1)$ we have $r(X,\delta) = r^{\max}(X,\underline{d})$ for every $X \in M^{\mathrm{alg}}(G)$.

Here is a summary of what we proved.

Let G be a binary graph of genus $g \geq 2$, $\underline{d} = (1,1)$ and $\delta \in \mathrm{Pic}(G)$ the class of $\underline{d}$. Pick $X \in M^{\mathrm{alg}}(G)$, then

$$r(X,\delta) = r^{\max}(X,\underline{d}) = \begin{cases} 1 & \text{if } X \text{ is special} \\ 0 & \text{otherwise.} \end{cases}$$

And if X is special there exists a unique $L \in \mathrm{Pic}^{(1,1)}(X)$ having rank 1.

2.2. Low genus cases. We use the following terminology. A vertex $v \in V(G)$ of weight zero and valency one is a *leaf-vertex*, and the edge $e \in E(G)$ adjacent to v is a *leaf-edge*. Note that a leaf-edge is a bridge.

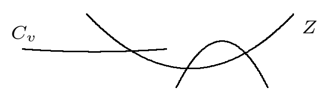
Let $\sigma : G \to \overline{G} = G/e$ be the contraction of a leaf-edge. By Proposition 1.12 the map $\sigma_* : \mathrm{Div}(G) \to \mathrm{Div}(\overline{G})$ induces an isomorphism

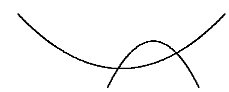
$$\sigma_* : \mathrm{Pic}(G) \xrightarrow{\cong} \mathrm{Pic}(\overline{G})$$

(abusing notation). Let $X \in M^{\mathrm{alg}}(G)$, then the component C_v corresponding to the leaf-vertex v is a smooth rational curve attached at a unique node; such components are called *rational tails*. Now, we have a natural surjection

$$M^{\mathrm{alg}}(G) \longrightarrow M^{\mathrm{alg}}(\overline{G}); \qquad X \mapsto \overline{X}$$

where $\overline{X}$ is obtained from X by removing C_v. Here is a picture, useful also for Lemma 2.4.

$X =$ 

$\overline{X} =$ 

LEMMA 2.4. *Let G be a graph and $\sigma : G \to \overline{G} = G/e$ the contraction of a leaf-edge. For every $\delta \in \mathrm{Pic}(G)$ and every $X \in M^{\mathrm{alg}}(G)$ we have, with the above notation,*

$$r(X, \delta) = r(\overline{X}, \sigma_*(\delta)).$$

In particular, Identity (2.2) *holds for G if and only if it holds for $\overline{G}$.*

PROOF. Let $v \in V(G)$ be the leaf-vertex of e and $C = C_v \subset X$ the corresponding rational tail; we write $X = C \cup Z$ with $Z \cong \overline{X}$, and identify $Z = \overline{X}$ from now on. Pick $\underline{d} \in \delta$ and set $c = \underline{d}(v)$; we define

$$\underline{d}^0 := \underline{d} + c\underline{t}_v$$

where $\underline{t}_v \in \mathrm{Prin}(G)$ was defined in (1.7). Hence $\underline{d}^0(v) = 0$ and $\underline{d}^0 \sim \underline{d}$. Notice that $\sigma_*(\underline{d}) = \sigma_*(\underline{d}^0)$. Now, since $C \cap Z$ is a separating node of X, there is a canonical isomorphism $\mathrm{Pic}\, X \cong \mathrm{Pic}(C) \times \mathrm{Pic}(Z)$ mapping L to the pair of its restrictions, (L_C, L_Z). Hence we have an isomorphism

$$\mathrm{Pic}^{\underline{d}^0}(X) \xrightarrow{\cong} \mathrm{Pic}^{\sigma_*(\underline{d}^0)}(\overline{X}); \qquad L \mapsto \overline{L} := L_Z,$$

as for any $L \in \mathrm{Pic}^{\underline{d}^0}(X)$ we have $L_C = \mathcal{O}_C$. Moreover, we have

$$r(X, L) = r(Z, L_Z) = r(\overline{X}, \overline{L})$$

by Remark 1.6. Therefore

$$r^{\max}(X, \underline{d}^0) = r^{\max}(\overline{X}, \sigma_*(\underline{d}^0)). \tag{2.3}$$

Now we claim that for every $\underline{d} \in \delta$ we have

$$r^{\max}(X, \underline{d}) \geq r^{\max}(X, \underline{d}^0). \tag{2.4}$$

This claim implies our statement. In fact it implies that $r(X, \delta)$ can be computed by looking only at representatives taking value 0 on C, i.e.

$$r(X, \delta) = \min\{r^{\max}(X, \underline{d}^0),\ \forall \underline{d}^0 \in \delta\};$$

now by (2.3) and the fact that $\sigma_* : \mathrm{Div}(X) \to \mathrm{Div}(\overline{X})$ is onto we get

$$r(X, \delta) = \min\{r^{\max}(\overline{X}, \overline{\underline{d}}),\ \forall \overline{\underline{d}} \in \sigma_*(\delta)\} = r(\overline{X}, \sigma_*(\delta))$$

and we are done.

We now prove (2.4). By what we said before, line bundles on X can be written as pairs (L_C, L_Z). Pick $L \in \operatorname{Pic}^{\underline{d}}(X)$ and set $L^0 := (\mathcal{O}_C, L_Z(cp))$ where $p = C \cap Z \in Z$ and $c = \deg_C L$ as before. Hence $L^0 \in \operatorname{Pic}^{\underline{d}^0}(X)$ and this sets up a bijection

$$\operatorname{Pic}^{\underline{d}}(X) \longrightarrow \operatorname{Pic}^{\underline{d}^0}(X); \qquad L \mapsto L^0.$$

We shall prove $r(X, L) \geq r(X, L^0)$ for every $L \in \operatorname{Pic}^{\underline{d}}(X)$, which clearly implies (2.4). If $c \geq 0$ we have

$$r(X, L) \geq r(C, \mathcal{O}(c)) + r(Z, L_Z) = c + r(Z, L_Z)$$

and

$$r(X, L^0) = r(Z, L_Z(cp)) \leq c + r(Z, L_Z);$$

combining the two inequalities we are done. If $c < 0$ we have

$$r(X, L) = r(Z, L_Z(-p)) \geq r(Z, L_Z(-|c|p)) = r(X, L^0).$$

The proof is finished. ■

Let G have genus $g \geq 2$ and let $\overline{G}$ be obtained after all possible leaf-edges contractions; then $\overline{G}$ is a semistable graph. By the previous result we can assume all graphs and curves of genus ≥ 2 semistable.

Corollary 2.5. *Conjecture* 1 *holds if* $g = 0$.

Proof. By Lemma 2.4 we can assume G has one vertex (of weight zero) and no edges, so that the only curve in $M^{\mathrm{alg}}(G)$ is $\mathbb{P}^1$. Now every $\delta \in \operatorname{Pic}^d(G)$, has a unique representative and $r_G(\delta) = \max\{-1, d\}$. On the other hand $\operatorname{Pic}^d(\mathbb{P}^1) = \{\mathcal{O}(d)\}$ and $r(\mathbb{P}^1, \mathcal{O}(d)) = \max\{-1, d\}$. ■

Another consequence of Lemma 2.4 is the following.

Proposition 2.6. *Conjecture* 1 *holds if* $g = 1$.

Proof. By Riemann-Roch we have, for every $\delta \in \operatorname{Pic}^d(G)$

$$r_G(\delta) = \begin{cases} d-1 & \text{if } d \geq 1 \\ 0 & \text{if } \delta = 0 \\ -1 & \text{otherwise.} \end{cases}$$

By Lemma 2.4 we can assume G has no leaves. If G consists of a vertex of weight 1 then a curve $X \in M^{\mathrm{alg}}(G)$ is smooth of genus 1, and the result follows from Riemann-Roch.

So we can assume G is a cycle with γ vertices, all 2-valent of weight zero, and γ edges. Now, we have $|\operatorname{Pic}^d(G)| = \gamma$ (as the complexity of G is obviously γ). Let us exhibit the elements of $\operatorname{Pic}^d(G)$ by suitable representatives:

$$\operatorname{Pic}^d(G) = \{[(d, \underline{0}_{\gamma-1})], [(d-1, 1, \underline{0}_{\gamma-2})], \ldots, [(d-1, \underline{0}_{\gamma-2}, 1)]\}$$

where we write $\underline{0}_i = (0, \ldots, 0) \in \mathbb{Z}^i$. We need to show the above γ multidegrees are not equivalent to one another; indeed the difference of any two of them is of type $\pm(\underline{0}_i, 1, \underline{0}_j, -1, \underline{0}_k)$ which has rank -1 (by Lemma 1.14 for example).

Pick now $X \in M^{\mathrm{alg}}(G)$. Assume $d \geq 1$. By Riemann-Roch $r(X, L) \geq d - 1$ for any line bundle L of degree d, so it suffices to show that every $\delta \in \operatorname{Pic}^d(G)$ has a representative $\underline{d}$ such that for some $L \in \operatorname{Pic}^{\underline{d}}(X)$ equality holds. Let $\underline{d}$ be any of the above representatives and pick $L \in \mathrm{Pic}^{\underline{d}}(X)$. It is easy to check directly that $r(X,L) = d-1$ (or, one can apply [**10**, Lm. 2.5]), so we are done.

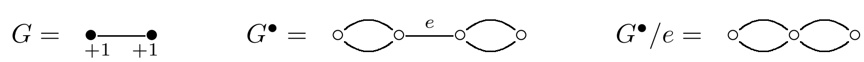
Suppose $d \leq 0$; by Lemma 2.2 we can assume $\delta \neq 0$. Let $\underline{d}$ again be any of the above representatives. One easily see that $r(X,L) = -1$ for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ (as a nonzero section of $\mathcal{O}_{\mathbb{P}^1}(1)$ cannot have two zeroes). Hence $r(X,\delta) = -1 = r_G(\delta)$ for every $X \in M^{\mathrm{alg}}(G)$. The result is proved. ∎

The proof of the next proposition contains some computations that could be avoided using later results. Nevertheless we shall give the direct proof, which explicitly illustrates previous and later topics.

Proposition 2.7. *Conjecture* 1 *holds for stable graphs of genus* 2*.*

Proof. Let G be a stable graph of genus 2 and $\delta \in \mathrm{Pic}^d(G)$. In some cases $r_G(\delta)$ is independent of G; namely if $d < 0$ then $r_G(\delta) = -1$, and if $d \geq 3$ then $r_G(\delta) = d-2$ by [**2**, Thm 3.6]. For the remaining cases we need to know G. As G is stable, it has at most two vertices; the case $|V(G)| = 1$ is treated just as for higher genus, so we postpone it to Corollary 2.11. If $|V(G)| = 2$ there are only two possibilities, which we shall treat separately. We shall use Remark 1.6 several times without mentioning it.

Case 1. G has only one edge and both vertices of weight 1. Below we have a picture of G together with its weightless model $G^\bullet$, and with a useful contraction of $G^\bullet$:



Clearly, we can identify $\mathrm{Pic}(G) = \mathbb{Z}$. Next denoting by e the bridge of $G^\bullet$, by Proposition 1.12 we have a rank preserving isomorphism

$$\mathrm{Pic}(G^\bullet) \cong \mathrm{Pic}(G^\bullet/e).$$

Finally, since there is an injection $\mathrm{Pic}(G) \hookrightarrow \mathrm{Pic}(G^\bullet)$ we also have

$$\mathrm{Pic}(G) \hookrightarrow \mathrm{Pic}(G^\bullet/e); \qquad [(d_1,d_2)] \mapsto [(0, d_1+d_2, 0)]$$

where we ordered the vertices from left to right using the picture.

For any $X \in M^{\mathrm{alg}}(G)$, we have $X = Z \cup Y$ with Z and Y smooth of genus 1, intersecting at one point.

If $d < 0$ we pick the representative $(0,d) \in \delta$. Then $r^{\max}(X,(0,d)) = -1$, hence $r(X,\delta) = -1$ and we are done. If $d \geq 3$ we pick $(d_1,d_2) \in \delta$ with $d_1 \geq 1$ and $d_2 \geq 2$ so that

$$r^{\max}(X,(d_1,d_2)) = d_1 - 1 + d_2 - 1 = d - 2 = r_G(\delta);$$

by (2.1) we are done. The case $\delta = 0$ is in 2.2. The remaining two cases, $d = 1, 2$ are done in the second and third column of the table below. The combinatorial rank is computed on $G^\bullet/e$. For the algebraic computations we used also the symmetry of the situation. The two consecutive rows starting with $r_G(\underline{d})$ and $r^{\max}(X,\underline{d})$ prove that $r(X,\delta) \leq r_G(\delta)$; the last row shows that equality holds.

$[\underline{d}] \in \mathrm{Pic}(G)$	$[(0,1)]$	$[(0,2)]$
$[\underline{d}^\bullet] \in \mathrm{Pic}(G^\bullet/e)$	$[(0,1,0)]$	$[(0,2,0)]$
$r_G(\underline{d}) =$	0	1
$r^{\max}(X, \underline{d}) =$	0	1
$\underline{d}' \sim \underline{d}$	$(a, 1-a)$	$(a, 2-a)$
$r^{\max}(X, \underline{d}') =$	$\begin{cases} a-1 \geq 1 & a \geq 2 \\ -a \geq 1 & a \leq -1 \end{cases}$	$\begin{cases} a-1 \geq 2 & a \geq 3 \\ 1 & a = 1 \\ 1-a \geq 2 & a \leq -1 \end{cases}$

Case 1 is finished.

Case 2. G is a binary graph, as in Example 2.3, with 3 edges. We have $\mathrm{Pic}^0(G) \cong \mathbb{Z}/3\mathbb{Z}$. If $d < 0$ or $d \geq 3$ we know $r_G(\delta)$; for the remaining cases we listed the rank of each class in the table below, with a choice of representatives making the computations trivial (by Lemma 1.14).

$d = 0$	$r_G(0,0) = 0$	$r_G(1,-1) = -1$	$r_G(2,-2) = -1$
$d = 1$	$r_G(0,1) = 0$	$r_G(1,0) = 0$	$r_G(2,-1) = -1$
$d = 2$	$r_G(0,2) = 0$	$r_G(1,1) = 1$	$r_G(2,0) = 0$

Let now $X \in M^{\mathrm{alg}}(G)$; we already described such curves in Example 2.3, where we proved the result for $\delta = [(1,1)]$, which we can thus skip, as well as $\delta = [(0,0)]$. We follow the rows of the table. If $d = 0$ and $a = 1, 2$ we have for any $L \in \mathrm{Pic}^{(a,-a)}(X)$,

$$r(X, L) = r(\mathbb{P}^1, \mathcal{O}(a-3)) = -1 = r_G(a, -a). \tag{2.5}$$

The case $d = 0$ is done. Next, $r^{\max}(X, (0,1)) \leq 0$, and it is clear if $L = \mathcal{O}(p)$, with p nonsingular point of X, we have $r(X, L) = 0$; hence $r^{\max}(X, (0,1)) = 0$. For the other multidegrees in $[(0,1)]$ we have

$$r(X, (3a, 1-3a)) = \begin{cases} r(\mathbb{P}^1, \mathcal{O}(3a-3) = 3a-3 \geq 0 & \text{if } a \geq 1 \\ r(\mathbb{P}^1, \mathcal{O}(-3a-2) = -3a-2 \geq 1 & \text{if } a \leq -1. \end{cases}$$

So $r(X, [(0,1)]) = 0 = r_G([(0,1)])$. As for the last class of degree 1, for every X and $L \in \mathrm{Pic}^{(2,-1)}(X)$ we have

$$r(X, L) = r(\mathbb{P}^1, \mathcal{O}(-1)) = -1 = r_G(2, -1)$$

hence this case is done.

We are left with $\delta = [(0,2)]$; we claim $r(X,\delta) = 0$ for every X. By Riemann-Roch $r(X,L) \geq 0$ for any $L \in \operatorname{Pic}^2(X)$, so we need to prove that for some $\underline{d} \in \delta$ equality holds for every $L \in \operatorname{Pic}^{\underline{d}}(X)$; choose $\underline{d} = (3,-1)$, then $r(X,L) = r(\mathbb{P}^1, \mathcal{O}(3-3)) = 0$ as claimed.

To finish the proof notice that $r(X,\delta) = -1$ if $d < 0$ (easily done arguing as for (2.5)). Finally, we claim $r(X,\delta) = d-2$ if $d \geq 3$. For this we pick for δ a representative (d_1,d_2) with $d_1 \geq 0$ and $d_2 \geq 3$; then one checks easily that $r^{\max}(X,(d_1,d_2)) = d-2$; by (2.1) we are done. ■

2.3. High degree divisors and irreducible curves. Recall that we can assume all graphs and curves semistable of genus at least 2. The following theorem states that if $d \geq 2g-2$ then Identity (2.2) is true in a stronger form. First we need the following.

DEFINITION 2.8. Let G be a semistable graph of genus $g \geq 2$, and let $\underline{d} \in \operatorname{Div}^d G$. We say that $\underline{d}$ is *semibalanced* if for every $Z \subset V(G)$ the following inequality holds

$$\underline{d}(Z) \geq \underline{k}_G(Z)d/(2g-2) - (Z \cdot Z^c)/2 \tag{2.6}$$

and if for every vertex v of weight zero and valency 2 we have $\underline{d}(v) \geq 0$.

We say that $\underline{d}$ is *balanced* if it is semibalanced and if for every vertex v of weight zero and valency 2 we have $\underline{d}(v) = 1$.

The reason for introducing this technical definition (the graph theoretic analogue of [**9**, Def. 4.6]) is that for line bundles of semibalanced multidegree we have extensions of Riemann's, and partially Clifford's, theorem, as we shall see in the proof of the next theorem.

THEOREM 2.9. *Let G be a semistable graph of genus g and assume $d \geq 2g-2$. Then for every $\delta \in \operatorname{Pic}^d(G)$ the following facts hold.*

(1) Conjecture 1 *holds.*
(2) There exists $\underline{d} \in \delta$ such that $r^{\max}(X,\underline{d}) = r_G(\underline{d})$ for every $X \in M^{\mathrm{alg}}(G)$.
(3) Every semibalanced $\underline{d} \in \delta$ satisfies part (2)*.*

PROOF. We have that every $\delta \in \operatorname{Pic} G$ admits a semibalanced representative (see [**9**, Prop. 4.12]). Therefore (3) implies (2), which obviously implies (1). We shall now prove (3).

If $d \geq 2g-1$, by [**2**, Thm 3.6] we have $r_G(\delta) = d-g$.

On the other hand, by the Riemann-Roch theorem for curves, we have $r(X,L) \geq d-g$ for every line bundle L of degree d.

Now, by the extension of Riemann's theorem to singular curves [**10**, Thm 2.3], for every balanced representative $\underline{d} \in \delta$, and for every $L \in \operatorname{Pic}^{\underline{d}}$, we have

$$r(X,L) = d-g. \tag{2.7}$$

Hence if $\underline{d}$ is balanced we are done. It remains to show that the theorem we just used extends to semibalanced multidegrees. A balanced multidegree $\underline{d}$ is defined as a semibalanced one, satisfying the extra condition $\underline{d}(v) = 1$ for any vertex v of weight zero and valency 2. Now it is simple to check that the proof of that theorem never uses the extra condition, hence (2.7) holds also for any L of semibalanced multidegree. This completes the proof in case $d \geq 2g-1$.

Now assume $d = 2g-2$. By Remark 1.10 we have $r_G(\delta) \leq g-1$ with equality if and only if δ is the canonical class. Let $\underline{d} \in \delta$ be semibalanced. By [**10**, Thm 4.4] (an extension of Clifford's theorem), if $\underline{d}$ is such that for every subcurve $Z \subsetneq X$ of arithmetic genus g_Z we have the following inequality

$$\underline{d}(Z) \geq 2g_Z - 1, \tag{2.8}$$

then we have $r^{\max}(X, \underline{d}) \leq g-1$ with equality if and only if $\underline{d} = \underline{\deg} K_X$; as $\underline{\deg} K_X = \underline{k}_G$ we will be done if (2.8) holds for every subcurve Z.

To prove that, we abuse notation writing $Z \subset V(G)$ for the set of vertices corresponding to the components of Z. As $\underline{d}$ is semibalanced we have

$$\underline{d}(Z) \geq \underline{k}_G(Z) - (Z \cdot Z^c)/2 = 2g_Z - 2 + (Z \cdot Z^c) - (Z \cdot Z^c)/2$$

as by (1.11) we have $\underline{k}_G(Z) = \deg_Z K_X = 2g_Z - 2 + (Z \cdot Z^c)$. Therefore

$$\underline{d}(Z) \geq 2g_Z - 2 + (Z \cdot Z^c)/2 \geq 2g_Z - 3/2,$$

(as $(Z \cdot Z^c) \geq 1$) which implies $\underline{d}(Z) \geq 2g_Z - 1$. So (2.8) holds and we are done. ∎

COROLLARY 2.10. *Conjecture* 1 *holds if* $d \leq 0$.
To prove Conjecture 1 *in all remaining cases it suffices to prove it for* $d \leq g-1$.

PROOF. For $\underline{d} \in \mathrm{Div}(G)$ set $\underline{d}^* = \underline{k}_G - \underline{d}$ so that $|\underline{d}^*| = 2g - 2 - d$. Then, by Riemann Roch, $r^{\max}(X, \underline{d}) = r_G(\underline{d})$ if and only if $r^{\max}(X, \underline{d}^*) = r_G(\underline{d}^*)$. Therefore the Conjecture holds for $[\underline{d}]$ if and ony if it holds for $[\underline{d}^*]$.

If $d \leq 0$ then $|\underline{d}^*| \geq 2g - 2$ and the Conjecture holds by Theorem 2.9. If $d \geq g$ then $|\underline{d}^*| \leq g - 2$, so we reduced to the required range. ∎

COROLLARY 2.11. *Conjecture* 1 *holds if* $|V(G)| = 1$*, i.e. if* $M^{\mathrm{alg}}(G)$ *parametrizes irreducible curves.*

PROOF. The graph G consists of a vertex v of weight h and $g-h$ loops attached to v, with $0 \leq h \leq g$; recall that we can assume $g \geq 2$. Let $\delta = [d] \in \mathrm{Pic}\, G$; we can assume $1 \leq d \leq g-1$. By [**2**, Lemma 3.7] we have $r_G(d) = \left\lfloor \frac{d}{2} \right\rfloor$.

Let now $X \in M^{\mathrm{alg}}(G)$; as X is irreducible Clifford's theorem holds, hence $r(X, L) \leq \left\lfloor \frac{d}{2} \right\rfloor$ for every $L \in \mathrm{Pic}^d(X)$. We must prove there exists $X \in M^{\mathrm{alg}}(G)$ admitting $L \in \mathrm{Pic}^d(X)$ for which equality holds. If $d = 1$ we take $L = \mathcal{O}_X(p)$ with p nonsingular point of X; then $r(X, \mathcal{O}_X(p)) = 0$. We are left with the case $g \geq 3$; it is well known that $M^{\mathrm{alg}}(G)$ contains a hyperelliptic curve, X, and that there exists $L \in \mathrm{Pic}^d(X)$ for which $r(X, L) = \left\lfloor \frac{d}{2} \right\rfloor$. So we are done. ∎

For convenience, we collect together all the cases treated in the paper.

SUMMARY 2.12. *Let G be a (finite, connectected, weighted) graph of genus g and let $\delta \in \mathrm{Pic}^d(G)$. Then Conjecture* 1 *holds in the following cases.*

(1) $g \leq 1$.
(2) $d \leq 0$ *and* $d \geq 2g - 2$.
(3) $|V(G)| = 1$.
(4) *G is a stable graph of genus 2.*

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Clay Mathematics Proceedings
Volume **18**, 2013

A tour of stable reduction with applications

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To Joe Harris.

Abstract. The stable reduction theorem for curves asserts that for a family of stable curves over the punctured disk, after a finite base change, the family can be completed in a unique way to a family of stable curves over the disk. In this survey we discuss stable reduction theorems in a number of different contexts. This includes a review of recent results on abelian varieties, canonically polarized varieties, and singularities. We also consider the semi-stable reduction theorem and results concerning simultaneous stable reduction.

Introduction

The stable reduction theorem for curves [**48, 101**] asserts that given a family of stable curves over the punctured disk, after a finite base change, the family can be completed in a unique way to a family of stable curves over the disk. In particular, the central fiber of the new family is determined, up to isomorphism, by the original family. This theorem plays a central role in the study of curves. A consequence is the fundamental result that the moduli space of stable curves is compact. Qualitatively, the theorem provides control over degenerations of smooth curves: when studying one-parameter degenerations, one may restrict to the case where the limit has normal crossing singularities.

In [**72**, §3.C] Harris–Morrison give a beautiful treatment of the stable reduction theorem from a computational perspective. They outline a proof of the theorem that provides the reader with a method of completing this process in particular examples, and importantly, of identifying the central fiber of the new family. The aim of this survey is to complement [**72**, §3.C] with stable reduction problems in other settings.

Roughly speaking, by a stable reduction problem we mean the problem of determining a class of degenerations so that a family over the punctured disk can, after a finite base change, be extended in a unique way to a family over the disk.

Typically the motivation will be a moduli problem, where one is given a particular class of geometric or algebraic objects that determine a non-compact moduli space. The stable reduction problem can be viewed as providing a modular compactification of the moduli space. In the language of stacks, stable reduction is equivalent to the valuative criterion of properness for the moduli stack (see §2). In

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2010 *Mathematics Subject Classification.* Primary 14H10, 14J10, 14K10.

The author was partially supported by NSF grant DMS-1101333.

§1, we work out an explicit motivating example. The main cases we will consider in the survey are stable reduction for abelian varieties §5, curves §6, and canonically polarized varieties §7.

Determining a class of degenerations that will provide a stable reduction theorem is often difficult. In this situation one can begin with the qualitative goal of obtaining some level of control over degenerations. For instance, one may focus on restricting the singularities of the central fiber, or controlling invariants such as monodromy §4.

In this direction, semi-stable reduction is the problem of filling in (possibly after a finite base change) a smooth family of schemes over the punctured disk to a family where the total space is smooth, and the central fiber is a reduced scheme with simple normal crossing singularities. Unlike the case of stable reduction, such a completion will not be unique. On the other hand, the singularities (and topology) of a semi-stable reduction will typically be much simpler than that of a stable reduction.

The main result in this context is a theorem of Mumford et al. [**82**] stating that semi-stable reductions exist in complete generality in characteristic 0 (see §3). This plays a central role in many stable reduction theorems. In particular, Kollár–Shepherd-Barron–Alexeev have developed an approach to the stable reduction problem using log canonical models of semi-stable reductions. We use the case of curves §6 and canonically polarized varieties §7.3 to discuss this.

One can also consider the question of extending families over higher dimensional bases. Given a stable reduction theorem one can then ask whether families over a dense open subset of a scheme of dimension 2 or more can be extended after a generically finite base change. We call this a simultaneous stable reduction problem. For moduli spaces that are proper Deligne–Mumford stacks, it is well known that simultaneous stable reductions always exist (Theorem 8.2, [**54**], [**50**]). However, in general, this is a delicate problem. Explicitly describing such a generically finite base change can be quite difficult.

We review simultaneous stable reduction in §8, where we focus on the cases of abelian varieties and curves. One recent motivation for considering this type of problem has to do with resolving birational maps between moduli spaces. The cases arising in the Hassett–Keel pogram for the moduli space of curves have received a great deal of attention recently; we review this in §8.5.

In light of the breadth of the topic, to prevent this survey from becoming too lengthy, we have chosen to focus on a few cases that have a historic connection to the stable reduction theorem for curves, capture the flavor of the topic in general, and which point to some of the recent progress in the field. We also include a number of examples. In the end, the material chosen reflects the author's exposure to the subject, and he apologizes to those people whose work was not included.

Acknowledgements. It is a pleasure to thank J. Achter, J. Alper, J. de Jong, M. Fedorchuk, D. Grant, C. Hall, D. Jensen, J. Kass, J. Kollár, S. Kovács, R. Laza, M. Lieblich, Z. Patakfalvi, R. Smith, R. Virk and J. Wise for discussions about various topics covered in the survey that have greatly improved the exposition. Special thanks are also due to J. Achter, J. Alper, D. Jensen, J. Kass, J. Kollár, R. Laza, Z. Patakfalvi, R. Smith, R. Virk and the referee for detailed comments on earlier drafts.

Notation and conventions.

1. A **family of schemes** $f : X \to B$ will be a flat, surjective, finite type morphism of schemes, of constant relative dimension. The scheme B will be called the **base** of the family and X the **total space** of the family. For a point $b \in B$, we denote by X_b the fiber of f over b.

2. We will typically use the following notation for spectrums of discrete valuation rings (DVRs). For a DVR R we will use the notation $K = K(R)$ for the fraction field, and $\kappa = \kappa(R)$ for the residue field. We will set $S = \operatorname{Spec} R$, with generic point $\eta = \operatorname{Spec} K$ and closed point $s = \operatorname{Spec} \kappa$.

3. If B is noetherian, and the family $f : X \to B$ is of constant relative dimension d, the **discriminant**, denoted Δ, is the 0-th Fitting scheme of the push-forward of the structure sheaf of the d-th Fitting scheme of the coherent sheaf $\Omega_{X/B}$. The discriminant parameterizes the singular fibers of the family. Typically, we consider this in the case where either the d-th fitting scheme of $\Omega_{X/B}$ is finite over B, or f is proper; in these cases Δ is a closed subscheme of B.

4. Let X be a scheme over an algebraically closed field k, which is regular in codimension one, and let D be an effective Weil divisor on X. We say D is in **étale (resp. Zariski or simple) normal crossing** position if X is regular along the support of D and for each closed point $x \in \operatorname{Supp}(D)$ there exists an étale morphism (resp. an open inclusion) $f : U \to X$ such that for any $u \in U$ with $f(u) = x$, there is a local system of parameters $u_1, \dots, u_n$ for $\mathscr{O}_{U,u}$ so that the pull-back of D via the composition $\operatorname{Spec} \mathscr{O}_{U,u} \to U \to X$, is defined by a product $u_1^{n_1} \cdots u_r^{n_r}$, for some $0 \le r \le n$ and some non-negative integers $n_1, \dots, n_r$. We will say a divisor is **nc**, (resp. **snc**) if it is in étale normal crossing (resp. simpe normal crossing) position.

5. A **modification** is a proper, birational morphism. An **alteration** is a generically finite, proper, surjective morphism.

6. A **germ** will be the spectrum of a complete local ring A and we will use the notation (X, x) with $X = \operatorname{Spec} A$ and x the maximal idea of A. A **(germ of a) singularity** will be a germ that is singular at x. We will typically focus on **hypersurface singularities**; by this we mean the case where $X = \operatorname{Spec} k[[x_1, \dots, x_n]]/(f)$, $f \in k[[x_1, \dots, x_n]]$ and k is a field. We will say a singularity is **isolated** if $\mathscr{O}_{X,x'}$ is a regular local ring for all $x' \in X$ with $x' \neq x$.

1. An example via elliptic curves

In this section we start by briefly reviewing a stable reduction for a 1-parameter family of elliptic curves degenerating to a cuspidal cubic, as described in Harris–Morison [**72**, Ch. 3.C]. We then turn to the case of simultaneous stable reduction, and give an explicit computation of a simultaneous stable reduction for a versal deformation of a cuspidal cubic. In other words, we analyze all 1-parameter degenerations at once. The presentation we give is a special case of a larger example described by Laza and the author in [**39**] (related to well known work of Brieskorn [**32**] and Tyurina [**121**, §3]; see also the recent work of Fedorchuk [**53**, §5], [**54**]) and can be viewed as an extension of the discussion in Harris–Morrison [**72**, p.129-30].

1.1. Stable reduction for a pencil of cubics. Stable reduction concerns one parameter degenerations. In this subsection we briefly review a stable reduction for a family of non-singular plane cubics degenerating to a plane cubic with a cusp. For brevity, we will leave out any computations. The reader is encouraged to read [**72**, Ch. 3.C], where this example is worked out in detail (see also §1.3 and §4.1.2).

Fix an algebraically closed field k with characteristic not equal to 2 or 3 and consider the family $X \to B = \mathbb{A}^1_k$ given by:

$$x_2^2 + x_1^3 + t_3 = 0,$$

where t_3 is the parameter on $\mathbb{A}^1_k$. The family is smooth away from $t_3 = 0$, and the central fiber has a cusp.

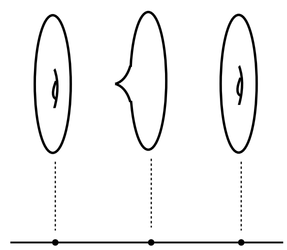

FIGURE 1. A degenerate family

The goal of stable reduction for curves is to replace the central fiber of $X \to B$ with a stable curve. We will see via the theory of monodromy, and by a direct computation, that this is not possible without a degree six base change. So let $B' = \operatorname{Spec} k[t_3'] \to \mathbb{A}^1_k$ be the degree six map given by $t_3' \mapsto (t_3')^6$. After base change we obtain a new family $X' = B' \times_B X \to B'$, which is also smooth away from the central fiber, and has a cuspidal cubic as the central fiber. Let $U = \operatorname{Spec} k[t_3']_{t_3'}$.

By an appropriate sequence of birational transformations of the surface X' (e.g. [**72**, p.122-129] for a slight variation), one can obtain a new family $\widehat{X} \to B'$ that is isomorphic to X' over U, and such that the central fiber $\widehat{X}_0$ of $\widehat{X}$ is a non-singular curve. The family $\widehat{X} \to B'$ is called a stable reduction of the family

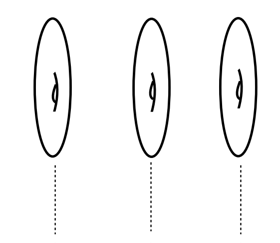

FIGURE 2. The stable reduction

$X \to B$. For an explicit computation with equations, see §4.1.2 (and also §1.3).

1.2. A 2-parameter family of cubics. We will next consider a concrete example of simultaneous stable reduction. While in general this is a more delicate question than that of stable reduction, in this example we will be able to make the simultaneous stable reduction completely explicit. We start by describing the family of curves, which can be viewed as a versal deformation of a cuspidal cubic.

Fix an algebraically closed field k with characteristic not equal to 2 or 3. Consider the family of curves

$$x_2^2 + x_1^3 + t_2 x_1 + t_3 = 0$$

with parameters t_2 and t_3. We denote the family by $X \to B$. One can easily check that the curve defined by the point (t_2, t_3) is non-singular if and only if $4t_2^3 - 27t_3^2 \neq 0$. The curve has a unique singularity, which is a node, if $4t_2^3 - 27t_3^2 = 0$ and $(t_2, t_3) \neq (0, 0)$, and the curve has a unique singularity, which is a cusp, if $(t_2, t_3) = (0, 0)$. Despite the family technically being none of the following, we will view it simultaneously as a family of projective curves of arithmetic genus one, a degenerate family of abelian varieties, and a deformation of a cusp.

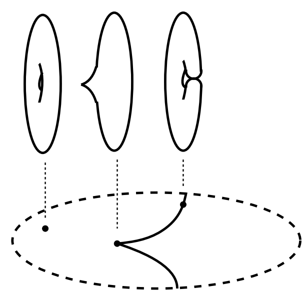

FIGURE 3. A degenerate family

REMARK 1.1. To make this discussion precise we should take

$$X = \operatorname{Proj}_{\mathbb{A}^2_k}\left(\frac{k[t_2,t_3][X_0,X_1,X_2]}{(X_0X_2^2 + X_1^3 + t_2X_0^2X_1 + X_0^3t_3)}\right) \subseteq \mathbb{P}^2_k \times \mathbb{A}^2_k,$$

$B = \mathbb{A}^2_k = \operatorname{Spec} k[t_2,t_3]$, $\pi : X \to \mathbb{A}^2_k$ the morphism induced by the second projection $\mathbb{P}^2_k \times \mathbb{A}^2_k \to \mathbb{A}^2_k$, and $\sigma_\infty : \mathbb{A}^2_k \to X$ the section at infinity given by the ring homomorphism

$$\frac{k[t_2,t_3][X_0,X_1,X_2]}{(X_0X_2^2 + X_1^3 + t_2X_0^2X_1 + X_0^3t_3)} \to k[t_2,t_3][X_0]$$

defined by the ideal (X_1, X_2). We then obtain a diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "\pi"] & \mathbb{P}^2 \times \mathbb{A}^2_k \arrow[d, "\pi_2"] \\
& \mathbb{A}^2_k. \arrow[ul, bend left, "\sigma_\infty"]
\end{tikzcd} \tag{1.1}$$

The morphism $\pi : X \to \mathbb{A}^2_k$ is a flat family of projective curves of arithmetic genus one. The section σ_∞ defines a group scheme structure and polarization on the generic fiber. This makes the generic fiber a principally polarized abelian scheme of dimension one. Restricting to germs, we obtain a deformation of a cusp.

Let us make a few more informal observations. Set

$$G = X - \{(0,0,t_2,t_3) : 4t_2^3 - 27t_3^2 = 0\}$$

(where here we are taking X to be the projective family). Then $\pi : G \to \mathbb{A}^2_k$ is a family of commutative groups. The group parameterized by (t_2, t_3) is a copy of $\mathbb{G}_m$, if $4t_2^3 - 27t_3^2 = 0$ and $(t_2,t_3) \neq (0,0)$. The group is a copy of $\mathbb{G}_a$, if $(t_2,t_3) = (0,0)$. These are the groups of line bundles of degree zero on the corresponding fibers. In fact $G/\mathbb{A}^2_k$ is the relative (connected component of the) Picard scheme $\mathbf{Pic}^0_{X/\mathbb{A}^2_k}$, and $X/\mathbb{A}^2_k$ is the compactified (connected component of the) Picard scheme $\overline{\mathbf{Pic}}^0_{X/\mathbb{A}^2_k}$ (see Altman–Kleiman [**18**]).

For the purpose of this discussion, we view it as pathological that the central fiber of the family $\pi : X \to \mathbb{A}^2_k$ has a cusp (and the central fiber of the family $\mathbf{Pic}^0_{X/\mathbb{A}^2_k} \to \mathbb{A}^2_k$ is an additive group). Our goal will be to modify the family so that we may replace the central fiber with a nodal curve (or a copy of $\mathbb{G}_m$ in the case of the family of groups, or a collection of smooth components meeting transversally in the case of a singularity).

The problem can also be stated in stack-theoretic language. Let $\overline{\mathcal{M}}_{1,1}$ be the moduli stack of Deligne–Mumford stable, one-pointed curves of arithmetic genus one, and let $\overline{M}_{1,1}$ be the coarse moduli space. The family $X/\mathbb{A}^2_k$ defines a rational map $\mathbb{A}^2_k \dashrightarrow \overline{\mathcal{M}}_{1,1}$ and we would like to give a resolution of this map.

1.3. Explicit simultaneous stable reduction. We now construct an explicit simultaneous stable reduction of the family. We will do this in several steps, and then discuss a monodromy computation that sheds light on the problem.

1.3.1. *Step 1: pulling back by a Weyl (group) cover.* Consider the map

$$\{a_1 + a_2 + a_3 = 0\} \to \mathbb{A}^2_k$$

$$(a_1, a_2, a_3) \mapsto (a_1a_2 + a_1a_3 + a_2a_3, -a_1a_2a_3).$$

The families obtained are given by the diagram below.

$$\begin{array}{ccc} \{a_1 + a_2 + a_3 = x_2^2 + \prod_{i=1}^3 (x_1 - a_i) = 0\} & \longrightarrow & \{x_2^2 + x_1^3 + t_2x_1 + t_3 = 0\} \\ \downarrow & & \downarrow \\ \{a_1 + a_2 + a_3 = 0\} & \longrightarrow & \mathbb{A}^2_k. \end{array} \tag{1.2}$$

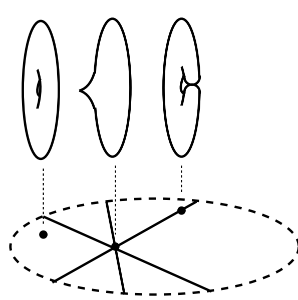

FIGURE 4. The Weyl cover

There is still a unique fiber that is cuspidal, but the discriminant has been replaced by a hyperplane arrangement of type A_2, given by the equation

$$\{(a_2 - a_3)^2(a_1 - a_3)^2(a_1 - a_2)^2 = 0\}.$$

Set $B' \to B$ to be the finite (Weyl) cover defined above, and set $X' \to B'$ to be the family obtained by pull-back. The Weyl group in this case is the group of type A_2; i.e. the permutation group Σ_3.

1.3.2. *Step 2: a wonderful blow-up.* It is a general principle that putting the discriminant locus into normal crossing position is beneficial (not only is a normal crossing divisor easier to understand, there is also the Borel Extension Theorem [**29**] for abelian varieties and the work of de Jong–Oort [**47**] and Cautis [**40**] for stable curves, all of which will be discussed in more detail in §5 and §6).

We put the discriminant in this example into nc position by blowing up the point that is the intersection of its three components. Explicitly, on one coordinate patch, we consider the map

$$\{1 + b_2 + b_3 = 0\} \to \{a_1 + a_2 + a_3 = 0\}$$
$$(b_1, b_2, b_3) \mapsto (b_1, b_1b_2, b_1b_3).$$

Pulling the family back by this map gives the new family:

$$\begin{array}{ccc} \{1 + b_2 + b_3 = x_2^2 + (x_1 - b_1)\prod_{i=1}^{2}(x_1 - b_1b_i) = 0\} & \longrightarrow & \dots \\ \downarrow & & \\ \{1 + b_2 + b_3 = 0\} & \longrightarrow & \dots . \end{array} \tag{1.3}$$

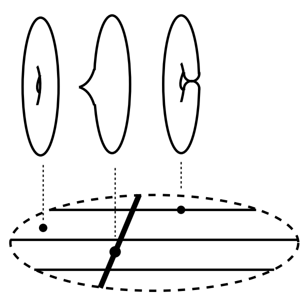

FIGURE 5. The wonderful blow-up

Denote the space obtained by this blow-up as $\tilde{B} \to B'$. We call this the wonderful blow-up. Let $\widetilde{X} \to \widetilde{B}$ be the family obtained by pull-back. This restricts to a family of cuspidal curves over the exceptional curve $\{b_1 = 0\}$. Note that the generic point of each irreducible component of the discriminant now parameterizes curves with a unique singularity of type A_1 or A_2.

REMARK 1.2. We will see below in §1.4 that over $\tilde{B}$ we can not replace the cuspidal curves with stable curves. In other words, there does not exist a morphism extending the rational map $\tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$. However, we can extend the map to the moduli scheme (see §1.3.6); i.e. there is a morphism extending the rational map $\tilde{B} \dashrightarrow \overline{M}_{1,1}$.

1.3.3. *Step 3: a double cover.* In order to obtain a family of stable curves, we will need to take a double cover of the base, branched along the exceptional locus. The double cover is not possible globally (the exceptional divisor does not admit a square root), so we proceed locally. Consider the map

$$\{1 + c_2 + c_3 = 0\} \to \{1 + b_2 + b_3 = 0\}$$

$$(c_1, c_2, c_3) \mapsto (c_1^2, c_2, c_3).$$

Pulling the family back by this map gives the new family:

$$\begin{array}{ccc} \{1 + c_2 + c_3 = x_2^2 + (x_1 - c_1^2)\prod_{i=1}^2 (x_1 - c_1^2 c_i) = 0\} & \longrightarrow & \dots \\ \downarrow & & \\ \{1 + c_2 + c_3 = 0\} & \longrightarrow & \dots . \end{array} \tag{1.4}$$

Let us denote this finite cover by $\tilde{B}' \to \tilde{B}$ and let $\widetilde{X}' \to \widetilde{B}'$ be the family obtained by pull-back.

1.3.4. *Step 4: blowing up the cusp locus in the total space.* There is a family of cuspidal curves lying over the locus $\{c_1 = 0\}$. In the total space $\widetilde{X}'$, the locus of cusps in the fibers is given as $\{c_1 = x_1 = x_2 = 0\}$. Our goal will be to perform a blow-up supported on this locus that will provide a family of semi-stable curves.

To do this, blow-up $\widetilde{X}'$ along the ideal

$$I = \left((c_1^2, x_1)^3, (c_1^3, c_1 x_1) \cdot (x_2), x_2^2\right).$$

Let us denote the resulting family as $\mathrm{Bl}_I \widetilde{X}' \to \widetilde{B}'$. The blow-up replaces the cuspidal curves with nodal curves consisting of two irreducible components: the desingularization of the cuspidal curve, which is a copy of $\mathbb{P}^1$ sitting inside of the blow-up $\mathrm{Bl}_{(x_1^3, x_2^2)} \mathbb{A}_k^2$, and a stable elliptic curve sitting inside of the weighted projective space $\mathbb{P}(1, 2, 3)$. We mention here that Hassett [**73**, §6.2] has determined the tails arising from a much more general class of singularities; we will discuss these results later in §10.

In short, we have locally (on the base) constructed an explicit semi-stable reduction of the cuspidal family, which is stable except in the fibers over the locus $\{c_1 = 0\}$, where it is nodal, but not stable.

1.3.5. *Step 5: the relative dualizing sheaf.* Finally, one can take the relative canonical model (obtained via the relative dualizing sheaf) for the family of nodal curves. Concretely, this will contract the extraneous $\mathbb{P}^1$s in the fibers, giving a family of stable curves. Let us denote this family by $\widehat{X} \to \widetilde{B}'$.

1.3.6. *Summary.* We now have a family $\widehat{X} \to \widetilde{B}'$ of stable curves extending the pull-back of the original family, where $\widetilde{B}'$ is an alteration of B. By definition, we obtain a morphism

$$\widetilde{B}' \to \overline{\mathcal{M}}_{1,1} \to \overline{M}_{1,1}.$$

The map $\widetilde{B}' \to \widetilde{B}$ is finite, so in fact there is a map $\widetilde{B} \to \overline{M}_{1,1}$ (e.g. [**40**, Lem. 2.4]). The locus $\{c_1 = 0\}$ can be identified with the λ-line, with corresponding family given by $y^2 = x(x-1)(x-\lambda)$. The points $\lambda = 0, 1, \infty$ correspond to the intersections of the strict transforms of the hyperplane arrangement (the discriminant after the Weyl cover). The restricted map $\{c_1 = 0\} \to \overline{M}_{1,1}$ can be identified with the map from the λ-line to the j-line.

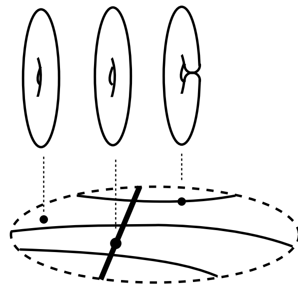

FIGURE 6. The simultaneous stable reduction

1.4. Obstructions. We have seen that there exists an extension of the rational map $\tilde{B} \dashrightarrow \overline{M}_{1,1}$ to the moduli scheme $\tilde{B} \to \overline{M}_{1,1}$. We now show the rational map $\tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ to the moduli stack does not extend; i.e. there is no family of stable curves over $\tilde{B}$ extending the pull-back of the original family.

We do this in the following way. Let S be the spectrum of a DVR with closed point s and generic point η. We will find a morphism $S \to \tilde{B}$ sending s to a closed point of the exceptional divisor (i.e. $\{c_1 = 0\}$, parameterizing the cuspidal locus) and sending η to the generic point of $\tilde{B}'$ (i.e. the smooth locus). Then we will show that the induced family of curves $X_S \to S$ does not extend to a family of stable curves; i.e. the composition $S \to \tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ does not extend to a morphism.

We will show in two ways that the general $S \to \tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ as above does not extend to a morphism. The first is via a monodromy computation. The second method is via a computation following an argument of Fedorchuk [**54**].

1.4.1. *The monodromy obstruction.* Consider the family $X' \to B'$ obtained via the Weyl cover, and the restriction $(X')_{|L} \to L$ of this family to a generic line L through the origin in B'. To show that there is no extension $\tilde{B} \to \overline{\mathcal{M}}_{1,1}$ to the moduli stack, it suffices to show that the restriction $(X')_{|L} \to L$ does not extend to a stable family of curves. To show this, observe that in the notation of §1.3.2, the restriction $(X')_{|L} \to L$ is a surface Z_{b_2} with equation (locally near the A_2 singularity):

$$x_2^2 + x_1^3 - (b_2^2 + b_2 + 1)b_1^2 x_1 - b_2(1 + b_2)b_1^3 = 0, \tag{1.5}$$

where b_1 is a parameter for L and b_2 is a (generic) fixed slope.

The surface Z_{b_2} has a D_4 singularity at the origin. This is also a cusp singularity for X_0, the central fiber of Z_{b_2} viewed as a family of curves. Recall that the standard resolution of a D_4 surface singularity $x^2 = f_3(y, z)$ is given by 4 blow-ups: First blow-up the D_4 singularity. This gives an exceptional divisor E_0. The D_4 singularity "splits" into three A_1 singularities corresponding to the three roots of f_3. Then blow-up each A_1 singularity. This introduces exceptional divisors E_1, E_2, E_3, giving the desired resolution. We associate to this a $\widetilde{D}_4$ graph (consisting of E_0 the central vertex, to which one attaches edges connecting the 4 vertices corresponding to the curves X_0, E_1, E_2 and E_3).

The monodromy obstruction can be identified via the theory of elliptic fibrations. From the $\widetilde{D}_4$ graph, we conclude that this is a type I_0^* degeneration in Kodaira's classification (see [**24**, §V.7, p.201]). It follows that the monodromy is $-\operatorname{Id}$ (see [**24**, p.210]). We also direct the reader to the discussion of the elliptic involution in [**72**, Ch. 2A], and to the monodromy computation made in §4.1.2. In conclusion, the monodromy not being unipotent, the family does not extend to a family of stable curves (this fact is reviewed in §5 and §6).

1.4.2. *An obstruction via a direct computation.* Again, our goal is to show that the general map $S \to \tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ with closed point sent to the cuspidal locus, and generic point sent to the smooth locus, does not extend to a morphism.

We follow an observation of Fedorchuk [**54**, Prop. 7.4]. Let S' be the spectrum of a DVR, which is a branched double cover of S admitting a morphism to $\tilde{B}'$. Pulling back the family $\widehat{X} \to \tilde{B}'$ we obtain a family of stable curves $\widehat{X}_{S'} \to S'$. Fedorchuk's observation is that it suffices to show that the total space $\widehat{X}_{S'}$ is smooth. Indeed, if there were a family of stable curves $\widehat{X}_S \to S$ extending the pull-back of the original family, then it would follow that $\widehat{X}_{S'}$ was equal to $S' \times_S \widehat{X}_S$. But then the total space $\widehat{X}_{S'}$ would have a singular point, giving a contradiction (the point of $\widehat{X}_S$ at the node of the central fiber, locally given by $xy - t^n$ with $n \geq 1$, would be replaced with a singular point $xy - t^{2n}$ of $\widehat{X}_{S'}$).

Fedorchuk's approach is to construct a particular one-parameter family of genus two curves degenerating to a cusp (his argument implies the result for families of curves of arbitrary genus [**54**, Prop. 7.4]). Alternatively, with the work we have done here in coordinates, one can show that the blow-up in the fourth step gives a smooth total space when restricted to the general S'. Since the total space is a smooth surface, and all of the curves blown down in the fifth step are (-1)-curves, this does not introduce singularities in the total space.

2. Stable reduction and the valuative criterion for properness

We now consider stable reduction more abstractly, in terms of the valuative criterion for properness for stacks. For readers not familiar with stacks, this is not strictly necessary for the material in the subsequent sections. The main focus here will be to review the fact that stable reduction is equivalent to the properness of a moduli stack, and that for separated, finite type Deligne–Mumford stacks, the properness of the coarse moduli space is equivalent to the properness of the stack. In order to return quickly to a more concrete setting, we postpone a discussion of simultaneous stable reduction in the language of stacks until §8.1.

2.1. The valuative criterion for properness of an algebraic stack. Fix once and for all a scheme Z, and consider the étale site $\mathrm{Sch}_Z^{\text{ét}}$ (e.g. [**52**, Exa. 2.3.1, p.27]). By a Z-sheaf, (resp. algebraic Z-space, resp. Z-stack), we will mean a sheaf (resp. algebraic space, resp. stack) on $\mathrm{Sch}_Z^{\text{ét}}$ (e.g. [**94**, Def. 1.1, Def. 3.1]).

An **algebraic (or Artin) Z-stack $\mathcal{M}$ over** $\mathrm{Sch}_Z^{\text{ét}}$ is a Z-stack such that the diagonal 1-morphism of Z-stacks $\Delta : \mathcal{M} \to \mathcal{M} \times_Z \mathcal{M}$ is representable, separated and quasi-compact, and there exists an algebraic Z-space U and a 1-morphism of Z-stacks $U \to \mathcal{M}$ that is surjective and smooth (e.g. [**94**, Def. 4.1]). Note that the diagonal Δ is in fact of finite type (e.g. [**94**, Lem. 4.2]). An algebraic Z-stack $\mathcal{M}$ is a **Deligne–Mumford (DM) Z-stack** if there is a an algebraic Z-space U' and a 1-morphism of Z-stacks $U' \to \mathcal{M}$ that is surjective and étale (e.g. [**94**, Def. 4.1]).

EXAMPLE 2.1. For a concrete example we will consider $\overline{\mathcal{M}}_g$, the DM $\mathbb{C}$-stack of genus $g \geq 2$ curves over $\mathrm{Sch}_{\mathbb{C}}^{\text{ét}}$. This is a category whose objects are families $X \to B$ of Deligne–Mumford stable, genus g curves, over a $\mathbb{C}$-scheme B (see §6), and whose morphisms are given by pull-back diagrams. Recall that the moduli stack "represents the moduli problem" in the following way: for a $\mathbb{C}$-scheme B, a morphism $B \to \overline{\mathcal{M}}_g$ is equivalent to a family $X \to B$ of stable, genus g curves, over $\mathbb{C}$.

We now state the valuative criteria for separateness and properness. We refer the reader to [**94**, Def. 7.6, Def. 7.11] (see also [**48**, Def. 4.7, Def. 4.11], [**123**, Def. 1.1]) for the respective definitions of separateness and properness, and we note only that separated and finite type are assumed in the definition of properness.

THEOREM 2.2 (Valuative Criterion for Separatedness). *Let $F : \mathcal{M} \to \mathcal{B}$ be a 1-morphism of algebraic Z-stacks. Then F is separated if and only if for every valuation ring R, with field of fractions K, and every 2-commutative diagram*

(2.1)
$$\begin{tikzcd} & \mathcal{M} \arrow[d, "F"] \\ \operatorname{Spec} R \arrow[ur, dashed, shift left, "x_1"] \arrow[ur, dashed, shift right, "x_2"'] \arrow[r] & \mathcal{B} \end{tikzcd}$$

any isomorphism between $x_1|_{\operatorname{Spec} K}$ and $x_2|_{\operatorname{Spec} K}$ can be extended to an isomorphism between x_1 and x_2. If moreover $\mathcal{B}$ is locally noetherian and F is locally of finite type, then one need only consider discrete valuation rings R.

We direct the reader to [**94**, Prop. 7.8] (see also [**48**, Thm. 4.18]).

REMARK 2.3. The criterion, and in particular the 2-commutivity, can be made more explicit as follows. For all $x_1, x_2 \in \operatorname{ob} \mathcal{M}_{\operatorname{Spec} R}$, all isomorphsims $\beta : F(x_1) \to F(x_2)$ in $\mathcal{B}_{\operatorname{Spec} R}$, and all isomorphisms $\alpha : (x_1)|_{\operatorname{Spec} K} \to (x_1)|_{\operatorname{Spec} K}$ in $\mathcal{M}_{\operatorname{Spec} K}$ such that $F(\alpha) = \beta|_{\operatorname{Spec} K}$, there exists at least one (and in fact a single) isomorphism $\tilde{\alpha} : x_1 \to x_2$ in $\mathcal{M}_{\operatorname{Spec} R}$ extending α (i.e. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$) and such that $F(\tilde{\alpha}) = \beta$. We also note that it suffices to take valuation rings that are complete, and have algebraically closed residue field (see [**94**, Prop. 7.8]).

THEOREM 2.4 (Valuative Criterion for Properness). *Let $F : \mathcal{M} \to \mathcal{B}$ be a separated, finite type 1-morphism of algebraic Z-stacks. Then F is proper if and only if for every discrete valuation ring R, with field of fractions K, and every 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{B} \end{tikzcd}$$

there exists a finite extension K' of K, so that taking R' to be the integral closure of R in K', there is a 2-commutative diagram

(2.2)
$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{B} \end{tikzcd}$$

extending the original diagram.

We direct the reader to [**94**, Thm. 7.10] (see also [**48**, Thm. 4.19]).

Remark 2.5. It suffices to consider DVRs that are complete, and have algebraically closed residue field (see [**94**, Thm. 7.10]). If one removes the hypothesis that F be separated, then the criterion (2.2) is equivalent to F being universally closed (e.g. [**94**, Thm 7.10]).

Example 2.6. Let us make these criteria more concrete in the example of $\overline{\mathcal{M}}_g$, $g \geq 2$. We will use the fact that $\overline{\mathcal{M}}_g$ is of finite type over $\mathbb{C}$ [**48**, Thm. 5.2] together with the fact that $\mathbb{C}$ is noetherian to conclude that we need only consider DVRs. Then using the fact that a morphism from a scheme to $\overline{\mathcal{M}}_g$ is the same as family of stable curves over the scheme, we may reinterpret the valuative criteria as follows.

Separatedness: Let R be a DVR with fraction field K. Set $S = \operatorname{Spec} R$. Suppose $X \to S$ (resp. $Y \to S$) is a family of stable, genus g curves over S, with restriction $X_K \to \operatorname{Spec} K$ (resp. $Y_K \to \operatorname{Spec} K$). Then the valuative criterion for separateness requires that any K-isomorphism $X_K \to Y_K$ extend to an S-isomorphism of $X/S \to Y/S$.

Properness: Let R be a DVR with fraction field K. Let $X_K \to \operatorname{Spec} K$ be a family of stable, genus g curves. Then the valuative criterion for properness requires that there exist a finite extension K' of K such that setting R' to be the integral closure of R in K', there exists a family of stable curves $X' \to \operatorname{Spec} R'$ extending the family $X_{K'} = X_K \times_{\operatorname{Spec} K} \operatorname{Spec} K' \to \operatorname{Spec} K'$ (in the sense that $X'|_{K'} \cong X_{K'}$).

In conclusion, the properness of $\overline{\mathcal{M}}_g$ is equivalent to the fact that any family of stable curves over the generic point of a DVR can be extended, possibly after a generically finite base change, to a family of stable curves, and this extension is unique up to isomorphism. This is exactly the statement of the Deligne–Mumford stable reduction theorem, which we will review in §6.

Remark 2.7. In practice, a natural moduli problem (over a scheme Z) will often lead to a separated, non-proper, algebraic Z-stack $\mathcal{M}$ of finite type over Z. A stable reduction theorem for the moduli problem then consists of finding a proper algebraic Z-stack $\overline{\mathcal{M}}$, containing $\mathcal{M}$ (ideally as a dense open substack). In general, finding such stacks has proven to be quite difficult. One approach is to use GIT to determine a (GIT-)stability condition, with the hope that the stable objects will provide the correct class to define $\overline{\mathcal{M}}$; we discuss this further in §11. Another approach, due to Kollár–Shepherd-Barron–Alexeev, which uses the Minimal Model Program (MMP), has had a great deal of success lately; we discuss this further in §7.

Remark 2.8. It may also happen that there are many such proper stacks $\overline{\mathcal{M}}$ from which to choose. Following Smyth (e.g. [**119**]), who has investigated this question extensively for the moduli of curves, a useful way to frame the problem is as follows. By considering all "degenerations" of objects in $\mathcal{M}$, one may obtain a "highly non-separated" algebraic Z-stack $\mathcal{U}$, which contains $\mathcal{M}$ as an open substack. Essentially by construction, the stack $\mathcal{U}$ should satisfy the valuative criterion (2.2). One is then in the situation of identifying proper substacks $\overline{\mathcal{M}}$ of $\mathcal{U}$ that contain $\mathcal{M}$. We direct the reader to Smyth [**119**] for more on this, especially for the case of curves (see also [**17**],[**16**] where a notion weaker than properness is considered).

2.2. Moduli spaces. A moduli space for a stack is an algebraic space or scheme that is as close as possible to the stack. More precisely, a **categorical moduli space** for an algebraic Z-stack $\mathcal{M}$ is a Z-morphism $\pi : \mathcal{M} \to M$ to an algebraic Z-space such that π is initial for Z-morphisms to algebraic Z-spaces. This means that given any Z-morphism $\Phi : \mathcal{M} \to Y$ to an algebraic Z-space, there is a unique Z-morphism $\eta : M \to Y$ making the following diagram commute

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & Y \\ M \arrow[ur, dashed, "\exists!\eta"'] & \end{tikzcd}$$

We will call M a **categorical moduli scheme** if M is a Z-scheme.

A **coarse moduli space (resp. scheme)** is a categorical moduli space (resp. scheme) satisfying the additional condition that for every algebraically closed field k, the induced map $|\mathcal{M}(k)| \to M(k)$ is a bijection, where $|\mathcal{M}(k)|$ is the set of isomorphism classes of the groupoid $\mathcal{M}_{\operatorname{Spec} k}$. For stacks with finite inertia there is the following theorem of Keel–Mori. Recall that for an algebraic Z-stack $\mathcal{M}$, the inertia stack $I_Z(\mathcal{M})$ is the fiber product $\mathcal{M} \times_{\mathcal{M}\times_Z\mathcal{M}} \mathcal{M}$, where both morphisms $\mathcal{M} \to \mathcal{M} \times_Z \mathcal{M}$ are the diagonal. An algebraic Z-stack is said to have finite inertia if $I_Z(\mathcal{M})$ is finite over $\mathcal{M}$. Note that by pull-back, a stack with finite diagonal, and hence a separated, finite type DM stack (e.g. [**123**, Lem. 1.13]), has finite inertia.

THEOREM 2.9 (Keel–Mori [**81**, Cor. 1.3]). *Let $\mathcal{M}$ be an algebraic Z-stack, locally of finite presentation, with finite innertia. Then there exists a coarse moduli space $\pi : \mathcal{M} \to M$, with π proper. If $\mathcal{M}/Z$ is separated, then M/Z is separated. If Z is locally notherian, then M/Z is locally of finite type. If Z is locally noetherian and $\mathcal{M}/Z$ is of finite type with finite diagonal, then $\mathcal{M}/Z$ is proper if and only if M/Z is proper.*

For a proof, we direct the reader to Keel–Mori [**81**] (see also Conrad [**44**, Thm. 1.1] and Olsson [**110**, Rem. 1.4.4]). We also direct the reader to the definition of a tame stack in Abramovich–Olsson–Vistoli [**4**, Def. 3.1].

REMARK 2.10. As a consequence of the theorem, one can prove a stable reduction theorem for a moduli problem (with a reasonable moduli stack) by showing that the moduli stack admits a proper moduli space. One standard approach to constructing a proper moduli space is via GIT (§11), where one will in fact typically obtain the stronger statement that the moduli space is projective. Note that alternatively, for a proper moduli space, one can use positivity results of Kollár [**84**] to establish the projectivity of the moduli space directly.

EXAMPLE 2.11. As $\overline{\mathcal{M}}_g$ is a proper DM $\mathbb{C}$-stack, the Keel–Mori theorem implies there is proper coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$. In fact, the moduli space is a projective variety over $\mathbb{C}$. In §11, we will sketch a GIT construction of $\overline{M}_g$ due to Gieseker [**63**]. By the valuative criteria, the existence of a projective coarse moduli space provides another proof of stable reduction. Note also, that as an application of the techniques in [**84**], Kollár gives an independent proof that $\overline{M}_g$ is projective ([**84**, Thm. 5.1]).

For more on algebraic stacks with positive dimensional stabilizers, the reader is directed to Alper [**15**]. See especially the definition [**15**, Def. 4.1] of a good moduli space. We point out that (under mild hypotheses) a good moduli space is a categorical moduli space ([**15**, Thm. 6.6, Thm. 4.16(vi)]). If $\pi : \mathcal{M} \to M$ is a good moduli space, then π may fail to be separated, but it does satisfy the valuative criterion (2.2); i.e. it is universally closed ([**15**, Thm. 4.16(ii)], see also [**17**, Prop. 2.17]). An illustrative example is the morphism $\pi : [\mathbb{A}^1_k/\mathbb{G}_m] \to \operatorname{Spec} k$, for an algebraically closed field k (see e.g. [**15**, Exa. 8.6]). This is a good (categorical) moduli space, which is not coarse, and such that π is universally closed, but not separated.

3. Semi-stable reduction

Semi-stable reduction is the process of filling in the central fiber of a family of smooth varieties over the punctured disk with a reduced scheme with normal crossing singularities. This is the natural generalization of filling in the central fiber in a family of smooth curves with a nodal curve. A semi-stable reduction provides a simple special fiber, which is useful from many points of view, such as Hodge theory, period maps, and monodromy. On the other hand, unlike a stable reduction, a semi-stable reduction is not unique.

In this section we discuss a theorem of Mumford et al. [**82**] that establishes the existence of semi-stable reductions in characteristic zero. In the next section, we will discuss the connection with monodromy, which plays a central role in the stable reduction theorem for abelian varieties. In §7.3 we will use the semi-stable reduction theorem in discussing an approach of Kollár–Shepherd-Barron–Alexeev to establishing stable reduction theorems.

3.1. Semi-stable Reduction Theorem. We begin by stating the semi-stable reduction theorem of Kempf–Knudsen–Mumford–Saint-Donat [**82**].

THEOREM 3.1 (Semi-stable Reduction Theorem [**82**, Thm. p.53]). *Assume that* $\operatorname{char}(k) = 0$ *and* $k = \overline{k}$. *Let* B *be on open subset of a non-singular curve over* k, *fix a point* $o \in B$, *and set* $U = B - \{o\}$. *Suppose that*

$$\pi : X \to B$$

is a surjective morphism of a variety X *onto* B *such that the restriction* $\pi_U : X|_U \to U$ *is smooth. Then there is a finite base change* $f : B' \to B$, *with* B' *non-singular and* $f^{-1}(o)$ *a single point* o', *a non-singular variety* X' *and a diagram*

$$\begin{tikzcd} X' \arrow[r, "p"'] \arrow[rr, bend left] \arrow[dr, "\pi'"'] & B' \times_B X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & B' \arrow[r, "f"] & B \end{tikzcd} \tag{3.1}$$

satisfying the properties below.

(1) *Setting* $U' = B' - \{o'\}$, p *is an isomorphism over* U'.
(2) $(\pi')^{-1}(o')$ *is a reduced scheme, which is an snc divisor on* X'.
(3) *The morphism* p *is projective, and given as a blow-up of an ideal sheaf* $\mathcal{I}$ *that is trivial away from the fiber over* o'.

This result is used so frequently in stable reduction arguments, and parts of the proof are so constructive, that it is worthwhile to sketch the outline here. One of the key points is the following example.

EXAMPLE 3.2. Consider the variety X in $\operatorname{Spec} k[\underline{x}, t] = \mathbb{A}^{r+1}_k$ defined by

$$t - x_1^{a_1} \cdots x_r^{a_r}.$$

We view X as a family $\pi : X \to B := \operatorname{Spec} k[t]$, with central fiber $D = \pi^{-1}(0)$. For each $d \in \mathbb{N}$, set $B_d = \operatorname{Spec} k[t]$, and $f_d : B_d \to B$ to be the map given by $t \mapsto t^d$.

We define X_d to be the normalization of the pull-back of X via the map $B_d \to B$. In other words, we have a diagram

$$\begin{tikzcd} X_d \arrow[r, "\nu_d"'] \arrow[rr, bend left] \arrow[dr, "\pi_d"'] & B_d \times_B X \arrow[r] \arrow[d] & X \arrow[d] \\ & B_d \arrow[r, "f_d"] & B \end{tikzcd}$$

In this example, we will assume that

$$d = \operatorname{lcm}(a_1, \ldots, a_r) \quad \textit{and} \quad \gcd(d, a_1, \ldots, a_r) = \gcd(a_1, \ldots, a_r) = 1,$$

and we will describe X_d and $\pi_d^{-1}(0)$. First, $X_{B_d} := B_d \times_B X$ is defined by

$$t^d - x_1^{a_1} \cdots x_r^{a_r}.$$

The assumption $\gcd(d, a_1, \ldots, a_r) = \gcd(a_1, \ldots, a_r) = 1$ implies that X_{B_d} is the image of the morphism

$$\operatorname{Spec} k[\underline{y}] = \mathbb{A}_k^r \to \mathbb{A}_k^{r+1} = \operatorname{Spec} k[\underline{x}, t] \tag{3.2}$$

given by $(y_1, \ldots, y_r) \mapsto (y_1^d, \ldots, y_r^d, y_1^{a_1} \cdots y_r^{a_r})$. The map (3.2) factors as

$$\mathbb{A}_k^r \to \mathbb{A}_k^r = \operatorname{Spec} k[\underline{z}] \to \mathbb{A}_k^{r+1}$$

where the first map is given by $(y_1, \ldots, y_r) \mapsto (y_1^{a_1}, \ldots, y_r^{a_r})$ and the second map is given by $(z_1, \ldots, z_r) \mapsto (z_1^{d/a_1}, \ldots, z_r^{d/a_r}, z_1 \cdots z_r)$. In short we have

$$\operatorname{Spec} k[\underline{y}] \to \operatorname{Spec} k[\underline{z}] \to X_{B_d} = \operatorname{Spec} \left(k[\underline{x}, t]/(t^d - x_1^{a_1} \cdots x_r^{a_r}) \right)$$

and the associated morphisms of rings are the inclusions:

$$k[y_1, \ldots, y_r] \supseteq k[y_1^{a_1}, \ldots, y_r^{a_r}] \supseteq k[y_1^d, \ldots, y_r^d, y_1^{a_1} \cdots y_r^{a_r}]. \tag{3.3}$$

Let us consider for a moment the special case where $\operatorname{Spec} k[\underline{z}] \to X_{B_d}$ is birational. This will be the case, for instance, if either $r = 2$, or, more generally, if $a_3 = \cdots = a_r = a_1 a_2$ (but will fail in general; e.g. the case $X = V(t - x_1^2 x_2 x_3)$). Then it follows from Zariski's main theorem that

$$\operatorname{Spec} k[\underline{z}] \to X_{B_d}$$

is the normalization $\nu_d : X_d \to X_{B_d}$. The divisor D corresponds to (t) in $k[\underline{x}, t]/(t^d - x_1^{a_1} \cdots x_r^{a_r})$, which corresponds to $z_1 \cdots z_r$ in $k[\underline{z}]$. In conclusion, in this special case, X_d is smooth and $\pi_d^{-1}(0)$ is a reduced, nc divisor.

In general, describing the normalization $\nu_d : X_d \to X_{B_d}$ is more complicated. From the ring on the right in (3.3), one readily obtains a toric description of X_{B_d}. The normalization can then be described in terms of associated semi-groups (see [**82**, p.101]). Using this approach, it is established in [**82**, Lem. 1, p.102, Lem. 2, p.103] that $\pi_d^{-1}(0)$ is reduced, and the pair X_d and $\pi^{-1}(0)$ give rise to a toroidal embedding without self-intersection. We discuss toroidal embeddings briefly in §9. In the case where X_d is non-singular, we point out that this implies that $\pi_d^{-1}(0)$ is nc.

Example 3.3. Again consider the variety X in $\operatorname{Spec} k[\underline{x}, t]$ defined by $t - x_1^{a_1} \cdots x_r^{a_r}$. Using the same notation as in the previous example, *we will keep the assumption that $d = \operatorname{lcm}(a_1, \ldots, a_r)$, but will discard the assumption that* $\gcd(d, a_1, \ldots, a_r) = \gcd(a_1, \ldots, a_r) = 1$. The fibered product $X_{B_d} := B_d \times_B X$

is defined by $t^d - x_1^{a_1}\cdots x_r^{a_r}$. Setting $e = \gcd(d, a_1, \ldots, a_r)$, this family decomposes as $\prod_{\zeta^e=1}\left(t^{d/e} - \zeta\prod_{i=1}^r x_1^{a_1/e}\cdots x_r^{a_r/e}\right)$. Since $d/e = \operatorname{lcm}(a_1/e, \ldots, a_r/e)$, and $\gcd(d/e, a_1/e, \ldots, a_r/e) = 1$, we see that we can reduce to the case of (e copies of) the previous example.

Example 3.4. Again consider the variety X in $\operatorname{Spec} k[\underline{x}, t]$ defined by $t - x_1^{a_1}\cdots x_r^{a_r}$. Using the same notation as above, set $\ell = \operatorname{lcm}(a_1, \ldots, a_r)$, *assume that* $d = n\cdot\ell$ *for some* $n \in \mathbb{N}$, *and again discard the assumption that* $\gcd(d, a_1, \ldots, a_r) = \gcd(a_1, \ldots, a_r) = 1$. One can show (see e.g. [**82**, Lemma 2, p.103]) that $X_d = B_d \times_{B_\ell} X_\ell$.

Remark 3.5. In summary, for the variety $X \subseteq \operatorname{Spec} k[\underline{x}, t]$ defined by $t - x_1^{a_1}\cdots x_r^{a_r}$, in the notation above if $d = n \cdot \operatorname{lcm}(a_1, \ldots, a_r)$ for some $n \in \mathbb{N}$, then X_d consists of $e = \gcd(a_1, \ldots, a_r)$ connected components. On each of these components, $\pi_d^{-1}(0)$ gives rise to a toroidal embedding without self-intersection. For surfaces, the singularities appearing on X_d will be at worst of type A (the definition of a type A singularity is recalled in §10).

We now briefly outline the Mumford et al. proof of the Semi-stable Reduction Theorem (see [**82**, pp.98-108] for more details).

Sketch of the proof of the Semi-stable Reduction Theorem. Let $\pi : X \to B$ be a morphism as in the statement of the theorem. Using the characteristic zero assumption, perform a log resolution of the pair $(X, \pi^{-1}(o))$. We obtain a new family $\tilde{\pi} : \tilde{X} \to B$, where $\tilde{\pi}^{-1}(o)$ is normal crossing (although it may not be reduced). Setting ℓ to be the lcm of the multplicities of the components of $\tilde{\pi}^{-1}(o)$, make a base change of degree ℓ, and then normalize the total space.

Call the space obtained $X^\nu \to B_\ell$. The claim is that X^ν satisfies the conditions of the theorem, with the possible exception that X^ν may fail to be smooth (in which case the central fiber may only induce a toroidal embedding without self intersection, rather than being nc). Indeed, the question is étale local, so to describe X^ν we may reduce to the examples we have already considered, where $\tilde{X}$ is defined by

$$t - \prod_{i=1}^r x_1^{a_1}\cdots x_r^{a_r}.$$

The remaining issue is to resolve the singularities of the total space of X^ν (while retaining the property that the central fiber induces a toroidal embedding without self-intersection). This is done in [**82**, pp.104-108]; note that a further base change may be required (see [**82**, p.107]). □

Remark 3.6. For the case of families of curves (where X is a surface), the total space of X^ν has type A singularities, and one can achieve the resolution in the final step above by a sequence of blow-ups introducing chains of rational curves.

4. Monodromy

The monodromy representation is a topological invariant associated to a family over a punctured disk. It is the essential invariant in the context of period maps (see Remark 4.6), as well as for abelian varieties. In this section, we briefly review the definition of monodromy, and then compute a few examples. We then state the monodromy theorem. While there is an algebraic monodromy representation for families over DVRs, for simplicity, we restrict to the case of monodromy in the analytic setting.

4.1. Preliminaries on monodromy. Let $X^\circ \to S^\circ$ be a smooth family of complex, projective varieties over the punctured disk. It is well known that for each $t_1, t_2 \in S^\circ$, the fiber X_{t_1} is diffeomorphic to the fiber X_{t_2} (see e.g. [**83**, Thm. 2.3, p.61]). In particular, the fibers are all homeomorphic, and the cohomology groups $H^\bullet(X_t, \mathbb{C})$ are isomorphic for all $t \in S^\circ$. Fix a base point $* \in S^\circ$ and consider a path $\gamma : [0,1] \to S^\circ$ that generates $\pi_1(S^\circ, *)$. The family of groups $H^\bullet(X_{\gamma(\tau)}, \mathbb{C})$, $\tau \in [0,1]$, determines an automorphism of $H^\bullet(X_*, \mathbb{C})$. The induced homomorphism

$$\pi_1(S^\circ, *) \to \operatorname{Aut} H^\bullet(X_*, \mathbb{C})$$

is called the (analytic) monodromy representation of the family.

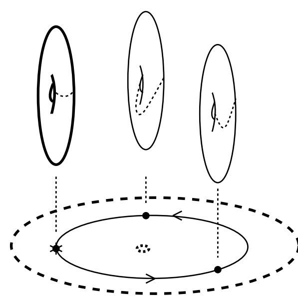

FIGURE 7. Monodromy

For a family that extends to a smooth family over S, the monodromy representation is trivial. In this sense, monodromy is an invariant that is meant to detect something about the singularities of the central fiber of a degeneration. For instance, ordinary double points (A_1 singularities) give rise to the so called Picard–Lefschetz transformations (see the example in §4.1.1). On the other hand, we note that it is not the case that trivial monodromy implies that a family can be extended to a smooth family over the disk. For an elementary example, see Remark 6.4. For a more interesting example, see Friedman [**56**].

Recall that an endomorphism T of a finite-dimensional vector space V is said to be **unipotent** (resp. **quasi-unipotent**) if there exist $M \geq 1$ (resp. $M, N \geq 1$) such that $(T - \mathrm{Id}_V)^M = 0$ (resp. $(T^N - \mathrm{Id}_V)^M = 0$).

4.1.1. *A family of stable curves.* Consider the family

$$x_2^2 - (x_1^2 - t)(x_1 - 1);$$

i.e. a family of smooth elliptic curves degenerating to a nodal cubic. Set $* = 1/2$ and let $\gamma : [0,1] \to S^\circ$ be a parameterization of the circle of radius $1/2$. The family of varieties lying over γ is a family of elliptic curves determined by the branch locus $\{-\sqrt{t}, \sqrt{t}, 1, \infty\}$. There is a basis of $H^1(X_*, \mathbb{C})$ for which the monodromy representation is given by

$$M_{A_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

See Carlson–Müler-Stach–Peters [**35**, p.18] for a detailed exposition of this. Note that this matrix is unipotent. This transformation is in fact a special case of the

Picard–Lefschetz theorem describing the monodromy transformations for degenerations to A_1 singularities (see e.g. [**19**, Ch.2, §1.5]).

4.1.2. *A family of cuspidal curves.* Consider the family

$$x_2^2 - x_1^3 - t;$$

i.e. a family of smooth elliptic curves degenerating to a cuspidal cubic. Again, set $* = 1/2$ and let $\gamma : [0,1] \to S^\circ$ be a paramaterization of the circle of radius $1/2$. There is a basis of $H^1(X_*, \mathbb{C})$ for which the monodromy representation is given by

$$M_{A_2} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Figure 8 shows the transformation of cycles on the copy of $\mathbb{P}^1$ lying below the elliptic

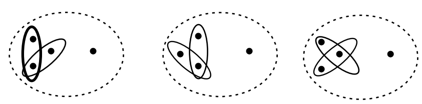

FIGURE 8. Monodromy for a cuspidal family

curve, with respect to the branch locus, for $t = \gamma(0) = 1/2$, $t = \gamma(1/2) = -1/2$ and $t = \gamma(1) = 1/2$, respectively. Considering lifts of these cycles, one arrives at the matrix above.

In this example, the monodromy representation is quasi-unipotent, but not unipotent. Note additionally that $M_{A_2}^3 = -\operatorname{Id}$. This implies that after pulling the family back by a triple cover, the monodromy will be given by $-\operatorname{Id}$. This new family gives a special case of the monodromy obstruction computation made in §1.4.1; in the notation of that section, this is the family given by taking $t_2 = 0$.

Note further that since $M_{A_2}^6 = \operatorname{Id}$, if we pull the family back by a six-fold cover, the monodromy becomes trivial. This can be seen directly in the following way. The family obtained after a six-fold cover is

$$x_2^2 - x_1^3 - t^6.$$

Changing coordinates by $x_1 \mapsto x_1 t^2$ and $x_2 \mapsto x_2 t^3$ gives the family

$$x_2^2 - x_1^3 - 1.$$

In other words, after the degree six base change, the family can be extended to a *trivial* family over S. (Note the j-invariant of the original family was equal to zero for all $t \in S^\circ$.)

REMARK 4.1. More generally, Kodaira has classified the degenerations of elliptic curves, and their associated monodromy representations. We direct the reader to [**24**, §V.7] for more details.

4.2. The monodromy theorem. The monodromy theorem is a general statement about the monodromy representation of a family of projective manifolds over the punctured disk. This will play an important role in regards to an extension theorem of Grothedieck for abelian varieties, which we discuss in §5.

THEOREM 4.2 (Monodromy Theorem). *Let $\pi^\circ : X^\circ \to S^\circ$ be a family of smooth, complex, projective manifolds of dimension n over the punctured disk. For each integer $0 \le k \le 2n$, the monodromy representation*

$$\pi_1(S^\circ, *) \to \operatorname{Aut} H^k(X_*, \mathbb{C})$$

is quasi-unipotent.

The reader is directed to Griffiths [**64**, Rem. 3.2, p.236] for references, including a discussion of the history of the theorem and a description of the many different methods of proof (see also Grothendieck [**68**, Thm. 1.2, p.6] for the algebraic statement).

Remark 4.3. The monodromy theorem implies that for a family of smooth, complex, projective manifolds over the punctured disk, after a finite base change the monodromy can be made unipotent. Indeed, if the generator of the monodromy representation is given by the automorphism T, then $(T^N - \mathrm{Id})^M = 0$ for some N, M. Thus after the base change given by $t \mapsto t^N$, the monodromy will be unipotent. We note that many of the proofs of the monodromy theorem provide bounds on N and M.

Remark 4.4. If $\pi : X \to S$ is a generically smooth family of complex projective varieties, such that $X_0 := \pi^{-1}(0)$ is an snc divisor in X, then the monodromy representation is unipotent (see the references in [**64**]). One can deduce the Monodromy Theorem from this using the Semi-Stable Reduction Theorem [**82**] (Theorem 3.1).

Remark 4.5. As is evident in the previous remark, if $\pi : X \to S$ is a generically smooth family of complex projective varieties, the topology of X_0 is related to the monodromy of the family. The Clemens–Schmid exact sequence makes this precise (e.g. Morrison [**100**, p.109]). There is also a notion of vanishing cohomology for isolated singularities on X_0. There is a monodromy operator on the vanishing cohomology, which is related to the monodromy of the family by an exact sequence. We direct the reader to [**68**, p.V, 7-9] and [**120**, (1.4)] for more details.

Remark 4.6. Monodromy is the essential invariant in the context of period maps. In particular, given a family $X^\circ \to S^\circ$ of smooth, projective varieties, then via Hodge theory one obtains a period map $S^\circ \to D/\Gamma$, where D is the period domain and Γ is an arithmetic group. The period map extends to a morphism $S \to D/\Gamma$ if and only if the monodromy representation is finite; i.e. generated by a root of the identity (e.g. [**35**, Thm. 13.4.5, p.355]).

5. Abelian varieties

In this section we return to the question of stable reduction, and consider the case of abelian varieties. Historically, this was one of the first places where questions about stable reduction were considered. The monodromy theorem discussed in the previous section plays a central role, essentially due to the equivalence of categories between abelian varieties and Hodge structures of weight 1.

Further motivation comes from the connection with the original proof of the stable reduction theorem for curves in positive characteristic, which we discuss further in the next section. In this section we also consider stable reduction in the context of Alexeev's moduli space of stable semiabelic pairs [**9**].

5.1. An example of stable reduction for a family of abelian varieties.

The family described in §1, viewed as a family of abelian varieties, gives a concrete example of stable reduction. Historically, however, one of the motivations for the development of the theory was to study abelian schemes over $\mathbb{Q}$ by reducing modulo a prime p. Viewing the abelian scheme over $\mathbb{Q}$ as a family over the generic point of $\operatorname{Spec}\mathbb{Z}$, problems concerning reduction modulo a prime can be translated into problems about extending abelian schemes over $\mathbb{Q}$ to schemes over $\operatorname{Spec}\mathbb{Z}$.

While a well known theorem of Fontaine [**55**, Cor., p.517] states there are no abelian schemes over $\operatorname{Spec}\mathbb{Z}$, so there can not be an extension to an abelian scheme over every prime, the stable reduction theorem addresses the question of extending over a particular prime after a finite base change.

With this as motivation, we consider the following example, which is closely related to the previous geometric examples, and emphasizes the connection between the two settings. We will use the terminology of group schemes, which we review in the next subsection.

Let $X \to \operatorname{Spec}\mathbb{Z}$ be the projective scheme defined by

$$y^2 - x^3 - 25\alpha x - 125\beta = 0,$$

with α and β integers such that $4\alpha^3 + 27\beta^2$ is not divisible by 5. Let $X_{\mathbb{Q}}$ be the scheme obtained by base change to $\operatorname{Spec}\mathbb{Q}$. There is the usual group law on $X_{\mathbb{Q}}$ induced by the point at infinity $(0:1:0)$. We are interested in understanding how this fails to extend to a group law on X over $\operatorname{Spec}\mathbb{Z}$, and how one might attempt to rectify this at a particular prime by taking a finite cover.

Concerning the group law, one can check directly that $X \to \operatorname{Spec}\mathbb{Z}$ fails to be smooth over (at least) the primes 2, 3 and 5, so that $X_{\mathbb{Q}}$ does not extend to a group scheme over those points. In this example, we focus on the issue of extension over (5). Let

$$X_{(5)} \to \operatorname{Spec}\mathbb{Z}_{(5)}$$

be the scheme obtained from X by base change. The fiber over the generic point (0) is $X_{\mathbb{Q}}$ and we are interested in extending $X_{\mathbb{Q}}$ to an abelian scheme over $\operatorname{Spec}\mathbb{Z}_{(5)}$.

Let $X_{\mathbb{F}_5}$ be the fiber of $X_{(5)}$ over the closed point. Then $X_{\mathbb{F}_5}$ is given by the equation

$$y^2 - x^3 = 0,$$

which is singular at $(0:0:1)$. We would like to describe a finite base change and a modification of the family that is smooth.

Consider the finite, degree 2 morphism

$$B' := \operatorname{Spec}\mathbb{Z}_{(5)}[\zeta]/(\zeta^2 - 5) \to \operatorname{Spec}\mathbb{Z}_{(5)}$$

induced by the extension $\mathbb{Q}(\sqrt[2]{5})/\mathbb{Q}$. Pulling back $X_{(5)}$ we obtain a family $X' \to B'$ defined by the equation

$$y^2 - x^3 - \alpha\zeta^4 x - \beta\zeta^6 = 0.$$

Making the change of coordinates $x \mapsto \zeta^2 x$, $y \mapsto \zeta^3 y$, we arrive at a family $\tilde{X} \to B'$ defined by

$$y^2 - x^3 - \alpha x - \beta = 0.$$

This is smooth over the closed point $(\zeta) \in B'$, and in fact there is a group law over B'. Thus, after a finite, degree two, base change, we have modified our family to give an abelian scheme over the base.

Remark 5.1. It is interesting to note the connection with the geometric case. The example above was constructed to be the analogue of the linear family over the Weyl cover (§1.4.1):

$$y^2 + x^3 - (b_2^2 + b_2 + 1)b_1^2 x - b_2(1 + b_2)b_1^3 = 0,$$

where, roughly speaking, we replaced x_2 with y, x_1 with $-x$, set $b_1 = 5$ and took b_2 general. We had seen that a degree two base change for the analytic family would allow for stable reduction, and this is exactly what we have found here in the arithmetic setting.

REMARK 5.2. For completeness, we mention that the discriminant Δ and j-invariant of X are

$$\Delta = -16(4\alpha^3 + 27\beta^2)(5^6) \neq 0 \quad \text{and} \quad j = -1728(4\alpha)^3 5^6/\Delta.$$

Note that j has non-negative valuation at 5. It is well known that from this data one can deduce that the family does not have abelian reduction at 5, but does have potentially abelian reduction there (see e.g. [**118**, VII Prop. 5.1, 5.5]).

5.1.1. *Monodromy.* Recall that the analytic monodromy representation of the analogous family was given by the negative of the identity (§1.4.1). Although we have not introduced the algebraic monodromy operator, we make the following observation. Since there is abelian reduction after a degree two base change, one can immediately conclude that the algebraic monodromy operator is a square root of the identity. One can then show the action on non-trivial, torsion points of sufficiently high order (relatively prime to 2 and 5) is non-trivial. Thus the monodromy operator is the negative of the identity, similar to the analogous analytic case. For more on degenerations of elliptic curves and monodromy, we direct the reader to the discussion of the Kodaira–Néron classification in [**118**, App. C.15].

5.2. Group scheme terminology. We now review some of the basic terminology of group schemes, directing the reader to [**31**, §4.1] and [**105**, Ch.6] for more details. For a scheme B, a **B-group scheme** is a group object in the category of B-schemes (Sch/B). A standard example, which we will use frequently, is $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$, with group law induced by the map

$$\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \quad \text{given by} \quad t \mapsto t \otimes t.$$

For an arbitrary scheme B, we define $\mathbb{G}_{m,B}$ by base extension, and the induced group law makes $\mathbb{G}_{m,B}$ a group object in the category of B-schemes. An **(affine) split B-torus T** is a B-group scheme that is isomorphic as a B-group scheme to a finite fibered product $\mathbb{G}_{m,B} \times_B \ldots \times_B \mathbb{G}_{m,B}$. An **(affine) B-torus T** is a B-group scheme that is étale locally on B a split torus. We define **B-subgroup schemes** in the obvious way (see e.g. [**31**, p.98]).

An **abelian scheme over** B is a B-group scheme that is smooth and proper over B with connected fibers. It follows from the Rigidity Lemma that the group law of an abelian scheme is commutative (see e.g. [**105**, Pro. 6.1, Cor. 6.4, p.115-6]). It is a well known result that an abelian scheme over a field is projective ([**124**]).

REMARK 5.3. In order to use the term variety consistently (within this paper), we reserve the term **abelian variety** for an abelian group scheme over an algebraicaly closed field. (This is not standard, in that one usually does not require the field to be algebraically closed.)

The best understood abelian schemes are Jacobians of curves. Recall that associated to a smooth curve X over an algebraically closed field, there is an abelian variety JX, called the Jacobian of X, parameterizing degree zero line bundles on X. We note in addition that associated to a family of smooth curves $X \to B$, there is an associated abelian scheme JX_B over B, called the (relative) Jacobian of X_B, with geometric fibers that are the Jacobians of the associated curves.

A **semi-abelian scheme** G_B over B is a smooth, separated, commutative B-group scheme such that each fiber $G_{B,b}$ over $b \in B$ is an extension of an abelian scheme A_b by an affine torus T_b:

$$0 \to T_b \to G_{B,b} \to A_b \to 0.$$

We direct the reader to [**51**, Cor. 2.11] for a statement on the global structure of a semi-abelian scheme. Extensions of an abelian variety A/k by a torus T/k are classified (up to isomorphism as extensions) by $\operatorname{Hom}\left(X(T), \widehat{A}\right)$, where $X(T) = \operatorname{Hom}(T, \mathbb{G}_{m,k})$ is the character group and $\hat{A} = \operatorname{Pic}^0(A)$ is the group of line bundles on A algebraically equivalent to zero (see e.g. [**115**, Thm. 6, p.184]).

We now come to the topic of reduction. Let R be a DVR, let K be its field of fractions, and let $S = \operatorname{Spec} R$. Let A_K be an abelian scheme over $\operatorname{Spec} K$. We say that A_K has **abelian (or good) reduction** (resp. **semi-abelian reduction**) if A_K can be extended to a smooth, separated S-group scheme G_S of finite type over S such that the the fiber over the closed point $s \in S$ is an abelian (resp. semi-abelian) scheme over s. We will say that A_K has **potentially abelian (or good) reduction** (resp. **potentially semi-abelian reduction**) if there is a finite extension K' of K, so that the abelian scheme $A_{K'}$ obtained by base change has abelian (resp. semi-abelian) reduction.

5.3. Néron models. Néron models provide a natural context for discussing the stable reduction theorem for abelian varieties. While the theory can be developed in more generality over Dedekind domains, we focus on the case of DVRs for simplicity.

As above, let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K. Let X_K be a smooth, separated K-scheme of finite type. A **Néron model** of X_K is an extension X_S of X_K over S that is a smooth, separated scheme of finite type, satisfying the following universal property: for any smooth S-scheme Y_S and any K-morphism $f_K : Y_K \to X_K$ there is a unique S-morphism $f_S : Y_S \to X_S$ extending f_K.

$$\begin{tikzcd}
 & X_K \arrow[rr] \arrow[dd] & & X_S \arrow[dd] \\
Y_K \arrow[ur, "f_K"] \arrow[rr] \arrow[dr] & & Y_S \arrow[ur, dashed, "f_S", "\exists!"'] \arrow[dr] & \\
 & \operatorname{Spec} K \arrow[rr] & & S.
\end{tikzcd}$$

If a Néron model exists, it is unique up to unique isomorphism. While Néron models do exist in a more general setting, we will focus here on the case of abelian schemes. The main theorem in this situation is:

THEOREM 5.4 (Néron [**108**]). *Let A_K be an abelian scheme over the field of fractions K of a DVR R. Then A_K admits a Néron model X_S over $S = \operatorname{Spec} R$.*

We direct the reader also to [**31**, Cor. 2, p.16, Pro. 6, p.14] and Artin [**21**].

Remark 5.5. From the universal property of the Néron model, it follows that the K-group scheme structure on A_K extends uniquely to a commutative S-group scheme structure on X_S. For group schemes, the condition that the Néron model be of finite type and separated is superfluous (e.g. [**31**, p.12, Rem. 7, p.14]). Finally, it is a result of Raynaud that the Néron model of an abelian scheme is quasi-projective [**112**, Thm. VIII.2, p.120].

Remark 5.6. The special fiber of a Néron model of an abelian scheme need not be connected. One such example is given by a smooth plane cubic degenerating to a nodal plane cubic that is the union of a line and a smooth conic. The special fiber $X_{S,s}$ of the Néron model can be computed using a result of Raynaud discussed in the remark below. One can show $X_{S,s}$ fits into an exact sequence

$$0 \to \mathbb{G}_m \to X_{S,s} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

(see e.g. Kass [**80**, §4.3]). There is always, however, an open S-subgroup scheme of a Néron model of an abelian scheme A_K that extends A_K and has connected central fiber. We will denote this by X_S°.

Remark 5.7. As with abelian varieties, the best understood Néron models are those associated to curves. One of the main tools is a theorem of Raynaud's, relating the Néron model of a Jacobian to the Picard functor. The following is a weaker version of the theorem, given in Deligne–Mumford [**48**, Thm. 2.5], which is used in the proof of the stable reduction theorem for curves. *Assume the residue field of R is algebraically closed. In the notation above, let C_S be a generically smooth family of nodal curves over S, with non-singular total space, and let A_K be the Jacobian of the generic fiber. Then the open S-subgroup scheme X_S° of the Néron model X_S of A_K, described above, represents the relative (connected component of the) Picard functor.*

Example 5.8. Assume the residue field of R is algebraically closed. Consider a family $C \to S$ of smooth curves degenerating to an irreducible, stable curve C_s with a single node. Let C_s^ν be the normalization of C_s, and assume that C_s is obtained from C_s^ν by attaching points $p, q \in C_s^\nu$. The family of curves C_K over K determines a principally polarized abelian scheme $X_K = JC_K$, the Jacobian of the curve. The special fiber of the open S-subgroup scheme X_S° of the Néron model of X_K is an extension

$$0 \to \mathbb{G}_m \to X_{S,s}^\circ \to JC_s^\nu \to 0$$

determined by the data of the line bundle $\mathscr{O}_{C_s^\nu}(p-q)$.

5.3.1. *The group structure of the central fiber of the Néron model.* In the notation above, we have seen that the Néron model of an abelian scheme A_K is a commutative group scheme over S. To get a handle on how Néron models are connected to the question of semi-abelian reduction, we will investigate the group structure on the central fiber of the Néron model using a few basic facts from the theory of algebraic groups (see also Serre [**115**]).

To begin, we recall Chevalley's theorem [**41, 113**] (see esp. [**31**] and [**43**, Thm. 1.1]): *Let K be a field and let G be a smooth, connected algebraic K-group. Then there exists a smallest (not necessarily smooth) connected linear subgroup L of G such that the quotient G/L is an abelian scheme over K. Moreover, if K is perfect, L is smooth and its formation is compatible with change of base field.*

In other words, if X_S is the Néron model of A_K, then the connected component of the identity in the central fiber $X^\circ_{S,s}$ fits into an exact sequence over s

$$0 \to L \to X^\circ_{S,s} \to A \to 0,$$

where L is a connected, commutative, linear s-group scheme and A is an abelian s-group scheme.

We now turn our attention to the structure of linear algebraic groups. Let us pull-back to the algebraic closure $\bar{k} = \overline{\kappa(s)}$, and denote the resulting group schemes by $\bar{L}$, $\overline{X}^\circ_{S,s}$ and $\bar{A}$ respectively. $\bar{L}$ being commutative, it is solvable (see e.g. [**30**, Def., p.59]). There is the following standard theorem (e.g. [**30**, Thm. 10.6, p.137]): *For a connected, solvable, linear algebraic group $\bar{L}$ over an algebraically closed field $\bar{k}$, the subset of unipotent elements $\bar{L}_u$ is a (closed) connected, normal $\bar{k}$-subgroup, and the quotient is an affine torus.* In the situation of the Néron model, this torus can be obtained by pull-back from a torus over $\kappa(s)$. Thus *the (subgroup $X^\circ_{S,s}$ of the) Néron model is a semi-abelian scheme, if and only if $\bar{L}_u$ is trivial.*

Remark 5.9. The standard way to assert that $\bar{L}_u$ is trivial is to assert that the unipotent radical of $\bar{L}$ is trivial. Indeed, for a connected, solvable group G over an algebraically closed field, the radical $\mathscr{R}G$ is equal to the group G (the radical is the largest connected, solvable, normal subgroup; e.g. [**30**, p.157]). Thus in this case the unipotent radical $(\mathscr{R}G)_u =: \mathscr{R}_uG$ (the set of unipotent elements of the radical) is equal to the set G_u.

5.4. The stable reduction theorem. The stable reduction theorem plays a central role in the study of abelian varieties, and also in the study of algebraic curves. In light of the results of Néron, and the basic structure theorems for algebraic groups, the stable reduction theorem states that after a generically finite base change, the unipotent radical of the central fiber of the Néron model can be made trivial.

As described in the introduction to [**48**], the stable reduction theorem was first proved independently by Grothendieck and Mumford in characteristic zero. Grothendieck's proof used the theory of étale cohomology, while Mumford's proof was derived from a stable reduction theorem for curves (in characteristic zero). Grothendieck then extended his proof to all characteristics in [**68**, Thm. 6.1, p.21] and Mumford provided an independent proof in characteristics other than 2 using the theory of theta functions.

Theorem 5.10 (Grothendieck–Mumford Stable Reduction Theorem). *Let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K. An abelian variety A_K over K has potential semi-abelian reduction over R.*

In fact the theorem can be stated more generally for the case where A_K is a semi-abelian scheme. We refer the reader also to [**31**, Thm. 1, p.180], and to [**51**, Thm. 2.6, p.9]. Grothendieck's proof relies on the following frequently cited result, which was the basis of the monodromy obstruction computation in §1.

Proposition 5.11 (Grothendieck [**68**, Prop. 3.5, p.350]). *In the notation above, A_K has abelian (resp. semi-abelian) reduction if and only if the monodromy representation is trivial (resp. unipotent).*

We direct the reader also to [**31**, Thm. 5, p.183]. The Grothendieck–Mumford Stable Reduction Theorem follows from the proposition and the Monodromy Theorem after the observation (see Remark 4.3) that quasi-unipotent monodromy can be made unipotent after a finite base change.

5.5. Alexeev's space of stable semiabelic pairs. One would like to derive from the Grothendieck–Mumford Stable Reduction Theorem a properness statement for a moduli space. This serves as motivation to introduce Alexeev's compactification [**9**] of the moduli space of principally polarized abelian varieties, where such a statement holds.

Let us recall some definitions from [**9**] (we also direct the reader to Olsson [**110**] for a related moduli problem). First a reduced scheme X is said to be **semi-normal** if given any proper, bijective morphism $f : X' \to X$ from a reduced scheme X' satisfying the property that $\kappa(f(x')) \to \kappa(x')$ is an isomorphism for all $x' \in X$, then f is an isomorphism (see e.g. [**85**, §7.2], [**9**, 1.1.6]). For instance, a nodal curve is semi-normal, while a cuspidal curve is not.

A **stable semiabelic variety** ([**9**, 1.1.5]) is a semi-normal, equidimensional, reduced scheme X over an algebraically closed field k, together with an action of a connected semi-abelian scheme G/k of the same dimension, such that there are only finitely many orbits for the G-action, and the stabilizer group scheme of every point of X is connected, reduced and lies in the toric part of G.

A **polarized stable semiabelic variety** ([**9**, 1.1.8]) is a projective stable semiabelic variety together with an ample invertible sheaf L. The degree of the polarization is defined as $h^0(L)$. A **stable semiabelic pair** (X, Θ) consists of a polarized stable semiabelic variety X with ample invertible sheaf L together with a section $\theta \in H^0(X, L)$ that does not vanish on any G-orbits. So in total, for a stable semiabelic pair, we have the data (X, G, L, θ). We take Θ to be the zero set of θ, and use the shorter notation (X, Θ) to indicate the connection to polarized abelian varieties.

We now make the relative definition. For a scheme B, a **stable semiabelic pair over** B, denoted (X_B, Θ_B), is the data

$$(X_B, G_B, L_B, \theta_B)$$

where $X_B \overset{\pi_B}{\to} B$ is a projective, flat morphism, G_B is a semi-abelian scheme over B acting on X_B, L_B is a relatively ample line bundle on X_B, $\theta_B \in H^0(B, \pi_* L_B)$, and the restriction of this data to every geometric point $\bar{b} \to B$ is a stable semiabelic pair over $\bar{b}$. ([**9**, p.617]). One can show that $\pi_* L_B$ is locally free and that this push forward commutes with arbitrary base change. The degree is defined to be the rank of $\pi_* L_B$.

For brevity, we do not give a precise definition of Alexeev's stack $\bar{\mathcal{A}}_g^A$, and say only that it is a substack of the stack of all stable semiabelic pairs of degree 1 and dimension g. The stack $\bar{\mathcal{A}}_g^A$ contains a component that has $\mathcal{A}_g$, the moduli stack of principally polarized abelian varieties of dimension g, as a dense open substack. Alexeev proves [**9**, Thm 5.10.1] that $\bar{\mathcal{A}}_g^A$ is a proper, algebraic (Artin) stack over $\mathbb{Z}$ with finite diagonal. Moreover, the stack admits a coarse moduli space, with a component that has normalization isomorphic to the second voronoi compactification $\bar{A}_g^{Vor}$ [**9**, Thm. 5.11.6, p. 701]. To establish properness, Alexeev proves the following stable reduction theorem for semiabelic pairs.

Theorem 5.12 (Alexeev [**9**, Thm. 5.7.1, p.692]). *Let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K. Let (X_K, Θ_K) be a stable semiabelic pair over K. Then there is a finite extension K' of K, so that taking R' to be the integral closure of R in K' and setting $S' = \operatorname{Spec} R'$, there exists a stable semiabelic pair $(X_{S'}, \Theta_{S'})$ over S' extending the pull-back $(X_{K'}, \Theta_{K'})$. Morover, the extension $(X_{S'}, \Theta_{S'})$ is unique up to isomorphism.*

Remark 5.13. As a consequence, the central fiber $(X_{s'}, \Theta_{s'})$ of $(X_{S'}, \Theta_{S'})$ is determined up to isomorphism by (X_K, Θ_K).

Example 5.14. Assume the residue field of R is algebraically closed. Consider a family $C \to S$ of smooth curves degenerating to an irreducible, stable curve C_s with a single node. Let C_s^ν be the normalization of C_s, and assume that C_s is obtained from C_s^ν by attaching points $p, q \in C_s^\nu$. The family of curves C_K over K determines a principally polarized abelian scheme (X_K, Θ_K), where $X_K = JC_K$ is the Jacobian of the curve. Let $(X_{S'}, \Theta_{S'})$ be a stable reduction of X_K.

The central fiber can be described as follows. The degree zero Picard functor applied to C_S determines a semi-abelian scheme over S. The central fiber is the semi-abelian scheme

$$0 \to \mathbb{G}_m \to G \to JC_s^\nu \to 0$$

determined by the data of the line bundle $\mathscr{O}_{C_s^\nu}(p-q)$. The group G can be completed to a $\mathbb{P}^1$-bundle over JC_s^ν, with sections σ_0 and σ_∞. The fiber $X_{s'}$ is obtained from this projective bundle by gluing the sections transversally, after shifting by $\mathscr{O}_{C_s^\nu}(p-q)$. Note that G acts on this space. We direct the reader to [**10**] for more details, as well as a description of the polarization (see also [**104**]).

6. Curves

In this section we consider stable reduction for curves. We begin with the Deligne–Mumford Stable Reduction Theorem. The main focus is on reviewing the connection between stable reduction for curves and stable reduction for abelian varieties. We also review a well known proof in characteristic zero using the semistable reduction theorem, to motivate work of Kollar–Shepherd-Barron–Alexeev, discussed later. Finally, we consider some recent "alternate" stable reduction theorems for curves, which have connections to the Hassett–Keel program.

6.1. Deligne–Mumford stable reduction. Recall that a stable curve X over an algebraically closed field is a pure dimension 1, reduced, connected, complete scheme of finite type, with at worst nodes as singularities, and with finite automorphism group. The genus is defined as $g = h^1(X, \mathscr{O}_X)$. For a scheme B, a stable curve X/B is a proper, flat morphism $X \to B$ whose geometric fibers are stable curves.

Theorem 6.1 (Deligne–Mumford Stable Reduction [**48**]). *Let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K. Let X_K be a stable curve over K. Then there is a finite extension K' of K, so that taking R' to be the integral closure of R in K' and setting $S' = \operatorname{Spec} R'$, there exists a stable curve $X_{S'}$ over S' extending the pull-back $X_{K'}$. Morover, the extension $X_{S'}$ is unique up to isomorphism.*

Remark 6.2. As a consequence, the central fiber $X_{s'}$ of $X_{S'}$ is determined up to isomorphism by X_K.

In characteristic 0, the theorem is due to Mayer–Mumford [**101**]. It appears that semi-stable reduction for curves in characteristic 0 was well known for some time (e.g. [**82**, p.VIII]). The class of stable curves was then defined in [**101**, Def., p.7] (see also [**105**, p.228]), and the properness of the associated moduli space was asserted there. The properness is equivalent to the stable reduction theorem, which one establishes from the existence of a semi-stable reduction, and the birational geometry of surfaces (we outline a well known argument below).

In positive characteristic, the first proof was due to Deligne–Mumford [**48**], and was made via the stable reduction theorem for abelian varieties. The outline of this proof is as follows. Take X_K smooth for simplicity and consider the Jacobian J_K/K. The Grothendieck–Mumford Stable Reduction theorem implies this extends to a family of semi-abelian varieties, at least after a finite base change. To complete the proof, it is then shown:

Theorem 6.3 (Deligne–Mumford [**48**, Thm. 2.4]). *A family of stable curves X_K/K extends to a family of stable curves over S if and only if the associated family of Jacobians J_K/K has semi-abelian reduction over S.*

A key point of the proof is the result of Raynaud's mentioned in Remark 5.7. The fact that stable reduction for the family of curves implies stable reduction for the family of Jacobians essentially follows directly from Raynaud's result. Using this half of Theorem 6.3, Mumford observed [**48**, p.75] that the stable reduction theorem for abelian varieties can be deduced from the stable reduction theorem for curves.

We direct the reader to [**31**, p.182] for a more detailed discussion. The outline of the argument is as follows. An abelian scheme A_K can be viewed as the quotient of a product of Jacobians; i.e.

$$0 \to A'_K \to J_K \to A_K \to 0$$

where A'_K is an abelian scheme, and J_K is a product of Jacobians (e.g. Serre [**115**, Cor. p.180]). One then shows that in general, for such an extension of abelian schemes, J_K has semi-abelian reduction if and only if A_K and A'_K do, completing the proof. Finally, we note that Artin–Winters have given another proof of stable reduction for curves in positive characteristic, which does not rely on the stable reduction theorem for abelian varieties [**22**].

Remark 6.4. While unipotent monodromy for a one-parameter family of smooth curves implies the family extends to a family of stable curves, trivial monodromy does not necessarily imply that the central fiber of the extension is smooth. For instance, a one-parameter family of smooth curves degenerating to a singular, stable curve of compact type will have associated Jacobian that is an abelian scheme over the base. Grothendieck's theorem implies that the monodromy of the family will be trivial. We direct the reader to Oda [**109**, Thm. 10] for a statement concerning a related monodromy invariant that detects when a one-parameter family can be extended to a smooth curve.

Remark 6.5. There is a stable reduction theorem for pointed curves as well. Pointed curves provide a natural introduction to the important topic of moduli spaces of pairs. For the sake of brevity, we have generally avoided this topic in the presentation here. It will, however, be of central importance in §7.3 on slc models, and it is worth noting this example as a precursor.

6.1.1. *Stable reduction in characteristic* 0. To motivate some of the other examples considered in this survey (following the approach of Kollár–Shepherd-Barron–Alexeev), it is instructive to sketch a proof of a special case of the stable reduction theorem for curves in characteristic 0, using the semi-stable reduction theorem. The goal is to emphasize the role of semi-stable reduction and relative canonical models.

Sketch of stable reduction for curves in characteristic 0. For simplicity we consider the case of a smooth family of curves

$$\pi_K : X_K \to \operatorname{Spec} K$$

(of genus g) over the generic point of $S = \operatorname{Spec} R$, the spectrum of a DVR. Complete this to a family of schemes $\pi : X \to S$. Applying the semi-stable reduction theorem, one obtains after a finite base change a family of nodal curves $\pi' : X' \to S'$. The relative canonical model

$$\operatorname{Proj}_{S'} \bigoplus_n \pi'_* \left(\omega_{X'/S'}^{\otimes n}\right) \to S'$$

is a family of stable curves extending the pull back of X_K. Let us denote this by $\pi^c : X^c \to S'$. Note that the relative canonical model of X^c/S' is again X^c/S'. (In short, we established the valuative criterion (2.2).)

Let us now show that the extension is unique up to isomorphism (i.e. that the valuative criterion of separateness (2.1) holds). To do this, suppose there were two stable reductions $\pi_1^c : X_1^c \to S'$ and $\pi_2^c : X_2^c \to S'$. The surfaces X_1^c and X_2^c are birational by construction. The claim is they are in fact isomorphic over S'. We outline the following standard proof of this in order to motivate similar statements in other settings.

Resolving the singularities of the surfaces, resolving the resulting birational map of smooth surfaces, and applying the Semi-stable Reduction Theorem again if necessary, we may assume there is a diagram:

$$\begin{tikzcd} & Z \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ X_1^c \arrow[dr, "\pi_1^c"'] & & X_2^c \arrow[dl, "\pi_2^c"] \\ & S' & \end{tikzcd}$$

where ϕ_1, ϕ_2 are sequences of blow-ups, Z is a smooth surface, and Z/S' is a family of nodal curves. One can show that $(\pi_1^c)_*\omega^{\otimes n}_{X_1^c/S'} \cong (\pi_1^c \circ \phi_1)_*\omega^{\otimes n}_{Z/S'}$ (see e.g. [**72**, Ex. 3.108, p.156, p.84]) and similarly for the other side of the diagram. It follows that

$$(\pi_1^c)_*\omega^{\otimes n}_{X_1^c/S'} \cong (\pi_1^c \circ \phi_1)_*\omega^{\otimes n}_{Z/S'} \cong (\pi_2^c \circ \phi_2)_*\omega^{\otimes n}_{Z/S'} \cong (\pi_2^c)_*\omega^{\otimes n}_{X_2^c/S'}.$$

Thus the relative canonical models of X_1^c/S' and X_2^c/S' agree, so in fact X_1^c and X_2^c are isomorphic over S'. □

6.2. Other stable reduction theorems for curves. Recently, especially in connection with the Hassett–Keel program (see §8.5), there has been interest in understanding alternate compactifications of (open subsets of) M_g. From the perspective of stacks, this is the question of determining alternate proper stacks $\overline{\mathcal{M}}_g^{Alt}$ that contain (open substacks of) $\mathcal{M}_g$ as an open substack. From the perspective of stable reduction, this is the problem of defining classes of curves for which a stable reduction theorem holds. We direct the reader to Smyth [**119**] for more details (see also [**17**],[**16**] where a notion weaker than properness is considered). Here we consider Schubert's space of pseudo-stable curves.

We recall the definitions from [**114**] (see also [**75**]). A pseudo-stable curve X over an algebraically closed field k is a pure dimension 1, reduced, connected, complete scheme of finite type, with at worst nodes and cusps as singularities, such that the canonical line bundle is ample and every sub-curve of genus 1 meets the rest of the curve in at least two points ([**114**, Def. p.297], [**75**, p.4473]). We note that for $g \geq 3$, pseudo-stable curves have finite automorphism groups ([**114**, p.312]). The genus is defined as $g = h^1(X, \mathscr{O}_X)$. For a scheme B, a pseudo-stable curve X/B is a proper, flat morphism $X \to B$ whose geometric fibers are pseudo-stable curves.

There is the following stable reduction theorem for this class of curves.

THEOREM 6.6 (Pseudo-Stable Reduction [**114**]). *Let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K. Let X_K be a genus $g \geq 3$, pseudo-stable curve over K. Then there is a finite extension K' of K, so that taking R' to be the integral closure of R in K' and setting $S' = \operatorname{Spec} R'$, there exists a pseudo-stable curve $X_{S'}$ over S' extending the pull-back $X_{K'}$. Morover, the extension $X_{S'}$ is unique up to isomorphism.*

For details we refer the reader to [**75**, §2] where the results in [**114**] are translated into the language of stacks. To be precise, for each $g \geq 3$, define a stack $\overline{\mathcal{M}}_g^{ps}$ whose objects are families of genus g, pseudo-stable curves, and whose morphisms are given by pull-back diagrams. The results in [**114**] establish that $\overline{\mathcal{M}}_g^{ps}$ is a proper DM stack.

7. Stable reduction in higher dimensions

We now consider the problem of stable reduction for higher dimensional varieties. In general, determining a proper moduli stack, or even a separated moduli stack, is quite difficult. The literature on this topic is vast, and we direct the reader to Viehweg [**122**], Alexeev [**7, 8**], Kollár–Shepherd-Baron [**91**] and Kollár [**84, 87, 88**].

The focus of this section is to outline the approach of Kollár–Shepherd-Barron–Alexeev (KSBA). The basic idea is to utilize relative log-canonical models of the semi-stable reduction to obtain the "stable" reduction. We discuss recent results of Kollár [**87**] in §7.3 that extend the results cited above to give a stable reduction theorem for canonically polarized varieties of any dimension. In §7.1 we discuss an example indicating a few of the difficulties that arise for varieties with negative kodaira dimension, and in §7.2 we discuss the case of $K3$ surfaces, for which a stable reduction theorem is not known.

7.1. Negative Kodaira dimension. It has long been understood that moduli spaces of non-canonically polarized schemes are in general, poorly behaved. In particular, in this subsection, we consider the case of varieties with negative Kodaira dimension.

One of the immediate problems that arises is the existence of varieties with non-discrete, affine automorphism groups. This immediately precludes the separateness of any moduli stack containing such varieties (since the diagonal of the stack could not be proper). We direct the reader to Kovács [**92**, §5.D] for a number of examples and for further discussion.

For the convenience of the reader, we recall the following elementary example ([**92**, Exa. 5.10]). Let us show in concrete terms, that any moduli stack containing $\mathbb{P}^1_k$ will fail the valuative criterion for separatedness. Let $X = Y = \operatorname{Proj}_{k[t]} k[X_0, X_1, t] = \left(\mathbb{P}^1_k \times \mathbb{A}^1_k\right)/\mathbb{A}^1_k$. Over the open set $U = \operatorname{Spec} k[t]_t$, there is an isomorphism $X_U \to Y_U$ given by $([a_0 : a_1], b) \mapsto ([ba_0 : a_1], b)$. This clearly does not extend to an isomorphism over $\mathbb{A}^1_k$. Passing to the DVR $R = k[t]_{(t)}$, one sees the valuative criterion for separateness fails. In fact, one can show that the valuative criterion for separateness fails in this example even if one considers polarizations (see [**92**, Exa. 5.10]).

For contrast, we direct the reader to Matsusaka–Mumford [**98**, Thm. 2] for a general result on separateness of moduli spaces of polarized manifolds that are not uni-ruled. Finally, we point out that there do exist separated moduli spaces of certain uni-ruled varieties. For instance, there are separated moduli spaces of Fano hypersurfaces of degree at least 3 ([**105**, Prop. 4.2, p.79]), and we direct the reader to Benoist [**26**, Thm. 1.6] for some recent results on separateness of moduli stacks of Fano complete intersections.

7.2. $K3$ **surfaces.** As another indication of the difficulties in establishing stable reduction theorems, we briefly discuss the case of $K3$ surfaces. We work over $\mathbb{C}$. Recall a $K3$ surface is a smooth, complex, projective surface X with $K_X \cong \mathscr{O}_X$ and $q := h^1(\mathscr{O}_X) = 0$. A polarized $K3$ surface is a pair (X, L) with L an ample line bundle. The degree of the polarization is defined to be $d := L.L$. Via Hodge theory, one can construct a moduli space of polarized $K3$ surfaces of degree d, which we will denote by F°_d, together with a period map

$$F^\circ_d \to \mathcal{D}_d/\Gamma_d,$$

where $\mathcal{D}_d$ is a 19-dimensional, symmetric homogeneous domain of type IV, and Γ_d is an arithmetic group. The morphism is injective (see [**96**]) and has image equal to the complement of the quotient of an arithmetic hyperplane arrangement. Note that including $K3$ surfaces with ADE singularities, one obtains a moduli space F_d isomorphic to $\mathcal{D}_d/\Gamma_d$ [**93, 111**]. We refer the reader to [**65**, §2.5, Thm. 2.9], which includes a concise overview of these results (and determines the Kodaira dimension of these spaces), and [**24**, VIII] for more details and references.

The Satake–Bailly–Borel compactification $F^*_d := (\mathcal{D}_d/\Gamma_d)^*$, as well as the toroidal compactifications $\overline{F}_d := \overline{\mathcal{D}_d/\Gamma_d}$, provide projective compactifications of the moduli of $K3$ surfaces. It is not known whether any of these are the coarse moduli space for some proper stack of degenerations of $K3$ surfaces.

The first step towards constructing such a proper moduli space would be a stable reduction theorem. A result in this direction is a refined semi-stable reduction theorem due to Kulikov and Persson–Pinkham.

THEOREM 7.1 (Kulikov [**93**], Persson–Pinkham [**111**, Thm., p.45]). *A family of $K3$ surfaces over the punctured disk $X^\circ \to S^\circ$ admits a semi-stable reduction $X \to S$ with central fiber X_0 (a reduced, snc scheme)* such that $K_{X_0} \equiv 0$.

An algebraic version due to Shepherd–Barron [**117**, Thm. 2, p.136] for families of polarized $K3$ surfaces provides a projective completion of the family, where the central fiber has slc (rather than snc) singularities. Some results due to Shah [**116**] using GIT constructions have provided some "weak" forms of stable reduction in some specific cases (in the sense of GIT; see §11). Unfortunately, none of these provide a stable reduction theorem for $K3$ surfaces, even in the polarized case. We refer the reader to [**58**] for a more extensive discussion of the topic.

REMARK 7.2. Recently, combining the Shepherd–Barron result [**117**, Thm. 2, p.136] with the KSBA strategy, Laza [**95**, Thm. 2.11] has constructed a proper moduli space of stable (slc) $K3$ pairs (X, Δ). Essentially, rigidifying the moduli problem further by choosing sections of the polarizations with mild singularities, a stable reduction theorem is possible. We refer the reader to Laza [**95**] for more details.

7.3. Canonically polarized varieties. We now consider stable reduction for canonically polarized varieties. The main result is a recent theorem of Kollár [**87**], which states that stable reduction holds for slc models. We begin by recalling some of the definitions.

7.3.1. *Preliminaries.* In this section, *all schemes will be taken to be reduced, of finite type over $\mathbb{C}$, and all points will be taken to be closed points, unless otherwise stated.* Recall a node of an equidimensional scheme X of dimension n is a point $x \in X$ such that $\widehat{\mathscr{O}}_{X,x} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1x_2)$ as $\mathbb{C}$-algebras. X is said to have at worst nodes in codimension 1 if there exists an open subset $V \subseteq X$ with $\operatorname{codim}_X(X - V) \geq 2$, with the property that for all $x \in V$, x is either a non-singular point, or a node. For scheme X that is S_2, X is nodal in codimension one if and only if, in codimension one, it is both semi-normal and Gorenstein (see Kollár [**89**, §5.1, p.196]).

We will want to discuss divisors on reducible, equidimensional, reduced schemes X. A Weil divisor D on such a scheme is a finite, formal, integral, linear combination of (not necessarily closed) points $E \in X$ such that $\mathscr{O}_{X,E}$ is a DVR. There is a notion of linear equivalence for such divisors obtained via Weil divisorial subsheaves; we direct the reader to Corti [**45**, (16.1.1), (16.2.2), p.171-2]. A $\mathbb{Q}$-divisor is defined similarly, with $\mathbb{Q}$-coefficients. A $\mathbb{Q}$-divisor D on X is said to be $\mathbb{Q}$-Cartier if there exists an $m \in \mathbb{N}$ such that mD is the Weyl divisor associated to a Cartier divisor.

For X a projective scheme, we denote by $\omega_X^\bullet$ the dualizing complex. We set $\omega_X := h^{-n}(\omega_X^\bullet)$, and call this the canonical sheaf of X. If X is Gorenstein in codimension one, then associated to ω_X is a linear equivalence class of Weil divisors (see e.g. [**45**, (16.3.3), p.173]). We denote this equivalence class by K_X and call it the canonical divisor (class). We direct the reader also to [**90**, §5.5] and [**89**, Def. 1.6, p.14] for more discussion.

REMARK 7.3. In order to limit the length of this survey, we have suppressed the notion of a pair in most of the topics covered. However, the utility of parameterizing varieties together with a distinguished divisor goes back at least to the case of principally polarized abelian varieties, where the canonical bundle is trivial, and one substitutes the theta divisor in its place to provide a natural rigidity to the problem. Recently it has become clear that in many other situations it can be beneficial to consider pairs (X, Δ) where X is a variety, and Δ is an effective divisor so that $K_X + \Delta$ is ample (see especially the work of Kollár and Alexeev cited above). The notion of pairs will be central in what follows.

7.3.2. *Semi log canonical models.* We start by recalling the definition of log canonical pairs. Let X *be a projective, reduced, equidimensional,* S_2 *scheme and let* Δ *be an effective* $\mathbb{Q}$*-divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier.* With these assumptions, we say that the pair (X, Δ) is **log canonical (lc)** if X is smooth in codimension one (or equivalently X is normal) and there exists a log resolution $f : Y \to X$ of (X, Δ) such that

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i, \tag{7.1}$$

where the E_i are f-exceptional divisors and $a_i \geq -1$ for every i. Note that the equality in (7.1) is $\mathbb{Q}$-equivalence of $\mathbb{Q}$-Cartier divisors. We say that X is **lc** if the pair $(X, 0)$ is lc, where 0 is the trivial divisor (see e.g. [**90**] for more discussion).

With the assumptions above (in italics), we say that the pair (X, Δ) is **semi log canonical (slc)** if X is nodal in codimension one (or equivalently, in codimension one X is seminormal and Gorenstein), $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and if $\nu : X^\nu \to X$ is the normalization of X and Θ is the $\mathbb{Q}$-Weil divisor on X given by

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta), \tag{7.2}$$

then the pair (X^ν, Θ) is lc. Note that the equality in (7.2) is an equivalence, for which we refer the reader to [**89**, (5.7.5), Def.-Lem. 5.10]. We say that X is **slc** if the pair $(X, 0)$ is slc, where 0 is the trivial divisor (see e.g. Abramovich–Fong–Kollár–McKernen [**2**], Fujino [**61**], Kollár [**87**] and Kollár [**89**, §5.2] for more discussion).

A **semi log canonical model** (slc model) is an slc scheme X such that K_X is ample ([**87**, Def. 15]). In particular, if X is smooth, then it is an slc model if and only if it is canonically polarized. A motivation for this definition also comes from the cases of curves and surfaces. A semi log canonical model of dimension one is a stable curve of genus $g \geq 2$. A result of Kollár–Shepherd-Barron [**91**, Cor. 5.7], Kollár [**84**, Cor. 5.6] and Alexeev [**6**] establishes that there is a proper moduli space of semi log canonical models of dimension two (with fixed invariants K_X^2 and $\chi(\mathscr{O}_X)$). The valuative criterion for properness of the moduli space can be established with an appropriate stable reduction theorem.

7.3.3. *Kollár's stable reduction theorem.* Recently Kollár has established a stable reduction theorem for semi log canonical models of any dimension. The full statement would require introducing the notion of relative semi log canonical models (and in particular the notion of reflexive hulls on non-normal schemes), which we omit (see [**87**, Def. 28, 29]). Below we state a weaker version of this stable reduction theorem, where the generic fiber of the family is lc. We sketch the parts of the proof that are formally similar to the proof of the stable reduction theorem that we sketched in the case of curves.

Theorem 7.4 (Kollár [**87**, 5.38]). *Let* B *be on open subset of a non-singular curve over* $\mathbb{C}$*, fix a point* $o \in B$*, and set* $U = B - \{o\}$*. Suppose that*

$$\pi : X \to B$$

is a flat, projective, morphism with connected fibers, such that the restriction $\pi_U : X_U = X|_U \to U$ *has lc fibers, and has* π_U*-ample relative dualizing sheaf* $\omega_{X_U/U}$*.*

Then there is a finite base change $f : B' \to B$, with B' non-singular and $f^{-1}(o)$ a single point o', and a scheme

$$\pi^c : X^c \to B'$$

such that X^c and $B' \times_B X$ are isomorphic over $U' := B' - o'$, and the fiber $X^c_{o'} = (\pi^c)^{-1}(o')$ is an slc model. Moreover, the extension X^c/B' is unique up to isomorphism.

Sketch of the proof. Let $\pi : X \to B$ be as in the statement of Theorem 7.4. From the Semi-stable Reduction Theorem we obtain a diagram as in (3.1) including a morphism $\pi' : X' \to B'$ satisfying the conclusions of Theorem 7.4, except that the central fiber of the morphism $\pi' : X' \to B'$, which is normal crossing, may not have ample canonical class (see Hacon–Xu [**71**] for the case where the general fiber is lc, rather than smooth). From this point, motivated in part by the case of curves, one considers

$$X^c := \mathrm{Proj}_{B'}\left(\bigoplus_{k=0}^{\infty} \pi'_*\left(\omega_{X'/B'}^{\otimes k}\right)\right).$$

A result of Birkar–Cascini–Hacon–McKernan [**28**, Thm. 1.2 (3)] implies that the sheaf $\bigoplus_{k=0}^{\infty} \pi'_*(\omega_{X'/B'}^{\otimes k})$ is finitely generated as an $\mathscr{O}_{B'}$-algebra (see also Hacon–Xu [**71**] for the case where the general fiber is lc). In other words the projection $\pi^c : X^c \to B'$ is a projective morphism that agrees with $\pi' : X' \to B'$ over U'. One can show that the central fiber of π^c is an slc model ([**87**]).

It remains to show that $\pi^c : X^c \to B'$ is unique up to isomorphism. One does this by establishing that any other projective morphism $\hat{\pi}^c : \hat{X}^c \to B'$ with $K_{\hat{X}^c}$ a $\mathbb{Q}$-Cartier divisor, which agrees with $\pi^c : X^c \to B'$ over U', and which has central fiber an slc model, is isomorphic to $\pi^c : X^c \to B'$ over B'. The proof in the case where the generic fiber is smooth is formally similar to the proof we sketched in the case of curves. We direct the reader to Kollár [Pro. 6, Def. 7, Def. 15, and pp.8-9] for more details (see also [**27**, Lem. 2.7]). □

Remark 7.5. It is well known that a smooth, projective variety of general type has a finite automorphism group (positive dimensional automorphism groups give rise to rational curves or abelian varieties covering the variety). More generally, it is a result of Iitaka that smooth, projective varieties of log general type have finite automorphism groups [**77**, Lem. 1, p.87, Def. p.71]. Consequently, considering log resolutions of each irreducible component of the normalization, one would expect from (7.1) and Iitaka's result that an slc model would have a finite automorphism group; this is in fact the case [**91**, p.328], [**89**, Cor. 10.69], [**27**, Lem. 2.5].

Example 7.6. It is interesting to note the importance of having a condition such as slc in the remark above. For instance, a plane quartic C consisting of two smooth conics meeting in a single point, which is a tacnode, has ample canonical bundle $\mathscr{O}_C(1)$. However, the automorphism group of the curve is not finite (see §11.3 below).

Remark 7.7. Fix a Hilbert function $H : \mathbb{Z} \to \mathbb{Z}$, and let $\overline{\mathcal{M}}_H$ be the associated category fibered in groupoids (over $\mathrm{Sch}_{\mathbb{C}}^{\text{ét}}$), with objects that are families of slc models with Hilbert function H, as defined in [**87**, Def. 29], and with morphisms given by pull-back diagrams. It is shown in [**27**, Thm. 2.8] (using recently announced results of Hacon–McKernan–Xu) that $\overline{\mathcal{M}}_H$ is a proper Deligne–Mumford $\mathbb{C}$-stack.

8. Simultaneous stable reduction

Simultaneous stable reduction can be viewed as the problem of stable reduction over bases other than a DVR; i.e. extending families over higher dimensional bases. In the language of stacks, it is the problem of resolving rational maps from schemes to stacks. Typically a generically finite base change is needed to do this. The problem is in general quite delicate, and closely related to the problem of resolving period maps to coarse moduli schemes.

We discuss some well known results of Faltings–Chai [**51**] for abelian varieties, and some more recent results of de Jong, de Jong–Oort, and Cautis for curves. An explicit example of simultaneous stable reduction was given in §1, and we start this section by discussing simultaneous stable reduction in the language of stacks.

8.1. Simultaneous stable reduction in the language of stacks. For a separated, finite type, algebraic Z-stack $\mathcal{M}$, the valuative criterion for properness can be viewed as a question about resolving rational maps from the spectrums of DVRs into $\mathcal{M}$. More precisely, $\mathcal{M}$ is proper if and only if for every DVR R, every rational map $S = \operatorname{Spec} R \dashrightarrow \mathcal{M}$ can be resolved after a generically finite base change.

Simultaneous stable reduction is the problem of resolving rational maps from higher dimensional schemes into the stack. Let us make this more precise. Given a Z-scheme B, a dense open subset $B^\circ \subseteq B$, and a Z-morphism

$$B^\circ \to \mathcal{M}$$

(which we will refer to loosely as a rational Z-map $B \dashrightarrow \mathcal{M}$) we will say $B^\circ \to \mathcal{M}$ (or $B \dashrightarrow \mathcal{M}$) admits a **simultaneous stable reduction** if there exists a Z-alteration $\widetilde{B} \to B$ and a Z-morphism $\widetilde{B} \to \mathcal{M}$ extending the original morphism from B° in the sense that the following diagram is 2-commutative:

$$\begin{tikzcd} \widetilde{B} \arrow[d] \arrow[rr, bend left] & \widetilde{B} \times_B B^\circ \arrow[l, hook'] \arrow[r] \arrow[d] & \mathcal{M} \\ B & B^\circ \arrow[l, hook'] \arrow[ur] & \end{tikzcd} \tag{8.1}$$

Example 8.1. Let us review the example in §1 in this language. Recall we were given a family $X \to B = \mathbb{A}^2_k$ of plane cubics (with a section), and an open set $U \subseteq B$, so that the restriction $X_U \to U$ was a family of elliptic curves. Such a family induces a morphism $U \to \overline{\mathcal{M}}_{1,1}$ (which we think of as a rational map $B \dashrightarrow \overline{\mathcal{M}}_{1,1}$). We explicitly described an alteration $\widetilde{B}' \to B$ with the property that setting $\widetilde{U}' = U \times_B \widetilde{B}'$ to be the preimage, the pull-back $X_U \times_U \widetilde{U}'$ extended to a family of stable marked curves $\widehat{X} \to \widetilde{B}'$. This gives a morphism $\widetilde{B}' \to \overline{\mathcal{M}}_{1,1}$ extending the original morphism $U \to \overline{\mathcal{M}}_{1,1}$ (in the sense of (8.1)).

For proper, algebraic Z-stacks over a noetherian scheme Z, with finite diagonal, simultaneous stable reductions exist quite generally. The following result, which seems to be well known, was pointed out to the author by Fedorchuk ([**54**, Rem. 7.3]).

Theorem 8.2 ([**54**, Rem. 7.3], [**50**, Thm. 2.7]). *Let Z be a noetherian scheme, and let $\mathcal{M}$ be a proper, algebraic Z-stack with finite diagonal $\mathcal{M} \xrightarrow{\Delta} \mathcal{M} \times_Z \mathcal{M}$.*

Then any rational Z-map $B \dashrightarrow \mathcal{M}$ from a quasi-compact, quasi-separated Z-scheme B (or Noetherian Z-scheme B) admits a simultaneous stable reduction; i.e. it can be resolved by an alteration.

REMARK 8.3. Recall that the diagonal morphism of an algebraic Z-stack is quasi-compact by assumption, and an algebraic Z-stack is Deligne–Mumford if and only if the diagonal is unramified (e.g. [**94**, Thm. 8.1]). A quasi-compact, unramified morphism of schemes is quasi-finite. In other words, Deligne–Mumford Z-stacks have quasi-finite diagonal (see also [**123**, Lem. 1.13 (i)], and [**50**, Rem. 2.5] for a converse in characteristic 0). Recall also that an algebraic Z-stack locally of finite type has diagonal that is locally of finite presentation (e.g. [**66**, Cor. 1.4.3.1]). Finally, note that a separated stack has proper diagonal by definition; thus, since a proper, quasi-finite morphism, locally of finite presentation, is finite ([**67**, Thm. 8.11.1]), a separated, locally finite type algebraic Z-stack with quasi-finite diagonal in fact has finite diagonal. In particular a separated, finite type, Deligne–Mumford stack $\mathcal{M}/Z$ has finite diagonal (see also [**123**, Lem. 1.13 (ii)]).

The theorem is an immediate consequence of the following lemma of Fedorchuk [**54**] and a theorem of Edidin–Hassett–Kresch–Vistoli [**50**].

LEMMA 8.4 (Fedorchuk [**54**, Rem. 7.3]). *Let Z be a noetherian scheme. Let $\mathcal{M}$ be an algebraic Z-stack, proper over Z, that admits a finite, surjective Z-morphism*

$$V \to \mathcal{M}$$

from a scheme V. Then any rational Z-map $B \dashrightarrow \mathcal{M}$ from a quasi-compact, quasi-separated Z-scheme B (or Noetherian Z-scheme B) admits a simultaneous stable reduction; i.e. it can be resolved by an alteration.

PROOF. The proof (following Fedorchuk [**54**]) is short and we include it here. Consider the finite morphism $V \to \mathcal{M}$ assumed in the statement of the lemma. Note we obtain that V is proper over Z, since V is finite (and hence proper) over $\mathcal{M}$ and we have assumed that $\mathcal{M}$ is proper over Z.

Let $B^\circ \to \mathcal{M}$ be the morphism inducing the rational map $B \dashrightarrow \mathcal{M}$. From the definition of an algebraic stack, the diagonal is representable. Consequently, $B^\circ \times_{\mathcal{M}} V$ is a scheme. We then have a commutative diagram

$$\begin{tikzcd} B^\circ \times_{\mathcal{M}} V \arrow[r] \arrow[d, "\text{finite}"] & V \arrow[d, "\text{finite}"] \\ B^\circ \arrow[r] & \mathcal{M}. \end{tikzcd} \tag{8.2}$$

Let $B' \to B$ be a finite morphism extending $B^\circ \times_{\mathcal{M}} V \to B^\circ$; to obtain this extension one can either use Zariski's Main Theorem [**67**, EGA IV.3 Thm. 8.12.6, p.45] or [**52**, Lem. 5.19, p.131] (in the latter case, one extends the push forward of the structure sheaf of $B^\circ \times_{\mathcal{M}} V$ to a coherent sheaf on B and then takes the relative spectrum). We thus obtain a rational map $B' \dashrightarrow V$.

Let $\widetilde{B}$ be the closure of the graph of $B^\circ \times_{\mathcal{M}} V \to V$ in $B' \times_Z V$. The morphism $B' \times_Z V \to B'$ is proper by base change, and a closed immersion is proper. It follows that $\widetilde{B} \to B'$ is proper, and also birational by construction. Thus the composition

$$\widetilde{B} \to B' \to B$$

gives an alteration that resolves the map to $\mathcal{M}$. $\square$

REMARK 8.5. The assumption that the scheme B be quasi-compact and quasi-separated, or that it be Noetherian, was used to ensure the existence of a finite cover of B extending the given finite cover of B°. Another approach could be to assume that B is covered by the spectrums of Japanese rings. In this case, the appropriate integral closures will be finitely generated, allowing for another construction of a finite cover.

Combining the lemma with the following theorem of Edidin–Hassett–Kresch–Vistoli [**50**] establishes Theorem 8.2.

THEOREM 8.6 (Edidin et al. [**50**, Thm. 2.7]). *Suppose Z is a noetherian scheme. Let $\mathcal{M}$ be an algebraic Z-stack of finite type over Z. Then the diagonal*

$$\mathcal{M} \to \mathcal{M} \times_Z \mathcal{M}$$

is quasi-finite if and only if there exists a finite, surjective Z-morphism $V \to \mathcal{M}$ from a (not necessarily separated) Z-scheme V.

From Theorem 8.2 we obtain the following corollary.

COROLLARY 8.7. *Suppose that $\mathcal{M}$ is one of the following stacks:*

(1) *$\bar{\mathcal{A}}_g^A$, the moduli of stable semi-abelic pairs degree 1 and dimension g;*
(2) *$\overline{\mathcal{M}}_g$ ($g \geq 2$), the moduli of stable, genus g curves;*
(3) *$\overline{\mathcal{M}}_g^{ps}$ ($g \geq 3$), the moduli of pseudo-stable, genus g curves;*
(4) *$\overline{\mathcal{M}}_H$, the moduli of slc models associated to a Hilbert function H;*
(5) *$\overline{\mathcal{P}}_d$, the moduli of degree d, stable slc $K3$ pairs.*

Then any rational map $B \dashrightarrow \mathcal{M}$ from a quasi-compact, quasi-separated scheme B (or Noetherian scheme B) admits a simultaneous stable reduction; i.e. it can be resolved by an alteration.

REMARK 8.8. In concrete terms, the corollary says the following. Given a dense open subset $U \subseteq B$, and a family $X_U \to U$, there exists an alteration $\widetilde{B} \to B$ so that the pull-back of the family can be extended to a family over all of $\widetilde{B}$.

REMARK 8.9. Part (2) of the corollary is a special case of a theorem of de Jong [**46**, Thm. 5.8]. We direct the reader there for more details, especially for a discussion of the total space of the family.

In many cases it can be useful to have an explicit description of an alteration giving a stable reduction. We will call this an **explicit simultaneous stable reduction**. Along these lines, one of the first questions one can ask is whether a rational map to a stack can be extended without an alteration. In particular, when B is non-singular, and $\Delta = B - B^\circ$ is an snc divisor, a theorem giving conditions for the rational map to extend to B will be called an **extension theorem**.

Finally, when a stack admits a coarse moduli scheme, it can also be interesting to consider the problem of resolving the induced rational map to the coarse moduli scheme. One place these types of problems show up naturally is in resolving rational (period) maps between coarse moduli schemes.

More precisely, suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are algebraic Z-stacks admitting coarse moduli schemes $\mathcal{M}_1 \to M_1$ and $\mathcal{M}_2 \to M_2$. Suppose there is an open dense subset $U_1 \subseteq M_1$, which admits morphisms $U_1 \to \mathcal{M}_1$ and $U_1 \to \mathcal{M}_2$. This induces a rational map $M_1 \dashrightarrow M_2$, and one may be interested in both a simultaneous stable reduction for $U_1 \to \mathcal{M}_2$ as well as a resolution of the rational map $M_1 \dashrightarrow M_2$. We will consider both types of problems in what follows.

8.2. Simultaneous stable reduction for abelian varieties. We begin by considering extension theorems for abelian varieties. That is we consider the case of extending families of abelian varieties over non-singular bases other than a DVR. The main result we mention is due to Faltings–Chai [**51**].

THEOREM 8.10 (Faltings–Chai Extension [**51**, Thm. 6.7, p.185]). *Let B be a regular scheme over a field of characteristic 0. Let $\Delta \subseteq B$ be an nc divisor. Let A_U be an abelian scheme over $U = B - \Delta$, which extends to a semi-abelian scheme A_V over an open subscheme V containing U and the generic points of Δ. Then A_U extends uniquely to a semi-abelian scheme A_B over B.*

REMARK 8.11. This fails in positive characteristic. A counter example when the characteristic of the generic points of B are positive is given in [**51**, p.192]. A counter example of Raynaud–Ogus–Gabber, when the characteristic of the generic points of B are zero (but where other points have positive characteristic), is given in de Jong–Oort [**47**, §6].

The Faltings–Chai theorem implies a special case of the Borel Extension Theorem. Recall that we use the notation $\mathcal{A}_g$ for the stack of principally polarized abelian varieties of dimension g. A morphism $U \to \mathcal{A}_g$ corresponds to a family $A_U \to U$ of principally polarized abelian varieties. We denote the coarse moduli space by A_g. We denote by A_g^* the Satake (Bailly–Borel) compactification, and by $\bar{A}_g$ any one of Mumford's toroidal compactifications. The most common toroidal compactification we will use is the second Voronoi, which we will denote by $\bar{A}_g^{Vor}$. We direct the reader to [**107**] for more details on compactifications of A_g.

THEOREM 8.12 (Borel Extension [**29**, Thm. A]). *Let B be a regular scheme over a field of characteristic 0. Let $\Delta \subseteq B$ be an nc divisor. Setting $U = B - \Delta$, then for any morphism $f : U \to \mathcal{A}_g$, the composition $U \to \mathcal{A}_g \to A_g$ extends to a morphism $B \to A_g^*$.*

Borel's proof uses hyperbolic complex analysis and holds more generally for (locally liftable) holomorphic maps into Baily–Borel compactifications of arithmetic quotients of bounded symmetric domains. Faltings–Chai [**51**, Cor. 6.11, p.191] also prove the related statement for maps into the moduli space $A_g[n]$ of principally polarized abelian varieties with level n-structure for $n \geq 3$. In this case they can use [**105**, Thm. 7.9, Thm. 7.10, p.139] to conclude that the coarse moduli space is quasi-projective and fine. In other words, in both this situation, as well as under the assumptions of the Borel extension theorem as stated above, one may assume there is a family of abelian varieties over U.

The argument from there is short. First, the extension statement is local. One can also show that it suffices to establish extension after a finite base change (e.g. [**40**, Lem. 2.4]). Thus we may take an étale base change, and assume we are in the situation where B is regular and Δ has support defined by $x_1 \cdots x_r$, where $x_1, \dots, x_r$ form part of a system of local parameters. Taking the finite cover $t_1 = x_1^{m_1}, \dots, t_r = x_r^{m_r}$ for appropriate values of $m_1, \dots, m_r$, one uses the monodromy theorem to get extension over the generic points of Δ. The result then follows from the Faltings–Chai Extension Theorem.

REMARK 8.13. The condition in the Borel Extension Theorem that there is a family of abelian varieties over U (or more generally that the holomorphic map is locally liftable to the bounded symmetric domain) is essential. More precisely, for

B and U as in the theorem, given a morphism $f : U \to A_g$, this need not extend to a morphism $B \to A_g^*$. An elementary example comes from the case of A_1 and A_1^*. We can identify A_1 as the quotient $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ of the upper half plane by the special linear group in the usual way, and it is well known that A_1^* is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. The map $(\mathbb{C}^*)^2 \to \mathbb{P}^1_{\mathbb{C}}$ given by $(\lambda_1, \lambda_2) \mapsto [\lambda_1 : \lambda_2]$ clearly does not extend to a morphism from $\mathbb{C}^2$.

Remark 8.14. It is natural to ask whether a statement like the Borel extension theorem could hold for the toroidal compactifications of A_g, and indeed there is an extension theorem due to Ash–Mumford–Rapaport–Tai [**23**] (see also Namikawa [**107**, Thm. 7.29, p.78]) giving explicit conditions for morphisms to extend over nc boundaries. In concrete examples these extension conditions can be difficult to establish. We discuss some particular examples below.

8.3. Examples of period maps to A_g. We now consider the related problem of resolving period maps into compactifications of the moduli scheme of abelian varieties. In this section we will work over $\mathbb{C}$. The most well known example is the Torelli map for curves; i.e. the morphism

$$\mathcal{T} : \mathcal{M}_g \to \mathcal{A}_g$$

that sends a curve C to its principally polarized Jacobian (JC, Θ_C). Let $T : M_g \to A_g$ be the associated morphism of coarse moduli spaces. Torelli's theorem states that T is injective.

The boundary Δ in $\overline{M}_g$ is (up to finite quotient singularities) an nc divisor. As a consequence of the Borel extension theorem, we obtain a morphism

$$T^* : \overline{M}_g \to A_g^*$$

extending T. For the toroidal compactifications of A_g, there are the general extension results mentioned above. In practice, these can be difficult to verify. It is a result of Mumford and Namikawa [**107**], [**106**, §18] that T extends to a morphism

$$\overline{T}^{Vor} : \overline{M}_g \to \bar{A}_g^{Vor}.$$

Caporaso–Viviani describe the fibers of the morphism in [**34**]. In addition, it is shown in Alexeev [**10**] that there is a morphism $\overline{\mathcal{T}}^{Vor} : \overline{\mathcal{M}}_g \to \bar{\mathcal{A}}_g^A$ extending $\mathcal{T}$. We direct the reader to Alexeev–Brunyate [**12**] for a proof that the Torelli map for stable curves extends to a morphism to the first Voronoi compactification (see also Gibney [**62**] for more on the image of the Torelli map to other toroidal compactificiations).

We now turn our attention to the Prym map. We denote by $\mathcal{R}_g$ the moduli stack of connected, étale double covers of non-singular curves of genus g. The Prym map

$$\mathcal{P}r : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

takes a double cover $\pi : \tilde{C} \to C$ to its principally polarized Prym variety (P, Ξ) (see Mumford [**102**] for more details). We denote by $Pr : R_g \to A_{g-1}$ the associated morphism of coarse moduli spaces. It is well known that the map is dominant for $g \leq 6$ (see esp. [**25**]), and in the other direction, Friedman–Smith [**59**] and Kanev [**78**] have shown that the map is generically injective for $g \geq 7$.

There is a compactification, $\overline{\mathcal{R}}_g$, due to Beauville [**25**], consisting of admissible double covers. The coarse moduli space $\overline{R}_g$ has (up to finite quotient singularities) an nc boundary. As a consequence, there is an extension

$$Pr^* : \overline{R}_g \to A^*_{g-1}.$$

On the other hand, Friedman–Smith [**60**] have shown that the Prym map does not extend to a morphism to $\bar{A}^{Vor}_{g-1}$ (or any reasonable Toroidal compactification). We direct the reader to Alexeev–Birkenhake–Hulek [**11**] for more details on the indeterminacy locus of the Prym map to $\bar{A}^{Vor}_{g-1}$.

The Clemens–Griffiths [**42**] period map for cubic threefolds provides another interesting example. Recall that a cubic threefold is a smooth cubic hypersurface $X \subseteq \mathbb{P}^4$. The intermediate Jacobian of X is the five dimensional complex torus $JX := H^{1,2}(X)/H^3(X,\mathbb{Z})$. This admits a principal polarization Θ_X, given by the hermitian form h on $H^{1,2}(X)$ defined by $h(\alpha,\beta) = 2i\int_X \alpha\wedge\bar{\beta}$. Letting M_{cub} be the moduli space of cubic threefolds, one obtains a morphism

$$J : M_{cub} \to A_5.$$

By virtue of the Clemens and Griffiths Torelli theorem [**42**] (see also Mumford [**102**]), J is injective. We denote the image by I, and we direct the reader to Casalaina-Martin–Friedman [**37**] for a geometric characterization of the abelian varieties parameterized by I.

The space M_{cub} admits a GIT compactification

$$\overline{M}_{cub} = \mathbb{P}H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(3)) /\!\!/ \mathrm{SL}(5),$$

(see Allcock [**13**], Yokoyama [**125**]) and it is natural to consider extensions of the period map J to $\overline{M}_{cub}$. Allcock–Carlson–Toledo [**14**] and Looijenga–Swierstra [**97**] have shown that M_{cub} can be identified with an open dense subset of a ten dimensional ball quotient $\mathcal{B}/\Gamma$. They show moreover, that the rational map $\overline{M}_{cub} \dashrightarrow (\mathcal{B}/\Gamma)^*$ to the Baily–Borel compactification, can be resolved by blowing up a single point. We call the resulting space $\widehat{M}_{cub}$.

Using the description of $\widehat{M}_{cub}$ given in [**14, 97**], Laza and the author describe an explicit blow-up $\widetilde{M}_{cub}$ of $\widehat{M}_{cub}$, with discriminant an nc divisor [**38**]. The process used to obtain the resolution is the same as that described for simultaneous stable reduction for ADE curves below (see §8.5). The Borel extension theorem then gives a morphism

$$J^* : \widetilde{M}_{cub} \to A_5^*.$$

Laza and the author use this extension of the period map together with results from [**102, 25, 37, 36**] and the explicit description of $\widetilde{M}_{cub}$ to describe the boundary of the image of J^* [**38**, Thm. 1.1].

Remark 8.15. An explicit resolution of the map $\overline{M}_{cub} \dashrightarrow \bar{A}_5^{Vor}$ is still not known. Certain components of the boundary of the (closure of the) image have been identified by Grushevsky–Salvati Manni [**70**] and Grushevsky–Hulek [**69**] via the theory of theta functions.

8.4. Simultaneous stable reduction for curves. We again begin by considering extension theorems. In analogy with the Faltings–Chai extension theorem for abelian varieties, we mention the following extension theorem of de Jong–Oort [**47**].

Theorem 8.16 (de Jong–Oort Extension [**47**, Thm. 5.1]). *Let B be a regular scheme and Δ an nc divisor on B. Set $U = B - \Delta$. A family of smooth curves of* *genus $g \geq 2$ over U extends to a family of stable curves over B if it extends to a family of stable curves over on open subset V containing each generic point of Δ.*

In fact the theorem is more general, in that one can allow for a generically stable family, so long as the topological type is locally constant on U. A similar result was proven by Moret-Bailly [**99**], where it is required that a generically smooth family extend to a *smooth* family over the generic points of Δ.

A consequence of the de Jong–Oort Extension Theorem is an analogue of the Borel Extension Theorem for stable curves. Before stating the result, let us first rephrase the previous theorem in the language of stacks. The theorem states that given a morphism to the stack $U \to \mathcal{M}_g$, there is an extension $B \to \overline{\mathcal{M}}_g$ if and only if there is an open set $V \subseteq B$ containing U and the generic points of Δ and an extension $V \to \overline{\mathcal{M}}_g$.

Corollary 8.17 (Cautis [**40**, Thm. A]). *Let B be a regular scheme and Δ an nc divisor on B. Set $U = B - \Delta$. Given a morphism $U \to \mathcal{M}_g$, there is an extension $B \to \overline{M}_g$.*

One obtains this corollary from the previous theorem in the same way as the analogous statement was proven for semi-abelian varieties (i.e. in the way the Borel Extension Theorem follows from the Faltings–Chai Extension Theorem).

An independent proof of the corollary was given by Cautis [**40**, Thm. A]. By virtue of $\overline{\mathcal{M}}_g$ being a separated Deligne–Mumford stack, it is immediate to prove the de Jong–Oort Extension Theorem from the corollary using the Abramovich–Vistoli purity lemma [**5**, Lemma 2.4.1].

8.5. Explicit simultaneous stable reduction for curves. Having established the existence of alterations resolving rational maps to $\overline{\mathcal{M}}_g$, one can ask for explicit alterations in specific settings. One place where this type of question arises naturally is in the Hassett–Keel program for the moduli space of curves.

We will not discuss the details of the Hassett–Keel program here (see [**75**]), but will simply note that in this program, projective varieties $\overline{M}_g(\alpha)$, $\alpha \in [0,1] \cap \mathbb{Q}$ arise, which are conjectured to parameterize curves of genus g with prescribed singularities (for $0 \ll \alpha \leq 1$ this has been established in Hassett–Hyeon [**75, 74**]). For "most" g and α there are birational maps

$$\overline{M}_g(\alpha) \dashrightarrow \overline{M}_g$$

to the moduli space of stable curves. It would be of interest to have explicit resolutions. In general, these birational maps will lift to rational maps to the stack $\overline{M}_g(\alpha) \dashrightarrow \overline{\mathcal{M}}_g$, and in this way we arrive at the related problem of simultaneous stable reduction.

With this as motivation, we will consider the following problem. *Given a generically smooth family of curves $X \to B$ with fibers having prescribed singularities, give an explicit description of a resolution of the rational map $B \dashrightarrow \overline{\mathcal{M}}_g$.*

The specific case we will consider is where the singular fibers have at worst ADE singularities (we review the definition of ADE singularities in §10). We call such curves ADE curves, and we will consider the question (étale) locally.

Laza and the author have given a solution to this problem in [**39**] and Fedorchuk has given an independent solution for singularities of type AD in [**54**]. Fedorchuk's proof is based on constructions of proper moduli spaces of hyperelliptic curves $\mathscr{H}[k,\ell]$, where the boundary consists of certain curves with AD singularities at worst of type A_k and D_ℓ. The proof provides modular descriptions of the spaces arising in the processes described below. We direct the reader to Fedorchuk [**54**] for more details.

Below is the version of the result in [**39**]. Since we consider the resolution question (étale) locally, it suffices to understand the case where $X \to B$ is a mini-versal deformation of an ADE curve X_0. The statement of the theorem uses the notion of a Weyl cover, and wonderful blow-up; these are explicit maps, which can be determined by the root systems associated to the singularities. We refer the reader to [**39**, §2,3] for more details. The Weyl cover and wonderful blow-up in §1 are examples. For the statement of the theorem, we note that the wonderful blow-up of the Weyl cover of B has the property that the pull-back of the discriminant is an nc divisor, with irreducible components corresponding to curves with fixed singularity type.

THEOREM 8.18 (Casalaina-Martin–Laza [**39**], Fedorchuk [**54**]). *Let $X \to B$ be a mini-versal deformation of an ADE curve X_0 with $p_a(X_0) = g \geq 2$. The wonderful blow-up of the Weyl cover of B resolves the rational moduli map to the moduli scheme $\overline{M}_g$, but fails to resolve the rational moduli map to the moduli stack $\overline{\mathcal{M}}_g$ along the A_{2n} locus of the discriminant ($n \in \mathbb{N}$). The addition of a stack structure (generically $\mathbb{Z}/2\mathbb{Z}$ stabilizers) along this locus resolves the moduli map to $\overline{\mathcal{M}}_g$.*

REMARK 8.19. Let us elaborate on the final statement in the theorem concerning stacks. There is a family of stable curves over the wonderful blow-up of the Weyl cover, except over the locus parameterizing curves with A_{2n} singularities. This locus is a collection of divisors, and there is an obstruction to extending the family over that locus. At the generic points, the obstruction becomes trivial after taking a branched double cover.

We direct the reader to [**39**] for the proof.

REMARK 8.20. As mentioned above in the section on period maps to the moduli space of abelian varieties, the method of proof of this theorem has applications to other situations including the study of the moduli space of cubic threefolds [**38**].

9. Simultaneous semi-stable reduction

The question of extending the Semi-stable Reduction Theorem to higher dimensional bases is of course very natural, and was asked already in the introduction of [**82**]. It has proven to be a difficult question; even the correct formulation of the problem is not immediately clear. We discuss some recent progress due to de Jong [**46**] and Abramovich–Karu [**3**].

9.1. A result of Abramovich–Karu. The first issue to address is what is meant by semi-stable reduction for higher dimensional bases. We take the following modification of the assumptions in the statement of the Semi-stable Reduction Theorem as the starting point. *We set B to be an open subset of a non-singular variety, set $U \subseteq B$ to be a non-empty open subset and suppose that $\pi : X \to B$ is a surjective, projective morphism of a variety X onto B so that the restriction $\pi_U : X|_U \to U$ is smooth.*

Our goal is to find a diagram as in (3.1) with B' nonsingular, f an alteration, p a projective modification, so that all of the geometric fibers of π' satisfy some natural conditions. For instance, at the very least, we would like all of the geometric fibers of π' to be reduced. Moreover, we could hope that all of the fibers have singularities that look at worst like smooth components meeting "transversally".

For instance, when $\dim(X) = \dim(B) + 1$, if one allows the total space X' to be singular, then it is a result of de Jong [**46**, Thm. 5.8] that such a semi-stable reduction exists. Moreover, de Jong shows that in this case if p is permitted to be an alteration, rather than a modification, then X' can be taken to be smooth. However, for families with higher dimensional fibers, it is not possible in general to obtain a "semi-stable reduction" where the fibers all have singularities that at worst look like smooth components meeting transversally. For instance, the two parameter family of surfaces defined by

$$(t_1 - x_1x_2, t_2 - x_3x_4) \tag{9.1}$$

precludes this (see Karu [**79**, Exa. 1.12, p.21]). Thus, in general, one needs a different definition of "semi-stable reduction" to get a reasonable result.

In light of the presentation in [**82**], and the family (9.1) above (which has fibers with at worst toric singularities), it is natural to change the focus to toroidal structures. Using this language, we state a theorem of Abramovich–Karu, and then discuss some definitions after the theorem.

Theorem 9.1 (Abramovich–Karu [**3**, Thm. 0.3]). *Assume* $\mathrm{char}(k) = 0$ *and* $k = \bar{k}$. *Let* $X \to B$ *be a surjective morphism of projective varieties over* k*, with geometrically integral generic fiber. There exists a diagram as in* (3.1) *with* X' *a projective variety,* B' *nonsingular,* f *a projective alteration,* p *a projective strict modification,* π' *a toroidal morphism, and all of the geometric fibers of* π' *equidimensional and reduced.*

A toroidal structure on a normal variety X is an open subset $U_X \subseteq X$, such that for each $x \in X$, there is a toric variety X_{σ_x}, a point $s \in X_{\sigma_x}$ and an isomorphism $\widehat{\mathscr{O}}_{X,x} \cong \widehat{\mathscr{O}}_{X_{\sigma_x},s}$ that maps the ideal of $X - U_X$ to the idea of $X_{\sigma_x} - T_{\sigma_x}$ where T_{σ_x} is the torus of X_{σ_x}. In other words, it is a variety together with an open set that étale locally looks like a toric variety together with its embedded torus. A toroidal morphism is defined in the obvious way (see e.g. [**3**, Def. 1.3, p.247]). We direct the reader to [**3**, p.45] for the definition of a strict modification; we note that in the case that $X \to B$ is flat, p will be a projective modification.

It is mentioned in [**3**, Rem. 1.1] that it may also be possible to address simultaneous semi-stable reduction using the language of log-structures, rather than toroidal morphisms. We also direct the reader to [**1**], which addresses the case of schemes over fields that are not algebraically closed. We conclude with the remark that roughly speaking, the theorem says that simultaneous semi-stable reduction is possible if one allows for toric singularities.

10. (Semi-)stable reduction for singularities

We now consider the (semi-)stable reduction problem locally and focus on singularities. The Mumford et al. Semi-stable Reduction Theorem for one-parameter families ensures the existence of a semi-stable reduction for (generically smooth) one-parameter families of singularities. The extensions to higher dimensional bases due to Abramovich–Karu establish a certain form of existence in the simultaneous case. Consequently, the problem we will consider here is describing in more detail semi-stable reductions for specific singularities.

Singularities of type ADE will appear frequently in what follows. Recall that these are the singularities (of dimension $n-1$, $n \geq 2$) defined by the polynomials:

$$\begin{array}{lcll} f_{A_k} & = & x_1^{k+1} + x_2^2 + \ldots + x_n^k & k \geq 1 \\ f_{D_k} & = & x_1(x_1^{k-2} + x_2^2) + x_3^2 + \ldots + x_n^2 & k \geq 4 \\ f_{E_6} & = & x_1^4 + x_2^3 + x_3^2 + \ldots + x_n^2 & \\ f_{E_7} & = & x_2(x_1^3 + x_2^2) + x_3^2 + \ldots + x_n^2 & \\ f_{E_8} & = & x_1^5 + x_2^3 + x_3^2 + \ldots + x_n^2. & \end{array}$$

10.1. Local stable reduction for curve singularities. In this section we discuss some recent work of Hassett [**73**] on local stable reduction for isolated, locally planar singularities. The main results are descriptions of the tails arising in the stable reduction process for curves.

10.1.1. *Preliminaries on local stable reduction.* A local stable reduction of an isolated, plane curve singularity (X_o, x) is defined as follows. We consider

$$\pi : X \to B$$

a one-parameter smoothing of (X_o, x), with $X_o = \pi^{-1}(o)$ for some $o \in B$; one can obtain such a smoothing by observing that the singularity (X_o, x) will arise on some plane curve, and the Hilbert scheme containing that curve is a projective space with generic point parameterizing a smooth curve. We then perform semi-stable reduction following Mumford et al. to obtain $\tilde{X} \to \tilde{B}$, where the central fiber is in nc position. Set $p : \tilde{X} \to \tilde{B} \times_B X$. Finally, take the log canonical model of $(\tilde{X}, \tilde{X}_o)$ relative to the morphism p.

We will denote the resulting family by $X^c \to \tilde{B}$; this is called the **local stable reduction** of the family $X \to B$. Note that X^c agrees with $\tilde{B} \times_B X$ away from the central fiber. By construction, the local stable reduction provides a local picture of the stable reduction for a one-parameter family of curves degenerating to a curve with a singularity (X_o, x).

We now review the definition of the tail of the local stable reduction. The central fiber of $X^c \to \widetilde{B}$, which we will denote X_o^c, can be decomposed as $X_o^c = X_o^\nu \cup X_o^T$, where X_o^ν is the normalization of X_o and $X_o^T := \overline{X_o^c - X_o^\nu}$. To fix notation, set $X_o^\nu \cap X_o^T = \{p_1, \ldots p_b\}$ where b is the number of branches of X_o. The pair $(X_o^\nu, \{p_1, \ldots, p_b\})$ depends only on X_o and not on the choice of smoothing. On the other hand, the pair $(X_o^T, \{p_1, \ldots, p_b\})$ may depend on the smoothing, and we call this the **tail of the local stable reduction** of the family $X \to B$.

10.1.2. *A result of Hassett.* We now mention Hassett's result that the tails arising in this process form subvarieties of the moduli space of curves. We will use the notation $\overline{M}_{g,(n)}$ for the moduli space of stable curves of genus g, with n unordered marked points.

Proposition 10.1 (Hassett [**73**, Prop. 3.2, p.176]). *Let (X_o, x) be a plane curve singularity with b branches. Let $\mathscr{T}_{X_o}$ be the set of tails obtained from the local stable reduction of each smoothing of X_o. The tails are connected, all of the same arithmetic genus γ, and $\mathscr{T}_{X_o}$ is naturally a (reduced) subscheme of $\overline{M}_{\gamma,(b)}$.*

In order to describe the subscheme $\mathscr{T}_{X_o}$ in more detail, Hassett considers the problem of deforming the pairs $(\operatorname{Spec} \mathbb{C}[[x,y]], X_o)$. He considers a process similar to that in the construction of the local stable reduction, performing semi-stable reduction for the pair $(\operatorname{Spec} \mathbb{C}[[x,y]], X_o)$ and then taking a log-canonical model. His results give explicit descriptions of tails that arise in stable reduction for a wide class of singularities, including the classes known as toric and quasi-toric singularities (which include ADE singularities). For the sake of space, we restrict to the special case of A_n singularities.

THEOREM 10.2 (Hassett [**73**, Thm. 6.2,6.3, p.185-6]). *Suppose that (X_o, x) is a plane curve singularity of type A_n. Then the scheme $\mathscr{T}_{X_o}$ is irreducible.*

(1) *If $n = 2k$, then $\mathscr{T}_{X_o}$ is the closure of the locus of hyperelliptic curves of genus k, with a marked Weierstrass point in $\overline{M}_{k,1}$.*
(2) *If $n = 2k+1$, then $\mathscr{T}_{X_o}$ is the closure of the locus of hyperelliptic curves of genus k with two conjugate marked points (i.e. interchanged by the hyperelliptic involution) in $\overline{M}_{k,(2)}$.*

REMARK 10.3. One application of these results is to the Hassett–Keel program. More precisely, the results can be used to provide a description of resolutions of rational maps among various moduli spaces that arise in the program. We direct the reader to [**39**, §4.2] for more discussion (see also §8.5 above).

10.2. Simultaneous resolution for simple surface singularities and the Weyl cover. We now turn our attention to surface singularities, again over $\mathbb{C}$. While in general one would consider questions of semi-stable reduction, for surface singularities of type ADE there is a result due to Brieskorn–Tyurina that one may in fact find simultaneous resolutions of singularities.

First let us recall what is meant by a simultaneous resolution of singularities. Let $\pi : X \to B$ be a flat morphism of schemes. A simultaneous resolution of singularities of π is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

such that p is proper, π' is smooth, and for every $b \in B$, the induced morphism $X'_b \to X_b$ is birational; i.e. it is a coherent way of resolving the singularities of the fibers of π.

Let us now make the following observation ([**90**, Exa. 4.27, p.128]): If B is a curve, and π is smooth over $B - \{o\}$ for some $o \in B$, then π does not admit a simultaneous resolution if X_o is a reduced curve, or $\dim X_o \geq 3$ and X_o has at worst isolated hypersurface singularities (see also Kollár–Shepherd-Barron [**91**] for more on surfaces singularities and Friedman [**57**] for more on threefolds). With this in mind, Brieskorn's theorem on surface singularities becomes quite surprising.

THEOREM 10.4 (Brieskorn–Tyurina). *Let $\pi : (X, x) \to (B, o)$ be a flat morphism of germs of singularities such that fiber (X_o, x) is an ADE surface singularity. Then there is a finite, surjective morphism $(B', o') \to (B, o)$ such that $\pi' : X' := B' \times_B X$ admits a simultaneous resolution of singularities.*

We direct the reader to Kollár–Mori [**90**, p.129] for a discussion of a number of techniques that can be used to prove the theorem, as well as some references. Brieskorn's [**33**] Weyl group cover of the mini-versal deformation space of an ADE singularity plays an important role. We briefly review this now. Let X_o be an ADE singularity of type T (i.e. $T = A_n$, D_n or E_n). Let B_T be a mini-versal deformation space of X_o with discriminant Δ_T. Define W_T to be the Weyl group of type T and R_T be the corresponding root system. Brieskorn shows there exists a Galois cover $f : B'_T \to B_T$ with covering group W_T and ramification locus Δ_T such that $f^*\Delta_T$ is an arrangement of hyperplanes determined by the root system R_T. The hyperplanes are in one-to-one correspondence with the roots in R_T considered up to ± 1. The morphism $f : B'_T \to B_T$ is called the **Weyl (group) cover**.

REMARK 10.5. For surfaces, ADE singularities are exactly the canonical singularities (see e.g. [**90**]). Thus this special case is enough to handle surface singularities in many circumstances. We direct the reader to [**91**] for a complete description of slc surface singularities.

11. Geometric invariant theory

One way of determining a class of objects that will provide a stable reduction theorem for a moduli problem is via GIT. Typically, one will rigidify the moduli problem to obtain a (projective) fine moduli scheme, and then take the quotient by a reductive group to return to a (projective) scheme parameterizing isomorphism classes of interest. The GIT semi-stable points naturally provide a class of objects where a weak form of stable reduction holds. We call this GIT semi-stable completion (or weak stable reduction for GIT) and discuss it in more detail in §11.2.

If there are no strictly semi-stable points, then one typically obtains a stable reduction theorem. Note also that the stability conditions obtained in this way will depend on the rigidification, as well as the choice of linearization. Different choices may lead to different stable reduction theorems. We discuss this further in §11.5.

11.1. Preliminaries on GIT. Let X be a projective variety over an algebraically closed field k. Let G be a linearly reductive algebraic group over k [**105**, Def. 1.4, p.26] acting on X, and let L be an ample G-linearized line bundle on X [**105**, Def. 1.6, p.30].

For $n \in \mathbb{N}$ and a section $s \in H^0(X, L^{\otimes n})$, we set

$$X_s = \{x \in X : s(x) \neq 0\}.$$

Recall from [**105**, Def. 1.7] that the set of semi-stable (resp. stable, resp. properly stable) points of X, denoted X^{ss} (resp. X^s, resp. X^s_0), is the set of points $x \in X$ such that there exists a natural number n and a G-invariant section $s \in H^0(X, L^{\otimes n})^G$ with $s(x) \neq 0$ (resp. $s(x) \neq 0$ and the action of G on X_s closed, resp. $s(x) \neq 0$, the action of G on X_s closed, and the dimension of the stabilizer of x is equal to 0). We denote the orbit of x by $G \cdot x$ and the stabilizer of x by G_x [**105**, p.3].

Mumford's theorem [**105**, Theorem 1.10] defines the GIT quotient of X under the group action. It states that there exists a (surjective) universally submersive [**67**, 15.7.8, p.245], G-invariant morphism of k-schemes

$$\phi : X^{ss} \to X /\!\!/_L G$$

that is a categorical quotient of X^{ss} by the action of G [**105**, Def. 0.5, p.3]. This satisfies the additional property that if x_1 and x_2 are closed points of X^{ss}, then $\phi(x_1) = \phi(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$ ([**105**, p.40]). In particular, the closed points of $X /\!\!/_L G$ are in bijection with closed orbits of closed points in X^{ss}. There is an open subset $(X /\!\!/_L G)^\circ \subseteq X /\!\!/_L G$ with the property that $X^s_0 = \phi^{-1}(X /\!\!/_L G)^\circ$ ([**105**, (1) p.37]), and the induced morphism

$$\phi^\circ : X^s_0 \to (X /\!\!/_L G)^\circ$$

is a geometric quotient; in particular the fibers over closed points are exactly the orbits of the closed points of X_0^s (see [**105**, Def. 0.6, p.4]). It is also shown that

$$X/\!\!/_L G = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G\right)$$

(see [**105**, p.40], [**49**, Prop. 8.1, p.120]) so that $X/\!\!/_L G$ is projective.

11.2. Weak stable reduction for GIT. As $X/\!\!/_L G$ is projective, any map from the generic point of a DVR to $X/\!\!/_L G$ extends to the whole DVR. We now consider the question of lifting such maps from $X/\!\!/_L G$ to X^{ss}.

More precisely, let R be a DVR over k, with fraction field $K = K(R)$, and with residue field $\kappa(R) = k$. Let $S = \operatorname{Spec} R$, let $\eta = \operatorname{Spec} K$ be the generic point, and let $s = \operatorname{Spec} \kappa(R)$ be the special point. We will assume we are given a morphism

$$f : S \to X/\!\!/_L G,$$

and we are interested in lifting f to X^{ss}. The following result is often referred to as semi-stable completion for GIT (Shah [**116**, Prop. 2.1, p.488], Mumford [**103**, Lem. 5.3, p.57]).

THEOREM 11.1 (Weak stable reduction for GIT). *Let $f : S \to X/\!\!/_L G$ be a morphism. There exists a finite extension K' of K, so that taking R' to be the integral closure of R in K' and setting $S' = \operatorname{Spec} R'$, there is a commutative diagram*

(11.1)
$$\begin{tikzcd} & & X^{ss} \arrow[d, "\phi"] \\ S' \arrow[urr, "g"] \arrow[r] & S \arrow[r, "f"'] & X/\!\!/_L G. \end{tikzcd}$$

Moreover, g may be chosen so that $g(s')$ lies in a closed orbit, where s' is the closed point of S'.

The essential point is the universal submersiveness of ϕ. Indeed, if one were only to require that $S' \to S$ be a surjective morphism of spectrums of DVRs (and not necessarily a finite morphism), then the existence of such a lift g would follow immediately from the definition of universal submersiveness (see e.g. [**86**, Rem. 3.7.6, p.50] for a well-known converse). The additional fact that $S' \to S$ can be taken to be finite follows from the proof of Mumford [**105**, Lem., p.14].

REMARK 11.2. Let us briefly consider weak stable reduction for GIT in the context of stacks. The analogous statement is that the natural map from the quotient stack $\pi : [X^{ss}/G] \to X/\!\!/_L G$ is universally closed. While this can be established by modifying the proof of weak stable reduction for GIT, it is also a consequence of the more general fact that $\pi : [X^{ss}/G] \to X/\!\!/_L G$ is a good (categorical) moduli space ([**15**, Thm. 4.16 (ii), §13, Thm. 6.6]). Concretely, given a map $f : S \to X/\!\!/_L G$ and a generic lift $g_\eta : \operatorname{Spec} K \to [X^{ss}/G]$, then after a generically finite base change, there is a lift $g : S' \to [X^{ss}/G]$ extending the pull back of g_η. Moreover, one may choose the lift so that the closed point of S' is sent to a closed point of $[X^{ss}/G]$ (corresponding to a closed orbit), and this point is unique (see [**17**, §2]).

REMARK 11.3. Since $X/\!\!/_L G$ is projective, it follows that $[X^{ss}/G]$ is universally closed and of finite type over k (Remark 11.2). It is not always the case, however, that $[X^{ss}/G]$ is separated. For instance, this will fail if $X^{ss} \neq X^s_0$, as there will be positive dimensional affine stabilizers preventing the diagonal from being proper (see also [**17**, Exa. 2.15]). On the other hand, if $[X^{ss}/G]$ is separated, it follows that it is also proper. Consequently, if $\mathcal{M}$ is a separated stack representing a moduli problem and $\mathcal{M} \cong [X^{ss}/G]$, then there is a stable reduction theorem for the moduli problem. (See also [**17**, Def. 2.1] for the more general notions of weakly separated and weakly proper morphisms.)

11.3. GIT stable reduction for plane curves. In this section we consider the example of plane quartic curves, worked out by Mumford [**105**, Ch.4 §2]. To do this, we start with the associated Hilbert scheme $X = \mathbb{P}H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4))$. There is a natural action of $\mathbb{P}GL(3)$ given by change of coordinates; as is typical, for the sake of simplicity we consider the action of $G = SL(3)$ instead via the isogeny $SL(3) \to \mathbb{P}GL(3)$. The Hilbert scheme, being a projective space, comes equipped with a polarization $L = \mathscr{O}(1)$ and a natural $SL(3)$-linearization. We set $\overline{M}_3^{GIT}$ to be the GIT quotient

$$\overline{M}_3^{GIT} := \mathbb{P}H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4))/\!\!/_{\mathscr{O}(1)} SL(3) = X/\!\!/_L G$$

Using the Hilbert–Mumford index, the following is worked out in [**105**, p.81-2] (see also [**20**, Lem. 1.4]). Let C be a plane quartic corresponding to a point $x \in \mathbb{P}H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4)) = X$.

(1) $x \in X^s_0$ if and only if C is non-singular, or C has only nodes and cusps as singularities.
(2) $x \in X^{ss} - X^s_0$ if and only if C is a double conic or C has a tacnode.
(3) $x \in X^{ss} - X^s_0$ *and has closed orbit* if and only if C is a double conic or C is the union of two conics, at least one of which is non-singular, and the conics meet tangentially.

While there is not a universal family of curves over $\overline{M}_3^{GIT}$, there is a universal family over X, and the weak stable reduction theorem for GIT implies the following. Given any one-parameter family of plane quartics over a punctured disk, with fibers of type (1)-(3) above, after a finite base change, the family can be filled in to a family over the complete disk, with central fiber of type (1) or (3). Moreover, the isomorphism class of such a central fiber is determined by the original family over the punctured disk.

11.4. Deligne–Mumford stable reduction revisited. Gieseker's construction of the moduli space of stable curves as a GIT quotient of a Hilbert scheme provides another proof of the Deligne–Mumford stable reduction theorem [**63**, p.i].

Let $g \geq 2$ and $\nu \geq 10$. Set $\mathrm{Hilb}_{g,\nu}$ to be the irreducible component of the Hilbert scheme containing the locus of ν-canonically embedded, genus g, non-singular curves. Let $H_{g,\nu} \subseteq \mathrm{Hilb}_{g,\nu}$ be the locus of (Deligne–Mumford) stable curves. Set $N = (2\nu - 1)(g - 1) - 1$ to be the dimension of ν-canonical space. The group $SL(N+1)$ acts on $\mathrm{Hilb}_{g,\nu}$ by change of basis. Gieseker has shown ([**63**, Ch.2]) that there exists an $SL(N+1)$-linearized polarization Λ on $\mathrm{Hilb}_{g,\nu}$ such that

$$H_{g,\nu} = (\mathrm{Hilb}_{g,\nu})^s_0 = \mathrm{Hilb}^{ss}_{g,\nu}\,. \tag{11.2}$$

Consequently, one obtains $\mathrm{Hilb}_{g,\nu} /\!\!/_\Lambda SL(N+1) \cong \overline{M}_g$ [**63**, Thm. 2.0.2]. A key point is the fact that a family $X \to B$ of (Deligne–Mumford) stable curves over a scheme B can (after possibly replacing B by an appropriate open subset) be embedded in $\mathbb{P}^N_B$ as a flat family parameterized by a morphism $B \to \mathrm{Hilb}_{g,\nu}$ (e.g. [**63**, p.13]). The Deligne–Mumford stable reduction theorem (over k, and up to the uniqueness statement) follows from (11.2) and the weak stable reduction theorem for GIT.

11.5. Comparing stability conditions. We now compare the GIT stable reduction theorems arising from §11.3 and §11.4, and discuss the connection with the spaces arising in the Hassett–Keel program.

Let us fix a DVR R, and consider a family $X \to S = \operatorname{Spec} R$ of smooth plane quartics degenerating to a quartic with a unique singularity, which is a tacnode (A_3). This is a family of GIT semi-stable curves in the sense of §11.3. However, the central fiber is not a curve with closed orbit, and the family is not a GIT semi-stable family in the sense of §11.4. The GIT stable reduction theorem states that after a generically finite base change $S' \to S$, one can complete the family in two different ways. In the sense of §11.3, one can complete the family so that the central fiber is the union of two plane conics meeting in two points, which are tacnodes (see [**76**, §3.4]). In the sense of §11.4, one can complete the family so that the central fiber consists of a reducible stable curve obtained as the union of an elliptic curve (the normalization of the tacnodal quartic) attached to another elliptic curve (the tail; also called an elliptic bridge) at two points.

In short, on the one hand, we are requiring the central fiber to be a plane quartic. On the other, we are requiring the central fiber to be a nodal curve (with finite automorphisms). Both conditions give a "weak" stable reduction theorem, albeit with very different central fibers. We direct the reader to Hassett–Hyeon [**75, 74**] for more discussion of GIT stability of curves with respect to different rigidifications, and linearizations. See also Smyth [**119**], Alper–Smyth–van der Wyck [**17**] and Alper–Smyth [**16**] for a stack theoretic approach to this type of problem.

In terms of the Hassett–Keel program (see §8.5), the space $\overline{M}_3^{GIT}$ of §11.3 is the space $\overline{M}_3(17/28)$ [**76**] (and $\overline{M}_3 = \overline{M}_3(1)$). The spaces are birational, as both contain dense open subsets corresponding to smooth, non-hyperelliptic curves. In fact, the family of curves over the subset U corresponding to non-singular curves with trivial automorphism group induces a rational map $\overline{M}_3^{GIT} \dashrightarrow \overline{\mathcal{M}}_3$. Resolving this map is closely related to the simultaneous stable reduction for curves with ADE singularities, discussed in §8.5 (see [**38**, §8] and [**39**, Cor. 3.6]).

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Clay Mathematics Proceedings
Volume **18**, 2013

The solvable monodromy extension property and varieties of log general type

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Dedicated to Joe Harris, with admiration on the occasion of his sixtieth birthday.

Abstract. We speculate on the relationship between the solvable monodromy extension (SME) property and log canonical models. A motivating example is the moduli space of smooth curves which, by earlier work, is known to have this SME property. In this case the maximal SME compactification is the moduli space of stable nodal curves which coincides with its log canonical model.

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1. Introduction

On a late afternoon a few years ago, on the way back from one of his water cooler trips, Joe Harris dropped by my alcove with the following problem on his mind:

Given a family of smooth curves over an open subscheme $U \subset S$, when does it extend to a family of stable curves over S?

We had not been discussing ideas directly along these lines, so this question took me a little by surprise. It seemed like a very natural problem and it subsequently became a part of my thesis. Using the language of moduli spaces, it can be restated as follows:

Given an open subscheme $U \subset S$ and a morphism $f : U \to \mathcal{M}_{g,n}$ when does it extend to a regular morphism $S \to \overline{\mathcal{M}}_{g,n}$?

The author is thankful for the support received through NSF grant DMS-1101439 and the Alfred P. Sloan foundation.

Here $\mathcal{M}_{g,n}$ denotes the moduli space of smooth genus g curves with n marked points and $\overline{\mathcal{M}}_{g,n}$ its Deligne-Mumford compactification. De Jong and Oort [**JO**] show that if $D := S \setminus U$ is a normal crossing divisor then $f : U \to \mathcal{M}_{g,n}$ extends to a regular morphism $S \to \overline{\mathcal{M}}_{g,n}$ assuming it does this over the generic points of D. Without this assumption one can still conclude that f extends to a map $S \to \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is the coarse moduli space.

In general, lifting a map $S \to \overline{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$ only requires taking a finite cover of S (*i.e.* one does not need to blow up at all). From this point of view the second answer above suffices. When working over $\mathbb{C}$, it turns out to have the following generalization.

THEOREM 1.1. [**C**, Thm. A] *Let $U \subset S$ be an open subvariety of an irreducible, normal variety S. A morphism $U \to \mathcal{M}_{g,n}$ extends to a regular map $S \to \overline{M}_{g,n}$ in a Zariski neighbourhood of $p \in S \setminus U$ if and only if the local monodromy around p is virtually abelian.*

1.1. The abelian/solvable monodromy extension property. Inspired by this result we introduced in [**C**, Sect. 2] the *abelian monodromy extension* (AME) property for a pair $(\mathcal{X}, \overline{X})$ consisting of a Deligne-Mumford stack $\mathcal{X}$ and a compactification $\overline{X}$ of its coarse scheme. Roughly, $\overline{X}$ is an AME compactification of X if $U \to \mathcal{X}$ extends to a regular map $S \to \overline{X}$ whenever the image of the induced map $\pi_1(U) \to \pi_1(\mathcal{X})$ on fundamental groups is virtually abelian. Here S is a small analytic neighbourhood of a point (so this is a local condition on the domain but a global condition on the target $(\mathcal{X}, \overline{X})$).

Among all AME compactifications of some X there is a unique maximal one which we denote X_{ame} [**C**, Cor. 3.7]. This means that for any other AME compactification $\overline{X}$ of X there exists a birational morphism $X_{\text{ame}} \to \overline{X}$. For example, $\overline{M}_{g,n}$ is the maximal AME compactification of $\mathcal{M}_{g,n}$ [**C**, Thm. 4.1].

Although the AME condition is quite strong, there are abundant examples of pairs satisfying the AME property. More precisely:

PROPOSITION 1.2. [**C**, Prop. 3.11] *Let $X \subset \overline{X}$ be a dense, open immersion such that $\overline{X}$ is a normal, complete variety. Then there exists an open $X^o \subset X$ such that $(X^o, \overline{X})$ has the AME property.*

In this paper we will consider the (very similar) *solvable monodromy extension* (SME) property instead of the AME property. The definition is precisely the same except that we replace "abelian" with "solvable" everywhere. Once again, if X has the SME property then there exists a maximal SME compactification X_{sme}. In this case, since any abelian group is solvable, X also has the AME property and there exists a regular morphism $X_{\text{ame}} \to X_{\text{sme}}$.

1.2. Relation to log canonical models. If C is a smooth curve, then it has the SME property if and only if it is stable (meaning that it has genus $g > 1$ or genus $g = 1$ with at least one puncture or genus $g = 0$ with at least three punctures). Notice how these are precisely the curves of log general type.

We make the following two (completely wild) speculations:

Speculation #1. *If X has the SME property then its log canonical model X_{lc} (assuming it exists) is a compactification of X. In particular, X is of log general type.*

Speculation #2. *If X has the SME property then there exists a regular morphism $X_{\mathrm{lc}} \to X_{\mathrm{sme}}$ extending the identity map on X.*

In the case of surfaces we prove in Propositions 4.3 and 4.5 that this is indeed true. It should also be possible to verify these speculations if X has a smooth log minimal model.

For general higher dimensional varieties we describe a possible (but very sketchy) approach to proving these two claims. Along the way we bring up some related questions which may be of independent interest.

The final section gives a simple example showing that varieties of log general type need not have the SME property. We also describe a potential application, explained to me by Sean Keel, to partial resolution of singularities.

1.3. Acknowledgments. I would like to thank Maksym Fedorchuk, Brendan Hassett, Johan de Jong and Sean Keel for helpful, interesting discussions and the anonymous referee for finding a mistake and suggesting additional improvements and references.

2. Preliminaries

We briefly discuss the two main concepts being related in this paper: the SME property and log canonical pairs.

2.1. The solvable monodromy extension (SME) property. The solvable monodromy extension property is a direct analogue of the abelian monodromy extension property from [**C**] (just take that definition and replace abelian with solvable).

2.1.1. *Local monodromy.* We summarize the definition of local monodromy from [**C**, Sec. 2.1]. Let X be an open subvariety of a normal variety $\overline{X}$. Next, consider an open subvariety $U \subset S$ of a normal variety S together with a morphism $U \to X$. Fix a connected, reduced, proper subscheme $T \subset S$. Now choose a sufficiently small analytic neighbourhood V of T. We define the *local monodromy around* T as the image of fundamental groups

$$\operatorname{Im}\left(\pi_1(V \cap U) \to \pi_1(X)\right).$$

Since T is connected $V \cap U$ is connected and so the image of $\pi_1(V \cap U)$ is defined (up to conjugation) without having to choose a base point. Most commonly we will take T to be a point $p \in S \setminus U$ to obtain the local monodromy around p.

2.1.2. *The SME property.* Given an open embedding of normal varieties $X \subset \overline{X}$, the pair $(X, \overline{X})$ has *the solvable monodromy extension (SME) property* if given any $U \subset S$ as above the morphism $U \to X$ extends to a regular map $S \to \overline{X}$ in a neighbourhood of p whenever the local monodromy around p is virtually solvable (recall that a group is *virtually solvable* if it contains a solvable subgroup of finite index).

In this case $\overline{X}$ is complete and we say that $\overline{X}$ is an *SME compactification* of X. We say X has the SME property if it has an SME compactification. One can also define the SME property for stacks (see [**C**, Sec. 2.2.1]) but for simplicity we will only consider varieties.

2.1.3. *Example: the moduli space of curves.* The main example from [**C**] of a variety with the AME property is the moduli space of curves (Theorem 1.1). It turns out the moduli space of curves also has the SME property.

COROLLARY 2.1. *$\overline{M}_{g,n}$ is the maximal SME compactification of $\mathcal{M}_{g,n}$.*

PROOF. By [**BLM**, Theorem B] every solvable subgroup of $\pi_1(\mathcal{M}_{g,n})$ is virtually abelian. This means that $\mathcal{M}_{g,n}^{\text{ame}} = \mathcal{M}_{g,n}^{\text{sme}}$ and the result follows from [**C**, Thm. 4.1] which states that $\overline{M}_{g,n}$ is the maximal AME compactification of $\mathcal{M}_{g,n}$. □

2.2. Log canonical pairs. Consider a normal variety X with a compactification $\overline{X}$ so that the boundary $\Delta := \overline{X} \setminus X$ is a normal crossing divisor. Note that in general one may have to first resolve X in order to find such an $\overline{X}$.

We say that X is of *log general type* if $(K_{\overline{X}} + \Delta)$ is big. Then, assuming finite generation of the log canonical ring (say via the log minimal model program), the *log canonical model* of X is

$$X_{\text{lc}} := \operatorname{Proj}\left(\bigoplus_{m\geq 0} H^0(\mathcal{O}_{\overline{X}}(m(K_{\overline{X}} + \Delta))\right).$$

If X_{lc} contains X as an open subscheme then we say that X_{lc} is the *log canonical compactification of X*. Note that X_{lc} does not depend on the choice of $\overline{X}$. By construction, assuming X has (at worst) log canonical singularities, it follows that X_{lc} also has (at worst) log canonical singularities.

In general, if C, C' are two curves inside some proper variety Y, we will say that C and C' are numerically equivalent (denoted $C \sim C'$) if $C \cdot D = C' \cdot D$ for any Cartier divisor $D \subset Y$.

2.3. Some properties of AME compactifications. In [**C**, Sect. 3] we proved various basic results about AME compactifications. The key facts we used was that a subgroup of an abelian group is abelian and that the image of an abelian group is abelian. These facts also hold for solvable groups. Subsequently, the results from [**C**, Sect. 3] still hold if we replace "AME" with "SME". We now state three such results which we will subsequently use.

LEMMA 2.2. [**C**, Cor. 3.3] *Suppose $(X, \overline{X})$ has the AME (resp. SME) property and let $Y \subset X$ be a closed, normal subscheme. If we denote by $\overline{Y}$ the normalization of the closure of Y in $\overline{X}$ then the pair $(Y, \overline{Y})$ also has the AME (resp. SME) property.*

LEMMA 2.3. [**C**, Prop. 3.10] *If X has the AME (resp. SME) property and $i : Y \to X$ is a locally closed embedding then there exists a regular morphism $Y_{\text{sme}} \to X_{\text{sme}}$ (resp. $Y_{\text{ame}} \to X_{\text{ame}}$) which extends $i : Y \to X$.*

LEMMA 2.4. *Suppose X has the SME property and let $\overline{X}$ be a compactification of X equipped with a regular map $\pi : \overline{X} \to X_{\text{sme}}$. If $C \subset \overline{X}$ is a curve so that the local monodromy around C is virtually solvable then π contracts C to a point.*

REMARK 2.5. In particular, this means that if X has the SME property then its fundamental group is not virtually solvable. The analogous result for AME varieties also holds.

Proof. Choose a surface $S \subset \overline{X}$, not contained in the boundary $\overline{X} \setminus X$, but which contains C. Then the local monodromy around $C \subset S$ is virtually solvable and, by Lemma 2.2, $S \cap X$ has the SME property. If we blow up S then the proper transform $\tilde{C}$ of C still has this property. Moreover, blowing up sufficiently we can assume that $\tilde{C}^2 < 0$.

Thus we end up with $\tilde{C} \subset S'$ where $\tilde{C}^2 < 0$ and a map $S' \to \overline{X}$ which does not contract $\tilde{C}$. Moreover, the local monodromy around $\tilde{C} \subset S'$ is virtually solvable. But, since $\tilde{C}^2 < 0$, we can blow down $\tilde{C}$ to a point $q \in S''$. Then the local monodromy around q is virtually solvable. This means that, in a neighbourhood of q, we get a regular map $S'' \to X_{\text{sme}}$. Subsequently, the composition $S' \to \overline{X} \xrightarrow{\pi} X_{\text{sme}}$ contracts $\tilde{C}$ to a point. This means that π contracts C to a point. □

3. A general property

The following results, which we will apply later, serve as some indication that varieties satisfying the SME property are of log general type.

Proposition 3.1. *Let X be a normal variety and $\overline{X}$ some compactification such that $\Delta := \overline{X} \setminus X$ is a divisor. If $C \subset \overline{X}$ is a rational curve such that*

- *$C \cap X \neq \emptyset$ and $C \subset \overline{X}_{\text{smooth}}$,*
- *$(K_{\overline{X}} + \Delta) \cdot C \leq 0$ and $\Delta \cdot C \geq 3$*

then there exists a curve B_0 and a map $F_0 : \mathcal{C}_0 := \mathbb{P}^1 \times B_0 \to \overline{X}$ such that

(i) *$F_0(\mathcal{C}_0)$ is a surface,*
(ii) *$F_0^{-1}(\Delta) = \{p_1, \ldots, p_n\} \times B_0$ for some fixed points $p_1, \ldots, p_n \in \mathbb{P}^1$,*
(iii) *there exists a section σ of $\mathcal{C}_0 \to B_0$ with $F_0(\sigma) = p$ for some $p \in X$.*

Proof. We will use the following non-trivial deformation theory result [**M**]. The deformation space of morphisms $f : \tilde{C} \to \overline{X}$ has dimension at least

$$-(K_{\overline{X}} \cdot C) + (1 - g_{\tilde{C}}) \cdot \dim(X)$$

where $\tilde{C}$ is the normalization of C and $g_{\tilde{C}}$ its genus. In our case $g_{\tilde{C}} = 0$ and $-K_{\overline{X}} \cdot C \geq \Delta \cdot C$ so the space of deformations has dimension at least $\dim(X) + \Delta \cdot C$.

Now consider such a family of deformations $\pi : \mathcal{C} \to B$ where $\dim(B) \geq \dim(X) + \Delta \cdot C$. Since the automorphism group of $\tilde{C}$ is 3-dimensional we can restrict $\mathcal{C}$ to a smaller family $\mathcal{C}' \to B' \subset B$ where $\dim(B') = \dim(B) - 3$ and such that for any fibre in $\mathcal{C}'$ there are only finitely many other fibres with the same image in $\overline{X}$. Notice that here we need $\Delta \cdot C \geq 3$ or else this family may be empty.

Suppose the general fibre of $\mathcal{C}'$ intersects Δ in $n \geq 1$ distinct points. If $n \geq 3$ then, after possibly blowing up B' and taking a finite cover, we obtain a map $B' \to \overline{M}_{0,n}$ to moduli space of genus zero curves with n marked points. Since $\dim(\overline{M}_{0,n}) = n - 3$ a generic fibre of this map has dimension at least

$$\dim(B') - n + 3 \geq \dim(X) + \Delta \cdot C - n \geq \dim(X).$$

The restriction of $\mathcal{C}'$ to such a fibre leaves us with a family of curves $\pi'' : \mathcal{C}'' \to B''$ and a map $F : \mathcal{C}'' \to \overline{X}$. If $n \leq 2$ then we take this to be the original family.

Since $\dim(\mathcal{C}'') = \dim(B'') + 1 \geq \dim(\overline{X}) + 1$ the general fibre of F has dimension at least one. Choose a general point $p \in F(\mathcal{C}'') \cap X$ and let $B''' := \pi''(F^{-1}(p)) \subset B''$. Restricting $\mathcal{C}''$ gives us a family $\pi''' : \mathcal{C}''' \to B'''$ with $\dim(B''') \geq 1$ with a map $F : \mathcal{C}''' \to \overline{X}$. After restricting even further we can assume for convenience that $\dim(B''') = 1$.

Now, by construction, there exists an open subset $B_0 \subset B'''$ such that the restriction of $\mathcal{C}'''$ to it is isomorphic to $\mathcal{C}_0 := \mathbb{P}^1 \times B_0$. We also have a map $F_0 : \mathcal{C}_0 \to \overline{X}$ which takes $\cup_i p_i \times B_0$ to Δ and the rest to X (here $p_1, \ldots, p_n$ denote our n marked points). Finally, after possibly pulling back to a finite cover of an open subset of B_0, there exists a section σ of $\mathcal{C}_0 \to B_0$ which is in the preimage of $p \in X$. $\square$

Corollary 3.2. *Suppose X is smooth, has the AME property and denote by $\overline{X}$ a simple, normal crossing compactification of X. If $C \subset \overline{X}$ is a rational curve not contained in the boundary $\Delta := \overline{X} \setminus X$, then $(K_{\overline{X}} + \Delta) \cdot C > 0$.*

Proof. By Lemma 2.4 we know $C \cdot \Delta \geq 3$. So suppose $(K_{\overline{X}} + \Delta) \cdot C \leq 0$ and consider the family $\mathcal{C}_0 = (\mathbb{P}^1, p_1, \ldots, p_n) \times B_0 \to B_0$ as in Proposition 3.1. This family comes equipped with a map $F_0 : \mathcal{C}_0 \setminus \{\cup_i p_i \times B_0\} \to X$ and a section σ such that $F_0(\sigma) = p \in X$. Compactify B_0 to some smooth curve B and $\mathcal{C}_0$ to the trivial product $\mathcal{C} = (\mathbb{P}^1, p_1, \ldots, p_n) \times B$.

Since Δ is simple, normal crossing, we have a regular map $f : \overline{X} \to X_{\text{ame}}$ which extends the identity map on X. By the AME property, $f \circ F_0 : \mathcal{C}_0 \to X_{\text{ame}}$ extends to a regular map $F : \mathcal{C} \to X_{\text{ame}}$. Then $F(\overline{\sigma}) = p$ which means that $\overline{\sigma}$ does not intersect any $p_i \times \overline{B}_0$ (here $\overline{\sigma}$ denotes the closure of σ). This means that the image of $\overline{\sigma}$ under the projection from $\mathcal{C}$ to $(\mathbb{P}^1, p_1, \ldots, p_n)$ is a single point q. Thus $\overline{\sigma}$ is just $\{q\} \times B$ which means $\overline{\sigma}^2 = 0$. But this is impossible because $F : \mathcal{C} \to X_{\text{ame}}$ contracts $\overline{\sigma}$ to a point. $\square$

It would be interesting (and useful) to generalize Proposition 3.1 to curves C of higher genus. In order to do that one needs a log version of the bend and break lemma (since then the analogue of Corollary 3.2 would almost say that $K_{\overline{X}} + \Delta$ is nef).

Question 1. *Is there a log version of the bend and break lemma?*

4. The case of surfaces

In this section suppose X is a normal surface and denote by $\overline{X}$ a simple normal crossing compactification of X. We denote by $\Delta \subset \overline{X}$ the boundary $\Delta := \overline{X} \setminus X$. In this case we have the following MMP for surfaces.

Theorem 4.1. [**F**, Thm. 3.3, 8.1] *There exists a sequence of contractions*

$$(\overline{X}, \Delta) = (\overline{X}_0, \Delta_0) \xrightarrow{\phi_0} (\overline{X}_1, \Delta_1) \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_{k-1}} (\overline{X}_k, \Delta_k) = (X^*, \Delta^*)$$

such that each $(\overline{X}_i, \Delta_i)$ is log-canonical and one of the following two things hold:

(i) *$K_{X^*} + \Delta^*$ is semi-ample or*
(ii) *there exists a morphism $g : X^* \to B$ such that $-(K_{X^*} + \Delta^*)$ is g-ample and $\dim(B) < 2$.*

Remark 4.2. The MMP states that in case (i) the divisor $K_{X^*} + \Delta^*$ is nef. Then by the abundance theorem for log-canonical surfaces, it is also semi-ample.

Proposition 4.3. *If X is a normal surface which has an AME compactification then X is of log general type.*

Proof. Since we are trying to prove X is of log general type we can remove the singular locus of X and assume it is smooth. Now, compactify X and consider a log minimal resolution $(\overline{X}, \Delta)$. First we show that applying Theorem 4.1 we end up in case (i).

Suppose to the contrary that we are in case (ii). Denote the composition of contractions $\phi : (\overline{X}, \Delta) \to (X^*, \Delta^*)$ and denote by E_ϕ the exceptional locus. There are two cases depending on whether $\dim(B) = 0$ or $\dim(B) = 1$.

If $\dim(B) = 0$ then $-(K_{X^*} + \Delta^*)$ is ample. By [**KeM**, Cor. 1.6], $X^* \setminus \Delta^*$ is $\mathbb{C}^\times$-connected which means that X is also $\mathbb{C}^\times$-connected. But then, by [**KeM**, Cor. 7.9], we find that $\pi_1(X)$ is virtually abelian. This is impossible since X has the AME property. See also [**Zh**] for a similar approach.

If $\dim(B) = 1$ then consider a connected component C of a general fibre of g. Since X^* is normal C is smooth. The image $\phi(E_\phi) \subset X^*$ has codimension 2 so $C \cap \phi(E_\phi) = \emptyset$. Hence C has a unique lift to $\overline{X}$ which we also denote C. Then

$$(K_{\overline{X}} + \Delta) \cdot C = (K_{X^*} + \Delta^*) \cdot C < 0 \tag{4.1}$$

because $-(K_{X^*} + \Delta^*)$ is g-ample. Since C is a fibre of g this means $C^2 = 0$ and hence

$$\deg(K_C) + \Delta \cdot C = (K_{\overline{X}} + C) \cdot C + \Delta \cdot C < 0.$$

Since C is not contained in Δ we have $\Delta \cdot C \geq 0$. Thus C has genus zero and $\Delta \cdot C \leq 1$. So C intersects Δ in at most one point, meaning that $\mathbb{A}^1 \subset C \setminus \Delta$. This is a contradiction since by Lemma 2.2, $(C, \Delta \cap C) \subset (\overline{X}, \Delta)$ is an AME pair.

So we must be in case (i). Consider the map $\pi : X^* \to B$ induced by $K_{X^*} + \Delta^*$. We must show that $\dim(B) = 2$. If $\dim(B) = 1$ then consider a general fibre C of π as above. Then

$$(K_{\overline{X}} + \Delta) \cdot C = (K_{X^*} + \Delta^*) \cdot C = 0$$

which is similar to equation (4.1). Then arguing as before, either C has genus one and $\Delta \cdot C = 0$ or it has genus zero and $\Delta \cdot C \leq 2$. In the first case C does not intersect Δ and we get a contradiction by Lemma 2.2. Similarly, in the second case C intersects Δ in at most two points, meaning that $\mathbb{C}^\times \subset C \setminus \Delta$ which is again a contradiction by Lemma 2.2.

If $\dim(B) = 0$ then $K_{X^*} + \Delta^*$ is trivial. We have

$$K_{\overline{X}} + \Delta = \phi^*(K_{X^*} + \Delta^*) + \sum_i a_i E_i$$

where $-1 \leq a_i \leq 0$ and $\{E_i\}$ are the irreducible, exceptional divisors. The left inequality follows since (X^*, Δ^*) is log canonical, while $a_i \leq 0$ is a consequence of the fact $(\overline{X}, \Delta)$ is a minimal log resolution (see [**Ko1**, Claim 66.3]).

Thus, either all a_i are zero and $\Delta = 0$ (which means $K_{\overline{X}}$ is trivial) or some $a_i < 0$ (which means $\kappa(\overline{X}) = -\infty$). The former cannot happen because the fundamental group of $X = \overline{X}$ would be virtually abelian. In the second case, we can take any rational curve $C \subset \overline{X}$ not contained in Δ. Then $(K_{\overline{X}} + \Delta) \cdot C = \sum_i a_i (E_i \cdot C) < 0$ which is a contradiction by Corollary 3.2.

Thus $\dim(B) = 2$ and X must be of log general type. □

Corollary 4.4. *Suppose X is a normal surface with at worst log canonical singularities which has the AME property. Then X_{lc} is a compactification of X.*

PROOF. Compactify X to $\overline{X}$ as in Theorem 4.1. By Proposition 4.3 $L^* := K_{X^*} + \Delta^*$ is semi-ample. It suffices to show that each ϕ_i as well as the map induced by L^* only contracts curves in the boundary.

Suppose ϕ_i contracts a curve C which does not lie in the boundary. Then $(K_{\overline{X}_i} + \Delta_i) \cdot C \leq 0$ and C is either rational or an elliptic curve which does not intersect Δ_i or $\mathrm{sing}(\overline{X}_i)$. The second case is not possible since the local monodromy around C would be abelian (contradicting the fact that X has the AME property). The first case is not possible by Corollary 3.2.

The fact that the map induced by L^* only contracts curves along the boundary follows similarly. □

PROPOSITION 4.5. *Suppose X is a normal surface with at worst log canonical singularities which has a maximal SME compactification X_{sme}. Then there is a regular morphism $X_{\mathrm{lc}} \to X_{\mathrm{sme}}$ extending the identity map on X.*

PROOF. Denote by $(\overline{X}, \Delta)$ a minimal simple normal crossing compactification of X and denote by $(X_{\mathrm{lc}}, \Delta_{\mathrm{lc}})$ its log canonical model. Then we have regular morphisms

$$X_{\mathrm{lc}} \xleftarrow{\pi_1} \overline{X} \xrightarrow{\pi_2} X_{\mathrm{sme}}.$$

Consider a connected, exceptional curve $E \subset \overline{X}$ of π_1. If $\pi_1(E)$ is a point which does not intersect the boundary of X_{lc} then the local monodromy around E, is the same as the local monodromy around $\pi_1(E)$ which, by Proposition 4.6 is virtually solvable. Thus, by Lemma 2.4, it follows that π_2 must contract E to a point and hence π_2 factors through π_1.

If $\pi_1(E)$ intersect the boundary of X_{lc} then the type of singularity at $\pi_1(E)$ is very restricted (see [**KoM**, Thm. 4.15] for a list of possible singularities). Locally around $\pi_1(E)$, the complement of the boundary looks like the quotient of $\mathbb{C}^2 \setminus \{x = 0\}$ or $\mathbb{C}^2 \setminus \{x = 0, y = 0\}$ by a finite group. Thus the monodromy is virtually abelian and, again as above, π_2 must contract E to a point and hence π_2 factors through π_1. □

PROPOSITION 4.6. *Let X be a normal surface and $x \in X$ a point. Then the following are equivalent:*

(i) *X has an at worst log canonical singularity at x,*
(ii) *the local fundamental group of X around x is solvable or finite.*

PROOF. This follows by combining the results in [**K**] and [**W**]. □

5. The case of higher dimensional varieties

5.1. Generalizing Proposition 4.3. Suppose X is a normal variety of arbitrary dimension which has an AME compactification. We would like to show that X is of log general type by imitating the proof of Proposition 4.3. For this purpose we can assume X is smooth and we compactify it to some $\overline{X}$ so that $\Delta := \overline{X} \setminus X$ is a simple normal crossing divisor.

The analogue of Theorem 4.1 in this case is the (partially conjectural) log MMP. It states that there exists a sequence of birational maps

$$(\overline{X}, \Delta) = (\overline{X}_0, \Delta_0) \overset{\phi_0}{\dashrightarrow} (\overline{X}_1, \Delta_1) \overset{\phi_1}{\dashrightarrow} \dots \overset{\phi_{k-1}}{\dashrightarrow} (\overline{X}_k, \Delta_k) = (X^*, \Delta^*)$$

such that each $(\overline{X}_i, \Delta_i)$ is a log-canonical pair and each ϕ_i is either a divisorial contraction or a flip. As before, we denote the composition ϕ and note that ϕ^{-1}

has no exceptional divisors. The resulting pair (X^*, Δ^*) satisfies one of the following two properties:

(i) $K_{X^*} + \Delta^*$ is nef or
(ii) there exists a morphism $g : X^* \to B$ such that $-(K_{X^*} + \Delta^*)$ is g-ample and $\dim(B) < \dim(X)$.

First we would like to argue that we must be in case (i). Assume instead that we are in case (ii). If $\dim(B) > 0$ consider a general fibre F and let $\Delta_F := \Delta^*|_F$. Then $K_F \cong K_{X^*}|_F$ and hence $K_F + \Delta_F = (K_{X^*} + \Delta^*)|_F$ is anti-ample. But the exceptional locus of ϕ^{-1} is codimension so there exists an open subset $F^o \subset F$ which is not of log general type but which sits inside X and hence has an AME compactification. By induction on the dimension of X this is impossible.

So we are left with considering the situation where $\dim(B) = 0$. In this case $-(K_{X^*} + \Delta^*)$ is ample. When X was a surface we used [**KeM**]. Two analogues of such a result in higher dimensions were posed, for instance, during an open problem session at the AIM workshop "Rational curves on algebraic varieties" (May 2007):

- Is the fundamental group of the smooth locus of a log Fano variety finite?
- Is a log Fano variety (X^*, Δ^*) log rationally connected? In other words, is there a rational curve passing through any two given points which intersects Δ^* only once?

For us, an affirmative answer to a strictly easier question would suffice:

Question 2. *If (X^*, Δ^*) is log Fano is there a rational curve which avoids any given locus L of codimension at least 2?*

Taking L to be the union of the exceptional locus of ϕ^{-1} and the singular locus we lift such a curve to $C \subset X$. Then, by Corollary 3.2, we have

$$(K_{X^*} + \Delta^*) \cdot C = (K_{\overline{X}} + \Delta) \cdot C > 0$$

which contradicts the fact that $-(K_{X^*} + \Delta^*)$ is ample.

So, assuming the answer to Question 2 is "Yes", we find that $K_{X^*} + \Delta^*$ is nef. The abundance conjecture, which was known in the case of surfaces (and also threefolds [**KMM**]), would then imply that $K_{X^*} + \Delta^*$ is semi-ample. So we can consider the induced map $\pi : X^* \to B$.

If $\dim(X) > \dim(B) > 0$ then choose a general fibre F. Then, proceeding as above, $K_F \cong K_{X^*}|_F$ and hence $K_F + \Delta_F$ is trivial (where $\Delta_F := \Delta^*|_F$). Thus, by induction on $\dim(X)$, this is a contradiction.

If $\dim(B) = 0$ then $K_{X^*} + \Delta^*$ is trivial. If X^* were smooth then we would get a contradiction as follows. If $\Delta^* = 0$ then X^* is Calabi-Yau and hence its fundamental group is virtually abelian. On the other hand, if $\Delta^* \neq 0$ then choose a point in X^* which is not in the image of the exceptional locus of ϕ. By the bend and break lemma we can find a rational curve C through this point. Then, since $(K_{X^*} + \Delta^*) \cdot C < 0$, one can find a map F_0 and in Proposition 3.1. Since C does not belong to the image of the exceptional locus of ϕ, this map can be lifted to a simple, normal crossing compactification of X where we get a contradiction as in the proof of Corollary 3.2.

To push this argument through when X^* is singular we could use an affirmative answer to the following question:

Question 3. *Suppose (X^*, Δ^*) is log Calabi-Yau, meaning that $K_{X^*}+\Delta^*$ is trivial:*

(i) *If* $\Delta^* = 0$*, is the fundamental group of the smooth locus of* X^* *virtually abelian?*
(ii) *If* $\Delta^* \neq 0$*, can you find a rational curve in* X^* *which avoids the singular locus and is not contained in the image of the exceptional locus of* ϕ*?*

Having ruled out the cases $\dim(X) > \dim(B)$ it follows that $\dim(X) = \dim(B)$ and X is of log general type.

5.2. Generalizing Proposition 4.5. Let $(X_{\mathrm{lc}}, \Delta_{\mathrm{lc}})$ be the log canonical model of X and resolve it to a simple normal crossing compactification $(\overline{X}, \Delta)$. Subsequently we have the following regular morphisms

$$X_{\mathrm{lc}} \xleftarrow{\pi_1} \overline{X} \xrightarrow{\pi_2} X_{\mathrm{sme}}.$$

We would like to factor π_1 as a sequence of extremal divisorial contractions

$$(\overline{X}, \Delta) \xrightarrow{\phi_0} (\overline{X}_1, \Delta_1) \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{k-1}} (\overline{X}_k, \Delta_k) = (X_{\mathrm{lc}}, \Delta_{\mathrm{lc}})$$

and to show by induction that:

(i) π_2 descends to a regular map $\overline{X}_i \xrightarrow{p_i} X_{\mathrm{sme}}$ and
(ii) any curve $C \subset \overline{X}_i$ contracted by ϕ_i is also contracted by p_i.

Notice that (ii) implies that p_i descends to a regular map $p_{i+1} : \overline{X}_{i+1} \to X_{\mathrm{sme}}$ (so it suffices to prove (ii)).

Unfortunately, it is not at all clear how to do this. One obvious approach is to hope that for a log canonical pair (X, Δ), the local fundamental group around a point $x \in X$ is virtually solvable. In the case of surfaces the answer to this question was "yes" but, as the referee points out, [**Ko2**, Thm. 2] gives a counter-example already in the case of 3-folds. In fact, [**Ko2**, Question 24] essentially asks if there are *any* natural restrictions on such local fundamental groups.

6. Some final remarks

6.1. Two examples. Choose four points on $\mathbb{P}^2$ in general position and denote by X the complement of the six lines through them. Then X has the SME property with X_{sme} isomorphic to the blowup of $\mathbb{P}^2$ at these four points. In fact, $X \cong M_{0,5}$ parametrizing five points on $\mathbb{P}^1$ and this blowup is just $X_{\mathrm{sme}} \cong \overline{M}_{0,5}$.

On the other hand, choose $n \geq 4$ lines in general position on $\mathbb{P}^2$ and let X be the complement of their union. Then by an old result of Zariski [**Za**] the fundamental group of X is abelian. Subsequently, X does not have the SME (or AME) property even though it is of log general type (with log canonical model $X_{\mathrm{lc}} = \mathbb{P}^2$).

6.2. Resolution of singularities. The following observation goes back to a conversation with Sean Keel a few years ago.

Denote by $X(r,n)$ the moduli space of n hyperplanes in $\mathbb{P}^{r-1}$. $X(r,n)$ has a particular compactification $\overline{X}(r,n)$ known as Kapranov's Chow quotient. Keel and Tevelev [**KT**] point out that the singularities of the boundary of $\overline{X}(r,n)$ are arbitrarily complicated.

On the other hand, one can show that $\overline{X}(r,n)$ is an SME compactification. Hence, if Speculations #1 and #2 are correct, we obtain a regular map $X(r,n)_{\mathrm{lc}} \to \overline{X}(r,n)$ which gives a canonical way to (partially) resolve arbitrarily bad singularities.

Question 4. *Is there a regular map* $X(r,n)_{\mathrm{lc}} \to \overline{X}(r,n)$*?*

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Clay Mathematics Proceedings
Volume **18**, 2013

Effective divisors on moduli spaces of curves and abelian varieties

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Dedicated to Joe Harris—master geometer, inspired teacher and valued friend—on the occasion of his 60th birthday

Introduction

The pseudo-effective cone $\mathrm{Eff}(X)$ of a smooth projective variety X is a fundamental, yet elusive invariant. On one hand, a few general facts are known: the interior of the effective cone is the cone of big divisors so, in particular, X is of general type if and only if $K_X \in \mathrm{int}(\mathrm{Eff}(X))$; less obviously [**4**], a variety X is uniruled if and only if K_X is not pseudo-effective and the dual of $\mathrm{Eff}(X)$ is the cone of movable curves; and, the effective cone is known to be polyhedral for Fano varieties. For further background, see [**60**]. On the other hand, no general structure theorem is known and the calculation of $\mathrm{Eff}(X)$ is a daunting task even in some of the simplest cases. For instance, the problem of computing the cone $\mathrm{Eff}(C^{(2)})$ for a very general curve C of genus g is known to be equivalent to Nagata's Conjecture, see [**16**].

The aim of this paper is to survey what is known about the effective cones of moduli spaces, with a focus on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves, $\overline{\mathcal{A}}_g$ of principally polarized abelian varieties and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps. Because related moduli spaces often have an inductive combinatorial structure and the associated families provide a rich cycle theory, the study of effective cones of moduli spaces has often proven more tractable and more applicable than that of general algebraic varieties.

For example, in the case of $\overline{\mathcal{M}}_g$, we may define, following [**48**], the slope $s(D)$ of a divisor class D of the form $a\lambda - b\delta - c_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_i c_i\delta_i$, with a and b positive, all the c's non-negative and at least one—in practice, almost always c_{irr}—equal to 0, to be $\frac{a}{b}$. We set $s(D) = \infty$ for divisors not of this form, for example, if $g \geq 3$, for the components Δ_{irr} and Δ_i. A fundamental invariant is then the slope $s(\overline{\mathcal{M}}_g) := \inf\{s(D) \,|\, D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)\}$. The Harris-Mumford theorem [**50**] on the Kodaira dimension of $\overline{\mathcal{M}}_g$, is equivalent to the inequality $s(\overline{\mathcal{M}}_g) < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}$, for $g \geq 24$. For a long time, the conjecture of [**48**] that the inequality $s(\overline{\mathcal{M}}_g) \geq$

2010 *Mathematics Subject Classification.* Primary 14D22, 14H51; Secondary 14E30, 14H10.
Key words and phrases. Algebraic geometry, moduli, stable curve, effective divisor.
The first author is partially supported by the NSF grant DMS-1200329.

$6+\frac{12}{g+1}$ holds, equality being attained only for the classical Brill-Noether divisors whose classes were also computed in [**50**], was widely believed. Counterexamples were provided in [**32**] for infinitely many g though all of these have slope greater than 6. On the other hand, all the methods (cf. [**48, 12, 76**]) for bounding $s(\overline{\mathcal{M}}_g)$ from below for large g, yield only bounds that tend to zero with g. This sets the stage for the following fundamental question:

Problem 0.1. Does the limit $s_\infty := \lim_{g\to\infty} s(\overline{\mathcal{M}}_g)$ exist, and, if so, what is it's value?

The authors know of no credible, generally accepted conjectural answer. The first tends to guess that $s_\infty = 0$, the second and third that $s_\infty = 6$. Hedging his guess, the third author has a dinner *bet* with the second, made at the 2009 MSRI Program in Algebraic Geometry: the former wins if $s_\infty = 0$, the latter if $s_\infty > 0$, and the bet is annulled should the limit not exist.

The argument for $s_\infty = 0$ is that the papers cited above. which compute the invariants of movable curves in $\overline{\mathcal{M}}_g$ using tools as diverse tools as Hurwitz theory, Teichmüller dynamics and Hodge integrals, do no better than $s(\overline{\mathcal{M}}_g) \geq O(\frac{1}{g})$. Intriguingly, the first two methods, though apparently quite different in character, suggest the *same* heuristic lower bound $\frac{576}{5g}$ for the slope; see Section 3 of this paper. Is this coincidence or evidence for the refined asymptotic $\liminf_{g\to\infty} g\, s(\overline{\mathcal{M}}_g) = \frac{576}{5}$ conjectured by the first author in [**12**], and hence that $s_\infty = 0$?

The argument for $s_\infty > 0$ is that effective divisors of small slope are known to have strong geometric characterizations: for instance, they must contain the locus $\mathcal{K}_g$ of curves lying on $K3$ surfaces. Constructing *any* such divisors, let alone ones of arbitrarily small slope, is notoriously difficult. In fact, for $g \geq 11$, not a single example of an effective divisor having slope less than $6+\frac{10}{g}$ is known. The current state of knowledge concerning divisors of small slope is summarized in Section 2 of the paper.

We invite the reader to take sides in this bet, or much better, settle it conclusively by computing s_∞. To encourage work that might enable him to win, the third author here announces the First Morrison Prize, in the amount of US$100, for the construction of any effective divisor on $\overline{\mathcal{M}}_g$ of slope less than 6, as determined by a jury consisting of the present authors. One further question is to what extent s_∞ has a modular meaning. As pointed out in [**48**, p. 323], the inequality $s_\infty > 0$ would imply a fundamental difference between the geometry of $\mathcal{M}_g$ and $\mathcal{A}_g$ and provide a new geometric approach to the Schottky problem.

We now describe the contents of the paper. Section 1 recalls the classical constructions of effective divisors on $\overline{\mathcal{M}}_g$, starting with Brill-Noether and Gieseker-Petri divisors. Then we discuss the cases $g \leq 9$, where a much better understanding of the effective cone is available and alternative Mukai models of $\overline{\mathcal{M}}_g$ are known to exist. In Section 2, we highlight the role of syzygy divisors in producing examples of divisors on $\overline{\mathcal{M}}_g$ of small slope and discuss the link to an interesting conjecture of Mercat [**64**] that suggests a stratification of $\mathcal{M}_g$ in terms of rank 2 vector bundles on curves. Special attention is paid to the interesting transition case $g = 11$, which is treated from the point of view both of Koszul cohomology and higher rank Brill-Noether theory.

Section 3 is devoted to finding lower bounds on $s(\overline{\mathcal{M}}_g)$ and the existing methods are surveyed. The common idea is to find a Zariski dense collection of 1-cycles

B_μ, so that any effective divisor must intersect one of these curves non-negatively, obtaining the bound $s(\overline{\mathcal{M}}_g) \geq \inf_\mu\left(\frac{B_\mu\cdot\delta}{B_\mu\cdot\lambda}\right)$. There are several methods of constructing these curves, e.g. by using simply-branched coverings of $\mathbb{P}^1$ and allowing a pair of branch points to come together [**48**], by imposing conditions on curves in projective spaces, especially canonical space [**19, 41**], as Teichmüller curves arising from branched covers of elliptic curves [**12**], or as complete intersection of nef tautological divisors on $\overline{\mathcal{M}}_g$, with intersection numbers evaluated via Gromov-Witten theory [**76**].

In Section 4, we turn to moduli of abelian varieties and discuss the recent paper [**34**] showing that the Andreotti-Mayer divisor N_0' of 5-dimensional ppav whose theta divisor is singular at a pair of points which are not two-torsion computes the slope of the perfect cone compactification $\overline{\mathcal{A}}_5$ of $\mathcal{A}_5$ as $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$.

Section 5 is devoted almost exclusively to moduli spaces of curves of genus $g = 0$. We begin with a few cases—the space $\widetilde{\mathcal{M}}_{0,n}$ that is the quotient of $\overline{\mathcal{M}}_{0,n}$ by the natural action of $\mathfrak{S}_n$ induced by permuting the marked points and the Kontsevich moduli spaces of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$—in which unpublished arguments of Keel make it easy to determine the effective cone completely. We then discuss more systematically the space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$, sketching the sharper results of Coskun, Harris and Starr [**19**] on their effective cones. We also review some of the results of the first author with Coskun and Crissman concerning the Mori program for these spaces, emphasizing the examples $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ where [**10**] completely works out the geometry of this program, giving an explicit chamber decomposition of the effective cone in terms of stable base loci, and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$ for which much, though not all, the geometry is worked out in [**13**].

The rest of Section 5 deals with results for $\overline{\mathcal{M}}_{0,n}$. For $n \leq 5$, the naive guess that the effective cone might be generated by the components of the boundary is correct, and we recall the argument for this. But for larger n new extremal rays appear. We first review the example of Keel and Vermeire [**88**] and the proof of Hassett and Tschinkel [**52**] that, for $n = 6$, there are no others. The main focus of this section is to give a brief guide to the ideas of Castravet and Tevelev [**8**] which show just how rapidly the complexity of these effective cones grows.

We conclude this introduction by citing some work on effective divisors that we have not reviewed. These include Rulla's extensions in [**79, 80**] of the ideas in §§5.1 to quotients by subgroups permuting only a subset of the marked points and Jensen's examples [**53**] for $\overline{\mathcal{M}}_{5,1}$ and $\overline{\mathcal{M}}_{6,1}$. In a very recent preprint, Cooper [**17**] studies the moduli spaces of stable quotients $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$ of Marian, Oprea and Pandharipande [**62**]. Because there is a surjection

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^{n-1}, d) \to \overline{Q}_{g,0}(\mathbb{P}^{n-1}, d),$$

this is relevant to §§5.2. In the case of $g = 1$ and $n = 0$ that Cooper considers, the target $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$ is smooth with a rank 2 Picard group and she is able to describe the effective (and nef) cones explicitly. In addition, we have not touched upon connections with the F-conjecture, including Pixton's exciting example [**78**] of an effective divisor on $\overline{\mathcal{M}}_{0,12}$ that intersects all topological 1-strata non-negatively yet is not equivalent to an effective sum of boundary divisors. Finally, as this paper is written the first author and Coskun are able to show that the effective cone of $\overline{\mathcal{M}}_{1,n}$ is not finitely generated for $n \geq 3$.

Conventions and notation To simplify notation, we will ignore torsion classes and henceforth use $\mathrm{Pic}(M)$ with no decoration for $\mathrm{Pic}(M)\otimes\mathbb{Q}$. We set $\mathrm{Eff}(M)$ and $\mathrm{Nef}(M)$ for the effective and nef cones of M. We denote by $\mathrm{Mov}(M)$ the cone of movable divisors on M parametrizing effective divisors whose stable base locus has codimension at least 2 in M. We write δ_{irr} for the class of the boundary component of irreducible nodal curves, and, when there is no risk of confusion, we simplify notation by omitting the limits of summations indexed by boundary components consisting of reducible curves. We work throughout over $\mathbb{C}$.

1. Geometric divisors on $\overline{\mathcal{M}}_g$

Any expression $D = a\lambda - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_i b_i\delta_i$ for an effective divisor D on $\overline{\mathcal{M}}_g$ (with all coefficients positive) provides an upper bound for $s(\overline{\mathcal{M}}_g)$. Chronologically, the first such calculations are those of the Brill-Noether divisors, which we briefly recall following [**50, 24**].

Definition 1.1. For positive integers $g, r, d \geq 1$ such that

$$\rho(g,r,d) := g-(r+1)(g-d+r) = -1,$$

we denote by $\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : W^r_d(C) \neq \emptyset\}$ the Brill-Noether locus of curves carrying a linear series of type $\mathfrak{g}^r_d$.

It is known [**25**] that $\mathcal{M}^r_{g,d}$ is an irreducible effective divisor. The class of its closure in $\overline{\mathcal{M}}_g$ has been computed in [**24**] and one has the formula

$$[\overline{\mathcal{M}}^r_{g,d}] = c_{g,r,d}\Big((g+3)\lambda - \frac{g+1}{6}\delta_{\mathrm{irr}} - \sum_i i(g-i)\delta_i\Big),$$

where $c_{g,r,d} \in \mathbb{Q}_{>0}$ is an explicit constant that can be viewed as an intersection number of Schubert cycles in a Grassmannian. Note that $s(\overline{\mathcal{M}}^r_{g,d}) = 6+\frac{12}{g+1}$, thus implying the upper bound $s(\overline{\mathcal{M}}_g) \leq 6+\frac{12}{g+1}$, for all g such that $g+1$ is composite, so that the diophantine equation $\rho(g,r,d) = -1$ has integer solutions. The initial Slope Conjecture [**48**] predicted that the Brill-Noether divisors are divisors of minimal slope. This turns out to be true only when $g \leq 9$ and $g = 11$.

Observe that remarkably, for various $r, d \geq 1$ such that $\rho(g,r,d) = -1$, the classes of the divisors $\overline{\mathcal{M}}^r_{g,d}$ are proportional. The proof given in [**24**] uses essential properties of Picard groups of moduli spaces of pointed curves and it remains a challenge to find an *explicit* rational equivalence linking the various Brill-Noether divisors on $\overline{\mathcal{M}}_g$. The first interesting case is $g = 11$, when there are two Brill-Noether divisors, namely $\overline{\mathcal{M}}^1_{11,6}$ and $\overline{\mathcal{M}}^2_{11,9}$. Note that when $g = 2$, the divisor Δ_1 has the smallest slope 10 in view of the relation $10\lambda = \delta_{\mathrm{irr}} + 2\delta_1$ on $\overline{\mathcal{M}}_2$, see for instance [**49**, Exercise (3.143)].

When $g = 3, 5, 7, 8, 9, 11$, there exist Brill-Noether divisors which actually determine the slope $s(\overline{\mathcal{M}}_g)$. This has been shown in a series of papers [**48, 9, 85, 36**] in the last two decades. Some cases have been recovered recently in [**19, 41**].

For $3 \leq g \leq 9$ and $g = 11$, it is well known [**66**] that a general curve of genus g can be realized as a hyperplane section H of a $K3$ surface S of degree $2g-2$ in $\mathbb{P}^g$. Consider a general Lefschetz pencil B in the linear system $|H|$. Blowing up the $2g-2$ base points of B, we get a fibration S' over B, with general fiber a smooth genus g curve. All singular fibers are irreducible one-nodal curves. From the relation

$$\chi_{\rm top}(S') = \chi_{\rm top}(B)\cdot\chi_{\rm top}(F) + \text{the number of nodal fibers},$$

where F is a smooth genus g curve, we conclude that

$$B\cdot\delta_{\rm irr} = 6g+18, \quad B\cdot\delta_i = 0 \text{ for } i>0.$$

Let ω be the first Chern class of the relative dualizing sheaf of S' over B. By the relation

$$12\lambda = \delta + \omega^2$$

and

$$\omega^2 = c_1^2(S') + 4(2g-2) = 6g-6,$$

we obtain that

$$B\cdot\lambda = g+1.$$

Consequently the slope of the curve B is given by

$$s_B = 6 + \frac{12}{g+1}.$$

Since the pencil B fills-up $\overline{\mathcal{M}}_g$ for $g\leq 9$ or $g=11$, we get the lower bound $s(\overline{\mathcal{M}}_g)\geq 6+\frac{12}{g+1}$ in this range. The striking coincidence between the slope of the Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ and that of Lefschetz pencil on a fixed $K3$ surface of genus g has a transparent explanation in view of Lazarsfeld's result [**59**], asserting that every nodal curve C lying on a $K3$ surface S such that $\mathrm{Pic}(S)=\mathbb{Z}[C]$, satisfies the Brill-Noether theorem, that is, $W^r_d(C)=\emptyset$ when $\rho(g,r,d)<0$. In particular, when $\rho(g,r,d)=-1$, the intersection of the pencil $B\subset\overline{\mathcal{M}}_g$ with the Brill-Noether divisor $\overline{\mathcal{M}}^r_{g,d}$ is *empty*, therefore also, $B\cdot\overline{\mathcal{M}}^r_{g,d}=0$. This confirms the formula

$$s(\overline{\mathcal{M}}^r_{g,d}) = s_B = 6+\frac{12}{g+1}.$$

This Lefschetz pencil calculation also shows [**36**] that any effective divisor $D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ such that $s(D)<6+\frac{12}{g+1}$ must necessarily contain the locus

$$\mathcal{K}_g := \{[C]\in\mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

In particular, effective divisors of slope smaller than $6+\frac{12}{g+1}$ have a strong geometric characterization, hence constructing them is relatively difficult. If one views a divisor on $\overline{\mathcal{M}}_g$ as being given in terms of a geometric condition that holds in codimension one in moduli, then in order for such a condition to lead to a divisor of small slope on $\overline{\mathcal{M}}_g$, one must search for geometric properties that single out sections of $K3$ surfaces among all curves of given genus. Very few such geometric properties are known at the moment, for curves on $K3$ surfaces are known to behave generically with respect to most geometric stratifications of $\mathcal{M}_g$, for instance those given by gonality or existence of special Weierstrass points.

For integers g such that $g+1$ is prime, various substitutes for the Brill-Noether divisors have been proposed, starting with the *Gieseker-Petri* divisors. Recall that the Petri Theorem asserts that for a line bundle L on a general curve C of genus g, the multiplication map

$$\mu_0(L): H^0(C,L)\otimes H^0(C,K_C\otimes L^\vee)\to H^0(C,K_C)$$

is injective. This implies that the scheme $G^r_d(C)$ classifying linear series of type $\mathfrak{g}^r_d$ is smooth of expected dimension $\rho(g,r,d)$ when C is general. The first proof of this statement was given by Gieseker whose argument was later greatly simplified in [**23**]. Eventually, Lazarsfeld [**59**] gave the most elegant proof, and his approach has the added benefit of singling out curves on very general K3 surfaces as the only collections of smooth curves of arbitrary genus verifying the Petri condition. The locus where the Gieseker-Petri theorem does not hold is the proper subvariety of the moduli space

$$\mathcal{GP}_g := \{[C] \in \mathcal{M}_g : \mu_0(L) \text{ is not injective for a certain } L \in \mathrm{Pic}(C)\}.$$

This breaks into subloci $\mathcal{GP}^r_{g,d}$ whose general point corresponds to a curve C such that $\mu_0(L)$ is not injective for some linear series $L \in W^r_d(C)$. The relative position of the subvarieties $\mathcal{GP}^r_{g,d}$ is not yet well-understood. The following elegant prediction was communicated to the second author by Sernesi:

Conjecture 1.2. *The locus $\mathcal{GP}_g$ is pure of codimension one in $\mathcal{M}_g$.*

Clearly there are loci $\mathcal{GP}^r_{g,d}$ of codimension higher than one. However, in light of Conjecture 1.2 they should be contained in other Petri loci in $\mathcal{M}_g$ that fill-up a codimension one component in moduli. Various partial results in this sense are known. Lelli-Chiesa [**61**] has verified Conjecture 1.2 for all $g \leq 13$. It is proved in [**33**] that whenever $\rho(g,r,d) \geq 0$, the locus $\mathcal{GP}^r_{g,d}$ carries at least a divisorial component. Bruno and Sernesi [**5**] show that $\mathcal{GP}^r_{g,d}$ is pure of codimension one for relatively small values of $\rho(g,d,r)$, precisely

$$0 < \rho(g,r,d) < g - d + 2r + 2.$$

The problem of computing the class of the closure $\overline{\mathcal{GP}}^r_{g,d}$ has been completely solved only when the Brill-Noether numbers is equal to 0 or 1. We quote from [**24**] (for the case $r = 1$) and [**32**] (for the general case $r \geq 1$).

Theorem 1.3. *Fix integers $r, s \geq 1$ and set $d := rs + r$ and $g := rs + s$, therefore $\rho(g,r,d) = 0$. The slope of the corresponding Gieseker-Petri divisor is given by the formula:*

$$s(\overline{\mathcal{GP}}^r_{g,d}) = 6 + \frac{12}{g+1} + \frac{6(s+r+1)(rs+s-2)(rs+s-1)}{s(s+1)(r+1)(r+2)(rs+s+4)(rs+s+1)}.$$

For small genus, one recovers the class of the divisor $\mathcal{GP}^1_{4,3}$ of curves of genus 4 whose canonical model lies on a quadric cone and then $s(\overline{\mathcal{GP}}^1_{4,3}) = \frac{17}{2}$. When $g = 6$, the locus $\mathcal{GP}^1_{6,4}$ consists of curves whose canonical model lies on a singular del Pezzo quintic surface and then $s(\overline{\mathcal{GP}}^1_{6,4}) = \frac{47}{6}$. In both cases, the Gieseker-Petri divisors attain the slope of the respective moduli space.

We briefly recall a few other divisor class calculations. For genus $g = 2k$, Harris has computed in [**47**] the class of the divisor $\mathfrak{D}_1$ whose general point corresponds to a curve $[C] \in \mathcal{M}_g$ having a pencil $A \in W^1_{k+1}(C)$ and a point $p \in C$ with $H^0(C, A(-3p)) \neq 0$. This led to the first proof that $\overline{\mathcal{M}}_g$ is of general type for even $g \geq 40$. This was superseded in [**24**], where with the help of Gieseker-Petri and Brill-Noether divisors, it is proved that $\overline{\mathcal{M}}_g$ is of general type for all $g \geq 24$.

Keeping $g = 2k$, if $\sigma : \overline{\mathcal{H}}_{g,k+1} \to \overline{\mathcal{M}}_g$ denotes the generically finite forgetful map from the space of admissible covers of genus g and degree $k+1$, then $\mathfrak{D}_1$ is the push-forward under σ of a boundary divisor on $\overline{\mathcal{H}}_{g,k+1}$, for the general point of the Hurwitz scheme corresponds to a covering with *simple* ramification. The other divisor appearing as a push-forward under σ of a boundary locus in $\overline{\mathcal{H}}_{g,k+1}$ is the divisor $\mathfrak{D}_2$ with general point corresponding to a curve $[C] \in \mathcal{M}_g$ with a pencil $A \in W^1_{k+1}(C)$ and two points $p, q \in C$ such that $H^0(C, A(-2p-2q)) \neq 0$. The class of this divisor has been recently computed by van der Geer and Kouvidakis [**86**].

An interesting aspect of the geometry of the Brill-Noether divisors is that for small genus, they are rigid, that is, $[\overline{\mathcal{M}}^r_{g,d}] \notin \mathrm{Mov}(\overline{\mathcal{M}}_g)$, see for instance [**33**]. This is usually proved by exhibiting a curve $B \subset \overline{\mathcal{M}}^r_{g,d}$ sweeping out $\overline{\mathcal{M}}^r_{g,d}$ such that $B \cdot \overline{\mathcal{M}}^r_{g,d} < 0$. Independently of this observation, one may consider the slope

$$s'(\overline{\mathcal{M}}_g) := \inf\{s(D) : D \in \mathrm{Mov}(\overline{\mathcal{M}}_g)\}$$

of the cone of movable divisors. For $g \leq 9$, the inequality $s'(\overline{\mathcal{M}}_g) > s(\overline{\mathcal{M}}_g)$ holds.

1.1. Birational models of $\overline{\mathcal{M}}_g$ for small genus. We discuss models of $\overline{\mathcal{M}}_g$ in some low genus cases, when this space is unirational (even rational for $g \leq 6$) and one has a better understanding of the chamber decomposition of the effective cone.

Example 1.4. We set $g = 3$ and let $B \subset \overline{\mathcal{M}}_3$ denote the family induced by a pencil of curves of type $(2, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. All members in this family are hyperelliptic curves. A standard calculation gives that $B \cdot \lambda = 3$ and $B \cdot \delta_{\mathrm{irr}} = 28$, in particular $B \cdot \overline{\mathcal{M}}^1_{3,2} = -1$. This implies not only that the hyperelliptic divisor $\overline{\mathcal{M}}^1_{3,2}$ is rigid, but also the inequality $s'(\overline{\mathcal{M}}_3) \geq s_B = \frac{28}{3}$. This bound is attained via the birational map

$$\varphi_3 : \overline{\mathcal{M}}_3 \dashrightarrow X_3 := |\mathcal{O}_{\mathbb{P}^2}(4)| /\!\!/ SL(3)$$

to the GIT quotient of plane quartics. Since φ_3 contracts the hyperelliptic divisor $\overline{\mathcal{M}}^1_{3,2}$ to the point corresponding to double conics, from the push-pull formula one finds that $s(\varphi_3^*(\mathcal{O}_{X_3}(1)) = \frac{28}{3}$. This proves the equality $s'(\overline{\mathcal{M}}_3) = \frac{28}{3} > 9 = s(\overline{\mathcal{M}}_3)$.

That $s'(\overline{\mathcal{M}}_g)$ is accounted for by a rational map from $\overline{\mathcal{M}}_g$ to an *alternative moduli space of curves* of genus g, also holds for a few higher genera, even though the geometry quickly becomes intricate.

Example 1.5. For the case $g = 4$, we refer to [**40**]. Precisely, we introduce the moduli space X_4 of $(3, 3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$, that is, the GIT quotient

$$X_4 := |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)| /\!\!/ SL(2) \times SL(2).$$

There is a birational map $\varphi : \overline{\mathcal{M}}_4 \dashrightarrow X_4$, mapping an abstract genus 4 curve C to $\mathbb{P}^1 \times \mathbb{P}^1$ via the two linear series $\mathfrak{g}^1_3$ on C. The Gieseker-Petri divisor is contracted to *the point* corresponding to triple conics. This shows that $[\overline{\mathcal{GP}}^1_{4,3}] \in \mathrm{Eff}(\overline{\mathcal{M}}_4)$ is an extremal point. By a local analysis, Fedorchuk computes in [**40**] that $s(\varphi_4^*(\mathcal{O}(1, 1))) = \frac{60}{9} = s'(\overline{\mathcal{M}}_4) > s(\overline{\mathcal{M}}_4)$. Furthermore, the model X_4 is one of the log-canonical models of $\overline{\mathcal{M}}_4$.

Mukai [**66, 69, 68**] has shown that general canonical curves of genus $g = 7, 8, 9$ are linear sections of a rational homogeneous variety

$$V_g \subset \mathbb{P}^{\dim(V_g)+g-2}.$$

This construction induces a new model X_g of $\overline{\mathcal{M}}_g$ having Picard number equal to 1, together with a birational map $\varphi_g : \overline{\mathcal{M}}_g \dashrightarrow X_g$. Remarkably, $s(\varphi_g^*(\mathcal{O}_{X_g}(1)) = s'(\overline{\mathcal{M}}_g)$. The simplest case is $g = 8$, which we briefly explain.

Example 1.6. Let $V := \mathbb{C}^6$ and consider $\mathbb{G} := G(2, V) \subset \mathbb{P}(\bigwedge^2 V)$. Codimension 7 linear sections of $\mathbb{G}$ are canonical curves of genus 8, and there is a birational map

$$\varphi_8 : \overline{\mathcal{M}}_8 \dashrightarrow X_8 := G(8, \bigwedge^2 V)/\!\!/SL(V)\,,$$

that is shown in [**69**] to admit a beautiful interpretation in terms of rank two Brill-Noether theory. The map φ_8^{-1} associates to a general projective 7-plane $H \subset \mathbb{P}(\bigwedge^2 V)$ the curve $[\mathbb{G} \cap H] \in \mathcal{M}_8$. In particular, a smooth curve C of genus 8 appears as a linear section of $\mathbb{G}$ if and only if $W^2_7(C) = \emptyset$. Observing that $\rho(X_8) = 1$, one expects that exactly five divisors get contracted under φ_8, and indeed—see [**37, 39**]—

$$\mathrm{Exc}(\varphi_8) = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \overline{\mathcal{M}}^2_{8,7}\}\,.$$

Using the explicit construction of φ_8 one can show that the Brill-Noether divisor gets contracted to a point. Thus X_8 can be regarded as a (possibly simpler) model of $\overline{\mathcal{M}}_8$ in which plane septimics are excluded.

1.2. Upper bounds on the slope of the moving cone. If $f : X \dashrightarrow Y$ is a rational map between normal projective varieties, then $f^*(\mathrm{Ample}(Y)) \subset \mathrm{Mov}(X)$. In order to get upper bounds on $s'(\overline{\mathcal{M}}_g)$ for arbitrary genus, a logical approach is to consider rational maps from $\overline{\mathcal{M}}_g$ to other projective varieties and compute pull-backs of ample divisors from the target variety. Unfortunately there are only few known examples of such maps, but recently two examples have been worked out. We begin with [**33**], where a map between two moduli spaces of curves is considered.

We fix an odd genus $g := 2a + 1 \geq 3$ and set $g' := \frac{a}{a+1}\binom{2a+2}{a} + 1$. Since $\rho(2a + 1, 1, a + 2) = 1$, we can define a rational map $\phi_a : \overline{\mathcal{M}}_g \dashrightarrow \overline{\mathcal{M}}_{g'}$ that associates to a curve C its *Brill-Noether curve* $\phi([C]) := [W^1_{a+2}(C)]$ consisting of pencils of minimal degree—that the genus of $W^1_{a+2}(C)$ is is g' follow from the Harris-Tu formula for Chern numbers of kernel bundles, as explained in [**24**]. Note that $\phi_1 : \overline{\mathcal{M}}_3 \dashrightarrow \overline{\mathcal{M}}_3$ is the identity map, whereas the map $\phi_2 : \overline{\mathcal{M}}_5 \dashrightarrow \overline{\mathcal{M}}_{11}$ has a rich and multifaceted geometry. For a general $[C] \in \mathcal{M}_5$, the Brill-Noether curve $W^1_4(C)$ is endowed with a fixed point free involution $\iota : L \mapsto K_C \otimes L^\vee$. The quotient curve $\Gamma := W^1_4(C)/\iota$ is a smooth plane quintic which can be identified with the space of singular quadrics containing the canonical image $C \hookrightarrow \mathbb{P}^4$. Furthermore, Clemens showed that the Prym variety induced by ι is precisely the Jacobian of C! This result has been recently generalized by Ortega [**73**] to all odd genera. Instead of having an involution, the curve $W^1_{a+2}(C)$ is endowed with a fixed point free correspondence

$$\Sigma := \Big\{(L, L') : H^0(L') \otimes H^0(K_C \otimes L^\vee) \to H^0(K_C \otimes L' \otimes L^\vee) \text{ is not injective}\Big\},$$

which induces a *Prym-Tyurin variety* $P \subset \mathrm{Jac}(W^1_{a+2}(C)$ of exponent equal to the Catalan number $\frac{(2a)!}{a!(a+1)!}$ and P is isomorphic to the Jacobian of the original curve C.

The main result of [**33**] is a complete description of the pull-back map ϕ_a^* at the level of divisors implying the slope evaluation:

THEOREM 1.7. *For any divisor class $D \in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$ having slope $s(D) = s$,*

$$s(\phi_a^*(D)) = 6 + \frac{8a^3(s-4) + 5sa^2 - 30a^2 + 20a - 8as - 2s + 24}{a(a+2)(sa^2 - 4a^2 - a - s + 6)} .$$

By letting s become very large, one obtains the estimate $s'(\overline{\mathcal{M}}_g) < 6 + \frac{16}{g-1}$.

A different approach is used by van der Geer and Kouvidakis [**87**] in even genus $g = 2k$. We consider once more the Hurwitz scheme $\sigma : \overline{\mathcal{H}}_{g,k+1} \to \overline{\mathcal{M}}_g$. Associate to a degree $k+1$ covering $f : C \to \mathbb{P}^1$ the *trace curve*

$$T_{C,f} := \{(x, y) \in C \times C : f(x) = f(y)\}.$$

For a generic choice of C and f, the curve $T_{C,f}$ is smooth of genus $g' := 5k^2 - 4k + 1$. By working in families one obtains a rational map $\chi : \overline{\mathcal{H}}_{g,k+1} \dashrightarrow \overline{\mathcal{M}}_{g'}$. Observe that as opposed to of the map ϕ_a from [**33**], the ratio $\frac{g'}{g}$ for the genera of the trace curve and that of the original curve is much lower. The map $\sigma_*\chi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$ is completely described in [**86**] and the estimate

$$s'(\overline{\mathcal{M}}_g) < 6 + \frac{18}{g+2}$$

is shown to hold for all even genera g. In conclusion, $\overline{\mathcal{M}}_g$ carries moving divisors of slope $6 + O\big(\frac{1}{g}\big)$ for any genus. We close by posing the following question:

PROBLEM 1.8. Is it true that $\liminf_{g\to\infty} s(\overline{\mathcal{M}}_g) = \liminf_{g\to\infty} s'(\overline{\mathcal{M}}_g)$?

2. Syzygies of curves and upper bounds on $s(\overline{\mathcal{M}}_g)$

The best known upper bounds on $s(\overline{\mathcal{M}}_g)$ are given by the Koszul divisors of [**30, 32**] defined in terms of curves having unexpected syzygies. An extensive survey of this material, including an alternative proof using syzygies of the Harris-Mumford theorem [**50**] on the Kodaira dimension of $\overline{\mathcal{M}}_g$ for odd genus $g > 23$, has appeared in [**31**]. Here we shall be brief and concentrate on the latest developments.

As pointed out in [**36**] as well as earlier in this survey, any effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ of slope $s(D) < 6 + \frac{12}{g+1}$ must necessarily contain the locus $\mathcal{K}_g$ of curves lying on $K3$ surfaces. It has been known at least since the work of Mukai [**67**] and Voisin [**89**] that a curve C lying on a $K3$ surface S carries special linear series that are not projectively normal. For instance, if $A \in W^1_{\lfloor \frac{g+3}{2} \rfloor}(C)$ is a pencil of minimal degree, then the multiplication map for the residual linear system

$$\mathrm{Sym}^2 H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C^{\otimes 2} \otimes A^{\otimes(-2)})$$

is not surjective. One can interpret projective normality as being the Green-Lazarsfeld property (N_0) and accordingly, stratify $\mathcal{M}_g$ with strata consisting of curves C that fail the higher properties (N_p) for $p \geq 1$, for a certain linear system $L \in W^r_d(C)$ with $h^1(C, L) \geq 2$. This stratification of $\mathcal{M}_g$ is fundamentally different from classical stratifications given in terms of gonality or Weierstrass points (for instance the Arbarello stratification). In this case, the locus $\mathcal{K}_g$ lies in the smallest stratum, that is, it plays the role of the hyperelliptic locus $\mathcal{M}^1_{g,2}$ in the gonality stratification! Observe however, that this idea, when applied to the canonical bundle K_C (when of course $h^1(C, K_C) = 1$), produces exactly the gonality stratification, see [**31, 45**] for details. Whenever the second largest stratum in the new Koszul stratification is of codimension 1, it will certainly contain $\mathcal{K}_g$ and is thus a good candidate for being a divisor of small slope. The main difficulty in carrying out this program lies not so much in computing the virtual classes of the Koszul loci, but in proving that they are divisors when one expects them to be so.

We begin by recalling basic definitions and refer to the beautiful book of Aprodu-Nagel [**1**] for a geometrically oriented introduction to syzygies on curves.

Definition 2.1. For a smooth curve C, a line bundle L and a sheaf $\mathcal{F}$ on C, we define the Koszul cohomology group $K_{p,q}(C; \mathcal{F}, L)$ as the cohomology of the complex

$$\bigwedge^{p+1} H^0(C,L) \otimes H^0(C, \mathcal{F} \otimes L^{\otimes (q-1)}) \stackrel{d_{p+1,q-1}}{\longrightarrow} \bigwedge^{p} H^0(C,L) \otimes H^0(C, \mathcal{F} \otimes L^{\otimes q}) \stackrel{d_{p,q}}{\longrightarrow}$$

$$\stackrel{d_{p,q}}{\longrightarrow} \bigwedge^{p-1} H^0(C,L) \otimes H^0(C, \mathcal{F} \otimes L^{\otimes (q+1)}).$$

It is a basic fact of homological algebra that the groups $K_{p,q}(C; \mathcal{F}, L)$ describe the graded pieces of the minimal resolution of the graded ring

$$R(\mathcal{F}, L) := \bigoplus_{q \geq 0} H^0(C, \mathcal{F} \otimes L^{\otimes q})$$

as an $S := \mathrm{Sym}\ H^0(C, L)$-module. Precisely, if $F_\bullet \to R(\mathcal{F}, L)$ denotes the minimal graded free resolution with graded pieces $F_p = \oplus_q S(-q)^{\oplus} b_{pq}$, then

$$\dim\ K_{p,q}(C; \mathcal{F}, L) = b_{pq}, \text{ for all } p, q \geq 0.$$

When $\mathcal{F} = \mathcal{O}_C$, one writes $K_{p,q}(C, L) := K_{p,q}(C; \mathcal{O}_C, L)$.

Example 2.2. Green's Conjecture [**45**] concerning the syzygies of a canonically embedded curve $C \hookrightarrow \mathbb{P}^{g-1}$ can be formulated as an equivalence

$$K_{p,2}(C, K_C) = 0 \Leftrightarrow p < \mathrm{Cliff}(C).$$

Despite a lot of progress, the conjecture is still wide open for arbitrary curves. Voisin has proved the conjecture for general curves of arbitrary genus in [**90, 91**]. In odd genus $g = 2p + 3$, the conjecture asserts that the resolution of a general curve $[C] \in \mathcal{M}_{2p+3}$ is pure and has precisely the form:

$$0 \to S(-g-1) \to S(-g+1)^{\oplus b_1} \to \cdots \to S(-p-3)^{\oplus b_{p+1}} \to S(-p-1)^{\oplus b_p} \to \cdots$$

$$\to S(-2)^{\oplus b_3} \to S(-2)^{\oplus b_1} \to R(K_C) \to 0.$$

The purity of the generic resolution in odd genus is reflected in the fact that the syzygy jumping locus

$$\{[C] \in \mathcal{M}_{2p+3} : K_{p,2}(C, K_C) \neq 0\}$$

is a virtual divisor, that is, a degeneracy locus between vector bundles of the same rank over $\mathcal{M}_{2p+3}$. It is the content of Green's Conjecture that set-theoretically, this virtual divisor is an honest divisor which moreover coincides with the Brill-Noether divisor $\mathcal{M}^1_{g,p+2}$.

One defines a *Koszul locus* on the moduli space as the subvariety consisting of curves $[C] \in \mathcal{M}_g$ such that $K_{p,2}(C, L) \neq 0$, for a certain special linear system $L \in W^r_d(C)$. The case when $\rho(g, r, d) = 0$ is treated in the papers [**30**] and [**32**]. For the sake of comparison with the case of positive Brill-Noether number, we quote a single result, in the simplest case $p = 0$, when the syzygy condition $K_{0,2}(C, L) \neq 0$ is equivalent to requiring that the embedded curve $C \stackrel{|L|}{\to} \mathbb{P}^r$ lie on a quadric hypersurface.

THEOREM 2.3. *Fix $s \geq 2$ and set $g = s(2s+1)$, $r = 2s$ and $d = 2s(s+1)$. The locus in moduli*

$$\mathcal{Z}_s := \big\{[C] \in \mathcal{M}_g : K_{0,2}(C, L) \neq 0 \text{ for a certain } L \in W^r_d(C)\big\}$$

is an effective divisor on $\mathcal{M}_g$. The slope of its closure in $\overline{\mathcal{M}}_g$ is equal to

$$s(\overline{\mathcal{Z}}_s) = \frac{a}{b_0} = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}.$$

This implies that $s(\overline{\mathcal{Z}}_s) < 6 + \frac{12}{g+1}$. In particular $s(\overline{\mathcal{M}}_g) < 6 + \frac{12}{g+1}$, for all genera of the form $g = s(2s + 1)$. In the case $s = 2$, one has the set-theoretic equality of divisors $\mathcal{Z}_2 = \mathcal{K}_{10}$ and $s(\overline{\mathcal{Z}}_2) = s(\overline{\mathcal{M}}_{10}) = 7$. This was the first instance of a geometrically defined divisor on $\overline{\mathcal{M}}_g$ having smaller slope than that of the Brill-Noether divisors, see [**36**].

The proof of Theorem 2.3 breaks into two parts, very different in flavor. First one computes the virtual class of $\overline{\mathcal{Z}}_s$, which would then equal the actual class $[\overline{\mathcal{Z}}_s]$, if one knew that $\mathcal{Z}_s$ was a divisor on $\mathcal{M}_g$. This first step has been carried out independently and with different techniques by the second author in [**32**] and by Khosla in [**57**]. The second step in the proof involves showing that $\mathcal{Z}_s$ is a divisor. It suffices to exhibit a single curve $[C] \in \mathcal{M}_g$ such that $K_{0,2}(C, L) = 0$, for every linear series $L \in W^r_d(C)$. By a standard monodromy argument, in the case $\rho(g, r, d) = 0$, this is equivalent to the seemingly weaker requirement that there exist both a curve $[C] \in \mathcal{M}_g$ and a *single* linear series $L \in W^r_d(C)$ such that $K_{0,2}(C, K_C) = 0$. This is proved by degeneration in [**32**].

The case of Koszul divisors defined in terms of linear systems with positive Brill-Noether number is considerably more involved, but the rewards are also higher. For instance, this approach is used in [**31**] to prove that $\overline{\mathcal{M}}_{22}$ is of general type.

We fix integers $s \geq 2$ and $a \geq 0$, then set

$$g = 2s^2 + s + a, \quad d = 2s^2 + 2s + a,$$

therefore $\rho(g, r, d) = a$. Consider the stack $\sigma : \mathfrak{G}^r_d \to \mathcal{M}_g$ classifying linear series $\mathfrak{g}^r_d$ on curves of genus g. Inside the stack $\mathfrak{G}^r_d$ we consider the locus of those pairs $[C, L]$ with $L \in W^r_d(C)$, for which the multiplication map

$$\mu_0(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is not injective. The expected codimension in $\mathfrak{G}^r_d$ of this cycle is equal to $a + 1$, hence the push-forward under σ of this cycle is a virtual divisor in $\mathcal{M}_g$. The case $a = 1$ of this construction will be treated in the forthcoming paper [**29**] from which we quote:

THEOREM 2.4. *We fix $s \geq 2$ and set $g = 2s^2 + s + 1$. The locus*

$$\mathfrak{D}_s := \{[C] \in \mathcal{M}_g : K_{0,2}(C, L) \neq 0 \text{ for a certain } L \in W^{2s}_{2s(s+1)+1}(C)\}$$

is an effective divisor on $\mathcal{M}_g$. *The slope of its closure inside* $\overline{\mathcal{M}}_g$ *equals*

$$s(\overline{\mathfrak{D}}_s) = \frac{3(48s^8 - 56s^7 + 92s^6 - 90s^5 + 86s^4 + 324s^3 + 317s^2 + 182s + 48)}{24s^8 - 28s^7 + 22s^6 - 5s^5 + 43s^4 + 112s^3 + 100s^2 + 50s + 12}.$$

We observe that the inequality

$$6 + \frac{10}{g} < s(\overline{\mathfrak{D}}_s) < 6 + \frac{12}{g+1},$$

holds for each $s \geq 3$. The case $s = 3$ of Theorem 2.4 is presented in [**31**] and it proves that $\overline{\mathcal{M}}_{22}$ is a variety of general type. Another very interesting case is $s = 2$, that is, $g = 11$. This case is studied by Ortega and the second author in [**35**] in connection with *Mercat's Conjecture* in higher rank Brill-Noether theory. In view of the relevance of this case to attempts of establishing a credible rank two Brill-Noether theory, we briefly explain the situation.

We denote as usual by $\mathcal{F}_g$ the moduli space parametrizing pairs $[S, H]$, where S is a smooth $K3$ surface and $H \in \mathrm{Pic}(S)$ is a primitive nef line bundle with $H^2 = 2g - 2$. Over $\mathcal{F}_g$ one considers the projective bundle $\mathcal{P}_g$ classifying pairs $[S, C]$, where S is a smooth $K3$ surface and $C \subset S$ is a smooth curve of genus g. Clearly $\dim(\mathcal{P}_g) = \dim(\mathcal{F}_g) + g = 19 + g$. Observe now that for $g = 11$ both spaces $\mathcal{M}_{11}$ and $\mathcal{P}_{11}$ have the same dimension, so one expects a general curve of genus 11 to lie on finitely many $K3$ surfaces. This expectation can be made much more precise.

For a general curve $[C] \in \mathcal{M}_{11}$, the rank 2 Brill-Noether locus

$$\mathcal{SU}_C(2, K_C, 7) := \{E \in \mathcal{U}_C(2, 20) : \det(E) = K_C,\ h^0(C, E) \geq 7\}$$

is a smooth $K3$ surface. Mukai shows in [**70**] that C lies on a unique $K3$ surface which can be realized as the Fourier-Mukai partner of $SU_C(2, K_C, 7)$. Moreover, this procedure induces a birational isomorphism

$$\phi_{11} : \mathcal{M}_{11} \dashrightarrow \mathcal{P}_{11}, \quad \phi_{11}([C]) := \left[\mathcal{SU}_C\widehat{(2, K_C}, 7), C\right]$$

and the two Brill-Noether divisors $\overline{\mathcal{M}}^1_{11,6}$ and $\overline{\mathcal{M}}^2_{11,9}$ (and likewise the Koszul divisor) are pull-backs by ϕ of Noether-Lefschetz divisors on $\mathcal{F}_{11}$.

Next we define the second Clifford index of a curve which measures the complexity of a curve in its moduli space from the point of view of rank two vector bundles.

DEFINITION 2.5. If $E \in \mathcal{U}_C(2, d)$ denotes a semistable vector bundle of rank 2 and degree d on a curve C of genus g, one defines its Clifford index as $\gamma(E) := \mu(E) - h^0(C, E) + 2$ and the *second Clifford index* of C by the quantity

$$\mathrm{Cliff}_2(C) := \min\bigl\{\gamma(E) : E \in \mathcal{U}_C(2, d),\quad d \leq 2(g-1),\quad h^0(C, E) \geq 4\bigr\}.$$

Mercat's Conjecture [**64**] predicts that the equality

$$\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C) \tag{2.6}$$

holds for *every* smooth curve of genus g. By specializing to direct sums of line bundles, the inequality $\mathrm{Cliff}_2(C) \leq \mathrm{Cliff}(C)$ is obvious. Lange and Newstead have proved the conjecture for small genus [**58**]. However the situation changes for $g = 11$ and the following result is proved in [**35**]:

THEOREM 2.7. *The Koszul divisor* $\mathfrak{D}_2$ *on* $\mathcal{M}_{11}$ *has the following realizations:*

(1) *(By definition)* $\{[C] \in \mathcal{M}_{11} : \exists L \in W^4_{13}(C)$ *such that* $K_{0,2}(C, L) \neq 0\}$.

(2) $\{[C] \in \mathcal{M}_{11} : \mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)\}$.
(3) $\phi_{11}^*(\mathcal{NL})$, *where* $\mathcal{NL}$ *is the Noether-Lefschetz divisor of (elliptic)* $K3$ *surfaces* S *with lattice* $\mathrm{Pic}(S) \supset \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, *where* $C^2 = 20$, $H^2 = 6$ *and* $C \cdot H = 13$.

The closure $\overline{\mathfrak{D}}_2$ *of* $\mathfrak{D}_2$ *in* $\overline{\mathcal{M}}_{11}$ *is a divisor of minimal slope*

$$s(\overline{\mathfrak{D}}_2) = s(\overline{\mathcal{M}}_{11}) = 7.$$

In [**35**], the second description in Theorem 2.7 is shown to imply that $\mathfrak{D}_2$ is the locus over which Mercat's Conjecture fails. Since $\mathfrak{D}_2 \neq \emptyset$, Mercat's Conjecture in its original form is false on $\mathcal{M}_{11}$. On the other hand, Theorem 2.7 proves equality (2.6) for general curves $[C] \in \mathcal{M}_{11}$. Proving Mercat's Conjecture for a general $[C] \in \mathcal{M}_g$, or understanding the loci in moduli where the equality fails, is a stimulating open question.

Remark 2.8. Note that in contrast with lower genus, for $g = 11$ we have $s(\overline{\mathcal{M}}_{11}) = s'(\overline{\mathcal{M}}_{11}) = 7$. Furthermore, the dimension of the linear system of effective divisors of slope 7 is equal to 19 (see [**36**]). The divisors $\overline{\mathcal{M}}^1_{11,6}$, $\overline{\mathcal{M}}^2_{11,9}$ and $\overline{\mathfrak{D}}_2$ are just three elements of this 19-dimensional linear system.

3. Lower bounds on $s(\overline{\mathcal{M}}_g)$

In this section we summarize several approaches towards finding lower bounds for $s(\overline{\mathcal{M}}_g)$ when g is large. The idea is simple. One has to produce one-dimensional collections of families $C \to B$ of genus g curves—in other words, curves $B \in \overline{\mathcal{M}}_g$—such that the union of all the curves B is Zariski dense. For example, a single moving curve B provides such a collection. No effective divisor D contain all such B and, when B does not lie in D, the inequality $B \cdot D \geq 0$ implies that the *slope* $s_B := \frac{B\cdot\delta}{B\cdot\lambda}$ is a lower bound for $s(D)$, and hence that the infimum of these slopes is a lower bound for $s(\overline{\mathcal{M}}_g)$. The difficulty generally arises in computing this bound. We discuss several constructions via covers of the projective line, via imposing conditions on space curves, via Teichmüller theory and via Gromov-Witten theory, respectively. Observe that when $\overline{\mathcal{M}}_g$ is a variety of general type, it carries no rational or elliptic moving curves.

3.1. Covers of $\mathbb{P}^1$ with a moving branch point. Harris and the third author [**48**] constructed moving curves in $\overline{\mathcal{M}}_g$ using certain Hurwitz curves of branched covers of $\mathbb{P}^1$. Consider a connected k-sheeted cover $f : C \to \mathbb{P}^1$ with $b = 2g-2+2k$ simply branch points $p_1, \ldots, p_b$. If γ_i is a closed loop around p_i separating it from the other p_j's, then, since p_i is a simple branch point, the monodromy around γ_i is a simple transposition τ_i in the symmetric group S_k on the points of a general fiber. The product of these transpositions (suitable ordered) must be the identity since a loop around all the p_i is nullhomotopic, the subgroup they generate must be transitive (since we assume that C is connected), and then covers with given branch points are specified by giving the τ_i up to simultaneous S_k-conjugation.

Varying p_b while leaving the others fixed, we obtain a one-dimensional Hurwitz space Z. When p_b meets another branch point, say p_{b-1}, the base $\mathbb{P}^1$ degenerates to a union of two $\mathbb{P}^1$-components glued at a node s, with $p_1, \ldots, p_{b-2}$ on one component and p_{b-1} and p_b on the other. The covering curve C degenerates to a nodal *admissible cover* accordingly, with nodes, say, $t_1, \ldots, t_n$. Such covers were introduced by Beauville for $k = 2$ [**3**] and by Harris and Mumford [**50**] for general k; for a non-technical introduction to admissible covers, see [**49**, Chapter 3.G].

Locally around t_i, the covering map is given by $(x_i, y_i) \to (u = x_i^{k_i}, v = y_i^{k_i})$, where x_i, y_i and u, v parameterize the two branches meeting at t_i and s, respectively. We still call $k_i - 1$ the order of ramification of t_i. The data $(n; k_1, \ldots, k_n)$ can be determined by the monodromy of the cover around p_{b-1} and p_b.

When p_b approaches p_{b-1}, consider the product $\tau = \tau_{b-1}\tau_b$ associated to the vanishing cycle β that shrinks to the node s as shown in Figure 1.

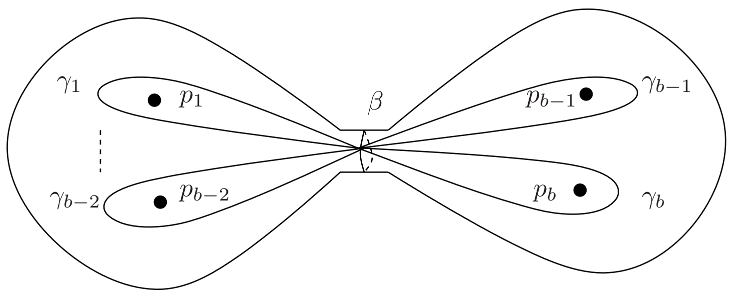

FIGURE 1. The target $\mathbb{P}^1$ degenerates when two branch points meet

Without loss of generality, suppose $\tau_b = (12)$, i.e. it switches the first two sheets of the cover, and suppose $\tau_{b-1} = (ij)$. There are three cases:

(1) If $\tau_{b-1} = \tau_b$, then $\tau = \mathrm{id}$. Consequently over the node s, we see k nodes $t_1, \ldots, t_k$ arising in the degenerate cover, each of which is unramified;

(2) If $|\{i, j\} \cap \{1, 2\}| = 1$, say $(ij) = (13)$, then $\tau = (123)$, i.e. it switches the first three sheets while fixing the others. We see $k - 2$ nodes $t_1, \ldots, t_{k-2}$ arising in the degenerate cover, such that t_1 has order of ramification $3-1 = 2$ and $t_2, \ldots, t_{k-2}$ are unramified;

(3) If $\{i, j\} \cap \{1, 2\} = \emptyset$, say $(ij) = (34)$, then $\tau = (12)(34)$. We see $k-3$ nodes $t_1, \ldots, t_{k-2}$ arising in the degenerate cover, such that t_1 and t_2 are both simply ramified and $t_3, \ldots, t_{k-2}$ are unramified.

Let $f : Z \to \overline{\mathcal{M}}_g$ be the moduli map sending a branched cover to (the stable limit of) its domain curve. The intersection $f_*Z \cdot \delta$ can be read off from the description of admissible covers. For instance, if an admissible cover belongs to case (1), it possesses k unramified nodes. Over the $\mathbb{P}^1$-component containing p_{b-1} and p_b, there are $k - 2$ rational tails that map isomorphically as well as a rational bridge that admits a double cover. Blowing down a rational tail gives rise to a smooth point of the stable limit, hence only the rational bridge contributes to the intersection with δ and the contribution is 2, since it is a (-2)-curve. The intersection $f_*Z \cdot \lambda$ can be deduced from the relation $12\lambda = \delta + \omega^2$, where ω is the first Chern class of the relative dualizing sheaf of the universal covering curve over Z, cf. [**49**, Chapter 6.C] for a sample calculation.

Using these ideas, Harris and the third author obtained a slope formula for s_Z [**48**, Corollary 3.15] in terms of counts of branched covers in each of the three cases, for which they provide only recursive formulae in terms of characters of the symmetric group. More generally, enumerating non-isomorphic branched covers with any fixed ramification is a highly non-trivial combinatorial *Hurwitz counting problem*.

Consider the case when $2k \geq g+2$. Since the Brill-Noether number $\rho(g,1,k) \geq 0$, a general curve of genus g admits a k-sheeted cover of $\mathbb{P}^1$. Therefore, Z is a moving curve in $\overline{\mathcal{M}}_g$. Assuming that, for g large, all ordered pairs of simple transpositions are equally likely to occur as (τ_{b-1}, τ_b)—which seems plausible to first order in k—leads to the estimate (cf. [**48**, Remark 3.23]) $s_Z \simeq \frac{576}{5g}$ (plus terms of lower order in g) as $g \to \infty$.

3.2. Linear sections of the Severi variety. A branched cover of $\mathbb{P}^1$ can be regarded as a map to a one-dimensional projective space. One way to generalize is to consider curves in $\mathbb{P}^2$. Let $\mathbb{P}(d) = \mathbb{P}^{\binom{d+2}{2}-1}$ be the space of plane curves of degree d. Consider the *Severi variety* $V^{d,n}_{\mathrm{irr}} \subset \mathbb{P}(d)$ defined as the closure of the locus parameterizing degree d, irreducible, plane nodal curves with n nodes. The dimension of $V^{d,n}_{\mathrm{irr}}$ is $N = 3d+g-1$, where $g = \binom{d-1}{2} - n$ is the geometric genus of a general curve in $V^{d,n}_{\mathrm{irr}}$. Let $H_p \subset \mathbb{P}(d)$ be a hyperplane parameterizing curves that pass through a point p in $\mathbb{P}^2$. Now fix $N-1$ general points $p_1, \ldots, p_{N-1}$ in the plane. Consider the one-dimensional section of $V^{d,n}_{\mathrm{irr}}$ cut out by these hyperplanes:

$$C^{d,n}_{\mathrm{irr}} = V^{d,n}_{\mathrm{irr}} \cap H_{p_1} \cap \cdots \cap H_{p_{N-1}}.$$

Normalizing the nodal plane curves as smooth curves of genus g, we obtain a moduli map from $C^{d,n}_{\mathrm{irr}}$ to $\overline{\mathcal{M}}_g$ (after applying stable reduction to the universal curve). The calculation for the slope of $C^{d,n}_{\mathrm{irr}}$ was carried out by Fedorchuk [**41**]. The intersection $C^{d,n}_{\mathrm{irr}} \cdot \delta$ can be expressed by the degree of Severi varieties. For instance, let $N^{d,n}_{\mathrm{irr}}$ be the degree of $V^{d,n}_{\mathrm{irr}}$ in $\mathbb{P}(d)$. Then $N^{d,n+1}_{\mathrm{irr}}$ corresponds to the number of curves in $C^{d,n}_{\mathrm{irr}}$ that possesses $n+1$ nodes. Each such node contributes 1 to the intersection with δ_{irr}. Therefore, we have

$$C^{d,n}_{\mathrm{irr}} \cdot \delta_{\mathrm{irr}} = (n+1) N^{d,n+1}_{\mathrm{irr}}.$$

Moreover, the degree of Severi varieties was worked out by Caporaso and Harris [**6**, Thereom 1.1], though again only a recursive formula is known.

The calculation of $C^{d,n}_{\mathrm{irr}} \cdot \lambda$ is much more involved. Based on the idea of [**6**], fix a line L in $\mathbb{P}^2$ and consider the locus of n-nodal plane curves whose intersections with L are of the same type, namely, intersecting L transversely at a_1 fixed points and at b_1 general points, tangent to L at a_2 fixed points and at b_2 general points, etc. The closure of this locus is called the *generalized Severi variety*. A hyperplane section of the Severi variety, as a cycle, is equal to a union of certain generalized Severi varieties [**6**, Theorem 1.2] and $C^{d,n}_{\mathrm{irr}}$ degenerates to a union of linear sections of generalized Severi varieties. The difficulty arising from this approach is that the surface of the total degeneration admits only a rational map to $\overline{\mathcal{M}}_g$, which does not a priori extend to a morphism, as would be the case over a one-dimensional base. Treating a plane curve as the image of a stable map, Fedorchuk was able to resolve the indeterminacy of this moduli map, take the discrepancy into account, and eventually express $C^{d,n}_{\mathrm{irr}} \cdot \lambda$ as a recursion [**41**, Theorem 1.11].

When the Brill-Noether number $\rho(g,2,d) = 3d-2g-6$ is non-negative, a general curve of genus g can be realized as a plane nodal curve of degree d. In this case $C^{d,n}_{\mathrm{irr}}$ yields a moving curve in $\overline{\mathcal{M}}_g$. Fedorchuk evaluated the slope of $C^{d,n}_{\mathrm{irr}}$ explicitly for $d \leq 16$ and $g \leq 21$, cf. [**41**, Table 1], which consequently serves as a lower bound for $s(\overline{\mathcal{M}}_g)$. In this range, the bounds decrease from 10 to 4.93 and Fedorchuk speculates that "even though we have nothing to say about the asymptotic behavior of the bounds produced by curves $C_{\mathrm{irr}}^{d,n}$, it would not be surprising if these bounds approached 0, as g approached ∞".

3.3. Imposing conditions on canonical curves. We have discussed covers of $\mathbb{P}^1$ and curves in $\mathbb{P}^2$ as means of producing moving curves in $\overline{\mathcal{M}}_g$. What about curves in higher dimensional spaces? A natural way of embedding non-hyperelliptic curves is via their *canonical model*. Coskun, Harris and Starr carried out this approach and obtained sharp lower bounds for $s(\overline{\mathcal{M}}_g)$ up to $g \leq 6$ [**19**, §§ 2.3].

Let us demonstrate their method for the case $g = 4$. A canonical genus 4 curve is a degree 6 complete intersection in $\mathbb{P}^3$, cut out by a quadric surface and a cubic surface. The dimension of the family of such canonical curves is equal to 24. Passing through a point imposes two conditions to a curve in $\mathbb{P}^3$ and intersecting a line imposes one condition. Now consider the one-dimensional family B parameterizing genus 4 canonical curves that pass through 9 general fixed points and intersect 5 general lines. Note that 9 general points uniquely determine a *smooth* quadric Q containing them, and 5 general lines intersect Q at 10 points. Let C be a curve in the family parameterized by B. If C is not contained in Q, then it has to intersect Q at $\geq 9+5 = 14$ points, contradicting that $C \cdot Q = 12$. Therefore, every curve in B is contained in Q. Recall that the Gieseker-Petri divisor $\overline{\mathcal{GP}}^1_{4,3}$ on $\overline{\mathcal{M}}_4$ parameterizes genus 4 curves whose canonical images lie in a quadric cone, and its slope is $\frac{17}{2}$. Therefore, the image of B in $\overline{\mathcal{M}}_4$ and the divisor $\overline{\mathcal{GP}}^1_{4,3}$ are disjoint. Moreover, since the points and lines are general, B is a moving curve in $\overline{\mathcal{M}}_4$. As a consequence, $s_B = \frac{17}{2}$ is a lower bound for $s(\overline{\mathcal{M}}_4)$. This bound is sharp and is attained by the Gieseker-Petri divisor of genus 4 curves lying on singular quadric surface in $\mathbb{P}^3$.

In general, the dimension of the Hilbert scheme of genus g canonical curves in $\mathbb{P}^{g-1}$ is g^2+3g-4. Since $g^2+3g-4 = (g+5)(g-2)+6$, we get a moving curve B, for $g \geq 9$ from the canonical curves that contain $g+5$ general points and intersect a general linear subspace $\mathbb{P}^{g-7}$. Several difficulties arise in trying to imitate the calculation of the slope of B. We have no detailed description of the geometry of canonical curves for large g, and especially of their enumerative geometry. However, this approach is sufficiently intriguing for us to propose the following:

Problem 3.1. Determine the lower bounds for $s(\overline{\mathcal{M}}_g)$ resulting from computing the characteristic numbers of canonical curves of arbitrary genus g.

3.4. Descendant calculation of Hodge integrals. We have seen a number of explicit constructions of moving curves in $\overline{\mathcal{M}}_g$. A rather different construction was investigated by Pandharipande [**76**] via Hodge integrals on $\overline{\mathcal{M}}_{g,n}$. Let ψ_i be the first Chern class of the cotangent line bundle on $\overline{\mathcal{M}}_{g,n}$ associated to the ith marked point. It is well known that ψ_i is a nef divisor class. Since a nef divisor class is a limit of ample classes, any curve class of type

$$\psi_1^{a_1}\cdots\psi_n^{a_n}, \quad \sum_{i=1}^{n} a_i = 3g-4+n$$

is a moving curve class in $\overline{\mathcal{M}}_{g,n}$. Pushing forward to $\overline{\mathcal{M}}_g$, we obtain a moving curve in $\overline{\mathcal{M}}_g$ whose slope is equal to

$$\frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}\cdot\delta}{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}\cdot\lambda}. \tag{3.2}$$

In general, such an integral given by the intersection of tautological classes on $\overline{\mathcal{M}}_{g,n}$ is called *Hodge integral*. Pandharipande evaluated (3.2) explicitly for $n = 1$ and $a_1 = 3g - 3$. The calculation was built on some fundamental results of Hodge integrals from [**28**]. For example, normalize a one-nodal, one-marked irreducible curve of arithmetic genus g to a smooth curve of genus $g - 1$ with three marked points corresponding to the original marked point and the inverse images of the node. Then we have

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-3} \cdot \delta_{\mathrm{irr}} = \frac{1}{2} \int_{\overline{\mathcal{M}}_{g-1,3}} \psi_1^{3g-3},$$

where the coefficient $\frac{1}{2}$ is because the normalization map $\overline{\mathcal{M}}_{g-1,3} \to \Delta_{\mathrm{irr}} \subset \overline{\mathcal{M}}_{g,1}$ is generically two to one. The Hodge integral on the right, as well as that in the denominator of (3.2) are calculated in [**28**].

Putting everything together, Pandharipande obtains the lower bound, for all $g \geq 2$,

$$s(\overline{\mathcal{M}}_g) \geq \frac{60}{g+4}.$$

It remains an interesting question to calculate (3.2) for general $a_1, \ldots, a_n$, but based on low genus experiments, it seems that the case $n = 1$ and $a_1 = 3g - 3$ provides the best lower bound (cf. [**76**, §5, Conjecture 1]). So, any new bound arising from this approach would be most likely of size $O(\frac{1}{g})$ as g tends to ∞.

3.5. Covers of elliptic curves and Teichmüller curves. Recall that [**48**] constructs moving curves using covers of $\mathbb{P}^1$. What about covers of curves of higher genera? Suppose the domain curve has genus g and the target has genus h. If $h > 1$, by the Riemann-Hurwitz formula, a d-sheeted cover satisfies $2g - 2 \geq d(2h - 2)$, hence there are only finitely many choices for d. An easy dimension count shows that the Hurwitz space parameterizing all such covers (under the moduli map) is a union of proper subvarieties of $\overline{\mathcal{M}}_g$. In principle there could exist an effective divisor containing all of those subvarieties.

This leaves the case $h = 1$, which was studied by the first author in [**11**]. Let $\mu = (m_1, \ldots, m_k)$ be a partition of $2g - 2$. Consider the Hurwitz space $\mathcal{T}_{d,\mu}$ parameterizing degree d, genus g, connected covers π of elliptic curves with a *unique* branch point b at the origin whose ramification profile is given by μ, i.e.

$$\pi^{-1}(b) = (m_1 + 1)p_1 + \cdots + (m_k + 1)p_k + q_1 + \cdots + q_l,$$

where p_i has order of ramification m_i and q_j is unramified. Over a fixed elliptic curve E, there exist finitely many non-isomorphic such covers. If we vary the j-invariant of E, the covering curves also vary to form a one-dimensional family, namely, the Hurwitz space $\mathcal{T}_{d,\mu}$ is a curve.

The images of $\mathcal{T}_{d,\mu}$ for all d form a countable union of curves in $\overline{\mathcal{M}}_g$. We will see below that $\mathcal{T}_{d,\mu}$ is a *Teichmüller curve* hence *rigid*. Nevertheless if $k \geq g - 1$, the union $\cup_d \mathcal{T}_{d,\mu}$ forms a Zariski dense subset in $\overline{\mathcal{M}}_g$ [**11**, Proposition 4.1]. In this case the lim inf of the slopes of $\mathcal{T}_{d,\mu}$ as d approaches ∞ still provides a lower bound for $s(\overline{\mathcal{M}}_g)$ and since effective divisor can contain only finitely many $\mathcal{T}_{d,\mu}$.

To calculate the slope of $\mathcal{T}_{d,\mu}$, note that an elliptic curve can degenerate to a one-nodal rational curve, by shrinking a vanishing cycle β to the node. The monodromy action associated to β determines the topological type of the admissible cover arising in the degeneration, which in principle indicates how to count the intersection number $\mathcal{T}_{d,\mu} \cdot \delta$. Moreover, using the relation $12\lambda = \delta + \omega^2$ associated to the universal covering map, one can also calculate $\mathcal{T}_{d,\mu} \cdot \lambda$. This leads to a formula, again recursive and difficult to unwind for the same reasons as the formulae in [**48**], for the slope of $\mathcal{T}_{d,\mu}$ in [**11**, Theorem 1.15].

However, $\mathcal{T}_{d,\mu}$ can be regarded as a special Teichmüller curve, and this provides a whole new perspective. Let $\mathcal{H}(\mu)$ parameterize pairs (X, ω) such that X is a genus g Riemann surface, ω is a holomorphic one-form on X and $(\omega)_0 = \sum m_i p_i$ for distinct points $p_i \in X$. Note that integrating ω along a path connecting two points defines a *flat* structure on X. In addition, integrating ω along a basis of the relative homology group $H_1(X; p_1, \ldots, p_k)$ realizes X as a plane polygon with edges identified appropriately under affine translation. The reader may refer to [**92**] for an excellent introduction to flat surfaces. Varying the shape of the polygon induces an $\mathrm{SL}_2(\mathbb{R})$ action on $\mathcal{H}(\mu)$. Project an orbit of this action to $\mathcal{M}_g$ by sending (X, ω) to X. The image is called a *Teichmüller curve* if it is algebraic. Teichmüller curves possess a number of fascinating properties. They are geodesics under the Kobayashi metric on $\mathcal{M}_g$. They are rigid [**63**], hence give infinitely many examples of rigid curves on various moduli spaces, in particular, on the moduli space of pointed rational curves [**12**].

Consider a branched cover $\pi : X \to E$ parameterized in $\mathcal{T}_{d,\mu}$, where E is the square torus with the standard one-form dz. The pullback $\omega = \pi^{-1}(dz)$ has divisor of zeros $\sum_i m_i p_i$. Therefore, (X, ω) yields a point in $\mathcal{H}(\mu)$. Alternatively, one can glue d copies of the unit square to realize X as a *square-tiled surface* endowed with flat structure; Figure 2 shows an example.

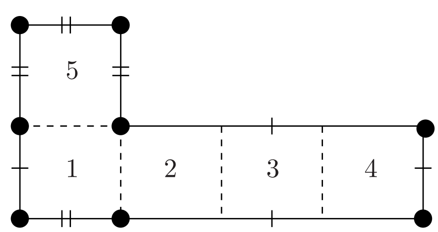

FIGURE 2. A square-tiled surface for $g = 2$, $d = 5$ and $\mu = (2)$

The $\mathrm{SL}_2(\mathbb{R})$ action amounts to deforming the square to a rectangle, i.e. changing the j-invariant of E, hence the Hurwitz curve $\mathcal{T}_{d,\mu}$ is an invariant orbit under this action. Indeed, $\mathcal{T}_{d,\mu}$ is called an *arithmetic Teichmüller curve*. Note that there exist Teichmüller curves that do not arise from a branched cover construction and their classification is far from complete. We refer to [**65**] for a survey on Teichmüller curves from the viewpoint of algebraic geometry.

As a square-tiled surface, X decomposes into horizontal cylinders with various heights and widths, which are bounded by horizontal geodesics connecting two zeros of ω. For instance, the top square of the surface in Figure 2 admits a horizontal cylinder of height and width both equal to 1, while the bottom four squares form another horizontal cylinder of height 1 and width 4. Suppose the vanishing cycle of E is represented by the horizontal core curve of the square. Then the core curve of a horizontal cylinder of height h and width l shrinks to h nodes, each of which contributes $\frac{1}{l}$ to the intersection $\mathcal{T}_{d,\mu} \cdot \delta$ by a local analysis. Denote by $\frac{h}{l}$ the *modulus* of such a cylinder. In general, the “average” number of modulus of horizontal cylinders in an $\mathrm{SL}_2(\mathbb{R})$-orbit closure is defined as the *Siegel-Veech area constant* c_μ associated to the orbit. See [**27**] for a comprehensive introduction to Siegel-Veech constants.

The slope of $\mathcal{T}_{d,\mu}$ has an expression involving its Siegel-Veech constant [**12**, Theorem 1.8], which also holds for any Teichmüller curve [**14**, §3.4]. Based on massive computer experiments, Eskin and Zorich believe that the Siegel-Veech constant c_μ approaches 2 as g tends to ∞ for Teichmüller curves in any (non-hyperelliptic) $\mathcal{H}(\mu)$. Assuming this expectation, the slope formula cited above implies that $s(\mathcal{T}_{d,\mu})$ grows as $\sim \frac{576}{5g}$ for $g \gg 0$ and $k \geq g-1$. We have seen this bound $\frac{576}{5g}$ in Section 3.1. But the curves used in [**48**] are moving while Teichmüller curves are rigid! It would be interesting to see whether this is only a coincidence having to do with some property of branched covers or whether the bound $\frac{576}{5g}$ has a more fundamental "hidden" meaning.

A final remark on Teichmüller curves is that their intersection numbers with divisors on $\overline{\mathcal{M}}_g$ can provide information about the $\mathrm{SL}_2(\mathbb{R})$ dynamics on $\mathcal{H}(\mu)$. For instance, about a decade ago Kontsevich and Zorich conjectured, based on numerical data, that for many low genus strata $\mathcal{H}(\mu)$ the sums of Lyapunov exponents are the same for *all* Teichmüller curves contained in that stratum. This conjecture has been settled by the first author and Möller [**14**]. The idea is that the three quantities—the slope, the Siegel-Veech constant and the sum of Lyapunov exponents—determine each other, hence it suffices to show the slopes are non-varying for all Teichmüller curves in a low genus stratum.

Consider, as an example, $\mathcal{H}(3,1)$ in genus $g = 3$. If a curve C possesses a holomorphic one-form ω such that $(\omega)_0 = 3p_1 + p_2$ for $p_1 \neq p_2$, then C is not hyperelliptic, since the hyperelliptic involution would switch the zeros. Consequently a Teichmüller curve in $\mathcal{H}(3,1)$ is disjoint from the divisor $\mathcal{M}^1_{3,2}$ of hyperelliptic curves in $\mathcal{M}_3$. Checking that this remains true for the respective closures of these two loci immediately implies that the slopes of all Teichmüller curves in $\mathcal{H}(3,1)$ are equal to $s(\overline{\mathcal{M}}^1_{3,2}) = 9$. For a detailed discussion of the interplay between Teichmüller curves and the Brill-Noether divisors, see [**14**]. Using similar ideas, the first author and Möller also settled the case of Teichmüller curves generated by quadratic differentials in low genus in [**15**].

3.6. Moduli spaces of k-gonal curves. We end this section by discussing various questions related to slopes on moduli spaces of k-gonal curves and we begin with the case of hyperelliptic curves. Let $\overline{H}_g := \overline{\mathcal{M}}^1_{g,2}$ be the closure of locus of genus g hyperelliptic curves in $\overline{\mathcal{M}}_g$. Alternatively, it is the admissible cover compactification of the space of genus g, simply branched double covers of $\mathbb{P}^1$. We have an injection $\iota : \overline{H}_g \hookrightarrow \overline{\mathcal{M}}_g$. The rational Picard group of $\overline{H}_g$ is generated by boundary components $\Xi_0, \ldots, \Xi_{[\frac{g-1}{2}]}$ and $\Theta_1, \ldots, \Theta_{[\frac{g}{2}]}$ (cf. [**49**, Chapter 6.C]) and [**56**] shows that these classes also generate $\mathrm{Eff}(\overline{H}_g)$. A general point of Ξ_i parameterizes a double cover of a one-nodal union $\mathbb{P}^1 \cup \mathbb{P}^1$ branched at $2i+2$ points in one component and $2g-2i$ in the other. A general point of Θ_i parameterizes a double cover of $\mathbb{P}^1 \cup \mathbb{P}^1$ branched at $2i+1$ points in one component and $2g-2i+1$ in the other. Cornalba and Harris [**18**] proved the following formulae:

$$\iota^*(\Delta_{\mathrm{irr}}) = 2 \sum_{i=0}^{\lfloor \frac{g-1}{2} \rfloor} \Xi_i, \quad \iota^*(\Delta_i) = \frac{1}{2}\Theta_{2i+1} \text{ for } i \geq 1,$$

$$\iota^*(\lambda) = \sum_{i=0}^{\lfloor \frac{g-1}{2} \rfloor} \frac{(i+1)(g-i)}{4g+2}\Xi_i + \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \frac{i(g-i)}{4g+2}\Theta_i.$$

Since the smallest boundary coefficient in the expression of $\iota^*(\lambda)$ is $\frac{g}{(4g+2)}$ for $\Xi_0 \equiv \iota^*\big(\frac{\Delta_{\mathrm{irr}}}{2}\big)$ (modulo higher boundary terms), if we impose formally the same slope problem to $\overline{H}_g$, the lower bound for slopes of effective divisors on $\overline{H}_g$ is

$$2 \cdot \frac{4g+2}{g} = 8 + \frac{4}{g}.$$

Phrasing it differently, if $B \subset \overline{H}_g$ is any one-dimensional family of genus g curves whose general member is smooth and hyperelliptic, then $(8g+4)B \cdot \lambda \geq gB \cdot \delta$, therefore $s_B \leq 8 + \frac{4}{g}$, cf. [**18**] or [**49**, Corollary 6.24]. Note that this bound converges to 8 as g approaches ∞. The maximum $8 + \frac{4}{g}$ can be achieved by considering a Lefschetz pencil of type $(2, g+1)$ on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Similar bounds were obtained for trigonal families by Stankova [**83**]. If $B \subset \overline{\mathcal{M}}^1_{g,3}$ is any one-dimensional family of trigonal curve with smooth generic member, then

$$s_B \leq \frac{36(g+1)}{5g+1}.$$

A better bound $s_B \leq 7 + \frac{6}{g}$ is known to hold for trigonal families $B \subset \overline{\mathcal{M}}_g$ not lying in the *Maroni locus* of $\overline{\mathcal{M}}^1_{g,3}$, that is, the subvariety of the trigonal locus corresponding to curves with unbalanced scroll invariants. It is of course highly interesting to find such bounds for the higher k-gonal strata $\overline{\mathcal{M}}^1_{g,k}$. A yet unproven conjecture of Harris, see [**83**, Conjecture 13.3], predicts that if $B \subset \overline{\mathcal{M}}^1_{g,k}$ is any 1-dimensional family with smooth generic member and not lying in a codimension one subvariety of $\overline{\mathcal{M}}^1_{g,k}$, then:

$$s_B \leq \Big(6 + \frac{2}{k-1}\Big) + \frac{2k}{g}.$$

To this circle of ideas belongs the following fundamental question:

Problem 3.3. Fix g sufficiently large, so that $\overline{\mathcal{M}}_g$ is of general type. Find the smallest integer $k_{g,\max} \leq \frac{g+2}{2}$ such that $\overline{\mathcal{M}}^1_{g,k}$ is a variety of general type for a $k \geq k_{g,\max}$. Similarly, find the largest integer $2 \leq k_{g,\min}$ such that $\overline{\mathcal{M}}^1_{g,k}$ is uniruled for all $k \leq k_{g,\min}$. Is it true that

$$\liminf_{g\to\infty}\ k_{g,\min} = \liminf_{g\to\infty}\ k_{g,\max}?$$

Obviously a similar question can be asked for the Severi varieties $\overline{\mathcal{M}}^2_{g,d}$, or indeed for all Brill-Noether subvarieties of the moduli space. To highlight our ignorance in this matter, Arbarello and Cornalba [**2**], using a beautiful construction of Beniamino Segre, showed that $\overline{\mathcal{M}}^1_{g,k}$ is unirational for $k \leq 5$, but it is not even known that $k_{g,\min} \geq 6$ for arbitrarily large g. The best results in this direction are due to Geiss [**44**] who proves the unirationality of $\overline{\mathcal{M}}^1_{g,6}$ for most genera $g \leq 45$.

4. The slope of $\mathcal{A}_g$

In view of the tight analogy with the case of $\overline{\mathcal{M}}_g$, we want to discuss questions related to the slope of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties (ppav) of dimension g. Let $\overline{\mathcal{A}}_g$ be the perfect cone, or first Voronoi, compactification of $\mathcal{A}_g$ and denote by $D := \overline{\mathcal{A}}_g - \mathcal{A}_g$ the irreducible boundary divisor. Then $\mathrm{Pic}(\overline{\mathcal{A}}_g) = \mathbb{Q}\cdot\lambda_1 \oplus \mathbb{Q}\cdot[D]$, where $\lambda_1 := c_1(\mathbb{E})$ is the first Chern class of the Hodge bundle. Sections of $\det(\mathbb{E})$ are weight 1 Siegel modular forms. Shepherd-Barron [**82**] showed that for $g \geq 12$, the perfect cone compactification is the canonical model of $\mathcal{A}_g$.

In analogy with the case of $\overline{\mathcal{M}}_g$, we define the slope of an effective divisor $E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)$ as

$$s(E) := \inf\left\{\frac{a}{b} : a, b > 0,\ a\lambda_1 - b[D] - [E] = c[D],\ c > 0\right\},$$

and then the slope of the moduli space as the quantity

$$s(\overline{\mathcal{A}}_g) := \inf_{E\in \mathrm{Eff}(\overline{\mathcal{A}}_g)} s(E).$$

Since $K_{\overline{\mathcal{A}}_g} = (g+1)\lambda_1 - [D]$, it follows that $\overline{\mathcal{A}}_g$ is of general type if $s(\overline{\mathcal{A}}_g) < g+1$, and $\overline{\mathcal{A}}_g$ is uniruled when $s(\overline{\mathcal{A}}_g) > g+1$. Mumford [**71**] was the first to carry out divisor class calculations in $\overline{\mathcal{A}}_g$. In particular, he studied the *Andreotti-Mayer divisor* N_0 on $\mathcal{A}_g$ consisting of ppav $[A, \Theta]$ having a singular theta divisor. Depending on whether the singularity occurs at a torsion point or not, one distinguishes between the components θ_{null} and N_0' of the Andreotti-Mayer divisor. The following scheme-theoretical equality holds:

$$N_0 = \theta_{\mathrm{null}} + 2N_0'.$$

The cohomology classes of the components of $\overline{N}_0$ are given are computed in [**71**]:

$$[\overline{N}_0'] = \Big(\frac{(g+1)!}{4} + \frac{g!}{2} - 2^{g-3}(2^g+1)\Big)\lambda_1 - \Big(\frac{(g+1)!}{24} - 2^{2g-6}\Big)[D],$$

respectively

$$[\overline{\theta}_{\mathrm{null}}] = 2^{g-2}(2^g+1)\lambda_1 - 2^{2g-5}[D].$$

Using these formulas coupled with Tai's results on the singularities of $\overline{\mathcal{A}}_g$, Mumford concluded that $\overline{\mathcal{A}}_g$ is of general type for $g \geq 7$. Note that at the time, the result had already been established for $g \geq 9$ by Tai [**84**] and by Freitag [**42**] for $g = 8$. On the other hand, it is well-known that $\mathcal{A}_g$ is unirational for $g \leq 5$. The remaining case is notoriously difficult. This time, the three authors refrain from betting on possible outcomes and pose the:

Problem 4.1. What is the Kodaira dimension of $\mathcal{A}_6$?

The notion of slope for $\overline{\mathcal{A}}_g$ is closely related to that of $\overline{\mathcal{M}}_g$ via the *Torelli map*

$$\tau : \overline{\mathcal{M}}_g \dashrightarrow \overline{\mathcal{A}}_g,$$

sending a curve to its (generalized) Jacobian. The restriction of τ to the union of $\mathcal{M}_g$ and the locus of one-nodal irreducible curves is an embedding. If a curve C consists of a one-nodal union of two lower genus curves C_1 and C_2, then $J(C) \cong J(C_1) \times J(C_2)$, which does not depend on the position of the node. Therefore, τ contracts Δ_i for $i > 0$ to a higher codimension locus in $\overline{\mathcal{A}}_g$. Moreover, we have that

$$\tau^*(\lambda_1) = \lambda, \quad \tau^*(D) = \delta_{\mathrm{irr}}.$$

Therefore, if F is an effective divisor on $\overline{\mathcal{A}}_g$ and that does not contain $\tau(\mathcal{M}_g)$, then $\tau^*(F)$ and F have the same slope. If we know that $s(\overline{\mathcal{M}}_g) \geq \epsilon$ for a positive number ϵ, then any modular form of weight smaller than ϵ has to vanish on $\tau(\mathcal{M}_g)$. This would provide a novel approach to understand which modular forms cut out $\mathcal{M}_g$ in $\mathcal{A}_g$, and thus give a solution to the geometric *Schottky problem.* By work of Tai [**84**] (explained in [**46**, Theorem 5.19]), the lower bound for slopes of effective divisors on $\overline{\mathcal{A}}_g$ approaches 0 as g tends to ∞, that is, $\lim_{g\to\infty} s(\overline{\mathcal{A}}_g) = 0$. In fact, there exists an effective divisor on $\overline{\mathcal{A}}_g$ whose slope is at most

$$\sigma_g := \frac{(2\pi)^2}{\big(2(g!)\zeta(2g)\big)^{1/g}}.$$

Since $\zeta(2g) \to 1$ and $(g!)^{1/g} \to \frac{g}{e}$ as $g \to \infty$, we find that $g\,\sigma_g \to 68.31\ldots$. Although this is a bit bigger than the unconditional lower bound of 60 for $g\,s(\overline{\mathcal{M}}_g)$ of subsection 3.4, this shows that the slope of this divisor is smaller than the heuristic lower bound $\frac{576}{5g}$ for $s(\overline{\mathcal{M}}_g)$ emerging from [**48**] and [**12**] for large values of g.

As for $\overline{\mathcal{M}}_g$, the slope of $\overline{\mathcal{A}}_g$ is of great interest. The first case is $g = 4$ and it is known [**81**] that $s(\overline{\mathcal{A}}_4) = 8$, and the minimal slope is computed by the divisor $\overline{\tau(\mathcal{M}_4)}$ of genus 4 Jacobians. In the next case $g = 5$, the class of the closure of the Andreotti-Mayer divisor is

$$[\overline{N}_0'] = 108\lambda_1 - 14D,$$

giving the upper bound $s(\overline{\mathcal{A}}_5) \leq \frac{54}{7}$. Very recently, a complete solution to the slope question on $\overline{\mathcal{A}}_5$ has been found in [**34**] by Grushevsky, Salvati-Manni, Verra and the second author. We spend the rest of this section explaining the following result:

THEOREM 4.2. *The slope of $\overline{\mathcal{A}}_5$ is attained by $\overline{N_0'}$. That is, $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, $\kappa(\overline{\mathcal{A}}_5, \overline{N}_0') = 0$, that is, the only effective divisors on $\overline{\mathcal{A}}_5$ having minimal slope are the multiples of $\overline{N}_0'$.*

The proof relies on the intricate geometry of the generically $27:1$ Prym map

$$P : \overline{\mathcal{R}}_6 \dashrightarrow \overline{\mathcal{A}}_5.$$

The map P has been investigated in detail in [**22**] and it displays some breathtakingly beautiful geometry. For instance the Galois group of P is the Weyl group of E_6, that is, the subgroup of $\mathfrak{S}_{27}$ consisting of permutations preserving the intersection product on a fixed cubic surface. The divisor N_0' is the branch locus of P, whereas the ramification divisor $\mathcal{Q}$ has three alternative realizations as a geometric subvariety of $\mathcal{R}_6$, see [**39**] and [**34**] for details. One should view this statement as a Prym analogue of the various incarnations of the $K3$ divisor $\mathcal{K}_{10}$ on $\mathcal{M}_{10}$, see [**36**]:

THEOREM 4.3. *The ramification divisor $\mathcal{Q}$ of the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ has the following geometric incarnations:*

(1) $\big\{[C,\eta] \in \mathcal{R}_6 : \mathrm{Sym}^2 H^0(C, K_C \otimes \eta) \xrightarrow{\ncong} H^0(C, K_C^{\otimes 2})\big\}$.
(2) $\big\{[C,\eta] \in \mathcal{R}_6 : C$ *has a sextic plane model with a totally tangent conic*$\big\}$.
(3) $\big\{[C,\eta] \in \mathcal{R}_6 : C$ *is a section of a Nikulin surface*$\big\}$.
(4) $\big\{[C,\eta] \in \mathcal{R}_6 : \mathrm{Sing}^{\mathrm{st}}(\Xi) \neq 0\big\}$.

The first realization is the most straightforward and it relies on the description of the differential of the Prym map via Kodaira-Spencer theory, see [**22**]. Description (4) refers to stable singularities of the theta divisor $\Xi \subset P(C,\eta)$ associated to the Prym variety. In particular, Ξ has a *stable singularity* if and only if the étale double cover $f : \tilde{C} \to C$ induced by the half-period $\eta \in \mathrm{Pic}^0(C)$, $\eta^{\otimes 2} = \mathcal{O}_C$, carries a line bundle L with $\mathrm{Nm}_f(L) = K_C$ with $h^0(\tilde{C}, L) \geq 4$. Description (3) concerns moduli spaces of $K3$ surfaces endowed with a symplectic involution (Nikulin surfaces): see [**39**]. The equivalence (1) $\Leftrightarrow$ (4) can be regarded as stating that $\mathcal{Q}$ is simultaneously the Koszul and the Brill-Noether divisors (in Prym sense) on the moduli space $\mathcal{R}_6$! By a local analysis, if $\pi : \overline{\mathcal{R}}_6 \to \overline{\mathcal{M}}_6$ is the morphism forgetting the half-period, one proves the following relation in $\mathrm{Pic}(\overline{\mathcal{R}}_6)$:

$$P^*(\overline{N}_0^{'}) = 2\overline{\mathcal{Q}} + \overline{\mathcal{U}} + 20\delta_{\mathrm{irr}}^{''}, \tag{4.4}$$

where $\mathcal{U} = \pi^*(\mathcal{GP}^1_{6,4})$ is the *anti-ramification* divisor of P and finally, $\delta_{\mathrm{irr}}^{''}$ denotes the boundary divisor class corresponding to Wirtinger coverings. Using the different parametrizations of $\mathcal{Q}$ provided by Theorem 4.3, one can construct a sweeping rational curve $R \subset \overline{\mathcal{Q}}$ such that

$$R \cdot \overline{\mathcal{U}} = 0, \ R \cdot \delta_{\mathrm{irr}}^{''} = 0 \ \text{ and } R \cdot \overline{\mathcal{Q}} < 0.$$

Via a simple argument, this shows that in formula (4.4), the divisor $\overline{\mathcal{Q}}$ does not contribute to the linear system $|P^*(\overline{N}_0^{'})|$. Similar arguments show that $\overline{\mathcal{U}}$ and $\delta_{\mathrm{irr}}^{''}$ do not contribute either, that is, N_0' is the only effective divisor in its linear system, or equivalently $\kappa(\overline{\mathcal{A}}_5, \overline{N}_0^{'}) = 0$. In particular $s(\overline{\mathcal{A}}_5) = s(\overline{N}_0^{'}) = \frac{54}{7}$. This argument shows that $\overline{N_0'}$ is rigid, hence $s'(\overline{\mathcal{A}}_5) > s(\overline{\mathcal{A}}_5)$, so we ask:

Problem 4.5. What is $s'(\overline{\mathcal{A}}_5)$?

A space related to both $\mathcal{M}_g$ and $\mathcal{A}_g$ is the universal theta divisors $\mathfrak{Th}_g \to \mathcal{M}_g$, which can be viewed as the universal degree $g-1$ symmetric product $\overline{\mathcal{M}}_{g,g-1}/\!\!/\mathfrak{S}_{g-1}$ over $\mathcal{M}_g$. The following result has been recently established by Verra and the second author in [**38**]:

Theorem 4.6. *$\mathfrak{Th}_g$ is a uniruled variety for genus $g \leq 11$ and of general type for $g \geq 12$.*

The proof gives also a description of the relative effective cone of $\mathfrak{Th}_g$ over $\overline{\mathcal{M}}_g$ as being generated by the boundary divisor $\tilde{\Delta}_{0,2}$ corresponding to non-reduced effective divisors of degree $g-1$ and by the universal ramification divisor of the Gauss map, that is, the closure of the locus of points $[C, x_1 + \cdots + x_{g-1}]$ for which the support of the 0-dimensional linear series $|K_C(-x_1 - \cdots - x_{g-1})|$ is non-reduced. The paper [**38**] also gives a complete birational classification of the universal symmetric product $\overline{\mathcal{M}}_{g,g-2}/\!\!/\mathfrak{S}_{g-2}$, showing that, once again, the birational character of the moduli space changes around genus $g = 12$.

We close this section with a general comment. Via the Torelli map, the moduli space of curves sits between $\overline{H}_g$ and $\overline{\mathcal{A}}_g$. In terms of lower bounds for slopes, $\overline{H}_g$ and $\overline{\mathcal{A}}_g$ behave totally different for large g: the former has lower bound converging to 8, whereas the latter approaches 0. The failure to find of effective divisors with small slope seems to suggest that $\overline{\mathcal{M}}_g$ is "closer" to $\overline{H}_g$, while the failure to find moving curves seems to suggest that $\overline{\mathcal{M}}_g$ is "closer" to $\overline{\mathcal{A}}_g$.

5. Effective classes on spaces of stable curves and maps of genus 0

5.1. Symmetric quotients in genus 0. On $\overline{\mathcal{M}}_{0,n}$ itself, the components of the boundary Δ are the loci Δ_I, indexed by $I \subset \{1, 2, \ldots, n\}$ (subject to the identification $\Delta_I = \Delta_{I^\vee}$ and to the stability condition that both $|I|$ and $|I^\vee|$ be at least 2), whose general point parameterizes a reducible curve having two components meeting in a single node and the marked points indexed by I on one side and those indexed by $I^\vee$ on the other. These generate, but not freely, $\mathrm{Pic}\big(\overline{\mathcal{M}}_{0,n}\big)$, see [**55**].

The starting point for the study of effective cones in genus 0 is the paper of Keel and McKernan [**56**]. In it, they consider the space $\widetilde{\mathcal{M}}_{0,n}$ that is the quotient of $\overline{\mathcal{M}}_{0,n}$ by the natural action of $\mathfrak{S}_n$ by permutations of the marked points. For instance $\widetilde{\mathcal{M}}_{0,2g+2}$ is isomorphic to the compactified moduli space of hyperelliptic curves of genus g already discussed in this survey. The boundary $\widetilde{\Delta}$ of $\widetilde{\mathcal{M}}_{0,n}$ has components $\widetilde{\Delta}_i$ that are simply the images of the loci Δ_i on $\overline{\mathcal{M}}_{0,n}$ defined as the union of all Δ_I with $|I| = i$ for i between from 2 and $\lfloor \frac{g}{2} \rfloor$.

Lemma 5.1. *Every $\mathfrak{S}_n$-invariant, effective divisor class D on $\overline{\mathcal{M}}_{0,n}$ is an effective sum of the boundary divisors Δ_i*

Corollary 5.2. *The cone $\mathrm{Eff}\big(\widetilde{\mathcal{M}}_{0,n}\big)$ is simplicial, and is generated by the boundary classes $\widetilde{\Delta}_i$.*

Proof of Lemma 5.1. Any $\mathfrak{S}_n$-invariant divisor D is clearly a linear combination $\sum b_i \Delta_i$ so the point is to show that, if D is effective, then we can take the b_i all non-negative, and this is shown by a pretty induction using test curves. We may assume that D contains no Δ_i since proving the result for the D' that results from subtracting all such contained components will imply the result for D.

As a base for the induction, pick an n-pointed curve $\big(C, [p_i, \ldots, p_n]\big)$ not in the support of D and form a test family with base $B \cong C$ by varying p_n, while fixing the other p_i. Since C is not in D, the curve B must meet D non-negatively. On the other hand, $B \cdot \Delta_2 = (n-1)$—there is one intersection each time p_n crosses one of the other p_i—and is disjoint from the other Δ_i. Hence, $b_2 \geq 0$.

Now assume inductively that $b_i \geq 0$. Choose a generic curve

$$C = \big(C', [p'_1, \ldots, p'_i]\big) \cup \big(C', [p''_1, \ldots, p''_{n-i}]\big)$$

in Δ_i in which q' on C' has been glued to q'' on C'' and form the family $B \cong C''$ by keeping q' and the marked points on both sides fixed but varying q'' (as in [**49**, Example (3.136)]). As above $B \cdot D \geq 0$, $B \cdot \Delta_j = 0$ unless j is either i or $i+1$. And, as above, $B \cdot \Delta_{i+1} = n - i$ (we get one intersection each time q'' crosses a p''_k), but now B lies *in* Δ_i so to compute $B \cdot \Delta_i$ we use the standard approach of [**49**, Lemma 3.94]. On the "left" side, the family over B is $C' \times C'$ and the section corresponding to q' has self-intersection 0. On the "right" side, the family is $C'' \times C'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ blown up at the points where the constant sections corresponding to the p''_k meet the diagonal section corresponding to q'' and hence the proper transform of that section has self-intersection $\big(2 - (n-i)\big)$. The upshot is that $B \cdot D = (n-i)b_{i+1} - (n-i-2)b_i$ completing the induction. $\square$

In fact, this proof shows quite a bit more. It immediately gives the first inequalities in Corollary 5.3 and the others follow by continuing the induction and using the identifications $\Delta_i = \Delta_{n-i}$.

COROLLARY 5.3 ([**56**, Lemma 4.8]). *If $D = \sum b_i\Delta_i$ is an effective divisor class on $\widetilde{\mathcal{M}}_{0,n}$ whose support does not contain any Δ_i (or, if D is nef), then $(n-i)b_{i+1} \geq (n-i-2)b_i$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and $ib_{i-1} \geq (i-2)b_i$ for $3 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

At this point, it is natural to hope that we might be able to replace the twiddles in Corollary 5.2 with bars with a bit more work. We will see in §§5.3 that this is far from the case.

Next, we give another application of Lemma 5.1, also due to Keel and kindly communicated to us by Jason Starr, this time to the Kontsevich moduli spaces of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. A general map f in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ has a smooth source curve C with linearly non-degenerate image $f(C) \subset \mathbb{P}^d$ of degree d and hence is nothing more than a rational normal curve. The space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, 1)$ is just the Grassmannian of lines in $\mathbb{P}^d$. The philosophy is to view $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ as a natural compactification of the family of such curves. For example, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ has an open stratum consisting of plane conics. One boundary divisor arises when the curve C becomes reducible, the map f has degree 1 on each component and image consists of a pair of transverse lines. But there is a second component, in which the map f degenerates to a double cover of a line in which the image is "virtually marked" with the two branch points. These intersect in a locus of maps from a pair of lines to a single line in which only the image of the point of intersection is "virtually marked".

This generalizes: $\mathrm{Pic}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)\big)$ is freely generated by effective classes Γ_i, the closure of the locus whose generic map has a domain with two components on which it has degrees i and $d-i$ with $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, and a class G, the degenerate locus where $f(C)$ lies in a proper subspace of $\mathbb{P}^d$—see [**75**, Theorem 1].

LEMMA 5.4. *A class $D = aG + \sum_i b_i\Gamma_i$ is effective if and only if $a \geq 0$ and each $b_i \geq 0$.*

PROOF. All we need to show is that effective classes have positive coefficients. We start with a. Choose a general map $g : \mathbb{P}^2 \to \mathbb{P}^d$ for which $g^*\big(\mathcal{O}(1)\big) = \mathcal{O}(d)$ (i.e. a generic $d+1$-dimensional vector space V of degree d polynomials in the plane). Then g sends a general pencil B of lines in $\mathbb{P}^2$ to a pencil of rational curves of degree d. The image of a general element of this pencil will be a rational normal curve of degree d, hence non-degenerate, so $g(B) \not\subset G$ and hence $g(B) \cdot G \geq 0$. No element of the pencil will be reducible, hence $g(B) \cdot \Gamma_i = 0$. Since we can make any rational normal curve a member of the pencil by suitably choosing V and B, this family of test curves must meet any effective divisor, in particular, D, non-negatively. So $a \geq 0$.

To handle the b_i, we use a remark of Kapranov [**54**] that the set K of maps $[f] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ whose image contains a fixed set of $d+2$ linearly general points is disjoint from G (by construction) and may be identified with $\overline{\mathcal{M}}_{0,d+2}$ (by using the points as the markings), so that points of $\Gamma_i \cap K$ correspond to those of Δ_{i+1}. We can choose K not to lie in D by taking the $(d+2)$-points to lie on a rational normal curve not in D so K must induce an effective class D_K on $\overline{\mathcal{M}}_{0,d+2}$. But K does not depend on the ordering of the $d+2$ points so D_K is $\mathfrak{S}_n$-invariant and the non-negativity of the b_i follows from Lemma 5.2. □

We also note that the argument about B in the first paragraph of the proof generalizes. If f is a stable map with domain C that is *not* in G or in any of the

Γ_i, then $f(C)$ is an irreducible, non-degenerate curve of degree d in $\mathbb{P}^d$, hence is a rational normal curve. The translations of f by $\mathbb{P}\mathrm{GL}(d+1)$ will thus be the locus of all rational normal curves, which is dense in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. Thus, we get the following, see [**19**, Lemma 1.8]:

COROLLARY 5.5. *If B is any reduced, irreducible curve in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ not lying in G or any of the Γ_i, then B is a moving curve.*

We will next look at sharpenings and extensions of these results.

5.2. Effective classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. We computed $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)\big)$ above in terms of the classes G and Γ_i in Lemma 5.4 above and we now want to discuss the extensions of Coskun, Harris and Starr [**19**] to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. Since there is no longer any risk of confusion between boundaries in $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, we will now write Δ_i for the Γ_i defined above[1]. We begin by introducing two other important effective classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

DEFINITION 5.6.
(1) Let H be the locus of maps whose image meets a fixed codimension 2 linear subspace $L \subset \mathbb{P}^d$.
(2) Let Δ_{wt} be the weighted total boundary defined by

$$\Delta_{\mathrm{wt}} = \sum_i \frac{i(d-i)}{d} \Delta_i\,.$$

By a test curve argument ([**19**, Lemma 2.1]) using the curves $B_k, 1 \le k \le \lfloor \frac{d}{2} \rfloor$ defined as the one-parameter families of maps whose images contain a fixed set of $d+2$ linearly general points, and meet a fixed subspace of dimension k (and, if $k > 1$, a second of dimension $d-k$), these classes are related by

$$2G = \frac{(d+1)}{d} H - \Delta_{\mathrm{wt}}\,. \tag{5.7}$$

By [**43**, Lemma 14], the general point of each B_k is a map with image a rational normal curve and hence, by Corollary 5.5, all the B_k are moving curves. Since, for $k > 1$, B_k, and only B_k, meets Δ_k, they are independent, and by construction, $\deg_{B_k}(G) = 0$ for all k. Hence the B_k are a set of moving curves spanning the null space of G in the cone of curves of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

We can use (5.7) to identify

$$u_{d,r} : V_d := \mathrm{span}_{\mathbb{C}}\Big\{H, \Delta_i, i = 1, \dots \lfloor \frac{d}{2} \rfloor\Big\} \to \mathrm{Pic}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big)$$

and view all the cones $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big)$ for a fixed d as living in V_d as well. The next lemma asserts that these cones are nested and stabilize.

LEMMA 5.8 ([**19**, Proposition 1.3]). *The inclusions*

$$\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big) \subset \mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)\big)$$

hold for all $r \ge 2$, with equality if $r \ge d$.

[1] Note that [**19**] uses $D_{\deg}$ and for our G and $\Delta_{i,d-i}$ for our Δ_i.

Informally, maps to the complement U of a point $p \in \mathbb{P}^{r+1}$ have codimension $r \geq 2$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)$ so projection from p induces a map

$$h : \mathrm{Pic}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big) \to \mathrm{Pic}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)\big)$$

sending effective divisors to effective divisors and compatible with the identifications $u_{d,r}$. This gives the inclusions. Equality follows for $r \geq d$ by producing inclusions $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big) \subset \mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)\big)$ as follows. If $D \in \mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)\big)$, then the map associated to a general point of D has image spanning a d-plane $W \subset \mathbb{P}^r$ and the pullback of D by any linear isomorphism $j : \mathbb{P}^d \to W$ is an effective class with the same coordinates in V_d. In the sequel, Lemma 5.8 lets us define new effective classes for a fixed small r and obtain classes for all larger values and, to simplify, we use the same notation for the prototypical class and its pullbacks.

We will refer the reader to [**19**, §3] for other complementary results—in particular, the construction of moving curves dual to the one-dimensional faces of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)\big)$, either exactly, assuming the Harbourne-Hirschowitz conjecture, or approximately to any desired accuracy without this assumption.

Because we have such an explicit description of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big)$, and likewise, from the sequel [**20**] of $\mathrm{Nef}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big)$, it is possible, at least for small values of r and d, to answer more refined questions. In particular, we can attempt to understand the chamber structure of the stable base locus decomposition of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)\big)$. The case $r = d$ is particularly interesting and we conclude this subsection by describing the results of Chen, Coskun and Crissman [**10, 13**] for $d = 3$ and $d = 4$ which reveal interesting relations with other moduli spaces.

To start we need the rosters of additional effective classes exhibited as geometric loci in Table 1. Then we need to give the coordinates of all these classes in terms of the basis consisting of H and the boundaries. In fact, they are all of the form $aH + b\Delta + b_{\mathrm{wt}}\Delta_{\mathrm{wt}}$. The coefficients, also given in the table, summarize cases worked out in §2 of [**13**] (where the coordinates of many tautological classes are also computed) and earlier results in [**21, 75, 74**], all obtained by standard test curve calculations.

Divisor	Least r	Description of general map with smooth domain	a	b	b_{wt}
T	r	$f(C)$ is tangent to a fixed hyperplane.	$\frac{d-1}{d}$	0	1
NL	2	$f(C)$ has a node lying on a fixed line.	$\frac{(d-1)(2d-1)}{2d}$	0	$-\frac{1}{2}$
TN	2	$f(C)$ has tacnode.	$\frac{3(d-1)(d-3)}{d}$	4	$d-9$
TR	2	$f(C)$ has triple point.	$\frac{(d-1)(d-2)(d-3)}{2d}$	-1	$-\frac{d-6}{2}$
NI	3	$f(C)$ is not an isomorphism; a generic $f(C)$ is irreducible rational, of degree d, with a single node.	$\frac{(d-1)(d-2)}{d}$	1	$-\frac{d}{2}$

TABLE 1. Other classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ defined as geometric loci

When $d = r = 3$, there is considerable collapsing. The loci TN and TR are empty, $NI = T$ and NL coincides with a class called F in [**10**] and defined as the closure of the locus of maps meeting a fixed plane in two points collinear with a fixed point. Writing $\overline{M}(\alpha) := \mathrm{Proj}\big(\bigoplus_{m\geq 0} H^0\big(m(H+\alpha\Delta)\big)\big)$ for the model of "slope" α, this leads —cf. [**10**, Theorem 1.2] to which we refer for further details—to the picture in Figure 3 of the chamber structure of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)\big)$. In this figure, each wall is labeled $D : \alpha$ with D a spanning class from the list above and α its slope. Each chamber is labeled with the model, defined below, that arises as $\overline{M}(\alpha)$ in its interior with brackets (or parentheses) used to indicate of whether this is (or is not) also the model on the corresponding wall.

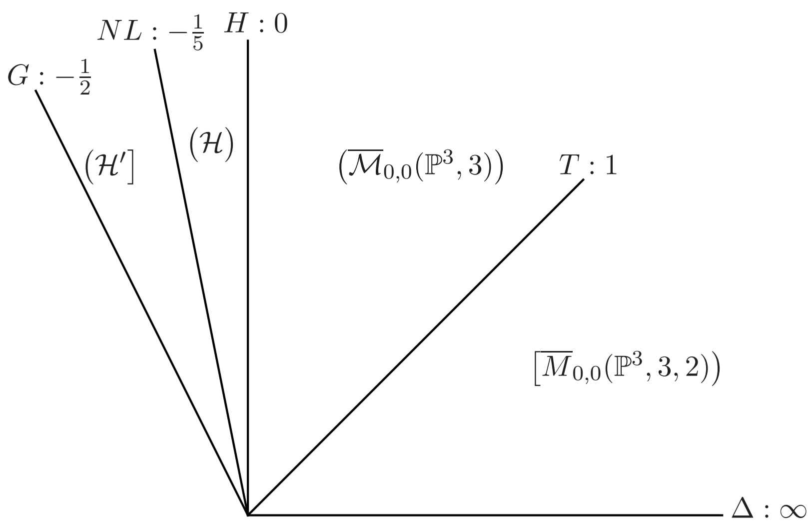

FIGURE 3. Chamber structure of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)\big)$

We will briefly describe the models and wall-crossing maps in the figure, referring to [**10**] for further details. We start at the bottom right with $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$, which is the space of 2-stable maps of Mustaţă and Mustaţă [**72**] in which maps whose source has a degree 1 "tail" are replaced by maps of degree 3 on the "degree 2" component that have a base point at the point of attachment of the tail; the map $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ is the contraction of Δ by forgetting the other end of the tail. The chamber bounded by T and H is, as is shown in [**20**], the Nef cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$.

The ray through H itself gives a morphism $\phi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \mathrm{Chow}(\mathbb{P}^3, 3)$ by sending a map f to the cycle class of $f(C)$ and this is the right side of a flip that contracts the 8 dimensional locus P_3 where f is a degree 3 cover of a line and the 9 dimensional locus $P_{1,2}$ on which f covers a pair of intersecting lines with degrees 1 and 2. The Hilbert scheme of twisted cubics contains a 12-dimensional component $\mathcal{H}$ whose general member is a twisted cubic (and a second 15-dimensional component—cf. [**77**]) and the left side of this flip is the cycle map $\psi : \mathcal{H} \to \mathrm{Chow}(\mathbb{P}^3, 3)$ which contracts the 9-dimensional locus $R \subset \mathcal{H}$ of curves possessing a non-reduced primary component.

Finally, by [**26**, Lemma 2], *every* point of $\mathcal{H}$ (and not just those coming from twisted cubics) is cut out by a unique net of quadrics, and hence there is a morphism $\rho : \mathcal{H} \to \mathcal{H}' \subset \mathbb{G}(3, 10)$ that contracts G.

Already, when $r = d = 4$, the picture gets substantially more complicated, and not simply because the dimension of this $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)\big)$ is 3, In this case, the stable base locus decomposition is no longer completely known. Again we refer to [**13**, §2] for proofs and further details about the claims that follow, and for less complete results about $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ for other values of r and d.

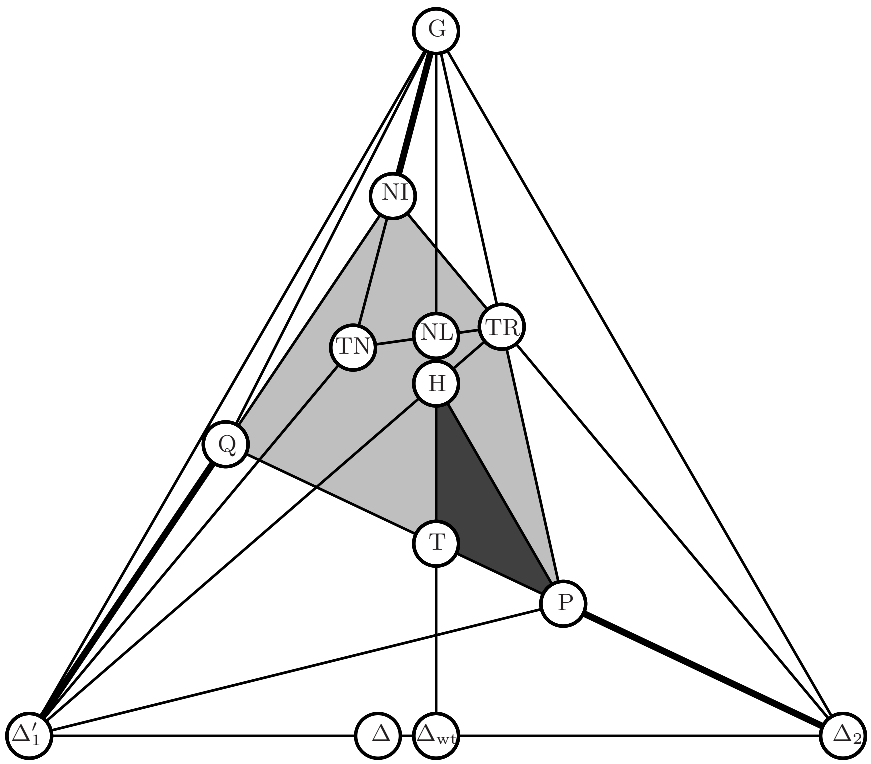

FIGURE 4. Chamber structure of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)\big)$

Figure 4 shows a slice in barycentric coordinates in the rays G, and the components $\Delta_1' := \frac{3}{4}\Delta_1$ and Δ_2 of the weighted boundary Δ_{wt} (which give a slightly more symmetric picture than using Δ_1 and Δ_2). Two extra classes appear as vertices. The first is the class $P = H + \Delta_1 + 4\Delta_2$, which is shown in [**20**, Remark 5.1] to be one of the 3 vertices of $\mathrm{Nef}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)\big)$—the other two are H and T. The second is the class $Q = 3H + 3\Delta_1 - 2\Delta_2$ defined (up to homothety) as the ray in which the Δ_2–P–T-plane meets the Δ_1'–NI-plane.

In the figure, the white circles label these classes and those in Table 1. There are no longer any coincidences between these classes but there quite a few coplanarities visible in the figure as collinearities. The lines show the boundaries of chambers in $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)\big)$ (not necessarily the full set of chambers corresponding to the stable base locus decomposition) in terms of the classes above. The central triangle shaded in dark gray is, as noted above, $\mathrm{Nef}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,4)\big)$.

A heavier line segment joins each of the three vertices of $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,4)\big)$ to an interior class. The two triangles formed by joining this edge to one of the two other vertices are the locus of divisors whose stable base locus contains the common vertex. For example, the triangle Δ_2-P-TR is the chamber whose stable base locus contains Δ_2 but *not* G or Δ_1. Together these triangles cover the complement of the light gray quadrilateral which therefore contains the cone of moving divisors; equality would follow if one could produce a locus with class Q and with no divisorial base locus, as the defining descriptions in Table 1 provide for the other vertices.

5.3. The combinatorial extremal rays of Castravet and Tevelev. We now focus on the space $\overline{\mathcal{M}}_{0,n}$ itself. Since the boundary divisors Δ_I are an effective set of generators of $\mathrm{Pic}\big(\overline{\mathcal{M}}_{0,n}\big)$, a natural question—rendered even more tempting by Corollary 5.2—is whether they generate $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,n}\big)$. This is trivial for $n=3$ and $n=4$ when $\overline{\mathcal{M}}_{0,n}$ is respectively, a point and $\mathbb{P}^1$, and easy for $n=5$.

Let us recall the argument from [**52**, Proposition 4.1] in the last case. Kapranov's construction [**54**] (or the weighted variant in [**51**]) exhibits $\overline{M}_{0,5}$, with the 5^{th} marked point distinguished, as the blowup of $\mathbb{P}^2$ in 4 general points p_i. Denote by L the class of a general line, by E_i the i^{th} exceptional divisor, and by E the sum of the E_i. We can then identify $\Delta_{\{i,5\}}$ with E_i and $\Delta_{\{i,j\}}$ with $L-E_i-E_j$ (these are the proper transform of the lines through p_i and p_j, the other (-1)-curves). With these identifications, the 5 maps $\overline{\mathcal{M}}_{0,5}\to\overline{\mathcal{M}}_{0,4}$ forgetting the i^{th} marked point, respectively correspond to the 5 (semiample) divisors $L-E_i$ and $2L-E$ and the 5 blow-downs to $\mathbb{P}^2$ correspond to $2L-E+E_i$ and to L.

Brute force calculation shows that, in the vector space spanned by L and the E_i, the cone $\mathcal{C}$ spanned by the 10 boundaries is dual to the cone spanned by these 10 semiample classes. Since the effective cone is the dual of the moving cone of curves and the latter lies inside the dual of the semiample cone, this shows that the boundaries generate $\mathrm{Eff}\big(\overline{\mathcal{M}}_{0,5}\big)$.

However, for any $n\geq 6$, examples due to Keel and Vermeire [**88**] show that there are effective classes F_σ that are not effective sums of the boundaries. For $n=6$, fix one of the 15 partitions σ of the marked points into three pairs—say $\sigma=(12)(34)(56)$ which we also view as determining an element in $\mathfrak{S}_n$. We can associate to this choice a divisor in two ways. The first is as the fixed locus F_σ of the involution of $\overline{\mathcal{M}}_{0,6}$ given by σ. There is a map $\phi:\overline{\mathcal{M}}_{0,6}\to\overline{\mathcal{M}}_3$ by identifying the points in each pair to obtain a 3-nodal irreducible rational curve, and we also obtain F_σ as $\phi^*(\overline{\mathcal{M}}^1_{3,2})$ where $\overline{\mathcal{M}}^1_{3,2}$ is the closure in $\overline{\mathcal{M}}_3$ of the hyperelliptic locus in $\mathcal{M}_3$.

To analyze F_σ, the starting point is again an explicit model of $\overline{\mathcal{M}}_{0,6}$ (one starts by blowing up 5 general points in $\mathbb{P}^4$, and then blows up the proper transforms of the 10 lines through 2 of these points). It is again straightforward to write down expressions for the Δ_I and for fixed loci like F_σ as combinations of classes L, E_i and E_{ij} defined in analogy with those above (cf. the table on p.79 of [**88**]). Vermeire then gives an essentially diophantine argument with these coefficients to show that F_σ is not an effective sum of boundaries. Pulling back F_σ by forgetful maps, produces effective classes on $\overline{\mathcal{M}}_{0,n}$ for any $n\geq 6$ that are not effective sums of boundaries.

In another direction, these examples are known be sufficient to describe only $\mathrm{Eff}(\overline{\mathcal{M}}_{0,6})$. Hassett and Tschinkel [**52**, Theorem 5.1] prove this by again showing that the dual of the cone generated by these classes lies in the moving cone of curves, for which the use of a tool like `Porta` which was convenient for $n = 5$ is now essential. Castravet [**7**] gives another argument that, though quite a bit longer, can be checked by hand, based on showing that the divisors of boundary components and the F_σ generate the Cox ring of $\overline{\mathcal{M}}_{0,6}$.

At this point, experts were convinced that $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$ probably had many non-boundary extremal rays but there was no clear picture of how they might be classified, indeed there were no new examples, until the breakthrough of Castravet and Tevelev [**8**] in 2009, which provides a recipe for constructing such rays from *irreducible hypertrees* Γ—combinatorial data whose definition we will give in a moment, along with the related notions of *generic* and *spherical duals*—that they conjecture yields them all.

THEOREM 5.9 ([**8**, Theorem 1.5, Lemma 7.8 and Lemma 4.11]). *Every hypertree Γ of order n determines an effective divisor $D_\Gamma \subset \overline{\mathcal{M}}_{0,n}$.*

(1) *If Γ is irreducible, then D_Γ is a non-zero, irreducible effective divisor satisfying:*
   (a) *D_Γ is an extremal ray of $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$ and meets $\mathcal{M}_{0,n}$.*
   (b) *There is a birational contraction $f_\Gamma : \overline{\mathcal{M}}_{0,n} \dashrightarrow X_\Gamma$ onto a normal projective variety X_Γ whose exceptional locus consists of D_Γ and components lying in Δ.*
   (c) *If Γ and Γ' are generic and $D_\Gamma = D_{\Gamma'}$, then Γ and Γ' are spherical duals.*

(2) *The pullback via a forgetful map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,m}$ of the divisor D_Γ on $\overline{\mathcal{M}}_{0,m}$ associated to any irreducible hypertree of order m, which when n is understood we will again (abusively) denote by D_Γ, also spans an extremal ray of $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$.*

(3) *If Γ is not irreducible, then every irreducible component of D_Γ—if this locus is nonempty—is pulled back via a forgetful map from the divisor $D_{\Gamma'} \subset \overline{\mathcal{M}}_{0,m}$ of an irreducible hypertree Γ' of order $m < n$.*

The table below shows the number $\mathrm{IH}(n)$ of irreducible hypertrees of order n, up to $\mathfrak{S}_n$-equivalence. For $n = 5$, this count must be 0 since all extremal rays are boundaries. For $n = 6$, the unique D_Γ yields the Keel-Vermeire divisors (cf. Figure 5).

n	5	6	7	8	9	10	11
$\mathrm{IH}(n)$	0	1	1	3	11	93	1027

As the table indicates, $\mathrm{IH}(n)$ grows rapidly with n. Empirically, most hypertrees are generic. So the upshot of Theorem 5.9 is to provide, as n increases, very large numbers of new extremal effective divisors. As a complement, [**8**, §9] gives examples of larger collections of non-generic irreducible hypertrees which give the same ray in $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$.

Based, as far as the authors can tell on the cases $n \leq 6$, Castravet and Tevelev propose a very optimistic converse conjecture. We quote from [**8**, Conjecture 1.1]:

CONJECTURE 5.10. *Every extremal ray of $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$ is either a boundary divisor or the divisor D_Γ of an irreducible hypertree Γ of order at most n.*

To our knowledge, more of the activity this has prompted has been devoted to searching for a counterexample than for a proof. Aaron Pixton informs us that he has an example (different from that of [**78**]) of a divisor D on $\overline{\mathcal{M}}_{0,12}$ that is effective and non-moving, and that is not equal to any irreducible hypertree divisor, but whether this divisor lies outside the cone spanned by the hypertree divisors is not, at the time of writing, clear.

We now turn to defining all the terms used above. We have taken the liberty of introducing a few new terms like triadic (see below) for notions used or referred to often, but not named in [**8**]. It will simplify our definitions to write $<n> := \{1, 2, \ldots, n\}$ (or any other model set of cardinality n).

Recall that a hypergraph Γ of order n consists of collection indexed by $j \in <d>$ of hyperedges $\Gamma_j \subset <n>$. We say that Γ' is a sub-hypergraph of Γ if each of the hyperedges Γ'_k is a subset of some hyperedge Γ_j. We start with the notion of *convexity* of a hypergraph.

Definition 5.11.

(1) For any set S of hyperedges, let $T_S = \bigcup_{j\in S} \Gamma_j$, $\tau_S = |T_S| - 2$ and $\sigma_S = \sum_{j\in S}\big(|\Gamma_j| - 2\big)$. We call Γ *convex* if for all $S \subset <d>$, $\tau_S \geq \sigma_S$. Taking S to be a singleton, this implies that every hyperedge contains at least 3 vertices. We call Γ *strictly convex* if this inequality is strict whenever $2 \leq |S| \leq (d-1)$.

(2) A hypertree is *triadic* if every hyperedge contains exactly 3 vertices (i.e. $\tau_S = \sigma_S$ for S any hyperedge). For such a hypertree, convexity simply says that any set S of hyperedges contains at least $|S| + 2$ vertices.

Now we turn to the notions of *hypertree* and of *irreducibility*.

Definition 5.12.

(1) A *hypertree* Γ of order n is a hypergraph satisfying:
   (a) Every vertex lies on at least 2 hyperedges.
   (b) (Convexity) Γ is convex. In particular, every hyperedge contains at least 3 vertices.
   (c) (Normalization) $\tau_\Gamma = \sigma_\Gamma$; that is, $n - 2 = \sum_{j\in<d>}\big(|\Gamma_j| - 2\big)$.

(2) A hypertree is *irreducible* if it is strictly convex.

Empirically, most hypertrees are triadic in which case the normalization condition simply says that $n = d + 2$ and irreducibility says that any proper subset of $e \geq 2$ hyperedges contains at least $e + 3$ vertices. The use of the term "tree" is motivated by the observation that a dyadic hypergraph (i.e. an ordinary graph) is a tree exactly when $n = d + 1$

Now we turn to the notion of *genericity*.

Definition 5.13. We let $\mathrm{Conv}(\Gamma)$ denote the set of all convex sub-hypergraphs of Γ and define the *capacity* of Γ by

$$\mathrm{cap}(\Gamma) := \max_{\Gamma'\in\mathrm{Conv}(\Gamma)} \sigma_{\Gamma'} .$$

Definition 5.14.

(1) A triple T of vertices that do not lie on any hyperedge of Γ but such that any two do lie on a hyperedge is called a *wheel*[2] of Γ.

[2]Here the term *triangle* might better capture the intuition.

(2) If Γ is a triadic hypertree of order n and T is a triple of vertices that is not a hyperedge, we can form a new triadic hypertree Γ_T of order $(n-2)$ by identifying the vertices in T and deleting any hyperedges containing 2 of these vertices.
(3) An irreducible triadic hypertree is *generic* if, whenever T is a triple that is neither a hyperedge nor a wheel, we have $\mathrm{cap}(\Gamma_T) = n - 4$.

An important source of generic triadic examples is provided by triangulations of the 2-sphere in which each vertex has even valence, or equivalently whose triangles can be 2-colored, say black and white, so that each edge has one side of each color.

Definition 5.15.
(1) For any 2-colorable spherical triangulation, the triangles of each color form the set of hyperedges of a triadic hypertree—called a *spherical* hypertree—on the full set of vertices, and we say this pair of hypertrees are *spherical duals* (or, in [**8**], the black and white hypertrees of an even triangulation of the sphere).
(2) Given a distinguished triangle in each of two *spherical* hypertrees, we may form their *connected sum* by choosing colorings which make one triangle white and the other black, deleting these two triangles, and glueing along the exposed edges.

Lemma 5.16. *A spherical hypertree is irreducible unless it is a connected sum.*

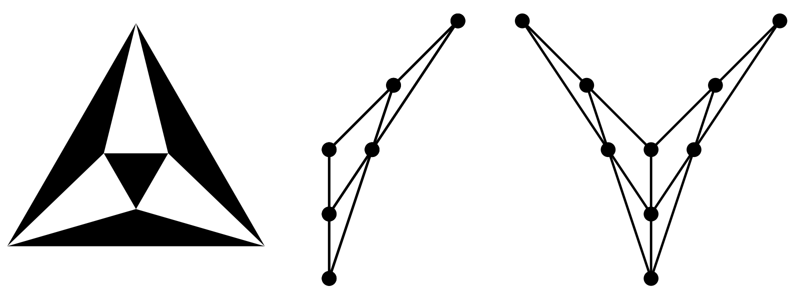

Figure 5. The Keel-Vermeire hypertree

At the left of Figure 5, we show the simplest 2-colorable spherical triangulation for which both the spherical duals give the irreducible triadic hypertree Γ_{KV} of order 6 shown in the center. The divisor $D_{\Gamma_{\mathrm{KV}}}$ of this hypertree is the Keel-Vermeire divisor. On the right, we show the hypertree given by taking the connected sum of the black and white spherical duals. This is not irreducible by taking S to be the set of hyperedges on the left or right side of the picture.

Proving Theorem 5.9 involves delicate combinatorial and geometric arguments that are far too involved to give here. All we will attempt to do is to sketch the main steps of the argument. Castravet and Tevelev [**8**] also prove many other complementary results that we will not even cite here.

A key motivating idea, though one whose proof comes rather late in the development, is that every irreducible hypertree has a *planar realization*, by which we mean an injection of $f_\Gamma : <n> \to \mathbb{P}^2$ so that the set of lines in the plane containing 3 or more points of the image of f_Γ is exactly the set of hyperedges of Γ. If so, and

π_p is the projection to $\mathbb{P}^1$ from a point p not on any of the lines through at least 2 of the points in this image, then the composition $\pi_p \circ f_\Gamma$ defines a set of n marked points on $\mathbb{P}^1$ and hence a point $[f_\Gamma, p] \in \overline{\mathcal{M}}_{0,n}$. A typical realization and projection for the Keel-Vermeire curve are shown in Figure 6.

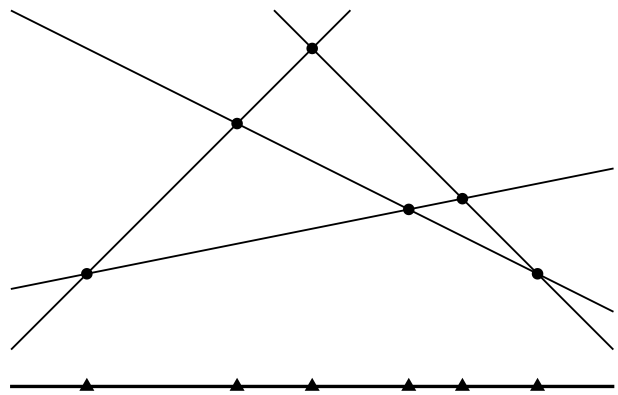

FIGURE 6. Planar realization and projection of the Keel-Vermeire hypertree

The closure of the locus of all such points is defined to be D_Γ and, by the time the non-emptiness of this locus can be established, the fact that it is an irreducible divisor has been established via a second description. Both planarity of hypertrees and irreducibility of the loci D_Γ are obtained as byproducts of a study of hypertree curves and associated Brill-Noether loci. We will sketch the ideas in the simpler case when Γ is triadic, simply hinting at the complications for general Γ.

The *hypertree curve* Σ_γ associated to a triadic hypertree Γ is obtained by taking a copy of a 3-pointed $\mathbb{P}^1$ for each hyperedge of Γ and gluing all the points corresponding to the vertex i to a single point p_i as a scheme-theoretic pushout (i.e. so that the branches look locally like the coordinate axes in an affine space of dimension equal to the valence of the vertex). Note that Σ_γ has genus $g = n-3 = d-1$ and the Picard scheme $\mathrm{Pic}^{\underline{1}}$ of line bundles of degree 1 on each component is (non-canonically) isomorphic to $(\mathbb{G}_m)^g$. For a hypertree whose vertices all have valence 2, this curve is already stable (as in Figure 6). In general, to get a stable model, it is necessary to replace each vertex of valence $v \geq 3$ by a v-pointed copy of $\mathbb{P}^1$ glued to the coincident components at its marked points, and to avoid adding moduli, to fix the choice of this curve in some arbitrary way.

Now the connection to Brill-Noether theory enters. For a general smooth curve Σ of genus $g \geq 2$, there is a birational morphism $\nu : G^1_{g+1} \to W^1_{g+1} \simeq \mathrm{Pic}^{g+1}(\Sigma)$ sending a pencil of divisors of degree $g+1$ to its linear equivalence class, and whose exceptional divisor E lies over the codimension 3 locus W^2_{g+1} of line bundles with $h^0(L) \geq 3$. The general pencil D in G^1_{g+1} and in E is globally generated, so the general pencil D in E can be obtained as the composition of the map to $\mathbb{P}^2$ associated to $\nu(E)$ with projection from a general point of $\mathbb{P}^2$.

The idea unifying all the steps above in [**8**] is to extend this picture to the genus 0 hypertree curves Σ_Γ above. Sticking to the simpler case of triadic Γ with all vertices of valence 2, a linear system on Σ_Γ is *admissible* if it is globally generated and sends the singularities to distinct points and an invertible sheaf is admissible if its complete linear series is. Define $\mathrm{Pic}^{\underline{1}}$ to be the set of admissible line bundles having degree 1 on each component, define the Brill-Noether locus $W^r \subset \mathrm{Pic}^{\underline{1}}$ to be the locus of admissible line bundles with $h^0(\Sigma, L) \geq r+1$, and define the locus G^r to be the pre-image of W^r under the natural forgetful map ν from pencils to line bundles. Again, there are extra complications if the hypertree is not triadic (because then the hypertree curves have moduli), or if there are vertices of higher valence (in which case, sheaves in $\mathrm{Pic}^{\underline{1}}$ are required to have degree 0 on the components inserted as each such vertex).

The main line of argument of [**8**] may then be sketched as follows. Theorem 2.4 identifies $\mathcal{M}_{0,n}$ with G^1 and shows that ν is birational with exceptional locus G^2 (and compactifies this picture when Γ is not triadic). After an interlude in §3 devoted to computing the dimensions of images of maps generalizing this compactification to hypergraphs that are not necessarily convex, Theorem 4.2 shows that the divisor D_Γ obtained by taking the closure of G^2 in $\overline{\mathcal{M}}_{0,n}$ is non-empty and irreducible and partially computes its class; a by product is the characterization of the components of D_Γ (Lemma 4.11) when Γ is not irreducible. Section 5 is another interlude proving Gieseker stability (with respect to the dualizing sheaf) of invertible sheaves in the (generalized) $\mathrm{Pic}^{\underline{1}}$ which is then applied to complete the construction of the birational contraction of Theorem 5.9 (cf. also Theorem 1.10). The reconciliation of the descriptions of D_Γ as the closure of the locus of plane realizations (in particular, the existence of such realizations) and as the closure of G^2 is carried out in §6 (Theorem 6.2). Finally, that the divisor D_Γ of a generic, irreducible Γ determines it (up to spherical duals) is proved in §7.

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Clay Mathematics Proceedings
Volume **18**, 2013

Rational Self Maps of Calabi-Yau Manifolds

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Abstract. We prove that a very general Calabi-Yau (CY) complete intersection in $\mathbb{P}^n$ does not admit a nontrivial dominant rational self map, assuming that the same holds for very general $K3$ surfaces of genus 3, 4 and 5. One of the crucial steps of our proof makes use of Mumford's result on the Chow ring of zero-dimensional cycles on a surface with nontrivial holomorphic 2-forms.

1. Introduction

1.1. Statements of results. The main purposes of this paper is to prove the following:

Theorem 1.1. *There is no dominant rational self map $\phi : X \dashrightarrow X$ of degree $\deg\phi > 1$ for a very general complete intersection X in $\mathbb{P}^n$ of dimension $\dim X \geq 2$ and of type $(d_1, d_2, ..., d_r)$ satisfying $d_1 + d_2 + ... + d_r \geq n + 1$.*

The proof is based on a degeneration argument by "splitting" X, introduced by P. Griffiths and J. Harris in [**G-H**], along with induction on $\dim X$. Eventually, it is reduced to the case $\dim X = 2$, i.e., the case of $K3$ surfaces. We assume the following:

Theorem 1.2. *There is no dominant rational self map $\phi : X \dashrightarrow X$ of degree $\deg\phi > 1$ for a very general projective $K3$ surface X of genus $3 \leq g \leq 5$, i.e., a very general complete intersection of type (4) in $\mathbb{P}^3$, $(2,3)$ in $\mathbb{P}^4$ or $(2,2,2)$ in $\mathbb{P}^5$.*

This was proved in [**Ch**] for $K3$ surfaces of all genus $g \geq 2$, although we only need $g = 3, 4, 5$ for Theorem 1.1. Indeed, most techniques needed for higher dimensions have already developed in [**Ch**]. However, this paper is self-contained with no other statement from [**Ch**] assumed except Theorem 1.2.

If we use the notations $\mathrm{Rat}(X) \supset \mathrm{Bir}(X) \supset \mathrm{Aut}(X)$ for the monoid of dominant rational self maps $\phi : X \dashrightarrow X$, the group of birational self maps $\varphi : X \dashrightarrow X$ and the automorphism group of X, respectively, the above theorem is equivalent to saying that

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) \tag{1.1}$$

for a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type.

2010 *Mathematics Subject Classification.* Primary 14J28; Secondary 14E05.
Key words and phrases. $K3$ surface, Calabi-Yau Manifolds, Rational maps.
Research partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

In addition, it is well known that a birational map $X \dashrightarrow X$ induces an isomorphism in codimension one, i.e., $\phi : X\backslash Z_1 \cong X\backslash Z_2$ is an isomorphism for some Z_1 and Z_2 of codimension ≥ 2 in X, since X is smooth and K_X is numerically effective (nef) (see e.g. [**Co**, 2, (2.5)]). It follows that ϕ induces an automorphism $\phi^* : \mathrm{Pic}(X) \cong \mathrm{Pic}(X)$ of the Picard group together with an isomorphism $\phi^* : \mathbb{P}H^0(L) \cong \mathbb{P}H^0(\phi^* L)$ of the corresponding linear systems for each $L \in \mathrm{Pic}(X)$. Combining this with the fact $\mathrm{Pic}(X) = \mathbb{Z}$ by Lefschetz, we conclude that $\phi^* = \mathrm{id}$ on $\mathrm{Pic}(X)$, i.e., $\phi^* L = L$ for all $L \in \mathrm{Pic}(X)$ and $\phi^* : \mathbb{P}H^0(L) \cong \mathbb{P}H^0(L)$ is an automorphism of $\mathbb{P}H^0(L)$. Consequently, when we embed X into $\mathbb{P}^N$ with a very ample L, ϕ is induced by an automorphism of $\mathbb{P}^N$ and hence $\phi \in \mathrm{Aut}(X)$. This proves $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ for a smooth projective variety with K_X nef and $\mathrm{Pic}(X) = \mathbb{Z}$. Hence

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) = \mathrm{Aut}(X) \tag{1.2}$$

for a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type.

Furthermore, since it is classically known that $\mathrm{Aut}(X)$ is trivial for "almost" all complete intersections [**M-M**], we may put Theorem 1.1 in the following more explicit form:

COROLLARY 1.3. *For a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type and* $\dim X \geq 2$*,*

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) = \mathrm{Aut}(X) = \{1\}. \tag{1.3}$$

1.2. Complete intersections of general type. Although our theorem is stated for both complete intersections of CY and general type, the only nontrivial part is the statement on CY complete intersections. The theorem is well known to be true for complete intersections of general type. We give a quick proof of this fact.

PROPOSITION 1.4. *Let X be a smooth projective variety of general type with Hodge group*

$$H^{1,1}(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X) \cong \mathbb{Q}. \tag{1.4}$$

Then $\deg \phi = 1$ *for every dominant rational map $\phi : X \dashrightarrow X$.*

PROOF. The indeterminacy of ϕ can be resolved by a sequence of blowups along smooth centers by Hironaka's resolution of singularities [**H**] (see also e.g. [**K**]). Let $f : Y \to X$ be the resulting birational regular map with the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "f"'] & X \\ X \arrow[ur, dashed, "\phi"'] & \end{tikzcd} \tag{1.5}$$

where Y is smooth and projective.

From this diagram, we derive the identity

$$K_Y = f^* K_X + \sum a(E_k, X) E_k = \varphi^* K_X + \sum \mu_l F_l \tag{1.6}$$

where K_X and K_Y are the canonical divisors of X and Y, respectively, $a(E_k, X)$ is the discrepancy of the exceptional divisor E_k with respect to X and F_l are the ramification divisors of φ.

By (1.6), we have

$$K_X = f_* \varphi^* K_X + \sum \mu_l f_* F_l \tag{1.7}$$

and hence

$$f_* \varphi^* K_X = \lambda K_X \tag{1.8}$$

in $H^2(X, \mathbb{Q})$ for some $0 < \lambda \leq 1$, by (1.4) and the fact $\mu_l > 0$. Thus

$$K_X \cdot \varphi_* f^* \xi = f_* \varphi^* K_X \cdot \xi = \lambda K_X \cdot \xi \tag{1.9}$$

for all $\xi \in H_2(X, \mathbb{Z})$. Clearly, this forces $\lambda = \deg \phi = 1$ since $K_X \neq 0$. $\square$

For a smooth complete intersection X of general type in $\mathbb{P}^n$, (1.4) holds trivially for $\dim X = 1$, for $\dim X \geq 3$ by Lefschetz and for $\dim X = 2$ and X very general by Noether-Lefschetz. Hence Theorem 1.1 actually holds for every smooth complete intersection $X \subset \mathbb{P}^n$ of general type and $\dim X \geq 3$.

1.3. Background. For the background on rational self maps of $K3$ surfaces and CY manifolds, please see the well-written paper [**D**].

Clearly, every variety X birational to a projective family of abelian varieties or finite quotients of abelian varieties over some base B admits nontrivial rational self maps given by the rational self maps of the generic fiber of X/B. Indeed, this observation is one of the main motivations of studying the rational self maps in the first place. As a consequence of Theorem 1.1, we see that a very general CY complete intersection in $\mathbb{P}^n$ is not birational to a fibration of abelian varieties, which is a known fact for $K3$ surfaces.

In higher dimensions, C. Voisin proved that a very general CY hypersurface $X \subset \mathbb{P}^n$ cannot be covered by abelian varieties of dimension $r \geq 2$ [**V3**], which is a stronger statement than X is not birational to a fibration of abelian varieties of dimension $r \geq 2$. However, the case $r = 1$ is not known, to the best of our knowledge. Indeed, we expect the following to hold:

Conjecture 1.5. *For a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type and* $\dim X \geq 3$,

$$\mathrm{Rat}(Y) = \mathrm{Bir}(Y) \tag{1.10}$$

for every projective variety Y dominating X via a generically finite rational map $Y \dashrightarrow X$.

Obviously, this conjecture implies that a very general CY complete intersection $X \subset \mathbb{P}^n$ of dimension ≥ 3 cannot be covered by elliptic curves, which is the weak Clemens' conjecture when X is a very general quintic 3-fold.

A bolder conjecture is that everything here including Theorem 1.1 and Conjecture 1.5 holds for a very general projective CY manifold. Certainly, the techniques developed here can be easily adapted to deal with other types of CY manifolds as long as these manifolds admit suitable degenerations. On the other hand, Voisin gives examples of CY varieties with Picard number one having dominant rational self maps of degree > 1 [**V2**, Sec. 4.2] (see also [**D-M**, Theorem 3.4]).

Correspondingly, there is a similar story for generalizations of Clemens' conjecture [**V1**, Remark 3.24].

1.4. Conventions and Acknowledgments. We work exclusively over $\mathbb{C}$ and with analytic topology wherever possible.

By a component of a variety, we mean an irreducible component unless we say a connected component of a variety, which is a connected component of the variety in the topological sense.

I am grateful to Prof. Keiji Oguiso for point it out to me the fact that $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ for a Calabi-Yau manifold X with Picard rank one. I would also like to thank the referee for many constructive suggestions.

2. Proof of Theorem 1.1

2.1. Degeneration. We start our proof by degenerating a complete intersection of type $(d_1, d_2, ..., d_r)$ to a union of two complete intersections of type $(d_1', d_2, ..., d_r)$ and $(d_1'', d_2, ..., d_r)$, respectively. More precisely, let $W \subset \Delta \times \mathbb{P}^n$ be a family of complete intersections of type $(d_1, d_2, ..., d_r)$ over the disk Δ with the properties that

- the central fiber $W_0 = S_1 \cup S_2$, where S_1 and S_2 are smooth complete intersections of type $(d_1', d_2, ..., d_{r-1}, d_r)$ and $(d_1'', d_2, ..., d_{r-1}, d_r)$, respectively, for some $d_1', d_1'' \in \mathbb{Z}^+$ satisfying $d_1' + d_1'' = d_1$;
- S_1 and S_2 meet transversely along $D = S_1 \cap S_2$, where D is a very general complete intersection of type $(d_1', d_1'', d_2, ..., d_{r-1}, d_r)$ in $\mathbb{P}^n$ and hence
$$\mathrm{Rat}(D) = \mathrm{Bir}(D) = \mathrm{Aut}(D) = \{1\} \tag{2.1}$$
by induction;
- W is smooth outside of a smooth complete intersection $\Lambda \subset D$ of type $(d_1', d_1'', d_1, d_2, ..., d_{r-1}, d_r)$ in $\mathbb{P}^n$ and is locally given by $xy = tz$ at every point $p \in \Lambda$.

We can resolve the singularities of W by blowing up W along S_1. Let $X \to W$ be the blowup. It is not hard to see that the central fiber of X/Δ is $X_0 = R_1 \cup R_2$, where R_1 is the blowup of S_1 at Λ and $R_2 \cong S_2$.

2.2. Resolution of indeterminacy. Suppose that there is a dominant rational map $\phi_t : X_t \dashrightarrow X_t$ of $\deg \phi_t > 0$ for every $t \neq 0$. We can extend it to a dominant rational map $\phi : X \dashrightarrow X$, after a base change, with the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d] & X \arrow[dl] \\ \Delta & \end{tikzcd} \tag{2.2}$$

Note that after a base change, X is locally given by
$$xy = t^m \tag{2.3}$$
for some positive integer m at every point $p \in D$. So X is $\mathbb{Q}$-factorial and has canonical singularities along D.

As in the proof of Proposition 1.4, we can resolve the indeterminacy of ϕ and arrive at the diagram (1.5), while preserving the base Δ. In addition, we can make Y_0 into a divisor with simple normal crossings after a further base change by the stable reduction theorem in [**KKMS**].

Let $\omega_{X/\Delta}$ and $\omega_{Y/\Delta}$ be the relative dualizing sheaves of X and Y over Δ, respectively. We have the family version of (1.6):

$$\omega_{Y/\Delta} = f^*\omega_{X/\Delta} + \sum a(E_k, X)E_k = \varphi^*\omega_{X/\Delta} + \sum \mu_l F_l \tag{2.4}$$

which plays a central role in our argument.

Since we have proved the theorem for X_t of general type, we assume that $X_t \subset \mathbb{P}^n$ is a CY complete intersection. In this case, we have $\sum a(E_k, X)E_k = \sum \mu_l F_l$ and (2.4) becomes

$$\begin{aligned} \omega_{Y/\Delta} &= f^*\omega_{X/\Delta} + \sum \mu(E)E = \sum \mu(E)E \\ &= \varphi^*\omega_{X/\Delta} + \sum \mu(E)E. \end{aligned} \tag{2.5}$$

where μ is the function defined by $\mu(E) = a(E, X)$ for all irreducible divisors $E \subset Y$ satisfying $f_*E = 0$. For convenience, we let $\mu(\widetilde{R}_i) = 0$ for $i = 1, 2$, where $\widetilde{R}_i \subset Y$ are the proper transforms of R_i under f.

Since X has at worse canonical singularities, we see that $\mu(E) \geq 0$ for all E. And we claim the following:

PROPOSITION 2.1. *For a component $E \subset Y_0$,*

$$\mu(E) > 0 \Rightarrow \varphi_*E = 0. \tag{2.6}$$

To see this, we apply the following simple observation.

LEMMA 2.2. *Let X/Δ and Y/Δ be two flat families of complex analytic varieties of the same dimension over the disk Δ. Suppose that X has reduced central fiber X_0 and Y is smooth. Let $\varphi : Y \to X$ be a proper surjective holomorphic map preserving the base. Let $S \subset Y_0$ be a reduced irreducible component of Y_0 with $\varphi_*S \neq 0$. Suppose that φ is ramified along S with ramification index $\nu > 1$. Then S has multiplicity ν in Y_0. In particular, Y_0 is nonreduced along S.*

PROOF. The problem is entirely local. Let $R = \varphi(S)$, q be a general point on S and $p = \varphi(q)$. Let U be an analytic open neighborhood of p in X and let V be the connected component of $\varphi^{-1}(U)$ that contains the point q. We may replace X and Y by U and V, respectively. Then we reduce it to the case that R and S are the only components of X_0 and Y_0, respectively, R and S are smooth and $\varphi : S \to R$ is an isomorphism, in which case the lemma follows easily. □

PROOF OF PROPOSITION 2.1. If $\varphi_*E \neq 0$, then φ is ramified along E with ramification index $\mu(E) + 1$ by (2.4) and Riemann-Hurwitz. This is impossible unless $\mu(E) = 0$ by the above lemma and the fact that Y_0 is of simple normal crossing. Consequently, (2.6) follows. □

We let $\mathcal{S} \subset Y_0$ be the union of components E with $\mu(E) = 0$, i.e.,

$$\mathcal{S} = \sum_{\mu(E)=0} E. \tag{2.7}$$

Then it follows from (2.6) that

$$\varphi_*\mathcal{S} = (\deg \phi)(R_1 + R_2). \tag{2.8}$$

Since X is smooth outside of D, $\mu(E) > 0$ if $f(E) \not\subset D$ and $f_*E = 0$. Consequently, we have $f(E) \subset D$ for every component $E \subset \mathcal{S}$ with $f_*E = 0$. Note that $\widetilde{R}_i \subset \mathcal{S}$.

Actually, we can arrive at a precise picture of $\mathcal{S}$ as follows.

2.3. Structure of $\mathcal{S}$. We may resolve the singularities of X by repeatedly blowing up X along R_1. By that we mean we first blow up X along R_1, then we blow up the proper transform of R_1 and so on. Let $\eta : X' \to X$ be the resulting resolution. We see that

$$X'_0 = P_0 \cup P_1 \cup ... \cup P_{m-1} \cup P_m \tag{2.9}$$

where P_0 and P_m are the proper transforms of R_1 and R_2, respectively, P_i are $\mathbb{P}^1$ bundles over D for $0 < i < m$ and $P_i \cap P_j \neq \emptyset$ if and only if $|i - j| \le 1$. Note that the relative dualizing sheaf of X'/Δ satisfies

$$\omega_{X'/\Delta} = \eta^* \omega_{X/\Delta} \tag{2.10}$$

and hence remains trivial.

We have the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, dashed, "\nu"'] \arrow[dr, "f"] & X \\ X' \arrow[r, "\eta"] & X \arrow[u, dashed, "\phi"'] \end{tikzcd} \tag{2.11}$$

where $\nu = \eta^{-1} \circ f$. By (2.10), $a(P_i, X) = 0$ for all i. Note that the discrepancy $a(P_i, X)$ does not depends on the birational model of X. So we necessarily have the proper transform $\nu_*^{-1}(P_i) \neq 0$. Otherwise, P_i would be the proper transform of an exceptional divisor of some birational regular map $Y' \to Y$ and hence $a(P_i, X) = a(P_i, Y) > 0$ since Y is smooth. Contradiction. Consequently, there exist $Q_i \subset Y_0$ which are the proper transforms of P_i under ν for $i = 0, 1, ..., m$.

On the other hand, for every component $Q \subset Y_0$ with $Q \notin \{Q_0, Q_1, ..., Q_m\}$, we have $\nu_* Q = 0$ and hence $a(Q, X) > 0$ by the same argument as above. Therefore, Q_i are the only components of Y_0 with $\mu(Q_i) = 0$. Consequently,

$$\mathcal{S} = Q_0 + Q_1 + ... + Q_{m-1} + Q_m, \tag{2.12}$$

$$f(Q_i) = D \text{ for } 0 < i < m \tag{2.13}$$

and

$$\varphi_* \mathcal{S} = \sum_{i=0}^{m} \varphi_* Q_i = (\deg \phi)(R_1 + R_2). \tag{2.14}$$

Obviously, Q_i is birational to $D \times \mathbb{P}^1$ for each $0 < i < m$.

Note that $Q_0 = \widetilde{R}_1$ and $Q_m = \widetilde{R}_2$.

Let T be a component of Y_0. Then by (2.5) and adjunction, we have

$$\omega_T = (\omega_{Y/\Delta} + T)\Big|_T = \sum \mu_E E\Big|_T - \sum_{\substack{E \neq T \\ E \subset Y_0}} E\Big|_T \tag{2.15}$$

where we write $\mu_E = \mu(E)$; here we use the fact that $T = -(Y_0 - T)$ as $\mathrm{Pic}(\Delta) = 0$. Hence

$$\sum_{E \not\subset Y_0} \mu_E E\Big|_T = \omega_T + \sum_{\substack{E \neq S \\ E \subset Y_0}} (1 + \mu_T - \mu_E) E\Big|_T \tag{2.16}$$

Suppose that $T = Q \subset \mathcal{S}$. Then (2.16) becomes

$$\sum_{\substack{E \neq Q \\ E \subset Y_0}} (1-\mu_E)E\Big|_Q = -\omega_Q + \sum_{E \not\subset Y_0} \mu_E E\Big|_Q \tag{2.17}$$

Suppose that $Q \neq Q_0, Q_m$. Let $F_p \cong \mathbb{P}^1$ be the fiber of $f : Q \to D$ over a general point $p \in D$. Clearly, we have

$$F_p \cdot \omega_Q = -2 \tag{2.18}$$

and hence

$$\sum_{\substack{E \neq Q \\ E \subset Y_0}} (1-\mu_E)E \cdot F_p \geq 2. \tag{2.19}$$

Therefore, each Q_j $(0 < j < m)$ meets at least two other Q_i $(0 \leq i \leq m)$ along rational sections of Q_j/D; and since Q_j is the proper transform of P_j, it cannot meet more than two among Q_i. So we see that Q_i form a "chain" in the same way as P_i do. More precisely, we have

- Q_i and Q_{i+1} meet transversely along a component D_i of $Q_i \cap Q_{i+1}$ satisfying $f(D_i) = D$ for $0 \leq i < m$;
- D_i, birational to D, are rational sections of $f : Q_i \to D$ for $1 \leq i \leq m-1$ and $f : Q_{i+1} \to D$ for $0 \leq i \leq m-2$.
- $f(Q_i \cap Q_j) \neq D$ for $|i-j| > 1$.

Next, we claim that

Proposition 2.3. *For each $0 \leq i \leq m$, we have*

$$\text{either } \varphi_* Q_i \neq 0 \text{ or } \varphi(Q_i) = D. \tag{2.20}$$

Namely, every Q_i either dominates one of R_1 and R_2 or is contracted onto D by φ. Since $\widetilde{R}_i$, being Fano, cannot be mapped onto D, which is a CY manifold, this implies that

$$\varphi_* Q_0 \neq 0 \text{ and } \varphi_* Q_m \neq 0. \tag{2.21}$$

So φ does not contract either $Q_0 = \widetilde{R}_1$ or $Q_m = \widetilde{R}_2$.

Note that if X were smooth, we would already have that $\varphi_* S \neq 0$ for all S with $\mu(S) = 0$ by (2.5) and Riemann-Hurwitz. However, things are a little more subtle here since X is singular.

Proof of Proposition 2.3. A natural thing to do is to resolve the indeterminacy of the rational map $\phi' = \eta^{-1} \circ \phi \circ \eta : X' \dashrightarrow X'$ with the diagram

$$\begin{tikzcd} Y' \arrow[d, "\nu'"'] \arrow[dr, "\varphi'"] & \\ X' \arrow[r, dashed, "\phi'"] \arrow[d, "\eta"'] & X' \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\phi"] & X \end{tikzcd} \tag{2.22}$$

where we can make Y'_0 into a divisor of simple normal crossing support [**H**]. Let $Q'_i \subset Y'$ be the proper transforms of P_i under ν'. Obviously, Q'_i are the proper transforms of Q_i under the birational map $\nu^{-1} \circ \nu' : Y' \dashrightarrow Y$. To show that (2.20) holds for Q_i, it suffices to show that the same thing holds for Q_i' when we map Y' to X via $\eta \circ \varphi'$.

We have

$$\omega_{Y'/\Delta} = (\nu')^* \eta^* \omega_{X/\Delta} + \sum a(E, X)E \tag{2.23}$$

where E runs through all exceptional divisors of $\eta \circ \nu'$. By (2.10), we see that $a(Q_i', X) = 0$ for all $0 \le i \le m$. Then we have $\varphi'_* Q_i' \neq 0$ by Riemann-Hurwitz and the fact that X' is smooth. So each Q_i' dominates some P_j via φ'. If Q_i' dominates P_0 or P_m, then Q_i' dominates R_1 or R_2 via $\eta \circ \varphi'$; if Q_i' dominates P_j for some $0 < j < m$, then $\eta(\varphi'(Q_i')) = D$. This proves (2.20). □

Corollary 2.4. *Let $0 \le i < j \le m$ be two integers with the properties that $\varphi_* Q_i \neq 0$, $\varphi_* Q_j \neq 0$ and $\varphi_* Q_k = 0$ for all $i < k < j$. Then*

$$\varphi_* D_k = D \text{ and } \varphi_{D_k} \circ f_{D_k}^{-1} = 1_D \tag{2.24}$$

for all $i \le k < j$, where $\varphi_{D_k} : D_k \to D$ and $f_{D_k} : D_k \to D$ are the restrictions of φ and f to D_k, respectively. In particular,

$$\deg \varphi_{D_k} = 1 \tag{2.25}$$

for all $0 \le k < m$, i.e., φ maps each D_k birationally onto D.

Remark 2.5. This is the only place where the induction hypothesis is needed: we need it to show that the dominant rational self map $\varphi_{D_k} \circ f_{D_k}^{-1} : D \dashrightarrow D$ is an identity, while $D \subset \mathbb{P}^n$ is a very general complete intersection CY manifold of lower dimension. Eventually, we will reduce Theorem 1.1 to the case of complete intersection $K3$ surfaces where Theorem 1.2 is required.

Proof of Corollary 2.4. If $\varphi_* Q_k = 0$, φ maps Q_k onto D by Proposition 2.3; since D is a CY manifold, φ must contract the fibers of $f : Q_k \to D$. Therefore, $\varphi(D_{k-1}) = \varphi(D_k) = D$. Hence $\varphi_{D_{k-1}} \circ f_{D_{k-1}}^{-1}$ and $\varphi_{D_k} \circ f_{D_k}^{-1}$ are dominant rational self maps of D; by induction hypothesis (2.1), they must be identity maps. Therefore, (2.24) follows easily when $j - i > 1$.

When $j - i = 1$, we will reduce it to the case $j - i > 1$ by applying a further base change. That is, for some $a > 1$, there is Y' with the commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\delta"] \arrow[d, "f'"'] & Y^{[a]} \arrow[dl, "f"] \\ X^{[a]} & \end{tikzcd} \tag{2.26}$$

and all the required properties, where $X^{[a]} = X \otimes \mathbb{C}[[\sqrt[a]{t}]]$ and $Y^{[a]} = Y \otimes \mathbb{C}[[\sqrt[a]{t}]]$. Correspondingly, $\mathcal{S}' = Q_0' + Q_1' + ... + Q_{ma}'$ with Q_{ka}' the proper transform of Q_k. Obviously, $\varphi'_* Q_{ia}' \neq 0$, $\varphi'_* Q_{ja}' \neq 0$ and $\varphi'_* Q_k' = 0$ for all $ia < k < ja$ and $\varphi' = \varphi \circ \delta$. It is also clear that (2.24) holds for (Y, f, φ) if it holds for (Y', f', φ'). Hence it again follows easily from Proposition 2.3 as $ja - ia > 1$. □

We might want to "get rid of" Q_i's that are contracted by φ. Let $\mathcal{X}$ be the variety obtained from X' by contracting all components P_i with $\varphi_* Q_i = 0$. After a possible further base change and stable reduction, we may assume that $f : Y \to X$ factors through $\mathcal{X}$; namely, we resolve the indeterminacy of $\theta^{-1} \circ f : Y \dashrightarrow \mathcal{X}$ and then apply stable reduction so that Y_0 remains a divisor with simple normal crossings, where θ is the map $\mathcal{X} \to X$ factored through by $\eta : X' \to X$.

Then we have the commutative diagram

$$\begin{tikzcd} X' \arrow[r] & \mathcal{X} \arrow[r, "\theta"] & X \arrow[r, dashed, "\phi"] \arrow[d] & X \arrow[d] \\ & Y \arrow[ul, dashed, "\nu"] \arrow[u, "\varepsilon"] \arrow[ur, "f"] \arrow[urr, "\varphi"'] \arrow[r] & W \arrow[r, dashed] & W \end{tikzcd} \tag{2.27}$$

The choice of $\mathcal{X}$ guarantees that

$$(\varphi \circ \varepsilon^{-1})_* \mathcal{X}_0 = (\deg \phi) X_0. \tag{2.28}$$

We continue to denote the components of $\mathcal{X}_0$ by P_i. So $\mathcal{X}_0$ is the unions of P_i for i satisfying $\varphi_* Q_i \neq 0$.

Using the diagram (2.27), we can prove the following:

PROPOSITION 2.6. *Let $Q = Q_i \subset \mathcal{S}$ be a component satisfying $\varphi_* Q \neq 0$ and let $\Gamma \subset Q$ be an irreducible subvariety of codimension one in Q such that $\varphi(\Gamma) \subset D$. Then either Γ is one of D_{i-1} and D_i or $\varphi_* \Gamma = 0$.*

PROOF. Otherwise, $\Gamma \neq D_{i-1}$, $\Gamma \neq D_i$ and $\varphi(\Gamma) = D$. So Γ has nonnegative Kodaira dimension and hence $\varepsilon_* \Gamma \neq 0$. Let $G = \varepsilon(\Gamma)$. Since $\varphi_* Q \neq 0$, $\varepsilon_* Q \neq 0$ by our construction of $\mathcal{X}$. So $P = \varepsilon(Q)$ is a component of $\mathcal{X}_0$. And since $\varepsilon : Q \to P$ is birational, $G \neq \varepsilon(D_{i-1}), \varepsilon(D_i)$. That is, $G \not\subset P'$ for all components $P' \neq P$ of $\mathcal{X}_0$. Therefore, $\varepsilon^{-1}(G)$ does not contain any component of $\mathcal{S}$.

Let Σ be the union of components of Y_0 that dominate G via ε and let q be a general point on Γ, $p = \varepsilon(q)$ and $J = \varepsilon^{-1}(p)$. By Zariski's main theorem, J is connected. If $\dim J = 0$, then $J = \{q\}$ and $\Sigma = \emptyset$.

Suppose that $\dim J = 1$. Since J is connected, Σ is connected. Let $J_1 \subset J$ be the component of J containing q and let $\Sigma_1 \subset \Sigma$ be the component of Y_0 containing J_1. Obviously, $\Gamma \subset \Sigma_1$ and hence $D \subset \varphi(\Sigma_1)$. And since $\Sigma_1 \not\subset \mathcal{S}$, $\varphi_* \Sigma_1 = 0$. Therefore, $\varphi(\Sigma_1) = D$. And since $J_1 \cong \mathbb{P}^1 \subset \Sigma_1$, $\varphi_* J_1 = 0$ and φ contracts Σ_1 onto D along the fibers of $\varepsilon : \Sigma_1 \to G$. Let $J_2 \neq J_1 \subset J$ be a component of J with $J_1 \cap J_2 \neq \emptyset$ and let $\Sigma_2 \subset \Sigma$ be the component of Y_0 containing J_2. Then Σ_2 meets Σ_1 along a rational multi-section of Σ_1/G. Therefore, $\varphi(\Sigma_2) = D$ and $\varphi_* J_2 = 0$ by the same argument as before. We can argue this way inductively that $\varphi_* J = 0$ and $\varphi(\Sigma^\circ) = D$ for every component $\Sigma^\circ \subset \Sigma$.

Let $r = \varphi(q)$ and K be the connected component of $\varphi^{-1}(r)$ containing the point q. Obviously, $J \subset K$. We claim that $J = K$. Otherwise, there is a component $K^\circ \subset K$ such that $K^\circ \not\subset J$ and $K^\circ \cap J \neq \emptyset$. Let T be a component of Y_0 containing K°. Obviously, $T \not\subset \Sigma$; otherwise, we necessarily have $\varepsilon(K^\circ) = p$ and $K^\circ \subset J$. Also we cannot have $T = Q$; otherwise, $K^\circ \subset Q$, $q \in K^\circ$ and $\varphi_* K^\circ = 0$, which is impossible for a general point $q \in \Gamma$. We cannot have $T = Q'$ for some $Q' \neq Q \subset \mathcal{S}$, either, since $p \in \varepsilon(K^\circ) \subset \varepsilon(T)$. Therefore, $T \not\subset \mathcal{S}$.

If $J = \{q\}$, then $q \in K^\circ$ since $K^\circ \cap J \neq \emptyset$; it follows that $\Gamma \subset T$ and $\varepsilon(T) = G$, which is impossible since $\Sigma = \emptyset$.

Otherwise, suppose that $\dim J = 1$. Again since $K^\circ \cap J \neq \emptyset$, $T \cap \Sigma \neq \emptyset$. If T and Σ meet along a rational multi-section of Σ/G, $\varepsilon(T) = G$, which is impossible as we have proved that $T \not\subset \Sigma$. Therefore, $T \cap \Sigma$ is contained in the fibers of Σ/G. And since $T \cap J \neq \emptyset$, $T \cap \Sigma$ contains a component of J, which is impossible for a general point $p \in G$. Therefore, $J = K$.

Let $U \subset X$ be an analytic open neighborhood of r in X and $V \subset Y$ be the connected component of $\varphi^{-1}(U)$ containing J. Since q is a general point of Γ,

$\varepsilon(J) = p \notin P_j$ for all $P_j \subset \mathcal{X}_0$ and $j \neq i$. Consequently, $V \cap \mathcal{S} = V \cap Q$. Therefore, $\varphi_* M = 0$ for all components M of V_0 with $M \neq V \cap Q$. So V cannot dominate U. Contradiction. □

2.4. Invariants α_i and β_i. Let Q_i be a component of $\mathcal{S}$. Suppose that Q_i dominates R_j via φ for some $1 \leq j \leq 2$. Let $\varphi_{Q_i} : Q_i \to R_j$ be the restriction of φ to Q_i. Proposition 2.6 tells us that D_{i-1} and D_i are the only components of $\varphi_{Q_i}^{-1}(D)$ that dominate D via φ.

Let α_i and β_{i-1} be the ramification indices of φ_{Q_i} along D_i and D_{i-1}, respectively, where we set $D_{-1} = D_m = \emptyset$, $\alpha_m = \beta_{-1} = 0$ and $\alpha_i = \beta_{i-1} = 0$ if $\varphi_* Q_i = 0$. Since $\varphi_{Q_i,*}\varphi_{Q_i}^* D = (\deg \varphi_{Q_i})D$,

$$\deg \varphi_{Q_i} = \alpha_i \deg \varphi_{D_i} + \beta_{i-1} \deg \varphi_{D_{i-1}} = \alpha_i + \beta_{i-1} \tag{2.29}$$

where $\deg \varphi_{D_{i-1}} = \deg \varphi_{D_i} = 1$ by (2.25).

Actually α_i and β_{i-1} are very explicitly determined as follows.

Proposition 2.7. *Let $0 \leq i < j \leq m$ be two integers with the properties that $\varphi_* Q_i \neq 0$, $\varphi_* Q_j \neq 0$ and $\varphi_* Q_k = 0$ for all $i < k < j$. Then*

$$\alpha_i = \beta_{j-1} = \frac{m}{j-i} \tag{2.30}$$

and $\varphi(Q_i) \neq \varphi(Q_j)$.

Proof. Let q be a general point on D_i and $U \subset X$ be an analytic open neighborhood of $\varphi(q)$ in X. Let $V \subset Y$ be the connected component of $\varphi^{-1}(U)$ containing q. Let $T \subset V_0$ be a component of $V \cap Y_0$ satisfying $T \not\subset \mathcal{S}$. Since q is a general point on D_i, it is easy to see that $\varepsilon(T) = P_i \cap P_j$. Indeed, T is a $\mathbb{P}^1$ bundle over $\varepsilon(T)$ and φ contracts the fibers of $T/\varepsilon(T)$ and maps T onto $D \cap U$. Therefore, $\varepsilon : V \to V' = \varepsilon(V)$ is proper and V' is open in $\mathcal{X}$. In addition, since T is contracted by φ along the fibers of $T/\varepsilon(T)$, the rational map $\varphi \circ \varepsilon^{-1} : V' \dashrightarrow U$ is actually regular.

Locally, at $\varepsilon(q) \in P_i \cap P_j$, $V' \subset \mathcal{X}$ is given by $xy = t^{j-i}$ in the polydisk $\Delta^N_{xyz...t}$ with $P_i = \{x = 0\}$ and $P_j = \{y = 0\}$. Similarly, at $\varphi(q) \in D$, $U \subset X$ is given by $xy = t^m$ in $\Delta^N_{xyz...t}$ with $R_1 = \{x = 0\}$ and $R_2 = \{y = 0\}$. The map $\varphi \circ \varepsilon^{-1}$ is regular and finite and sends $V' = \{xy = t^{j-i}\}$ onto $U = \{xy = t^m\}$ while preserving the base $\Delta = \{|t| < 1\}$. It has to be the map sending $(x, y, z, ..., t)$ to $(x^a, y^a, z, ..., t)$ or $(y^a, x^a, z, ..., t)$ with $a = m/(j-i)$. It follows that $\alpha_i = \beta_{j-1} = a$ and $\varphi(Q_i) \neq \varphi(Q_j)$. □

Corollary 2.8. *The following holds:*

- $\alpha_i \neq 1$ *and* $\beta_{i-1} \neq 1$ *for all* $0 < i < m$.
- *If* $\deg \varphi_{Q_0} = 1$ *or* $\deg \varphi_{Q_m} = 1$, *then* $\deg \varphi = 1$.

Proof. The first statement follows directly from Proposition 2.7.

If $\deg \varphi_{Q_0} = 1$, then $\alpha_0 = \deg \varphi_{D_0} = 1$. By Proposition 2.7, we must have $\beta_{m-1} = 1$ and $\varphi_* Q_k = 0$ for all $0 < k < m$. Hence $\deg \varphi_{D_m} = \deg \varphi_{D_0} = 1$ and $\deg \varphi_{Q_m} = 1$ by (2.29). It follows that $\deg \varphi = 1$.

Similarly, we can show that $\deg \varphi = 1$ if $\deg \varphi_{Q_m} = 1$. □

Corollary 2.9. *The following are equivalent:*

- $\alpha_0 = 1$.
- $\beta_{m-1} = 1$.

- $\alpha_i = 1$ *for some* $0 \le i \le m-1$.
- $\beta_j = 1$ *for some* $0 \le j \le m-1$.
- $\varphi_* Q_i = 0$ *for all* $1 \le i \le m-1$.
- $\deg \phi_{R_1} = \deg \phi$, *where* ϕ_{R_1} *is the restriction of* ϕ *to* R_1.
- $\deg \phi_{R_2} = \deg \phi$, *where* ϕ_{R_2} *is the restriction of* ϕ *to* R_2.
- $\deg \phi = 1$.

In particular, at least one of Q_i $(i \ne 0, m)$ *is not contracted by* φ *if* $\deg \phi > 1$.

PROOF. This is more or less trivial. □

2.5. The case of hypersurfaces. Suppose that $\deg \phi > 1$. Then there exists $1 \le i \le m-1$ such that $Q = Q_i$ dominates R_j via φ for some $1 \le j \le 2$ by Corollary 2.9. To make our life a little easier, we can set $j = 2$ by replacing ϕ with ϕ^2 if necessary and applying the following observation:

PROPOSITION 2.10. *Suppose that* $\deg \phi > 1$. *For each* $j = 1, 2$, *there exists* $1 \le i \le m-1$ *such that* $\varphi(Q_i) = R_j$ *if and only if*

$$\phi_*(R_1 + R_2) \ne (\deg \phi) R_j. \tag{2.31}$$

PROOF. It is easy to see by (2.14) that (2.31) holds if $\varphi(Q_i) = R_j$ for some $1 \le i \le m-1$.

On the other hand, suppose that (2.31) holds. By Proposition 2.7, $\varphi(Q_i) = R_j$ for some $1 \le i \le m-1$ if $\phi(R_1) \ne \phi(R_2)$ or $\phi(R_1) = \phi(R_2) \ne R_j$.

If $\phi(R_1) = \phi(R_2) = R_j$, then it follows from (2.14) again that $\varphi(Q_i) = R_j$ for some $1 \le i \le m-1$. □

Suppose that (2.31) fails for $j = 2$. That is,

$$\phi_*(R_1 + R_2) = (\deg \phi) R_2. \tag{2.32}$$

Let $\phi^2 = \phi \circ \phi$. It is easy to see that

$$(\phi^2)_*(R_1 + R_2) = (\deg \phi)(\deg \phi_{R_2}) R_2 \ne (\deg \phi^2) R_2 \tag{2.33}$$

since $\deg \phi_{R_2} \ne \deg \phi$. So if (2.31) fails, it will hold for ϕ^2. In conclusion, by replacing ϕ by ϕ^2 if necessary, we can always find $1 \le i \le m-1$ such that $Q = Q_i$ dominates R_2 via φ.

So far we have all the geometric facts about $\varphi_Q = \varphi_{Q_i} : Q \to R_2$ that we need to complete our proof:

A1. let $F_p = f_Q^{-1}(p)$ be the fiber of $f_Q : Q \to D$ over a general point $p \in D$, $p_{i-1} = F_p \cap D_{i-1}$ and $p_i = F_p \cap D_i$; then

$$\varphi(p_{i-1}) = \varphi(p_i) = p \tag{2.34}$$

due to (2.24);

A2. D_{i-1} and D_i are the only components of $\varphi_Q^{-1}(D)$ that dominates D via φ; consequently, there is a subvariety $Z \subset D$ of codimension ≥ 1, independent of p, such that

$$\varphi_{Q,*} F_p \cdot D = \alpha_i \varphi(p_i) + \beta_{i-1} \varphi(p_{i-1}) + \Sigma = (\alpha_i + \beta_{i-1}) p + \Sigma \tag{2.35}$$

with Σ supported on Z, where $\varphi_{Q,*}$ is the push-forward induced by $\varphi_Q : Q \to R_2$.

In summary, for $p \in D$ general, i.e. p outside of a proper closed subvariety $Z'' \subset D$, one has

$$p = \frac{1}{\alpha_i + \beta_{i-1}}(\varphi_{Q,*}F_p \cdot D) - \frac{1}{\alpha_i + \beta_{i-1}}\Sigma \tag{2.36}$$

in $\mathrm{CH}_0(D, \mathbb{Q})$ by (2.35), where Σ lies in the image of $\mathrm{CH}_0(Z, \mathbb{Q}) \to \mathrm{CH}_0(D, \mathbb{Q})$.

If $\varphi_{Q_i,*}F_p$ lies in the same class of $\mathrm{CH}_1(R_2, \mathbb{Q})$ for all p, which happens when

$$\mathrm{CH}_1(R_2, \mathbb{Q}) = H_2(R_2, \mathbb{Q}) = \mathbb{Q}, \tag{2.37}$$

then $\varphi_{Q,*}F_p$ is a constant class in $\mathrm{CH}_1(R_2, \mathbb{Q})$ for all p and hence

$$c_0 = \frac{1}{\alpha_i + \beta_{i-1}}(\varphi_{Q,*}F_p \cdot D) \tag{2.38}$$

is a constant class in $\mathrm{CH}_0(D, \mathbb{Q})$. We can certainly choose Z'' such that c_0 lies in the image of $\mathrm{CH}_0(Z'', \mathbb{Q}) \to \mathrm{CH}_0(D, \mathbb{Q})$. It follows that the closed immersion $i : Z' = Z \cup Z'' \hookrightarrow D$ induces a surjection

$$i_* : \mathrm{CH}_0(Z', \mathbb{Q}) \to \mathrm{CH}_0(D, \mathbb{Q}) \tag{2.39}$$

with $\operatorname{codim}_D Z' \geq 1$. This cannot happen for a CY manifold D by Roĭtman's generalization of Mumford's classical results on Chow rings of zero-dimensional cycles on surfaces [**R1**] (see also [**Mu**], [**R2**] and [**B-S**]):

Theorem 2.11 (Mumford, Roĭtman, Bloch-Srinivas). *Let X be a smooth projective variety of dimension n. If there exists $i : Y \hookrightarrow X$ such that $\dim Y < n$ and*

$$i_* : \mathrm{CH}_0(Y, \mathbb{Q}) \to \mathrm{CH}_0(X, \mathbb{Q}) \tag{2.40}$$

is surjective, then $h^{n,0}(X) = 0$.

This gives us a quick proof for hypersurfaces, in particular, quintic 3-folds in $\mathbb{P}^4$, since for hypersurfaces of degree d_1 in $\mathbb{P}^n$, we can do the splitting $d_1' = d_1 - 1$ and $d_1'' = 1$. That is, $R_2 \cong \mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^n$ and hence (2.37) holds.

2.6. Completion of the proof. For complete intersections, we cannot guarantee (2.37). Indeed, (2.37) is only known for $d_2, ..., d_r$ sufficiently small (we can always take $d_1'' = 1$), in which case $\mathrm{CH}_1(R)$ is generated by the lines on $R = R_2$.

However, we can get around the problem by taking advantage of the fact that $D \subset R$ is a general member of the complete linear series $|\mathcal{O}_R(D)|$. Let $\mathcal{D} \subset |\mathcal{O}_R(D)| \times R$ be the incidence correspondence

$$\mathcal{D} = \{(D', p) : D' \in |\mathcal{O}_R(D)|, p \in D'\}. \tag{2.41}$$

We observe that R is Fano and hence rationally connected and the projection $\mathcal{D} \to R$ gives $\mathcal{D}$ the structure of a $\mathbb{P}^{N-1}$-bundle over R, since $|\mathcal{O}_R(D)| = \mathbb{P}^N$ is base point free. Therefore, $\mathcal{D}$ is rationally connected and hence

$$\mathrm{CH}_0(\mathcal{D}) = \mathbb{Z}. \tag{2.42}$$

Since we can find $Q_{D'}$ dominating R with the properties A1-A2 for a general member $D' \in |\mathcal{O}_R(D)|$, there exist a dominant and generically finite map $\rho : U \to$

$|\mathcal{O}_R(D)|$ and a smooth projective variety $\mathcal{Q}$ with the commutative diagram

$$
\begin{tikzcd}
 & \mathcal{Q} \arrow[r, "\varphi"] \arrow[d, "f"] & R \\
(\mathcal{D} \times_\rho U) \times \mathbb{P}^1 \arrow[r] \arrow[ur, dashed, "\sim"] & \mathcal{D} \times_\rho U &
\end{tikzcd}
\tag{2.43}
$$

where φ maps a general fiber Q of $\pi_U \circ f : \mathcal{Q} \to \mathcal{D} \times_\rho U \to U$ to R with the properties A1-A2 and the existence of a birational map between $(\mathcal{D} \times_\rho U) \times \mathbb{P}^1$ and $\mathcal{Q}$ is due to the fact that the fiber of $\pi_U \circ f$ over a general point $u \in U$ is a $\mathbb{P}^1$-bundle Q_u over $D_u := \pi_U^{-1}(u) \in |\mathcal{O}_R(D)|$. Here we use the notations $\pi_{\mathcal{D}}$ and π_U for the projections $\mathcal{D} \times_\rho U \to \mathcal{D}$ and $\mathcal{D} \times_\rho U \to U$, respectively.

Let $\overline{p} = (D', p)$ be a general point on $\mathcal{D}$. The pre-image $(\pi_{\mathcal{D}} \circ f)^{-1}(\overline{p})$ consists of N copies of $\mathbb{P}^1$, say, $\Gamma_1, \Gamma_2, ..., \Gamma_N$, where $N = \deg \rho$. That is,

$$(\pi_{\mathcal{D}} \circ f)^* \overline{p} = \Gamma_1 + \Gamma_2 + ... + \Gamma_N. \tag{2.44}$$

For each Γ_k, we have

$$\varphi_* \Gamma_k \cdot D' = (\alpha_i + \beta_{i-1}) p + \Sigma_k \tag{2.45}$$

with Σ_k supported on Z, where Z is a subvariety of D' of codimension ≥ 1, independent of p. Therefore,

$$\varphi_*((\pi_{\mathcal{D}} \circ f)^* \overline{p}) \cdot D' = N(\alpha_i + \beta_{i-1}) p + \Sigma \tag{2.46}$$

with Σ supported on Z. By (2.42), we still have a surjection (2.39) on D'. Since the above argument works for a general member $D' \in |\mathcal{O}_R(D)|$, it holds for D and we are done.

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Clay Mathematics Proceedings
Volume **18**, 2013

Variations on Nagata's Conjecture

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Abstract. Here we discuss some variations of Nagata's conjecture on linear systems of plane curves. The most relevant concerns non-effectivity (hence nefness) of certain rays, which we call *good rays*, in the Mori cone of the blow-up X_n of the plane at $n \geq 10$ general points. Nagata's original result was the existence of a good ray for X_n with $n \geq 16$ a square number. Using degenerations, we give examples of good rays for X_n for all $n \geq 10$. As with Nagata's original result, this implies the existence of counterexamples to Hilbert's XIV problem. Finally we show that Nagata's conjecture for $n \leq 89$ combined with a stronger conjecture for $n = 10$ implies Nagata's conjecture for $n \geq 90$.

Contents

Introduction

A fundamental problem in algebraic geometry is understanding which divisor classes on a given variety have effective representatives. One of the simplest contexts for this problem is that of curves in the plane, and here already it is of substantial interest, and not only in algebraic geometry. For example, given n sufficiently general points $x_1, \ldots, x_n$ in the complex plane $\mathbb{C}^2$, nonnegative integers $m_1, \ldots, m_n$ and an integer d, when is there a polynomial $f \in \mathbb{C}[x, y]$ of degree d vanishing to order at least m_i at each point x_i? Although there is a conjectural answer to this

2010 *Mathematics Subject Classification.* Primary 14C20, secondary 13E15, 14H50, 14D06.

The first author was partially supported by MIUR and INdAM. Harbourne was partially supported by the National Security Agency under Grant/Cooperative agreement H98230-11-1-0139, and thus the United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice. Roé was partially supported by the Spanish Ministerio de Economía y Competitividad grant MTM 2009-10359.

question (the (SHGH) Conjecture; see Conjecture 1.1 and also [**19**]), the conjecture remains open after more then a half century of attention by many researchers.

This problem is closely related to the question of what self-intersections occur for reduced irreducible curves on the surface X_n obtained by blowing up the projective plane at the n points x_i. Blowing up the points introduces rational curves (infinitely many, in fact, when $n > 8$) of self-intersection -1. Each curve C on X_n corresponds to a projective plane curve D_C of some degree d vanishing to orders m_i at the points x_i; the self-intersection C^2 is $d^2 - m_1^2 - \cdots - m_n^2$. An example of a curve D_C corresponding to a curve C of self-intersection -1 on X_n is the line through two of the points x_i, say x_1 and x_2; in this case, $d = 1$, $m_1 = m_2 = 1$ and $m_i = 0$ for $i > 2$, so we have $d^2 - m_1^2 - \cdots - m_n^2 = -1$. According to the (SHGH) Conjecture, these (-1)-curves should be the only reduced irreducible curves of negative self-intersection (see Conjecture 2.3) but proving that there are no others turns out to be itself very hard and is still open.

One could hope that a weaker version of this problem might satisfy the criterion Hilbert stated in his address to the International Congress in Paris in 1900, of being difficult enough "to entice us, yet not completely inaccessible so as not to mock our efforts" ("uns reizt, und dennoch nicht völlig unzugänglich, damit es unserer Anstrengung nicht spotte"). In fact, Nagata, in connection with his negative solution of the 14-th of the problems Hilbert posed in his address, made such a conjecture, Conjecture 2.1. It is weaker than Conjecture 2.3 yet still open for every non-square $n \geq 10$. Nagata's conjecture does not rule out the occurrence of curves of self-intersection less than -1, but it does rule out the worst of them. In particular, Nagata's conjecture asserts that if there is a curve of degree d with $n \geq 10$ sufficiently general points of multiplicities $m_1, \ldots, m_m$, then $d^2 \geq nm^2$ must hold, where $m = (m_1 + \cdots + m_n)/n$. Thus perhaps there are curves with $d^2 - m_1^2 - \cdots - m_n^2 < 0$, such as the (-1)-curves mentioned above, but $d^2 - m_1^2 - \cdots - m_n^2$ is (conjecturally) only as negative as is allowed by the condition that after averaging the multiplicities m_i for $n \geq 10$ one must have $d^2 - nm^2 \geq 0$.

What our results here show is that in order to prove Nagata's Conjecture for all $n \geq 10$ it is enough to prove it only for $n < 90$, if one can verify a slightly stronger conjecture for $n = 10$. But what we hope is to persuade the reader that it satisfies Hilbert's criteria of being both enticing and challenging, and at least not *completely* inaccessible!

Acknowledgements: The authors thank B. Totaro for useful comments and the referee for his careful reading of the paper.

1. Linear systems on general blow-ups of the plane

1.1. Generalities. Fix n points $x_1, \ldots, x_n$ in the complex projective plane $\mathbb{P}^2$ (which will be often assumed to be in *very general* position and called *general*) and nonnegative integers $d, m_1, \ldots, m_n$. We denote by $\mathcal{L}(d; m_1, \ldots, m_n)$, or simply by $(d; m_1, \ldots, m_n)$, the linear system of plane curves of *degree* d having *multiplicity* at least m_i at the *base point* x_i, for $1 \leq i \leq n$. Often we will use exponents to denote repetition of multiplicities. Sometimes, we may simply denote $\mathcal{L}(d; m_1, \ldots, m_n)$ by $\mathcal{L}$.

The linear system $(d; m_1, \ldots, m_n)$ is the projective space corresponding to a vector subspace $\mathfrak{a}_d \subset H^0(\mathcal{O}_{\mathbb{P}^2}(d))$, and $\mathfrak{a} = \oplus_{i=0}^n \mathfrak{a}_d$ is the homogeneous ideal, in the coordinate ring $S = \oplus_{i=0}^{n} H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ of $\mathbb{P}^2$, of the scheme $\mathrm{Proj}(S/\mathfrak{a})$ denoted by $\sum_{i=1}^{n} m_i x_i$, usually called a *fat points scheme*.

The *expected dimension* of $(d; m_1, \ldots, m_n)$ is defined to be

$$e(d; m_1 \ldots, m_n) = max\,\{-1, v(d; m_1 \ldots, m_n)\}$$

where

$$v(d; m_1 \ldots, m_n) = \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$$

is the *virtual dimension* of the system. The system is said to be *special* if

$$h(d; m_1 \ldots, m_n) > e(d; m_1 \ldots, m_n)$$

where $h(d; m_1 \ldots, m_n) := \dim(\mathcal{L}(d; m_1, \ldots, m_n))$ is its *true dimension*. In particular, an empty linear system is never special.

We record the following definitions:

- $(d; m_1, \ldots, m_n)$ is *asymptotically non-special* (ANS) if there is an integer y such that for all nonnegative integers $x \geq y$ the system $(xd; xm_1, \ldots, xm_n)$ is non-special;
- the *multiplicities vector* $(m_1 \ldots, m_n)$ of nonnegative integers is *stably non-special* (SNS) if the linear system $(d; xm_1, \ldots, xm_n)$ is non-special for all positive integers d, x.

Consider the *Cremona–Kantor (CK) group* $\mathcal{G}_n$ generated by quadratic transformations based at n general points $x_1, \ldots, x_n$ of the plane and by permutations of these points (see [**12**]). The group $\mathcal{G}_n$ acts on the set of linear systems of the type $(d; m_1, \ldots, m_n)$. All systems in the same *(CK)-orbit* (or *(CK)-equivalent*) have the same expected, virtual and true dimension. A linear system $(d; m_1, \ldots, m_n)$ is *Cremona reduced* if it has minimal degree in its (CK)-orbit. We note that (CK)-orbits need not contain a Cremona reduced element if they are orbits of empty linear systems, but orbits of non-empty linear systems always contain Cremona reduced members. It is a classical result, which goes back to Max Noether (see, e.g., [**3**]), that a non–empty system $(d; m_1, \ldots, m_n)$ with general base points is Cremona reduced if and only if the sum of any pair or triple of distinct multiplicities does not exceed d. In this case the system is called *standard* and we may assume $m_1 \geq \ldots \geq m_n$.

1.2. General rational surfaces. Consider the blow-up $f : X_n \to \mathbb{P}^2$ of the plane at $x_1, \ldots, x_n$, which we call a *general rational surface*. The *Picard group* $\mathrm{Pic}(X_n)$ is the abelian group freely generated by:

- the *line class*, i.e., total transform $L = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$;
- the classes of the *exceptional divisors* $E_1, \ldots, E_n$ which are contracted to $x_1, \ldots, x_n$.

More generally we may work in the $\mathbb{R}$-vector space $N_1(X_n) = \mathrm{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{R}$.

We will often abuse notation, identifying divisors on X_n with the corresponding line bundles and their classes in $\mathrm{Pic}(X_n)$, thus passing from additive to multiplicative notation. We will use the same notation for a planar linear system $\mathcal{L} = (d; m_1, \ldots, m_n)$ and its *proper transform*

$$\mathcal{L} = dL - \sum_{i=1}^{n} m_i E_i$$

on X_n. With this convention the integers $d, m_1, \ldots, m_n$ are the components with respect to the ordered basis $(L, -E_1, \ldots, -E_n)$ of $N_1(X_n)$. The *canonical divisor* on X_n is $K_n = (-3; -1^n)$ (denoted by K if there is no danger of confusion) and, if $(d; m_1, \ldots, m_n)$ is an ample line bundle on X_n, then $(d; m_1, \ldots, m_n)$ is ANS.

Using the intersection form on $N_1(X_n)$, one can *intersect* and *self-intersect* linear systems $(d; m_1, \ldots, m_n)$. Given a linear system $\mathcal{L} = (d; m_1, \ldots, m_n)$, one has

$$v(d; m_1, \ldots, m_n) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K}{2}$$

and, if $d \geq 0$, Riemann-Roch's theorem says that

$$\mathcal{L} \text{ is special if and only if } h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) > 0. \tag{1.1}$$

1.3. Special effects. Though (1.1) says that speciality is a cohomological property, the only known reason for speciality comes from geometry in the following way.

Assume we have an *effective* linear system $\mathcal{L}$, i.e. $h^0(\mathcal{L}) > 0$, and suppose there is an irreducible curve C of arithmetic genus g on X_n such that:

- $h^2(\mathcal{L}(-C)) = 0$, e.g. $h^0(\mathcal{L}(-C)) > 0$;
- $h^1(\mathcal{L}_{|C}) > 0$, e.g. $\mathcal{L} \cdot C \leq g - 1$ if $g \geq 2$ and $\mathcal{L} \cdot C \leq g - 2$ if $g \leq 1$.

Then the *restriction exact sequence*

$$0 \to \mathcal{L}(-C) \to \mathcal{L} \to \mathcal{L}_{|C} \to 0$$

implies that

$$h^1(\mathcal{L}) \geq h^1(\mathcal{L}_{|C}) > 0$$

hence $\mathcal{L}$ is special. In this case C is called a *special effect curve* for $\mathcal{L}$ (see [**2**]). For example, C is a special effect curve for $\mathcal{L}$ if $g = 0$ and $\mathcal{L} \cdot C \leq -2$, in which case C sits in the base locus of $\mathcal{L}$. But then $C^2 < 0$ and therefore, if $x_1, \ldots, x_n$ are general points, one has $C^2 = -1$, i.e. C is a (-1)*–curve* (see [**10**]). In this case $\mathcal{L}$ is said to be (-1)*-special* and contains $-(\mathcal{L} \cdot C)C$ in its fixed part. Thus, any (-1)–curve C gives rise to infinitely many (-1)–special linear systems, e.g. all systems of the type $|nC|$, with n a positive integer. Note that for $1 \leq n \leq 8$ there are only finitely many (-1)–curves on X_n, whereas if $n \geq 9$ there are infinitely many (-1)–curves on X_n.

1.4. The Segre–Harbourne–Gimigliano–Hirschowitz Conjecture. The only known examples of special linear systems on a general rational surface X_n are (-1)-special. This motivates the conjecture (see [**27, 18, 15, 20, 7**], quoted in chronological order):

CONJECTURE 1.1 (Segre–Harbourne–Gimigliano–Hirschowitz (SHGH)). *A linear system $\mathcal{L}$ on X_n is special if and only if it is (-1)-special.*

It goes back to Castelnuovo that Conjecture 1.1 holds if $n \leq 9$ (see [**4**]; more recent treatments can be found in [**24, 15, 18, 17**]). The general conjecture remains open.

Since standard linear systems are not (-1)-special (see [**17, 20**]), an equivalent formulation of the (SHGH) conjecture is: *a standard system of plane curves with general base points is not special.*

Recall that a linear system $\mathcal{L}$ is *nef* if $\mathcal{L} \cdot \mathcal{L}' \geq 0$ for all effective $\mathcal{L}'$. Since the (CK)-orbit of a nef divisor always contains a Cremona reduced element and hence a standard element (see [**17**]), the (SHGH) conjecture implies the following conjecture, which we regard as a weak form of (SHGH), a point-of-view justified by Proposition 2.6(ii) below.

Conjecture 1.2. *A* nef *linear system $\mathcal{L}$ on X_n is not special.*

This conjecture is also open in general. The notion of nefness extends to elements in $N_1(X_n)$ and $\xi \in N_1(X_n)$ is nef if and only if $\lambda\xi$ is nef for all $\lambda > 0$. Given a nonzero $\xi \in N_1(X_n)$, the set $[\xi] = \{\lambda\xi : \lambda > 0\}$ is called the *ray* generated by ξ. Thus it makes sense to talk of *nef rays*.

2. Hilbert's 14-th problem and Nagata's conjecture

2.1. Hilbert's 14-th problem. Let k be a field, let $t_1, \dots, t_n$ be indeterminates over k and let $\mathbb{K}$ be an *intermediate field* between k and $k(t_1, \dots, t_n)$, i.e.

$$k \subseteq \mathbb{K} \subseteq k(t_1, \dots, t_n).$$

Hilbert's 14-th problem asks: *is* $\mathbb{K} \cap k[t_1, \dots, t_n]$ *a finitely generated k-algebra?* Hilbert had in mind the following situation coming from invariant theory. Let G be a subgroup of the *affine group* or of $\mathrm{GL}(n, k)$. Then G acts as a set of automorphisms of the k-algebra $k[t_1, \dots, t_n]$, hence on $k(t_1, \dots, t_n)$, and we let $\mathbb{K} = k(t_1, \dots, t_n)^G$ be the field of G-invariant elements. Then the question is: *is*

$$k[t_1, \dots, t_n]^G = \mathbb{K} \cap k[t_1, \dots, t_n]$$

a finitely generated k-algebra?

In [**24**], Nagata provided counterexamples to the latter formulation of Hilibert's problem. To do this he used the nefness of a certain line bundle of the form $(d; m_1, \dots, m_n)$ (see §2.2 below).

Hilbert's problem has trivially an affirmative answer in the case $n = 1$. The answer is also affirmative for $n = 2$, as proved by Zariski in [**30**]. Nagata's *minimal counterexample* has $n = 32$ and $\dim(G) = 13$. Several other counteraxamples have been given by various authors, too long a story to be reported on here (see, e.g., [**14, 23, 29**]).

2.2. Nagata's Conjecture. In his work on Hilbert's 14-th problem, Nagata made the following conjecture:

Conjecture 2.1 (Nagata's Conjecture (N)). *If $n \geq 9$ and $(d; m_1, \dots, m_n)$ is an effective linear system on X_n, then*

$$\sqrt{n} \cdot d \geq m_1 + \cdots + m_n \tag{2.1}$$

and strict inequality holds if $n \geq 10$.

Using a degeneration argument, Nagata proved the following result, on which his counterexamples to Hilbert's 14-th problem rely:

Proposition 2.2. *(N) holds if $n = k^2$, with $k \geq 3$.*

Taking this into account, it is clear that (N) is equivalent to saying that the *Nagata class* $N_n = (\sqrt{n}, 1^n)$, or the *Nagata ray* $\nu_n = [\sqrt{n}, 1^n]$ it generates, is nef if $n \geq 9$. Note that it suffices to verify (N) for linear systems containing prime divisors.

Let C be an irreducible curve of genus g on X_n. If (SHGH) holds, then the virtual dimension of $\mathcal{O}_{X_n}(C)$ is nonnegative, which reads

$$C^2 \geq g - 1. \tag{2.2}$$

In particular, (SHGH) implies the following conjecture:

Conjecture 2.3. *If C is a prime divisor on X_n, then $C^2 \geq -1$ with $g = 0$ when $C^2 = -1$.*

Lemma 2.4. *Conjecture 2.3 implies (N).*

Proof. Suppose C is a prime divisor in $(d; m_1, \ldots, m_n)$ violating (N), i.e., $n \geq 9$ and $\sqrt{n} \cdot d < m_1 + \cdots + m_n$. Then, for $m = (m_1 + \cdots + m_n)/n$ and using Cauchy–Schwartz inequality $m^2 \leq (m_1^2 + \cdots + m_n^2)/n$, we have

$$d^2 < n\frac{(m_1 + \cdots + m_n)^2}{n^2} = nm^2 \leq m_1^2 + \cdots + m_n^2.$$

Thus $C^2 < 0$ and therefore $C^2 = -1$ and C has genus 0 by Conjecture 2.3. But now we have the contradiction $1 = -C \cdot K_n = 3d - (m_1 + \cdots + m_n) \leq \sqrt{n}d - (m_1 + \cdots + m_n) < 0$. □

Hence Conjecture 2.3 can be regarded as a strong form of (N). The aforementioned result in [**10**] (that $C^2 \geq -1$ if $C \subset X_n$ is irreducible and rational) yields (2.2) if $g = 0$. If $g = 1$ and $C^2 = 0$, then C is (CK)-equivalent to $(3; 1^9, 0^{n-9})$ (see [**7**]). Thus the following conjecture (see [**16**, Conjecture 3.6]) is at least plausible.

Conjecture 2.5 (Strong Nagata's Conjecture (SN)). *If C is an irreducible curve of genus $g > 0$ on X_n, then $C^2 > 0$ unless $n \geq 9$, $g = 1$ and C is (CK)–equivalent to $(3; 1^9, 0^{n-9})$, in which case $C^2 = 0$.*

Proposition 2.6. *We have the following:*

(i) *(SHGH) implies (SN) (in particular (SN) holds for $n \leq 9$);*
(ii) *(SHGH) holds if and only if (SN) and Conjecture 1.2 both hold;*
(iii) *(SN) implies (N).*

Proof. Part (i), hence also the forward implication of (ii), is clear. As for the reverse implication in (ii), note thet (SN) implies Conjecture 2.3, and Conjecture 2.3 together with Conjecture 1.2 is the formulation of (SHGH) given in [**18**]. Finally we prove part (iii), hence we assume $n \geq 9$. Let C be an irreducible curve in $(d; m_1, \ldots, m_n)$ on X_n. If $C^2 \geq 0$ then $d^2 \geq m_1^2 + \cdots + m_n^2$. By the Cauchy–Schwartz inequality this implies (2.1). Equality holds if and only if $m_1 = \cdots = m_n = m$ and $d = m\sqrt{n}$, hence n is a square, which is only possible if $n = 9$ by Proposition 2.2. If $C^2 < 0$, then $g = 0$ and $C^2 = -1$, so that

$$C \cdot N_n = C \cdot (N_n - K_n) - C \cdot K_n = (\sqrt{n} - 3)C \cdot L + 1 \geq 1.$$

□

3. The Mori cone viewpoint

3.1. Generalities. A class $\xi \in N_1(X_n)$ is *integral* [resp. *rational*] if it sits in $\mathrm{Pic}(X_n)$ [resp. in $\mathrm{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{Q}$]. A ray in $N_1(X_n)$ is *rational* if it is generated by a rational class. A rational ray in $N_1(X_n)$ is *effective* if it is generated by an effective class.

The *Mori cone* $\overline{\mathrm{NE}}(X_n)$ is the closure in $N_1(X_n)$ of the set $\mathrm{NE}(X_n)$ of all effective rays, and it is the dual of the *nef cone* $\mathrm{Nef}(X_n)$ which is the closed cone described by all nef rays.

A (-1)-*ray* in $N_1(X_n)$ is a ray generated by a (-1)-*curve*, i.e., a smooth, irreducible, rational curve C with $C^2 = -1$ (hence $C \cdot K_n = -1$).

Mori's Cone Theorem says that

$$\overline{\mathrm{NE}}(X_n) = \overline{\mathrm{NE}}(X_n)^{\succcurlyeq} + R_n$$

where $\overline{\mathrm{NE}}(X_n)^{\succcurlyeq}$ [resp. $\overline{\mathrm{NE}}(X_n)^{\preccurlyeq}$] is the subset of $\overline{\mathrm{NE}}(X_n)$ described by rays generated by nonzero classes ξ such that $\xi \cdot K_n \geq 0$ [resp. $\xi \cdot K_n \leq 0$] and

$$R_n \;=\; \sum_{\rho \text{ a } (-1)-\text{ray}} \rho \;\subseteq\; \overline{\mathrm{NE}}(X_n)^{\preccurlyeq}.$$

We will denote by $\overline{\mathrm{NE}}(X_n)^{\succ}$ [resp. $\overline{\mathrm{NE}}(X_n)^{\prec}$] the interior of $\overline{\mathrm{NE}}(X_n)^{\succcurlyeq}$ [resp. $\overline{\mathrm{NE}}(X_n)^{\preccurlyeq}$].

Concerning (SHGH), the situation is well understood for classes in R_n, in view of this result (see [**22**]):

THEOREM 3.1. *A nef linear system on X_n with class in R_n is non-special.*

The *nonnegative cone* $\mathcal{Q}_n$ in $N_1(X_n)$ is the cone of classes ξ such that $\xi \cdot L \geq 0$ and $\xi^2 \geq 0$, whose boundary, which is a quadric cone, we denote by $\partial\mathcal{Q}_n$. By Riemann-Roch's theorem one has $\mathcal{Q}_n \subseteq \overline{\mathrm{NE}}(X_n)$. We will use the obvious notation $\mathcal{Q}_n^{\succcurlyeq}, \mathcal{Q}_n^{\preccurlyeq}, \mathcal{Q}_n^{\succ}, \mathcal{Q}_n^{\prec}$ to denote the intersection of $\mathcal{Q}_n$ with $\overline{\mathrm{NE}}(X_n)^{\succcurlyeq}$ etc., and similarly for $\partial\mathcal{Q}_n$.

The situation is quite different according to the values of n:

(i) The *Del Pezzo case* $n \leq 8$. Here $-K_n$ is ample, hence $\overline{\mathrm{NE}}(X_n) = R_n$. There are only finitely many (-1)-curves on X_n, hence $\overline{\mathrm{NE}}(X_n)$ is polyhedral and $\overline{\mathrm{NE}}(X_n) \subseteq \overline{\mathrm{NE}}(X_n)^{\prec}$. If $\kappa_n = [3, 1^n]$ is the *anticanonical ray*, then κ_n is in the interior of $\mathcal{Q}_n$.

(ii) The *quasi Del Pezzo case* $n = 9$. Here $-K_9$ is an irreducible curve with self-intersection 0. Hence κ_9 is nef, sits on $\partial\mathcal{Q}_9$, and the tangent hyperplane to $\partial\mathcal{Q}_9$ at κ_9 is the hyperplane $\kappa_9^{\perp}$ of classes ξ such that $\xi \cdot K_9 = 0$. Then $\overline{\mathrm{NE}}(X_9)^{\succcurlyeq} = \kappa_9$ and $\overline{\mathrm{NE}}(X_9) = \kappa_9 + R_9 \subseteq \overline{\mathrm{NE}}(X_9)^{\preccurlyeq}$. There are infinitely many (-1)-curves on X_9, and κ_9 is the only limit ray of (-1)-rays. The anticanonical ray κ_9 coincides with the Nagata ray ν_9.

(iii) The *general case* $n \geq 10$. Here $-K_n$ is not effective, and has negative self-intersection $9 - n$. Hence κ_n lies off $\mathcal{Q}_n$, which in turn has non-empty intersection with both $\overline{\mathrm{NE}}(X_n)^{\succ}$ and $\overline{\mathrm{NE}}(X_n)^{\prec}$. There are infinitely many (-1)-curves on X_n, whose rays lie in $\overline{\mathrm{NE}}(X_n)^{\prec}$ and their limit rays lie at the intersection of $\partial\mathcal{Q}_n$ with the hyperplane $\kappa_n^{\perp}$. The Nagata ray ν_n sits on $\partial\mathcal{Q}_n^{\succ}$. The plane joining the rays κ_n and ν_n is the *homogeneous slice*, formed by the classes of *homogeneous linear systems* of the form $(d; m^n)$, with $d \geq 0$.

For information on the homogeneous slice, and relations between (N) and (SHGH) there, see [**9**].

3.2. More conjectures. The following conjecture is in [**11**]. Taking into account the aforementioned result in [**10**], it would be a consequence of (SN).

CONJECTURE 3.2. *If $n \geq 10$, then*

$$\overline{\mathrm{NE}}(X_n) = \mathcal{Q}_n^{\succcurlyeq} \ + \ R_n. \tag{3.1}$$

Let $D_n = (\sqrt{n-1}, 1^n) \in N_1(X_n)$ be the *de Fernex* point and $\delta_n = [\sqrt{n-1}, 1^n]$ the corresponding ray. One has $D_n^2 = -1$, $D_n \cdot K_n = n - 3\sqrt{n-1} = \frac{n^2-9n+9}{n+3\sqrt{n-1}} > 0$ for $n \geq 8$ and, if $n = 10$, $D_n = -K_n$. We will denote by $\Delta_n^{\succcurlyeq}$ [resp. $\Delta_n^{\preccurlyeq}$] the set of classes $\xi \in N_1(X_n)$ such that $\xi \cdot D_n \geq 0$ [resp. $\xi \cdot D_n \leq 0$].

One has (see [**11**]):

THEOREM 3.3. *If $n \geq 10$ one has:*

(i) *all (-1)-rays lie in the cone $\mathcal{D}_n := \mathcal{Q}_n - \delta_n$;*
(ii) *if $n = 10$, all (-1)-rays lie on the boundary of the cone $\mathcal{D}_n$;*
(iii) *if $n > 10$, all (-1)-rays lie in the complement of the cone $\mathcal{K}_n := \mathcal{Q}_n - \kappa_n$;*
(iv) *$\overline{\mathrm{NE}}(X_n) \subseteq \mathcal{K}_n + R$;*
(v) *if Conjecture 3.2 holds, then*

$$\overline{\mathrm{NE}}(X_n) \cap \Delta_n^{\preccurlyeq} = \mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}. \tag{3.2}$$

REMARK 3.4. As noted in [**11**], Conjecture 3.2 does not imply that $\overline{\mathrm{NE}}(X_n)^{\succcurlyeq} = \mathcal{Q}_n^{\succcurlyeq}$, unless $n = 10$, in which case this is exactly what it says (see Theorem 3.3(v)). Conjecture 3.2 implies (N) but not (SN).

Consider the following:

CONJECTURE 3.5 (The Δ-conjecture (ΔC)). *If $n \geq 10$ one has*

$$\partial\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq} \subset \mathrm{Nef}(X_n). \tag{3.3}$$

PROPOSITION 3.6. *If (ΔC) holds, then*

$$\overline{\mathrm{NE}}(X_n) \cap \Delta_n^{\preccurlyeq} = \mathrm{Nef}(X_n) \cap \Delta_n^{\preccurlyeq} = \mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}. \tag{3.4}$$

PROOF. By (3.3) and by the convexity of $\mathrm{Nef}(X_n)$ one has

$$\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq} \subseteq \mathrm{Nef}(X_n) \cap \Delta_n^{\preccurlyeq}.$$

Moreover $\mathrm{Nef}(X_n) \cap \Delta_n^{\preccurlyeq} \subseteq \overline{\mathrm{NE}}(X_n) \cap \Delta_n^{\preccurlyeq}$. Finally (3.3) implies (3.2) because $\overline{\mathrm{NE}}(X_n)$ is dual to $\mathrm{Nef}(X_n)$. □

The following proposition indicates that Nagata-type conjectures we are discussing here can be interpreted as asymptotic forms of the (SHGH) conjecture.

PROPOSITION 3.7. *Let $n \geq 10$.*

(i) *If (ΔC) holds, then all classes in $\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq} - \partial\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}$ are ample and therefore, if integral, they are (ASN);*
(ii) *If (SN) holds, then a rational class in $\mathcal{Q}_n^{\succcurlyeq} - \partial\mathcal{Q}_n^{\succcurlyeq}$ is (ASN) unless it has negative intersection with some (-1)-curve.*

PROOF. Part (i) follows from Proposition 3.6 and the fact that the ample cone is the interior of the nef cone (by Kleiman's theorem, see [**21**]).

As for part (ii), if $\xi \in \mathcal{Q}_n^{\succcurlyeq} - \partial\mathcal{Q}_n^{\succcurlyeq}$ is nef, then it is also big. If C is an irreducible curve such that $\xi \cdot C = 0$, then $C^2 < 0$ by the index theorem, hence C is a (-1)-curve. Contract it, go to X_{n-1} and take the class $\xi_1 \in N_1(X_{n-1})$ which pulls back to ξ. Repeat the argument on ξ_1, and go on. At the end we find a class $\xi_i \in N_1(X_{n-i})$ for some $i \leq n$, which is ample by Nakai-Moishezon criterion, and the (ASN) follows for ξ.

If ξ is not nef and C is an irreducible curve such that $\xi \cdot C < 0$, then $C^2 < 0$ hence C is a (-1)-curve. □

One can give a stronger form of (ΔC).

Lemma 3.8. *Any rational, non-effective ray in $\partial\mathcal{Q}_n$ is nef and it is extremal for both $\overline{\mathrm{NE}}(X_n)$ and $\mathrm{Nef}(X_n)$. Moreover it lies in $\partial\mathcal{Q}_n^{\succcurlyeq}$.*

Proof. Let ξ be a generator of the ray and let $\xi = P + N$ be the Zariski decomposition of ξ. Since the ray is not effective, one has $P^2 = 0$. Since $\xi^2 = 0$, then $N^2 = 0$, hence $N = 0$, proving that ξ is nef.

Suppose that $\xi = \alpha + \beta$, with $\alpha, \beta \in \overline{\mathrm{NE}}(X_n)$. Then $\xi^2 = 0$, $\xi \cdot \alpha \geq 0$ and $\xi \cdot \beta \geq 0$, imply $\alpha^2 = -\alpha \cdot \beta = \beta^2$ which yields that α and β are proportional. This shows that the ray is extremal for $\overline{\mathrm{NE}}(X_n)$. The same proof shows that it is extremal also for $\mathrm{Nef}(X_n)$.

The final assertion follows by the Mori's Cone theorem. □

A rational, non-effective ray in $\partial\mathcal{Q}_n$ will be called a *good ray*. An irrational, nef ray in $\partial\mathcal{Q}_n$ will be called a *wonderful ray*. No wonderful ray has been detected so far. The following is clear:

Lemma 3.9. *Suppose that $(\delta; m_1, \dots, m_n)$ generates either a good or wonderful ray. If $(d; m_1, \dots, m_n)$ is an effective linear system then*

$$d > \delta = \sqrt{\sum_{i=1}^{n} m_i^2}.$$

The following conjecture implies (ΔC).

Conjecture 3.10 (The strong Δ-conjecture (SΔC)). *If $n > 10$, all rational rays in $\partial\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}$ are non-effective. If $n = 10$, a rational ray in $\mathcal{Q}_{10} \cap \Delta_{10}^{\preccurlyeq} = \mathcal{Q}_{10}^{\succcurlyeq}$ is non–effective, unless it is generated by a curve (CK)–equivalent to $(3; 1^9, 0)$.*

Proposition 3.11. *For $n = 10$, (SΔC) is equivalent to (SN).*

Proof. If (SΔC) holds then clearly (SN) holds. Conversely, assume (SN) holds, consider a rational effective ray in $\partial\mathcal{Q}_{10}^{\succcurlyeq}$ and let C be an effective divisor in the ray. Then $C = n_1C_1 + \cdots + n_hC_h$, with $C_1, \dots, C_h$ distinct irreducible curves and $n_1, \dots, n_h$ positive integers. One has $C_i \cdot C_j \geq 0$, hence $C_i \cdot C_j = 0$ for all $1 \leq i \leq j \leq h$. This clearly implies $h = 1$, hence the assertion. □

By the proof of Proposition 3.5, any good ray gives a constraint on $\overline{\mathrm{NE}}(X_n)$, so it is useful to find good rays. Even better would be to find wonderful rays. We will soon give more reasons for searching for such rays (see §4).

Example 3.12. Consider the family of linear systems

$$\mathcal{B} = \{B_{q,p} := (9q^2 + p^2; 9q^2 - p^2, (2qp)^9) : (q,p) \in \mathbb{N}^2, q \leq p\}$$

generating rays in $\partial\mathcal{Q}_{10}^{\succcurlyeq}$. Take a sequence $\{(q_n, p_n)\}_{n\in\mathbb{N}}$ such that $\lim_n \frac{p_n+q_n}{p_n} = \sqrt{10}$. For instance take $\frac{p_n+q_n}{p_n}$ to be the convergents of the periodic continued fraction expansion of $\sqrt{10} = [3; \overline{6}]$, so that

$$p_1 = 2,\ p_2 = 13,\ p_3 = 80, \dots \quad q_1 = 1,\ q_2 = 6,\ q_3 = 37, \dots.$$

The sequence of rays $\{[B_{q_n,p_n}]\}_{n\in\mathbb{N}}$ converges to the Nagata ray ν_{10}. If we knew that the rays of this sequence are good, this would imply (N) for $n = 10$.

A way of searching for good rays is the following (see §5). Let $(m_1 \dots, m_n)$ be a (SNS) multiplicity vector, with $d = \sqrt{\sum_{i=1}^n m_i^2}$ an integer such that $3d < \sum_{i=1}^n m_i$. Then $[d; m_1 \dots, m_n]$ is a good ray. We will apply this idea in §5.5.

4. Good rays and counterexamples to Hilbert's 14-th problem

In this section we show that any good or wonderful ray for $n \geq 10$ provides a counterexample to Hilbert's 14-th problem. The proof follows Nagata's original argument in [**24**], which we briefly recall.

Let $\mathbb{F}$ be a field. Let $\mathbf{X} = (x_{ij})_{1\leq i\leq 3; 1\leq j\leq n}$ be a matrix of indeterminates over $\mathbb{F}$ and consider the field $k = \mathbb{F}[\mathbf{X}] := \mathbb{F}[x_{ij}]_{1\leq i\leq 3; 1\leq j\leq n}$ (we use similar vector notation later). The points $x_j = [x_{1j}, x_{2j}, x_{3j}] \in \mathbb{P}^2_k$, $1 \leq j \leq n$, may be seen as n general points of $\mathbb{P}^2_{\mathbb{F}}$. The subspace $V \subset k^n$ formed by all vectors $\mathbf{b} = (b_1, \dots, b_n)$ such that $\mathbf{X} \cdot \mathbf{b}^t = \mathbf{0}$, is said to be *associated* to $x_1, \dots, x_n$.

Fix a multiplicities vector $\mathbf{m} = (m_1, \dots, m_n)$ of positive integers and consider the subgroup H of the multiplicative group $(k^*)^n$ formed by all vectors $\mathbf{c} = (c_1, \dots, c_n)$ such that $\mathbf{c}^{\mathbf{m}} := c_1^{m_1} \cdots c_n^{m_n} = 1$. We set $\delta = \sqrt{\sum_{i=1}^n m_i^2}$.

Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be vectors of indeterminates over k, and consider $k[\mathbf{u}, \mathbf{v}]$. The group $G = H \times V$ acts on the k-algebra $k[\mathbf{u}, \mathbf{v}]$ in the following way: if $\sigma = (\mathbf{c}, \mathbf{b})$ and $c = c_1 \cdots c_n$, then

$$\sigma(u_i) = \frac{c_i}{c}(u_i + b_i v_i), \ \sigma(v_i) = c_i v_i \text{ for } 1 \leq i \leq n.$$

Theorem 4.1. *If $(\delta; m_1, \dots, m_n)$ generates a good or a wonderful ray, then the k-algebra $A = k[\mathbf{u}, \mathbf{v}]^G$ is not finitely generated.*

Proof. The elements $t := \mathbf{v}^{\mathbf{m}}$ and

$$w_i = \sum_{j=1}^n x_{ij}(v_1 \cdots v_{j-1} u_j v_{j+1} \cdots v_n), \text{ for } 1 \leq i \leq 3$$

are in A. Set $\mathbf{w} = (w_1, w_2, w_3)$, which is a vector of indeterminates on k. Then $S := k[\mathbf{w}]$ is the homogeneous coordinate ring of $\mathbb{P}^2_k$. Imitating the argument in [**24**, Lemma 2], one proves that $A = k[\mathbf{u}, \mathbf{v}] \cap k(\mathbf{w}, t)$ and, as a consequence (see [**24**, Lemma 3]), that A consists of all sums $\sum_{i\in\mathbb{Z}} a_i t^{-i}$, such that $a_i \neq 0$ for finitely many $i \in \mathbb{Z}$, $a_i \in S$ for all $i \in \mathbb{Z}$, and $a_i \in \mathfrak{b}_i := \bigcap_{j=1}^n \mathfrak{p}_j^{im_j}$ where $\mathfrak{p}_j$ is the homogeneous ideal of the point x_j.

By [**24**, Lemma 3], to prove that A is not finitely generated it suffices to show that

(4.1) for all positive $m \in \mathbb{Z}$, there is a positive $\ell \in \mathbb{Z}$ such that $\mathfrak{b}_m^\ell \neq \mathfrak{b}_{m\ell}$.

This is proved as in [**24**, Lemma 1]. Indeed, let $\alpha(\mathfrak{q})$ be the minimum degree of a polynomial in a homogeneous ideal $\mathfrak{q}$ of S. Since

$$v(d; mm_1, \dots, mn_n) = \frac{d^2 - m^2\delta^2}{2} + \dots,$$

where $\dots$ denote lower degree terms, we have $\lim_{m\to\infty} \frac{\alpha(\mathfrak{b}_m)}{m} \leq \delta$. But because $(\delta; m_1, \dots, m_n)$ is nef, we have $\frac{\alpha(\mathfrak{b}_m)}{m} \geq \delta$ for all positive integers m. Hence $\lim_{m\to\infty} \frac{\alpha(\mathfrak{b}_m)}{m} = \delta$. By Lemma 3.9, one has

$$\frac{\alpha(\mathfrak{b}_m^\ell)}{m\ell} = \frac{\alpha(\mathfrak{b}_m)}{m} > \delta$$

from which (4.1) follows. □

5. Existence of good rays

5.1. The existence theorem.

THEOREM 5.1.1 (Existence Theorem (ET)). *For every $n \geq 10$, there are good rays $[d; m_1, \ldots, m_n] \in \overline{\mathrm{NE}}(X_n)$, with $m_i > 0$ for all $1 \leq i \leq n$.*

The proof goes by induction on n (see §5.6). The *induction step* is based on the following proposition:

PROPOSITION 5.1.2. *Set $n = s + t - 1$, with s, t positive integers. Assume $D = (\delta; \mu_1, \ldots, \mu_s) \in \mathrm{Nef}(X_s)$ with $\mu_1, \ldots, \mu_s$ nonnegative, rational numbers. Let $\nu_1, \ldots, \nu_s$ be nonnegative rational numbers such that $m_1 = \sum_{i=1}^{s} \mu_i \nu_i$ is an integer, and let $d, m_2, \ldots, m_t$ be nonnegative rational numbers such that $C = (d; m_1, \ldots, m_t)$ generates a non-effective ray in $N_1(X_t)$. Then for every rational number $\eta \geq \delta$, $C_\eta = (d; \eta\nu_1, \ldots, \eta\nu_s, m_2, \ldots, m_t)$ generates a non-effective ray in $N_1(X_n)$.*

The proof of Proposition 5.1.2 relies on a degeneration argument introduced in [**6**] (see also [**5, 9**]), which is reviewed in §5.2. The next corollary shows how Proposition 5.1.2 may be applied to inductively prove Theorem 5.1.1.

COROLLARY 5.1.3. *Same setting as in Proposition* 5.1.2. *Assume that:*

(i) *C generates a good ray in $N_1(X_t)$;*
(ii) *$D \in \mathrm{Nef}(X_s)$ and $D^2 = 0$.*

Then $C_\delta \in \partial\mathcal{Q}_n^{\succ}$ is nef.

PROOF. We use the vector notation $\mu = (\mu_1, \ldots, \mu_s)$, $\nu = (\nu_1, \ldots, \nu_s)$, and denote by $|| \quad ||_p$ the ℓ^p norm of vectors. For $\eta \geq \delta$, one has

$$\begin{aligned} C_\eta \cdot K_n - C \cdot K_t = \eta||\nu||_1 - m_1 = \eta||\nu||_1 - \mu \cdot \nu \geq \eta||\nu||_1 - ||\nu||_2||\mu||_2 = \\ = \eta||\nu||_1 - \delta||\nu||_2 \geq \delta(||\nu||_1 - ||\nu||_2) \geq 0. \end{aligned}$$

Since $C \cdot K_t \geq 0$, then also $C_\eta \cdot K_n \geq 0$, hence $C_\delta \cdot K_n \geq 0$. Moreover

$$\begin{aligned} C_\eta^2 = C^2 + m_1^2 - \eta^2 \sum_{i=1}^{s} \nu_i^2 &= C^2 + \big((\sum_{i=1}^{s} \mu_i^2) - \eta^2\big) \sum_{i=1}^{s} \nu_i^2 \\ &= C^2 + \big(\delta^2 - \eta^2\big) \sum_{i=1}^{s} \nu_i^2 \leq C^2 = 0, \end{aligned}$$

in particular $C_\delta^2 = 0$.

Assume C_δ is not nef, hence there is an irreducible curve E such that $C_\delta \cdot E < 0$ and $E^2 < 0$. Take $\eta \geq \delta$ close to δ and rational. Set $E_\epsilon = \epsilon E + C_\eta$, with $\epsilon \in \mathbb{R}$. One has $E_\epsilon^2 = \epsilon^2 E^2 + 2\epsilon(C_\eta \cdot E) + C_\eta^2$ and $(C_\eta \cdot E)^2 - C_\eta^2 \cdot E^2 > 0$ because it is close to $(C_\delta \cdot E)^2 > 0$. Then

$$\tau = \frac{-(C_\eta \cdot E) - \sqrt{(C_\eta \cdot E)^2 - C_\eta^2 \cdot E^2}}{E^2}$$

is negative, close to 0 and such that $E_\tau^2 = 0$, and $E_\epsilon^2 > 0$ for $\epsilon < \tau$ and close to τ. Then for these values of ϵ the class $C_\eta = E_\epsilon - \epsilon E$ would generate an effective ray, a contradiction. □

REMARK 5.1.4. We will typically apply Proposition 5.1.2 and Corollary 5.1.3 with $\mu_1 = \ldots = \mu_s = 1$, $\delta = \sqrt{s}$ and $\nu_1 = \ldots = \nu_s = \frac{m_1}{s}$. If either $s = 4, 9$ or $s \geq 10$ and (N) holds, then hypothesis (ii) of Corollary 5.1.3 holds. Hence, if $C = (d; m_1, \ldots, m_t)$ generates a good ray in $N_1(X_t)$, then $C_{\frac{s}{\eta}} = (d; (\frac{m_1}{\eta})^s, m_2, \ldots, m_t)$ generates a noneffective ray for all rational numbers $\eta \leq \sqrt{s}$, therefore the ray $[d; (\frac{m_1}{\sqrt{s}})^s, m_2, \ldots, m_t]$ is either good or wonderful, in particular it is nef.

In this situation, if C is standard and $s \geq 9$, then C_δ is also standard. The same holds for $s = 4$ if $2d \geq 3m_1$. This will be the case for the examples we will provide to prove Theorem 5.1.1, so all of them will be standard.

The *base of the induction*, consists in exhibiting SNS multiplicity vectors for $10 \leq n \leq 12$, giving rise to good rays as indicated at the end of § 3.2. They will provide the starting points of the induction for proving Theorem 5.1.1 (see §5.6). This step is based on a slight improvement of the same degeneration technique used to prove Proposition 5.1.2 (see §5.4).

REMARK 5.1.5. To the best of our knowledge, it is only for a square number of points that SNS multiplicity vectors and good rays were known so far: i.e., $[d; 1^{d^2}]$ is a good ray (see [**24**]) and (1^{d^2}) is an SNS multiplicity vector for $d \geq 4$ (see [**8, 13, 25**]).

EXAMPLE 5.1.6. Using the goodness of $[d; 1^{d^2}]$ and applying Corollary 5.1.3, we see that all rays of the form $[dh; h^{d^2-\ell}, 1^{\ell h^2}]$, with $d \geq 4$, $h \geq 1$ and $0 \leq \ell \leq d^2$ integers, are good.

5.2. The basic degeneration. We briefly recall the degeneration we use to prove Theorem 5.1.1 (see [**5, 6, 9**] for details).

Consider $Y \to \mathbb{D}$ the family obtained by blowing up the trivial family $\mathbb{D} \times \mathbb{P}^2 \to \mathbb{D}$ over a disc $\mathbb{D}$ at a point in the central fiber. The general fibre Y_u for $u \neq 0$ is a $\mathbb{P}^2$, and the central fibre Y_0 is the union of two surfaces $V \cup Z$, where $V \cong \mathbb{P}^2$ is the exceptional divisor and $Z \cong \mathbb{F}_1$ is the original central fibre blown up at a point. The surfaces V and Z meet transversally along a rational curve E which is the negative section on Z and a line on V.

Choose s general points on V and $t-1$ general points on Z. Consider these $n = s + t - 1$ points as limits of n general points in the general fibre Y_u and blow these points up in the family Y, getting a new family. We will abuse notation and still denote by Y this new family. The blow-up creates n exceptional surfaces R_i, $1 \leq i \leq n$, whose intersection with each fiber Y_u is a (-1)-curve, the exceptional curve for the blow-up of that point in the family. The general fibre Y_u of the new family is an X_n. The central fibre Y_0 is the union of V blown-up at s general points, and Z blown-up at $t-1$ general points. We will abuse notation and still denote by V and Z the blown-up surfaces which are now isomorphic to X_s, X_t respectively.

Let $\mathcal{O}_Y(1)$ be the pullback on Y of $\mathcal{O}_{\mathbb{P}^2}(1)$. Given a vector $(m_1, \ldots, m_n)$ of multiplicities, a degree d and a *twisting integer* a, consider the line bundle

$$\mathcal{L}(a) = \mathcal{O}_Y(d) \otimes \mathcal{O}_Y(-m_1R_1) \otimes \cdots \otimes \mathcal{O}_Y(-m_nR_n)) \otimes \mathcal{O}_Y(-aV).$$

Its restriction to Y_u for $u \neq 0$ is $(d; m_1, ..., m_n)$. Its restrictions to V and Z are $\mathcal{L}_V = (a; m_1, ..., m_s)$, $\mathcal{L}_Z = (d; a, m_{s+1}, ..., m_n)$ respectively. Every *limit line bundle* of $(d; m_1, ..., m_n)$ on Y_t is the restriction to $Y_0 = V \cup Z$ of $\mathcal{L}(a)$ for an integer a.

We will say that a line bundle $\mathcal{L}(a)$ is *centrally effective* if its restriction to both V and Z is effective.

THEOREM 5.2.7 (The Basic Non-Effectivity Criterion (BNC), (see [**9**])). *If there is no twisting integer a such that $\mathcal{L}(a)$ is centrally effective, then $(d; m_1, ..., m_n)$ is non-effective.*

5.3. The proof of the induction step. In this section we use (BNC) to give the:

PROOF OF PROPOSITION 5.1.2. . We need to prove that

$$(xd; x\eta\nu_1, \ldots, x\eta\nu_s, xm_2, \ldots, xm_t)$$

is not effective for all positive integers x.

In the setting of §5.2, fix the multiplicities on V to be $x\eta\nu_1, \ldots, x\eta\nu_s$ and on Z to be $xm_2, \ldots, xm_t$. We argue by contradiction, and assume there is a central effective $\mathcal{L}(a)$. Then $\mathcal{L}_V = (a; x\eta\nu_1, \ldots, x\eta\nu_s)$ is effective hence $D \cdot \mathcal{L}_V \geq 0$, i.e. $a\delta \geq x\eta m_1$, therefore $a \geq xm_1$. Since $\mathcal{L}_Z = (xd; a, xm_2, \ldots, xm_t)$ is effective, so is $(xd; xm_1, xm_2, \ldots, xm_t)$ which contradicts $C = (d; m_1, \ldots, m_t)$ not being effective. □

5.4. 2-throws. To deal with the base of the induction, we need to analyse the *matching* of the sections of the bundles $\mathcal{L}_V$ and $\mathcal{L}_Z$ on the double curve. For this we need a modification of the basic degeneration, based on the concept of a 2-*throw*, described in [**9**], which we will briefly recall now. In doing this we will often abuse notation, which we hope will create no problems for the reader.

Consider a degeneration of surfaces over a disc, with central fibre containing two components X_1 and X_2 meeting transversally along a double curve R. Let E be a (-1)-curve on X_1 that intersects R transversally at two points. Blow it up in the threefold total space of the degeneration. The exceptional divisor $T \cong \mathbb{F}_1$ meets X_1 along E, which is the negative section of T. The surface X_2 is blown up twice, with two exceptional divisors G_1 and G_2.

Now blow-up E again, creating a double component $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ of the central fibre that meets X_1 along E and T along its negative section. The blow-up affects X_2, by creating two more exceptional divisors F_1 and F_2 which are (-1) curves on X_2, while G_1 and G_2 become (-2)-curves. Blowing S down by the other ruling contracts E on the surface X_1. The curve R becomes nodal, and T changes into a $\mathbb{P}^2$. The surface X_2 becomes non-normal, singular along the identified (-1)-curves F_1, F_2.

On X_2 we introduced two pairs of *infinitely near points* corresponding to the (-1)-curves F_i and $F_i + G_i$, which is also a curve with self-intersection -1, and we call F_i and $F_i + G_i$ a pair of *infinitely near* (-1)-curves, with $1 \leq i \leq 2$. We denote the assignment of multiplicities to a pair of infinitely near points as above by $[a, b]$, indicating a multiple point a and an infinitely near multiple point b, namely $-a(F_i + G_i) - bF_i$.

5.5. The base of the induction. The above discussion is general. In order to deal with the base of the induction, we will now apply it to the degeneration $V \cup Z$ described in section 5.2, with $n = 10$ (for the cases $11 \leq n \leq 12$ the basic degeneration, plus some more care on the matching, suffices). The proofs here are quite similar to the ones in [**5, 9**], hence we will be brief.

5.5.1. *The $n = 10$ case.*

PROPOSITION 5.5.8. *The multiplicity vector $(5, 4^9)$ is (SNS). In particular $B_{1,2} = (13; 5, 4^9)$ generates a good ray.*

PROOF. It suffices to prove that, for every positive integer x, $(13x; 5x, (4x)^9)$ is non-effective and $(13x + 1; 5x, (4x)^9)$ is non-special.

We will show that $(15x; 6x, (4x)^3, (5x)^5, 4x)$ ((CK)-equivalent to $(13x; 5x, (4x)^9)$) is not effective. We assume by contradiction that the linear system is effective for some x.

Consider first the basic degeneration with $s = 4$, $t = 7$, endowed with the line bundles $\mathcal{L}(a)$ as in §5.2 (whose restrictions to V and Z are $\mathcal{L}_V = (a; 6x, (4x)^3)$, $\mathcal{L}_Z = (15x; a, (5x)^5, 4x)$). Then perform the 2-throw of the (-1)-curve $E = (3; 2, 1^6)$ on Z (see §5.4). The normalization of V is a 8-fold blow up of $\mathbb{P}^2$, two of the exceptional divisors being identified in V. More precisely, the normalization of V is the blow-up of the plane at 8 points: 4 of them are in general position, 4 lie on a line, and two of them are infinitely near. It is better to look at the surface Z before blowing down E. Then $Z \cong X_7$. Finally, by executing the 2-throw we introduce a plane T.

We record that the pencil $P_V = (5; 3, 2^3, [1, 1]^2)$ on the normalization of V and the pencil $P_Z = (3; 2, 1^5, 0)$ on Z are nef.

We abuse notation and still denote by $\mathcal{L}(a)$ the pullback of this bundle to the total space of the family obtained by the double blow-up of E (see §5.4). For each triple of integers (a, b_1, b_2), we can consider the bundle $\mathcal{L}(a, b_1, b_2) = \mathcal{L}(a) \otimes \mathcal{O}_Y(-b_1 T - (b_1 + b_2)S)$. We will still denote by $\mathcal{L}(a, b_1, b_2)$ the pushout of this bundle to the total space of the 2-throw family. Every limit line bundle of $(15x; 6x, (5x)^5, (4x)^4)$ has the form $\mathcal{L}(a, b_1, b_2)$.

We are interested in those $\mathcal{L}(a, b_1, b_2)$ which are centrally effective. Such systems were studied in [**5**, section 2], whose notation had $\mu = a, y = u = b_1, y' = v = b_2$, $h = \sigma/2 = -\mathcal{L}(a) \cdot E/2 = -8x + a$ and several other parameters x_i, z_i corresponding to 6 additional curves that were thrown, which we can safely equate to 0. The effectivity of the restriction to S and T implies that $b_1 \geq -8x + a$, $b_1 + b_2 \geq -16x + 2a$, and the effectivity of the restriction to V, that $-8x + a \geq x > 0$. Then the computation of [**5**] shows that for every b_1, b_2 satisfying the preceding inequalities, the restriction of $\mathcal{L}(a, b_1, b_2)$ to Z and V are subsystems of the restriction of $\mathcal{L}(a, -8x + a, -8x + a)$, namely

$$\mathcal{L}_Z = (63\,x - 6\,a; 32\,x - 3\,a, (21\,x - 2\,a)^5, 20\,x - 2\,a)$$

and

$$\mathcal{L}_V = (a; 6x, (4x)^3, [-8\,x + a, -8\,x + a]^2).$$

It suffices to see that there is no value of a which makes both $\mathcal{L}_Z$ and $\mathcal{L}_V$ effective, and for which there are divisors in these two systems which agree on the double curve.

For $\mathcal{L}(a, b_1, b_2)$ to be centrally effective one needs

$$\mathcal{L}_Z \cdot P_Z = 20x - 2a \geq 0, \quad \mathcal{L}_V \cdot P_V = a - 10x \geq 0.$$

This forces $a = 10x$ and the restriction of $\mathcal{L}(a, b_1, b_2)$ to Z and V are *equal* to

$$\mathcal{L}_Z = (3x; 2x, x^5, 0), \quad \mathcal{L}_V = (10x; 6x, (4x)^3, [2x, 2x]^2),$$

which means $b_1 = b_2 = -8x + a = 2x$, hence the restriction of the line bundle to T is trivial, and the systems $\mathcal{L}_Z$ and $\mathcal{L}_V$ are composed with the pencils P_V and P_Z, thus $\dim(\mathcal{L}_Z) = x$, $\dim(\mathcal{L}_V) = 2x$.

Focusing on $\mathcal{L}_V$, we only need to consider the subspace of sections that match along F_1 and F_2. Since the identification $F_1 = F_2$ is done via a sufficiently general projectivity, this vector space has dimension 1 (see [**9**, §8] for details). Then, by transversality on $Z \cap V = R$ (see [**6**, §3]), and since $\mathcal{L}_{Z|R}$ has dimension x and degree $2x$, no section on Z matches the one on V to create a section on X_0.

Now consider $(13x+1; 5x, (4x)^9)$ and its (CK)-equivalent system $(15x+3; 6x+2, (4x)^3, 5x, (5x+1)^4, 4x)$. A similar analysis as before, using $a = 10x+1$, leads to the following limit systems on Z and V (trivial on T)

$$\mathcal{L}_Z = (3x+12; 2x+7, (x+4)^4, x+3, 3), \quad \mathcal{L}_V = (10x+1; 6x+2, (4x)^3, [2x-1, 2x-2]^2).$$

The system $\mathcal{L}_Z$ is (CK)-equivalent to $(x+5; x+2, 2, 1^4)$ so it is nef, non-empty of the expected dimension (see [**17**]). The system $\mathcal{L}_V$ is also non-empty of the expected dimension: it consists of three lines plus a residual system (CK)-equivalent to the nef system $(2x+3; 3, 2, 1^2, 2x-2)$. Thus to compute the dimension of the limit system as in [**6**], it remains to analyse the restrictions to R (or rather, the *kernel systems* $\hat{\mathcal{L}}_Z$, $\hat{\mathcal{L}}_V$ of such restrictions). Since both surfaces are anticanonical, this can be done quite easily, showing that they are non-special with $\dim(\hat{\mathcal{L}}_Z) = 2x+4$ and $\dim(\hat{\mathcal{L}}_V) = 10x-7$. Thus [**6**, 3.4, (b)], applies and non-speciality of $(15x+3; 6x+2, (4x)^3, 5x, (5x+1)^4, 4x)$ follows. □

5.5.2. *The $n = 11$ case.*

Proposition 5.5.9. *The multiplicity vector $(3, 2^{10})$ is (SNS). In particular $(7; 3, 2^{10})$ generates a good ray.*

Proof. We prove that $(7x+\delta; (2x)^{10}, 3x)$ is non-effective for all x and $\delta = 0$ and non-special for $\delta = 1$.

Consider the basic degeneration as in §5.2, with $s = 4$, $t = 8$. Then $\mathcal{L}(a)$ restrict as

$$\mathcal{L}_V = (a; (2x)^4), \quad \mathcal{L}_Z = (7x+\delta; a, (2x)^6, 3x).$$

Look at the case $\delta = 0$, where we want to prove non-effectivity. (BNC) does not suffice for this, so we will compute the dimension of a limit system as in [**6**]. To do this, pick $a = 4x$. The systems $\mathcal{L}_V$ and $\mathcal{L}_Z$ are composed with the pencils $P_V = (2; 1^4)$ and $P_Z = (7; 4, (2)^6, 3)$ respectively (note that $(7; 4, (2)^6, 3)$ is (CK)-equivalent to a pencil of lines), and $\dim(\mathcal{L}_V) = 2x, \dim(\mathcal{L}_Z) = x$. The restriction to R has degree $4x \geq 2x + x + 1$, so by transversality of the restricted systems [**6**, §3], the limit linear system consists of the kernel systems $\hat{\mathcal{L}}_V = (4x-1; (2x)^4)$ and $\hat{\mathcal{L}}_Z = (7x; 4x+1, (2x)^6, 3x)$. These are non-effective, because they meet negatively P_V and P_Z respectively. So $(7x; (2x)^{10}, 3x)$ is non-effective.

For $\delta = 1$ pick again $a = 4x$. Then $\mathcal{L}_V$ is the same, $\mathcal{L}_Z = (7x+1; 4x, (2x)^6, 3x)$ and the kernel systems are both nef, hence they are non-special by [**17**]. Moreover the restriction of $\mathcal{L}_Z$ to R is the complete series of degree $4x$. Again by transversality as in [**6**, §3], the claim follows. □

5.5.3. *The $n = 12$ case.*

Proposition 5.5.10. *The multiplicity vector $(2^8, 1^4)$ is (SNS). In particular $(6; 2^8, 1^4)$ generates a good ray.*

PROOF. We prove that $(6x+\delta; x^4, (2x)^8)$ is empty for all x and $\delta = 0$ and non-special for $\delta = 1$.

Consider the degeneration of §5.2, with $s = 4$, $t = 9$. Then $\mathcal{L}(a)$ restrict as

$$\mathcal{L}_V = (a; (x)^4),\ \ \mathcal{L}_Z = (6x; a, (2x)^8).$$

Let us analyze the case $\delta = 0$ for $a = 2x$. The system $\mathcal{L}_Z$ consists of $2x$ times the unique cubic E through the 9 points and $\mathcal{L}_V$ is composed with the pencil $P_V = (2, 1^4)$, and its restriction to R is composed with a general pencil of degree 2. By transversality it does not match the divisor cut out by $2xE$ on R and the limit system is formed by the kernel systems. An elementary computation shows that they are not effective.

For $\delta = 1$ and $a = 2x$, $\mathcal{L}_V$ is the same, whereas $\mathcal{L}_Z = (6x+1; (2x)^9)$ and the kernel $\hat{\mathcal{L}}_Z = (6x+1; 2x+1, (2x)^8)$ are both nef, hence they are non-special by [**17**]. As before, the restriction of $\mathcal{L}_Z$ to R is the complete series of degree $2x$, and transversality gives the claim. $\square$

REMARK 5.5.11. (i) In Example 5.1.6 we saw that $[dh; h^{d^2-\ell}, 1^{\ell h^2}]$, with $0 \le \ell \le h$, and $d \ge 4$, $h \ge 1$, is a good ray. Proposition 5.5.10 shows that if $d = 3$, $h = 2$, $\ell = 1$, the ray is still good. By Corollary 5.1.3, this implies that if $d = 3$, $h = 2$, $1 \le \ell \le 9$, the ray is still good. With a similar argument, one sees that all cases $d = 3$, $h \ge 1$, also give rise to good rays. We leave this to the reader.

(ii) For any integer $d \ge 6$, take positive integers r, s such that $d^2 = 4s + r$. The ray generated by $(d; 2^s, 1^r)$ on X_n is nef. Indeed, one has $(d-1)(d-2) \ge 2s$, so there exists an irreducible curve C of degree d with exactly s nodes $p_1, \ldots, p_s$ ([**28**], Anhang F). On the blow–up of the s nodes and r other points $q_1, \ldots, q_r$ of the curve, the proper transform of the curve is a prime divisor of selfintersection zero, thus nef.

If $d = 2k$ is even then $r = 4k^2 - 4s$ is a multiple of four and $(d; 2^s, 1^r) = (2k; 2^{k^2-\ell}, 1^{4\ell})$ generates a good ray with $\ell = k^2 - s$ by example 5.1.6 (because of (i) we may assume $d > 6$). This suggests that that the ray $[d; 2^s, 1^r]$ may always be good. To prove it, taking into account Corollary 5.1.3, it would suffice to show that $[2k+1; 2^{k^2+k}, 1]$ is good for all $k \ge 3$.

5.6. The proof of the ET. For $10 \le n \le 12$ the problem is settled by Propositions 5.5.8, 5.5.9 and 5.5.10. To cover all $n \ge 13$, we apply Corollary 5.1.3 with $s = 4$, $D = (2; 1^4)$ and $\nu_1 = \ldots = \nu_4 = \frac{m_1}{4}$ (see Remark 5.1.4). For instance one finds the good rays

$$\begin{array}{ll}
[13 \cdot 2^h; 5^4, (5\cdot 2)^3, \ldots, (5 \cdot 2^{h-1})^3, (2^{h+2})^9] & \text{if } n = 10 + 3h, \text{ for } h \ge 1,\\
[7 \cdot 2^h; 3^4, (3\cdot 2)^3, \ldots, (3 \cdot 2^{h-1})^3, (2^{h+1})^{10}] & \text{if } n = 11 + 3h, \text{ for } h \ge 1,\\
[6 \cdot 2^{h-1}; (2)^3, \ldots, (2^{h-1})^3, (2^h)^7, 1^8] & \text{if } n = 12 + 3h, \text{ for } h \ge 1.
\end{array}$$

6. An application

PROPOSITION 6.1. *If (SN) holds for $n = 10$ and (N) holds for all $n \le 89$ then (N) holds for all $n \ge 90$.*

The proof is based on the following:

LEMMA 6.2. *Assume (SN) holds for $n = 10$. Let $n = s_1 + \cdots + s_{10}$, where $s_1, \ldots, s_{10}$ are positive integers such that the Nagata ray ν_{s_i} is nef for $1 \leq i \leq 10$ and*

$$3\sqrt{n} \leq \sum_{i=1}^{10} \sqrt{s_i}. \tag{6.1}$$

Then ν_n is nef.

PROOF. Consider the ray $[\sqrt{n}, \sqrt{s_1}, \ldots, \sqrt{s_{10}}]$ which, by the hypotheses, is in $\partial\mathcal{Q}_{10}^{\succcurlyeq}$. We can approximate it by a sequence $\{[d_h, m_{1,h}, \ldots, m_{10,h}]\}_{h\in\mathbb{N}}$ of good rays (see Proposition 3.11). Make now an obvious multiple application of Corollary 5.1.3 with $C = [d_h, m_{1,h}, \ldots, m_{10,h}]$ and D each of the Nagata points ν_{s_i}, for $1 \leq i \leq 10$ (see Remark 5.1.4). We obtain that $[d_h, (\frac{m_{1,h}}{\sqrt{s_1}})^{s_1}, \ldots, (\frac{m_{10,h}}{\sqrt{s_{10}}})^{s_{10}}]$ is nef for all $h \in \mathbb{N}$. Since this ray tends to ν_n for $h \to \infty$, the assertion follows. $\square$

PROOF OF PROPOSITION 6.1. We argue by induction. Let $n \geq 90$, and write $n = 9h + k$, with $9 \leq k \leq 17$ and $h \geq 9$. By induction both ν_h and ν_k are nef. Moreover (6.1) is in this case $3\sqrt{9h+k} \leq 9\sqrt{h} + \sqrt{k}$, which reads $h \geq \frac{16}{81}k$, which is verified because $k \leq 17$ and $h \geq 9$. Then ν_n is nef by Lemma 6.2. $\square$

REMARK 6.3. Lemma 6.2 is reminiscent of the results in [**1**] and [**26**].

The hypotheses in Proposition 6.1 can be weakened. For instance, Lemma 6.2 implies that, if (SN) holds for $n = 10$, then ν_{13} is nef. Actually, it suffices to know that $[\sqrt{13}; 2, 1^9]$ is nef. As in Example 3.12, we may take a sequence $\{(q_n, p_n)\}_{n\in\mathbb{N}}$ such that $\frac{p_n+2q_n}{p_n}$ are the convergents of the periodic continued fraction expansion of $\sqrt{13} = [3; \overline{1^3, 6}]$, so that

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 20, \ldots \quad q_1 = 1,\ q_2 = 2,\ q_3 = 3,\ q_4 = 13 \ldots.$$

The sequence of rays $\{[B_{q_n,p_n}]\}_{n\in\mathbb{N}}$ converges to ν_{13}. If we knew that the rays of the sequence are good, this would imply (N) for $n = 13$. Note that $B_{q_1,p_1} = (13; 5, 4^9)$ generates a good ray by Proposition 5.5.8.

Similarly, if (SN) holds for $n = 10$, then ν_n is nef for $n = 10h^2$, etc. We do not dwell on these improvements here.

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Clay Mathematics Proceedings
Volume **18**, 2013

Symplectic restriction varieties and geometric branching rules

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To Joe, with gratitude, in celebration of his sixtieth birthday.

Abstract. In this paper, we introduce new, combinatorially defined subvarieties of isotropic Grassmannians called *symplectic restriction varieties*. We study their geometric properties and compute their cohomology classes. In particular, we give a positive, combinatorial, geometric branching rule for computing the map in cohomology induced by the inclusion $i : SG(k,n) \to G(k,n)$. This rule has many applications in algebraic geometry, symplectic geometry, combinatorics, and representation theory. In the final section of the paper, we discuss the rigidity of Schubert classes in the cohomology of $SG(k,n)$. Symplectic restriction varieties, in certain instances, give explicit deformations of Schubert varieties, thereby showing that the corresponding classes are not rigid.

1. Introduction

Specialization has been a fruitful technique since the beginning of enumerative geometry. Enumerative geometry studies the problem of determining the number of geometric objects (such as curves or linear spaces) satisfying constraints (such as being incident to general linear spaces). Determining these invariants is often very hard. However, if the constraints are in a special position, the problem may become easier. The specialization technique consists of finding a special configuration of constraints for which the answer to the enumerative problem becomes evident and then relating the original problem to this simpler problem.

In the last three decades, Joe Harris has been a master at using specialization to answer long standing problems of algebraic geometry. For example, Griffiths and Harris in their celebrated paper [**GH1**], using an ingenious specialization, proved the Brill-Noether Theorem by showing that Schubert cycles, which parameterize linear spaces that intersect general secant lines of a rational normal curve, intersect dimensionally properly. Later, Eisenbud and Harris, by specializing to a g-cuspidal rational curve, gave a simple proof of the Gieseker-Petri Theorem [**EH1**], [**EH2**].

2010 *Mathematics Subject Classification.* Primary 14M15, 14N35, 32M10.

Key words and phrases. Isotropic Grassmannian, cohomology, rigidity, the restriction problem.

During the preparation of this article the author was partially supported by the NSF grant DMS-0737581, the NSF CAREER grant DMS-0950951535, and an Alfred P. Sloan Foundation Fellowship.

More importantly, they developed the theory of limit linear series to systematically study limits of linear systems under certain specializations [**EH3**]. The theory led to rapid advances in Brill-Noether theory and our understanding of the moduli space of curves [**EH4**]. In a parallel development, Griffiths and Harris used specializations to give a new proof of the Noether-Lefschetz Theorem independent of Hodge theory [**GH2**]. Harris also effectively used specialization to study the geometry of the Severi varieties of nodal plane curves of degree d and genus g. He proved their irreducibility [**H**] and, in joint work with Caporaso, computed their degrees [**CH**].

Inspired by Griffiths and Harris' proof of the Brill-Noether Theorem, in the last decade, Vakil and the author have used similar specializations to systematically compute the structure constants of the cohomology of Grassmannians and flag varieties [**V**], [**C3**], [**C5**], [**CV**]. Specializations have also been successfully applied to study the cohomology of other homogeneous varieties. For example, the author has computed the restriction coefficients and proved geometric branching rules for Type B and D flag varieties [**C2**]. Given the prominent role that the specialization technique has played in the work of Joe Harris and his students, it seems fitting to include a paper calculating enumerative invariants by specialization in a volume celebrating Joe Harris and his work.

The purpose of this paper is to compute the restriction coefficients and prove geometric branching rules for Type C Grassmannians using specializations. The extension to Type C flag varieties is straightforward, but in order to keep the exposition in this paper short, we postpone the discussion to the companion paper [**C4**].

Let V be an n-dimensional vector space over the complex numbers $\mathbb{C}$. Let Q be a non-degenerate skew-symmetric form on V. Since Q is non-degenerate, n must be even, say $n = 2m$. A linear space $W \subset V$ is called *isotropic* with respect to Q if for every $w_1, w_2 \in W$, $w_1^T Q w_2 = 0$. The *symplectic isotropic Grassmannian* $SG(k, n)$ parameterizes k-dimensional subspaces of V that are isotropic with respect to Q.

The isotropic Grassmannian $SG(k, n)$ naturally includes in the Grassmannian $G(k, n)$. This inclusion i induces a map on cohomology

$$i^* : H^*(G(k, n), \mathbb{Z}) \to H^*(SG(k, n), \mathbb{Z}).$$

The cohomology groups of both $G(k, n)$ and $SG(k, n)$ have integral bases given by Schubert classes. Given a Schubert class σ_κ in $H^*(G(k, n), \mathbb{Z})$, $i^*\sigma_\kappa$ can be expressed as a non-negative linear combination

$$i^*\sigma_\kappa = \sum_{\lambda,\mu} c^\kappa_{\lambda;\mu} \sigma_{\lambda;\mu}$$

of the Schubert classes $\sigma_{\lambda;\mu}$ in $H^*(SG(k, n), \mathbb{Z})$. The coefficients $c^\kappa_{\lambda;\mu}$ are called *symplectic restriction* or *branching coefficients*. These coefficients carry a lot of geometric, combinatorial and representation theoretic information. For example, they are closely related to computing moment polytopes and restrictions of representations of $SL(n)$ to $Sp(n)$ (see [**BS**], [**C2**], [**GS**], [**He**], and [**P**]). The main technical theorem of this paper gives a positive, geometric rule for computing restriction coefficients.

THEOREM 1.1. *Algorithm* 3.29 *gives a positive, geometric rule for computing the symplectic restriction coefficients.*

More importantly, we will introduce a new set of varieties called *symplectic restriction varieties*. These varieties parameterize isotropic subspaces that satisfy rank conditions with respect to a not-necessarily isotropic flag. In Section 4, we will specify the conditions that these flags need to satisfy and carefully define these varieties. The reader may informally think of these varieties as varieties that interpolate between the restrictions of general Schubert varieties in $G(k,n)$ to $SG(k,n)$ and Schubert varieties in $SG(k,n)$.

The proof of Theorem 1.1 will proceed by a specialization. We will specialize the flag defining a Schubert variety in $G(k,n)$ successively until we arrive at an isotropic flag. We will show that at each stage of the specialization, the corresponding restriction varieties break into a union of restriction varieties, each occurring with multiplicity one. In Section 3, we will develop combinatorial objects called *symplectic diagrams* to record the result of these specializations.

In earlier work, Pragacz gave a positive rule for computing restriction coefficients for Lagrangian Grassmannians [**Pr1**], [**Pr2**]. It is also possible to compute restriction coefficients (in a non-positive way) by first computing the pullbacks of the tautological bundles from $G(k,n)$ to $SG(k,n)$ and then using localization or the theory of Schubert polynomials to express the Chern classes of these bundles in terms of Schubert classes. To the best of the author's knowledge, Algorithm 3.29 is the first positive, geometric rule for computing the restriction coefficients for all isotropic Grassmannians $SG(k,n)$.

While the combinatorics of symplectic restriction coefficients can be very complicated, the beauty of the approach is that the computation depends on four very simple geometric principles. We now explain these principles. Let Q_d^r denote a d-dimensional vector space such that the restriction of Q has corank r. Let $\mathrm{Ker}(Q_d^r)$ denote the kernel of the restriction of Q to Q_d^r. Let L_j denote an isotropic subspace of dimension j with respect to Q. Let $L_j^{\perp}$ denote the set of $w \in V$ such that $w^T Q v = 0$ for all $v \in L_j$.

Evenness of rank. The rank of a non-degenerate skew-symmetric form is even. Hence, $d-r$ is even for Q_d^r. Furthermore, if $d=r$, then Q_d^r is isotropic.

The corank bound. Let $Q_{d_1}^{r_1} \subset Q_{d_2}^{r_2}$ and let $r_2' = \dim(\mathrm{Ker}(Q_{d_2}^{r_2}) \cap Q_{d_1}^{r_1})$. Then $r_1 - r_2' \leq d_2 - d_1$. In particular, $d + r \leq n$ for Q_d^r.

The linear space bound. The dimension of an isotropic subspace of Q_d^r is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an m-dimensional linear space L satisfies $\dim(L \cap \mathrm{Ker}(Q_d^r)) \geq m - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be an $(s+1)$-dimensional isotropic space such that $\dim(L \cap \mathrm{Ker}(Q_d^r)) = s$. If an isotropic linear subspace M of Q_d^r intersects $L - \mathrm{Ker}(Q_d^r)$, then M is contained in $L^{\perp}$.

These four principles dictate the order of the specialization and determine the limits that occur. Given a flag, we will specialize the smallest dimensional non-isotropic subspace Q_d^r, whose corank can be increased subject to the corank bound, keeping all other flag elements unchanged. We will replace Q_d^r with $\tilde{Q}_d^{r+2}$. The branching rule simply says that under this specialization, the limit L' of a linear space L satisfying rank conditions with respect to the original flag satisfies the same rank conditions with the unchanged flag elements and either $\dim(L' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(L \cap \mathrm{Ker}(Q_d^r))$ or $\dim(L' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(L \cap \mathrm{Ker}(Q_d^r)) + 1$. Furthermore, both of these cases occur with multiplicity one unless the latter leads to a smaller dimensional variety or the former violates the linear space bound. See Sections 3 and 5 for an explicit statement of the rule and for examples.

The organization of this paper is as follows. In Section 2, we will recall basic facts concerning the geometry of isotropic Grassmannians. In Section 3, we will introduce the algorithm in combinatorial terms without reference to geometry. In Section 4, we will define symplectic restriction varieties and explain the combinatorics in geometric terms. In Section 5, we will describe the specialization and prove that the combinatorial game introduced in Section 3 computes the restriction coefficients. In the last section, we will give an application of symplectic restriction varieties to questions of rigidity.

Acknowledgements: The germs of the ideas in this paper date back to my conversations with Joe Harris while I was in graduate school. I would like to thank him for his guidance and unfailing support.

2. Preliminaries

In this section, we recall basic facts concerning the geometry of isotropic Grassmannians.

Let $n = 2m$ be a positive, even integer. Let V be an n-dimensional vector space over $\mathbb{C}$. Let Q be a non-degenerate, skew-symmetric form on V. By Darboux's Theorem, we can choose a basis for V such that in this basis Q is expressed as $\sum_{i=1}^{m} x_i \wedge y_i$. A subspace W of V is called *isotropic* if $w^T Q v = 0$ for any two vectors $v, w \in W$. The dimension of an isotropic subspace of V is at most m. Given a vector space W, the orthogonal complement $W^\perp$ of W is defined as the set of $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If the dimension of W is k, then the dimension of $W^\perp$ is $n-k$ and the restriction of Q to $W^\perp$ has rank $n-2k$ (or, equivalently, corank k).

The Grassmannian $SG(k,n)$ parameterizing k-dimensional isotropic subspaces of V is a homogeneous variety for the symplectic group $Sp(n)$. The Grassmannian $SG(m,n)$ parameterizing maximal isotropic subspaces has dimension

$$\dim(SG(m,n)) = \frac{m(m+1)}{2}.$$

This can be seen inductively. The dimension of $SG(1,2) \cong \mathbb{P}^1$ is one since every vector is isotropic with respect to Q. Consider the incidence correspondence

$$I = \{(w, W) \mid w \in \mathbb{P}(W) \text{ and } [W] \in SG(m,n)\}$$

parameterizing a pair of a maximal isotropic subspace W and a point w of $\mathbb{P}(W)$. The first projection of the incidence correspondence I maps to $\mathbb{P}(V)$ with fibers isomorphic to $SG(m-1, n-2)$. The second projection maps the incidence correspondence to $SG(m,n)$ with fibers isomorphic to $\mathbb{P}(W)$. By the Theorem on the Dimension of Fibers [**S**, I.6.7] and induction, we conclude that the dimension of $SG(m,n)$ is $\frac{m(m+1)}{2}$.

The dimension of the isotropic Grassmannian $SG(k,n)$ is

$$\dim SG(k,n) = \frac{m(m+1)}{2} + \frac{(m-k)(3k-m-1)}{2} = nk - \frac{3k^2-k}{2}.$$

To see this, consider the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \in SG(k,n), W_2 \in SG(m,n), W_1 \subset W_2\}$$

parameterizing two-step flags consisting of a k-dimensional isotropic space contained in a maximal isotropic space. Since every k-dimensional isotropic space can be completed to a maximal isotropic space, the first projection is onto $SG(k,n)$. The fibers of the first projection are isomorphic to the isotropic Grassmannian $SG(m-k, n-2k)$. The second projection is onto $SG(m,n)$ with fibers isomorphic to $G(k,m)$. The Theorem on the Dimension of Fibers [**S**, I.6.7] and the previous paragraph imply the claim.

More generally, we will need to study spaces parameterizing k-dimensional linear spaces isotropic with respect to a degenerate skew form Q_n^r of corank r on an n-dimensional vector space. Naturally, $n-r$ needs to be even. Since the restriction of Q_n^r to a linear space complementary to its kernel is non-degenerate, we conclude that the largest dimensional isotropic subspace has dimension $r + \frac{n-r}{2}$. Set $h = \frac{n-r}{2}$. Then the space of $(r+h)$-dimensional isotropic linear spaces with respect to Q_n^r is isomorphic to $SG(h, 2h)$ and has dimension $\frac{h(h+1)}{2}$. Considering the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \subset W_2 \text{ isotropic with respect to } Q_n^r,$$

$$\dim(W_1) = k, \text{ and } \dim(W_2) = h + r\},$$

we see that the space of k-dimensional isotropic subspaces of Q_n^r has dimension $\frac{h(h+1)}{2} + k(h+r-k)$ if $k \geq h$ and $\frac{h(h+1)}{2} + k(h+r-k) - \frac{(h-k)(h-k+1)}{2}$ if $k < h$.

By Ehresmann's Theorem [**E**] (see [**Bo**, IV.14.12]), the cohomology of $SG(k,n)$ is generated by the classes of Schubert varieties. Let $0 \leq s \leq k$ be a non-negative integer. Let $\lambda_\bullet : 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq m$ be a sequence of increasing positive integers. Let $\mu_\bullet : m > \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$ be a sequence of decreasing non-negative integers such that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq k$. Then the Schubert varieties in $SG(k,n)$ may be indexed by pairs of admissible sequences $(\lambda_\bullet; \mu_\bullet)$. Fix an isotropic flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots F_m \subset F_{m-1}^\perp \subset \cdots F_1^\perp \subset V.$$

The Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}(F_\bullet)$ is defined as the Zariski closure of the set of linear spaces

$$\{W \in SG(k,n) \mid \dim(W \cap F_{\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(W \cap F_{\mu_j}^\perp) = j \text{ for } s < j \leq k\}.$$

In the literature, it is customary to denote Schubert classes in the cohomology of $SG(m,n)$ by strictly decreasing partitions $m \geq a_1 > a_2 > \cdots > a_s > 0$ of length $s \leq m$. In our notation, the sequence $a_\bullet$ translates to the sequence $\lambda_\bullet$ by setting $a_i = m + 1 - \lambda_i$. Note that when $n = 2m$, the sequence $\lambda_\bullet$ determines the sequence $\mu_\bullet$ by the requirement that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq m$. Therefore, it is common to omit the sequence $\mu_\bullet$ from the notation. We will not follow this convention. In Schubert calculus, many authors prefer to record Schubert classes so that the codimension will be easily accessible. Our notation has the advantage that it is preserved under natural maps between Grassmannians arising from linear embeddings between ambient vector spaces.

We will index Schubert classes in the cohomology of the Grassmannian $G(k,n)$ by increasing sequences of non-negative integers $a_\bullet : 0 < a_1 < a_2 < \cdots < a_k \leq n$. The Schubert variety $\Sigma_{a_\bullet}(F_\bullet)$ with respect to a flag $F_\bullet$ parameterizes k-dimensional subspaces W of V that satisfy $\dim(W \cap F_{a_i}) \geq i$ for $1 \leq i \leq k$.

3. A combinatorial game

In this section, we will introduce a combinatorial game that computes the symplectic restriction coefficients. The purpose of this section is to explain the mechanics of the rule without reference to geometry. In the next two sections, we will interpret the game in geometric terms and prove that it computes the symplectic restriction coefficients. The geometrically minded reader may wish to look ahead at the next two sections.

Notation 3.1. Let $0 \leq s \leq k$ be an integer. A *sequence of n natural numbers of type s* for $SG(k,n)$ is a sequence of n natural numbers such that every number is less than or equal to $k-s$. We write the sequence from left to right with a small gap to the right of each number in the sequence. We refer to the gap after the i-th number in the sequence as the *i-th position*. For example, 1 1 2 0 0 0 0 0 and 3 0 0 2 0 1 0 0 are two sequences of 8 natural numbers of types 1 and 0, respectively, for $SG(3,8)$.

Definition 3.2. Let $0 \leq s \leq k$ be an integer. A *sequence of brackets and braces of type s* for $SG(k,n)$ consists of a sequence of n natural numbers of type s, s brackets] ordered from left to right and $k-s$ braces } ordered from right to left such that:

1. (1) Every bracket or brace occupies a position and each position is occupied by at most one bracket or brace.
2. (2) Every bracket is to the left of every brace.
3. (3) Every positive integer greater than or equal to i is to the left of the i-th brace.
4. (4) The total number of integers equal to zero or greater than i to the left of the i-th brace is even.

Example 3.3. 11]200}0}00 and 300}20}10}0 are typical examples of sequences of brackets and braces for $SG(3,8)$ that have the two examples from Notation 3.1 as their sequences of natural numbers. When writing a sequence of brackets and braces, we often omit the gaps not occupied by a bracket or a brace.

Example 3.4. Let us give several non-examples to clarify Definition 3.2. The first condition disallows diagrams such as]0000} (the first bracket is not in a position), 0]]000, 000}}0, 00]}00 (two brackets, two braces, or a bracket and a brace occupy the same position, respectively). The second condition disallows diagrams such as 00}0]000 (a brace cannot be to the left of a bracket). The third condition disallows diagrams such as 100}30}20}0 (3 is to the right of the third brace and 2 is to the right of the second brace). The fourth condition disallows diagrams such as 1]2000}0}00 (the number of zeros to the left of the second brace, and the number of zeros and twos to the left of the first brace are odd).

Notation 3.5. By convention, the brackets are indexed from left to right and the braces are indexed from right to left. We write $]^i$ and $\}^i$ to denote the i-th bracket and i-th brace, respectively. Their positions are denoted by $p(]^i)$ and $p(\}^i)$.

The position of a bracket or a brace is equal to the number of integers to its left. For notational convenience, we declare that, in a sequence of brackets and braces of type s for $SG(k,n)$, the brace $\}^{k-s+1}$ denotes $]^s$ and an integer in the sequence equal to $k-s+1$ should be read as 0. Let $l(i)$ denote the number of integers in the sequence that are equal to i. Let r_i be the total number of positive integers less than or equal to i that are to the left of $\}^i$. For $0 < j < i$, let $\rho(i,j) = p(\}^j) - p(\}^i)$ and let $\rho(i,0) = n - p(\}^i)$. Equivalently, $\rho(i,0)$ (respectively, $\rho(i,j)$) denotes the number of integers to the right of the i-th brace (respectively, to the right of the i-th brace and to the left of the j-th brace).

EXAMPLE 3.6. For the sequence of brackets and braces 300}20}10}0 for $SG(3,8)$, the positions are $p(\}^3) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_i = l(i) = 1$, for $1 \leq i \leq 3$, $\rho(i,i-1) = 2$, for $2 \leq i \leq 3$, and $\rho(1,0) = 1$.

EXAMPLE 3.7. For the sequence of brackets and braces 1]22]00}00}0 for $SG(4,8)$, the positions are $p(]^1) = 1, p(]^2) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_1 = l(1) = 1$, $l(2) = 2$, and $r_2 = 3$. Moreover, $\rho(2,1) = 2$ and $\rho(1,0) = 1$.

DEFINITION 3.8. Two sequences of brackets and braces are *equivalent* if the lengths of their sequence of numbers are equal, the brackets and braces occur at the same positions, and the collection of digits that occur between any consecutive brackets and/or braces are the same up to reordering.

EXAMPLE 3.9. The sequences 1221]00200}000}00, 1122]20000}000}00 and the sequences 003}02}01}0, 300}20}10}0 are equivalent pairs of sequences. We can depict an equivalence class of sequences by the representative where the digits are listed so that between any two consecutive brackets and/or braces the positive integers precede the zeros and are listed in non-decreasing order. We will always use this *canonical representative* and often blur the distinction between the equivalence class and this representative.

DEFINITION 3.10. A sequence of brackets and braces is *in order* if the sequence of numbers consists of a sequence of non-decreasing positive integers followed by zeros except possibly for one i immediately to the right of $\}^{i+1}$ for $1 \leq i < k-s$. Otherwise, we say that the sequence is *not in order*. A sequence is *in perfect order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros.

EXAMPLE 3.11. The sequences 11]00]100}000, 1]20000}1}0}00, 122]100}00}00 are not in order. The sequences 300}20}10}000, 11]22]00}00}00, 1]33]0000}200}0}0 are in order. Furthermore, 11]22]00}00}00 is in perfect order.

DEFINITION 3.12. A sequence of brackets and braces is *saturated* if $l(i) = \rho(i,i-1)$ for $1 \leq i \leq k-s$.

EXAMPLE 3.13. The sequences 11]22]00}00}00 and 1]22]100}00}00 are saturated, whereas, 22]00}00}00 and 1]0000}00}000 are not.

The next definition is a technical definition that plays a role in the proof and is a consequence of the order in which the game is played. The reader can define a symplectic diagram as a sequence of brackets and braces that occurs in the game and refer to the conditions only when necessary.

Definition 3.14. A *symplectic diagram* for $SG(k,n)$ is a sequence of brackets and braces of type s for $SG(k,n)$ for some $0 \leq s \leq k$ such that:

(S1) $l(i) \leq \rho(i, i-1)$ for $1 \leq i \leq k-s$.

(S2) Let τ_i be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Then
$$2\tau_i \leq p(\}^i) + r_i.$$

(S3) Either the sequence is in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of $i < \eta$ after $\}^{i+1}$.

(S4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If an integer i occurs to the left of all the zeros, then either $i = 1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j, j-1) = l(j)$ for $j_0 \neq j > \min(i, \eta)$ and $\rho(j_0, j_0 - 1) \leq l(j_0) + 2 - \xi_{j_0}$. Moreover, any integer η violating order occurs to the right of $\}^{j_0}$.

Remark 3.15. Conditions (S1) and (S2) are necessary to guarantee that symplectic diagrams represent geometrically meaningful objects. Conditions (S3) and (S4) are consequences of the order the game is played and describe the most complicated possible diagrams that can occur. The reader can ignore these conditions. They are necessary to carry out the dimension counts and to prove that the algorithm is defined at each step. They are not needed in order to run the algorithm.

Example 3.16. Let us give some examples to clarify Definition 3.14. Condition (S1) allows for diagrams such as 11]22]2]00}000}00 but disallows 22]3300}2}00}000 (there are two 3's and three 2's in the sequence but $\rho(2,3) = 1$ and $\rho(1,2) = 2$). Condition (S2) disallows diagrams such as 000]10}0 ($r_1 = 1$, $\tau_1 = 4$, but $2 \cdot 4 > 5 + 1$). Condition (S3) allows for 2344]300}00}00}10}0 (a non-decreasing sequence of positive integers 2344 followed by a sequence consisting of one 3, one 1 and zeros), but disallows 22]110000}2200}0000}00 (there are two 1s and two 2s following the non-decreasing sequence 22) or 22]133]00}00}00}0 (there are two 3s following the non-decreasing sequence 22). Condition (S4) allows for diagrams such as 11]3300}00}1}000, 1]1]33]00}00}00}00, however, it disallows diagrams such as 144]00}00}00}00}0 (1 occurs in the initial non-decreasing part of the sequence, but 2 and 3 do not occur. 1 is not followed by a bracket and $l(3) = 0 \neq \rho(3,2) = 2$, $l(2) = 0 \neq \rho(2,1) = 2$).

The next definition is crucial for the game and the reader should remember these conditions.

Definition 3.17. A symplectic diagram is called *admissible* if it satisfies the following additional conditions.

(A1) The two integers to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is one.

(A2) Let x_i be the number of brackets $]^h$ such that every integer to the left of $]^h$ is positive and less than or equal to i. Then
$$x_i \geq k - i + 1 - \frac{p(\}^i) - r_i}{2}.$$

EXAMPLE 3.18. Condition (A1) disallows diagrams such as 11]23]00}00}00}00 (the digits preceding the second bracket are not equal), 2]200}00}00 (there is a bracket in position 1, but the first digit is not 1). Condition (A2) is hard to visualize without resorting to counting. Let p be the position of the rightmost bracket such that every digit to the left of p is positive and less than or equal to i. In words, condition (A2) says that the total number of zeros and integers greater than i in the sequence is at least twice the number of brackets and braces in positions $p+1$ through $p(\}^i)$. The following diagrams violate condition (A2): 22}00}00 ($x_2 = 0$, $p(\}^2) = r_2 = 2$, but $0 < 1$), 200}2}00} (the number of braces up to $p(\}^2) = 4$ is 2; the number of zeros is 2, but $2 < 2 \cdot 2$), 11]33]00}00}1}000 (the total number of brackets and braces between positions 3 and $9 = p(\}^1)$ is 4. The number of zeros and integers greater than 1 is 6, but $2 \cdot 4 > 6$).

REMARK 3.19. The admissible symplectic diagrams are the main combinatorial objects in this paper. They represent symplectic restriction varieties, which are the main geometric objects of the paper and will be defined in the next section. The symplectic diagram records a non-necessarily isotropic flag. The corresponding symplectic restriction variety parameterizes isotropic spaces that satisfy certain rank conditions with respect to this flag. The definition of an admissible symplectic diagram reflects the basic facts about isotropic subspaces discussed in the introduction, as we will see in the next section.

DEFINITION 3.20. The *symplectic diagram $D(\sigma_{\lambda;\mu})$ associated to the Schubert class* $\sigma_{\lambda;\mu}$ in $SG(k,n)$ is the saturated symplectic diagram in perfect order, where the brackets occur at positions $\lambda_1, \dots, \lambda_s$ and the braces occur at positions $n - \mu_{s+1}, \cdots, n - \mu_k$.

EXAMPLE 3.21. The symplectic diagram associated to $\sigma_{2,4;4,2}$ in $SG(4,10)$ is 11]22]00}00}00.

LEMMA 3.22. *The diagram $D(\sigma_{\lambda;\mu})$ is an admissible symplectic diagram.*

PROOF. Let $n = 2m$. Since $0 < \lambda_1 < \cdots < \lambda_s \leq m < n - \mu_{s+1} < \cdots < n - \mu_k$, the brackets and braces occur in different positions and the brackets are to the left of the braces. Since the sequence is saturated and in perfect order, the number of integers in the sequence equal to i is $\mu_{k-i+1} - \mu_{k-i+2} \leq \mu_{s+1} < m$ (with the convention that $\mu_{k+1} = 0$), for $1 \leq i \leq k-s$ and occur to the left of $\}^{k-s}$. Finally, the number of integers equal to zero or greater than or equal to i to the left of $\}^i$ is $n - 2\mu_{k-i+1} = 2(m - \mu_{k-i+1})$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all 4 conditions in Definition 3.2.

By definition, $D(\sigma_{\lambda;\mu})$ is saturated, so $l(i) = \rho(i, i-1)$ and conditions (S1) and (S4) hold. Since the diagram is in perfect order, (S3) holds and

$$\tau_i = \max(\lambda_s, \mu_{s+1}) \leq m.$$

On the other hand, $p(\}^i) + r_i = n - \mu_{k-i+1} + \mu_{k-i+1} = n = 2m \geq 2\tau_i$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all the conditions in Definition 3.14.

Finally, since $\lambda_j \neq \mu_i + 1$ for any i, j, the two integers preceding a bracket must be equal. Furthermore, if $\lambda_1 = 1$, $\mu_1 \geq 1$. Hence, condition (A1) holds. For $1 \leq i \leq k-s$, $k-i+1-(p(\}^i)-r_i)/2 = k-i+1+\mu_{k-i+1}-m$. From the sequence $0, 1, \dots, m-1$, remove the integers $\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_s - 1$ to obtain a sequence $\alpha_m < \alpha_{m-1} < \cdots < \alpha_{s+1}$. By assumption $\mu_{k-i+1} = \alpha_j$ for some $j \geq k-i+1$. Hence, $k-i+1+\mu_{k-i+1}-m \leq \alpha_j - (m-j) = x_i$. To see the last equality, observe that x_i is the number of integers λ_h that are less than or equal to $\mu_{k-i+1} = \alpha_j$. This number is equal to the number of integers $(\alpha_j - (m-j))$ between 0 and α_j that do not occur in the sequence $\alpha_m, \ldots, \alpha_j$. Hence, condition (A2) holds. We conclude that $D(\sigma_{\lambda;\mu})$ is an admissible symplectic diagram. □

The game is defined on admissible symplectic diagrams. We will see in the next section that saturated admissible diagrams in perfect order represent Schubert varieties in $SG(k,n)$. The goal of the algorithm is to transform every admissible symplectic diagram to a collection of saturated admissible diagrams in perfect order. Given an admissible symplectic diagram D, we will associate to it one or two sequences D^a and/or D^b of brackets and braces. Initially, neither D^a nor D^b has to be admissible. We will shortly describe an algorithm that modifies D^a and D^b so that they become admissible. The game records a degeneration of the flag elements represented by D.

Definition 3.23. Let D be an admissible symplectic diagram of type s for $SG(k,n)$. For the purposes of this definition, read any mention of $k-s+1$ as 0 and any mention of $\}^{k-s+1}$ as $]^s$.

(1) If D is not in order, let η be the integer in condition (S3) violating the order.

- (i) If every integer $\eta < i \leq k-s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta+1$ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by interchanging η and ν.
- (ii) If an integer $\eta < i \leq k-s$ does not occur to the left of η, let ν be the leftmost integer equal to $i+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{i+1}$ not equal to ν and changing ν to i.

(2) If D is in order but is not a saturated admissible diagram in perfect order, let κ be the largest index for which $l(i) < \rho(i, i-1)$.

- (i) If $l(\kappa) < \rho(\kappa, \kappa-1) - 1$, let ν be the leftmost digit equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by changing ν and the leftmost 0 to the right of $\}^{k+1}$ not equal to ν to κ.
- (ii) If $l(\kappa) = \rho(\kappa, \kappa-1) - 1$, let η be the integer equal to $\kappa-1$ immediately to the right of $\}^{\kappa}$.
  - (a) If κ occurs to the left of η, let ν be the leftmost integer equal to κ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by changing ν to $\kappa-1$ and η to zero.
  - (b) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{\kappa+1}$ not equal to ν and changing ν to κ.

Let p be the position in D immediately to the right of ν. If there exists a bracket at a position $p' > p$ in D^a, let $q > p$ be the minimal position occupied by a bracket in D^a. Let D^b be the diagram obtained from D^a by moving the bracket at position q to position p. Otherwise, D^b is not defined.

Example 3.24. Let $D = 2300\}10\}0\}0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}20\}0\}0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]200\}00\}$, then the second 2 violates the order and $D^a = 220]000\}00\}$, $D^b = 22]0000\}00\}$.

Let $D = 124400\}00\}1\}0\}00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii) and $D^a = 123400\}10\}0\}0\}00$.

Let $D = 22]00\}00\}00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1,0)-1$, we are in case (2)(i) and $D^a = 12]00\}10\}00$ and $D^b = 1]200\}10\}00$.

Let $D = 3300\}200\}0\}$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3,2)-1$, we are in case (2)(ii)(a) and $D^a = 2300\}000\}0\}$.

Finally, let $D = 330000\}00\}1\}0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2,1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b) and $D^a = 230000\}10\}0\}0$.

We will soon check that both D^a and D^b are symplectic diagrams; however, they do not have to be admissible. We now describe algorithms for turning them into admissible diagrams.

ALGORITHM 3.25. If D^a is not an admissible symplectic diagram, perform the following steps to turn it into an admissible diagram.

Step 1. If D^a does not satisfy condition (A2), let i be the maximal index for which condition (A2) fails. Define a new diagram D^c as follows. Let the two rightmost integers equal to i in D^a be in the places $\pi_1 < \pi_2$. Delete $\}^i$ and move the i in place π_2 to place $\pi_1 + 1$. Slide the integers in places $\pi_1 < \pi < \pi_2$ and brackets and braces in positions $\pi_1 < p < \pi_2$ one to the right. Add a bracket at position $\pi_1 + 1$. Subtract one from the integers $i < h \leq k - s$; and if $i = k - s$, change the integers equal to $k - s$ to 0. Let D^c be the resulting diagram and replace D^a with D^c. If D^a satisfies condition (A2), proceed to the next step.

Step 2. If D^a fails condition (A1), let $]^j$ be the smallest index bracket for which it fails and let i be the integer preceding $]^j$. Change this i to $i-1$ ($k-s$ if $i=0$) and move $\}^{i-1}$ ($\}^{k-s}$ if $i = 0$) one position to the left. Repeat this procedure until the sequence of brackets and braces satisfies condition (A1). Let the resulting sequence be D^c. In both steps, we refer to D^c as a *quadric diagram derived from* D^a.

ALGORITHM 3.26. If D^b does not satisfy condition (A1), run Step 2 of Algorithm 3.25 on D^b. Explicitly, let $]^j$ be the minimal index bracket for which (A1) fails. Let i be the integer immediately to the left of $]^j$. Replace i with $i - 1$ and move $\}^{i-1}$ one position to the left. As long as the resulting sequence does not satisfy condition (A1), repeat this process either until the resulting sequence is an admissible symplectic diagram (in which case, this is *the symplectic diagram derived from* D^b) or two braces occupy the same position. In the latter case, no admissible symplectic diagrams are derived from D^b.

EXAMPLE 3.27 (Examples of Algorithm 3.25). Let $D = 22]33]00\}00\}00\}00$. Then the diagram $D^a = 12]33]00\}00\}10\}00$ fails condition (A2) since $x_1 = 0 < 1 = 5 - (10-2)/2$. Hence, according to Step 1 of Algorithm 3.25, we replace D^a with $11]1]22]00\}00\}000$ (delete $\}^1$, move the 1 in position 9 to position 2 and slide everything in positions 2-8 one position to the right, add a bracket in position 2, and subtract 1 from the integers greater than 1). The latter is an admissible diagram.

Let $D = 00\}00\}00$. Then $D^a = 22\}00\}00$ fails condition (A2) since $x_2 = 0 < 1 - (2-2)/2$. Hence, Step 1 of Algorithm 3.25 replaces D^a with $00]00\}00$ (delete $\}^2$ and add a bracket in position 2), which is admissible.

Similarly, if $D = 11]33]00\}00\}00\}00$, then the diagram $D^a = 11]23]00\}20\}00\}00$ fails condition (A2) since $x_2 = 1 < 2$. Hence, according to Step 1 of Algorithm 3.25, we replace D^a with $11]22]2]00\}000\}00$, which is admissible.

If $D = 22]2]200\}0000\}00$, then the diagram $D^a = 12]2]200\}1000\}00$ is not admissible since it fails condition (A1) for $]^1$. Step 2 of Algorithm 3.25 replaces D^a first with $11]2]200\}100\}000$ (change the 2 preceding $]^1$ to 1 and move $\}^1$ one position to the right). Note that this diagram fails condition (A1) for $]^2$. Hence, Step 2 replaces it with $11]1]200\}10\}0000$ (change the 2 preceding $]^2$ to 1 and move $\}^1$ one position to the left). This diagram is admissible, hence it is the diagram derived from D^a.

Example 3.28 (Examples of Algorithm 3.26). Let $D = 11]33]00\}00\}00\}00$, then $D^b = 11]2]300\}20\}00\}00$ fails condition (A1). Algorithm 3.26 replaces it with $11]1]300\}20\}0\}000$, which is admissible.

Let $D = 00]0000\}00\}00\}$, then $D^b = 3]30000\}00\}00\}$ does not satisfy condition (A1) since the digit to the left of $]^1$ has to be 1. Algorithm 3.26 replaces D^b first with $2]30000\}0\}000\}$, which still fails condition (A1). Hence, Algorithm 3.26 replaces this diagram with $1]30000\}0\}00\}0$, which is admissible.

If $D = 00]0000\}2\}0\}$, then $D^a = 30]2000\}0\}0\}$ and $D^b = 3]20000\}0\}0\}$. They both fail condition (A1). When we run Algorithm 3.26 on D^b, we turn the 3 into 2 and slide $\}^2$ one position to the left. In that case, we obtain $1]30000\}\}00\}$. Since two braces occupy the same position, no diagrams are derived from D^b in this case. When we run Algorithm 3.25 on D^a, we obtain the admissible diagram $33]200\}00\}0\}$.

Let D be an admissible symplectic diagram and let ν be as in Definition 3.23. Let $\pi(\nu)$ denote the place of ν in the sequence of integers. If $p(]^s) > \pi(\nu)$, then $]^{x_{\nu-1}+1}$ is the first bracket to the right of ν. If the integer to the immediate left of $]^{x_{\nu-1}+1}$ is positive, let $y_{x_{\nu-1}+1}$ be this integer. Otherwise, let $y_{x_{\nu-1}+1} = k - s + 1$. The condition $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ plays an important role. In words, this condition says that the number of values larger than ν or equal to zero that the integers to the left of $]^{x_{\nu-1}+1}$ attain is one more than the cardinality of the set of integers consisting of zero and integers larger than ν occurring to the left of $]^{x_{\nu-1}+1}$. In view of conditions (S3), (S4) and (A1), a sequence satisfying this equality looks like

$$\cdots \nu\ \nu+1 \cdots \nu+l-1\ \nu+l\ \nu+l] \cdots \quad \text{or} \quad \cdots \nu\ \nu+1\ \cdots \nu+l\ 00] \cdots,$$

where we have drawn the part of the sequence starting with the left most ν and ending with $]^{x_{\nu-1}+1}$. We are now ready to state the algorithm.

Algorithm 3.29. Let D be an admissible, symplectic diagram of type s for $SG(k,n)$. If D is saturated and in perfect order, return D and stop. Otherwise, let D^a and D^b be defined as in Definition 3.23.

(1) If $p(]^s) \leq \pi(\nu)$ or $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D, then return the admissible symplectic diagrams that are derived from D^a.
(2) Otherwise, return the admissible symplectic diagrams that are derived from both D^a and D^b.

We run the algorithm on two symplectic diagrams.

EXAMPLE 3.30.

$$\begin{tikzcd} 00\}00\}00 \arrow[r] & 00]00\}00 \arrow[r] \arrow[d] & 00]0]000 \\ & 1]100\}00 & \end{tikzcd}$$

In this example, first $D^a = 22\}00\}00$ is not admissible since the diagram fails condition (A2). Therefore, we replace it by $00]00\}00$. Next, $D^a = 10]10\}00$ and $D^b = 1]100\}00$. D^a is not admissible since it does not satisfy condition (A2). Hence, we replace it by the admissible diagram $00]0]000$. D^b is admissible. Note that the last two diagrams are saturated and in perfect order, so the algorithm terminates. We will soon see that this calculation shows $i^*\sigma_{2,4} = \sigma_{2,3;} + \sigma_{1;2}$ in $SG(2,6)$.

Finally, we give a larger example in $SG(3,10)$ that illustrates the inductive structure of the game.

EXAMPLE 3.31.

$$\begin{tikzcd} 300\}20\}10\}000 \arrow[r] & 200\}00\}10\}000 \arrow[r] & 200]00\}10\}000 \arrow[r] \arrow[d] & 1]0000\}00\}000 \arrow[d] \\ & & 100]00\}00\}000 \arrow[dl] \arrow[d] & 1]2200\}00\}000 \arrow[d] \\ & 100]0]000\}000 \arrow[dl] \arrow[d] & 11]200\}0\}0000 \arrow[d] & 1]1200\}10\}000 \arrow[d] \\ \underline{000]0]0]00000} & 11]00]100\}000 \arrow[dl] \arrow[d] & \underline{11]11]00\}0000} & 1]1100\}00\}000 \arrow[d] \\ \underline{11]1]0000\}000} & \underline{11]11]00\}0000} & & \underline{1]1100]00\}000} \end{tikzcd}$$

We will see that this calculation shows $i^*\sigma_{3,5,7} = \sigma_{3,4,5;} + \sigma_{2,3;3} + 2\sigma_{2,4;4} + \sigma_{1,5;3}$ in $H^*(SG(3,10),\mathbb{Z})$.

DEFINITION 3.32. A *degeneration path* is a sequence of admissible symplectic diagrams

$$D_1 \to D_2 \to \cdots \to D_r$$

such that D_{i+1} is one of the outcomes of running Algorithm 3.29 on D_i for $1 \leq i < r$.

The main theorem of this paper is the following.

THEOREM 3.33. *Let D be an admissible symplectic diagram for $SG(k,n)$. Let $V(D)$ be the symplectic restriction variety associated to D. Then, in terms of the Schubert basis of $SG(k,n)$, the cohomology class $[V(D)]$ can be expressed as*

$$[V(D)] = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu},$$

where $c_{\lambda;\mu}$ is the number of degeneration paths starting with D and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

Theorem 1.1 stated in the introduction is a corollary of Theorem 3.33.

Definition 3.34. Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then $i^*\sigma_{a_\bullet} = 0$ and we do not associate a symplectic diagram to $\sigma_{a_\bullet}$. Suppose that $a_j \geq 2j-1$ for $1 \leq j \leq k$. Let u be the number of i such that $a_i = 2i-1$. For j such that $a_j \neq 2j-1$, let u_j be the number of integers $i < j$ such that $a_i = 2i-1$. Let v_j be the number of integers $i > j$ such that $a_i = 2i-1$. Then the diagram $D(a_\bullet)$ associated to $i^*\sigma_{a_\bullet}$ is a diagram consisting of u brackets at positions $1, 2 \cdots, u$ and a brace for each $a_j > 2j-1$ at position $a_j - u_j + v_j$. The sequence of integers consists of u integers equal to 1 followed by zeros except for one integer equal to $k-j-v_j+1$ immediately following the first bracket or brace to the right of $\}^{k-j-v_j+1}$ (or in the first position if $j + v_j = 1$) for each odd $a_j > 2j-1$.

Example 3.35. The diagram $D(\sigma_{3,5,7})$ in $SG(3,8)$ is 300}20}10}0. The diagram $D(\sigma_{1,3,6,7,10})$ in $SG(5,10)$ is 1]1]1]00}00}000. The diagram $D(\sigma_{1,3,7,8,9,12})$ in $SG(6,14)$ is 1]1]1]300}0}00}00000.

Remark 3.36. The reader will notice that $D(\sigma_{a_\bullet})$ is the diagram obtained by running Algorithm 3.25 on the diagram that has a brace at positions a_j and whose sequence consists of zeros except for one $k-j+1$ immediately to the right of $\}^{k-j+2}$ when a_j is odd.

Lemma 3.37. *If $a_j \geq 2j-1$ for $1 \leq j \leq k$, then $D(a_\bullet)$ is an admissible symplectic diagram.*

Proof. The brackets occur at positions $1, \dots, u$. Let a_j and a_{j+l} be two consecutive integers in the sequence $a_\bullet$ satisfying $a_i > 2i-1$. Then the positions of the corresponding braces are $a_j - u_j + v_j$ and $a_{j+l} - u_{j+l} + v_{j+l}$. Since $u_{j+l} = u_j + l - 1$ and $v_{j+l} = v_j - l + 1$, the positions of the two braces differ by the quantity $\beta = a_{j+l} - a_j - 2l + 2$. If $l = 1$, $\beta > 0$. If $l > 1$, then $a_j < a_{j+1} = 2j+1$. Since $a_{j+l} \geq 2j + 2l$, β is also positive. The first brace corresponds to the smallest index j_0 such that $a_{j_0} > 2j_0 - 1$ and occurs at position $a_{j_0} - (j_0 - 1) + (u - j_0 + 1) = u + a_{j_0} - 2j_0 + 2 \geq u + 2$. The number of positive integers less than or equal to $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is u (respectively, $u+1$) if a_j is even (respectively, odd). Hence, the number $a_j - u_j + v_j - u(-1) = a_j - 2u_j(-1)$ (where -1 occurs if a_j is odd) of integers equal to zero or greater $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is even. Therefore, conditions (1)-(4) of Definition 3.2 hold.

By construction, $l(i) \leq 1$ for $i > 1$ and $l(1) = u(+1)$ depending on whether the largest $a_j > 2j-1$ is even (or odd). In either case, one easily sees that $l(1) \leq \rho(1,0)$. The number of positive integers to the left of $\}^{k-j-v_j+1}$ is equal to u plus the number o_j of odd $a_l < a_j$ such that $a_l > 2l-1$. Since $2(u_j + o_j) \leq 2j \leq a_j$, we have that $2(u+o_j) \leq a_j - u_j + v_j + u = a_j + 2v_j$ and condition (S2) holds. The sequence is in order and the only integers other than $k-u$ occurring in the initial part of the sequence are ones, which are followed by brackets. We conclude that all the conditions in Definition 3.14 hold.

Since any bracket is preceded by 1, condition (A1) holds. Finally, for $\}^{k-j-v_j+1}$, the quantity $j + v_j - \frac{a_j - u_j + v_j - u(-1)}{2} = j + u - \frac{a_j(-1)}{2} \leq u$ (where -1 occurs if a_j is odd) since $a_j > 2j-1$. We conclude that $D(a_\bullet)$ is an admissible symplectic diagram. $\square$

The precise formulation of Theorem 1.1 is given by the following corollary.

COROLLARY 3.38. *Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then set $i^*\sigma_{a_\bullet} = 0$. Otherwise, let $D(\sigma_{a_\bullet})$ be the diagram associated to $\sigma_{a_\bullet}$. Express*

$$i^*\sigma_{a_\bullet} = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu}$$

in terms of the Schubert basis of $SG(k,n)$. Then $c_{\lambda;\mu}$ is the number of degeneration paths starting with $D(\sigma_{a_\bullet})$ and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

PROOF. In Lemma 4.20, we will prove that the intersection of $SG(k,n)$ with a general Schubert variety in $G(k,n)$ with class $\sigma_{a_\bullet}$ is a restriction variety of the form $V(D(\sigma_{a_\bullet}))$. The corollary is immediate from this lemma and Theorem 3.33. $\square$

We conclude this section by proving that Algorithm 3.29 is well-defined and terminates. The proof of Theorem 3.33 is geometric and will be taken up in the next two sections.

PROPOSITION 3.39. *Algorithm* 3.29 *replaces an admissible symplectic diagram with one or two admissible symplectic diagrams.*

PROOF. If D is a saturated symplectic diagram in perfect order, then the algorithm returns D and there is nothing further to check. We will first check that D^a and D^b are (not necessarily admissible) symplectic diagrams. The diagram D^b is obtained from D^a by moving a bracket to the left. Conditions (2), (3), (4) of Definition 3.2 and conditions (S1), (S2), (S3) and (S4) of Definition 3.14 are preserved under moving a bracket to the left. Since $\nu \neq 1$ is the leftmost integer in D equal to a given integer, by condition (A1) for D, there cannot be a bracket at position p in D or D^a. Hence, condition (1) is satisfied for D^b. We conclude that if D^a is a symplectic diagram, then D^b is also a symplectic diagram. We will now check that D^a is a symplectic diagram in each case.

In case (1)(i), by condition (S3) for D, let η be the unique integer that violates the order. Since η is violating the order, η is to the left of $\}^{\eta+1}$. D^a is obtained by swapping η and ν, the leftmost integer equal to $\eta+1$. This operation does not change the positions of the brackets and braces and keeps $l(i)$ fixed for every i. After the swap, every integer i is still to the left of $\}^i$ for every i since η was to the left of $\}^{\eta+1}$. Furthermore, the operation also preserves or decreases τ_i for every i. We thus conclude that conditions (1) through (4) of Definition 3.2 and condition (S1), (S2) and (S4) of Definition 3.14 hold for the diagram D^a. After the swap, η is part of the non-decreasing initial sequence in D^a. Hence, the diagram D^a is either in order or $\eta+1$ is the only integer violating the order. Condition (S3) holds for D^a. We conclude that D^a is a symplectic diagram.

In case (1)(ii), let η be the unique integer that violates the order. Assume that $\eta < i \leq k-s$ does not occur to the left of η. Then i does not occur anywhere in the sequence and, in condition (S4) for D, $i = j_0$. We claim that the i-th and $(i-1)$-st braces in D must look like $\cdots\}^i\eta\}^{i-1}\cdots$. By conditions (S3) and (S4) for D, η is to the right of $\}^i$ and to the left of $\}^{\eta+1}$. If η is between $\}^{i+h}$ and $\}^{i+h-1}$ for $h \neq -1$, then since $\rho(i+h, i+h-1) = l(i+h)$ by condition (S4), the parity in condition (4) is violated for $\}^{i+h-1}$. We conclude that η is between $\}^i$ and $\}^{i-1}$. Furthermore, $1 \leq \rho(i,i-1) \leq l(i) + 2 - \xi_i = 1$ by condition (S4). The formation of D^a does not affect conditions (1) through (3) in Definition 3.2. Condition (4) holds for D^a since the formation of D^a changes the number of integers that are equal to zero or greater than j to the right of $\}^j$ only when $j = i$ and for $\}^i$ it changes the number by two. Since the formation of D^a only increases $l(i)$ by one and decreases or preserves $l(j)$ for $j \neq i$, D^a satisfies (S1). Similarly, τ_i increases by one and all other τ_j remain fixed or decrease. On the other hand, r_i increases by two, hence D^a satisfies condition (S2). There is one exception. If $i = k - s$ and every integer to the left of $]^s$ is positive, τ_{k-s} increases by two. Then, $\tau_{k-s} = r_{k-s}$, hence $2\tau_{k-s} \leq p(\}^{k-s}) + r_{k-s}$ and D^a satisfies (S2). The diagram D^a is either in order or η is still the only integer violating the order, hence D^a satisfies (S3). Finally, the formation of D^a changes $l(i) = 1$ and decreases $l(i+1)$ by one. Hence, $l(i) = \rho(i, i-1)$ for D^a. By condition (S4) for D, we have that $\rho(j, j-1) = l(j)$ in D^a for any j for which the equality held for D except for $j = i + 1$. Furthermore, $\xi_{i+1} = 1$ in D^a, so $\rho(i+1, i) = l(i+1) + 1 = l(i+1) + 2 - \xi_{i+1}$ in D^a. Hence (S4) holds for D^a. We conclude that D^a is a symplectic diagram.

From now on assume that D is in order. Then there cannot be $i \geq \kappa$ such that i is immediately to the right of $\}^{i+1}$. Suppose there exists such an i. The number $\chi(i)$ and $\chi(i+1)$ of zeros and integers greater than i, respectively $i+1$, to the left of $\}^i$, respectively $\}^{i+1}$, has to be even. However, $\chi(i) = \chi(i+1) + l(i+1) + \rho(i+1, i) - 1$. Since by assumption $\rho(i+1, i) = l(i+1)$, we conclude that either $\chi(i)$ or $\chi(i+1)$ cannot be even leading to a contradiction.

In case (2)(i), changing ν to κ and the first zero to the right of $\}^{\kappa+1}$ does not change the positions of brackets and braces, it decreases $l(\kappa+1)$ by one and increases $l(\kappa)$ by two. Furthermore, the sequence D^a is still in order, unless $\kappa = k - s$ and there are zeros to the left of $]^s$. In the latter case, the κ to the right of $]^s$ is the unique integer violating order. Since by assumption $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ in D, $l(\kappa) \leq \rho(\kappa, \kappa - 1)$ in D^a. The parity of the integers equal to zero or greater than i also remains constant for all $1 \leq i \leq k - s$. We conclude that conditions (1) through (4) in Definition 3.2 and conditions (S1) and (S3) in Definition 3.14 hold for D^a. The quantity τ_i remains constant for $i > \kappa$ and increases by one for $i \leq \kappa$ unless $\kappa = k-s$, $l(k-s) \geq p(]^s)$ and τ_{k-s} increases by two. In the latter case, τ_{k-s} is less than or equal to both r_{k-s} and $p(\}^{k-s})$ and (S2) holds. In the former case, r_κ increases by two, hence (S2) holds for the index κ. Since $\rho(\kappa, \kappa-1) < l(\kappa) - 1$, (S2) also holds for indices $i < \kappa$. If there exists an index $i < \kappa$ in D such that i is not a 1 followed by a bracket, then in condition (S4) for D, we have that $j_0 = \kappa$. Furthermore, $\rho(\kappa, \kappa-1) = l(\kappa) + 2$. Hence, the formation of D^a preserves the equalities in condition (S4) except for $j = \kappa$ or $\kappa + 1$. In D^a, we have that $\rho(\kappa, \kappa-1) = l(\kappa)$ and $\rho(\kappa+1, \kappa) = l(\kappa+1) + 1 = l(\kappa+1) + 2 - \xi_{\kappa+1}$. We conclude that condition (S4) holds for D^a. Therefore, D^a is a symplectic diagram.

Finally, the argument showing that D^a is a symplectic diagram in case (2)(ii)(a) is identical to the argument in case (1)(i) and the argument in case (2)(ii)(b) is identical to the case (1)(ii), so we leave them for the reader. We conclude that both D^a and D^b are symplectic diagrams. However, they need not be admissible. We now check that Algorithms 3.25 and 3.26 preserve the fact that the resulting sequences are symplectic diagrams and output admissible symplectic diagrams.

D^a may fail to be admissible either because it fails condition (A1) or (A2) in Definition 3.17. The formation of D^a from D does not change the quantities x_h, $p(\}^h)$. In cases (1)(i) and (2)(ii)(a) the quantity r_h either remains the same or decreases. Hence, in these cases D^a satisfies condition (A2). In case (1)(ii), r_h remains the same or decreases except for r_i, which increases by two. Hence, the inequality in condition (A2) can only be violated for the index i by one. If it is violated, we conclude that in D, we have $x_i = k-i+1-\frac{p(\}^i)-r_i}{2}$. Recall that in this case D looks like $\cdots \}^i \eta \}^{i-1} \cdots$. Since i does not appear in D, $x_i = x_{i-1}$. Writing the inequality in (A2) for D and the index $i-1$ and noting that $r_{i-1} = r_i + 1$ and $p(\}^{i-1}) = p(\}^i) + 1$, we see that $x_i = x_{i-1} \geq k - i + 2 - \frac{p(\}^i)-r_i}{2} = x_i + 1$. Since D satisfies (A2), this is a contradiction. We conclude that D^a satisfies (A2) also in the case (1)(ii). By similar reasoning, in cases (2)(i) and (2)(ii)(b), D^a can violate the inequality in (A2) only for the index κ by one. After Step 1 of Algorithm 3.25, all the inequalities in condition (A2) remain unchanged or improve and $\}^\kappa$ is eliminated. We conclude that after Step 1, the resulting diagram satisfies (A2). When the inequality in (A2) is violated for D^a, it is violated for the index κ by at most 1. When we form D^b in cases (2)(i) and (2)(ii)(b), x_κ also increases by one. Hence, D^b, when it exists, always satisfies (A2).

Observe that the operation in Step 1 of Algorithm 3.25 preserves the fact that D^a is a symplectic diagram. By construction, conditions (1)-(4) and (S1) and (S2) hold. The diagram resulting after Step 1 is in order, hence (S3) holds. The operation renames $l(i)$ as $l(i-1)$ for $i > \kappa+1$ and $\rho(i+1,i)$ as $\rho(i,i-1)$ for $i > \kappa+1$. The operation does not change the quantities $l(i)$ and $\rho(i,i-1)$ when $i < \kappa$ and replaces $l(\kappa)$ and $l(\kappa+1)$ with their sum under the name $l(\kappa)$. The quantities $\rho(\kappa,\kappa-1)$ and $\rho(\kappa+1,\kappa)$ are replaced by $\rho(\kappa,\kappa-1)+\rho(\kappa+1,\kappa)-1$ and renamed $\rho(\kappa,\kappa-1)$. Hence, the equalities in condition (S4) are preserved. Since (A1) also holds for the resulting diagram D^c, we conclude that if D^a fails condition (A2), then Step 1 of Algorithm 3.25 produces an admissible symplectic diagram.

Observe that changing a digit to the left of a bracket and moving a brace one unit to the left, increases x_i and r_i by one and decreases $p(\}^i)$ by one. Hence, it preserves the inequality in condition (A2). It also preserves the conditions (1) through (4) and (S1) through (S4), with the possible exception of (1) in case $p(\}^{i+1}) = p(\}^i) - 1$. Condition (A1) is violated for D^a when there is a bracket in position $p(\nu)+1$ and it is violated only for that bracket. After l applications of Step 2 of Algorithm 3.25, Condition (A1) is still violated if there exists brackets at positions $p(\nu)+1, p(\nu)+2, \cdots, p(\nu)+l$. Since there are a finite number of brackets, this process stops and the resulting diagram satisfies condition (A1). In this case, the only brace that moves is $\}^{\nu-1}$. Since $l(\nu) \leq \rho(\nu,\nu-1)$ in D, the intermediate sequences and the resulting sequence all satisfy condition (1). If D^b does not satisfy condition (A1), then the only bracket that can violate it is the one in position $p(\nu)$. In this case, Algorithm 3.26 successively decreases the integer to the right of the bracket in $p(\nu)$ by one until it either becomes equal to the integer to its right or to one in case there isn't an integer to its right. Hence, this algorithm terminates in finitely many steps. A diagram might violate condition (1) in the process, but in that case the diagram is discarded. Hence, after finitely many steps either the diagram is discarded or results in an admissible symplectic diagram. We conclude that Algorithm 3.29, replaces D with one or two admissible symplectic diagrams. □

PROPOSITION 3.40. *After finitely many applications of Algorithm* 3.29*, every admissible symplectic diagram is transformed to a collection of admissible symplectic diagrams in perfect order.*

PROOF. If the diagram D is not in order, after one application of the algorithm either the diagram is in order or the integer violating the order increases or the position of the integer violating the order in the sequence decreases. Since these steps cannot go on indefinitely, after finitely many steps, the diagram is in order. Furthermore, during the process either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$ in the initial part of the sequence remains constant or increases. If the diagram is in order, then at each application of the algorithm either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$, in the initial part of the sequence increases. Since these cannot go on indefinitely, we conclude that repeated applications of the algorithm transform an admissible symplectic diagram into a collection of admissible symplectic diagrams in perfect order. Hence, the algorithm terminates in finitely many steps. ☐

4. Symplectic restriction varieties

In this section, we interpret admissible symplectic diagrams geometrically. We introduce symplectic restriction varieties and discuss their basic geometric properties.

Recall that Q denotes a non-degenerate skew-symmetric form on a vector space V of dimension n. Let L_{n_j} denote an isotropic subspace of Q of dimension n_j. Let $Q_{d_i}^{r_i}$ denote a linear space of dimension d_i such that the restriction of Q to it has corank r_i. Let K_i denote the kernel of the restriction of Q to $Q_{d_i}^{r_i}$.

Definition 4.1. A *sequence* $(L_\bullet, Q_\bullet)$ is a partial flag of linear spaces $L_{n_1} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q_{d_{k-s}}^{r_{k-s}} \subsetneq \cdots \subsetneq Q_{d_1}^{r_1}$ such that

- $\dim(K_i \cap K_h) \geq r_i - 1$ for $h > i$.
- $\dim(L_{n_j} \cap K_i) \geq \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}}) - 1)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$.

The main geometric objects of this paper will be sequences satisfying further properties.

Definition 4.2. A sequence is *in order* if

- $K_i \cap K_h = K_i \cap K_{i+1}$, for all $h > i$ and $1 \leq i \leq k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}}))$, for $1 \leq j \leq s$ and $1 \leq i < k-s$.

A sequence $(L_\bullet, Q_\bullet)$ is *in perfect order* if

- $K_i \subseteq K_{i+1}$, for $1 \leq i < k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, r_i)$ for all i and j.

Definition 4.3. A sequence $(L_\bullet, Q_\bullet)$ is called *saturated* if $d_i + r_i = n$, for $1 \leq i \leq k-s$.

The next definition is the analogue of Definition 3.14 and is a consequence of the order of specialization.

Definition 4.4. A sequence $(L_\bullet, Q_\bullet)$ is called a *symplectic sequence* if it satisfies the following properties.

(GS1) The sequence $(L_\bullet, Q_\bullet)$ is either in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that

$$K_i \subseteq K_h \quad \text{for} \quad h > i > \eta \quad \text{and} \quad K_i \cap K_h = K_i \cap K_{i+1} \quad \text{for} \quad i < \eta \quad \text{and} \quad h > i.$$

Furthermore, if $K_\eta \subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}})) \quad \text{for} \quad i < \eta \quad \text{and}$$

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) - 1) \text{ for } i \geq \eta.$$

If $K_\eta \not\subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}})) \text{ for all } i.$$

(GS2) If $\alpha = \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) > 0$, then either $i = 1$ and $n_\alpha = \alpha$ or there exists at most one j_0 such that, for $j_0 \neq j > \min(i, \eta)$, $r_j - r_{j-1} = d_{j-1} - d_j$. Furthermore,

$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap Q^{r_{j_0}}_{d_{j_0}})$$

and $K_\eta \not\subset Q^{r_{j_0}}_{d_{j_0}}$.

Remark 4.5. Given a sequence $(L_\bullet, Q_\bullet)$, the basic principles concerning skew-symmetric forms imply inequalities among the invariants of a sequence. The evenness of rank implies that $d_i - r_i$ is even for every $1 \leq i \leq k - s$. The corank bound implies that $r_i - \dim(Q^{r_i}_{d_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The linear space bound implies that $2(n_s + r_i - \dim(K_i \cap L_{n_s})) \leq r_i + d_i$ for every $1 \leq i \leq k - s$. These inequalities are implicit in the sequence $(L_\bullet, Q_\bullet)$.

Remark 4.6. For a symplectic sequence $(L_\bullet, Q_\bullet)$, the invariants n_j, r_i, d_i together with the dimensions $\dim(L_{n_j}, K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$ determine the sequence $(L_\bullet, Q_\bullet)$ up to the action of the symplectic group. This will become obvious when we construct these sequences by choosing bases.

Definition 4.7. A symplectic sequence $(L_\bullet, Q_\bullet)$ is *admissible* if it satisfies the following additional conditions:

(GA1) $n_j \neq \dim(L_{n_j} \cap K_i) + 1$ for any $1 \leq j \leq s$ and $1 \leq i \leq k - s$.

(GA2) Let x_i denote the number of isotropic subspaces L_{n_j} that are contained in K_i. Then

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}.$$

The translation between sequences and symplectic diagrams. Symplectic sequences can be represented by symplectic diagrams introduced in §3. An isotropic linear space L_{n_j} is represented by a bracket] in position n_j. A linear space $Q^{r_i}_{d_i}$ is represented by a brace } in position d_i such that there are exactly r_i positive integers less than or equal to i to the left of the i-th brace. Finally, $\dim(L_{n_j} \cap K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$, $h > i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively.

Example 4.8. 11]200}0}00 records a sequence $L_2 \subset Q^3_5 \subset Q^2_6$, where $L_2 \subset \mathrm{Ker}(Q^2_6)$. In the diagram, there is one bracket that occurs in position 2. There are two braces that occur in positions 5 and 6. We thus conclude that the sequence contains one isotropic subspace of dimension 2 (L_2) and two non-isotropic subspaces of dimensions 5 (Q_5) and 6 (Q_6). There are two integers equal to 1 and one integer equal to 2 in the sequence. Hence, the corank of the restriction of Q to the six (respectively, five) dimensional subspace Q^2_6 (Q^3_5) is two (three). Finally, since every integer to the left of the bracket is equal to one, we conclude that $L_2 \subset \mathrm{Ker}(Q^2_6)$.

More explicitly, given a symplectic sequence $(L_\bullet, Q_\bullet)$, the corresponding symplectic diagram $D(L_\bullet, Q_\bullet)$ is determined as follows: The sequence of integers begins with $\dim(L_{n_1} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_1} \cap K_i) - \dim(L_{n_1} \cap$

$K_{i-1})$ integers equal to i, for $2 \leq i \leq k-s$, in increasing order, followed by $n_1 - \dim(L_{n_1} \cap K_{k-s})$ integers equal to 0. The sequence then continues with $\dim(L_{n_j} \cap K_1) - \dim(L_{n_{j-1}} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_j} \cap K_i) - \max(\dim(L_{n_{j-1}} \cap K_i), \dim(L_{n_j} \cap K_{i-1}))$ integers equal to i in increasing order, followed by $n_j - \max(n_{j-1}, \dim(L_{n_j} \cap K_{k-s}))$ zeros for $j = 2, \dots, s$ in increasing order. The sequence then continues with $\dim(Q_{d_{k-s}}^{r_{k-s}} \cap K_1) - \dim(L_{n_s} \cap K_1)$ integers equal to 1, followed by $\dim(Q_{d_{k-s}}^{r_{k-s}} \cap K_i) - \max(\dim(Q_{d_{k-s}}^{r_{k-s}} \cap K_{i-1}), \dim(L_{n_s} \cap K_i))$ integers equal to i in increasing order, followed by zeros until position d_{k-s}. Between positions d_i and d_{i-1} $(i > k-s)$, the sequence has $\dim(Q_{d_{i-1}}^{r_{i-1}} \cap K_1) - \dim(Q_{d_i}^{r_i} \cap K_1)$ integers equal to 1, followed by $\dim(Q_{d_{i-1}}^{r_{i-1}} \cap K_h) - \max(\dim(Q_{d_i}^{r_i} \cap K_h), \dim(Q_{d_{i-1}}^{r_{i-1}} \cap K_{h-1}))$ integers equal to h in increasing order, for $h \leq i-1$, followed by zeros until position d_{i-1}. Finally, the sequence ends with $n - d_1$ zeros. The brackets occur at positions n_j and the braces occur at positions d_i.

PROPOSITION 4.9. *The diagram $D(L_\bullet, Q_\bullet)$ is a symplectic diagram of type s for $SG(k,n)$. Furthermore, if $(L_\bullet, Q_\bullet)$ is admissible, then $D(L_\bullet, Q_\bullet)$ is admissible.*

PROOF. By construction each bracket or brace occupies a position. Since $n_1 < n_2 < \cdots < n_s < d_{k-s} < \cdots < d_1$, a position is occupied by at most one bracket or brace. Since $n_j < d_i$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$, every bracket occurs to the left of every brace. By construction, it is clear that $\dim(L_{n_j} \cap K_i)$ and $\dim(Q_{d_h}^{r_h} \cap K_i)$, for $h \geq i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively. Hence, every integer equal to i occurs to the left of $\}^i$. Finally, the total number of integers equal to zero or greater than i to the left of $\}^i$ is equal to the rank of the restriction of Q to $Q_{d_i}^{r_i}$. Since this rank is necessarily even, the total number of integers equal to zero or greater than i to the left of $\}^i$ is even. This shows that we have a sequence of brackets and braces of type s.

The sequence of brackets and braces is a symplectic diagram. The corank bound implies that $r_i - \dim(Q_{d_i}^{r_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The left hand side of the inequality is represented by the number of integers equal to i in the sequence. The right hand side of the inequality is equal to the number of integers between $\}^i$ and $\}^{i-1}$. We thus get the inequality $l(i) \leq \rho(i, i-1)$ required by Condition (S1) in Definition 3.14. By the linear space bound, the largest dimensional linear space contained in $Q_{d_i}^{r_i}$ has dimension bounded by $(d_i + r_i)/2$. The invariant r_i is equal to both the number of positive integers less than or equal to i contained to the left of $\}^i$ and $\dim(K_i)$. The span of L_{n_s} and the kernels K_h for $h \geq i$ is an isotropic subspace of $Q_{d_i}^{r_i}$. The dimension of this subspace is denoted by τ_i and is equal to the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Hence, $2\tau_i \leq p(\}^i) + r_i$ and condition (S2) of Definition 3.14 holds.

If the sequence is in (perfect) order, then the corresponding sequence of brackets and braces is in (perfect) order. Assume the sequence is not in order. The definition of a sequence implies that, for $i < k-s$, there can be at most one i which is not to the left of $\}^{k-s}$. Suppose the sequence satisfies Condition (GS1). Then, there exists an integer η such that for $i > \eta$ those integers that are not to the left of $\}^{k-s}$ are to the immediate left of $\}^{i+1}$. Furthermore, condition (GS1) implies that the positive numbers up to η are in non-decreasing order and η is the only integer violating the order. Thus condition (S3) is satisfied. Finally, condition (GS2) directly translates to condition (S4). We conclude that the sequence of brackets and braces is a symplectic diagram.

If the sequence $(L_\bullet, Q_\bullet)$ is admissible, then the corresponding symplectic diagram is also admissible. Let i be the minimal index such that $L_{n_j} \subset K_i$. If there isn't such an index, let $i = k - s + 1$. If $i > 1$, then condition (GA1) implies that $\dim(L_{n_j} \cap K_{i-1}) \leq n_j - 2$. Hence, the two integers preceding $]^j$ are equal to i (or 0 if $i = k - s + 1$). If $i = 1$, then all the integers preceding $]^j$ are equal to 1. Furthermore, if $n_j = 1$, condition (GA1) implies that $L_{n_j} \subset K_i$ for all $1 \leq i \leq k - s$. We conclude that condition (A1) holds. The invariant x_i is equal to both the number of isotropic subspaces L_{n_j} contained in K_i and the number of brackets such that every integer to the left of it is positive and less than or equal to i. Since $d_i = p(\}^i)$, conditions (A2) and (GA2) are exactly the same. This concludes the proof of the proposition. □

Remark 4.10. Proposition 4.9 also explains the definition of a symplectic diagram in geometric terms. Condition (4) of Definition 3.2 is implied by the evenness of rank and simply states that $d_i - r_i$ has to be even. As discussed in the proof of Proposition 4.9, condition (S1) is a translation of the corank bound and condition (S2) is implied by the linear space bound.

Conversely, we can associate an admissible sequence to every admissible symplectic diagram. By Darboux's Theorem, we can take the skew-symmetric form to be defined by $\sum_{i=1}^m x_i \wedge y_i$. Let the dual basis for x_i, y_i be e_i, f_i such that $x_i(e_j) = \delta_i^j$, $y_i(f_j) = \delta_i^j$ and $x_i(f_j) = y_i(e_j) = 0$. Given an admissible symplectic diagram, we associate $e_1, \dots, e_{p(]^s)}$ to the integers to the left of $]^s$ in order. We then associate $e_{p(]^s)+1}, \dots, e_{r'}$ to the positive integers to the right of $]^s$ and left of $\}^{k-s}$ in order. Let $e_{i_1}, \dots, e_{i_l}$ be vectors that have so far been associated to zeros. Then associate $f_{i_1}, \dots, f_{i_l}$ to the remaining zeros to the left of $\}^{k-s}$ in order. If there are any zeros to the left of $\}^{k-s}$ that have not been assigned a basis vector, assign them $e_{r'+1}, f_{r'+1}, \dots, e_{r''}, f_{r''}$ in pairs in order. Continuing this way, if there is a positive integer between $\}^{i+1}$ and $\}^i$, associate to it the smallest index basis element e_α that has not yet been assigned. Assume that the integers equal to $i+1$ have been assigned the vectors $e_{j_1}, \dots, e_{j_l}$. Assign to the zeros between $\}^{i+1}$ and $\}^i$, the vectors $f_{j_1}, \dots, f_{j_l}$. If there are any zeros between $\}^{i+1}$ and $\}^i$ that have not been assigned a vector, assign them $e_{\alpha+1}, f_{\alpha+1}, \dots, e_\beta, f_\beta$ in pairs until the zeros are exhausted. Let L_{n_j} be the span of the basis elements associated to the integers to the left of $]^j$. Let $Q_{d_i}^{r_i}$ be the span of the basis elements associated to the integers to the left of $\}^i$. We thus obtain a sequence $(L_\bullet, Q_\bullet)$ whose associated symplectic diagram is D.

Example 4.11. To 11]233]0000}00}0}00 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_6, e_7, f_7, f_4, f_5, f_3, f_1, f_2.$$

Then L_2 is the span of e_1, e_2, L_5 is the span of e_1 through e_5, Q_9^5 is the span of e_1 through e_7 and f_6, f_7, Q_{11}^3 is the span of e_1 through e_7 and f_4 through f_7. Finally, Q_{12}^2 is the span of Q_{11}^3 and f_3.

To 22]33]0000}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, f_5, e_6, f_6, f_3, f_4, e_7, f_1, f_2, f_7.$$

L_2 is the span of e_1, e_2, L_4 is the span of e_1 through e_4, Q^4_8 is the span of e_1 through e_6 and f_5, f_6, Q^2_{10} is the span of Q^4_8 and f_3, f_4 and Q^1_{13} is the span of Q^2_{10} and e_7, f_1, f_2.

Finally, to 22]300]300}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_4, f_5, f_3, f_6, e_7, f_1, f_2, f_7.$$

Then L_2 is the span of e_1 and e_2, L_5 is the span of e_1 through e_5, Q^4_8 is the span of e_1 through e_5 and f_4, f_5, Q^2_{10} is the span of Q^4_8 and f_3, f_6. Finally, Q^1_{13} is the span of all the vectors but f_7.

Remark 4.12. Notice that equivalent symplectic diagrams correspond to permutations of the basis elements that do not change the vector spaces in $(L_\bullet, Q_\bullet)$.

Remark 4.13. The construction of a symplectic sequence $(L_\bullet, Q_\bullet)$ from a symplectic diagram D is well-defined. By condition (S2), the number of zeros to the left of $]^s$ is less than or equal to the number of zeros between $]^s$ and $\}^{k-s}$. Hence, we can choose vectors $f_{i_1}, \dots, f_{i_l}$ corresponding to the vectors $e_{i_1}, \dots, e_{i_l}$. Similarly, if there does not exist a positive integer between $\}^{i+1}$ and $\}^i$, then by condition (S1), $l(i+1) \leq \rho(i+1, i)$. We can, therefore, associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. If there exists a positive integer between $\}^{i+1}$ and $\}^i$, then there is only one positive integer between them by condition (S3). If $l(i+1) = \rho(i+1, i)$, then condition (4) is violated. Hence, $l(i+1) < \rho(i+1, i)$ and we can associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. Thus the construction of the sequence makes sense. It is now straightforward to check that the sequence associated to an admissible symplectic diagram is an admissible sequence. Furthermore, the two constructions are inverses of each other.

We are now ready to define symplectic restriction varieties.

Definition 4.14. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $SG(k, n)$. Then the *symplectic restriction variety* $V(L_\bullet, Q_\bullet)$ is the Zariski closure of the locus in $SG(k, n)$ parameterizing

$$\{W \in SG(k,n) \mid \dim(W \cap L_{n_j}) = j \text{ for } 1 \leq j \leq s, \dim(W \cap Q^{r_i}_{d_i}) = k - i + 1 \\ \text{and } \dim(W \cap K_i) = x_i \text{ for } 1 \leq i \leq k - s\}.$$

Remark 4.15. The geometric reasons for imposing conditions (A1) and (A2) in Definition 3.17 are now clear. Condition (A1) is an immediate consequence of the kernel bound. If $\dim(L_{n_j} \cap K_i) = n_j - 1$ and a linear space of dimension $k-i+1$ intersects n_j in dimension j and K_i in dimension $j - 1$, then the linear space is contained in $L^\perp_{n_j}$. Hence, we need to impose condition (A1).

The inequality

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}$$

is an easy consequence of the linear space bound. We require the k-dimensional isotropic subspaces to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k - i + 1$ and to intersect the singular locus of $Q^{r_i}_{d_i}$ in a subspace of dimension x_i. By the linear space bound, any linear space of dimension $k - i + 1$ has to intersect the singular locus in a subspace of dimension at least $k - i + 1 - \frac{d_i - r_i}{2}$, hence the inequality in condition (A2) holds.

Example 4.16. The two most basic examples of symplectic restriction varieties are:

(1) A Schubert variety $\Sigma_{\lambda;\mu}$ in $SG(k,n)$, which is the restriction variety associated to a symplectic diagram $D(\sigma_{\lambda;\mu})$, and
(2) The intersection $\Sigma_{a_\bullet} \cap SG(k,n)$ of a general Schubert variety in $G(k,n)$ with $SG(k,n)$, which is the restriction variety associated to $D(a_\bullet)$.

In general, symplectic restriction varieties interpolate between these two examples.

LEMMA 4.17. *A symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence is a Schubert variety in $SG(k,n)$. Conversely, every Schubert variety in $SG(k,n)$ can be represented by such a sequence.*

PROOF. Let $F_1 \subset \cdots \subset F_1^\perp \subset V$ be an isotropic flag. If $\Sigma_{\lambda,\mu}$ is a Schubert variety defined with respect to this flag, then the symplectic restriction variety defined with respect to the sequence $L_{n_j} = F_{\lambda_j}$ and $Q_{d_i}^{r_i} = F_{\mu_{k-i+1}}^\perp$ is a saturated and perfectly ordered admissible sequence.

Conversely, suppose that the sequence $(L_\bullet, Q_\bullet)$ is a saturated and perfectly ordered admissible sequence. Since the sequence is saturated, we have that $Q_{d_i}^{r_i} = K_i^\perp$. Since the sequence is in perfect order, we have that $\dim(L_{n_j} \cap \mathrm{Ker}(Q_{d_i}^{r_i})) = \min(r_i, n_j)$. Consequently, the set of linear spaces $\{L_{n_j}, \mathrm{Ker}(Q_{d_i}^{r_i})\}$ can be ordered by inclusion, or equivalently, by dimension. Then the resulting partial flag can be extended to an isotropic flag. By condition (GA1) of the definition of an admissible sequence, we have that $n_j \neq r_i + 1$ for any i, j. Hence, the symplectic restriction variety defined with respect to $(L_\bullet, Q_\bullet)$ is the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}$, where $\lambda_j = n_j$, for $1 \leq j \leq s$, and $\mu_i = r_{k-i+1}$, for $s < i \leq k$. ∎

REMARK 4.18. By Lemma 4.17, the saturated symplectic diagrams in perfect order represent Schubert varieties.

Next, we show that the intersection of a general Schubert variety Σ with the symplectic Grassmannian $SG(k,n)$ (when non-empty) is a restriction variety.

LEMMA 4.19. *Let Σ be the Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$. Then $\Sigma \cap SG(k,n) \neq \emptyset$ if and only if $a_i \geq 2i-1$ for $1 \leq i \leq k$.*

PROOF. Suppose $a_i < 2i-1$ for some i. If $[W] \in \Sigma \cap SG(k,n)$, then $W \cap F_{a_i}$ is an isotropic subspace of $Q \cap F_{a_i}$ of dimension at least i. Since F_{a_i} is general, the corank of $Q \cap F_{a_i}$ is 0 or 1 and equal to a_i modulo 2. By the linear space bound, the largest dimensional isotropic subspace of $Q \cap F_{a_i}$ has dimension less than or equal to $i-1$. Therefore, W cannot exist and $\Sigma \cap SG(k,n) = \emptyset$.

Conversely, let $a_i = 2i-1$ for every i. Then $G_1 = F_1$ is isotropic, $G_2 = F_1^\perp$ in F_3 is the unique two-dimensional isotropic subspace of $Q \cap F_3$ containing G_1. By induction, we see that $G_i = G_{i-1}^\perp$ is the unique subspace of dimension i isotropic with respect to $Q \cap F_{2i-1}$ that contains G_{i-1}. Continuing this way, we construct a unique isotropic subspace W of dimension k contained in $\Sigma \cap SG(k,n)$. If $a_i \geq 2i-1$, the vector space W just constructed is still contained in $\Sigma \cap SG(k,n)$, hence this intersection is non-empty. ∎

LEMMA 4.20. *Let Σ be the Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$ such that $a_i \geq 2i-1$. Then $\Sigma \cap SG(k,n) = V(D(a_\bullet))$.*

PROOF. Let $a_i = 2i-1$, then since F_{a_i} is general, the restriction of Q to F_{a_i} has a one-dimensional kernel K_i. By the linear space bound, any i-dimensional isotropic subspace W contained in F_{a_i} contains K_i. For each j such that $a_j > 2j-1$, recall that u_j is the number of $i < j$ such that $a_i = 2i-1$ and v_j is the number of $i > j$ such that $a_i = 2i-1$. Let K be the span of one-dimensional kernels K_i for each $a_i = 2i-1$. Then $\dim(K) = u$ and any k-dimensional subspace W contained in $\Sigma \cap SG(k,n)$ contains K. For j such that $a_j > 2j-1$, let $G_{j+v_j} = \mathrm{Span}(F_{a_j}, K) \cap K^{\perp}$. The dimension of G_{j+v_j} is $a_j - u_j + v_j$. The corank of the restriction of Q to G_{j+v_j} is $u + \delta(a_j)$, where $\delta(a_j) = 0(1)$ if a_j is even (odd). Furthermore, any isotropic linear space contained in $\Sigma \cap SG(k,n)$ intersects G_{j+v_j} in a subspace of dimension at least $j + v_j$. From this description and the definition of $V(D(a_\bullet))$, it is now clear that $\Sigma \cap SG(k,n) = V(D(a_\bullet))$. ☐

PROPOSITION 4.21. *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. Then $V(L_\bullet, Q_\bullet)$ is an irreducible subvariety of $SG(k,n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i - 1 - 2k + 2i + x_i). \tag{4.1}$$

PROOF. The proof is by induction on k. When $k = 1$, if the sequence consists of an isotropic linear space L_{n_1}, then the corresponding symplectic restriction variety is $\mathbb{P}L_{n_1}$ hence it is irreducible of dimension $n_1 - 1$. If the sequence consists of one non-isotropic subspace $Q_{d_1}^{r_1}$, then the corresponding symplectic restriction variety is also projective space of dimension $d_1 - 1$. In both cases, the varieties are irreducible of the claimed dimension. This proves the base case of the induction.

If the sequence does not contain any skew-symmetric forms, then the corresponding restriction variety is isomorphic to a Schubert variety in the ordinary Grassmannian $G(k,n)$. In that case, it is well known that Schubert varieties are irreducible and have dimension $\sum_{j=1}^{k}(n_j - j)$ [**C3**].

Observe that omitting $Q_{d_1}^{r_1}$ from an admissible sequence $(L_\bullet, Q_\bullet)$ for $SG(k,n)$ gives rise to an admissible sequence $(L'_\bullet, Q'_\bullet)$ for $SG(k-1,n)$. There is a natural surjective morphism $f : V^0(L_\bullet, Q_\bullet) \to V^0(L'_\bullet, Q'_\bullet)$ that sends a vector space W to $W \cap Q_{d_2}^{r_2}$ (or $W \cap L_{n_{k-1}}$ if $s = k-1$). By induction, $V(L'_\bullet, Q'_\bullet)$ is irreducible of dimension $\sum_{j=1}^{s}(n_j - j) + \sum_{i=2}^{k-s}(d_i - 1 - 2k + 2i + x_i)$. The fibers of the morphism f over a point W' correspond to choices of isotropic k-planes W that contain W' and are contained in $Q_{d_1}^{r_1}$. This is a Zariski dense open subset of projective space of dimension $d_1 - 2(k-1) - 1 + x_1$. Hence, by the Theorem on the Dimension of Fibers [**S**, I.6.7], $V(L_\bullet, Q_\bullet)$ is irreducible of the claimed dimension. This concludes the proof of the proposition. ☐

5. The geometric explanation of the combinatorial game

In this section, we will prove the combinatorial rule by interpreting it geometrically. The transformation from an admissible diagram D to D^a records a one-parameter specialization of the restriction variety $V(D)$. The algorithm describes the flat limit of this specialization.

The specialization. We now explain the specialization. There are several cases depending on whether D is in order and whether $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ or not. In the previous section, given an admissible quadric diagram D, we associated an admissible sequence by defining each of the vector spaces $(L_\bullet, Q_\bullet)$ as a union of basis elements that diagonalize the skew-symmetric form Q. All our specializations will replace exactly one of the basis elements $v = e_u$ or $v = f_u$ for some $1 \leq u \leq m$ with a vector $v(t) = e_u(t)$ or $v(t) = f_u(t)$ varying in a one-parameter family. For $t \neq 0$, the resulting set of vectors will be a new basis for V, but when $t = 0$ two of the basis elements will become equal. Since each linear space in $(L_\bullet, Q_\bullet)$ is a union of basis elements, we get a one-parameter family of vector spaces $(L_\bullet(t), Q_\bullet(t))$ by replacing every occurrence of the vector v with $v(t)$ for $t \neq 0$. Correspondingly, we have a one-parameter family of restriction varieties $V(L_\bullet(t), Q_\bullet(t))$. Since these varieties are projectively equivalent as long as $t \neq 0$, we obtain a flat one-parameter family. Our task is to describe the limit when $t = 0$.

In case (1)(i), D is not in order, η is the unique integer violating the order, and ν is the leftmost integer equal to $\eta+1$. Suppose that under the translation between symplectic diagrams and sequences of vector spaces, e_u is the vector associated to η and e_v is the vector associated to ν. Then consider the one-parameter family obtained by changing e_v to $e_v(t) = te_v + (1-t)e_u$ and keeping every other vector fixed. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains e_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is the span of the same basis elements except that e_v is replaced with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (1)(ii), D is not in order, η is the unique integer violating the order, $i > \eta$ does not occur in the sequence to the left of η and ν is the leftmost integer equal to $i+1$. Let e_u be the vector associated to η and let e_v be the vector associated to ν. Consider the one-parameter family obtained by changing f_v to $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is the span of the same basis elements except that f_v is replaced with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (2)(i), D is in order and $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Suppose that e_v is the vector associated to ν, the leftmost $\kappa+1$. Let e_u and f_u be two vectors associated to zeros between $\}^\kappa$ and $\}^{\kappa-1}$. These exist since $l(\kappa) < \rho(\kappa, \kappa-1)-1$. Consider the one-parameter specialization replacing f_v with $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (2)(ii)(a), D is in order and $l(\kappa) = \rho(\kappa, \kappa-1)-1$. Let ν be the leftmost integer equal to κ and suppose that e_v is the vector associated to ν. Let e_u be the vector associated to the $\kappa - 1$ following $\}^\kappa$. Then let $e_v(t) = te_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains e_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

Finally, in case (2)(ii)(b), D is in order, $l(\kappa) = \rho(\kappa, \kappa-1) - 1$ and there does not exist an integer equal to κ to the left of κ. Let e_v be the vector associated to ν, the leftmost integer equal to $\kappa+1$ and let e_u be the vector associated to $\kappa - 1$ to the right of $\}^\kappa$. Then let $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

The flat limits of the vector spaces are easy to describe. If L_{n_j} or $Q_{d_i}^{r_i}$ does not contain the vector v, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all $t \neq 0$. Hence, the flat limit $L_{n_j}(0) = L_{n_j}$ and $Q_{d_i}^{r_i}(0) = Q_{d_i}^{r_i}$. Similarly, if L_{n_j} or $Q_{d_i}^{r_i}$ contains both of the basis elements spanning $v(t)$, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all $t \neq 0$. Then in the limit $L_{n_j}(0) = L_{n_j}$ and $Q_{d_i}^{r_i}(0) = Q_{d_i}^{r_i}$. A vector space changes under the specialization only when it contains the vector with coefficient t and does not contain the vector with coefficient $(1-t)$. In this case, in the limit $t=0$, the flat limit $L_{n_j}(0)$ or $Q_{d_i}^{r_i}(0)$ is obtained by replacing in L_{n_j} or Q_{d_i} the basis element with coefficient t with the basis element with coefficient $(1-t)$.

Notice that in each of these cases, the set of limiting vector spaces is depicted by the symplectic diagram D^a. In case (1)(i), if η is between $\}^a$ and $\}^{a-1}$ and ν is between $]^b$ and $]^{b+1}$ (respectively, between $]^s$ and $\}^{k-s}$), the vector spaces L_{n_j} for $j \leq b$ (respectively, $j \leq s$) and $Q_{d_i}^{r_i}$ for $i < a$ are unaffected. In all the other vector spaces, e_v is replaced by e_u. The effect on symplectic diagrams is to switch η and ν as in the definition of D^a. In case (1)(ii), assume that η is between $\}^i$ and $\}^{i-1}$. The linear spaces other than $Q_{d_i}^{r_i}$ remain unchanged under the degeneration. In $Q_{d_i}^{r_i}$ the vector f_v is replaced by e_u. Note that this increases the corank of the restriction of Q to $Q_{d_i}^{r_i}(0)$ by two since now both vectors e_u and e_v in the kernel. This has the effect of changing ν to i and a zero between $\}^{i+1}$ and $\}^i$ to η as in the definition of D^a. In case (2)(i), all the vector spaces but $Q_{d_\kappa}^{r_\kappa}$ remain unchanged. The degeneration replaces f_v in $Q_{d_\kappa}^{r_\kappa}$ by e_u. This increases the corank of the restriction of Q to $Q_{d_\kappa}^{r_\kappa}(0)$ by two since both e_u and e_v are now contained in the kernel of the restriction. The corresponding symplectic diagram is obtained by changing ν and a zero between $\}^{\kappa+1}$ and $\}^\kappa$ to κ as in the definition of D^a. The cases (2)(ii)(a) and (b) are analogous to the cases (1)(i) and (1)(ii), respectively.

For the rest of the paper, we use the specialization just described.

Example 5.1. For concreteness, consider the restriction variety associated to $200\}000\}00$ in $SG(2,8)$ parameterizing isotropic subspaces that intersect $A = \mathrm{Span}(e_1, e_2, f_2)$ and are contained in $B = \mathrm{Span}(e_i, f_i)$, $1 \leq i \leq 3$. The first specialization is given by $tf_2 + (1-t)e_3$. In the limit, $A_1 = A(0) = \mathrm{Span}(e_1, e_2, e_3)$ and $B(0) = B$. This changes the diagram to $000]000\}00$. The corresponding restriction variety parameterizes linear spaces that intersect $A(0)$ and are contained in B. The next specialization is given by $tf_1 + (1-t)e_4$. In the limit, $A_1(0) = A_1$ and $B_1 = B(0) = \mathrm{Span}(e_1, e_2, e_3, e_4, f_2, f_3)$. This changes the diagram to $100]100\}00$. The corresponding restriction variety parameterizes linear spaces that intersect A_1 and are contained in B_1. The final specialization is given by $te_2 + (1-t)e_4$. In the limit, $A_2 = A_1(0) = \mathrm{Span}(e_1, e_4, e_3)$ and $B_1(0) = B_1$. This changes the diagram to $110]000\}00$. The flat limit of the restriction varieties has two components. The linear spaces may intersect $\mathrm{Span}(e_1, e_4)$, in which case we get the restriction variety associated to the diagram $11]0000\}00$. Otherwise, by the kernel bound, the linear spaces have to be contained in $A_2^\perp$. In this case, we get the restriction variety associated to the diagram $111]00\}000$. The reader should convince themselves that this is precisely the outcome of Algorithm 3.29.

We are now ready to state and prove the main geometric theorem.

Theorem 5.2. *(The Geometric Branching Rule) The flat limit of the specialization of $V(D)$ is supported along $\bigcup V(D_i)$, where $V(D_i)$ is a symplectic restriction variety associated to a diagram D_i obtained by running Algorithm* 3.29 *on D. Furthermore, the flat limit is generically reduced along each $V(D_i)$. In particular, the equality*

$$[V(D)] = \sum [V(D_i)]$$

holds between the cohomology classes of symplectic restriction varieties.

PROOF OF THEOREM 3.33 ASSUMING THEOREM 5.2. By Proposition 3.39, Algorithm 3.29 replaces each admissible symplectic diagram by one or two admissible symplectic diagrams. Hence, the algorithm can be repeated. By Proposition 3.40, after finitely many steps, the algorithm terminates leading to a collection of saturated admissible symplectic diagrams in perfect order. By Lemma 4.17, each of these diagrams represent a Schubert variety. Therefore, Theorem 3.33 is an immediate corollary of Theorem 5.2. □

PROOF OF THEOREM 5.2. The proof of Theorem 5.2 has two steps. First, we interpret the algorithm as the specialization described in the beginning of this section. Let $V(D)$ denote the initial symplectic restriction variety. Let $V(D(t))$ denote the one-parameter family of restriction varieties described in the specialization and let $V(D(0))$ be the flat limit at $t = 0$. We show that $V(D(0))$ is supported along the union of restriction varieties $V(D_i)$, where D_i are the admissible symplectic diagrams derived from D via Algorithm 3.29. In the second step, we verify that the support of the flat limit contains each $V(D_i)$ and the flat limit is generically reduced along each $V(D_i)$. This suffices to prove the theorem.

We now analyze the specialization to conclude that the support of $V(D(0))$ is the union of symplectic restriction varieties $V(D_i)$. The proof is by a dimension count. In order to restrict the possible irreducible components of $V(D(0))$, we find conditions that the linear spaces parameterized by $V(D(0))$ have to satisfy. We then observe that these conditions already cut out the symplectic varieties $V(D_i)$ and that each $V(D_i)$ has the same dimension as $V(D)$. The following observation puts strong restrictions on the support of the flat limit.

OBSERVATION 5.3. The linear spaces parameterized by $V(D(t))$ intersect the linear spaces $L_{n_j}(t)$ (respectively, $Q_{d_i}^{r_i}(t)$) in a subspace of dimension at least j (respectively, $k - i + 1$). Similarly, they intersect $\mathrm{Ker}(Q_{d_i}^{r_i}(t))$ in a linear space of dimension at least x_i. Since intersecting a proper variety in at least a given dimension is a closed condition, the linear spaces parameterized by $V(D(0))$ have to intersect the linear spaces $L_{n_j}(0)$ (respectively, $Q_{d_i}^{r_i}(0)$) in a subspace of dimension at least j (respectively, $k - i + 1$). Furthermore, they intersect $\mathrm{Ker}(Q_{d_i}^{r_i}(0))$ in a subspace of dimension at least x_i.

Let Y be an irreducible component of $V(D(0))$. We can construct a sequence of vector spaces $F_{u_1} \subset \cdots \subset F_{u_k}$ such that the locus Z parameterizing linear spaces with $\dim(W \cap F_{u_j}) \geq j$ contains Y. We have already seen that the linear spaces $L_{n_j}(0)$ and $Q_{d_i}^{r_i}(0)$ are the linear spaces recorded by the symplectic diagram D^a. Let $z_1, \dots, z_n$ be the ordered basis of V obtained by listing the basis elements associated to D^a from left to right. Let F_u be the linear space spanned by the basis elements $z_1, \cdots, z_u$. Let $F_{u_1} \subset \cdots \subset F_{u_k}$ be the jumping linear spaces for Y, that is the linear spaces of the form F_u such that $\dim(W \cap F_u) > \dim(W \cap F_{u-1})$ for the general isotropic space W parameterized by Y. Observation 5.3 translates to the inequalities $u_j \leq n_j$ for $j \leq s$ and $u_i \leq d_{k-i+1}$ for $s < i \leq k$. Hence, we can obtain a sequence depicting the linear spaces $F_{u_1}, \dots, F_{u_k}$ by moving the braces and brackets in the diagram D^a to the left one at a time. By the proof of Proposition 4.21, Equation (4.1) gives an upper bound on the dimension of the locus Z (note that we used the fact that the sequence is admissible in the proof only to deduce the equality).

We now estimate the dimension of Z. Let $(L^a_\bullet, Q^a_\bullet)$ denote the linear spaces depicted by the diagram D^a. We obtain the sequence defining Z by replacing linear spaces in $(L^a_\bullet, Q^a_\bullet)$ by smaller dimensional ones.

- If we replace a linear space $L^a_{n_i}$ of dimension n_i in $(L^a_\bullet, Q^a_\bullet)$ with a linear space F_{u_i} not contained in $(L^a_\bullet, Q^a_\bullet)$ but containing $L^a_{n_{i-1}}$, then according to Equation (4.1) the dimension changes as follows. Let y^a_i be the index of the smallest index linear space $Q^{r_l}_{d_l}$ such that $L^a_{n_i} \subset K_l$. Similarly, let y^u_i be the smallest l such that $F_{u_i} \subset K_l$. The left sum in Equation (4.1) changes by $u_i - n^a_i$. The quantities x_l increase by one for $y^u_i \leq l < y^a_i$. Hence, the sum on the right increases by $y^a_i - y^u_i$. Hence, the total change in dimension is $u_i - n^a_i + y^a_i - y^u_i$. By condition (S4) of Definition 3.14 for D^a and condition (A1) for D, in D^a, there is at most one missing integer among the positive integers to the left of the brackets and the two integers preceding all brackets but possibly $]^{x_{\nu-1}+1}$ are equal. We conclude that if we move any bracket to the left except for $]^{x_{\nu-1}+1}$, we strictly decrease the dimension. Furthermore, if we move $]^{x_{\nu-1}+1}$ to the left, we strictly decrease the dimension unless in D we have the equality $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$, so that the decrease in the position resulting by shifting the bracket in D^a is equal to the increase in the number of linear spaces $Q^{r_l}_{d_l}$ containing F_{u_i} in their kernel.
- If we replace the linear space $Q^{r_i,a}_{d_i}$ of dimension d^a_i in $(L^a_\bullet, Q^a_\bullet)$ with a non-isotropic linear space $F_{u_{k-i+1}}$ of dimension d^u_i containing $Q^{r_{i-1},a}_{d_{i-1}}$, then, by Equation (4.1), the dimension changes as follows. Let x^u_i be the number of linear spaces that are contained in the kernel of the restriction of Q to $F_{u_{k-i+1}}$. Then the dimension changes by $d^u_i - d^a_i - x^a_i + x^u_i$. We have that $d^u_i - d^a_i - x^a_i + x^u_i \leq 0$ with strict inequality unless the number of linear spaces contained in the kernel of $F_{u_{k-i+1}}$ increases by an amount equal to $d^a_i - d^u_i$. The latter can only happen if condition (A1) is violated for the diagram so that increasing the dimension of the kernel by one can increase the number of linear spaces contained in the kernel.
- Finally, if we replace the linear space $Q^{r_{k-s},a}_{d_{k-s}}$ of dimension d^a_{k-s} in $(L^a_\bullet, Q^a_\bullet)$ with an isotropic linear space $F_{u_{s+1}}$ containing L_{n_s}, then the first sum in Equation (4.1) changes by $u_{s+1} - s - 1$. The second sum changes by $-d^a_{k-s} + y^u_{s+1} - x^a_{k-s} + (2s+1)$, where y^u_{s+1} denotes the number of non-isotropic subspaces containing $F_{u_{s+1}}$ in the kernel of the restriction of Q. Hence, the total change is
$$-d^a_{k-s} + u_{s+1} - x^a_{k-s} + y^u_{s+1} + s.$$
If $x^a_{k-s} = s - j < s$, then $y^u_{s+1} = 0$. Since by the linear space bound $u_{s+1} + j + 1 \leq d_{k-s}$, we conclude that the dimension strictly decreases. If $x_{k-s} = s$, then the change is strictly negative unless $r_{k-s} = d_{k-s}$ and $d_{k-s} = u_{s+1}$.

The dimension count shows that $V(D)$ and $V(D^a)$ have the same dimension. When $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D, $V(D^b)$ and $V(D^a)$ have the same dimension. Furthermore, Step 2 of Algorithm 3.25 and Algorithm 3.26 preserve the dimension of the variety. By Equation (4.1), Step 1 of Algorithm 3.25 also preserves the dimension. If condition (A2) is violated for D^a for the index i, then by Proposition 3.39, we have that $2x_i = 2k - 2i - d_i + r_i$. On the other hand, the operation in Step 1 of Algorithm 3.25 changes the left sum in Equation (4.1) by $r_i + (s - x_i) - s - 1 = r_i - x_i - 1$, since it adds a new bracket of size r_i and increases the positions of the brackets with index $x_i + 1, \dots, s$. It changes the left sum by $-d_i + 1 - x_i + 2k - 2(k-s) + 2(k-s-i)$ since it removes the brace with index i and increases the positions and x_l for the braces with indices $l = i+1, \dots, k-s$. We conclude that the change in dimension is $r_i - 2x_i - d_i + 2k - 2i = 0$. We conclude that every variety $V(D_i)$ associated to $V(D)$ by Algorithm 3.29 has the same dimension as $V(D)$.

We can now determine the support of the flat limit of the specialization. Since in flat families the dimension of the fibers are preserved, Y has the same dimension as $V(D)$. Hence, our dimension calculation puts very strong restrictions on Z. First, suppose that either $x_{\nu-1} = s$ or $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D. If D^a is admissible, then by our dimension counts, replacing an isotropic or non-isotropic linear space in $(L^a_\bullet, Q^a_\bullet)$ with a smaller dimensional linear space produces a strictly smaller dimensional locus. We conclude that the general linear space parameterized by Y satisfies exactly the rank conditions imposed by $(L^a_\bullet, Q^a_\bullet)$. Hence, Y is contained in $V(D^a)$. Since both are irreducible varieties of the same dimension, we conclude that $Y = V(D^a)$. If D^a is not admissible, then it either violates condition (A1) or (A2). If D^a fails condition (A2), then $x_i < k - i + 1 - \frac{d_i - r_i}{2}$ for some i. Since the linear spaces parameterized by Y have to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k - i + 1$, by the linear space bound, we conclude that these linear spaces have to intersect K_i in a subspace of dimension at least $x_i + 1$. In D^a, there is only one integer i that is not in the beginning non-decreasing part of the sequence of integers. Geometrically, the linear spaces $L^a_{n_j}$ or $Q^{r_j,a}_{d_j}$ either contain or are contained in K_i or intersect K_i in a codimension one linear space. Let $F_{a_1} \subset F_{a_2} \subset \cdots \subset F_{a_l}$ be a partial flag such that F_{a_h} intersects M in a codimension one subspace of M. Let $M = G_{a_0+1} \subset G_{a_1+1} \subset \cdots \subset G_{a_l+1}$ be the partial flag where G_{a_h+1} is the span of F_{a_h} and M for $h \geq 1$. The locus of linear spaces of dimension $x_i + l + 1$ that intersect F_{a_h} in a subspace of dimension at least $x_i + h$ and intersect M in a subspace of dimension at least $x_i + 1$ is equivalent to the locus of linear spaces that intersect the vector spaces G_{a_h+1} in subspaces of dimension at least $x_i + 1 + h$. Notice that the diagram D^c formed in Step 1 of the Algorithm 3.25 depicts the linear spaces

$$L_{n_1}, \dots, L_{n_{x_i}}, K_i, \mathrm{Span}(K_i, L_{n_{x_i+1}}), \cdots, \mathrm{Span}(K_i, Q^{r_{i+1}}_{d_{i+1}}), Q^{r_{i-1}}_{d_{i-1}}, \cdots, Q^{r_1}_{d_1}.$$

Hence, by the linear space bound Y must be contained in $V(D^c)$. By Proposition 3.39, D^c is an admissible symplectic diagram. Hence, $V(D^c)$ is an irreducible variety that has the same dimension as Y. We conclude that $Y = V(D^c)$. On the other hand, if D^a satisfies condition (A2) but fails condition (A1), then it fails it for the bracket with index $x_{\nu-1} + 1$ and the index ν. By the kernel bound, any linear space that intersects $L_{n_{x_{\nu-1}+1}}$ in a subspace away from the kernel of Q restricted to $Q^{r_{\nu-1}}_{d_{\nu-1}}$ has to be contained in $L^\perp_{n_{x_{\nu-1}+1}}$. The latter vector space is depicted in a symplectic diagram by changing ν to $\nu - 1$ and shifting $\}^{\nu-1}$ one unit to the right as in Step 2 of Algorithm 3.25. This argument applies as long as condition (A1) fails for the resulting sequence. We conclude that Y has to be contained in $V(D^c)$. Since Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$.

Now suppose that $x_{\nu-1} < s$ and $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D. Then, by our dimension count, replacing the linear space $L_{n_{x_{\nu-1}+1}}$ by a linear space $F_{u_{x_{\nu-1}+1}}$ corresponding to a bracket of the form

$$\cdots a\ a+1 \ldots \nu-1\ \nu\ \nu+2 \ldots \nu+l-1\ \nu+l\ \nu+l] \cdots \to$$
$$\cdots a]a+1 \ldots \nu-1\ \nu\ \nu+2 \ldots \nu+l-1\ \nu+l\ \nu+l \cdots$$

produces a locus Z that has the same dimension as Y. Replacing any other linear space results in a smaller dimensional locus. However, unless $F_{u_{x_{\nu-1}+1}} = \mathrm{Ker}(Q^{r_\nu}_{d_\nu}) \cap L_{n_{x_{\nu-1}+1}}$ not all linear spaces parameterized by Z can be in the flat limit. Observe that $W^\perp(t)$ intersects $L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a})$ in a subspace of dimension at least $\pi(a)+1$ for every $W(t) \in V(D(t))$. By upper semi-continuity, the same has to hold of the flat limit at $t=0$. Hence, unless $a=\nu$, we obtain a smaller dimensional variety. We conclude that $Y \subset V(D^b)$. If D^b is admissible, then both varieties are irreducible of the same dimension and we conclude that $Y = V(D^b)$. If D^b is not admissible, then by Proposition 3.39, D^b satisfies condition (A2) but fails condition (A1). Furthermore, it fails condition (A1) only for the bracket $\cdots a\ \nu] \cdots$. By the kernel bound, the linear spaces parameterized of dimensions $k-a, k-a+1, \ldots, k-\nu+2$ contained in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively, are contained in $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^\perp$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$. Algorithm 3.26 replaces the linear spaces $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$ with $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^\perp$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively. Hence, Y is contained in $V(D^c)$. Finally, if during the process two braces occupy the same position, then the resulting locus Z has strictly smaller dimension by our dimension counts so does not lead to a locus Z containing Y. Since in all other cases Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$. This completes the proof that the support of the flat limit of the specialization is contained in the union of $V(D_i)$, where D_i are the admissible symplectic diagrams associated to D by Algorithm 3.29.

Finally, there remains to check that each of the irreducible components occur with multiplicity one. This is an easy local calculation. The point here is that taking the option D^a at each stage of the algorithm leads to a Schubert variety. Similarly, taking the option D^b at all allowed places in the algorithm leads to a Schubert variety. The classes of these two Schubert varieties occur in the class of $V(D)$ with multiplicity one. Therefore, by intersecting $V(D)$ with the dual of these Schubert varieties, we can tell the multiplicity of $V(D^a)$ and $V(D^b)$.

First, in each of the five cases we can assume that $\eta = 1$. Let U be the Zariski open set of our family of restriction varieties parameterizing linear spaces $W(t)$ such that $\dim(W(t) \cap Q^{r_\eta(t)}_{d_\eta}(t)) = k - \eta + 1$. Let Z be the family of symplectic restriction varieties obtained by applying the specialization to the admissible sequence $(L'_\bullet, Q'_\bullet)$ (represented by D') obtained from $(L_\bullet, Q_\bullet)$ by omitting the linear spaces $Q^{r_1}_{d_1}, \ldots, Q^{r_{\eta-1}}_{d_{\eta-1}}$. Then there exists a natural morphism $f : U \to Z$ sending $W(t)$ to $W(t) \cap Q^{r_\eta(t)}_{d_\eta}(t)$, which is smooth at the generic point of each of the irreducible components of the fiber of Z at $t=0$. The fibers f over $W' \in Z$ is the linear spaces of dimension k that contain W' and satisfy the appropriate rank conditions with respect to the linear spaces $Q^{r_1}_{d_1}, \ldots, Q^{r_{\eta-1}}_{d_{\eta-1}}$. Notice that running Algorithm 3.29 on D' results in the same outcome as running in D and removing the braces with indices $i < \eta$. Hence, we can do the multiplicity calculation for the family Z. We may, therefore, assume that $\eta = 1$.

In all the cases, the argument is almost identical with very minor variations. We will give it in the hardest case, case (2)(i), and leave the minor modifications in the other cases to the reader. In case (2)(i), by a similar argument, we may further assume that $\kappa = 1$, $d_\kappa + r_\kappa = n-2$, $x_\kappa = 0$ and $s \leq 1$. The most interesting case is when $s = 1$ and $2d_{k-s} \geq n$. Let y_1 be the minimal index l such that L_{n_1} is contained in $\mathrm{Ker}(Q_{d_l}^{r_l})$. We will check that the multiplicities are one by finding a cycle that intersects $V(D)$ in one point and exactly one of the limits in one point. If D^a is admissible, then consider the Schubert variety Σ defined with respect to a general isotropic flag with the following invariants

$$\lambda_i = n - d_i + 2 \ \text{ for } \ \kappa = 1 \leq i \leq l-1, \ \lambda_i = n - d_i + 1 \ \text{ for } \ l \leq i \leq k-1,$$

$$\text{and } \ \mu_k = n - n_1 + 1.$$

If D^a satisfies condition (A2) but not (A1), change the definition of λ_1 so that $\lambda_1 = n - d_1 + 2$. If D^a fails condition (A2), change the definition of Σ so that

$$\lambda_i = n - d_{i+1} + 1 \ \text{ for } \ 1 \leq i \leq l-2, \ \lambda_i = n - d_{i+1} \ \text{ for } \ l-1 \leq i \leq k-2,$$

$$\text{and } \ \mu_{k-1} = n - n_1, \mu_k = n - r_\kappa + 1.$$

By Kleiman's Transversality Theorem [**K1**], it is immediate that both $\Sigma \cap V(D)$ and $\Sigma \cap V(D^a)$ consist of a single reduced point, whereas $\Sigma \cap V(D^b)$ is empty. Since Σ requires the k-plane to be contained in a linear space of dimension $n - n_1 + 1$ and $V(D^b)$ requires the linear space to intersect a linear space of dimension less than n_1, these conditions cannot be simultaneously satisfied for general choices of linear spaces. Hence, $\Sigma \cap V(D^b)$ is empty. On the other hand, the intersection $L_{n_1} \cap F_{\mu_k}^{\perp}$ consists of a one-dimensional vector space W_1 and $Q_{d_i}^{r_i} \cap F_{\lambda_i}$ consist of one-dimensional linear spaces contained in $W_1^{\perp}$ when $l \leq i \leq k-1$ and two-dimensional linear spaces not contained in $W_1^{\perp}$ when $1 \leq i \leq l-1$. Since any linear space contained in $V(D) \cap \Sigma$ or $V(D^a) \cap \Sigma$ must intersect all these linear spaces in one-dimensional subspaces, we conclude that the k-dimensional linear space satisfying conditions imposed by $V(D)$ and Σ or $V(D^a)$ and Σ are uniquely determined. It follows that the multiplicity of $V(D^a)$ is one.

Similarly, if $p(]^1) - \pi(2) - 1 = y_1 - 2$, then D^b is admissible. Let Ω be the Schubert variety defined with respect to a general isotropic flag with the following invariants:

$$\lambda_i = n - d_i + 1 \ \text{ for } \ 1 \leq i \leq k-1, \ \ \mu_k = r_1.$$

By Kleiman's Transversality Theorem [**K1**], it is immediate that both $\Omega \cap V(D)$ and $\Omega \cap V(D^b)$ consist of a single reduced point, whereas $\Omega \cap V(D^a)$ is empty. The conditions imposed by Ω and $V(D^a)$ cannot be simultaneously satisfied, hence $\Omega \cap V(D^a)$ is empty. On the other hand, $F_{\lambda_i} \cap Q_{d_i}^{r_i}$ by construction are one-dimensional subspaces that need to be contained in any W contained in $\Omega \cap V(D)$ or $\Omega \cap V(D^b)$. These determine $(k-1)$-dimensional subspace W' of W. $L_{n_1}^b \cap F_{\mu_1}^{\perp}$ is also a one-dimensinonal subspace Λ that needs to be contained in W. Since $\Lambda \subset (W')^{\perp}$, this uniquely constructs $W \in \Omega \cap V(D^b)$. Similarly, $L_{n_1} \cap F_{\mu_1}^{\perp}$ is a y_1-dimensional linear space. However, the intersection of this linear space with $(W')^{\perp}$ is one-dimensional and must be contained in W. This uniquely constructs W in $V(D) \cap \Omega$. We leave the minor modifications necessary in the other cases to the reader (see [**C2**] for more details in the orthogonal case). This concludes the proof of the theorem. □

6. Rigidity of Schubert classes

In this section, as an application of Algorithm 3.29, we discuss the rigidity of Schubert classes in $SG(k,n)$. Let G be an algebraic group and let P be a parabolic subgroup. Let $X = G/P$ be the corresponding homogeneous variety. A Schubert class c in the cohomology of X is called *rigid* if the only projective subvarieties of X representing c are Schubert varieties. A Schubert class c in the cohomology of X is called *multi rigid* if the only projective subvarieties of X representing kc, for any positive integer k, are unions of k Schubert varieties. For details about rigidity of Schubert classes we refer the reader to the papers [**B**], [**Ho1**], [**RT**], [**C1**] and [**C6**].

In many cases, symplectic restriction varieties provide explicit deformations of Schubert classes showing that the corresponding Schubert classes are not rigid. The following example is typical.

Example 6.1. The Grassmannian $SG(1,n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, all the Schubert varieties $\mathbb{P}L_{n_j}$ are linear spaces. However, note that not all linear spaces are Schubert varieties. Points and codimension one linear spaces are always Schubert varieties. The restriction of Q to a codimension one linear space has a one-dimensional kernel W, hence it is of the form $W^{\perp}$. We conclude that points and codimension one linear spaces are rigid. Linear spaces $\mathbb{P}M$ with $1 < \dim(M) < n-1$ do not have to be isotropic, hence the corresponding Schubert classes are not rigid since they can be deformed to non-isotropic linear spaces.

The following theorem generalizes this example.

Theorem 6.2. *Let $\sigma_{\lambda_\bullet;\mu_\bullet}$ be a Schubert class in the cohomology of $SG(k,n)$.*

(1) *If $s = 0$ and $\mu_j > k-j+1$ for some j, then $\sigma_{\lambda_\bullet;\mu_\bullet} = \sigma_{\mu_\bullet}$ is not rigid.*
(2) *If $s \geq 1$ and $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$, then $\sigma_{\lambda_\bullet;\mu_\bullet}$ is not rigid.*

Proof. In both cases, we find a symplectic restriction variety that has the same class as the Schubert variety but is not a Schubert variety. First, suppose that $s = 0$. Consider a general Schubert variety $\Sigma_{a_\bullet}$ in $G(k,n)$ with class $\sigma_{a_\bullet}$, where $a_j = n-\mu_j$. Then the cohomology class of the restriction variety $V(D_{a_\bullet})$ is the Schubert class $\sigma_{\mu_\bullet}$. To prove this run Algorithm 3.29 on the diagram $D(a_\bullet)$. Since $n-\mu_j \geq m+j > 2j-1$, the diagram $D(a_\bullet)$ does not have any brackets. Furthermore, since the Schubert variety already satisfies condition (A2), any intermediate diagram satisfies (A2). Hence, there are no brackets in any of the intermediate diagrams (a position of the bracket cannot be larger than m). Therefore, the intermediate diagrams automatically satisfy condition (A1). We conclude that the algorithm only produces D^a and at each stage D^a is admissible. The formation of D^a does not change the position of the braces. Hence, when D^a becomes saturated in perfect order, $V(D^a)$ equals a Schubert variety with class $\sigma_{\mu_\bullet}$. If $\mu_j > k-j+1$ for some j, then $V(D_{a_\bullet})$ is not a Schubert variety. Let j be the largest index such that $\mu_j > k-j+1$. If $j = k$, then the span of the linear spaces parameterized by $V(D^a)$ is not isotropic. Hence, $V(D^a)$ cannot be a Schubert variety. If $j < k$, then the linear space $Q^{r_j}_{d_j}$ is distinguished and is not isotropic. Hence, $V(D^a)$ is not a Schubert variety.

Now assume that $s \geq 1$ and $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$. Consider the following partial flag:

$$F_{\lambda_1} \subset \cdots \subset F_{\lambda_{s-1}} \subset Q^{\lambda_s - 2}_{\lambda_s} \subset F^{\perp}_{\mu_{s+1}} \subset \cdots \subset F^{\perp}_{\mu_k},$$

where F_i are isotropic subspaces and $Q^{\lambda_s-2}_{\lambda_s}$ is a non-isotropic space contained in $F^{\perp}_{\lambda_{s-1}}$. By our assumption that $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$ such a sequence exists. Let Y be the Zariski closure of the locus of k-dimensional isotropic subspaces W that satisfy $\dim(W \cap F_{\lambda_i}) = i$ for $1 \leq i < s$, $\dim(W \cap F^{\perp}_{\mu_j}) = j$ for $s < j \leq k$ and $\dim(W \cap Q^{\lambda_s-2}_{\lambda_s}) = s$. Then the cohomology class of Y is $\sigma_{\lambda_\bullet;\mu_\bullet}$, but Y is not a Schubert variety. To calculate the class of Y, we run Algorithm 3.29 on the partial flag defining Y. If we omit the linear spaces $F^{\perp}_{\mu_{s+1}}, \ldots, F^{\perp}_{\mu_k}$, we obtain an admissible sequence. The sequence is in order, so we are in case (2)(i) with $\kappa = k - s + 1$. The Algorithm only produces D^a, which does not satisfy condition (A2). Step 1 of Algorithm 3.25, replaces $Q^{\lambda_s-2}_{\lambda_s}$ with F_{λ_s} and the result is a Schubert variety. We conclude that the class of Y is $\sigma_{\lambda_\bullet,\mu_\bullet}$. However, since $Q^{\lambda_s-2}_{\lambda_s}$ is not isotropic, Y is not a Schubert variety. This concludes the proof. □

COROLLARY 6.3. (1) *If the Schubert class $\sigma_{n-\mu_1,\ldots,n-\mu_k}$ in the cohomology of $G(k,n)$ can be represented by a smooth subvariety of $G(k,n)$, then the Schubert class $\sigma_{;\mu_1,\ldots,\mu_k}$ can also be represented by a smooth subvariety of $SG(k,n)$.*

(2) *If there exists an index $i < k$ such that $m - i - 1 > \mu_i > \mu_{i+1} + 2$ or if there exists an index $1 < i < k$ such that $m - i > \mu_{i-1} = \mu_i + 1 > \mu_{i+1} + 2$, then $\sigma_{;\mu_1,\ldots,\mu_k}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*

(3) *If the Schubert class $\sigma_{\lambda_1,\ldots,\lambda_k}$ in the cohomology of $G(k,m)$ can be represented by a smooth subvariety of $G(k,m)$, then the Schubert class $\sigma_{\lambda_1,\ldots,\lambda_k;}$ in the cohomology of $SG(k,n)$ can be represented by a smooth subvariety of $SG(k,n)$.*

(4) *If there exists an index $i < k$ such that $i < \lambda_i < \lambda_{i+1} + 2$ or an index $1 < i < k-1$ such that $i - 1 < \lambda_{i-1} = \lambda_i - 1 < \lambda_{i+1} - 2$, then $\sigma_{\lambda_1,\ldots,\lambda_k;}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*

PROOF. By Theorem 6.2, the Schubert class $\sigma_{;\mu_1,\ldots,\mu_k}$ is the class of the restriction variety $D(\sigma_{n-\mu_1,\ldots,n-\mu_k})$. If $\sigma_{n-\mu_1,\ldots,n-\mu_k}$ can be represented by a smooth subvariety Y of $G(k,n)$, then, by Kleiman's Transversality Theorem [**K1**], for a general translate of Y, $Y \cap SG(k,n)$ is a smooth subvariety of $SG(k,n)$ representing the Schubert class $\sigma_{;\mu_1,\ldots,\mu_k}$. This proves (1).

A Schubert variety in $SG(k,n)$ with class $\sigma_{\lambda_1,\ldots,\lambda_k;}$ parameterizes k-dimensional subspaces of a maximal isotropic space W, hence it is also a subvariety of $G(k,W) = G(k,m)$ with class $\sigma_{\lambda_1,\ldots,\lambda_k}$. If the latter class can be represented by a smooth subvariety Y of $G(k,W)$, then Y also represents the class $\sigma_{\lambda_1,\ldots,\lambda_k;}$ in $SG(k,n)$. This proves (3).

The Schubert variety Σ parameterizing k-dimensional isotropic subspaces contained in a fixed maximal isotropic space W is a smooth subvariety of $SG(k,n)$ isomorphic to $G(k,m)$ that has cohomology class $\sigma_{m-k+1,\ldots,m;}$. If Y is a smooth subvariety representing $\sigma_{;\mu_1,\ldots,\mu_k}$, then, by Kleiman's Transversality Theorem, the intersection of Σ with a general translate of Y is a smooth subvariety of $G(k,m)$ representing the class $\sigma_{m-\mu_1,\ldots,m-\mu_k}$. Therefore, Theorem 1.6 of [**C1**] implies (2).

Under the inclusion $i : SG(k,n) \to G(k,n)$, a Schubert variety Σ in $SG(k,n)$ with class $\sigma_{\lambda_1,\dots,\lambda_k;}$ is a Schubert variety of $G(k,n)$ with class $\sigma_{\lambda_1,\dots,\lambda_k}$. If the former class can be represented by a smooth subvariety Y of $SG(k,n)$, then $i(Y)$ is a smooth subvariety that represents the latter class in $G(k,n)$. Hence, if the latter class cannot be represented by a smooth subvariety of $G(k,n)$, then Y cannot exist. Therefore, Theorem 1.6 of [**C1**] implies (4).

$\square$

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Clay Mathematics Proceedings
Volume **18**, 2013

Riemann-Roch for Deligne-Mumford stacks

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It is a pleasure to dedicate this article to my teacher, Joe Harris.

Abstract. We give a simple proof of the Riemann-Roch theorem for Deligne-Mumford stacks using the equivariant Riemann-Roch theorem and the localization theorem in equivariant K-theory, together with some basic commutative algebra of Artin local rings.

1. Introduction

The Riemann-Roch theorem is one of the most important and deep results in mathematics. At its essence, the theorem gives a method to compute the dimension of the space of sections of a vector bundle on a compact analytic manifold in terms of topological invariants (Chern classes) of the bundle and manifold.

Beginning with Riemann's inequality for linear systems on curves, work on the Riemann-Roch problem spurred the development of fundamental ideas in many branches of mathematics. In algebraic geometry Grothendieck viewed the classical Riemann-Roch theorem as an example of a transformation between K-theory and Chow groups of a smooth projective variety. In differential geometry Atiyah and Singer saw the classical theorem as a special case of their celebrated index theorem which computes the index of an elliptic operator on a compact manifold in terms of topological invariants.

Recent work in moduli theory has employed the Riemann-Roch theorem on Deligne-Mumford stacks. A version of the theorem for complex V-manifolds was proved by Kawasaki [**Kaw**] using index-theoretic methods. Toen [**Toe**] also proved a version of Grothendieck-Riemann-Roch on Deligne-Mumford stacks using cohomology theories with coefficients in representations. Unfortunately, both the statements and proofs that appear in the literature are quite technical and as a result somewhat inaccessible to many working in the field.

The purpose of this article is to state and prove a version of the Riemann-Roch theorem for Deligne-Mumford stacks based on the equivariant Riemann-Roch theorem for schemes and the localization theorem in equivariant K-theory. Our motivation is the belief that equivariant methods give the simplest and least technical proof of the theorem. The proof here is based on the author's joint work with W. Graham [**EG2, EG3, EG4**] in equivariant intersection theory and equivariant K-theory. It requires little more background than some familiarity with Fulton's intersection theory [**Ful**] and its equivariant analogue developed in [**EG1**].

2010 *Mathematics Subject Classification.* Primary 14C40, 19L10, 14A20.

The contents of this article are as follows. In Section 2 we review the algebraic development of the Riemann-Roch theorem from its original statement for curves to the version for arbitrary schemes proved by Baum, Fulton and MacPherson. Our main reference for this materia, with some slight notational changes, is Fulton's intersection theory book [**Ful**].

In Section 3 we explain how the equivariant Riemann-Roch theorem [**EG2**] easily yields a Grothendieck-Riemann-Roch theorem for representable morphisms of smooth Deligne-Mumford stacks.

Section 4 is the heart of the article. In it we prove the Hirzebruch-Riemann-Roch theorem for smooth, complete Deligne-Mumford stacks. Using the example of the weighted projective line stack $\mathbb{P}(1,2)$ as motivation, we first prove (Section 4.2) the result for quotient stacks of the form $[X/G]$ with G diagonalizable. This proof combines the equivariant Riemann-Roch theorem with the classical localization theorem in equivariant K-theory and originally appeared in [**EG3**]. In Section 4.3 we explain how the non-abelian localization theorem of [**EG4**] is used to obtain the general result. We also include several computations to illustrate how the theorem can be applied.

In Section 5 we briefly discuss the Grothendieck-Riemann-Roch theorem for Deligne-Mumford stacks and illustrate its use by computing the Todd class of a weighted projective space.

For the convenience of the reader we also include an Appendix with some basic definitions used in the theory.

Acknowledgment: The author is grateful to the referee for a very thorough reading of an earlier version of this article.

2. The Riemann-Roch theorem for schemes

The material in Sections 2.1 - 2.3 is well known and further details can be found in the book [**Ful**].

2.1. Riemann-Roch through Hirzebruch. The original Riemann-Roch theorem is a statement about curves. If D is a divisor on a smooth complete curve C then the result can be stated as:

$$l(D) - l(K_C - D) = \deg D + 1 - g$$

where K_C is the canonical divisor and $l(D)$ indicates the dimension of the linear series of effective divisors equivalent to D. Using Serre duality we can rewrite this as

$$\chi(C, L(D)) = \deg D + 1 - g.$$

where $L(D)$ is the line bundle determined by D. Or, in slightly fancier notation

$$\chi(C, L(D)) = \deg c_1(L(D)) + 1 - g. \tag{1}$$

The Hirzebruch-Riemann-Roch theorem extends (1) to arbitrary smooth complete varieties.

Theorem 2.1 (Hirzebruch-Riemann-Roch). *Let X be a smooth projective variety and let V be a vector bundle on X. Then*

$$\chi(X, V) = \int_X \mathrm{ch}(V)\,\mathrm{Td}(X) \tag{2}$$

where $\mathrm{ch}(V)$ *is the Chern character of* V, $\mathrm{Td}(X)$ *is the Todd class of the tangent bundle and* $\int_X$ *is refers to the degree of the 0-dimensional component in the product.*

The Hirzebruch version of Riemann-Roch yields many useful formulas. For example, if X is a smooth algebraic surface then the arithmetic genus can be computed as

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}\int_X c_1^2 + c_2 = \frac{1}{12}(K^2 + \chi) \tag{3}$$

where χ is the topological genus.

2.2. The Grothendieck-Riemann-Roch theorem. This theorem extends the Hirzebruch-Riemann-Roch theorem to the relative setting. Rather than considering Euler characteristics of vector bundles on smooth, complete varieties we consider the relative Euler characteristic for proper morphisms of smooth varieties.

Let $f\colon X \to Y$ be a proper morphism of smooth varieties. The Chern character defines homomorphisms $\mathrm{ch}\colon K_0(X) \to \mathrm{Ch}^* X \otimes \mathbb{Q}$, and $\mathrm{ch}\colon K_0(Y) \to \mathrm{Ch}^* Y \otimes \mathbb{Q}$. Likewise, there are two pushforward maps: the relative Euler characteristic $f_*\colon K_0(X) \to K_0(Y)$ and proper pushforward $f_*\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(Y)$. Since we have 4 groups and 4 natural maps we obtain a diagram - which which does not commute!

$$\begin{array}{ccc} K_0(X) & \stackrel{\mathrm{ch}}{\to} & \mathrm{Ch}^*(X) \otimes \mathbb{Q} \\ f_* \downarrow & & f_* \downarrow \\ K_0(Y) & \stackrel{\mathrm{ch}}{\to} & \mathrm{Ch}^*(Y) \otimes \mathbb{Q} \end{array} \tag{4}$$

The Grothendieck-Riemann-Roch theorem supplies the correction that makes (4) commutative. If $\alpha \in K_0(X)$ then

$$\mathrm{ch}(f_*\alpha)\,\mathrm{Td}(Y) = f_*\left(\mathrm{ch}(\alpha)\,\mathrm{Td}(X)\right) \in \mathrm{Ch}^*(Y) \otimes \mathbb{Q}. \tag{5}$$

In other words the following diagram commutes:

$$\begin{array}{ccc} K_0(X) & \stackrel{\mathrm{ch}\,\mathrm{Td}(X)}{\to} & \mathrm{Ch}^*(X) \otimes \mathbb{Q} \\ f_* \downarrow & & f_* \downarrow \\ K_0(Y) & \stackrel{\mathrm{ch}\,\mathrm{Td}(Y)}{\to} & \mathrm{Ch}^*(Y) \otimes \mathbb{Q} \end{array} \tag{6}$$

Since $\mathrm{Td}(Y)$ is invertible in $\mathrm{Ch}^*(Y)$ we can rewrite equation (5) as

$$\mathrm{ch}(f_*\alpha) = f_*\left(\mathrm{ch}(\alpha)\,\mathrm{Td}(T_f)\right) \tag{7}$$

where $T_f = [TX] - [f^*TY] \in K_0(X)$ is the relative tangent bundle.

Example 2.2. Equation (7) can be seen as a relative version of the Hirzebruch-Riemann-Roch formula, but it is also more general. For example, it can also be applied when $f\colon X \to Y$ is a regular embedding of codimension d. In this case a more refined statement holds. If N is the normal bundle of f and V is a vector bundle of rank r on X then the equation

$$c(f_*V)) = 1 + f_*P(c_1(V), \ldots, c_r(V), c_1(N), \ldots c_d(N))$$

holds in $\mathrm{Ch}^*(Y)$ where $P(T_1, \ldots, T_d, U_1, \ldots, U_d)$ is a universal power series with integer coefficients.

This result is known as Riemann-Roch without denominators and was conjectured by Grothendieck and proved by Grothendieck and Jouanolou.

2.3. Riemann-Roch for singular schemes. If $Z \subset X$ is a subvariety of codimension k then $\mathrm{ch}[\mathcal{O}_Z] = [Z] + \beta$ where β is an element of $\mathrm{Ch}^*(X)$ supported in codimension strictly greater than k. Since $\mathrm{Td}(X)$ is invertible in $\mathrm{Ch}^*(X)$ the Grothendieck-Riemann-Roch theorem can be restated as follows:

THEOREM 2.3. *The map* $\tau_X \colon K_0(X) \to \mathrm{Ch}^*(X) \otimes \mathbb{Q}$ *defined by*

$$[V] \mapsto \mathrm{ch}(V)\,\mathrm{Td}(X)$$

is covariant for proper morphisms of smooth schemes[1] *and becomes an isomorphism after tensoring* $K_0(X)$ *with* $\mathbb{Q}$.

The Riemann-Roch theorem of Baum, Fulton and MacPherson generalizes previous Riemmann-Roch theorems to maps of arbitrary schemes. However, the Grothendieck group of vector bundles $K_0(X)$ is replaced by the Grothendieck group of coherent sheaves $G_0(X)$.

THEOREM 2.4. [**Ful**, Theorem 18.3, Corollary 18.3.2] *For all schemes X there is a homomorphism $\tau_X \colon G_0(X) \to \mathrm{Ch}^*(X) \otimes \mathbb{Q}$ satisfying the following properties:*

(a) τ_X is covariant for proper morphisms.

(b) If V is a vector bundle on X then $\tau_X([V]) = \mathrm{ch}(V)\tau_X(\mathcal{O}_X)$.

(c) If $f \colon X \to Y$ is an lci morphism with relative tangent bundle T_f then for every class $\alpha \in G_0(Y)$ $\tau_X f^*\alpha = \mathrm{Td}(T_f) \cap f^*\tau(\alpha)$.

(d) If $Z \subset X$ is a subvariety of codimension k then $\tau(\mathcal{O}_Z) = [Z] + \beta$ where $\beta \in \mathrm{Ch}^*(X)$ is supported in codimension strictly greater than k.

(e) The map τ_X induces an isomorphism $G_0(X) \otimes \mathbb{Q} \to \mathrm{Ch}^*(X) \otimes \mathbb{Q}$.

REMARK 2.5. If X is smooth then $K_0(X) = G_0(X)$ and using (c) we see that $\tau_X(\mathcal{O}_X) = \mathrm{Td}(X)$ and thereby obtain the Hirzebruch and Grothendieck Riemann-Roch theorems. In [**Ful**] the Chow class $\tau_X(\mathcal{O}_X)$ is called the *Todd class* of X.

REMARK 2.6. Theorem 2.4 is proved by a reduction to the (quasi)-projective case via Chow's lemma. Since Chow's lemma also holds for algebraic spaces, the Theorem immediately extends to the category of algebraic spaces.

3. Grothendieck Riemann-Roch for representable morphisms of quotient Deligne-Mumford stacks

The goal of this section explain how the equivariant Riemann-Roch theorem 3.1 yields a Grothendieck-Riemann-Roch theorem for *representable* morphisms of Deligne-Mumford quotient stacks.

3.1. Equivariant Riemann-Roch. If G is an algebraic group acting on a scheme X then there are equivariant versions of K-theory, Chow groups and Chern classes (see the appendix for definitions). Thus it is natural to expect an equivariant Riemann-Roch theorem relating equivariant K-theory with equivariant Chow groups. Such a theorem was proved in [**EG2**] for the arbitrary action of an algebraic group G on a separated algebraic space X. Before we state the equivariant Riemann-Roch theorem we introduce some further notation.

The equivariant Grothendieck group of coherent sheaves, $G_0(G, X)$, is a module for both $K_0(G, X)$, the Grothendieck ring of G-equivariant vector bundles, and

[1] This means that if $f \colon X \to Y$ is a proper morphism of smooth schemes then $f_* \circ \tau_X = \tau_Y \circ f_*$ as maps $K_0(X) \to \mathrm{Ch}^*(Y) \otimes \mathbb{Q}$.

$R(G) = K_0(G, \mathrm{pt})$, the Grothendieck ring of G-modules. Each of these rings has a distinguished ideal, the augmentation ideal, corresponding to virtual vector bundles (resp. representations) of rank 0. A result of [**EG2**] shows that the two augmentation ideals generate the same topology on $G_0(G, X)$ and we denote by $\widehat{G_0(G, X)}$ the completion of $G_0(G, X)_{\mathbb{Q}}$ with respect to this topology.

The equivariant Riemann-Roch theorem generalizes Theorem 2.4 as follows:

THEOREM 3.1. *There is a homomorphism $\tau_X : G_0(G, X) \to \prod_{i=0}^{\infty} \mathrm{Ch}^i_G(X) \otimes \mathbb{Q}$ which factors through an isomorphism $\widehat{G_0(G, X)} \to \prod_{i=0}^{\infty} \mathrm{Ch}^i_G(X) \otimes \mathbb{Q}$. The map τ_X is covariant for proper equivariant morphisms and when X is a smooth scheme and V is a vector bundle then*

$$\tau_X(V) = \mathrm{ch}(V)\,\mathrm{Td}(TX - \mathfrak{g}) \tag{8}$$

where $\mathfrak{g}$ is the adjoint representation of G.

REMARK 3.2. The K-theory class $TX - \mathfrak{g}$ appearing in (8) corresponds to the tangent bundle of the quotient stack $[X/G]$. If G is finite then $\mathfrak{g} = 0$ and if G is diagonalizable (or more generally solvable) then $\mathfrak{g}$ is a trivial representation of G and the formula $\tau_X(V) = \mathrm{ch}(V)\,\mathrm{Td}(TX)$ also holds.

EXAMPLE 3.3. If $X = \mathrm{pt}$ and $G = \mathbb{C}^*$ then $R(G)$ is the representation ring of G. Since G is diagonalizable the representation ring is generated by characters and $R(G) = \mathbb{Z}[\xi, \xi^{-1}]$ where ξ is the character of weight one. If we set $t = c_1(\xi)$ then the map τ_X is simply the exponential map $\mathbb{Z}[\xi, \xi^{-1}] \to \mathbb{Q}[[t]]$, $\xi \mapsto e^t$. The augmentation ideal of $R(G)$ is $\mathfrak{m} = (\xi - 1)$. If we tensor with $\mathbb{Q}$ and complete at the ideal $\mathfrak{m}$ then the completed ring $\widehat{R(G)}$ is isomorphic to the power series ring $\mathbb{Q}[[x]]$ where $x = \xi - 1$. The map τ_X is the isomorphism sending x to $e^t - 1 = t(1 + t/2 + t^2/3! + \ldots)$.

3.2. Quotient stacks and moduli spaces.

DEFINITION 3.4. A quotient stack is a stack $\mathcal{X}$ equivalent to the quotient $[X/G]$ where $G \subset \mathrm{GL}_n$ is a linear algebraic group and X is a scheme (or more generally an algebraic space[2]).

A quotient stack is Deligne-Mumford if the stabilizer of every point is finite and geometrically reduced. Note that in characteristic 0 the second condition is automatic.

A quotient stack $\mathcal{X} = [X/G]$ is *separated* if the action of G on X is proper - that is, the map $\sigma\colon G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is proper. Since G is affine σ is proper if and only if it is finite. In characteristic 0 any separated quotient stack is automatically a Deligne-Mumford stack.

The hypothesis that a Deligne-Mumford stack is a quotient stack is not particularly restrictive. Indeed, the author does not know any example of a separated Deligne-Mumford stack which is not a quotient stack. Moreover, there are a number of general results which show that "most" Deligne-Mumford stacks are quotient stacks [**EHKV, KV**]. For example if $\mathcal{X}$ satisfies the resolution property - that is, every coherent sheaf is the quotient of a locally free sheaf then $\mathcal{X}$ is quotient stack.

It is important to distinguish two classes of morphisms of Deligne-Mumford stacks, *representable* and *non-representable* morphisms. Roughly speaking, a morphism of Deligne-Mumford stacks $\mathcal{X} \to \mathcal{Y}$ is representable if the fibers of f are

[2]The fact that X is an algebraic space as opposed to a scheme makes little difference in this theory.

schemes. Any morphism $X' \to \mathcal{X}$ from a scheme to a Deligne-Mumford stack is representable. If $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ are quotient stacks and $f\colon \mathcal{X} \to \mathcal{Y}$ is representable then $\mathcal{X}$ is equivalent to a quotient $[Z/H]$ (where $Z = Y \times_{\mathcal{Y}} \mathcal{X}$)and the map of stacks $\mathcal{X} \to \mathcal{Y}$ is induced by an H-equivariant morphism $Z \to Y$. Thus, for quotient stacks we may think of representable morphisms as those corresponding to G-equivariant morphisms.

The non-representable morphisms that we will encounter are all morphisms from a Deligne-Mumford stack to a scheme or algebraic space. Specifically we consider the structure map from a Deligne-Mumford stack to a point and the map from a stack to its coarse moduli space.

Every Deligne-Mumford stack $\mathcal{X}$ is *finitely parametrized*. This means that there is finite surjective morphism $X' \to \mathcal{X}$ where X is a scheme. Thus we can say that a separated stack $\mathcal{X}$ is *complete* if it is finitely parametrized by a complete scheme.

A deep result of Keel and Mori [**KM**] implies that every separated Deligne-Mumford stack $\mathcal{X}$ has a *coarse moduli* space M in the category of algebraic spaces. Roughly speaking, this means that there is a proper surjective (but not representable) morphism $p\colon \mathcal{X} \to M$ which is a bijection on geometric points and satisfies the universal property that any morphism $\mathcal{X} \to M'$ with M' an algebraic space must factor through p. When $\mathcal{X} = [X/G]$ then the coarse moduli space M is the geometric quotient in the category of algebraic spaces. When $X = X^s$ is the set of stable points for the action of a reductive group G then M is the geometric invariant theory quotient of [**MFK**].

The map $\mathcal{X} \to M$ is not finite in the usual scheme-theoretic sense, because it is not representable, but it behaves like a finite morphism in the sense that if $f\colon X' \to \mathcal{X}$ is a finite parametrization then the composite morphism $X' \to M$ is finite. Note, however, that if we define $\deg p$ by requiring $\deg p \deg f = \deg X'/M$ then $\deg p$ may be fractional (see below).

Since p is a bijection on geometric points, some of the geometry of the stack $\mathcal{X}$ can be understood by studying the coarse space M. Note, however, that when $\mathcal{X}$ is smooth the space M will in general have finite quotient singularities.

3.2.1. *K-theory and Chow groups of quotient stacks.* If $\mathcal{X}$ is a stack then we use the notation $K_0(\mathcal{X})$ to denote the Grothendieck group of vector bundles on $\mathcal{X}$ and we denote by $G_0(\mathcal{X})$ the Grothendieck group of coherent sheaves on $\mathcal{X}$. If $\mathcal{X}$ is smooth and has the resolution property then the natural map $K_0(\mathcal{X}) \to G_0(\mathcal{X})$ is an isomorphism.

If $\mathcal{X} = [X/G]$ then $K_0(\mathcal{X})$ (resp. $G_0(\mathcal{X})$) is naturally identified with the equivariant Grothendieck ring $K_0(G, X)$ (resp. equivariant Grothendieck group $G_0(G, X)$.

Chow groups of Deligne-Mumford stacks were defined with rational coefficients by Gillet [**Gil**] and Vistoli [**Vis**] and with integral coefficients by Kresch [**Kre**]. When $\mathcal{X} = [X/G]$ Kresch's Chow groups agree integrally with the equivariant Chow groups $\mathrm{Ch}^*_G(X)$ defined in [**EG1**]. The proper pushforward of rational Chow groups $p\colon \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q} \to \mathrm{Ch}^*(M) \otimes \mathbb{Q}$ is an always an isomorphism [**Vis, EG1**]. In particular this means that if $\mathcal{X} = [X/G]$ is a Deligne-Mumford stack then every equivariant Chow class can be represented by a G-invariant cycle on X (as opposed to $X \times \mathbf{V}$ where $\mathbf{V}$ is a representation of G). Consequently $\mathrm{Ch}^k(\mathcal{X}) \otimes \mathbb{Q} = 0$ for $k > \dim \mathcal{X}$.

The theory of Chern classes in equivariant intersection theory implies that a vector bundle V on $\mathcal{X} = [X/G]$ has Chern classes $c_i(V)$ which operate on $\mathrm{Ch}^*(\mathcal{X})$. If $\mathcal{X}$ is smooth then we may again view the Chern classes as elements of $\mathrm{Ch}^*(\mathcal{X})$. If $\mathcal{X}$ is smooth and Deligne-Mumford the Chern character and Todd class are again maps $K_0(\mathcal{X}) \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$.

Every smooth Deligne-Mumford stack has a tangent bundle. If $\mathcal{X} = [X/G]$ is a quotient stack then the map $X \to [X/G]$ is a G-torsor so the tangent bundle to $\mathcal{X}$ corresponds to the quotient $TX/\mathfrak{g}$ where $\mathfrak{g}$ is the adjoint representation of G. In particular under the identification of $\mathrm{Ch}^*(\mathcal{X}) = \mathrm{Ch}^*_G(X)$, $c(T\mathcal{X}) = c(TX)c(\mathfrak{g})^{-1}$. If G is finite or diagonalizable then $\mathfrak{g}$ is a trivial representation so $c_t(\mathfrak{g}) = 1$. Thus, the Chern classes of $T\mathcal{X}$ are just the equivariant Chern classes of TX in these cases.

3.2.2. *Restatement of equivariant Riemann-Roch for Deligne-Mumford quotient stacks.* As already noted, when G acts properly then $\mathrm{Ch}^i_G(X)_{\mathbb{Q}} = 0$ for $i > \dim[X/G]$ so the infinite direct product in Theorem 3.1 is just $\mathrm{Ch}^*(\mathcal{X})$ where $\mathcal{X} = [X/G]$. A more subtle fact proved in [**EG2**] is that if G acts with finite stabilizers (in particular if the action is proper) then $G_0(G,X) \otimes \mathbb{Q}$ is supported at a finite number of points of $\mathrm{Spec}(R(G) \otimes \mathbb{Q})$. It follows that $\widehat{G_0(G,X)}$ is the same as the localization of the $R(G) \otimes \mathbb{Q}$-module $G_0(G,X) \otimes \mathbb{Q}$ at the augmentation ideal in $R(G) \otimes \mathbb{Q}$. For reasons that will become clear in the next section we denote this localization by $G_0(G,X)_1$ (or $K_0(G,X)_1$). Identifying equivariant K-theory with the K-theory of the stack $\mathcal{X} = [X/G]$ we will also write $K_0(\mathcal{X})_1$ and $G_0(\mathcal{X})_1$ respectively. Theorem 3.1 implies the following result about smooth Deligne-Mumford quotient stacks.

THEOREM 3.5. *There is a homomorphism $\tau_X \colon G_0(\mathcal{X}) \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$ which factors through an isomorphism $G_0(\mathcal{X})_1 \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$. The map τ_X is covariant for proper representable morphisms and when $\mathcal{X}$ is a smooth and V is a vector bundle then*

$$\tau_X(V) = \mathrm{ch}(V)\,\mathrm{Td}(\mathcal{X}) \tag{9}$$

4. Hirzebruch Riemann-Roch for quotient Deligne-Mumford stacks

At first glance, Theorem 3.5 looks like the end of the Riemann-Roch story for Deligne-Mumford stacks, since it gives a stack-theoretic version of the Grothendieck-Riemann-Roch theorem for representable morphisms and also explains the relationship between K-theory and Chow groups of a quotient stack. Unfortunately, the theorem cannot be directly used to compute the Euler characteristic of vector bundles or coherent sheaves on complete Deligne-Mumford stacks.

The problem is that the Euler characteristic of a vector bundle V on $\mathcal{X}$ is the K-theoretic direct image $f_!V := \sum(-1)^i H^i(\mathcal{X}, V)$ under the projection map $f \colon K_0(\mathcal{X}) \to K_0(\mathrm{pt}) = \mathbb{Z}$. However, the projection map $\mathcal{X} \to \mathrm{pt}$ is not representable - since if it were then $\mathcal{X}$ would be a scheme or algebraic space.

A Hirzebruch-Riemann-Roch theorem for a smooth, complete, Deligne-Mumford stack $\mathcal{X}$ should be a formula for the Euler characteristic of a bundle in terms of degrees of Chern characters and Todd classes. In this section, which is the heart of the paper, we show how to use Theorem 3.5 and generalizations of the localization theorem in equivariant K-theory to obtain such a result. Henceforth, we will work exclusively over the complex numbers $\mathbb{C}$.

4.1. Euler characteristics and degrees of 0-cycles. If V is a coherent sheaf on $\mathcal{X} = [X/G]$ then the cohomology groups of V are representations of G and we make the following definition.

Definition 4.1. If V is a G-equivariant vector bundle on X then Euler characteristic of V viewed as a bundle on $\mathcal{X} = [X/G]$ is $\sum_i (-1)^i \dim H^i(X,V)^G$ where $H^i(X,V)^G$ denotes the invariant subspace. We denote this by $\chi(\mathcal{X}, V)$.

Note that, if $\dim G > 0$ then X will never be complete, so $H^i(X,V)$ need not be finite dimensional. Nevertheless, if $\mathcal{X}$ is complete then $H^i(X,V)^G$ is finite dimensional as it can be identified with the cohomology of the coherent sheaf $H^i(M, p_*V)$ under the proper morphism $p\colon \mathcal{X} \to M$ from $\mathcal{X}$ to its coarse moduli space.

If G is linearly reductive (for example if G is diagonalizable) then the cohomology group $H^i(X,V)$ decomposes as direct sum of G-modules and $H^i(X,V)^G$ is the trivial summand. In this case it easily follows that the assignment $V \mapsto \sum_i (-1)^i \dim H^i(X,V)^G$ defines an Euler characteristic homomorphism $K_0(G,X) \to \mathbb{Z}$. The identification of vector bundles on $\mathcal{X}$ with G-equivariant bundles on X yields an Euler characteristic map $\chi\colon K_0(\mathcal{X}) \to \mathbb{Z}$. When the action of G is free and $\mathcal{X}$ is represented by a scheme, this is the usual Euler characteristic.

However, even if G is not reductive but acts properly on X then the assignment $V \mapsto \sum_i (-1)^i \dim H^i(X,V)^G$ still defines an Euler characteristic map $\chi\colon K_0(\mathcal{X}) \to \mathbb{Z}$. This follows from Keel and Mori's description of the finite map $[X/G] \to M = X/G$ as being étale locally in M a quotient $[V/H] \to V/H$ where V is affine and H is finite (and hence reductive because we work in characteristic 0).

The above reasoning also applies to G-linearized coherent sheaves on X and we also obtain an Euler characteristic map $\chi\colon G_0(\mathcal{X}) \to \mathbb{Z}$. These maps can be extended by linearity to maps $\chi\colon K_0(\mathcal{X}) \otimes F \to F$ (resp. $G_0(\mathcal{X}) \otimes F \to F$) where F is any coefficient ring.

Example 4.2. If G is a finite group let $BG = [\mathrm{pt}/G]$ be the classifying stack parametrizing algebraic G coverings. The identity morphism $pt \to pt$ factors as $pt \to BG \to pt$ where the first map is the universal G-covering and which associates to any scheme T the trivial covering $G \times T \to T$. The map $BG \to pt$ is the coarse moduli space map and associates to any G-torsor $Z \to T$ to the ground scheme T.

The map $\mathrm{pt} \to BG$ is representable and the pushforward in map $K_0(\mathrm{pt}) \to K_0(BG)$ is the map $\mathbb{Z} \to R(G)$ which sends the a vector space V to the representation $V \otimes \mathbb{C}[G]$ where $\mathbb{C}[G]$ is the regular representation of G.

Since the $\mathbb{C}[G]$ contains a copy of the trivial representation with multiplicity one, it follows that, with our definition, the composition of pushforwards $\mathbb{Z} = K_0(pt) \to R(G) = K_0(BG) \to \mathbb{Z} = K_0(pt)$ is the identity - as expected.

4.1.1. *The degree of a 0-cycle.* Some care is required in understanding 0-cycles on a Deligne-Mumford stack. The reason is that a closed 0-dimensional integral substack η is not in general a closed point but rather a *gerbe*. That is, it is isomorphic after étale base change to BG for some finite group G. Assuming that the ground field is algebraically closed then the degree of $[\eta]$ is defined to be $1/|G|$.

If $\mathcal{X} = [X/G]$ is a complete Deligne-Mumford quotient stack then 0-dimensional integral substacks correspond to G-orbits of closed points and we can define for a closed point $x \in X$ $\deg[Gx/G] = 1/|G_x|$ where G_x is the stabilizer of x.

Example 4.3. The necessity of dividing by the order the stabilizer can be seen by again looking at the factorization of the morphism $\mathrm{pt} \to BG \to \mathrm{pt}$ when G is a finite group. The map $\mathrm{pt} \to BG$ has degree $|G|$ so the map $BG \to \mathrm{pt}$ must have degree $\frac{1}{|G|}$.

4.2. Hirzebruch-Riemann-Roch theorem for quotients by diagonalizable groups. The goal of this section is to understand the Riemann-Roch theorem in an important special case: separated Deligne-Mumford stacks of the form $\mathcal{X} = [X/G]$ where X is a smooth variety and $G \subset (\mathbb{C}^*)^n$ is a diagonalizable group. We will develop the theory using a very simple example - the weighted projective line stack $\mathbb{P}(1,2)$.

4.2.1. *Example: The weighted projective line stack* $\mathbb{P}(1,2)$*, Part I.* Consider the weighted projective line stack $\mathbb{P}(1,2)$. This stack is defined as the quotient of $[\mathbb{A}^2 \smallsetminus \{0\}/\mathbb{C}^*]$ where $\mathbb{C}^*$ acts with weights $(1,2)$; i.e., $\lambda(v_0, v_1) = (\lambda v_0, \lambda^2 v_1)$. Because $X = \mathbb{A}^2 \smallsetminus \{0\}$ is an open set in a two-dimensional representation, every equivariant vector bundle on X is of the form $X \times V$ where V is a representation of $\mathbb{C}^*$. In this example we consider two line bundles on $\mathbb{P}(1,2)$ - the line bundle L associated to the weight one character ξ of $\mathbb{C}^*$ and the line bundle $\mathcal{O}$ associated to the trivial character.

Direct calculation of $\chi(\mathbb{P}(1,2),\mathcal{O})$ and $\chi(\mathbb{P}(1,2),L)$: It is easy to compute $\chi(\mathbb{P}(1,2),L)$ and $\chi(\mathbb{P}(1,2),\mathcal{O})$ directly. The coarse moduli space of $\mathbb{P}(1,2)$ is the geometric quotient $(\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{C}^*$. Even though $\mathbb{C}^*$ no longer acts freely the quotient is still $\mathbb{P}^1$ since it has a covering by two affines $\operatorname{Spec}\mathbb{C}[x_0^2/x_1]$ and $\operatorname{Spec}\mathbb{C}[x_1/x_0^2]$, where x_0 and x_1 are the coordinate functions on $\mathbb{A}^2$. The Euler characteristic pushforward $K_0(\mathbb{P}(1,2)) \to K_0(\mathrm{pt}) = \mathbb{Z}$ factors through the proper pushforward $K_0(\mathbb{P}(1,2)) \to K_0(\mathbb{P}^1)$. Consequently, we can compute $\chi(\mathbb{P}(1,2),L)$ and $\chi(\mathbb{P}(1,2),\mathcal{O})$ by identifying the images of these bundles on $\mathbb{P}^1$. A direct computation using the standard covering of $\mathbb{A}^2 \smallsetminus \{0\}$ by $\mathbb{C}^*$ invariant affines shows that both L and $\mathcal{O}$ pushforward to the trivial bundle on $\mathbb{P}^1$. Hence

$$\chi(\mathbb{P}(1,2),L) = \chi(\mathbb{P}(1,2),\mathcal{O}) = 1$$

An attempt to calculate $\chi(\mathbb{P}(1,2),\mathcal{O})$ and $\chi(\mathbb{P}(1,2),L)$ using Riemann-Roch methods: Following Hirzebruch-Riemann-Roch for smooth varieties we might expect to compute $\chi(\mathbb{P}(1,2),L)$ as $\int_{\mathbb{P}(1,2)} \operatorname{ch}(L)\operatorname{Td}(\mathbb{P}(1,2))$. To do that we will use the presentation of $\mathbb{P}(1,2)$ as a quotient by $\mathbb{C}^*$. The line bundle L corresponds to the pullback to $\mathbb{A}^2$ of the standard character ξ of $\mathbb{C}^*$ and the tangent bundle to the stack $\mathbb{P}(1,2)$ fits into a weighted Euler sequence

$$0 \to \mathbf{1} \to \xi + \xi^2 \to T\mathbb{P}(1,2) \to 0$$

where $\mathbf{1}$ denotes the trivial character of $\mathbb{C}^*$ and again ξ is the character of $\mathbb{C}^*$ of weight 1. If we let $t = c_1(\xi)$ then

$$\operatorname{ch}(L)\operatorname{Td}(\mathbb{P}(1,2)) = (1+t)(1+3t/2) = 1 + 5t/2$$

Now the Chow class t is represented by the invariant cycle $[x=0]$ on $\mathbb{A}^2$ and the corresponding point of $\mathbb{P}(1,2)$ has stabilizer of order 2. Thus

$$\int_{\mathbb{P}(1,2)} \operatorname{ch}(L)\operatorname{Td}(\mathbb{P}(1,2)) = 1/2(5/2) = 5/4$$

which is 1/4 too big. On the other a hand then again $\chi(\mathbb{P}(1,2),\mathcal{O}) = 1$ but

$$\int_{\mathbb{P}(1,2)} \operatorname{ch}(\mathcal{O})\operatorname{Td}(\mathbb{P}(1,2)) = 3/4$$

is too small by 1/4. In particular

$$\int_{\mathbb{P}(1,2)} \operatorname{ch}(\mathcal{O}+L)\operatorname{Td}(\mathbb{P}(1,2)) = 2 \tag{10}$$

which is indeed equal to $\chi(\mathbb{P}(1,2), \mathcal{O}+L)$.

Equation (10) may seem unremarkable but is in fact a hint as to how to obtain a Riemann-Roch formula that works for all bundles on $\mathbb{P}(1,2)$.

4.2.2. *The support of equivariant K-theory.* To understand why (10) holds we need to study $K_0(\mathbb{P}(1,2)$ as an $R(\mathbb{C}^*)$-module. Precisely,

$$K_0(\mathbb{P}(1,2)) = K_0(\mathbb{C}^*, \mathbb{A}^2 \smallsetminus \{0\}) = \mathbb{Z}[\xi, \xi^{-1}]/(\xi^2-1)(\xi-1).$$

This follows from the fact that $\mathbb{A}^2$ is a representation of $\mathbb{C}^*$ so $K_0(\mathbb{C}^*, \mathbb{A}^2) = R(\mathbb{C}^*) = \mathbb{Z}[\xi, \xi^{-1}]$ where again ξ denotes the weight one character of $\mathbb{C}^*$. Because we delete the origin we must quotient by the ideal generated by the K-theoretic Euler class of the tangent space to the origin. With our action, $\mathbb{A}^2$ is the representation $\xi + \xi^2$ so the tangent space of the origin is also $\xi+\xi^2$. The Euler class of this representation is $(1-\xi^{-1})(1-\xi^{-2})$ which generates the ideal $(\xi^2-1)(\xi-1)$.

From the above description we see that $K_0(\mathbb{C}^*, \mathbb{A}^2 \smallsetminus \{0\}) \otimes \mathbb{C}$ is an Artin ring supported at the points 1 and -1 of $\operatorname{Spec} R(G) \otimes \mathbb{C} = \mathbb{C}^*$. The vector bundle $\mathcal{O}+L$ on $\mathbb{P}(1,2)$ corresponding to the element $1+\xi \in R(\mathbb{C}^*)$ is supported at $1 \in \mathbb{C}^*$ and the formula

$$\chi(\mathbb{P}(1,2), \mathcal{O}+L) = \int_{\mathbb{P}(1,2)} (\operatorname{ch}(\mathcal{O}+L)\operatorname{Td}(\mathbb{P}(1,2))$$

is correct. On the other hand the class of the bundle $\mathcal{O}$ decomposes as $[\mathcal{O}]_1 + [\mathcal{O}]_{-1}$ where $[\mathcal{O}]_1 = 1/2(1+\xi)$ is supported at 1 and $[\mathcal{O}]_{-1} = 1/2(1-\xi)$ is supported at -1. In this case the integral $\int_{\mathbb{P}(1,2)} \operatorname{ch}(\mathcal{O})\operatorname{Td}(\mathbb{P}(1,2))$ computes $\chi(\mathbb{P}(1,2), [\mathcal{O}]_1)$.

This phenomenon is general. If $\alpha \in K_0(G,X)\otimes\mathbb{Q}$, denote by α_1 the component supported at the augmentation ideal of $R(G)$.

Corollary 4.4. [**EG4**, cf. Proof of Theorem 6.8] *Let G be a linear algebraic group (not necessarily diagonalizable) acting properly on smooth variety X. Then if $\alpha \in K_0(\mathcal{X}) \otimes \mathbb{Q}$*

$$\int_{\mathcal{X}} \operatorname{ch}(\alpha)\operatorname{Td}(\mathcal{X}) = \chi(\mathcal{X}, \alpha_1). \tag{11}$$

Proof. Since the equivariant Chern character map factors through $K_0(G,X)_1$ it suffices to prove that

$$\int_{\mathcal{X}} \operatorname{ch}(\alpha)\operatorname{Td}(\mathcal{X}) = \chi(\mathcal{X}, \alpha) \tag{12}$$

for $\alpha \in K_0(G,X)_1$. To prove our result we use the fact that every Deligne-Mumford stack $\mathcal{X}$ is finitely parametrizable. Translated in terms of group actions this means that there is a finite, surjective G-equivariant morphism $X' \to X$ such that G acts freely on X' and the quotient $\mathcal{X}' = [X'/G]$ is represented by a scheme. (This result was first proved by Seshadri in [**Ses**] and is the basis for the finite parametrization theorem for stacks proved in [**EHKV**].) The scheme X' is in general singular[3], but the equivariant Riemann-Roch theorem implies the following proposition.

[3]If the quotient X/G is quasi-projective then a result of Kresch and Vistoli [**KV**] shows that we can take X' to be smooth, but this is not necessary for our purposes.

Proposition 4.5. *Let G act properly on X and let $f\colon X' \to X$ be a finite surjective G-equivariant map. Then the proper pushforward $f_*\colon G_0(G,X') \to G_0(G,X)$ induces a surjection $G_0(G,X')_1 \to G_0(G,X)_1$, where $G_0(G,X)_1$ (resp. $G_0(G,X)_1$) denotes the localization of $G_0(G,X)\otimes\mathbb{Q}$ (resp. $G_0(G,X')\otimes\mathbb{Q}$) at the augmentation ideal of $R(G)\otimes\mathbb{Q}$.*

Proof of Proposition 4.5. Because G acts properly on X and $X' \to X$ is finite (hence proper) it follows that G acts properly on X'. Thus $\mathrm{Ch}^*_G(X')\otimes\mathbb{Q}$ and $\mathrm{Ch}^*_G(X)\otimes\mathbb{Q}$ are generated by G-invariant cycles. Since f is finite and surjective any G-invariant cycle on X is the direct image of some rational G-invariant cycle on X'; i.e., the pushforward of Chow groups $f_*\colon \mathrm{Ch}^*_G(X') \to \mathrm{Ch}^*_G(X)$ is surjective after tensoring with $\mathbb{Q}$. Hence by Theorem 3.5 the corresponding map $f_*\colon G_0(G,X')_1 \to G_0(G,X)_1$ is also surjective. □

Now G acts freely on X' so $G_0(G,X')\otimes\mathbb{Q}$ is supported entirely at the augmentation ideal of $R(G)\otimes\mathbb{Q}$. Therefore we have a surjection $G_0(G,X')\otimes\mathbb{Q} \to G_0(G,X)_1$. Since X is smooth, we can also identify $K_0(G,X)_1$ with $G_0(G,X)_1$ and express the class $\alpha \in K_0(G,X)_1$ as $\alpha = f_*\beta$. Since f is finite we see that $\chi(\mathcal{X}',\alpha) = \chi(\mathcal{X},\beta)$. Since $\mathcal{X}'$ is a scheme, we know by the Riemann-Roch theorem for the singular schemes that $\chi(\mathcal{X}',\beta) = \int_{\mathcal{X}'} \tau_{\mathcal{X}'}(\beta)$. Applying the covariance of the equivariant Riemann-Roch map for proper equivariant morphisms we conclude that

$$\int_{\mathcal{X}} \mathrm{ch}(\alpha)\,\mathrm{Td}(\mathcal{X}) = \int_{\mathcal{X}'} \tau_{\mathcal{X}'}(\beta) = \chi(\mathcal{X},\beta) = \chi(\mathcal{X},\alpha).$$

□

4.2.3. *The localization theorem in equivariant K-theory.* Corollary 4.4 tells us how to deal with the component of $G_0(G,X)$ supported at the augmentation ideal. We now turn to the problem of understanding what to do with the rest of equivariant K-theory. The key tool is the *localization theorem*.

The correspondence between diagonalizable groups and finitely generated abelian groups implies that if G is a complex diagonalizable group then $R(G)\otimes\mathbb{C}$ is the coordinate ring of G. Since the $R(G)\otimes\mathbb{Q}$-module $G_0(G,X)\otimes\mathbb{Q}$ is supported at a finite number of closed points of $\operatorname{Spec} R(G)\otimes\mathbb{Q}$ it follows that $G_0(G,X)\otimes\mathbb{C}$ is also supported at a finite number of closed points of $G = \operatorname{Spec} R(G)\otimes\mathbb{C}$. If $h \in G$ then we denote by $G_0(G,X)_h$ the localization of $G_0(G,X)\otimes\mathbb{C}$ at the corresponding maximal ideal of $R(G)\otimes\mathbb{C}$. In the course of the proof of [**Tho3**, Theorem 2.1] Thomason showed that $G_0(G,X)_h = 0$ if h acts without fixed point on X. Hence $h \in \operatorname{Supp} G_0(G,X)$ implies that $X^h \neq \emptyset$. Since G is assumed to act with finite stabilizers (because it acts properly) it follows that h must be of finite order if $h \in \operatorname{Supp} G_0(G,X)$.

If X is a smooth scheme then we can identify $G_0(G,X) = K_0(G,X)$ and the discussion of the above paragraph applies to the Grothendieck ring of vector bundles.

Let X^h be the fixed locus of $h \in G$. If X is smooth then X^h is a smooth closed subvariety of X so the inclusion $i_h\colon X^h \to X$ is a regular embedding. Since the map i_h is G-invariant the normal bundle N_h of $X^h \to X$ comes with a natural G-action. The key to understanding what happens to the summand $G_0(G,X)_h$ is the localization theorem:

THEOREM 4.6. *Let G be a diagonalizable group acting on a smooth variety X. The pullback $i_h^*\colon G_0(G,X) \to G_0(G,X^h)$ is an isomorphism after tensoring with $\mathbb{C}$ and localizing at h. Moreover, the Euler class of the normal bundle, $\lambda_{-1}(N_h^*)$, is invertible in $G_0(G,X^h)_h$ and if $\alpha \in G_0(G,X)$ then*

$$\alpha = (i_h)_* \left(\frac{i_h^* \alpha}{\lambda_{-1}(N_h^*)} \right)$$

REMARK 4.7. The localization theorem in equivariant K-theory was originally proved by Segal in [**Seg**]. The version stated above is essentially [**Tho3**, Lemma 3.2].

4.2.4. *Hirzebruch-Riemann-Roch for diagonalizable group actions.* The localization theorem implies that if $\alpha \in G_0(G,X)_h$ then

$$\chi(\mathcal{X},\alpha) = \chi([X^h/G], \frac{i_h^*\alpha}{\lambda_{-1}N_h^*}).$$

Thus if $\alpha \in G_0(G,X)_h$ then we can compute $\chi([X/G],\alpha)$ by restricting to the fixed locus X^h. This is advantageous because there is an automorphism of $G_0(G,X^h)$ which moves the component of a K-theory class supported at h to the component supported at 1 without changing the Euler characteristic.

DEFINITION 4.8. Let V be a G-equivariant vector bundle on a space Y and suppose that an element $h \in G$ of finite order acts trivially on Y. Let H be the cyclic group generated by h and let $X(H)$ be its character group. Then V decomposes into a sum of h-eigenbundles $\oplus_{\xi \in X(H)} V_\xi$ for the action of H on the fibres of $V \to Y$. Because the action of H commutes with the action of G (since G is abelian) each eigenbundle is a G-equivariant vector bundle. Define $t_h([V]) \in K_0(G,Y) \otimes \mathbb{C}$ to be the class of the virtual bundle $\sum_{\xi \in X(H)} \xi(h) V_\xi$. A similar construction for G-linearized coherent sheaves defines an automorphism $t_h\colon G_0(G,Y) \otimes \mathbb{C} \to G_0(G,Y) \otimes \mathbb{C}$.

The map t_h is compatible with the automorphism of $R(G) \otimes \mathbb{C}$ induced by the translation map $G \to G$, $k \mapsto kh$ and maps the localization $K_0(G,Y)_h$ to the localization $K_0(G,Y)_1$. The analogous statement also holds for the corresponding localizations of $G_0(G,Y) \otimes \mathbb{C}$.

The crucial property of t_h is that it preserves invariants.

PROPOSITION 4.9. *If G acts properly on Y and Y/G is complete then*

$$\chi([Y/G],\beta) = \chi([Y/G], t_h(\beta)).$$

PROOF. Observe that if $V = \oplus_{\xi \in X(H)} V_\xi$ then the invariant subbundle V^G is contained in the H-weight 0 submodule of V. Since $t_h(E)$ fixes the 0 weight submodule we see that the invariants are preserved. □

Combining the localization theorem with Proposition 4.9 we obtain Hirzebruch-Riemann-Roch for actions of diagonalizable groups.

THEOREM 4.10. [**EG3**, cf. Theorem 3.1] *Let G be a diagonalizable group acting properly on smooth variety X such that the quotient stack $\mathcal{X} = [X/G]$ is complete. Then if V is an equivariant vector bundle on X*

$$\chi(\mathcal{X},V) = \sum_{h \in \operatorname{Supp} K_0(G,X)} \int_{[X^h/G]} \operatorname{ch}\left(t_h(\frac{i_h^* V}{\lambda_{-1}(N_h^*)}) \right) \operatorname{Td}([X^h/G]). \tag{13}$$

4.2.5. *Conclusion of the* $\mathbb{P}(1,2)$ *example.* Since $K_0(\mathbb{P}(1,2)) = \mathbb{Z}[\xi]/(\xi^2-1)(\xi-1)$, we see that K-theory is additively generated by the class $1, \xi, \xi^2$. We use Theorem 4.10 to compute $\chi(\mathbb{P}(1,2), \xi^l)$. First

$$\begin{aligned}\chi(\mathbb{P}(1,2), \xi_1^l) &= \int_{\mathbb{P}(1,2)} \mathrm{ch}(\xi^1)\,\mathrm{Td}(\mathbb{P}(1,2)) = \int_{\mathbb{P}(1,2)} (1+lt)(1+3/2t) \\ &= \int_{\mathbb{P}(1,2)} (l+3/2)t = \frac{(2l+3)}{4}.\end{aligned}$$

Now we must calculate the contribution from the component supported at -1. If we let $X = \mathbb{A}^2 \smallsetminus \{0\}$ then X^{-1} is the linear subspace $\{(0,a) | a \neq 0\}$. Because $\mathbb{C}^*$ acts with weight 2 on X^{-1} the stack $[X^{-1}/\mathbb{C}^*]$ is isomorphic to the classifying stack $B\mathbb{Z}_2$ and $K_{\mathbb{C}^*}(X^{-1}) = \mathbb{Z}[\xi]/(\xi^2-1)$ while $\mathrm{Ch}^*_{\mathbb{C}^*}(X^{-1}) = \mathbb{Z}[t]/2t$ where again $t = c_1(\xi)$ and $\int_{[X^{-1}/\mathbb{C}^*]} 1 = 1/2$. Using our formula we see that

$$\chi(\mathbb{P}(1,2), \xi_{-1}^l) = \int_{[X^{-1}/\mathbb{C}^*]} \mathrm{ch}\left(\frac{(-1)^l \xi^l}{1+\xi^{-1}}\right) \mathrm{Td}([X^{-1}/\mathbb{C}^*]).$$

Since $c_1(\xi)$ is torsion, the only contribution to the integral on the 0-dimensional stack $[X^{-1}/\mathbb{C}^*]$ is from the class 1 and we see that $\chi(\mathbb{P}(1,2), \xi_{-1}^l) = (-1)^l/4$, so we conclude that

$$\chi(\mathbb{P}(1,2), \xi^l) = \frac{2l+3+(-1)^l}{4}.$$

In particular, $\chi(\mathbb{P}(1,2), \mathcal{O}) = \chi(\mathbb{P}(1,2), L) = 1$. Note however that $\chi(\mathbb{P}(1,2), L^2) = 2$.

Exercise 4.11. You should be able to work things out for arbitrary weighted projective stacks. The stack $\mathbb{P}(4,6)$ is known to be isomorphic to the stack of elliptic curve $\overline{\mathcal{M}}_{1,1}$ and so $K_0(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}[\xi]/(\xi^4-1)(\xi^6-1)$. Hence $K_0(\overline{\mathcal{M}}_{1,1})$ is supported at $\pm 1, \pm i, \omega, \omega^{-1}, \eta, \eta^{-1}$ where $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/6}$. Use Theorem 4.10 to compute $\chi(\overline{\mathcal{M}}_{1,1}, \xi^k)$. This computes the dimension of the space of level one weight k-modular forms. The terms in the sum will be complex numbers but the total sum is of course integral.

4.2.6. *Example: The quotient stack* $[(\mathbb{P}^2)^3/\mathbb{Z}_3]$. To further illustrate Theorem 4.10 we consider Hirzebruch-Riemann-Roch on the quotient stack $\mathcal{X} = [(\mathbb{P}^2)^3/\mathbb{Z}_3]$ where $\mathbb{Z}_3$ acts on $(\mathbb{P}^2)^3$ by cyclic permutation. This example will serve as a warm-up for Section 4.3.1 where we consider the stack $[(\mathbb{P}^2)^3/S_3]$.

Our goal is to compute $\chi(\mathcal{X}, L)$ where $L = \mathcal{O}(m) \boxtimes \mathcal{O}(m) \boxtimes \mathcal{O}(m)$ viewed as a $\mathbb{Z}_3$-equivariant line bundle on $(\mathbb{P}^2)^3$. To make this computation we observe that $\mathrm{Ch}^*(\mathcal{X}) = \mathrm{Ch}^*_{\mathbb{Z}_3}((\mathbb{P}^2)^3)$ is generated by $\mathbb{Z}_3$ invariant cycles. It follows that every element $\mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$ is represented by a symmetric polynomial (of degree at most 6) in the variables H_1, H_2, H_3, where H_i is the hyperplane class on the i-th copy of $\mathbb{P}^2$.

As before we have that

$$\chi(\mathcal{X}, L_1) = \int_{\mathcal{X}} \mathrm{ch}(L)\,\mathrm{Td}(\mathcal{X}). \tag{14}$$

Since $\mathcal{X} \to (\mathbb{P}^2)^3$ is a $\mathbb{Z}_3$ covering we can identify $T\mathcal{X}$ with $T((\mathbb{P}^2)^3)$ viewed as $\mathbb{Z}_3$-equivariant vector bundle. Using the standard formula for the Todd class of projective space we can rewrite equation (14) as

$$\chi(\mathcal{X}, L_1) = \int_{\mathcal{X}} \prod_{i=1}^{3} (1 + mH_i + m^2H_i^2/2)(1 + 3H_i/2 + H_i^2). \tag{15}$$

The only term which contributes to the integral on the right-hand side of (15) is $(H_1H_2H_3)^2$. Now if $P \in \mathbb{P}^3$ is any point then $(H_1H_2H_3)^2$ is represented by the invariant cycle $[P \times P \times P]$. Since $\mathbb{Z}_3$ fixes this cycle we see that $\int_{\mathcal{X}}[P \times P \times P] = 1/3$ and conclude that

$$\chi(X, L_1) = 1/3 \left(\text{coefficient of } (H_1H_2H_3)^2\right). \tag{16}$$

Expanding the product in (15) shows that

(17)

$$\chi(\mathcal{X}, L_1) = 1/3 \left(1 + 9m/2 + 33m^2/4 + 63m^3/8 + 33m^4/8 + 9m^5/8 + m^6/8\right).$$

Since $R(\mathbb{Z}_3) \otimes \mathbb{C} = \mathbb{C}[\xi]/(\xi^3 - 1)$ we may identify Spec $R(\mathbb{Z}_3) \otimes \mathbb{C}$ as the subgroup $\mu_3 \subset \mathbb{C}^*$ and compute the contributions to $\chi(\mathcal{X}, L)$ from the components of L supported at $\omega = e^{2\pi i/3}$ and ω^2.

For both ω and ω^2 the fixed locus of the corresponding element of $\mathbb{Z}_3$ is the diagonal $\Delta_{(\mathbb{P}^2)^3} \stackrel{\Delta}{\hookrightarrow} (\mathbb{P}^2)^3$. The group $\mathbb{Z}_3$ acts trivially on the diagonal so $K_{\mathbb{Z}_3}(\Delta_{(\mathbb{P}^2)^3}) = K_0(\mathbb{P}^2) \otimes R(\mathbb{Z}_3)$. Under this identification, the pullback of the tangent bundle of $(\mathbb{P}^2)^3$ is $T\mathbb{P}^2 \otimes V$ where V is the regular representation of $\mathbb{Z}_3$ corresponding to the action of $\mathbb{Z}_3$ on a 3-dimensional vector space by cyclic permutation. Hence

$$\Delta^*(T(\mathbb{P}^2)^3)) = T\mathbb{P}^2 \otimes \mathbf{1} + T\mathbb{P}^2 \otimes \xi + T\mathbb{P}^2 \otimes \xi^2$$

where ξ is the character of $\mathbb{Z}_3$ with weight $\omega = e^{2\pi i/3}$. The $\mathbb{Z}_3$-fixed component of this $\mathbb{Z}_3$ equivariant bundle is the tangent bundle to fixed locus $\Delta_{(\mathbb{P}^2)^3}$ and its complement is the normal bundle. Thus $T\Delta_{(\mathbb{P}^2)^3} = T\mathbb{P}^2 \otimes \mathbf{1}$ and $N_\Delta = (T\mathbb{P}^2 \otimes \xi) + (T\mathbb{P}^2 \otimes \xi^2)$. Computing the K-theoretic Euler characteristic gives:

$$\begin{aligned} \lambda_{-1}(N_\Delta^*) &= \lambda_{-1}(T^*\mathbb{P}^2 \otimes \xi^2)\lambda_{-1}(T^*\mathbb{P}^2 \otimes \xi) \\ &= (1 - T^*_{\mathbb{P}^2} \otimes \xi^2 + K_{\mathbb{P}^2} \otimes \xi)(1 - T^*_{\mathbb{P}^2} \otimes \xi + K_{\mathbb{P}^2} \otimes \xi^2). \end{aligned}$$

(Here we use the fact that $\xi^* = \xi^{-1} = \xi^2$ in $R(\mathbb{Z}_3)$.) Because the above expression is symmetric in ξ and ξ^2, applying the twisting operator for either ω or ω^2 yields

$$t(\lambda_{-1}(N_\Delta^*)) = (1 - \omega^2 T^*\mathbb{P}^2 \otimes \xi^2 + \omega K_{\mathbb{P}^2} \otimes \xi)(1 - \omega T^*\mathbb{P}^2 \otimes \xi + \omega^2 K_{\mathbb{P}^2}).$$

Expanding the product in K-theory gives:

$$t(\lambda_{-1})(N_\Delta^*) = 1 + K_{\mathbb{P}^2}^2 + (T^*\mathbb{P}^2)^2 - (T^*\mathbb{P}^2 - K_{\mathbb{P}^2} + T^*\mathbb{P}^2 K_{\mathbb{P}^2}) \otimes (\omega\xi + \omega^2\xi). \tag{18}$$

Expression (18) simplifies after taking the Chern character because the Chern classes of any representation are torsion. Precisely,

$$\operatorname{ch}(t(\lambda_{-1}(N_\Delta^*))) = 9 - 27H + 99H^2/2.$$

where H is the hyperplane class on $\Delta_{(\mathbb{P}^2)^3}$. Also note that $\Delta^* L = \mathcal{O}(3m) \otimes \mathbf{1}$ where $\mathbf{1}$ denotes the trivial representation of $\mathbb{Z}_3$. Hence $t(\Delta^* L) = \Delta^* L$ and

$$\begin{aligned} \chi(\mathcal{X}, L_\omega) &= \int_{[\Delta_{(\mathbb{P}^2)^3}/\mathbb{Z}_3]} \operatorname{ch}(\mathcal{O}(3m) \operatorname{ch}(t(\lambda_{-1}(N_\Delta^*)^{-1} \operatorname{Td}(\mathbb{P}^2) \\ &= 1/3 \left(\text{ coefficient of } H^2\right) \\ &= 1/3(1 + 3m/2 + m^2/2) \end{aligned} \tag{19}$$

with the same answer for $\chi(\mathcal{X}, L_{\omega^2})$. Putting the pieces together we see that

$$\chi(\mathcal{X}, L) = 1 + 5m/2 + 37m^2/12 + 21m^3/8 + 11m^4/8 + 3m^5/8 + m^6/24. \tag{20}$$

REMARK 4.12. Note that we have quick consistency check for our computation - namely that $\chi(\mathcal{X}, L)$ is an integer-valued polynomial in m. The values of $\chi(\mathcal{X}, L)$ for $m = 0, 1, 2, 3$ are $1, 11, 76, 340$.

4.3. Hirzebruch Riemann-Roch for arbitrary quotient stacks. We now turn to the general case of quotient stacks $\mathcal{X} = [X/G]$ with X smooth and G an arbitrary linear algebraic group acting properly[4] on X. Again $G_0(\mathcal{X}) \otimes \mathbb{C}$ is a module supported a finite number of closed points of $\operatorname{Spec} R(G) \otimes \mathbb{C}$. For a general group G, $R(G) \otimes \mathbb{C}$ is the coordinate ring of the quotient of G by its conjugation action. As a result, points of $\operatorname{Spec} R(G) \otimes \mathbb{C}$ are in bijective correspondence with conjugacy classes of semi-simple (i.e. diagonalizable) elements in G. An element $\alpha \in G_0(G, X)$ decomposes as $\alpha = \alpha_1 + \alpha_{\Psi_2} + \ldots + \alpha_{\Psi_r}$ where α_{Ψ_r} is the component supported at the maximal ideal corresponding to the semi-simple conjugacy class $\Psi_r \subset G$. Moreover, if a conjugacy class Ψ is in $\operatorname{Supp} G_0(\mathcal{X}) \otimes \mathbb{C}$ then Ψ consists of elements of finite order.

By Corollary 4.4 if $\mathcal{X} = [X/G]$ is complete then $\chi(\mathcal{X}, \alpha_1) = \int_{\mathcal{X}} \operatorname{ch}(\alpha) \operatorname{Td}(\mathcal{X})$. To understand what happens away from the identity we use a non-abelian version of the localization theorem proved in [**EG4**]. Before we state the theorem we need to introduce some notation. If Ψ is a semi-simple conjugacy class let $S_\Psi = \{(g, x) | gx = x, g \in \Psi\}$. The condition the G acts properly on X implies that S_Ψ is empty for all but finitely many Ψ and the elements of these Ψ have finite order. In addition, if S_Ψ is non-empty then the projection $S_\Psi \to X$ is a finite unramified morphism.

REMARK 4.13. Note that the map $S_\Psi \to X$ need not be an embedding. For example if $G = S_3$ acts on $X = \mathbb{A}^3$ by permuting coordinates and Ψ is the conjugacy class of two-cycles, then S_Ψ is the disjoint union of the three planes $x = y$, $y = z$, $x = z$.

If we fix an element $h \in \Psi$ then the map $G \times X^h \to S_\Psi$, $(g, x) \mapsto (ghg^{-1}, gx)$ identifies S_Ψ as the quotient $G \times_Z X^h$ where $Z = \mathcal{Z}_G(h)$ is the centralizer of the semi-simple element $h \in G$. In particular $G_0(G, S_\Psi)$ can be identified with $G_0(Z, X^h)$. The element h is central in Z and if $\beta \in G_0(G, S_\Psi)$ we denote by β_{c_Ψ} the component of β supported at $h \in \operatorname{Spec} Z$ under the identification described above. It is relatively straightforward [**EG4**, Lemma 4.6] to show that β_{c_Ψ} is in fact independent of the choice of representative element $h \in \psi$, and thus we obtain a distinguished "central" summand $G_0(G, S_\Psi)_{c_\Psi}$ in $G_0(G, S_\Psi)$.

THEOREM 4.14 (Non-abelian localization theorem). [**EG4**, Theorem 5.1] *The pullback map $f_\Psi^* \colon G_0(G, X) \to G_0(G, S_\Psi)$ induces an isomorphism between the localization of $G_0(G, X)$ at the maximal ideal $m_\Psi \in \operatorname{Spec} R(G) \otimes \mathbb{C}$ corresponding to the conjugacy class Ψ and the summand $G_0(G, S_\Psi)_{c_\Psi}$ in $G_0(G, S_\Psi)$. Moreover,*

[4]Because we work in characteristic 0, the hypothesis that G acts properly implies that the stabilizers are linearly reductive since they are finite. In addition every linear algebraic group over $\mathbb{C}$ has a Levy decomposition $G = LU$ with L reductive and U unipotent and normal. If G acts properly then U necessarily acts freely because a complex unipotent group has no non-trivial finite subgroups. Thus, if we want, we can quotient by the free action of U and reduce to the case that G is reductive.

the Euler class of the normal bundle, $\lambda_{-1}(N_{f_\Psi^*})$ *is invertible in* $G_0(G,S_\Psi)_{c_\Psi}$ *and if* $\alpha \in G_0(G,X)_{m_\Psi}$ *then*

$$\alpha = f_{\Psi *}\left(\frac{f^*\alpha_{c_\Psi}}{\lambda_{-1}(N_f^*)}\right). \tag{21}$$

The theorem can be restated in way that is sometimes more useful for calculations. Fix an element $h \in \Psi$ and again let $Z = \mathcal{Z}_G(h)$ be the centralizer of h in G. Let $\iota^! : G_0(G,X) \to G_0(Z,X^h)$ be the composition of the restriction of groups map $G_0(G,X) \to G_0(Z,X)$ with the pullback $G_0(Z,X) \xrightarrow{i_h^*} G_0(Z,X^h)$. Let β_h denote the component of $\beta \in G_0(Z,X^h)$ in the summand $G_0(Z,X^h)_{m_h}$. Let $\mathfrak{g}$ (resp. $\mathfrak{z}$) be the adjoint representation of G (resp. Z). The restriction of the adjoint representation to the subgroup Z makes $\mathfrak{g}$ a Z-module, so $\mathfrak{g}/\mathfrak{z}$ is a Z-module. Since $S_\Psi = G \times_Z X^h$, under the identification $G_0(G,S_\Psi) = G_0(Z,X^h)$ the class of the conormal bundle of the map f_Ψ is identified with $N_{i_h}^* - \mathfrak{g}/\mathfrak{z}^*$. Thus we can restate the non-abelian localization theorem as follows:

Corollary 4.15. *Let* $\iota_!$ *be the composite of* $f_{\Psi *}$ *with the isomorphism*

$$G_0(Z,X^h) \to G_0(G,S_\Psi).$$

Then for $\alpha \in G_0(G,X)_{m_\Psi}$

$$\alpha = \iota_!\left(\frac{(\iota^!\alpha)_h \cdot \lambda_{-1}(\mathfrak{g}/\mathfrak{z}^*)}{\lambda_{-1}(N_{i_h}^*)}\right). \tag{22}$$

The element $h \in Z(h)$ is central, and as in the abelian case we obtain a twisting map $t_h \colon G_0(Z,X^h) \to G_0(Z,X^h)$ which maps the summand $G_0(Z,X^h)_h$ to the summand $G_0(Z,X^h)_1$ and also preserves invariants.

We can then obtain the Riemann-Roch theorem in the general case. Let $1_G = \Psi_1,\ldots,\Psi_n$ be conjugacy classes corresponding to the support of $G_0(G,X)$ as an $R(G)$ module. Choose a representative element $h_r \in \Psi_r$ for each r. Let Z_r be the centralizer of h in G and let $\mathfrak{z}_r$ be its Lie algebra.

Theorem 4.16. *Let* $\mathcal{X} = [X/G]$ *be a smooth, complete Deligne-Mumford quotient stack. Then for any vector bundle* V *on* $\mathcal{X}$

$$\chi(\mathcal{X},V) = \sum_{r=1}^n \int_{[X^{h_r}/Z_r]} \mathrm{ch}\left(t_{h_r}\Big(\frac{[i_r^*V]\cdot\lambda_{-1}(\mathfrak{g}^*/\mathfrak{z}_r^*)}{\lambda_{-1}(N_{i_r}^*)}\Big)\right)\mathrm{Td}([X^{h_r}/Z_r]) \tag{23}$$

where $i_r \colon X^{h_r} \to X$ *is the inclusion map.*

4.3.1. *A computation using Theorem* 4.16*: The quotient stack* $[(\mathbb{P}^2)^3/S_3]$. We now generalize the calculation of Section 4.2.6 to the quotient $\mathcal{Y} = [Y/S_3]$ where the symmetric group S_3 acts on $Y = (\mathbb{P}^2)^3$ by permutation. Again we will compute $\chi(\mathcal{Y},L)$ where $L = \mathcal{O}(m)\boxtimes\mathcal{O}(m)\boxtimes\mathcal{O}(m)$ viewed as an S_3-equivariant line bundle on $(\mathbb{P}^2)^3$. As was the case for the $\mathbb{Z}_3$ action the S_3-equivariant rational Chow group is generated by symmetric polynomials in H_1,H_2,H_3 where H_i is the hyperplane class on the i-th copy of $\mathbb{P}^2$. The calculation of $\chi(\mathcal{Y},L_1)$ is identical to the one we did for the stack $\mathcal{X} = [(\mathbb{P}^2)/\mathbb{Z}_3]$ except that the cycle $[P\times P\times P]$ has stabilizer S_3 which has order 6. Thus,

$$\chi(\mathcal{X},L_1) = 1/6\left(1 + 9m/2 + 33m^2/4 + 63m^3/8 + 33m^4/8 + 9m^5)/8 + m^6/8\right). \tag{24}$$

Now $\operatorname{Spec} R(S_3) \otimes \mathbb{C}$ consists of 3 points, corresponding to the conjugacy classes of $\{1\}$, $\Psi_2 = \{(12),(13),(23)\}$ and $\Psi_3 = \{(123),(132)\}$. We denote the components of L at the maximal ideal corresponding to Ψ_2 and Ψ_3 by L_2 and L_3 respectively, so that $L = L_1 + L_2 + L_3$.

The computation of $\chi(\mathcal{Y}, L_3)$ is identical to the computation of $\chi(\mathcal{X}, L_\omega)$ in Section 4.2.6. If we choose the representative element $\omega = (123)$ in Ψ_3 then $\mathcal{Z}_{S_3}(\omega) = \langle\omega\rangle = \mathbb{Z}_3$. Again $Y^\omega = \Delta_{(\mathbb{P}^2)^3}$ and the tangent bundle to $(\mathbb{P}^2)^3$ restricts to the $\mathbb{Z}_3$-equivariant bundle $T\mathbb{P}^2 \otimes V$ where V is the regular representation. Hence (see (19))

$$\chi(\mathcal{Y}, L_3) = 1/3(1 + 3m/2 + m^2/2) \tag{25}$$

To compute $\chi(\mathcal{Y}, L_2)$ choose the representative element $\tau = (12)$ in the conjugacy class $\Psi = (12)$. Then $\mathcal{Z}_{S_3}(\tau) = \langle\tau\rangle = \mathbb{Z}_2$ and the fixed locus of τ is $Y^\tau = \Delta_{(\mathbb{P}^2)^2} \times \mathbb{P}^2 \overset{\Delta_{12}}{\hookrightarrow} (\mathbb{P}^2)^3$ where $\Delta_{(\mathbb{P}^2)^2} \subset (\mathbb{P}^2)^2$ is the diagonal. The action of $\mathbb{Z}_2$ is trivial and the tangent bundle to $(\mathbb{P}^2)^3$ restricts to $(T_{\mathbb{P}^2} \otimes V) \boxtimes T\mathbb{P}^2$ where V is now the regular representation of $\mathbb{Z}_2$ so $N_{\Delta_{12}} = (T\mathbb{P}^2 \otimes \xi) \boxtimes T\mathbb{P}^2$ where ξ is the non-trivial character of $\mathbb{Z}_2$. Since ξ is self-dual as a character of $\mathbb{Z}_2$ we see that

$$\lambda_{-1}(N^*_{\Delta_{12}}) = (1 - (T^*\mathbb{P}^2 \otimes \xi) + K_{\mathbb{P}^2}) \tag{26}$$

Applying the twisting operator yields

$$t(\lambda_{-1}(N^*_{\Delta_{12}}) = (1 + T^*\mathbb{P}^2 \otimes \xi + K_{\mathbb{P}^2}) \tag{27}$$

Taking the Chern character we have

$$\operatorname{ch}(t(\lambda_{-1}(N^*_{\Delta_{12}}))) = 4 - 6H + 6H^2$$

where H is the hyperplane class on the diagonal $\mathbb{P}^2$. The restriction of L to Y^τ is the line bundle $(\mathcal{O}(2m) \otimes \mathbf{1}) \boxtimes \mathcal{O}(m)$. Thus,

$$\begin{aligned}
\chi(\mathcal{X}, L_2) &= \int_{[X^\tau/\mathbb{Z}_2]} \operatorname{ch}(\mathcal{O}(2m) \boxtimes \mathcal{O}(m) \operatorname{ch}(t(\lambda_{-1}(N^*_\Delta)^{-1} \operatorname{Td}(Y^\tau)) \\
&= 1/2 \left(\text{ coefficient of } H^2 H_3^2 \right) \\
&= 1/2(1 + 3m + 13m^2/4 + 3m^3/2 + m^4/4)
\end{aligned}$$

Adding the Euler characteristics of L_1, L_2, L_3 gives

$$\chi(\mathcal{Y}, L) = 1 + 11m/4 + 19m^2/6 + 33m^3/16 + 13m^4/16 + 3m^5/16 + m^6/48$$

which is again an integer-valued polynomial in m.

4.3.2. *Statement of the theorem in terms of the inertia stack.* The computation of $\chi(\mathcal{X}, \alpha)$ does not depend on the choice of the representatives of elements in the conjugacy classes and Theorem 4.16 can be restated in terms of the S_Ψ and correspondingly in terms of the inertia stack.

Definition 4.17. Let $IX = \{(g,x) | gx = x\} \subset G \times X$ be the inertia scheme. The projection $IX \to X$ makes IX into a group scheme over X. If the stack $[X/G]$ is separated then IX is finite over X.

The group G acts on IX by $g(h,x) = (ghg^{-1}, gx)$ and the projection $IX \to X$ is G-equivariant with respect to this action. The quotient stack $I\mathcal{X} := [IX/G]$ is called the *inertia stack* of the stack $\mathcal{X} = [X/G]$ and there is an induced morphism of stacks $I\mathcal{X} \to \mathcal{X}$. Since G acts properly on X then the map $I\mathcal{X} \to \mathcal{X}$ is finite and unramified.

Since G acts with finite stabilizers a necessary condition for (g, x) to be in IX is for g to be of finite order.

PROPOSITION 4.18. *If Ψ is a conjugacy class of finite order then S_Ψ is closed and open in IX and consequently there is a finite G-equivariant decomposition $IX = \coprod_\Psi S_\Psi$.*

Since IX has a G-equivariant decomposition into a finite disjoint sum of the S_Ψ we can define a twisting automorphism $t \colon G_0(G, IX) \otimes \mathbb{C} \to G_0(G, IX) \otimes \mathbb{C}$ and thus a corresponding twisting action on $G_0(I\mathcal{X})$. If V is a G-equivariant vector bundle on IX then its fiber at a point (h, x) is $\mathcal{Z}_G(h)$-module $V_{h,x}$ and $t(V)$ is the class in $G_0(G, IX) \otimes \mathbb{C}$ whose "fiber" at the point (h, x) is the virtual $\mathcal{Z}_G(h)$-module $\oplus_{\xi \in X(H)} \xi(h)(V_{h,x})_\xi$ where H is the cyclic group generated by h.

The Hirzebruch-Riemann-Roch theorem can then be stated very concisely as:

THEOREM 4.19. *Let $\mathcal{X} = [X/G]$ be a smooth, complete Deligne-Mumford quotient stack and let $f \colon I\mathcal{X} \to \mathcal{X}$ be the inertia map. If V is a vector bundle on $\mathcal{X}$ then*

$$\chi(\mathcal{X}, V) = \int_{I\mathcal{X}} \operatorname{ch}\left(t\left(\frac{f^*V}{\lambda_{-1}(N_f^*)}\right)\right) \operatorname{Td}(I\mathcal{X})$$

5. Grothendieck Riemann-Roch for proper morphisms of Deligne-Mumford quotient stacks

In the final section we state the Grothendieck-Riemann-Roch theorem for arbitrary proper morphisms of quotient Deligne-Mumford stacks.

5.1. Grothendieck-Riemann-Roch for proper morphisms to schemes and algebraic spaces. The techniques used to prove the Hirzebruch-Riemann-Roch for proper Deligne-Mumford stacks actually yield a Grothendieck-Riemann-Roch result for arbitrary separated Deligne-Mumford stacks relative to map $\mathcal{X} \to M$ where M is the moduli space of the quotient stack $\mathcal{X} = [X/G]$.

THEOREM 5.1. [**EG4**, Theorem 6.8] *Let $\mathcal{X} = [X/G]$ be a smooth quotient stack with coarse moduli space $p \colon \mathcal{X} \to M$. Then the following diagram commutes:*

$$\begin{array}{ccc} G_0(\mathcal{X}) & \stackrel{I\tau_{\mathcal{X}}}{\to} & \operatorname{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\ p_* \downarrow & & p_* \downarrow \\ G_0(M) & \stackrel{\tau_M}{\to} & \operatorname{Ch}^*(M) \otimes \mathbb{C} \end{array} .$$

Here $I\tau_{\mathcal{X}}$ is the isomorphism that sends the class in $G_0(\mathcal{X})$ of a vector bundle V to $\operatorname{ch}\left(t(\frac{f^*V}{\lambda_{-1}(N_f^*)})\right) \operatorname{Td}(I\mathcal{X})$ and τ_M is the Fulton-MacPherson Riemann-Roch isomorphism.

REMARK 5.2. If $\mathcal{X}$ is satisfies the resolution property then every coherent sheaf on $\mathcal{X}$ can be expressed as a linear combination of classes of vector bundles.

Using the universal property of the coarse moduli space and the covariance of the Riemann-Roch map for schemes and algebraic spaces we obtain the following Corollary.

COROLLARY 5.3. *Let $\mathcal{X} = [X/G]$ be a smooth quotient stack and let $\mathcal{X} \to Z$ be a proper morphism to a scheme or algebraic space. Then the following diagram commutes:*

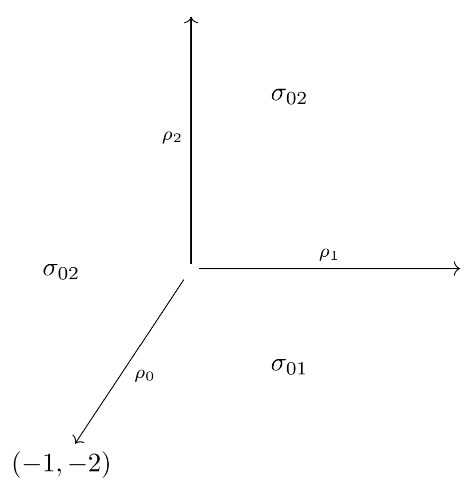
$$\begin{array}{ccc} G_0(\mathcal{X}) & \stackrel{I\tau_{\mathcal{X}}}{\to} & \mathrm{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\ p_* \downarrow & & p_* \downarrow \\ G_0(Z) & \stackrel{\tau_Z}{\to} & \mathrm{Ch}^*(Z) \otimes \mathbb{C}. \end{array}$$

5.1.1. *Example: The Todd class of a weighted projective space.* If X is an arbitrary scheme we define the *Todd class*, $\mathrm{td}(X)$, of X to be $\tau_X(\mathcal{O}_X)$ where τ_X is the Riemann-Roch map of Theorem 2.4. If X is smooth, then $\mathrm{td}(X) = \mathrm{Td}(TX)$, and for arbitrary complete schemes $\chi(X, V) = \int_X \mathrm{ch}(V)\,\mathrm{td}(X)$ for any vector bundle V on X.

In this section we explain how to use Theorem 5.1 to give a formula for the Todd class of the singular weighted projective space $\mathbf{P}(1,1,2)$. The method can be extended to any simplicial toric variety, complete or not, [**EG3**]. (See also [**BV**] for a computation of the equivariant Todd class of complete toric varieties using other methods.)

The singular variety $\mathbf{P}(1,1,2)$ is the quotient of $X = \mathbb{A}^3 \smallsetminus \{0\}$ where $\mathbb{C}^*$ acts with weights $(1,1,2)$. This variety is the coarse moduli space of the corresponding smooth stack $\mathbb{P}(1,1,2)$. A calculation similar to that of Section 4.2.2 shows that $K_0(\mathbb{P}(1,1,2)) = \mathbb{Z}[\xi]/(\xi-1)^2(\xi^2-1)$ and $\mathrm{Ch}^*(\mathbb{P}(1,1,2)) = \mathbb{Z}[t]/2t^3$ where $t = c_1(\xi)$.

The stack $\mathbb{P}(1,1,2)$ is a toric Deligne-Mumford stack (in the sense of [**BCS**]) and the weighted projective space $\mathbf{P}(1,1,2)$ is the toric variety $X(\Sigma)$ where Σ is the complete 2-dimensional fan with rays by $\rho_0 = (-1,-2), \rho_1 = (1,0), \rho_2 = (0,1)$. This toric variety has an isolated singular point P_0 corresponding to the cone σ_{01} spanned by ρ_0 and ρ_1.



Each ray determines a Weil divisor D_{ρ_i} which is the image of the fundamental class of the hyperplane $x_i = 0$. With the given action, $[x_0 = 0] = [x_1 = 0] = t$ and $[x_2 = 0] = 2t$. Since the action of $\mathbb{C}^*$ on $\mathbb{A}^3$ is free on the complement of a set of codimension 2, the pushforward defines an isomorphism of integral Chow groups $\mathrm{Ch}^1(\mathbb{P}(1,1,2)) = \mathrm{Ch}^1(\mathbf{P}(1,1,2))$. Thus, $\mathrm{Ch}^1(X(\Sigma) = \mathbb{Z}$ and $D_{\rho_0} \equiv D_{\rho_1}$ while $D_{\rho_2} \equiv 2D_{\rho_0}$. Also, $\mathrm{Ch}^2(X(\Sigma)) = \mathbb{Z}$ is generated by the class of the singular point P_0 and $[P_0] = 2[P]$ for any non-singular point P.

The tangent bundle to $\mathbb{P}(1,1,2)$ fits into the Euler sequence

$$0 \to \mathbf{1} \to 2\xi + \xi^2 \to T\mathbb{P}(1,1,2) \to 0$$

so $c_1(T\mathbb{P}(1,1,2)) = 4t$ and $c_2(T\mathbb{P}(1,1,2)) = 15t^2$. Thus

$$\mathrm{Td}(\mathbb{P}(1,1,2)) = 1 + 2t + 21/12t^2$$

and pushing forward to $\mathbf{P}(1,1,2)$ gives a contribution of $1 + 2D_{\rho_0} + 21/24P_0$ to $\mathrm{td}(\mathbf{P}(1,1,2))$.

Now we must also consider the contribution coming from the fixed locus of (-1) acting on $\mathbb{A}^3 \setminus \{0\}$. The fixed locus is the line $x_0 = x_1 = 0$ and the normal bundle has K-theory class 2ξ. After twisting by -1 we obtain a contribution of

$$p_*\left[\mathrm{ch}\left(\frac{1}{(1+\xi^{-1})^2}\right)\mathrm{Td}([X^{-1}/\mathbb{C}^*])\right] \tag{28}$$

Since $[X^{-1}/\mathbb{C}^*]$ is 0-dimensional and has a generic stabilizer of order 2 we obtain an additional contribution of $1/2\,\mathrm{rk}(1/(1+\xi^{-1})^2)[P_0] = (1/2 \times 1/4)[P_0] = 1/8[P_0]$ to $\mathrm{td}(\mathbf{P}(1,1,2))$. Combining the two contributions we conclude that:

$$\mathrm{td}(\mathbf{P}(1,1,2)) = 1 + 2D_{\rho_0} + [P_0]$$

in $\mathrm{Ch}^*(\mathbf{P}(1,1,2))$.

5.2. Grothendieck-Riemann-Roch theorem for Deligne-Mumford quotient stacks. Suppose that $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ are smooth Deligne-Mumford quotient stacks and $f\colon \mathcal{X} \to \mathcal{Y}$ is a proper, but not-necessarily representable morphism. The most general Grothendieck-Riemann-Roch result we can write down is the following:

THEOREM 5.4. [**EK**] *The following diagram of Grothendieck groups and Chow groups commutes:*

$$\begin{array}{ccc} G_0(\mathcal{X}) & \stackrel{I\tau_{\mathcal{X}}}{\to} & \mathrm{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\ f_* \downarrow & & f_* \downarrow \\ G_0(\mathcal{Y}) & \stackrel{I\tau_{\mathcal{Y}}}{\to} & \mathrm{Ch}^*(I\mathcal{Y}) \otimes \mathbb{C} \end{array}$$

REMARK 5.5. A proof of this result using the localization methods of [**EG3, EG4**] will appear in [**EK**]. A version of this Theorem (which also holds in some prime characteristics) was proved by Bertrand Toen in [**Toe**]. However, in that paper the target of the Riemann-Roch map is not the Chow groups but rather a "cohomology with coefficients in representations." Toen does not explicitly work with quotient stacks, but his hypothesis that the stack has the resolution property for coherent sheaves implies that the stack is a quotient stack.

In [**EK**] we will also give a version of Grothendieck-Riemann-Roch for proper morphisms of arbitrary quotient stacks.

6. Appendix on K-theory and Chow groups

In this section we recall some basic facts about K-theory and Chow groups both in the non-equivariant and equivariant settings. For more detailed references see [**Ful, FL, Tho1, EG1**].

6.1. K-theory of schemes and algebraic spaces.

DEFINITION 6.1. Let X be an algebraic scheme. We denote by $G_0(X)$ the Grothendieck group of coherent sheaves on X and $K_0(X)$ the Grothendieck group of locally free sheaves; i.e vector bundles.

There is a natural map $K_0(X) \to G_0(X)$ which is an isomorphism when X is a smooth scheme. The reason is that if X is smooth every coherent sheaf has a finite resolution by locally free sheaves. For a proof see [**Ful**, Appendix B8.3].

DEFINITION 6.2. If $X \to Y$ is a proper morphism then there is a pushforward map $f_*\colon G_0(X) \to G_0(Y)$ defined by $f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}]$. When $Y = \mathrm{pt}$, then $G_0(Y) = \mathbb{Z}$ and $f_*(\mathcal{F}) = \chi(X, \mathcal{F})$.

The Grothendieck group $K_0(X)$ is a ring under tensor product and the map $K_0(X) \otimes G_0(X) \to G_0(X)$, $([V], \mathcal{F}) \mapsto \mathcal{F} \otimes V$ makes $G_0(X)$ into a $K_0(X)$-module. If $f\colon X \to Y$ is an arbitrary morphism of schemes then pullback of vector bundles defines a ring homomorphism $f^*\colon K_0(Y) \to K_0(X)$.

When $f\colon X \to Y$ is proper, the pullback for vector bundles and the pushforward for coherent sheaves are related by the projection formula. Precisely, if $\alpha \in K_0(Y)$ and $\beta \in G_0(X)$ then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta$$

in $G_0(Y)$.

There is large class of morphisms $X \xrightarrow{f} Y$, for which there are pullbacks $f^*\colon G_0(Y) \to G_0(X)$ and pushforwards $f_*\colon K_0(X) \to K_0(Y)$. For example, if f is flat, the assignment $[\mathcal{F}] \mapsto [f^*\mathcal{F}]$ defines a pullback $f^*\colon G_0(Y) \to G_0(X)$.

Suppose that every coherent sheaf on Y is the quotient of a locally free sheaf (for example if Y embeds into a smooth scheme). If $f\colon X \to Y$ is a regular embedding then the direct image f_*V of a locally free sheaf has a finite resolution $W_\cdot$ by locally free sheaves. Thus we may define a pushforward $f_*\colon K_0(X) \to K_0(Y)$ by $f_*[V] = \sum_i (-1)^i [W_i]$ in this case. Also, if X and Y are smooth then there is a pushforward $f_*\colon K_0(X) \to K_0(Y)$. When X and Y admit ample line bundles then there are pushforwards $f_*\colon K_0(X) \to K_0(Y)$ for any proper morphism of finite Tor-dimension.

DEFINITION 6.3. The Grothendieck ring $K_0(X)$ has an additional structure as a λ-ring. If V is a vector bundle of rank r set $\lambda^k[V] = [\Lambda^k V]$. If t is parameter define $\lambda_t(V) = \sum_{k=0}^r \lambda^k[V] t^k \in K_0(X)[t]$ where t is a parameter. The class $\lambda_{-1}(V^*) = 1 - [V^*] + [\Lambda^2 V^*] + \ldots + (-1)^r [\Lambda^r V^*]$ is called the *K-theoretic Euler class* of V.

Although, $K_0(X)$ is simpler to define and is functorial for arbitrary morphisms, it is actually much easier to prove results about the Grothendieck group $G_0(X)$. The reason is that G-functor behaves well with respect to localization. If $U \subset X$ is open with complement Z then there is an exact sequence

$$G_0(Z) \to G_0(X) \to G_0(U) \to 0.$$

The definitions of $G_0(X)$ and $K_0(X)$ also extend to algebraic spaces as does the basic functoriality of these groups. However, even if X is a smooth algebraic space there is no result guaranteeing that X satisfies the *resolution property* meaning that every coherent sheaf is the quotient of a locally free sheaf. Thus it is not possible to prove that the natural map $K_0(X) \to G_0(X)$ is actually an isomorphism. (Note however, that there no known examples of smooth separated algebraic spaces where the resolution property provably fails, c.f. [**Tot**].) In this case one can either replace $K_0(X)$ with the Grothendieck group of perfect complexes or work exclusively with $G_0(X)$.

6.2. Chow groups of schemes and algebraic spaces.

Definition 6.4. If X is a scheme (which for simplicity we assume to be equidimensional) we denote by $\mathrm{Ch}^i(X)$ the Chow group of codimension i-dimensional cycles modulo rational equivalence as in [**Ful**] and we set $\mathrm{Ch}^*(X) = \oplus_{i=0}^{\dim X} \mathrm{Ch}^i(X)$.

As was the case for the Grothendieck group $G_0(X)$, if $f\colon X \to Y$ is proper then there is a pushforward $f_*\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(Y)$. The map is defined as follows:

Definition 6.5. If $Z \subset X$ is a closed subvariety let $W = f(Z)$ with its reduced scheme structure

$$f_*[Z] = \left\{ \begin{array}{cc} [K(Z) : K(W)][W] & \text{if } \dim W = \dim Z \\ 0 & \text{otherwise} \end{array} \right\}$$

where $K(Z)$ (resp. $K(W)$) is the function field of Z (resp. W).

If X is complete then we denote the pushforward map $\mathrm{Ch}^* X \to \mathrm{Ch}^*(\mathrm{pt}) = \mathbb{Z}$ by $\int_X$.

Because we index our Chow groups by codimension, the map f_* shifts degrees. If $f\colon X \to Y$ has (pure) relative dimension d then $f_*(\mathrm{Ch}^k(X)) \subset \mathrm{Ch}^{k+d}(Y)$.

There is again a large class of morphisms $X \stackrel{f}{\to} Y$ for which there are pullbacks $f^*\colon \mathrm{Ch}^*(Y) \to \mathrm{Ch}^*(X)$. Some of the most important examples are flat morphisms where the pullback is defined by $f^*[Z] = [f^{-1}(Z)]$, regular embeddings and, more generally, local complete intersection morphisms.

We again have a localization exact sequence which can be used for computation. If $U \subset X$ is open with complement Z then there is a short exact sequence

$$\mathrm{Ch}^*(Z) \to \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(U) \to 0$$

Definition 6.6. If X is smooth (and separated) then the diagonal $\Delta\colon X \to X \times X$ is a regular embedding. Pullback along the diagonal allows us to define an intersection product on $\mathrm{Ch}^*(X)$ making it into a graded ring, called the Chow ring. If $[Z] \subset \mathrm{Ch}^k(X)$ and $[W] \subset \mathrm{Ch}^l(X)$ then we define $[Z] \cdot [W] = \Delta^*([Z \times W]) \in \mathrm{Ch}^{k+l}(X)$.

Any morphism of smooth varieties is a local complete intersection morphism, so if $f\colon X \to Y$ is a morphism of smooth varieties then we have a pullback $f^*\colon \mathrm{Ch}^* Y \to \mathrm{Ch}^* X$ which is a homomorphism of Chow rings.

The theory of Chow groups carries through completely to algebraic spaces [**EG1**, Section 6.1].

6.3. Chern classes and operations. Associated to any vector bundle V on a scheme X are Chern classes $c_i(V)$, $0 \leq i \leq \mathrm{rk}\, V$. Chern classes are defined as operations on Chow groups. Specifically $c_i(V)$ defines a homomorphism $\mathrm{Ch}^k X \to \mathrm{Ch}^{k+i} X$, $\alpha \mapsto c_i(V)\alpha$, with c_0 taken to be the identity map and denoted by 1. Chern classes are compatible with pullback in the following sense: If $f\colon X \to Y$ is a morphism for which there is a pullback of Chow groups then $c_i(f^*V)f^*\alpha = f^*(c_i(V)\alpha)$.

Chern classes of a vector bundle V may be viewed as elements of the *operational Chow ring* $A^*X = \oplus_{i=0} A^iX$ defined in [**Ful**, Definition 17.3]. An element of $c \in A^iX$ is a collection of homomorphisms $c\colon \mathrm{Ch}^*(X') \to \mathrm{Ch}^{*+k}(X')$ defined for any morphism of schemes $X' \to X$. These homomorphisms should be compatible with pullbacks of Chow groups and should also satisfy the projection formula $f_*(c\alpha) = cf_*\alpha$ for any proper morphism of X-schemes $f\colon X'' \to X'$ and class $\alpha \in \mathrm{Ch}^*(X'')$. Composition of morphisms makes A^*X into a graded ring and it can be shown that $A^kX = 0$ for $k > \dim X$.

If X is smooth, then the map $A^*X \to \mathrm{Ch}^*X$, $c \mapsto c([X])$ is an isomorphism of rings where the product on Ch^*X is the intersection product. In particular, if X is smooth then the Chern class $c_i(V)$ is completely determined by $c_i(V)[X] \in \mathrm{Ch}^i(X)$ so in this way we may view $c_i(V)$ as an element of $\mathrm{Ch}^i(X)$.

The total Chern class $c(V)$ of a vector bundle is the sum $\sum_{i=0}^{\mathrm{rk}\, V} c_i(V)$. Since $c_0 = 1$ and $c_i(V)$ is nilpotent for $i > 0$ the total Chern class $c(V)$ is invertible in A^*X. Also, if $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of vector bundles then $c(V) = c(V')c(V'')$, so the assignment $[V] \mapsto c(V)$ defines a homomorphism from the Grothendieck group $K_0(X)$ to the multiplicative group of units in A^*X.

6.3.1. *Splitting, Chern characters and Todd classes.* If V is a vector bundle on a scheme X, then the *splitting construction* ensures that there is a scheme X' and a smooth, proper morphism $f\colon X' \to X$ such that $f^*\colon \mathrm{Ch}^*X \to \mathrm{Ch}^*X'$ is injective and f^*V has a filtration $0 = E_0 \subset E_1 \subset \ldots E_r = f^*V$ such that the quotients $L_i = E_i/E_{i-i}$ are line bundles. Thus $c(f^*V)$ factors as $\prod_{i=1}^r (1 + c_1(L_i))$. The classes $\alpha_i = c_1(L_i)$ are Chern roots of V and any symmetric expression in the α_i is the pullback from Ch^*X of a unique expression in the Chern classes of V.

Definition 6.7. If V is a vector bundle on X with Chern roots $\alpha_1, \ldots \alpha_r \in A^*X'$ for some $X' \to X$ then the *Chern character* of V is the unique class $\mathrm{ch}(V) \in A^*X \otimes \mathbb{Q}$ which pulls back to $\sum_{i=0}^r \exp(\alpha_i)$ in $A^*(X') \otimes \mathbb{Q}$. (Here exp is the exponential series.)

Likewise the *Todd class* of V is the unique class $\mathrm{Td}(V) \in A^*X \otimes \mathbb{Q}$ which pulls back to $\prod_{i=0}^r \frac{\alpha_i}{1-\exp(-\alpha_i)}$ in $A^*(X') \otimes \mathbb{Q}$.

The Chern character can be expressed in terms of the Chern classes of V as

$$\mathrm{ch}(V) = \mathrm{rk}\, V + c_1 + (c_1^2 - c_2)/2 + \ldots \tag{29}$$

and the Todd class as

$$\mathrm{Td}(V) = 1 + c_1/2 + (c_1^2 + c_2)/12 + \ldots \tag{30}$$

Because $A^k(X) = 0$ for $k > \dim X$ the series for $\mathrm{ch}(V)$ and $\mathrm{Td}(X)$ terminate for any given scheme X and vector bundle V.

If V and W are vector bundles on X then $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$ and $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)$ so the Chern character defines a homomorphism of rings $\mathrm{ch}\colon K_0(X) \to A^*X \otimes \mathbb{Q}$. We also have that $\mathrm{Td}(V \oplus W) = \mathrm{Td}(V)\,\mathrm{Td}(W)$ so we obtain a homomorphism $\mathrm{Td}\colon K_0(X) \to (A^*X \otimes \mathbb{Q})^\star$ from the additive Grothendieck group to the multiplicative group of units in $A^*X \otimes \mathbb{Q}$.

When X is smooth we interpret the target of the Chern character and Todd class to be Ch^*X.

6.4. Equivariant K-theory and equivariant Chow groups. We now turn to the equivariant analogues of Grothendieck and Chow groups.

6.4.1. *Equivariant K-theory.* Most of the material on equivariant K-theory can be found in [**Tho1**] while the material on equivariant Chow groups is in [**EG1**].

Definition 6.8. Let X be a scheme (or algebraic space) with the action of an algebraic group G. In this case we define $K_0(G, X)$ to be the Grothendieck group of G-equivariant vector bundles and $G_0(G, X)$ to be the Grothendieck group of G-linearized coherent sheaves.

As in the non-equivariant case there is pushforward of Grothendieck groups $G_0(G, X) \to G_0(G, Y)$ for any proper G-equivariant morphism. Similarly, there is a pullback $K_0(G, Y) \to K_0(G, X)$ for any G-equivariant morphism $X \to Y$. There are also pullbacks in G-theory for equivariant regular embeddings and equivariant lci morphisms. There is also a localization exact sequence associated to a G-invariant open set U with complement Z.

The Grothendieck group $K_0(G, X)$ is a ring under tensor product and $G_0(G, X)$ is a module for this ring. The equivariant Grothendieck ring $K_0(G, \mathrm{pt})$ is the representation ring $R(G)$ of G. Since every scheme maps to a point, $R(G)$ acts on both $G_0(G, X)$ and $K_0(G, X)$ for any G-scheme X. The $R(G)$-module structure on $G_0(G, X)$ plays a crucial role in the Riemann-Roch theorem for Deligne-Mumford stacks.

If V is a G-equivariant vector bundle then $\Lambda^k V$ has a natural G-equivariant structure. This means that wedge product defines a λ-ring structure on $K_0(G, X)$. In particular we define the equivariant Euler class of a rank r bundle V by the formula

$$\lambda_{-1}(V^*) = 1 - [V^*] + [\Lambda^2 V^*] - \ldots + (-1)^r [\Lambda^r V^*].$$

Results of Thomason [**Tho2**, Lemmas 2.6, 2.10, 2.14] imply that if X is normal and quasi-projective or regular and separated over the ground field (both of which implies that X has the resolution property) and G acts on X then X has the G-equivariant resolution property. It follows that if X is a smooth G-variety then every G-linearized coherent sheaf has a finite resolution by G-equivariant vector bundles. Hence $K_0(G, X)$ and $G_0(G, X)$ may be identified if X is a smooth scheme.

The Grothendieck groups $G_0(G, X)$ and $K_0(G, X)$ are naturally identified with the corresponding Grothendieck groups of the categories of locally free and coherent sheaves on the quotient stack $\mathcal{X} = [X/G]$.

Remark 6.9 (Warning). If X is complete then there are pushforward maps $K_0(G, X) \to K_0(G, \mathrm{pt}) = R(G)$ and $G_0(G, X) \to K_0(G, \mathrm{pt}) = R(G)$ that associate to a vector bundle V (resp. coherent sheaf $\mathcal{F}$) the virtual representation $\sum (-1)^i H^i(X, V)$ (resp. $\sum (-1)^i H^i(X, \mathcal{F})$.). Although V may be viewed as a vector bundle on the quotient stack $\mathcal{X} = [X/G]$ the virtual representation $\sum (-1)^i H^i(X, V)$ *is not* the Euler characteristic of V as a vector bundle on $\mathcal{X}$.

6.4.2. *Equivariant Chow groups.* The definition of equivariant Chow groups requires more care and is modeled on the Borel construction in equivariant cohomology. If G acts on X then the i-th equivariant Chow group is defined as $\mathrm{Ch}^i(X_G)$ where X_G is any quotient of the form $(X \times U)/G$ where U is an open set in a representation $\mathbf{V}$ of G such that G acts freely on U and $\mathbf{V} \smallsetminus U$ has codimension more than i. In [**EG1**] it is shown that such pairs $(U, \mathbf{V})$ exist for any algebraic group and that the definition of $\mathrm{Ch}^i_G(X)$ is independent of the choice of U and $\mathbf{V}$.

Because equivariant Chow groups are defined as Chow groups of certain schemes, they enjoy all of the functoriality of ordinary Chow groups. In particular, if X

is smooth then pullback along the diagonal defines an intersection product on $\mathrm{Ch}^*_G(X)$.

Remark 6.10. Intuitively an equivariant cycle may be viewed as a G-invariant cycle on $X \times \mathbf{V}$ where $\mathbf{V}$ is some representation of G. Because representations can have arbitrarily large dimension $\mathrm{Ch}^i(X)$ can be non-zero for all i.

If G acts freely then a quotient X/G exists as an algebraic space and $\mathrm{Ch}^i_G(X) = \mathrm{Ch}^i(X/G)$. More generally, if G acts with finite stabilizers then elements of $\mathrm{Ch}^i_G(X) \otimes \mathbb{Q}$ are represented by G-invariant cycles on X and consequently $\mathrm{Ch}^i_G(X) = 0$ for $i > \dim X - \dim G$.

As in the non-equivariant case, an equivariant vector bundle V on a G-scheme defines Chern class operations $c_i(V)$ on $\mathrm{Ch}^*_G(X)$. The Chern class naturally live in the equivariant operational Chow ring $A^*_G(X)$ and as in the non-equivariant case the map $A^*_G(X) \to \mathrm{Ch}^*_G(X)$, $c \mapsto c[X]$ is a ring isomorphism if X is smooth.

We can again define the Chern character and Todd class of a vector bundle V. However, because $\mathrm{Ch}^i_G(X)$ can be non-zero for all i, the target of the Chern character and Todd class is the infinite direct product $\Pi_{i=0}^{\infty} \mathrm{Ch}^i_G(X) \otimes \mathbb{Q}$.

When G acts on X with finite stabilizers then $\mathrm{Ch}^i_G(X) \otimes \mathbb{Q}$ is 0 for $i > \dim X - \dim G$ so in this case the target of the Chern character and Todd class map is $\mathrm{Ch}^*_G(X)$.

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Clay Mathematics Proceedings
Volume **18**, 2013

The Regularity of the Conductor

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Dedicated to Joe Harris, who has taught us so much, on the occasion of his 60th Birthday

Abstract. We bound the Castelnuovo-Mumford regularity and syzygies of the ideal of the singular set of a plane curve, and more generally of the conductor scheme of certain projectively Gorenstein varieties.

1. Introduction

This note was inspired by a letter from Remke Kloosterman asking whether the following result (now essentially Proposition 3.6 in Kloosterman [**2013**]) was known:

Theorem 1.1. *Suppose that $C \subset \mathbb{P}^2_{\mathbf{C}}$ is a reduced plane curve of degree d over the complex numbers, and suppose that the only singularities of C are ordinary nodes and cusps, i.e., have local analytic equations $xy = 0$ or $y^2 - x^3 = 0$. If $\Gamma \subset \mathbb{P}^2_{\mathbf{C}}$ denotes the set of points at which C has nodes, then $\operatorname{reg} I_\Gamma \leq d-1$, and the minimal number of homogeneous generating syzygies of degree d is precisely the number of irreducible components of C minus 1.*

In case C is irreducible this result simply says that the regularity of the set of nodal points of C is bounded by $d-2$. Since the regularity of the set of nodal points is bounded by the regularity of the set of all the singular points, this is a consequence of the classical "completeness of the adjoint series" (see Section 4).

In the general case, Kloosterman's proof is based on delicate arguments of Dimca [**1990**] about the mixed Hodge theory of singular hypersurfaces. Aside from the application to arithmetic geometry that Kloosterman makes, his result seemed to us interesting and important because it sheds some light on the famous problem of understanding the restrictions on the positions of the nodes of a plane curve, about which little is known (see Section 8).

It is the purpose of this note to give a simple expression for the regularity of the conductor ideal that extends Kloosterman's result in a way not limited to characteristic zero or to curves; it is a statement about any finite birational extension of a quasi-Gorenstein ring by a Cohen-Macaulay ring. The proof, given in the next section, involves only considerations of duality.

2010 *Mathematics Subject Classification.* Primary: 13B22, 13D02; Secondary: 14H50, 14E05.

The first author was supported by NSF grants 1001867 and 1002171.
The second author was supported by NSF grants 0901613 and 1205002.

Specializing to the case of plane curves, we can use the extra strength of our result to extend Theorem 1.1 to curves with arbitrary singularities (Corollaries 3.3 and 3.4). Here is a special case of that result:

COROLLARY 1.2. *Suppose that $C \subset \mathbb{P}^2_K$ is a reduced plane curve of degree d over an algebraically closed field K. If Γ denotes any set of singular points of C, then $\operatorname{reg} I_\Gamma \leq d-1$. If, moreover, Γ contains the points (if any) at which distinct irreducible components of C meet and all these points of intersection are ordinary nodes, then the minimal number of homogeneous generating syzygies of I_Γ of degree d is precisely the number of irreducible components of C minus 1.*

Theorem 1.1 follows because the set Γ in the Theorem contains the points at which distinct components of the curve meet.

Another ideal related to the singular set and the conductor is the Jacobian ideal of the plane curve. This ideal is an "almost complete intersection" in characteristic zero. If the curve has only ordinary nodes as singularities, then the conductor ideal is the saturation of the Jacobian ideal. We prove a general result (Proposition 5.2) about the syzygies of an almost complete intersection with perfect saturation that implies, in the situation of a degree d plane curve $f = 0$ with only nodes, that the syzygies of the partial derivatives of f have degree at least $2d-3$, and a little more (Corollary 5.1); this generalizes a result of Dimca and Sticlaru [**2011**] that, again, was originally proven by Hodge theory.

Besides the conductor one can measure the difference between a standard graded algebra A and a (partial) normalization B by the size of the A-module B/A. We also give bounds on the regularity of this module (and on the regularity of B as an A-module) in the case where both A and B are Cohen-Macaulay and A is reduced (Proposition 3.8).

The fact that the number of components of a plane curve appears in a formula for the regularity of the conductor suggests that there might be a simple relation between the conductor of a reducible hypersurface and the conductors of its components; such a relation is given in Proposition 6.1.

2. Notation and Conventions

Throughout we let $S = K[x_0, \dots, x_r]$ be a polynomial ring over a field K. If M is a finitely generated graded S-module, we write $\operatorname{reg} M$ for the (Castelnuovo-Mumford) regularity of M and indeg M for the infimum of $\{i \mid M_i \neq 0\}$. Let X be a subscheme of $\mathbb{P}^r_K$ with saturated homogeneous ideal I_X and homogeneous coordinate ring $A = S/I_X$. We define $\operatorname{reg} X := \operatorname{reg} I_X = \operatorname{reg} A + 1$. If $A \subset B$ is a ring extension, we denote by

$$\mathfrak{C}_{B/A} := \operatorname{ann}_A(B/A) \subset A \text{ and}$$
$$\mathfrak{C}'_{B/A} := \operatorname{ann}_S(B/A) \subset S\,,$$

the *conductor* of B into A, regarded as an ideal of A or of S, respectively. Note that $A/\mathfrak{C}_{B/A} = S/\mathfrak{C}'_{B/A}$.

When X is reduced and $B = \overline{A}$ is the normalization of A, we write $\mathfrak{C}_X$ or $\mathfrak{C}'_X$ instead of $\mathfrak{C}_{B/A}$ or $\mathfrak{C}'_{B/A}$. These are homogeneous ideals of codimension at least 1 and $1 + \operatorname{codim} X$, respectively.

3. Regularity

Recall that a homogeneous ideal $I \subset S$ is called *perfect* if $S/I \neq 0$ is a Cohen-Macaulay ring or, equivalently, if the projective dimension of the S-module S/I is $\operatorname{codim} I$.

THEOREM 3.1. *Let $X \subset \mathbb{P}^r_K$ be a reduced scheme of codimension c with homogeneous coordinate ring A, and assume that A is Gorenstein. Let $\overline{A}$ be the normalization of A. Write $S = K[x_0, \dots, x_r]$ for the homogeneous coordinate ring of $\mathbb{P}^r_K$. Suppose B is a graded Cohen-Macaulay ring with $A \subsetneq B \subset \overline{A}$.*

The conductor ideal $\mathfrak{C}'_{B/A} \subset S$ is perfect of codimension $c+1$ and

$$\operatorname{reg} \mathfrak{C}'_{B/A} = \operatorname{reg} X - 1 - \operatorname{indeg}(B/A)\,;$$

in particular, if X is geometrically reduced and irreducible, then $\operatorname{reg} \mathfrak{C}'_{B/A} < \operatorname{reg} X - 1$.

Moreover, $\operatorname{Tor}^S_c(\mathfrak{C}'_{B/A}, K)_{\operatorname{reg} \mathfrak{C}'_{B/A}+c}$ *is K-dual to* $(B/A)_{\operatorname{indeg}(B/A)}$; *in particular, the c-th syzygy module of $\mathfrak{C}'_{B/A}$ has precisely* $\dim_K(B/A)_{\operatorname{indeg}(B/A)}$ *homogeneous minimal generators of highest degree.*

Under mild hypotheses we can use Theorem 3.1 to extract information about the codimension 1 components of the singular locus of X. The module $S/\mathfrak{C}'_X$ is supported precisely on these components. If the codimension 1 components are generically only nodes and cusps (that is, after we localize at such a component, complete, and extend the residue field to its algebraic closure, they become nodes and cusps), then the conductor ideal is generically radical; and since the conductor ideal is perfect, it is equal to the reduced ideal of the union of the codimension 1 components of the singular locus; thus this reduced scheme has regularity $\leq \operatorname{reg} X - 1$.

We remind the reader that regularity has a simple interpretation for perfect ideals:

PROPOSITION 3.2. *Suppose that $I \subset S$ is a homogeneous perfect ideal of codimension c with $1 \leq c \leq r$, and let $\mathcal{I}$ be the corresponding ideal sheaf on $\mathbb{P}^r_K$. The following statements are equivalent*:

1. $\operatorname{reg} S/I \leq m$;
2. *The c-th syzygies of S/I are generated in degrees $\leq m + c$;*
3. $H^{r-c+1}(\mathcal{I}(m + c - r)) = 0$;
4. *The value of the Hilbert function $\dim_K(S/I)_e$ is equal to the corresponding value of the Hilbert polynomial of S/I for all $e \geq m + c - r$.*

Moreover, if $m = \operatorname{reg} S/I$, the number of highest syzygies of S/I (or I) of highest degree can be computed in terms of cohomology via a natural isomorphism

$$\operatorname{Tor}^S_c(S/I, K)_{m+c} \cong H^{r-c+1}(\mathcal{I}(m + c - r - 1)).$$

PROOF. The equivalence of the first four statements is standard (see for example Eisenbud [**2005**], Section 4). For the last statement, note that under the given hypothesis the minimal homogeneous free S-resolution of S/I has length c, and the highest degree of a c-th homogeneous generating syzygy of S/I is $m + c$. Writing

$-^*$ for K-duals, we obtain that

$$\begin{aligned}\operatorname{Tor}_c^S(S/I,K)_{m+c} &\cong ((K \otimes_S \operatorname{Ext}_S^c(S/I,S))_{-m-c})^* \\ &\cong (\operatorname{Ext}_S^c(S/I,S)_{-m-c})^* \\ &\cong (\operatorname{Ext}_S^{c-1}(I,S)_{-m-c})^* .\end{aligned}$$

Since $S(-r-1)$ is the canonical module of S, local duality shows that the last of these is naturally isomorphic to $H^{r-c+1}(\mathcal{I}(m+c-r-1))$. □

If X is a reduced curve in $\mathbb{P}^2_K$, then X is arithmetically Gorenstein, and the normalization of X is Cohen-Macaulay. Moreover, there are many Cohen-Macaulay rings between the homogeneous coordinate ring of X and its normalization. We can exploit this situation to improve Theorem 1.1:

Corollary 3.3. *If $X \subset \mathbb{P}^2_K$ is a reduced singular plane curve of degree d over a perfect field K, then*

$$\operatorname{reg} \mathfrak{C}'_X = d-1-\inf\{m \mid h^0(\mathcal{O}_{\overline{X}}(m)) > \binom{m+2}{2}\},$$

and the minimal number of homogeneous generating syzygies of $\mathfrak{C}'_X$ of degree d is one less than the number of components of X over the algebraic closure of K. Thus $\operatorname{reg} \mathfrak{C}'_X < d-1$ if and only if X is geometrically irreducible, and in particular

$$\operatorname{reg}((\operatorname{Sing} X)_{\mathrm{red}}) \le d-1,$$

with strict inequality when X is geometrically irreducible.

In case K is algebraically closed and X is reducible, the result becomes attractively simple:

Corollary 3.4. *Suppose that K is algebraically closed and that $X \subset \mathbb{P}^2_K$ is a reduced but reducible curve. Let $I \subset S = k[x_0,x_1,x_2]$ be the saturated ideal of the subscheme of the conductor scheme where at least two components meet. If J is any unmixed ideal in S with $\mathfrak{C}'_X \subset J \subset I$, then J has regularity $d-1$ and the minimal number of homogeneous generating syzygies of J of degree d is one less than the number of components of X.*

In general the length δ of the conductor scheme of a plane curve of degree d ranges from 0 to $\binom{d}{2}$—the latter being the case when the curve is a union of lines. The regularity of a set of δ points in $\mathbb{P}^2_K$ ranges from δ down to about $\sqrt{2\delta}$. Thus the statement that the regularity of the conductor is bounded by $d-1$ (or $d-2$ in the case of geometrically irreducible curves) is quite strong.

Theorem 3.1 says in particular that the reduced ideal of a set of singular points on a reduced plane curve of degree d has regularity at most $d-1$, and regularity at most $d-2$ if the curve is geometrically reduced and irreducible. Here is another version of this statement:

Corollary 3.5. *If $I \subset K[x_0,x_1,x_2]$ is the reduced ideal of a finite set of points in $\mathbb{P}^2_K$, then the symbolic square $I^{(2)}$ contains no nonzero reduced form of degree $\le \operatorname{reg} I$ and no geometrically reduced and irreducible form of degree $\le \operatorname{reg} I + 1$.*

This consequence of Corollary 3.3 seems to beg for generalization. What can one say for higher symbolic powers, or more variables? A famous conjecture of Nagata states that if J is the defining ideal of δ *general* points in $\mathbb{P}^2$ then the smallest degree of a form contained in $J^{(m)}$ is at least $m\sqrt{\delta}$ (see for example Harbourne [**2001**] for a discussion and recent results). Our result for $m = 2$ deals only with reduced forms, but with arbitrary sets of points.

EXAMPLE 3.6. Let $S = K[x_0, x_1, x_2]$ be the homogeneous coordinate ring of $\mathbb{P}^2_K$, and let M be a generic map

$$M : S(-7) \oplus S(-8) \longrightarrow S(-5)^3.$$

Let I be the ideal generated by the three 2×2 minors of M, which have degree 5. From the formulas in Eisenbud [**2005**], Chapter 3 one sees that I is the homogeneous ideal of a set Γ of 19 reduced points. Further, M is the matrix of syzygies on the ideal I, and thus $\operatorname{reg} I = 7$.

Using Macaulay2 one can check that the smallest degree of a curve passing doubly through the points of Γ is 10, and that there is such a curve X of degree 10 whose singularities consist of ordinary nodes at the 19 points of I. From Corollary 3.3 we see that X is irreducible and that the normalization $\overline{X}$ has $h^0(\mathcal{O}_{\overline{X}}(1)) = 3$; that is, X is not the projection of a nondegenerate curve of degree 10 in $\mathbb{P}^3_K$.

We can prove a weaker version of Theorem 3.1 without the Cohen-Macaulay assumption on A. Recall that a positively graded Noetherian K-algebra A with homogeneous maximal ideal A_+ and graded canonical module ω_A is called *quasi-Gorenstein* if $\omega_A \cong A(a)$ for some $a \in \mathbb{Z}$, called the a-invariant of A. By local duality, $a = \operatorname{reg} H^{\dim A}_{A_+}(A)$, so, in case A is generated in degree 1, we have $a + \dim A \leq \operatorname{reg} A$, with equality when A is Gorenstein.

THEOREM 3.7. *Let $X \subset \mathbb{P}^r_K$ be a geometrically reduced scheme with homogeneous coordinate ring A. Suppose that B is a graded Cohen-Macaulay ring with $A \subsetneq B \subset \overline{A}$.*

If A is quasi-Gorenstein, then $\mathfrak{C}_{B/A}$ is a Cohen-Macaulay A-module and an unmixed ideal in A of codimension 1. Moreover, $\operatorname{reg} \mathfrak{C}'_{B/A} \leq \operatorname{reg} X$.

We can also say something about the regularity of the ring B, considered as a graded module over A; see also Ulrich and Vasconcelos [**1990**], the proof of 2.1(b).

PROPOSITION 3.8. *Let $X \subset \mathbb{P}^r_K$ be a reduced scheme with homogeneous coordinate ring A. Suppose that B is a graded Cohen-Macaulay ring with $A \subsetneq B \subset \overline{A}$.*

If A is Cohen-Macaulay, then the A-module B/A is Cohen-Macaulay of codimension 1. Furthermore,

$$\operatorname{reg}(B/A) \leq \operatorname{reg} X - 2 \quad \text{and} \quad \operatorname{reg} B \leq \operatorname{reg} X - 1\,.$$

We now proceed to the proofs.

PROOF OF THEOREM 3.1. To simplify the notation we write $\mathfrak{C}'$ for $\mathfrak{C}'_{B/A}$ and $\mathfrak{C}$ for $\mathfrak{C}_{B/A}$. We have $\mathfrak{C} \cong \operatorname{Hom}_A(B, A)$. Since B is a finite Cohen-Macaulay A-module and $\dim_A B = \dim A$, we have $\operatorname{Ext}^1_A(B, A) = 0$. Therefore, applying the long exact sequence in $\operatorname{Ext}^{\cdot}_A(-, A)$ to the short exact sequence

$$0 \to A \longrightarrow B \longrightarrow B/A \to 0\,,$$

we obtain a homogenous isomorphism $A/\mathfrak{C} \cong \operatorname{Ext}^1_A(B/A, A)$. Since A is Gorenstein, we have $\omega_A = A(a)$. Thus

$$A/\mathfrak{C} \cong \operatorname{Ext}^1_A(B/A, \omega_A)(-a)\,.$$

As A is a Cohen-Macaulay S-module of codimension c, local duality (or the change of rings spectral sequence) in turn gives

$$\begin{aligned}\operatorname{Ext}^1_A(B/A,\omega_A) &\cong \operatorname{Ext}^{1+c}_S(B/A,\omega_S)\\ &\cong \operatorname{Ext}^{1+c}_S(B/A,S)(-r-1)\,,\end{aligned}$$

and we conclude that

$$A/\mathfrak{C} \cong \operatorname{Ext}^{1+c}_S(B/A,S)(-a-r-1)\,.$$

From the above short exact sequence it also follows that B/A is a Cohen-Macaulay S-module of codimension $c+1$ and hence has a minimal homogeneous free S-resolution $F.$ of length $c+1$. Now the isomorphism

$$S/\mathfrak{C}' \cong A/\mathfrak{C} \cong \operatorname{Ext}^{1+c}_S(B/A,S)(-a-r-1)$$

implies that dualizing $F.$ into $S(-a-r-1)$ gives a minimal homogeneous free S-resolution of $S/\mathfrak{C}'$. In particular, $\mathfrak{C}'$ is a perfect ideal of codimension $c+1$ and

$$\operatorname{Tor}^S_c(\mathfrak{C}',K) \cong \operatorname{Tor}^S_{c+1}(S/\mathfrak{C}',K) \cong \operatorname{Hom}_K(B/A\otimes_S K,K)(-a-r-1)\,.$$

Since A is Cohen-Macaulay, the degree shift can be rewritten as

$$\begin{aligned}-a-r-1 &= -a-\dim A - c\\ &= -\operatorname{reg} A - c\\ &= -\operatorname{reg} X + 1 - c\,.\end{aligned}$$

It follows that the two graded vector spaces

$$\operatorname{Tor}^S_c(\mathfrak{C}',K)(\operatorname{reg} X - 1 + c) \quad\text{and}\quad (B/A)\otimes_A K$$

are dual to one another.

The maximal generator degree of the former controls the regularity of $\mathfrak{C}'$ because this ideal is perfect. Therefore

$$\begin{aligned}\operatorname{reg}\mathfrak{C}' &= \operatorname{maxdeg}\ \operatorname{Tor}^S_c(\mathfrak{C}',K) - c\\ &= -\operatorname{indeg}(B/A) + \operatorname{reg} X - 1\,,\end{aligned}$$

as claimed. If X is geometrically reduced and irreducible, then $A_0 = K = (\overline{A})_0$ and therefore $\operatorname{indeg}(B/A) > 0$. The remaining assertions of the Theorem are now clear as well. □

Proof of Corollary 3.3. The last statements follow immediately from the formulas for the regularity of $\mathfrak{C}'_X$ and the number of its degree d syzygies, so we prove those.

Let A be the homogeneous coordinate ring of X. For $m < d$ we have $\dim_K A_m = \binom{m+2}{2}$, so the formula for $\operatorname{reg}\mathfrak{C}'_X$ follows at once from the one given in Theorem 3.1.

If $B = \overline{A}$, then the number of homogeneous generating syzygies of $\mathfrak{C}'_X$ of degree d is $\dim_K(B/A)_0$, again by Theorem 3.1. On the other hand, $(B/A)_0 = H^0(\pi_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X)$ and the dimension of this K-vector space is one less than the number of components of X over the algebraic closure of K. □

Proof of Corollary 3.4. Suppose that X has e irreducible components. Let A be the homogeneous coordinate ring of X and let B' be the direct product of the homogeneous coordinate rings of the irreducible components of X, a "partial normalization" of A. Notice that $I \subset \mathfrak{C}'_{B'/A}$. By Theorem 3.1 the ideal $\mathfrak{C}'_{B'/A}$ has regularity $d-1$ and its syzygy module has exactly $e-1$ homogeneous basis elements of degree d. The same holds true for the conductor $\mathfrak{C}'_X$. Proposition 3.2 implies that the regularity and the number of syzygies of top degree are both monotonic in the sequence of ideals $\mathfrak{C}'_X \subset J \subset I \subset \mathfrak{C}'_{B'/A}$, yielding the desired formulas for the regularity and the number of degree d syzygies of J. □

Proof of Corollary 1.2. From Corollary 3.3 we know that $\operatorname{reg} I_\Gamma \leq d-1$ and that this inequality is strict for irreducible curves. If the curve is reducible and its components meet in ordinary nodes, then the part of the conductor scheme concentrated at the points of intersection is the reduced scheme of the intersection points itself, so we may apply Corollary 3.4. □

Before proving Theorem 3.7 we need a lemma on conductors, see also Kunz [**2005**], 17.6. Generalizing our previous notation, we write $\mathfrak{C}_{B/A} := A :_A B$ for the conductor of any ring extension $A \subset B$. Again, $\mathfrak{C}_{B/A}$ is the unique largest B-ideal contained in A, and, whenever A is Noetherian, this ideal contains a non zerodivisor of B if and only if the extension $A \subset B$ is finite and birational.

Lemma 3.9. *Let $A \subset B \subset C$ be extensions of rings. If $\mathfrak{C}_{B/A} = Bu$ for some non zerodivisor of C, then*

$$\mathfrak{C}_{C/A} = \mathfrak{C}_{C/B}\,\mathfrak{C}_{B/A}\,.$$

Proof. For any rings $A \subset B \subset C$ the inclusion $\mathfrak{C}_{C/B}\,\mathfrak{C}_{B/A} \subset \mathfrak{C}_{C/A}$ follows from the definition of the conductor. If $\mathfrak{C}_{B/A} = Bu$, for some non zerodivisor $u \in C$, then $\mathfrak{C}_{C/A} \subset \mathfrak{C}_{B/A} = Bu$ and we may write $\mathfrak{C}_{C/A} = \mathfrak{C}u$ for some subset $\mathfrak{C}$ of B. This subset is a C-ideal because $\mathfrak{C}_{C/A}$ is a C-ideal and u is a non zerodivisor of C. It follows that $\mathfrak{C} \subset \mathfrak{C}_{C/B}$ and therefore $\mathfrak{C}_{C/A} \subset \mathfrak{C}_{C/B}\,u \subset \mathfrak{C}_{C/B}\,\mathfrak{C}_{B/A}$. □

Proof of Theorem 3.7. Since X is geometrically reduced, we may extend the ground field and assume that it is algebraically closed. We may then choose a homogeneous Noether normalization inside A over which the total ring of quotients of A is separable, and thus birational to a homogeneous hypersurface ring $A' \subset A$ that contains the Noether normalization.

Write $\omega_A = A(a)$ and $\omega_{A'} = A'(a')$. We claim that $\mathfrak{C}_{A/A'}$ is generated by a homogeneous non zerodivisor u on B of degree $a'-a$. This is because

$$\mathfrak{C}_{A/A'} \cong \operatorname{Hom}_{A'}(A, A') \cong \operatorname{Hom}_{A'}(A, \omega_{A'}(-a')) \cong \omega_A(-a') \cong A(a-a').$$

It follows from Lemma 3.9 that $\mathfrak{C}_{B/A'} = \mathfrak{C}_{B/A}\,u$. By Theorem 3.1 the A'-module $A'/\mathfrak{C}_{B/A'}$ is Cohen-Macaulay of codimension 1, so $\mathfrak{C}_{B/A'}$ is a maximal Cohen-Macaulay A'-module. Therefore $\mathfrak{C}_{B/A}$ is also a maximal Cohen-Macaulay module over A' and thus over A. This also shows that $\mathfrak{C}_{B/A}$ is an unmixed ideal in A of codimension 1, because ω_A and hence A satisfies Serre's condition S_2.

For the second statement, notice that

$$\operatorname{reg}\mathfrak{C}'_{B/A} = \operatorname{reg}(S/\mathfrak{C}'_{B/A}) + 1 = \operatorname{reg}(A/\mathfrak{C}_{B/A}) + 1\,.$$

The exact sequence

$$0 \to \mathfrak{C}_{B/A} \longrightarrow A \longrightarrow A/\mathfrak{C}_{B/A} \to 0$$

in turn shows that

$$\operatorname{reg}(A/\mathfrak{C}_{B/A}) \leq \max\{\operatorname{reg}\mathfrak{C}_{B/A} - 1, \operatorname{reg} A\}\,.$$

Since $\mathfrak{C}_{B/A'} = \mathfrak{C}_{B/A}\, u$ with u a homogeneous non zerodivisor on B of degree $a' - a$, we see that $\operatorname{reg} \mathfrak{C}_{B/A} = \operatorname{reg} \mathfrak{C}_{B/A'} + a - a'$. Combining this with the two displayed inequalities we obtain

$$\operatorname{reg} \mathfrak{C}'_{B/A} \leq \max\{\operatorname{reg} \mathfrak{C}_{B/A'} + a - a',\ \operatorname{reg} A + 1\}\,.$$

Again, the exact sequence

$$0 \to \mathfrak{C}_{B/A'} \longrightarrow A' \longrightarrow A'/\mathfrak{C}_{B/A'} \to 0$$

gives

$$\operatorname{reg} \mathfrak{C}_{B/A'} \leq \max\{\operatorname{reg}(A'/\mathfrak{C}_{B/A'}) + 1,\ \operatorname{reg} A'\} = \max\{\mathfrak{C}'_{B/A'},\ a' + \dim A'\}\,.$$

Finally, we apply Theorem 3.1 to the extension $A' \subset B \subset \overline{A'} = \overline{A}$ and obtain

$$\operatorname{reg} \mathfrak{C}'_{B/A'} \leq a' + \dim A'\,.$$

Combining the last three displayed inequalities we deduce

$$\operatorname{reg} \mathfrak{C}'_{B/A} \leq \max\{a + \dim A,\ \operatorname{reg} A + 1\} = \operatorname{reg} A + 1 = \operatorname{reg} X\,,$$

as desired. $\square$

Proof of Proposition 3.8. One sees immediately, as in the proof of Theorem 3.1, that B/A is a Cohen-Macaulay A-module of codimension 1. As for the other claims, we may assume that the ground field K is infinite. In this case A admits a homogeneous Noether normalization A'. Since the Cohen-Macaulay rings A and B are finitely generated graded modules over the polynomial ring A', it follows that they are maximal Cohen-Macaulay modules and hence free. Thus

$$0 \to A \longrightarrow B$$

is a homogeneous free resolution of minimal length of the A'-module B/A. As the latter module is Cohen-Macaulay, its regularity can be read from the last module in the minimal homogeneous free resolution. It follows that $\operatorname{reg}_{A'} B/A \leq \operatorname{reg}_{A'} A - 1$. Since A is finite over A', the regularity of an A-module is the same as its regularity as an A'-module. Therefore $\operatorname{reg}_A B/A \leq \operatorname{reg} A - 1$, which also implies $\operatorname{reg}_A B \leq \operatorname{reg} A$. $\square$

4. Completeness of the Adjoint Series

If X is irreducible, then Theorem 1.1 can be deduced from a classical result known as the "completeness of the adjoint series". We will explain how below, but first we explain the meaning of the terms.

An *adjoint of degree e* is a form of degree e satisfying an "adjoint condition" at each singularity. For example, at an ordinary node or cusp the adjoint condition is simply to vanish at the point; if the singularity is an ordinary k-fold point, then the adjoint condition is vanishing to order $k-1$. In general the adjoint condition is to be contained in the local conductor ideal at the point. Thus, from our point of view, an adjoint is simply a form of degree e contained in $\mathfrak{C}_X$.

The adjoint conditions at the singular points of X give rise to a divisor D on $\overline{X}$, namely the zero locus of the pull-backs of all the (local) functions satisfying the adjoint conditions. The *completeness of the adjoint series* is the statement that the natural inclusion of $(\mathfrak{C}_X)_e$, the space of adjoints of degree e, to $H^0(\mathcal{O}_{\overline{X}}(e)(-D))$, is an isomorphism. From our point of view this is just the statement that $\mathfrak{C}_X$ is an ideal of both A, the coordinate ring of X, and $B = \oplus_e H^0(\mathcal{O}_{\overline{X}}(e))$, so that the natural inclusion $\mathfrak{C} \subset \mathfrak{C}B$ is in fact an equality.

We can now deduce the case of Theorem 1.1 when the curve X is irreducible. First, it suffices to show that the set Γ of all singular points of X has regularity $\leq d-2$, since then any subset has regularity $\leq d-2$ as well. To do this we must show that $H^1(\mathcal{I}_\Gamma(d-3)) = 0$. So it suffices to show that the nodes and cusps impose independent conditions on the forms of degree $d-3$.

Under the hypothesis of the Theorem that X has only ordinary nodes and cusps, $\mathcal{I}_\Gamma = \widetilde{\mathfrak{C}'_X}$. In this case, the "completeness of the adjoint series" says that the vector space of forms of degree $d-3$ vanishing on Γ gives the complete canonical series on $\overline{X}$, and thus has dimension genus(X), which is

$$\binom{d-1}{2} - \deg\Gamma = \dim S_{d-3} - \deg\Gamma;$$

that is, the conditions imposed by Γ are independent, as required.

5. Syzygies of an Almost Complete Intersection

In the case where X is a plane curve with only ordinary nodes as singularities, the conductor ideal is the unmixed part of the Jacobian ideal, the ideal of partial derivatives of the defining equation of X, and we can derive a surprising consequence for the syzygies of these partial derivatives:

Corollary 5.1. *Let $X \subset \mathbb{P}^2_K$ be a reduced curve of degree d over an algebraically closed field whose characteristic is not a divisor of d. Write $S = K[x_0, x_1, x_2]$ for the homogeneous coordinate ring of $\mathbb{P}^2_K$. Assume that the defining equation of X is $F(x_0, x_1, x_2) = 0$ and that X has only ordinary nodes as singularities. If*

$$\sum_{i=0}^{2} G_i \frac{\partial F}{\partial x_i} = 0$$

is a homogeneous relation in S, then $\deg G_i \geq d-2$ whenever $G_i \neq 0$, and equality is possible if and only if X is reducible.

This result was proven independently, in characteristic zero, in Dimca and Sticlaru [**2011**] (Theorem 4.1), using Hodge theory. We deduce it from a much more general statement about almost complete intersections given in Proposition 5.2.

Proposition 5.2. *Let $S = K[x_0, \dots, x_r]$ be a polynomial ring over a field and let $I \subset S$ be a homogeneous perfect ideal of codimension g. Let $f_1, \dots, f_{g+1}$ be homogeneous elements of I such that $f_1, \dots, f_g$ form a regular sequence. Set $\mathfrak{a} := (f_1, \dots, f_g) \subset J := (f_1, \dots, f_{g+1})$. If I is the unmixed part of J, meaning the intersection of the primary components of J having codimension g, then*

$$\omega_{S/I} = ((\mathfrak{a} : f_{g+1})/\mathfrak{a})(-r-1+\sum_{i=1}^{g} \deg f_i)\,.$$

Furthermore,

$$\operatorname{reg} S/I = \operatorname{reg} S/\mathfrak{a} - \operatorname{indeg}((\mathfrak{a} : f_{g+1})/\mathfrak{a})\,,$$

and if f_{g+1} has maximal degree among the f_i, this can be written as

$$\operatorname{reg} S/I = \operatorname{reg} S/\mathfrak{a} - \operatorname{indeg}(I_1(\phi)/\mathfrak{a})\,,$$

where ϕ is a matrix of homogeneous syzygies of $f_1, \dots, f_{g+1}$ and $I_1(\phi)$ denotes the ideal generated by the entries of ϕ.

Proof. We have

$$\operatorname{reg} S/I = -\operatorname{indeg}\ \omega_{S/I} + \dim S/I$$

since S/I is Cohen-Macaulay, and by linkage

$$\omega_{S/I} = ((\mathfrak{a} : I)/\mathfrak{a})(-r - 1 + \sum_{i=1}^{g} \deg f_i)\,.$$

Thus, using the fact that $\operatorname{reg} S/\mathfrak{a} = -g + \sum_{i=1}^{g} \deg f_i$, we obtain

$$\operatorname{reg} S/I = \operatorname{reg} S/\mathfrak{a} - \operatorname{indeg}((\mathfrak{a} : I)/\mathfrak{a})\,.$$

Since $\mathfrak{a}$ is unmixed,

$$\mathfrak{a} : I = \mathfrak{a} : f_{g+1}\,,$$

proving the first two assertions.

If f_{g+1} has maximal degree among the f_i, then the non-zero elements in the last row of the syzygy matrix ϕ, which are the generators of $\mathfrak{a} : f_{g+1}$, have the lowest degree among the non-zero entries of their respective column. The columns with zero last entry, on the other hand, are relations on the regular sequence $f_1, \dots, f_g$ and hence have entries in $\mathfrak{a}$. It follows that

$$\operatorname{indeg}((\mathfrak{a} : f_{g+1})/\mathfrak{a}) = \operatorname{indeg}(I_1(\phi)/\mathfrak{a})\,.$$

□

We are now ready to prove Corollary 5.1.

Proof. Our assumption on the characteristic implies that F is contained in the ideal of S generated by $\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}$. If X is smooth, this ideal has codimension at least 3 and X is irreducible. Otherwise we apply Proposition 5.2 with $r = 2$, $I = \mathfrak{C}'_X$, and $J = (\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2})$. One has $J \subset I$, and equality holds locally at every prime ideal of codimension two because all singularities of X are ordinary nodes. Since I is unmixed, we conclude that I is the unmixed part of J. After a linear change of variables we may assume that $\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}$ is a regular sequence of two forms of degree $d - 1$. Thus the Proposition shows that

$$\operatorname{reg} S/I = 2(d-2) - \operatorname{indeg}(I_1(\phi)/\mathfrak{a})\,.$$

Finally, according to Corollary 3.3 one has $\operatorname{reg} S/I \le d - 2$, and equality holds if and only if X is reducible. □

6. The Conductor of a Reducible Hypersurface

The estimates obtained in Section 3 point to a difference between the reducible and the irreducible case. In the present section we study this phenomenon by relating the conductor ideal of a reducible subscheme to the conductor ideals of its irreducible components. This is done in the next proposition. Our proof is an adaptation of arguments in Kunz [**2005**], 17.6, 17.11, 17.12, where the case of local rings of plane curve singularities is treated.

PROPOSITION 6.1. *If $X \subset \mathbb{P}^r_K$ is a reduced hypersurface with distinct irreducible components X_i, then*

$$\mathfrak{C}'_X = \sum_i \mathfrak{C}'_{X_i} \prod_{j \neq i} I_{X_j} \, .$$

PROOF. We write F, F_i for the defining homogeneous polynomials of X, X_i in the polynomial ring $S := K[x_0, \dots, x_r]$, $G_i := \prod_{j \neq i} F_j$, and A, A_i for the homogeneous coordinate rings $S/(F)$, $S/(F_i)$. Consider the ring extensions

$$A \subset B := \times_i A_i \subset C := \overline{A} = \overline{B} = \times_i \overline{A_i} \, .$$

Further, write e_i for the ith idempotent of B and notice that $A_i e_i = A e_i = S e_i$, $B = \sum_i S e_i$, $1_A = \sum_i e_i$.

Clearly $\mathfrak{C}_{C/B} = \times_i \mathfrak{C}_{\overline{A_i}/A_i} = \sum_i \mathfrak{C}'_{X_i} e_i$.

Next, we claim that $\mathfrak{C}_{B/A} = B(\sum_i G_i e_i)$. Indeed, the image of an element $H \in S$ in A belongs to $\mathfrak{C}_{B/A}$ if and only if $S H e_i \subset A$ for every i, which means that for every i there exists $H_i \in S$ so that $H_i \equiv H \bmod F_i$ and $H_i \equiv 0 \bmod G_i$. This is equivalent to $H e_i \in S G_i e_i$. Hence indeed $\mathfrak{C}_{B/A} = \sum_i S G_i e_i = B(\sum_i G_i e_i)$, where the last equality holds because the e_i are orthogonal idempotents.

Since the B-ideal $\mathfrak{C}_{B/A}$ is generated by a single non zerodivisor of C, it follows that $\mathfrak{C}_{C/A} = \mathfrak{C}_{C/B}\, \mathfrak{C}_{B/A}$ according to Lemma 3.9. Therefore

$$\begin{aligned}
\mathfrak{C}'_X \, 1_A = \mathfrak{C}_{C/A} &= (\textstyle\sum_i G_i e_i)(\displaystyle\sum_i \mathfrak{C}'_{X_i} e_i) \\
&= \sum_i G_i \mathfrak{C}'_{X_i} e_i \\
&= (\sum_i G_i \mathfrak{C}'_{X_i})(\sum_i e_i) \\
&= (\sum_i G_i \mathfrak{C}'_{X_i}) \, 1_A \, .
\end{aligned}$$

We deduce that $\mathfrak{C}'_X = \sum_i G_i \mathfrak{C}'_{X_i}$ because the ideals on both sides of the equation contain F. ☐

COROLLARY 6.2. *In the setting of Proposition* 6.1 *we write $S := K[x_0, \dots, x_r]$ and $d := \deg X$, $d_i := \deg X_i$. For every i there is an exact sequence*

$$0 \to S(-d) \longrightarrow \mathfrak{C}'_{\cup_{j \neq i} X_j}(-d_i) \oplus \mathfrak{C}'_{X_i}(-d + d_i) \longrightarrow \mathfrak{C}'_X \to 0 \, .$$

In particular,

$$\operatorname{reg} \mathfrak{C}'_X \leq \max_i \{d - 1, \operatorname{reg} \mathfrak{C}'_{X_i} + d - d_i\}.$$

PROOF. We write $F = F_i G_i$ as in the previous proof. We will construct the desired sequence in the more precise form

$$0 \to SF \longrightarrow F_i \mathfrak{C}'_{\cup_{j \neq i} X_j} \oplus G_i \mathfrak{C}'_{X_i} \longrightarrow \mathfrak{C}'_X \to 0 \, .$$

Proposition 6.1 gives $F_i \mathfrak{C}'_{\cup_{j \neq i} X_j} + G_i \mathfrak{C}'_{X_i} = \mathfrak{C}'_X$, so we may take the right hand map in the desired sequence to be addition, and it suffices to show that

$$F_i \mathfrak{C}'_{\cup_{j \neq i} X_j} \cap G_i \mathfrak{C}'_{X_i} = SF.$$

The right hand side of this equation is contained in the left hand side because $\mathfrak{C}'_{\cup_{j\neq i}X_j} \ni G_i$ and $\mathfrak{C}'_{X_i} \ni F_i$. The reverse inequality holds because F_i and G_i are relatively prime, and thus

$$F_i\mathfrak{C}'_{\cup_{j\neq i}X_j} \cap G_i\mathfrak{C}'_{X_i} \subset F_iS \cap G_iS = F_iG_iS = FS\,.$$

The assertion about the regularity of $\mathfrak{C}'_X$ follows from the exact sequence, using induction on the number of irreducible components of X. □

Corollary 6.3. *Let $X \subset \mathbb{P}^2_K$ be a reduced plane curve over an algebraically closed field with distinct irreducible components X_i and write $\mathcal{J}_{X_i}$ for the saturated ideals defining the reduced singular sets $(\operatorname{Sing} X_i)_{\text{red}} \subset \mathbb{P}^2_K$. If X has only ordinary nodes and cusps as singularities, then the ideal*

$$\sum_i \mathcal{J}_{X_i} \prod_{j\neq i} I_{X_j}$$

is the saturated ideal defining the reduced singular set $(\operatorname{Sing} X)_{\text{red}} \subset \mathbb{P}^2_K$.

Proof. Because of the assumption on the singularities, the conductor ideals $\mathfrak{C}'_X$ and $\mathfrak{C}'_{X_i}$ are reduced. Now apply Proposition 6.1. □

7. Examples of Resolutions of Singular Sets

We exhibit two situations in which we can specify the resolution of the singular ideal of a plane curve completely. The third example shows that the bound coming from Bézout's Theorem can be much less sharp than that of Corollary 3.3. In this section $X \subset \mathbb{P}^2_K$ will always be a reduced curve over an algebraically closed field K.

Example 7.1. Suppose that $X \subset \mathbb{P}^2_K$, given by $F = 0$, is the union of more than 1 distinct smooth components X_i meeting transversely; in other words, X has smooth irreducible components and only ordinary nodes as singularities. Suppose that the equation of X_i is $F_i = 0$, so that $F = \prod_{i=1}^{\ell} F_i$. According to Corollary 6.3 for instance, the reduced singular set of X has homogeneous ideal generated by the ℓ products

$$G_i := \prod_{j\neq i} F_j\,.$$

It follows, in particular, that the ideal I generated by the G_i has codimension 2.

Hence this ideal is the flat specialization of the perfect ideal

$$(\{g_i := \prod_{j\neq i} x_j \mid 1 \le i \le \ell\}) \subset K[x_1, \dots, x_\ell]\,.$$

Since all the products x_ig_i are equal to $\prod_{j=1}^{\ell} x_j$, we have $\ell - 1$ syzygies of the form $x_{i+1}g_{i+1} - x_ig_i$ on the g_i, and these generate all the syzygies. Thus I is also perfect and its homogeneous minimal free resolution over the polynomial ring S in 3 variables has the form

$$0 \to \oplus_{i=1}^{\ell-1} S(-d) \longrightarrow \oplus_{i=1}^{\ell} S(-d + d_i) \longrightarrow I \to 0\,,$$

where $d = \deg X$ and $d_i = \deg X_i$. In particular, we see directly that $\operatorname{reg} I = d - 1$ as claimed.

Example 7.2. Let $X \subset \mathbb{P}^2_K$ be a rational curve of degree $d \geq 2$ with only ordinary nodes and cusps as singularities, so that $(\operatorname{Sing} X)_{\text{red}}$ consists of $\binom{d-1}{2}$ distinct points. We claim that the minimal free resolution of the saturated ideal $I := I_{(\operatorname{Sing} X)_{\text{red}}}$ has the form

$$0 \to S(-d+1)^{d-2} \longrightarrow S(-d+2)^{d-1} \longrightarrow I \to 0\,,$$

and thus $\operatorname{reg} I = d-2$. As $\operatorname{reg} \mathfrak{C}'_X \geq \operatorname{reg} I$ and X is irreducible, it also follows that $\operatorname{reg} \mathfrak{C}'_X = d-2$ and $h^0(\mathcal{O}_{\overline{X}}(1)) > 3$, according to Corollary 3.3.

To see the claim about the resolution, we first note that no curve of degree $d-3$ can pass through all the nodes of X. This is because such a curve would meet X in a scheme of degree at least $2\binom{d-1}{2} = (d-1)(d-2)$, whereas by Bézout's Theorem the intersection scheme could only be of length $d(d-3) < (d-1)(d-2)$.

Because I is the ideal of a reduced set of points, any linear form x that vanishes at none of the points is a non zerodivisor modulo I. If we reduce I modulo x we get a homogeneous ideal of finite colength $\binom{d-1}{2}$, not containing any form of degree $d-3$, in the polynomial ring in 2 variables. The only such ideal is the $(d-2)$-nd power of the maximal homogeneous ideal, and this has minimal free resolution as above. Since reducing modulo a linear non zerodivisor preserves the shape of the resolution, we are done.

Example 7.3. In some cases, the regularity bound given in Corollary 3.3 can be deduced simply from Bézout's Theorem. For example, suppose that $X \subset \mathbb{P}^2_K$ is an irreducible curve of degree d and the singular set of X is the transverse complete intersection of curves E and F of degrees $e < f$, respectively. In this case Corollary 3.3 asserts in particular that $\operatorname{reg}(E \cap F) = e + f - 1 \leq d - 2$, that is, $e + f < d$. By Bézout's Theorem, the degree of $E \cap X$ is de. But E meets X with multiplicity at least 2 at each of the ef points of $E \cap F$, so $2ef \leq de$, or $e + f < 2f \leq d$ as claimed.

On the other hand, suppose $I \subset K[x_0, x_1, x_2]$ is the ideal generated by the $m \times m$ minors of a generic $m + 1 \times m$ matrix M whose first column consists of generic forms of degree $2m - 1$ and whose other entries are generically chosen quadratic forms. From the formulas in Eisenbud [**2005**], Chapter 3 we see that

- I is generated by $m + 1$ forms of degree $4m - 3$;
- I is the ideal of a set Δ consisting of
$$\delta = 20\binom{m-1}{2} + 18m - 17$$
reduced points;
- $\operatorname{reg} I = 6m - 5$.

If X is an irreducible curve singular at all the points of Δ, then we can find a linear combination of the $m+1$ generators of I defining a curve that meets X in a scheme of length at least $m + 2\delta$, so Bézout's Theorem shows that the degree d of any such curve X satisfies

$$d \geq \frac{m + 2\delta}{4m - 3}\,,$$

which, after substituting the value of δ, becomes $d \geq 5m-2$. However, Corollary 3.3 shows that in fact we must have $d \geq \operatorname{reg} I + 2 = 6m-3$. (In experiments, the minimal degree in these circumstances—given that the matrix M is chosen generally—for the first values of $m \geq 2$ seems actually to be equal to $8m - 6$.)

8. Other results on the situation of the nodes

Coolidge [**1939**] says that "One of the most important unsolved probems in the whole theory of plane curves [is] the situation of the permissible singular points." But we know of very few results shedding light on this problem. In fact, other than the results of Kloosterman and of this paper the only references of which we are aware are:

- On pp 389 ff Coolidge gives some results for rational curves of degrees 6 and 7 in $\mathbb{P}^2_K$.
- A result of Arbarello and Cornalba [**1981**] shows that vanishing doubly at δ nodes imposes independent conditions on forms of degree d whenever $\binom{d+2}{2} \geq 3\delta$.

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Clay Mathematics Proceedings
Volume **18**, 2013

Stability of genus five canonical curves

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Dedicated to our advisor, Joe Harris, on his sixtieth birthday.

Abstract. We analyze GIT stability of nets of quadrics in $\mathbb{P}^4$ up to projective equivalence. Since a general net of quadrics defines a canonically embedded smooth curve of genus 5, the resulting quotient $\overline{M}^G := \mathbb{G}(3,15)^{ss}/\!\!/\mathrm{SL}(5)$ gives a birational model of $\overline{M}_5$. We study the geometry of the associated contraction $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$, and prove that f is the final step in the log minimal model program for $\overline{M}_5$.

1. Introduction

A canonically embedded non-hyperelliptic, non-trigonal smooth curve of genus 5 is a complete intersection of 3 quadrics in $\mathbb{P}^4$ [**ACGH85**, Ch.V]. Thus, the Grassmannian of nets in $\mathbb{P}\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \simeq \mathbb{P}^{14}$ gives a natural compactification of the open Hilbert scheme of non-hyperelliptic, non-trigonal smooth canonical curves of genus 5, and the corresponding GIT quotient

$$\overline{M}^G := \mathbb{G}(3,15)^{ss}/\!\!/\,\mathrm{SL}(5)$$

is a projective birational model of $\overline{M}_5$. In this paper, we study the geometry of $\overline{M}^G$ and show that the natural birational contraction $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ represents the final stage of the log minimal model program for $\overline{M}_5$.

The main portion of the paper is devoted to a GIT stability analysis of nets of quadrics in $\mathbb{P}^4$. The GIT stability analysis for *pencils* of quadrics in $\mathbb{P}^4$ appears in [**AM99**], where it is shown that a pencil of quadrics in $\mathbb{P}^4$ is semi-stable if and only if the associated discriminant binary quintic is non-zero and has no triple roots, and in [**MM93**]. More generally a pencil of quadrics in $\mathbb{P}^n$ is semi-stable if and only if the associated discriminant binary $(n+1)$-form is non-zero and is GIT-semi-stable with respect to the natural SL(2)-action [**AM99**, Theorem 5]. The GIT analysis for nets of quadrics turns out to be more involved. In particular, as Remark 3.21 shows, there is no natural correspondence between SL(5)-stability of a net and SL(3)-stability of the associated discriminant quintic curve.

We prove that a semi-stable net defines a locally planar curve of genus 5 embedded in $\mathbb{P}^4$ by its dualizing sheaf, and give a description of the singularities occurring on such curves.

2010 *Mathematics Subject Classification.* Primary 14H10, 14L24, 14E30.

The first author was partially supported by NSF grant DMS-1259226. The second author was partially supported by NSF grant DMS-0901095.

MAIN THEOREM 1. *A net is semi-stable if and only if it defines a locally planar genus* 5 *curve satisfying one of the following conditions:*

(1) C is a reduced quadric section of a smooth quartic del Pezzo in $\mathbb{P}^4$, but C is not *one of the following:*
   - *(a) A union of an elliptic quartic curve and two conics meeting in a pair of triple points.*
   - *(b) A union of two elliptic quartics meeting along an A_5 and an A_1 singularities.*
   - *(c) A curve with a 4-fold point with two lines as its two branches.*
   - *(d) A union of two tangent conics and an elliptic quartic meeting the conics in a D_6 singularity and two nodes.*
   - *(e) A curve with a D_5 singularity such that the hyperelliptic involution of the normalization exchanges the points lying over the D_5 singularity.*
   - *(f) C contains a conic meeting the residual genus 2 component in an A_7 singularity and the attaching point of the genus 2 component is a Weierstrass point.*
   - *(g) A degeneration of one of the six curves listed above.*

(2) C is non-reduced and it degenerates isotrivially to one of the following curves:
   - *(a) The balanced ribbon, defined by*

$$(ac - b^2, ae - 2bd + c^2, ce - d^2). \tag{1.1}$$

   - *(b) A double twisted cubic meeting the residual conic in two points, defined by*

$$(ad - bc, ae - c^2 + L^2, be - cd), \tag{1.2}$$

     where L is a general linear form.
   - *(c) A double conic meeting the residual rational normal quartic in three points, defined by*

$$(ad - bc, ae - c^2 + bL_1 + dL_2, be - cd), \tag{1.3}$$

     where L_1 and L_2 are general linear forms. In particular, we have a semi-stable triple conic with two lines

$$(ad - bc, ae + bd - c^2, be - cd). \tag{1.4}$$

   - *(d) Two double lines joined by two conics, defined by*

$$(ad, ae + bd - c^2, be). \tag{1.5}$$

We should make a comment on the shortcomings of this result. While this theorem gives in principle a complete characterization of the singularities arising on curves in $\overline{M}^G$, it does not give a satisfactory description of the functor represented by $\overline{M}^G$. The difficulty is that a complete characterization of the functor of semi-stable curves necessarily involves the global geometry of the curves in question in a way that defies uniform description. For example, A_1 and A_5 singularities are generally allowed, except in the unique case when two elliptic quartics meet in A_1 and A_5 singularities. Similarly, D_5 singularities are generally allowed, except in the case when the hyperelliptic involution on the normalization exchanges points lying over the singularity.

As a by-product of our GIT analysis, we obtain a good understanding of the geometry of the birational map $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$. To state our first result in this direction, let us define the $A_5^{\{1\}}$-locus to be the locus of curves in $\overline{M}^G$ which can be expressed as the union of a genus 3 curve and a smooth rational curve meeting along an A_5 singularity. The significance of these curves lies in the fact that their stable limits are precisely curves in $\Delta_2 \subset \overline{M}_5$ with a genus 2 component attached at a non-Weierstrass point. Our main results regarding the birational geometry of f can now be summarized in the following theorem.

MAIN THEOREM 2. *The birational map $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ is a rational contraction, contracting the following divisors:*

(1) f contracts Δ_1 and exhibits the generic point of Δ_1 as a fibration over the A_2-locus in $\overline{M}^G$.
(2) f contracts Δ_2 and exhibits the generic point of Δ_2 as a fibration over the $A_5^{\{1\}}$-locus in $\overline{M}^G$.
(3) f contracts the trigonal divisor $\mathrm{Trig}_5 \subset \overline{M}_5$ *to the single point given by Equation* (1.4).

In addition, f flips various geometrically significant loci in the boundary of $\overline{M}_5$ to associated equisingular strata in $\overline{M}^G$, as summarized in Table 1. Detailed proofs of the assertions made in this table would take us rather far afield into the intricacies of stable reduction; thus, we leave these assertions without proof and merely offer the table as a guide to future exploration of f. We refer the reader to [**HM98**] for a beautiful introduction to stable reduction, [**Has00**] for the results concerning stable reduction of planar curve singularities, and the recent survey [**CM12**] for an in-depth guide to stable reduction.

Finally, we study the contraction $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ from the perspective of the minimal model program. Recall that

$$\overline{M}_5(\alpha) := \mathrm{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_5, \lfloor m(K_{\overline{\mathcal{M}}_5} + \alpha\delta)\rfloor), \quad \alpha \in [0,1]\cap \mathbb{Q},$$

and that as α decreases from 1, the corresponding birational models constitute the log minimal model program for $\overline{M}_5$ [**Has05, HH09, HH13**]. Our final result interprets the contraction $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ as the final step of this program.

MAIN THEOREM 3.
(1) The moving slope of $\overline{M}_5$ is 33/4*, realized by the divisor*

$$f^*\mathcal{O}(1) \sim 33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2,$$

where $\sim$ denotes the numerical proportionality.

(2) There is a natural isomorphism $\overline{M}^G \simeq \overline{M}_5(\alpha)$, for all $\alpha \in (3/8, 14/33] \cap \mathbb{Q}$, identifying $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ with the final step of the log MMP for $\overline{M}_5$. In particular, $\overline{M}_5(3/8)$ is a point.

In Table 1, we have listed the α-invariants, as defined in [**AFS10**], of some singularities appearing on curves parameterized by $\overline{M}_5(14/33)$ in order to indicate the anticipated threshold values of α at which the transformations should occur in the course of the log MMP for $\overline{M}_5$. The reader should also refer to [**AFS10**] for the definition of the notations $A_5^{\{1\}}$, $D_6^{\{1,2\}}$, $A_{3/4}$, $A_{3/5}$.

Table 1. Conjectural outline of the log MMP for $\overline{M}_5$.

α	Singularity Type	Locus in $\overline{M}_5(\alpha+\epsilon)$
9/11	A_2	elliptic tails attached nodally
7/10	A_3	elliptic bridges attached nodally
2/3	A_4	genus 2 tails attached nodally at a Weierstrass point
19/29	$A_5^{\{1\}}$	genus 2 tails attached nodally
19/29	$A_{3/4}$	genus 2 tails attached tacnodally at a Weierstrass point
12/19	$A_{3/5}$	genus 2 tails attached tacnodally
17/28	A_5	genus 2 bridges attached nodally at conjugate points
5/9	D_4	elliptic triboroughs attached nodally
5/9	D_5	genus 2 bridges attached nodally at a Weierstrass and free point
5/9	$D_6^{\{1,2\}}$ double lines	genus 2 bridges attached nodally at two free points
25/44	A_{10}, A_{11} ribbons	hyperelliptic curves
1/2	double twisted cubics, $D_8^{\{1,2\}}$	irreducible nodal curves with hyperelliptic normalization
14/33	4-fold point	genus 3 triboroughs
14/33	triple conics	trigonal curves

Note that while we now have a nearly complete description of the log MMP for $\overline{M}_4$ [**HL10, Fed12, CMJL12**], we have no construction of the intermediate models $\overline{M}_5(\alpha)$ for $14/33 < \alpha \leq 2/3$. Table 1 gives a rather ominous indication of the potential complexity of this task.

Let us now give a roadmap of the paper. In Section 2, we describe GIT-unstable nets of quadrics. We first describe a complete, finite set $\{\rho_i\}_{i=1}^{12}$ of destabilizing one-parameter subgroups (Theorem 2.1), and then provide geometric descriptions of the nets of quadrics destabilized by ρ_i for each $1 \leq i \leq 12$ (Theorem 2.10). In Section 3, we combine a geometric study of quartic surfaces in $\mathbb{P}^4$ with Theorem 2.10 to obtain a positive description of semi-stable nets of quadrics (Theorems 3.1 and 3.2). Finally, in Section 4 we use our semi-stability results to give proofs of Main Theorems 1, 2, 3.

Acknowledgements. We are profoundly grateful to Joe Harris for introducing us to the subject of algebraic geometry, for generously sharing his love of the discipline and his unique creative style. Our understanding and appreciation of Geometric Invariant Theory also owes a great deal to the papers and lectures of Brendan Hassett, David Hyeon, and Ian Morrison. We wish to thank each of these individuals, as well as Jarod Alper, Anand Deopurkar, David Jensen, Radu Laza,

and David Swinarski, for numerous conversations and suggestions relating to the contents of this paper.

2. GIT analysis

2.1. GIT preliminaries. Set $V := \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}(1))$ and let $W := \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}(2)) \simeq \mathrm{Sym}^2 V$ be the vector space of quadratic forms. To a net of quadrics $\Lambda = (Q_1, Q_2, Q_3)$ in W, we associate its *Hilbert point*

$$[\Lambda] := [Q_1 \wedge Q_2 \wedge Q_3] \in \mathbb{G}(3, W) \subset \mathbb{P} \bigwedge^3 W.$$

We denote by $\widetilde{[\Lambda]} := Q_1 \wedge Q_2 \wedge Q_3$ a lift of $[\Lambda]$ to $\bigwedge^3 W$.

Recall that Λ is said to be *semi-stable* if $0 \notin \mathrm{SL}(5) \cdot \widetilde{[\Lambda]}$, and *stable* if in addition $\mathrm{SL}(5) \cdot \widetilde{[\Lambda]}$ is closed. GIT gives a projective quotient $\mathbb{G}(3, W)^{ss} /\!/ \mathrm{SL}(5)$, where $\mathbb{G}(3, W)^{ss} \subset \mathbb{G}(3, W)$ is the open locus of semi-stable nets, and the main objective of this paper is to give a geometric description of $\mathbb{G}(3, W)^{ss}$.

The standard tool for such analysis is the Hilbert-Mumford numerical criterion [**MFK94**, Theorem 2.1]. In our situation, the statement of the numerical criterion may be formulated as follows: Let $\rho\colon \mathbb{C}^* \to \mathrm{SL}(5)$ be a one-parameter subgroup (1-PS), acting diagonally on a basis $\{a, b, c, d, e\}$ of V with weights $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}\}$, satisfying:

- $\bar{a} + \bar{b} + \bar{c} + \bar{d} + \bar{e} = 0$,
- $\bar{a} \geq \bar{b} \geq \bar{c} \geq \bar{d} \geq \bar{e}$,
- Not all weights $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}\}$ are 0.

We call such an action *normalized*. The basis $\{a, b, c, d, e\}$ induces a basis of $\bigwedge^3 W$, with Plücker coordinates as basis elements. The *ρ-weight* of a quadratic monomial $m = xy$ is $w_\rho(m) = \bar{x} + \bar{y}$ and the ρ-weight of a Plücker coordinate $m_1 \wedge m_2 \wedge m_3$ is $\sum_{i=1}^3 w_\rho(m_i)$. We say that a net Λ is *ρ-semi-stable* (resp., *ρ-stable*) if there exists a Plücker coordinate that does not vanish on $[\Lambda]$ with *non-negative* (resp., *positive*) ρ-weight. With this notation, the numerical criterion simply asserts that Λ is semi-stable (resp., stable) if and only if Λ is ρ-semi-stable (resp., ρ-stable) for all 1-PS's.

A priori, the numerical criterion requires one to check ρ-semi-stability for all 1-PS's. However, there necessarily exists a *finite* set of numerical types of 1-PS's $\{\rho_i\}_{i=1}^N$ such that the union of the ρ_i-unstable points is $\mathbb{G}(3, 15) \setminus \mathbb{G}(3, 15)^{ss}$. The first main result of this section, Theorem 2.1, describes such a set of 1-PS's explicitly. The second main result of this section, Theorem 2.10, gives geometric characterizations of the nets destabilized by each ρ_i in our list. Finally, in Section 3, we use this result to describe the semi-stable locus $\mathbb{G}(3, 15)^{ss} \subset \mathbb{G}(3, 15)$ explicitly.

2.2. Notation and conventions. Throughout this section, we use the following notation. Given a basis $\{a, b, c, d, e\}$ of V, we consider two orderings on the set of quadratic monomials in W. There is the lexicographic ordering, which is complete, and which we denote by $\succ_{lex}$. Then there is the ordering, denoted by $\geqslant$, according to which $m_1 \geqslant m_2$ if and only if $w_\rho(m_1) \geq w_\rho(m_2)$ for any normalized 1-PS acting diagonally on $\{a, b, c, d, e\}$. Note that

$$m_1 \geqslant m_2 \Longrightarrow m_1 \succeq_{lex} m_2.$$

Finally, given a normalized 1-PS acting on $\{a, b, c, d, e\}$, there is a complete ordering $\succ_\rho$ on the quadratic monomials in W defined as follows: $m_1 \succ_\rho m_2$ if and only if one of the following conditions hold:

(1) $w_\rho(m_1) > w_\rho(m_2)$
(2) $w_\rho(m_1) = w_\rho(m_2)$ and $m_1 \succ_{lex} m_2$.

For any quadric $Q \in W$, we let $in_{lex}(Q)$ denote the initial monomial of Q with respect to $\succ_{lex}$ and, if ρ is a normalized 1-PS acting on $\{a, b, c, d, e\}$, we let $in_\rho(Q)$ denote the initial monomial of Q with respect to $\succ_\rho$.

For any net Λ, we may choose a basis $\{Q_1, Q_2, Q_3\}$ such that $in_{lex}(Q_1) \succ_{lex} in_{lex}(Q_2) \succ_{lex} in_{lex}(Q_3)$. We call such a basis of Λ *normalized.*

Finally, given a basis $\{a, b, c, d, e\}$ of V, we define the *distinguished flag* $O \subset L \subset P \subset H \subset \mathbb{P}V$ as follows:

$$\begin{aligned} O &: b = c = d = e = 0, \\ L &: c = d = e = 0, \\ P &: d = e = 0, \\ H &: e = 0. \end{aligned}$$

2.3. Classification of destabilizing subgroups.

THEOREM 2.1. *Suppose that Λ is semi-stable with respect to every one-parameter subgroup of the following numerical types:*

(1) $\rho_1 = (1, 1, 1, 1, -4)$.
(2) $\rho_2 = (2, 2, 2, -3, -3)$.
(3) $\rho_3 = (3, 3, -2, -2, -2)$.
(4) $\rho_4 = (4, -1, -1, -1, -1)$.
(5) $\rho_5 = (3, 3, 3, -2, -7)$.
(6) $\rho_6 = (4, 4, -1, -1, -6)$.
(7) $\rho_7 = (9, 4, -1, -6, -6)$.
(8) $\rho_8 = (7, 2, 2, -3, -8)$.
(9) $\rho_9 = (12, 7, 2, -8, -13)$.
(10) $\rho_{10} = (9, 4, -1, -1, -11)$.
(11) $\rho_{11} = (14, 4, -1, -6, -11)$.
(12) $\rho_{12} = (13, 8, 3, -7, -17)$.

Then Λ is semi-stable.

REMARK 2.2. In fact, our proof gives a slightly stronger statement. Namely, Λ is semi-stable with respect to a fixed torus T if and only if it is semi-stable with respect to all one-parameter subgroups in T of the numerical types $\{\rho_i\}_{i=1}^{12}$.

Preliminary observations. Fix a net Λ which is ρ_i-semi-stable for each $\{\rho_i\}_{i=1}^{12}$. By the numerical criterion, to prove that Λ is semi-stable, it suffices to show that Λ is semi-stable with respect to an arbitrary 1-PS $\chi\colon \mathbb{C}^* \to \mathrm{SL}(5)$. Without loss of generality, we can assume that χ is normalized, acting diagonally on the basis $\{a, b, c, d, e\}$ with weights $(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e})$, satisfying $\bar{a} \geq \bar{b} \geq \bar{c} \geq \bar{d} \geq \bar{e}$. To prove the theorem, we must exhibit a Plücker coordinate that does not vanish on Λ and has non-negative χ-weight. More explicitly, if (Q_1, Q_2, Q_3) is a normalized basis of Λ, we must exhibit non-zero quadratic monomials m_1, m_2, m_3 in the variables $\{a, b, c, d, e\}$ which appear with non-zero coefficient in $Q_1 \wedge Q_2 \wedge Q_3$ and satisfy $w_\chi(m_1) + w_\chi(m_2) + w_\chi(m_3) \geq 0$. We begin with two preparatory lemmas.

Lemma 2.3. *The normalized basis* (Q_1, Q_2, Q_3) *of* Λ *satisfies the following:*

(i) $Q_3 \notin (e^2)$,
(ii) $(Q_1, Q_2, Q_3) \not\subset (d, e)$, *and either* $(Q_2, Q_3) \notin (d, e)$ *or* $Q_3 \notin (d, e)^2$.
(iii) $(Q_2, Q_3) \not\subset (c, d, e)^2$, *and either* $(Q_1, Q_2, Q_3) \not\subset (c, d, e)$ *or* $Q_3 \notin (c, d, e)^2$.
(iv) $in_{lex}(Q_1) = a^2$ *or* $in_{lex}(Q_1), in_{lex}(Q_2) \in (a)$.

Proof. (i), (ii), (iii), (iv) follow immediately from ρ_1, ρ_2, ρ_3, ρ_4-semi-stability of Λ, respectively. □

Lemma 2.4. *If* $\bar{b} \leq 0$, *then* Λ *is* χ*-semi-stable.*

Proof. First, suppose $in_{lex}(Q_1) = a^2$. Then $in_{lex}(Q_2) \geqslant be$ and $in_{lex}(Q_3) \geqslant de$ by Lemma 2.3(iii) and (i), respectively. In addition, ρ_2-semi-stability implies that if $Q_3 \in (d, e)^2$, then $Q_2 \notin (d, e)$. Thus, either $in_{lex}(Q_3) \geqslant ce$ or Q_2 contains a term $\geqslant c^2$. In the latter case, we obtain a Plücker coordinate of weight at least $2\bar{a} + 2\bar{c} + \bar{d} + \bar{e} = -2\bar{b} - \bar{d} - \bar{e} > 0$, since $\bar{b} \leq 0$ and $\bar{e} < 0$. In the former case, ρ_1-semi-stability implies that we cannot have $(Q_2, Q_3) \subset (e)$, so Q_2 or Q_3 contains a term $\geqslant d^2$. We obtain a Plücker coordinate of weight at least $2\bar{a} + 2\bar{d} + \bar{c} + \bar{e} = -2\bar{b} - \bar{c} - \bar{e} > 0$. Thus, Λ is χ-stable.

Next, suppose $in_{lex}(Q_1) \neq a^2$. Then we have $in_{lex}(Q_1) \geqslant ad$, $in_{lex}(Q_2) \geqslant ae$, $in_{lex}(Q_3) \geqslant de$ by Lemma 2.3(i) and (iv). If $in_{lex}(Q_2) \geqslant ad$, we are done since we have a Plücker coordinate of weight $2\bar{a} + \bar{c} + 2\bar{d} + \bar{e} = -2\bar{b} - \bar{c} - \bar{e} > 0$. Assume $in_{lex}(Q_2) = ae$. Then ρ_7-semi-stability implies that either Q_3 contains a term $\geqslant ce$ or $in_{lex}(Q_1) \geqslant ac$. In either case, we obtain a Plücker coordinate of weight at least $2\bar{a} + \bar{c} + \bar{d} + 2\bar{e} = -2\bar{b} - \bar{c} - \bar{d} \geq 0$. We conclude that Λ is χ-semi-stable. □

We can now begin the proof of the main theorem.

Proof of Theorem 2.1. We consider separately the following three cases:

(I) O is not in the base locus of Λ;
(II) O is in the base locus of Λ but L is not;
(III) L is in the base locus of Λ.

Case I: O is not a base point. We have $in_{lex}(Q_1) = a^2$. Lemma 2.3(iii) implies that $in_{lex}(Q_2) \geqslant be$. If Q_3 has a term $\geqslant cd$, then Λ is χ-stable because $2\bar{a} + (\bar{b} + \bar{e}) + (\bar{c} + \bar{d}) = \bar{a} > 0$. We assume that Q_3 has no term $\geqslant cd$. By ρ_5-semi-stability, Q_2 has a term $\geqslant cd$. Now, if Q_3 has a term $\geqslant be$, then Λ is again χ-stable. So we assume that Q_3 has no term $\geqslant be$.

First, assume Q_2 has a term $\geqslant bd$. If $in_{lex}(Q_3) = ce$, then Λ is χ-stable since $2\bar{a} + (\bar{b} + \bar{d}) + (\bar{c} + \bar{e}) = \bar{a} > 0$. Otherwise, $in_{lex}(Q_3) \in \{d^2, de\}$ and, by Lemma 2.3(ii), Q_2 must have a term $\geqslant c^2$. Thus, if Λ is not χ-semi-stable, we must have $2\bar{a} + (\bar{b} + \bar{d}) + (\bar{d} + \bar{e}) = \bar{a} + \bar{d} - \bar{c} < 0$ and $2\bar{a} + 2\bar{c} + (\bar{d} + \bar{e}) = \bar{a} + \bar{c} - \bar{b} < 0$. This is clearly impossible.

From now on, we suppose Q_2 has no term $\geqslant bd$ and $Q_3 \in (ce, d^2, de, e^2)$. Since Λ is ρ_6-semi-stable, Q_3 contains a d^2 term. If, in addition, $in_{lex}(Q_3) = ce$, then recalling that Q_2 has a term $\geqslant cd$, we have Plücker coordinates of weights at least

$$
\begin{aligned}
2\bar{a} + (\bar{c} + \bar{d}) + (\bar{c} + \bar{e}) &= \bar{a} + \bar{c} - \bar{b}, \\
2\bar{a} + (\bar{b} + \bar{e}) + 2\bar{d} &= \bar{a} + \bar{d} - \bar{c}, \\
2\bar{a} + (\bar{b} + \bar{e}) + (\bar{c} + \bar{e}) &= \bar{a} + \bar{e} - \bar{d}.
\end{aligned}
$$

The three expressions cannot be simultaneously non-positive, so Λ is χ-stable.

It remains to consider the case $in_{lex}(Q_3) = d^2$. By Lemma 2.3(ii), Q_2 contains a term $\geqslant c^2$. Thus, if Λ is not χ-semi-stable, we have $2\bar{a}+(\bar{b}+\bar{e})+2\bar{d} = \bar{a}+\bar{d}-\bar{c} < 0$ and $2\bar{a}+2\bar{c}+2\bar{d} = -2(\bar{b}+\bar{e}) < 0$. This is clearly impossible.

Case II: O is a base point but L is not in the base locus.

CLAIM 2.5. *Without loss of generality, we may assume Q_1, Q_2, Q_3 satisfy the following conditions:*

(1) $in_{lex}(Q_1), in_{lex}(Q_2) \in \{ab, ac, ad, ae\}$ and $Q_3 \in (b, c, d, e)^2$.
(2) Q_1 has a term $\geqslant b^2$, but Q_2, Q_3 have no term $\geqslant b^2$.
(3) $Q_3 \in (be, ce, d^2, de, e^2)$.
(4) Q_2 has a term $\geqslant cd$.

PROOF OF CLAIM. Indeed, (1) is immediate from Lemma 2.3(iv) using the assumption that O is a basepoint. If $Q_3 \notin (b, c, d, e)^2$, then $in_{lex}(Q_3) \geqslant ae$ and Λ is χ-stable as $(\bar{a}+\bar{c})+(\bar{a}+\bar{d})+(\bar{a}+\bar{e}) > 0$.

For (2), the assumption that L is not in the base locus implies that Q_1, Q_2, or Q_3 must have a term $\geqslant b^2$. We now deal with the case when Q_2 or Q_3 has a term $\geqslant b^2$. If Q_3 has a term $\geqslant b^2$, then Λ is χ-stable since $(\bar{a}+\bar{e})+(\bar{a}+\bar{d})+(2\bar{b}) = \bar{a}+\bar{b}-\bar{c} \geq \bar{a} > 0$. If Q_2 has a term $\geqslant b^2$, we consider two cases: If Q_3 has a term $\geqslant ce$, then we are done by Lemma 2.4 since $(\bar{a}+\bar{d})+2\bar{b}+(\bar{c}+\bar{e}) = \bar{b}$. Otherwise, $Q_3 \in (d, e)^2$. If $in_{lex}(Q_1) = ad$, then this contradicts ρ_7-semi-stability. Thus $in_{lex}(Q_1) \geqslant ac$. We are now done by Lemma 2.4 since $(\bar{a}+\bar{c})+(2\bar{b})+(\bar{d}+\bar{e}) = \bar{b}$.

Finally, to prove (3), we recall that $Q_3 \in (b, c, d, e)^2$. By Lemma 2.4, Q_3 is χ-semi-stable if it has a term $\geqslant cd$, as $(2\bar{b})+(\bar{a}+\bar{e})+(\bar{c}+\bar{d}) = \bar{b}$. $\square$

We subdivide the further analysis into six cases according to the initial monomials of $in_{lex}(Q_1)$ and $in_{lex}(Q_2)$.

Case II.1: $in_{lex}(Q_1) = ad$ and $in_{lex}(Q_2) = ae$. By (2) Q_2 has no term $\geqslant b^2$. Since Λ is $\rho_{11} = (14, 4, -1, -6, -11)$-semi-stable, we see that $Q_3 \notin (ce, d^2, de, e^2)$. It follows by (3) that $in_{lex}(Q_3) = be$.

By $\rho_8 = (7, 2, 2, -3, -8)$-semi-stability, Q_2 has a term $\geqslant c^2$. We now consider two subcases, according to whether Q_3 has a d^2 term.

If Q_3 has a d^2 term, we have Plücker coordinates of χ-weights

$$\begin{aligned}(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+(\bar{b}+\bar{e}) &= \bar{a}+\bar{e}-\bar{c},\\ (\bar{a}+\bar{d})+2\bar{c}+(\bar{b}+\bar{e}) &= \bar{c},\\ 2\bar{b}+2\bar{c}+2\bar{d} &= -2(\bar{a}+\bar{e}).\end{aligned}$$

Evidently, these expressions cannot be simultaneously negative, so Λ is χ-semi-stable.

If Q_3 has no term $\geqslant d^2$, then by $\rho_{10} = (9, 4, -1, -1, -11)$-semi-stability, Q_2 has a term $\geqslant bd$. Thus, we have Plücker coordinates with χ-weights at least

$$\begin{aligned}(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+(\bar{b}+\bar{e}) &= \bar{a}+\bar{e}-\bar{c},\\ (\bar{a}+\bar{d})+2\bar{c}+(\bar{b}+\bar{e}) &= \bar{c},\\ (\bar{a}+\bar{d})+(\bar{b}+\bar{d})+(\bar{b}+\bar{e}) &= \bar{b}+\bar{d}-\bar{c}.\end{aligned}$$

Evidently, these expressions cannot be simultaneously negative so Λ is χ-semi-stable.

Case II.2: $in_{lex}(Q_1) = ac$ and $in_{lex}(Q_2) = ae$. Since Q_2 has no b^2 term, ρ_7-semi-stability implies $in_{lex}(Q_3) \geqslant ce$.

Suppose first Q_2 has a term $\geqslant bd$. By ρ_{10}-semi-stability, either Q_3 has a be term or Q_3 has a d^2 term. If Q_3 has a be term, then Λ is semi-stable by Lemma 2.4 since $(\bar{a}+\bar{c})+(\bar{b}+\bar{d})+(\bar{b}+\bar{e}) = \bar{b}$.

If Q_3 has a d^2 term, then we have Plücker coordinates of weights at least

$$\begin{aligned}(\bar{a}+\bar{c})+(\bar{a}+\bar{e})+(\bar{c}+\bar{e}) &= -2(\bar{b}+\bar{d}),\\ (\bar{a}+\bar{c})+(\bar{b}+\bar{d})+(\bar{c}+\bar{e}) &= \bar{c},\\ 2\bar{b}+(\bar{a}+\bar{e})+2\bar{d} &= \bar{b}+\bar{d}-\bar{c}.\end{aligned}$$

Evidently, these cannot all be negative so Λ is χ-semi-stable.

Suppose now Q_2 has no term $\geqslant bd$. Then by ρ_5-semi-stability Q_2 has a term $\geqslant cd$ and by ρ_{10}-semi-stability Q_3 has a d^2 term.

Finally, by $\rho_{12} = (13, 8, 3, -7, -17)$-semi-stability, either Q_2 has a c^2 term or Q_3 has be term.

If Q_3 has be term, recalling that Q_1 has a b^2 term, we have Plücker coordinates of weights at least

$$\begin{aligned}(\bar{a}+\bar{c})+(\bar{a}+\bar{e})+(\bar{b}+\bar{e}) &= \bar{a}+\bar{e}-\bar{d},\\ (\bar{a}+\bar{c})+(\bar{a}+\bar{e})+(2\bar{d}) &= \bar{a}+\bar{d}-\bar{b},\\ (\bar{a}+\bar{c})+(\bar{c}+\bar{d})+(\bar{b}+\bar{e}) &= \bar{c}.\\ 2\bar{b}+(\bar{a}+\bar{e})+(2\bar{d}) &= \bar{b}+\bar{d}-\bar{c}.\end{aligned}$$

One of these is non-negative so Λ is χ-semi-stable.

If Q_2 has a c^2 term, then we have Plücker coordinates of weights at least

$$\begin{aligned}(\bar{a}+\bar{c})+(\bar{a}+\bar{e})+(\bar{c}+\bar{e}) &= -2(\bar{b}+\bar{d}),\\ 2\bar{b}+(\bar{a}+\bar{e})+2\bar{d} &= \bar{b}+\bar{d}-\bar{c},\\ 2\bar{b}+2\bar{c}+2\bar{d} &= -2(\bar{a}+\bar{e}).\end{aligned}$$

One of these is non-negative so Λ is χ-semi-stable.

Case II.3: $in_{lex}(Q_1)=ac$ and $in_{lex}(Q_2)=ad$. By ρ_7-semi-stability, $in_{lex}(Q_3) \geqslant ce$. Hence, there is a Plücker coordinate of weight at least $2\bar{b}+(\bar{a}+\bar{d})+(\bar{c}+\bar{e}) = \bar{b}$ and we are done by Lemma 2.4.

Case II.4: $in_{lex}(Q_1) = ab$ and $in_{lex}(Q_2) = ae$. By ρ_5-semi-stability, Q_2 has a term $\geqslant cd$. If $in_{lex}(Q_3) = be$, then we are done by Lemma 2.4 as $(\bar{a}+\bar{b})+(\bar{c}+\bar{d})+(\bar{b}+\bar{e}) = \bar{b}$. Assume $Q_3 \in (ce, d^2, de, e^2)$. We consider the following three subcases cases:

Suppose Q_3 has d^2 term but no ce term. Then Q_2 has a term $\geqslant c^2$ by Lemma 2.3(ii). We have Plücker coordinates with weights at least $(\bar{a}+\bar{b})+(\bar{a}+\bar{e})+2\bar{d} = \bar{a}+\bar{d}-\bar{c}$ and $(\bar{a}+\bar{b})+2\bar{c}+2\bar{d} = \bar{c}+\bar{d}-\bar{e} \geq \bar{c}$. Evidently, these cannot both be negative.

Suppose Q_3 has both d^2 and ce terms. We have Plücker coordinates with weights $(\bar{a}+\bar{b})+(\bar{a}+\bar{e})+2\bar{d} = \bar{a}+\bar{d}-\bar{c}$ and $(\bar{a}+\bar{b})+(\bar{c}+\bar{d})+(\bar{c}+\bar{e}) = \bar{c}$. Evidently, these cannot both be negative.

Finally, suppose Q_3 has no d^2 term. Then by ρ_6-semi-stability Q_2 has a term $\geqslant bd$. If Q_3 has a ce term, then we are done by Lemma 2.4 as $(\bar{a}+\bar{b})+(\bar{b}+\bar{d})+(\bar{c}+\bar{e}) = \bar{b}$. We may now assume $Q_3 = de$. By Lemma 2.3(ii), Q_2 has a term $\geqslant c^2$. Then we have Plücker coordinates with weights at least

$$(\bar{a}+\bar{b})+(\bar{a}+\bar{e})+(\bar{d}+\bar{e})=\bar{a}+\bar{e}-\bar{c},$$
$$(\bar{a}+\bar{b})+2\bar{c}+(\bar{d}+\bar{e})=\bar{c},$$
$$(\bar{a}+\bar{b})+(\bar{b}+\bar{d})+(\bar{d}+\bar{e})=\bar{b}+\bar{d}-\bar{c}.$$

Evidently, these cannot be simultaneously negative.

Case II.5: $in_{lex}(Q_1)=ab$ and $in_{lex}(Q_2)=ad$. If Q_3 has a term $\geqslant ce$, then Λ is χ-stable since $(\bar{a}+\bar{b})+(\bar{a}+\bar{d})+(\bar{c}+\bar{e})=\bar{a}>0$. Assume $Q_3\in(d,e)^2$. By Lemma 2.3(ii), Q_2 has a term $\geqslant c^2$. So we have Plücker coordinates of weights at least $(\bar{a}+\bar{b})+2\bar{c}+(\bar{d}+\bar{e})=\bar{c}$ and $(\bar{a}+\bar{b})+(\bar{a}+\bar{d})+(\bar{d}+\bar{e})=\bar{a}+\bar{d}-\bar{c}$. Evidently, these cannot be simultaneously negative.

Case II.6: Finally, if $in_{lex}(Q_1)=ab$ and $in_{lex}(Q_2)=ac$, then Λ is χ-stable since $(\bar{a}+\bar{b})+(\bar{a}+\bar{c})+(\bar{d}+\bar{e})=\bar{a}>0$.

Case III: L is in the base locus.

CLAIM 2.6. *Without loss of generality, we may assume Q_1,Q_2,Q_3 satisfy the following conditions:*

(1) $in_{lex}(Q_1),in_{lex}(Q_2)\in\{ac,ad,ae\}$.
(2) $in_{lex}(Q_3)\in\{bd,be\}$.

PROOF. Indeed, (1) follows from Lemma 2.3(iv) using $Q_1,Q_2\in(c,d,e)$. For (2), note that Lemma 2.3(iii) implies $Q_3\notin(c,d,e)^2$, and that Λ is χ-stable if either $in_{lex}(Q_3)\geqslant bc$ or $in_{lex}(Q_3)\geqslant ae$. □

We consider three cases according to the initial monomials $in_{lex}(Q_1)$ and $in_{lex}(Q_2)$:

Case III.1: Suppose $in_{lex}(Q_1)=ac$, $in_{lex}(Q_2)=ad$. Then Λ is χ-stable since $(\bar{a}+\bar{c})+(\bar{a}+\bar{d})+(\bar{b}+\bar{e})=\bar{a}>0$.

Case III.2: Suppose $in_{lex}(Q_1)=ac$, $in_{lex}(Q_2)=ae$. If $in_{lex}(Q_3)=bd$, then Λ is χ-stable. Assume $in_{lex}(Q_3)=be$. By ρ_6-semi-stability, Q_2 has a term $\geqslant bd$. Since $(\bar{a}+\bar{c})+(\bar{b}+\bar{d})+(\bar{b}+\bar{e})=\bar{b}$, we are done by Lemma 2.4.

Case III.3: $in_{lex}(Q_1)=ad$, $in_{lex}(Q_2)=ae$. We consider separately two subcases: $in_{lex}(Q_3)=bd$ and $in_{lex}(Q_3)=be$.

Case III.3(a): If $in_{lex}(Q_3)=bd$, there is a Plücker coordinate of weight $(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+(\bar{b}+\bar{d})=\bar{a}+\bar{d}-\bar{c}$. Thus, Λ will be semi-stable if we can find a Plücker coordinate of weight at least $\bar{c}$. By Lemma 2.3(ii), one of Q_1,Q_2,Q_3 contains a term $\geqslant c^2$. If Q_3 contains a term $\geqslant c^2$, then we have a Plücker coordinate of weight at least $(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+2\bar{c}=\bar{a}-\bar{b}+\bar{c}\geq\bar{c}$. If Q_2 contains a term $\geqslant c^2$, then we have a Plücker coordinate of weight $(\bar{a}+\bar{d})+2\bar{c}+(\bar{b}+\bar{d})=\bar{c}+\bar{d}-\bar{e}\geq\bar{c}$. If Q_1 contains a term $\geqslant c^2$, then we have a Plücker coordinate of weight $(2\bar{c})+(\bar{a}+\bar{e})+(\bar{b}+\bar{d})=\bar{c}$. We are done.

Case III.3(b): If $in_{lex}(Q_3)=be$, we use ρ_6-semi-stability to see that Q_2 has a term $\geqslant bd$. Therefore, there is a Plücker coordinate of weight $(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+(\bar{b}+\bar{e})=\bar{a}+\bar{e}-\bar{c}$ and a Plücker coordinate of weight at least $(\bar{a}+\bar{d})+(\bar{b}+\bar{d})+(\bar{b}+\bar{e})=\bar{b}+\bar{d}-\bar{c}$. To prove semi-stability, it suffices to exhibit a Plücker coordinate of weight at least $\bar{c}$.

By Lemma 2.3(ii), one of Q_1,Q_2,Q_3 contains a term $\geqslant c^2$. If Q_3 contains a term $\geqslant c^2$, then we have a Plücker coordinate of weight $(\bar{a}+\bar{d})+(\bar{a}+\bar{e})+2\bar{c}=\bar{a}+\bar{c}-\bar{b}\geq\bar{c}$. If Q_2 contains a term $\geqslant c^2$, then we have a Plücker coordinate of weight

$(\bar{a}+\bar{d})+(2\bar{c})+(\bar{b}+\bar{e})=\bar{c}$. It remains to consider the case when only Q_1 has a term $\geqslant c^2$. By ρ_8-semi-stability, Q_3 has a term $\geqslant cd$, and by ρ_9-semi-stability Q_1 has a term $\geqslant bc$. We obtain a Plücker coordinate with weight $(\bar{b}+\bar{c})+(\bar{a}+\bar{e})+(\bar{c}+\bar{d})=\bar{c}$. At last, we are done. □

2.4. Classification of Unstable Points. In this section, we give geometric description of the strata in $\mathbb{G}(3,15)$ destabilized by each of the 1-PS's enumerated in Theorem 2.1. For ease of exposition, we analyze the first four 1-PS's separately from the final eight.

LEMMA 2.7. *A net Λ is destabilized by one of $\{\rho_i\}_{i=1}^4$ iff it satisfies one of the following conditions with respect to a distinguished flag $O\subset L\subset P\subset H\subset\mathbb{P}^4$.*

(1) $\rho_1=(1,1,1,1,-4)$*:*
   (a) A pencil of Λ contains H, or
   (b) An element of the net is singular along H.
(2) $\rho_2=(2,2,2,-3,-3)$*:*
   (a) Λ contains P, or
   (b) A pencil of Λ contains P, and an element of the pencil is singular along P.
(3) $\rho_3=(3,3,-2,-2,-2)$*:*
   (a) Λ contains L, and an element of Λ is singular along L, or
   (b) A pencil of Λ is singular along L.
(4) $\rho_4=(4,-1,-1,-1,-1)$*:*
   (a) Λ contains O, and a pencil of Λ is singular at O.

PROOF. In case (4), the only triple of initial ρ_4-weights with negative sum is $(3,-2,-2)$ and $(-2,-2,-2)$. However, the stratum of nets with initial weights $(-2,-2,-2)$ is in the closure of the stratum of nets with initial weights $(3,-2,-2)$. Evidently, any quadric of weight 3 contains O, while any quadric of weight -2 is singular at O. Thus, the net with initial ρ_4-weights $(3,-2,-2)$ has a base point at O and contains a pencil of quadrics singular at O. The proofs of cases (1)–(3) are similar. □

On the basis of this partial analysis, we may already conclude the important fact that a semi-stable net has a pure one-dimensional intersection, and hence defines a connected curve with local complete intersection singularities.

COROLLARY 2.8. *If a net of quadrics in $\mathbb{P}^4$ is semi-stable, then the corresponding intersection is connected and purely one-dimensional.*

PROOF. Fulton-Hansen connectedness theorem [**FH79**] gives the first statement. If the intersection fails to be purely one-dimensional, then either a pencil of quadrics in the net contains a hyperplane, in which case the net is destabilized by ρ_1, or we may choose a basis $\{Q_1,Q_2,Q_3\}$ of the net, such that $S:=Q_1\cap Q_2$ is a quartic surface and one of its irreducible components is contained entirely in Q_3. Because $R_S:=\mathbb{C}[a,b,c,d,e]/(Q_1,Q_2)$ has dimension 3, R_S is Cohen-Macaulay and so the ideal (Q_1,Q_2) is saturated. Thus S cannot lie entirely inside Q_3. We conclude that there must be an irreducible component $S'\subset S$ of degree at most 3 which is contained in Q_3. If $\deg S'=1$, then the net contains a plane and is thus destabilized by ρ_2. If $\deg S'=2$, then the span of S' is a hyperplane and a pencil of quadrics in the net contains this hyperplane. Such a net is destabilized by ρ_1.

Finally if $\deg S' = 3$, then the classical classification of surfaces of minimal degree due to del Pezzo [**dP85**] implies that S' is a *rational normal scroll*; see [**EH87**] for a modern proof of this result and [**Har92**] for an introduction to scrolls. We have two cases to consider. If S' is smooth, then the net is projectively equivalent to $(ad-bc, ae-bd, ce-d^2)$ (see [**Har92**, Lecture 9]) and is destabilized by ρ_3. If S' is singular, then it must be a cone over a rational normal cubic curve. If O denotes the vertex of the cone, then we must have a pencil of quadrics singular at O. Such a net is destabilized by ρ_4.

□

The following lemma is an unenlightening but straightforward combinatorial stepping stone to the geometric analysis in Theorem 2.10.

LEMMA 2.9. *Suppose Λ is ρ_k-unstable for $k \in \{5, \dots, 12\}$ but is ρ_j-semi-stable for $1 \leq j \leq k-1$. Let m_1, m_2, m_3 be the initial monomials of Λ with respect to ρ_k. Then $(w_{\rho_k}(m_1), w_{\rho_k}(m_2), w_{\rho_k}(m_3))$ must be one of the following triples:*

(5) $\rho_5 = (3, 3, 3, -2, -7)$:
- $(6, -4, -4)$

(6) $\rho_6 = (4, 4, -1, -1, -6)$:
- $(8, -2, -7)$
- $(3, -2, -2)$

(7) $\rho_7 = (9, 4, -1, -6, -6)$:
- $(8, 3, -12)$

(8) $\rho_8 = (7, 2, 2, -3, -8)$:
- $(4, -1, -6)$

(9) $\rho_9 = (12, 7, 2, -8, -13)$:
- $(4, -1, -6)$

(10) $\rho_{10} = (9, 4, -1, -1, -11)$:
- $(8, -2, -7)$

(11) $\rho_{11} = (14, 4, -1, -6, -11)$:
- $(8, 3, -12)$

(12) $\rho_{12} = (13, 8, 3, -7, -17)$:
- $(16, -4, -14)$

PROOF. The proof is purely algorithmic. Consider $\rho_k = (\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e})$ for $5 \leq k \leq 12$ and suppose $w_1 \geq w_2 \geq w_3$ is the triple of ρ_k-initial weights of a ρ_k-unstable net Λ. Lemma 2.3 translates into the following conditions satisfied by w_1, w_2, w_3:

(C1) If $\bar{d} \neq \bar{e}$, then $w_3 > 2\bar{e}$.
(C2) $w_1 \geq 2c$. Moreover, if $w_2 < 2\bar{c}$, then $w_3 \geq \bar{c} + \bar{e}$.
(C3) $w_2 \geq \bar{b} + \bar{e}$. Moreover, if $w_1 < 2\bar{b}$, then $w_2 \geq \bar{a} + \bar{e}$ and $w_3 \geq \bar{b} + \bar{e}$.
(C4) If $w_1 \neq 2\bar{a}$, then $w_1 \geq \bar{a} + \bar{d}$ and $w_2 \geq \bar{a} + \bar{e}$.

Now for each ρ_k, we list all triples of ρ_k-initial weights that have negative sum and satisfy (C1)–(C4). We will do only Case (12), by far the most involved, and leave the rest as an exercise to the reader.

The set of possible $\rho_{12} = (13, 8, 3, -7, -17)$-weights of quadratic monomials is

$$\{26, 21, 16, 11, 6, 1, -4, -9, -14, -24, -34\}.$$

Suppose $w_1 \geq w_2 \geq w_3$ are initial ρ_{12}-weights of ρ_{12}-unstable Λ and Λ is ρ_i-stable for $i = 1, \dots, 4$. By (C1), $w_3 \geq -24$. If $w_1 = 26$, then the triples with negative sum are $(26, -14, -14)$, which violates (C3), and $(26, -4, -24)$, which violates (C2).

Suppose $w_1 < 26$. Then $w_1 \geq 6$ and $w_2 \geq -4$ by (C4). The triples with negative sum satisfying these conditions are

- $(21, 1, -24)$, which violates (C2);
- $(16, -4, -14)$;
- $(16, 6, -24)$;
- $(11, -4, -9)$;
- $(11, 1, -14)$, which violates (C3);
- $(6, -4, -4)$;
- $(6, 1, -9)$;
- $(6, 6, -14)$, which violates (C3).

Finally, one can easily check that the following statements hold: A net with ρ_{12}-initial weights $(16, 6, -24)$ is $\rho_7 = (9, 4, -1, -6, -6)$-unstable. A net with ρ_{12}-initial weights $(11, -4, -9)$ is $\rho_8 = (7, 2, 2, -3, -8)$-unstable. A net with ρ_{12}-initial weights $(6, -4, -4)$ is $\rho_9 = (12, 7, 2, -8, -13)$-unstable. A net with ρ_{12}-initial weights $(6, 1, -9)$ is destabilized by $\rho_8 = (7, 2, 2, -3, -8)$ after the coordinate change $c \leftrightarrow b$. $\square$

THEOREM 2.10. *A curve C which is a complete intersection of three quadrics is unstable if and only if it is (a degeneration of) one of the following curves:*

(1) C is a double structure on an elliptic quartic curve in $\mathbb{P}^3$.
(2) C consists of a union of a double conic and two conics.
(3) C has a non-reduced structure along a line L and the residual curve C' meets L in at least $\deg C' - 2$ points.
(4) C has a point O with a three dimensional Zariski tangent space, i.e. C is not locally planar.
(5) C contains a degenerate double structure on a conic, i.e. the double structure is contained in $\mathbb{P}^3$.
(6)
   (a) C consists of a union of an elliptic quartic curve and two conics meeting along a pair of triple points.
   (b) C contains a double line that meets the residual arithmetic genus one component in three points.
(7) C consists of two elliptic quartics meeting in an A_5 singularity and a node.
(8) C contains a planar 4-fold point whose two branches are lines.
(9) C contains a double line and the residual genus two curve is tangent to it.
(10) C consists of two tangent conics and an elliptic quartic meeting the conics in a D_6 singularity and two nodes.
(11) C has D_5 singularity and the hyperelliptic involution on the normalization of C exchanges the points lying over the D_5 singularity.
(12) C contains a conic meeting the residual genus two component in an A_7 singularity and the attaching point is a Weierstrass point on the genus two component.

PROOF. For each of $\{\rho_i\}_{i=1}^{12}$, we shall give a geometric description of the ρ_i-unstable stratum. Suppose C is a 1-dimensional complete intersection of three quadrics and let Λ be its homogeneous ideal. The analysis for $\{\rho_i\}_{i=1}^{4}$ proceeds via Lemma 2.7 (note that parts (1a) and (2a) of the lemma do not apply as they describe intersections with a higher-dimensional component).

(1) By Lemma 2.7 (1b), a curve C is ρ_1-unstable if and only if Λ contains a double hyperplane. If Λ contains a double hyperplane, then C is a non-reduced curve with a double structure along an elliptic quartic curve in $\mathbb{P}^3$. Conversely, given a curve C with a double structure along a necessarily degenerate elliptic quartic, let H be the hyperplane containing C_{red}. Since the restriction of Λ to H is at most two dimensional, we must have an element $Q \in \Lambda$ which contains H. If $\operatorname{rank} Q = 2$, then our curve would be a reducible union of two degenerate quartic curves. We conclude that $\operatorname{rank} Q = 1$, and so Λ contains a double hyperplane.

(2) By Lemma 2.7 (2b), C is ρ_2-unstable if and only if a pencil of Λ contains a plane P and an element of the pencil is singular along P. Let Q_2, Q_3 generate the pencil, with Q_3 singular along P. We have $Q_3 = H_1 \cup H_2$, a union of two hyperplanes, with $P = H_1 \cap H_2$. Since $P \subset Q_2$, we must have $Q_2 \cap H_1 = P \cup P_1$ and $Q_2 \cap H_2 = P \cup P_2$ where P_1 and P_2 are planes. In sum, $Q_2 \cap Q_3 = P_1 \cup P \cup P_2$, where the plane P occurs in the intersection with multiplicity two. It follows that $C = Q_1 \cap Q_2 \cap Q_3$ consists of the union of two conics ($Q_1 \cap P_1$ and $Q_1 \cap P_2$) and a double conic ($Q_1 \cap P$). Conversely, given such a curve, if we let H_1 and H_2 denote the hyperplanes spanned by each reduced conic with the double conic, then $H_1 \cup H_2$ contains the curve, so we recover an element Q_3 of the net singular along P, the span of the double conic. Furthermore, since all elements of the net contain the double conic, the quadrics containing P form a pencil.

(3) By Lemma 2.7 (3), C is ρ_3-unstable in two cases:

(a) The net contains a line L and there is a quadric singular along L.
(b) There is a pencil of quadrics singular along a line L.

Suppose there is a pencil of quadrics singular along a line L. Let $O' \in L \cap Q_3$. Then O' is a base point of the net and a pencil of the net is singular at O'. It follows by Lemma 2.7 (4) that the net is destabilized by ρ_4. Hence, it suffices to consider the case when the net contains L and there is a quadric Q_3 singular along L. Then C is generically non-reduced along L. Let C' be the subcurve of C residual to L. Note that every hyperplane containing L intersects Q_3 in two planes and the intersection of $Q_1 \cap Q_2$ with each of these planes is the union of L and at most a single other point. It follows that every hyperplane containing L intersects C' in at least $\deg C' - 2$ points lying on L. Therefore, C' meets L in $\deg C' - 2$ points.

Conversely, suppose C has a multiple structure along L and the residual curve C' intersects L in $\deg C' - 2$ points. Then the projection of C' away from L is a conic. It follows that $C' \setminus L$ lies on a rank 3 quadric Q singular along L. Finally, a non-reduced component supported on L lies on Q. Since C is Cohen-Macaulay, it follows that C lies on Q.

(4) By Lemma 2.7 (4), C is ρ_4-unstable if and only if O is a base point of the net, and the net contains a pencil of quadrics singular at O. If we choose generators Q_1, Q_2, Q_3 with Q_2, Q_3 singular at O, then $T_O C = T_O Q_1 \cap T_O Q_2 \cap T_O Q_3 = T_O Q_1$ implies $\dim T_O C \geq 3$. Conversely, if $O \in C$ such that $\dim T_O C \geq 3$, then O is a base point of the net and there is a pencil of quadrics singular at O.

Consider now $\{\rho_i\}_{i=5}^{12}$. For each triple of ρ_i-initial weights from Lemma 2.9, the locus of nets having these initial weights is an irreducible locally closed set. In what follows we describe the generic point of each of them.

(5) $\rho_5 = (3, 3, 3, -2, -7)$. By Lemma 2.9, it suffices to consider a ρ_5-unstable net with initial ρ_5-weights $(6, -4, -4)$. Such a net has generators (Q_1, Q_2, Q_3) such

that $Q_2 = d^2 + eL$, where L is a linear form, and $Q_3 \in (e)$. Evidently, C contains the double conic (Q_1, e, d^2) contained in the hyperplane (e).

Conversely, if C contains a double structure on a conic contained in a hyperplane H, then we may take (e) to be the ideal of H and (d, e) to be the ideal of the plane spanned by the underlying conic. It is clear that the restriction of Λ to H must contain the double plane d^2. Thus the ideal of C contains a quadric in (e) and a linearly independent quadric in $(d^2) + (e)$. It follows that C is destabilized by ρ_5.

(6) $\rho_6 = (4, 4, -1, -1, -6)$. There are two possible triples of initial ρ_6-weights: $(8, -2, -7)$ and $(3, -2, -2)$.

If the initial weights are $(8, -2, -7)$, then there is a basis of Λ of the form (Q_1, Q_2, Q_3), where $Q_2 \in (c, d)^2 + (e)$ and $Q_3 = eL(c, d, e)$. If we let H' be the hyperplane $L(c, d, e) = 0$, then $H' \cap Q_1 \cap Q_2$ is an elliptic normal curve, while $H \cap Q_1 \cap Q_2$ is a pair of conics. All three components meet in the two points of $L \cap Q_1$.

Conversely, given a curve C of this form, let H' be the hyperplane spanned by the elliptic normal curve, and H the hyperplane spanned by the pair of conics. Let $d = 0$ and $e = 0$ be the equations of H' and H, respectively. Then $Q_3 := de \in \Lambda$, and the restriction of Λ to H contains a rank 2 quadric singular along a line contained in H'. Thus we can choose the coordinate c so that the ideal of C can be written as (Q_1, Q_2, Q_3), where $Q_2 \in (c, d)^2 + (e)$. Thus C is destabilized by ρ_6.

If the initial weights are $(3, -2, -2)$, then the net is generated by quadrics $Q_1 \in (c, d, e)$ and $Q_2, Q_3 \in (c, d)^2 + (e)$. For a general such net, we can choose coordinates so that $(Q_2, Q_3) = (ae + c^2, be + d^2)$. This pencil cuts out a Veronese quadric with a double line along $c = d = e = 0$ (cf. Lemma 3.5). Being a quadric section of this Veronese, C must be a union of a double line and an elliptic sextic meeting the double line in three points.

Conversely, suppose C has a double line component meeting the residual component of arithmetic genus one in three points. Take a quadric in the net with a vertex on the double line and let $e = 0$ be the tangent hyperplane to this quadric. Then the scheme-theoretic intersection of C with $e = 0$ is a double line in $\mathbb{P}^3$. Assuming that the line is $c = d = e = 0$, we conclude that in appropriately chosen coordinates the net is (Q_1, Q_2, Q_3), where $Q_1 \in (c, d, e)$, $Q_2 = ae + c^2$, and $Q_3 = be + d^2$. Such a net is destabilized by ρ_6.

(7) $\rho_7 = (9, 4, -1, -6, -6)$. By Lemma 2.9, we have only need to consider initial weights $(8, 3, -12)$. Then the generators of the net can be written as

$$\begin{aligned} Q_1 &= ac + b^2 + c^2 \mod (d, e), \\ Q_2 &= ad + bc \mod (c, d, e)^2, \\ Q_3 &= de. \end{aligned}$$

The two elliptic quartics are $Q_1 = Q_2 = d = 0$ and $Q_1 = Q_2 = e = 0$. Dehomogenizing with respect to a, we see that locally at O, we have $c = b^2 + R_1$, $d = bc + R_2 = b^3 + bR_1 + R_2$, and $de = 0$. This translates into $(e - nb^3)e = 0$ locally at O, which is an A_5 singularity. Restricting to $d = e = 0$, we see that the two elliptic quartics intersect at O and one other point. The claim follows.

(8) $\rho_8 = (7, 2, 2, -3, -8)$. A general net with ρ_8-initial weights $(4, -1, -6)$ has generators $Q_1 = ad + \overline{Q}_1(b, c, d, e)$, $Q_2 = ae + dL_1(b, c, d) + eL_2(b, c, d, e)$, and

$Q_3 = eL_3(b,c,d,e) + d^2$. After an appropriate change of variables, the generators can be rewritten as

$$Q_1 = ad + \overline{Q}_1(b,c,d,e),$$
$$Q_2 = ae + bd,$$
$$Q_3 = ce + d^2.$$

Note that $\{d = e = 0\} \cap C = \{\overline{Q}_1(b,c) = 0\}$ is the union of two lines meeting at O. From the above, we deduce that $d = R(b,c)$ and $e = bd = bR(b,c)$ for some power series $R(b,c)$ with a quadratic initial form. Now $ce + d^2 = 0$ translates into

$$R(b,c)(bc + R(b,c)) = 0,$$

which defines an ordinary 4-fold planar point.

Conversely, suppose Λ is a net defining a curve C with a 4-fold planar point O whose two branches are lines. Let L_1, L_2 be the lines and C' be the residual sextic. Then C' has geometric genus one and a node at O. Let $\pi\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from O, Set $C'' = \pi(C')$ and $p_i = \pi(L_i)$ for $i = 1,2$. Let p_3 and p_4 the images of the tangent lines to the branches of C' at O. Then C'' is a genus one quartic in $\mathbb{P}^3$ and p_1, p_2 are points lying on its chord $\overline{p_3p_4}$. Being a genus one quartic, C'' lies on a pencil of quadrics in $\mathbb{P}^3$ and hence there is a quadric in $\mathbb{P}^3$ containing C'' together with the 4 collinear points p_1, p_2, p_3, p_4. This gives a rise to a singular quadric Q_3 in $\mathbb{P}^4$ that has a vertex at O, contains C'', and contains the plane P spanned by L_1 and L_2. Since the quadrics in Λ containing P form a pencil, we conclude that there is a quadric $Q_2 \in \Lambda$ that contains P and which is linearly independent with Q_3.

Summarizing, we can choose coordinates so that P is given by $d = e = 0$ and find a basis of Λ consisting of $Q_1 = ad + \overline{Q}_1(b,c,d,e), Q_2 = ae + bd, Q_3 = dL + eM$, where L and M are linear forms in (b,c,d,e). The resulting local analytic equation at O is $R(b,c)(bM(b,c,d,e) - L(b,c,d,e)) = 0$, where $R(b,c)$ is power series with a quadratic initial form. This equation defines a triple point unless $L(b,c,d,e) \in (d,e)$. Therefore $L(b,c,d,e) \in (d,e)$, and Λ is destabilized by $\rho_8 = (7,2,2,-3,-8)$.

(9) $\rho_9 = (12,7,2,-8,-13)$. By Lemma 2.9, we have to consider nets with initial weights $(4,-1,-6)$. Such a net is generated by

$$Q_1 = ad + c^2 \mod (c,d,e)^2,$$
$$Q_2 = ae + bd \mod (d,e)(c,d,e),$$
$$Q_3 = be + cd \mod (d,e)^2.$$

Restricting to P, we see that the net has a double structure along L and L meets the residual genus 2 curve D in a single point $O : b = c = d = e = 0$. Dehomogenizing with respect to a, we see that the singularity at O is locally analytically $c^2(b^2 - c) = 0$. Thus L is tangent to D at O.

Conversely, if a locally planar complete intersection C contains a double line tangent to the residual genus 2 component, then after an appropriate change of coordinates, the ideal of C is (Q_1, Q_2, Q_3), where $Q_1 = ad + c^2 \mod (c,d,e)^2$, $Q_2 = ae + bd$, and $Q_3 = be + cd$. Such a net is destabilized by $(12,7,2,-8,-13)$.

(10) $\rho_{10} = (9,4,-1,-1,-11)$. Consider nets with initial ρ_{10}-weights $(8,-2,-7)$. The generators for such a net can be chosen to be $Q_1 = ac + b^2 + Q_1(b,c,d,e)$, $Q_2 = ae + Q_2(c,d,e)$, $Q_4 = be$. The net defines a reducible curve. Along the plane

$b = e = 0$, the two components meet in three points defined by $Q_2(c,d) = ac + Q_1(c,d) = 0$. Restricting to $e = 0$, we obtain a reducible quartic $Q_2(c,d) = Q_1 = 0$, which is a union of two conics. Restricting to $b = 0$, we obtain an elliptic quartic meeting the two conics in a D_6 singularity and two nodes.

(11) $\rho_{11} = (14, 4, -1, -6, -11)$. By Lemma 2.9, the only relevant triple of initial weights is $(8, 3, -12)$. The general net with ρ_{11}-initial weights $(8, 3, -12)$ is generated, after an appropriate change of coordinates, by

$$\begin{aligned} Q_1 &= ad - b^2 - R_1(c,d,e) \\ Q_2 &= ae - bc - R_2(c,d,e) \\ Q_3 &= ce - d^2. \end{aligned}$$

Dehomogenizing with respect to a, we can write the first two equations as $d = b^2 + R_1(c,d,e)$ and $e = bc + R_2(c,d,e)$.

Now we plug into the equation Q_3 to get a local equation for the plane curve singularity at O:

$$bc^2 + cR_2(c,d,e) - b^4 - 2b^2R_1(c,d,e) - R_1^2(c,d,e) = 0,$$

which defines a D_5 singularity.

Furthermore, the hyperelliptic involution induced on the normalization of C by the projection away from L interchanges the points lying over the singularity.

(12) $\rho_{12} = (13, 8, 3, -7, -17)$. By Lemma 2.9, the only possible triple of ρ_{12}-initial weights is $(16, -4, -14)$. After an appropriate coordinate change, the net is generated by

$$\begin{aligned} Q_1 &= ac + b^2 + aR_1(d,e) + R_2(c,d,e), \\ Q_2 &= ae + cd, \\ Q_3 &= ce + d^2, \end{aligned}$$

where R_1 is a linear form and R_2 is a quadratic form. The resulting curve has a conic component C_1 in the plane P and C_1 meets the residual component C_2 at the point O in a singularity with the local analytic equation $d(b^4 - d) = 0$, that is an A_7 singularity. By setting $c = -t^3, d = t^2, e = t$, we see that C_2 is given by the equation

$$b^2 - t^3 + R_1(t^2, t) + R_2(t^3, t^2, t) = 0.$$

In other words, the projection away from L realizes C_2 as the genus two double cover of $\mathbb{P}^1$ ramified at O. $\square$

3. Geometry of semi-stable curves

Notation. To a net of quadrics in $\mathbb{P}^4$ and a choice of its basis (Q_1, Q_2, Q_3), we associate the quintic polynomial $\det(xQ_1 + yQ_2 + zQ_3)$. The PGL(3)-orbit of the corresponding quintic plane curve is an invariant of the net, which we call the *discriminant quintic.*

Since a semi-stable net defines a complete intersection by Corollary 2.8, we will use words "net" and "curve" interchangeably. In particular, the discriminant $\Delta(C)$ of a semi-stable curve C is the discriminant quintic of its defining net of quadrics.

3.1. Main Results. In this section, we use the instability results of the previous section to give an explicit description of semi-stable curves. Our main results are the following two theorems classifying reduced and non-reduced semi-stable curves.

THEOREM 3.1. *A reduced semi-stable curve is a quadric section of a smooth quartic del Pezzo in $\mathbb{P}^4$. Conversely, a quadric section C of a smooth quartic del Pezzo in $\mathbb{P}^4$ is unstable if and only if*

(1) C is non-reduced, or
(2) C is a union of an elliptic quartic and two conics meeting in a pair of triple points, or
(3) C is a union of two elliptic quartics meeting along an A_5 and an A_1 singularities, or
(4) C has a 4-fold point with two lines as its two branches, or
(5) C is a union of two tangent conics and an elliptic quartic meeting the conics in a D_6 singularity and two nodes, or
(6) C has a D_5 singularity with pointed normalization $(\widetilde{C}, p_1, p_2)$ and p_1 is conjugate to p_2 under the hyperelliptic involution of $\widetilde{C}$, or
(7) C contains a conic meeting the residual genus 2 component in an A_7 singularity and the attaching point of the genus 2 component is a Weierstrass point, or
(8) C is a degeneration of curves in (1)–(7).

THEOREM 3.2 (Non-reduced semi-stable). *Let $N \subset \overline{M}^G$ be the image of the locus of non-reduced semi-stable curves. Then N has the following decomposition into irreducible components:*

$$N = N_1 \cup N_2 \cup N_3 \cup N_4,$$

where

(1) N_1 consists of a single point parameterizing the balanced genus 5 ribbon described by Equation (1.1).
(2) N_2 parameterizes curves with a double twisted cubic meeting the residual conic in two points described by Equation (1.2).
(3) N_3 parameterizes curves with a double conic component meeting the residual rational normal quartic in three points described by Equation (1.3).
(4) N_4 consists of a single point parameterizing the semi-stable curve with two double lines joined by conics described by Equation (1.5).

Our analysis proceeds by investigation of the discriminant quintic and is motivated by the following easy result on the relationship between a curve and its discriminant:

LEMMA 3.3. *Let C be a complete intersection of three quadrics in $\mathbb{P}^4$. Then $\Delta(C)$ is reduced if and only if C lies on a smooth quartic del Pezzo in $\mathbb{P}^4$.*

PROOF. Let Λ be the net of quadrics containing C. If $\Delta(C)$ is reduced, then C lies on a smooth quartic del Pezzo, defined by any pencil $\ell \subset \Lambda$ transverse to $\Delta(C)$. Conversely, if C lies on a smooth quartic del Pezzo P in $\mathbb{P}^4$, then the pencil of quadrics containing P has a reduced discriminant. It follows that $\Delta(C)$ is reduced. $\square$

In Corollary 3.17 and Proposition 3.18, we will show that a semi-stable curve C is reduced if and only if its discriminant $\Delta(C)$ is reduced. This, together with Theorem 2.10, leads to a fairly concrete description of reduced semi-stable curves as divisors on smooth del Pezzos given in Theorem 3.1. On the other hand, if C is non-reduced our analysis breaks into two cases, according to whether $\Delta(C)$ has a double line or a double conic. In each case, we find a distinguished quartic surface containing C, which enables us to describe C rather explicitly. The surfaces arising in this analysis are described in Section 3.2.

3.2. Special quartic surfaces in $\mathbb{P}^4$. Four quartic surfaces, each a complete intersection of two quadrics in $\mathbb{P}^4$, play a special role in our analysis of semi-stable curves. Before describing them, let us briefly recall the classification of pencils of quadrics, or, equivalently quartic del Pezzo surfaces, by their Segre symbols [**HP52, AM99**]:

DEFINITION 3.4. Let $\ell = \{Q(t) \mid t \in \mathbb{P}^1\}$ be a pencil of quadrics in $\mathbb{P}^4$, not all singular. Suppose ℓ has exactly k singular elements $Q_1, \dots, Q_k$. The *Segre symbol* of ℓ is a double array

$$\Sigma = \big((a_{ij})_{1\le j\le m_i}\big)_{1\le i\le k},$$

where $\sum_{j\ge r} a_{ij}$ is the minimum order of vanishing at $[Q_i]$ of $(6-r)\times(6-r)$ minors of ℓ, considered as a function of t; in particular, $\sum_{j=1}^{m_i} a_{ij}$ is the multiplicity of $[Q_i]$ in the discriminant $\Delta(\ell)$.

Two quartic del Pezzo surfaces with projectively equivalent discriminants are projectively equivalent if and only if their Segre symbols are equal; see [**AM99**, Theorem 2] or [**HP52**, p.278]. Therefore, if Σ is a Segre symbol, we can speak of a del Pezzo surface $P(\Sigma)$.

3.2.1. *Special del Pezzos.* We consider two special del Pezzo surfaces $P_0 := P(1,(1,1),(1,1))$ and $P_1 := P(1,2,2)$. We recall from [**AM99**, Lemma 3] that P_1 is the anti-canonical embedding of the blow-up of $\mathbb{P}^2$ at points $\{p, q_1, r_1, q_2, r_2\}$ on a smooth conic, with r_i infinitesimally close to q_i, for $i = 1, 2$; and that P_0 is the anti-canonical embedding of the blow up of $\mathbb{P}^2$ at points $\{p, q_1, r_1, q_2, r_2\}$, where r_i is infinitesimally close to q_i, for $i = 1, 2$, and p is the intersection of the lines $\overline{q_i r_i}$.

Note that P_1 isotrivially specializes to P_0. Indeed, if we choose coordinates x, y, z on $\mathbb{P}^2$ so that $z = 0$ is the line $\overline{q_1 q_2}$ and $x = 0$ (resp., $y = 0$) is the line $\overline{q_1 r_1}$ (resp., $\overline{q_2 r_2}$), then the degeneration can be realized by the one-parameter subgroup of PGL(3) acting on $\mathbb{P}^2$ via $t \cdot [x : y : z] = [tx : ty : t^{-2}z]$.

3.2.2. *Veronese quartic.* The third quartic surface of interest is described in the following lemma.

LEMMA 3.5 (Non-linearly normal Veronese). *Let $V \subset \mathbb{P}^4$ be the surface defined by the ideal $(ac - b^2, ce - d^2)$. Then V is a projection of a Veronese surface in $\mathbb{P}^5$. Moreover, V has two pinch point singularities (local equation $uv^2 = w^2$) at* $[0 : 0 : 0 : 0 : 1]$ *and* $[1 : 0 : 0 : 0 : 0]$ *as well as simple normal crossing along the line $b = c = d = 0$. If an irreducible double quartic curve C on V is cut out by a quadric, then C is either a double hyperplane section or C is projectively equivalent to the balanced ribbon $I_R = (ac - b^2, ce - d^2, ae - 2bd + c^2)$.*

PROOF. Evidently, V is the image of $[x : y : z] \mapsto [x^2 : xy : y^2 : yz : z^2]$, which is a projection of the Veronese in $\mathbb{P}^5$. The statement about singularities follows from a local computation.

Every irreducible double quartic curve on V must be the image of a double conic on $\mathbb{P}^2$. Suppose that the conic has equation $f(x,y,z)=0$. The double quartic is a quadric section if and only if $f^2(x,y,z) \in \operatorname{Sym}^2 \mathbb{C}[x^2,xy,y^2,yz,z^2]$. In particular, $f^2(x,y,z)$ cannot have x^3z and xz^3 monomials. Thus either $f(x,y,z)$ has no xz term or it has no x^2 and z^2 terms. In the former case, $f(x,y,z)=0$ is a hyperplane section of V. Suppose now $f(x,y,z)$ has no x^2 or z^2 term but has xz term. Then we can write $f(x,y,z)=xz+yL(x,z)+\lambda y^2$. After a linear change of variables on $\mathbb{P}^2$ inducing a compatible linear change of variables in $\mathbb{P}^4$, we can assume that $f(x,y,z)=xz+\lambda y^2$. If $\lambda=0$, then $f^2(x,y,z)=ae$ and it defines a union of two double conics. This contradicts the irreducibility assumption. Thus, we can assume that $f(x,y,z)=xz-y^2$, so that $(xz-y^2)^2=x^2z^2-2xzy^2+y^4=ae-2bd+c^2$. $\square$

3.2.3. *Reducible quartic.* The final quartic of special interest to us is the reducible union of a plane with a cubic scroll, which arises in Part (1) of the following lemma.

LEMMA 3.6. *Suppose ℓ is a pencil of quadrics containing a common plane and with no common singular points. Then ℓ is one of the following up to projectivity:*

(1) $\ell=(ad-bc, be-cd)$, defining a union of a plane with a cubic scroll. The vertices of the quadrics in ℓ trace out the conic $b=d=ae-c^2=0$.

(2) $\ell=(ad-\mu b^2, be-cd)$, where $\mu\in\mathbb{C}$. The vertices of the rank 4 quadrics in ℓ trace out the line $b=d=e=0$.

PROOF. Suppose a pencil of quadrics contains a plane $b=d=0$. Then the general form of the pencil is $bL_1-dL_2=bL_3-dL_4=0$. Since the pencil does not have a common singular point, the set given by $b=d=L_1=L_2=L_3=L_4=0$ is empty. Without loss of generality, we can assume that b,d,L_2,L_3,L_4 are linearly independent and choose coordinates so that $L_2=a$, $L_3=e$, $L_4=c$. Thus $\ell=(ad-bL_1, be-cd)$. Changing coordinates, we can assume that $L_1=\lambda c+\mu b$. If $\lambda\neq 0$, a further change of coordinates: $a':=\lambda a$, $c':=\lambda c+\mu b$, $b':=\lambda b$, $e':=e+(\mu/\lambda)d$, $d':=d$ gives $I=(a'd'-b'c', b'e'-c'd')$.

Finally, if $\lambda=0$, then the pencil is $(ad-\mu b^2, be-cd)$. $\square$

3.3. Non-reduced discriminants of semi-stable curves. We proceed to give a complete classification of semi-stable curves with non-reduced discriminants. An important implication of our analysis is the fact that a semi-stable curve with a non-reduced discriminant is itself non-reduced.

3.3.1. *Discriminants of pencils and nets of quadrics in $\mathbb{P}^4$.* We begin with a series of simple lemmas, whose proofs we omit.

LEMMA 3.7. *Suppose Q_1 is a rank 4 quadric. Then $\det(Q_1+tQ_2)$ has a root of multiplicity two at $t=0$, or is identically zero, if and only if Q_2 vanishes at the vertex of Q_1.*

LEMMA 3.8. *A pencil ℓ of quadrics in $\mathbb{P}^4$ consists of singular quadrics only if:*

(A) Quadrics in ℓ have a common singular point; or

(B) Quadrics in ℓ contain a common plane; or

(C) Restricted to a common hyperplane, the quadrics in ℓ are singular along a line;

Furthermore, if ℓ satisfies (C) but not (A) or (B), then up to projectivity $\ell=(ac-b^2, ce-d^2)$, defining the Veronese quartic V.

LEMMA 3.9. *A pencil ℓ of quadrics in $\mathbb{P}^3$ consists of singular quadrics if and only if:*

(A) Quadrics in ℓ have a common singular point; or
(B) Restricted to a common plane, quadrics in ℓ contain a double line. The general such ℓ is, up to projectivity, $(be, ce - d^2)$.

Next, we analyze the possibilities for non-reduced discriminants of semi-stable curves.

PROPOSITION 3.10. *A discriminant quintic of a semi-stable net Λ has a double line if and only if (up to projectivity) $(ae - bd, ad - bc) \subset \Lambda$ and Λ is of the form*

$$(ad - bc, be - cd, ae - c^2 + bL_1 + dL_2)$$

In particular, such Λ contains a double conic.

PROOF. Suppose ℓ is a double line in $\Delta(C)$. The analysis proceeds according to possibilities for ℓ enumerated in Lemma 3.8.

(A) Suppose all elements of ℓ are singular at a point O. By Lemma 3.7, ℓ can be a double line of $\Delta(C)$ in two cases: either O is a base point of Λ or all quadrics in ℓ have rank ≤ 3. The former case is impossible by Lemma 2.7 (4). In the latter case Lemma 3.9 says that either all quadrics in ℓ are singular along a line, in which case Λ is destabilized by Lemma 2.7 (3) or ℓ is up to projectivity (a degeneration of) $(be, ce - d^2)$, in which case Λ is destabilized by $\rho_5 = (3, 3, 3, -2, -7)$.

(B) Suppose ℓ is a pencil of quadrics containing a plane, say $b = d = 0$, and having no common singular points. Then by Lemma 3.6 either $\ell = (ad - bc, be - cd)$ or $\ell = (ad - b^2, be - cd)$, up to projectivity. However, if $\ell = (ad - b^2, be - cd)$, then the singular points of quadrics in ℓ trace out the line $b = d = e = 0$, which must then fall in the base locus of the net because ℓ is a double line of $\Delta(C)$. Such a net is destabilized by the 1-PS with weights $(4, -1, 4, -6, -1)$.

If $\ell = (ad - bc, be - cd)$, then $b = d = ae - c^2 = 0$ is the conic along which elements of ℓ are singular, so this conic must be in the base locus of Λ. The claim follows.

(C) Suppose that we are not in the cases (A) or (B). Then $\ell = (ae - c^2, be - d^2)$, up to projectivity. Since the generic quadric in ℓ has rank 4 and the vertices of quadrics in ℓ vary along $c = d = e = 0$, we deduce that $c = d = e = 0$ must be in the base locus of the net. Such Λ is destabilized by $\rho_6 = (4, 4, -1, -1, -6)$. $\square$

The converse to the above result is the following.

PROPOSITION 3.11. *A semi-stable curve with a double conic component is projectively equivalent to the intersection of the quadrics $Q_1 = ad - bc, Q_2 = be - cd, Q_3 = ae - c^2 + bL_1 + dL_2$, where L_i are not simultaneously zero.*

PROOF. Let C be a semi-stable curve with a double structure supported on a smooth conic X. Denote by Λ the associated net of C. Let $\ell \subset \Lambda$ be the pencil of quadrics containing the plane P_X spanned by X. Note that any quadric not in ℓ intersects P_X in X, and so is smooth along X. Since every point of X is a singular point of C, for every point of X there must be a quadric in ℓ singular at that point. It follows that X is traced out by vertices of quadrics in ℓ. (We note in passing that this implies that ℓ appears with multiplicity 2 in Δ.) By Lemma 3.6, we must have $\ell = (ad - bc, be - cd)$. Then the singular points of quadrics in ℓ trace out the conic $b = d = ae - c^2 = 0$. It follows that $\Lambda = (ad - bc, be - cd, ae - c^2 + bL_1 + dL_2)$. $\square$

REMARK 3.12. In Proposition 3.11, the intersection of Q_1 and Q_2 is the union of the plane $b = d = 0$ and the cubic scroll $(ad-bc, be-cd, ae-c^2)$. The scroll is the blow-up of $\mathbb{P}^2$ at a point and is embedded in $\mathbb{P}^4$ by $2H - E$, where H is the class of a line and E is the class of the (-1)-curve. The scroll meets the plane $b = d = 0$ in the conic $ae - c^2 = 0$. The quadric Q_3 intersects the scroll in this conic (of class H on the scroll) and in a curve of class $3H-2E$, which is a rational normal quartic in $\mathbb{P}^4$ meeting the conic $b = d = ae-c^2 = 0$ in three points. Thus any curve described by Proposition 3.11 looks like a double conic meeting a rational normal quartic in three points.

PROPOSITION 3.13. *The discriminant quintic of a semi-stable curve C has a double conic only in the following cases:*

(1) Up to projectivity, the net of quadrics containing C is

$$I_R := (ac - b^2, ae - 2bd + c^2, ce - d^2).$$

(2) Up to projectivity, the net of quadrics containing C is

$$I_{DL} := (ad, ae + bd - c^2, be).$$

(3) The curve C has a double line meeting the residual genus 2 curve in 2 points; such C isotrivially degenerates to the curve defined by I_{DL}.

(4) The curve C contains a double twisted cubic and is defined by Equation (1.2).

PROOF. Let C be a curve with a discriminant $\Delta(C) = 2q + \ell$, where q is a conic and ℓ is the residual pencil.

Case 1: Suppose first that the generic quadric in q has rank 3. The further analysis breaks into the cases enumerated by Lemma 3.8:

(A) All quadrics in ℓ have a common singular point; or

(B) All quadrics in ℓ contain a common plane; or

(C) $\ell = (ac - b^2, ce - d^2)$, defining the Veronese quartic V.

Observation: Before proceeding with a case-by-case analysis, we make an elementary observation about conics in Λ. Namely, suppose $q \subset \Lambda$ is a conic. Suppose $[Q_1]$ and $[Q_2]$ are two distinct point on q and let $[Q_3] \in \Lambda$ be the point of the intersection of the tangent lines to q at $[Q_1]$ and $[Q_2]$. Then q can be explicitly parameterized by $\mathbb{P}^1$ via $[s:t] \mapsto [s^2Q_1 + t^2Q_2 + stQ_3]$. We also note that Q_1, Q_2, and Q_3 obtained by this construction span the net.

Case (A): The quadrics in ℓ have a common singular point $b = c = d = e = 0$. We can assume that $\ell = (Q_2, Q_3)$ and $Q_1 = a^2 + \overline{Q}_1(b, c, d, e)$. Note that with this choice of coordinates, every quadric in the net can be written as a linear combination of a^2 and a quadric in variables b, c, d, e. Since elements of q have generic rank 3, all elements of q have form $\lambda a^2 + R(b, c, d, e)$, where $\lambda \in \mathbb{C}$ and $R(b, c, d, e)$ are generically rank 2 quadrics with no common singular point. In particular, it follows that quadrics corresponding to $q \cap \ell$ are rank 2 quadrics in variables b, c, d, e.

Given any two points $[Q_1], [Q_2] \in q$, we can choose coordinates b, c, d, e so that $Q_1 = bc \pmod{a^2}$ and $Q_2 = de \pmod{a^2}$. By the observation above every element of q can be written as $s^2Q_1 + t^2Q_2 + stQ_3$, where $[s:t] \in \mathbb{P}^1$, and where $[Q_3]$ is the point of intersection of the tangent lines to q at $[Q_1]$ and $[Q_2]$. Set $\overline{Q}_3 := Q_3 \pmod{a}$. Then $s^2bc + t^2de + st\overline{Q}_3$ is a rank 2 quadric for all $[s:t] \in \mathbb{P}^1$.

Claim 3.14. *$\overline{Q}_3 = \mu be + \frac{1}{\mu}cd$, possibly after an appropriate renaming of variables.*

Proof. Let M be the symmetric matrix associated to $s^2bc + t^2de + st\overline{Q}_3$. Analyzing t^5s and s^5t terms of the 3×3 minors of M, we see that $\overline{Q}_3 \in (bd, be, cd, ce)$. Writing $\overline{Q}_3 = xbd + ybe + zcd + wce$, the upper-left 3×3 minor of M is $2xzs^4t^2$. Hence $xz = 0$. Similarly, one shows that $xy = zw = yw = 0$. Without loss of generality, we can assume $x = w = 0$. Computing the remaining 3×3 minors we obtain $y(1-yz) = z(1-yz) = 0$. The claim follows. □

We now consider separately two subcases:

Case (A.1): ℓ is tangent to q. We take $[Q_1]$ to be the point of tangency and $[Q_2]$ to be any point on the conic. By above, we can assume that $Q_1 = de$, $Q_2 = bc + a^2$, and $Q_3 = be + cd$. (The last equality comes from $\overline{Q}_3 = be + cd$ and $Q_3 \in \ell$). This net is destabilized by $\rho_2 = (2, 2, 2, -3, -3)$.

Case (A.2): ℓ is not tangent to q. Let $q \cap \ell = \{[Q_1], [Q_2]\}$. As above, we let $[Q_3]$ be the point of intersection of tangents to q at $[Q_1]$ and $[Q_2]$. Letting $Q_1 = bc$ and $Q_2 = de$, we can write $Q_3 = \mu be + \frac{1}{\mu}cd + a^2$ by Claim 3.14. By appropriately renaming and scaling the variables, we obtain the net I_{DL}.

Case (B): The quadrics in ℓ contain a plane P and have no common singular points. If ℓ intersects q in two distinct points, then we are in case (A). (Indeed, the quadrics in $q \cap \ell$ have rank 3 and any two rank 3 quadrics containing a common plane have a common singular point.)

Suppose ℓ is tangent to q. Let $[Q_1]$ be the point of tangency. Then Q_1 has rank at most 3 and contains the plane P. We can choose coordinates so that P has equation $d = e = 0$ and $Q_1 = ce + \lambda d^2$, for some $\lambda \in \mathbb{C}$. Let $[Q_2] \in q \smallsetminus \ell$ and let $[Q_3] \in \ell$ be the point of intersection of the tangents to q at $[Q_1]$ and $[Q_2]$ (note that the tangent line to q at $[Q_1]$ is ℓ). Then $Q_3 = dL_1 + eL_2$, for some linear forms L_1, L_2. Since Q_1 and Q_3 have no common singular points, the system of equations $c = d = e = L_1 = L_2 = 0$ has no solutions. Thus we can we can assume $L_1 = b$ and $L_2 = a$.

All quadrics in q have form $s^2Q_1 + t^2Q_2 + stQ_3$, where $[s : t] \in \mathbb{P}^1$ and have rank 3 by our assumption. The analysis of t^7s terms in the vanishing 4×4 minors of the symmetric matrix of $s^2Q_1 + t^2Q_2 + stQ_3$ shows that $Q_2 \in (c, d, e)$.

It follows that $\Lambda = (Q_1, Q_2, Q_3)$, where $Q_1 = ce + \lambda d^2, Q_2 = ae + bd$, and $Q_3 \in (c, d, e)$. Thus Λ is destabilized by $\rho_3 = (3, 3, -2, -2, -2)$.

Case (C): Suppose $\ell = (ac - b^2, ce - d^2)$. By the observation above, there exists Q_3 such that $s^2(ac-b^2) + t^2(ce-d^2) + stQ_3$ has rank 3 for s and t. The vanishing of the 4×4 minors of the symmetric matrix of $s^2(ac-b^2) + t^2(ce-d^2) + stQ_3$ implies that $Q_3 = ae - 2bd + c^2$. Hence the net is $I_R = (ac - b^2, ae - 2bd + c^2, ce - d^2)$.

Case 2: We now consider the case when the generic quadric in q has rank 4. Taking a general pencil in Λ, we see that C lies on the del Pezzo $P(1, 2, 2)$. Furthermore, by Lemma 3.7 the vertices of quadrics in q form a curve D in the base locus of the net. In particular, C is non-reduced.

We now proceed to consider different cases according to the degree of D.

• If D is a line, then we are done by Proposition 3.15 below, which describes all semi-stable curves containing a double line on $P(1, 2, 2)$.

• If D is a conic, then we are done by Proposition 3.11, which describes all semi-stable curves containing double conics.

• If D is a twisted cubic, then the residual component of C is a conic. Let $\ell \subset \Lambda$ be the pencil of quadrics containing the plane spanned by this conic. The intersection of the quadrics in ℓ is a union of a plane and a cubic scroll. The general such ℓ has equation $(ad-bc, be-cd)$ (cf. Lemma 3.6) with the scroll being $(ad-bc, be-cd, ae-c^2)$. Since every double twisted cubic on a scroll is a double hyperplane section, we conclude that every semi-stable curve with a double twisted cubic component is a degeneration of a curve defined by Equation (1.2).

• If D is a quartic, then by Theorem 2.10 (1) D must be a rational normal quartic. We now consider the possibilities for ℓ enumerated in Lemma 3.8. Case (B) is clearly impossible. In Case (A), all quadrics in ℓ are singular at some point O. In this case, D lies on a cone over $E \subset \mathbb{P}^3$ with a vertex at O, where E is a complete intersection of two quadrics in $\mathbb{P}^3$. Since the arithmetic genus of E is 1 and E is a projection of a rational normal quartic, we see that E is singular. We conclude that O lies on a chord (or a tangent line) of D. This immediately leads to a contradiction, as D is a complete intersection of three quadrics not passing through O.

Finally, in Case (C) D lies on the Veronese $(ac-b^2, ce-d^2)$. By Lemma 3.5 the ideal of C is I_R. This finishes the proof. □

3.3.2. *Double lines.* Let $\pi\colon S \to \mathbb{P}^2$ be the blow up of a plane at five points $\{p, q_1, r_1, q_2, r_2\}$, with r_i infinitesimally close to q_i for $i = 1, 2$. Set $E_0 := \pi^{-1}(0)$ and let $\pi^{-1}(q_i) = F_i \cup G_i$, where F_1, F_2 are the (-2)-curves and G_1, G_2 are the (-1)-curves on S. Denote by H the class of a line on S. If $\phi\colon S \to \mathbb{P}^4$ is the anti-canonical map, then $\phi(S)$ is the del Pezzo P_1 described in Section 3.2.1.

PROPOSITION 3.15. *Let C be a semi-stable curve on P_1 with a double line component. Then C isotrivially degenerates to the curve defined by*

$$I_{DL} = (ad, ae + bd - c^2, be).$$

PROOF. Suppose $2L + R \in |-2K_S|$ is a divisor on S such that $\phi(2L + R)$ is a semi-stable curve and $\phi(L)$ is a line. Then L is a (-1)-curve and R meets L in four points, counting multiplicities. But by Theorem 2.10 (3) $\phi(L)$ cannot meet $\phi(R)$ in four or more points. It follows that one of the irreducible components of R is a (-2)-curve meeting L. The only (-1)-curves meeting (-2)-curves on S are, up to symmetries:

(i) $L = G_1$ meeting the (-2) curve F_1.

(ii) $L = H - F_1 - G_1 - F_2 - G_2$ meeting both (-2)-curves F_1 and F_2.

However, in case (i), we see that $R - F_1$ has arithmetic genus one and meets $L + F_1$ in three points. It follows that $\phi(R) = \phi(R - F_1)$ meets the double line $\phi(L) = \phi(L + F_1)$ in three points. It follows that C is unstable by Theorem 2.10 (6)(b).

In case (ii), we see that $R - F_1 - F_2 = 4H - 2E_0 - F_1 - 2G_1 - F_2 - 2G_2$ is a genus two curve meeting $L + F_1 + F_2$ in 2 points. Recall that there is an isotrivial degeneration of P_1 to P_0. Under this degeneration, L is fixed and the limit of the residual genus two component is

$$\underbrace{(H - F_1 - 2G_1 - E_0)}_{\text{(-2)-curve}} + \underbrace{(H - F_2 - 2G_2 - E_0)}_{\text{(-2)-curve}} + 2E_0 + (H - E_0) + (H - E_0).$$

It follows that under the degeneration of P_1 to P_0, C is a union of two double lines of class E_0 and $H - F_1 - G_1 - F_2 - G_2$, respectively, and two conics of class $H - E_0$. A simple computation shows that in appropriate coordinates C is given by the ideal $I_{DL} = (ad, ae + bd - c^2, be)$. □

We summarize the discussion of this section in the following result.

PROPOSITION 3.16. *Suppose C is a semi-stable curve. If $\Delta(C)$ is non-reduced, then C is non-reduced.*

PROOF. $\Delta(C)$ is non-reduced if and only if it has a double line or a double conic. The result now follows immediately from Propositions 3.10 and 3.13. □

COROLLARY 3.17. *A reduced semi-stable curve lies on a smooth quartic del Pezzo.*

PROOF. By Proposition 3.16, the discriminant of a reduced semi-stable curve is reduced and so the curve lies on a smooth quartic del Pezzo by Lemma 3.3. □

Conversely, we now prove Part (1) of Theorem 3.1 stating that a semi-stable curve on a smooth quartic del Pezzo is necessarily reduced.

PROPOSITION 3.18. *Suppose C is a semi-stable curve. If C lies on a smooth quartic del Pezzo, then C is reduced.*

PROOF. Suppose C lies on a smooth quartic del Pezzo S and let ℓ be the pencil of quadrics containing S. To prove that C is reduced, we argue by contradiction. Suppose that C has a non-reduced irreducible component D. Then for every point $p \in D$, there is a quadric in the ideal of C that is singular at p. Since quadrics in ℓ cannot be singular along D, we deduce that the ideal of C contains a single quadric that is singular along all of D. This leads to a contradiction using Lemma 2.7. Indeed, if D is a line, then C is destabilized by $\rho_3 = (3, 3, -2, -2, -2)$; if D is a conic, then C is destabilized by $\rho_2 = (2, 2, 2, -3, -3)$; if D is a twisted cubic, then C is destabilized by $\rho_1 = (1, 1, 1, 1, -4)$. □

3.4. Proofs of Theorem 3.1 and 3.2.

PROOF OF THEOREM 3.1. To finish the proof of Theorem 3.1, we observe that Parts (2)–(7) follow from Theorem 2.10 (6)(a), (7), (8), (10), (11), (12), respectively. □

PROOF OF THEOREM 3.2. To prove Theorem 3.2, we note that a non-reduced semi-stable curve has a non-reduced discriminant by Proposition 3.18 and Lemma 3.3. It follows from Propositions 3.10 and 3.13 that a non-reduced semi-stable curve degenerates isotrivially to a curve in $N_1 \cup N_2 \cup N_3 \cup N_4$.

We note that N_1 and N_4 each consist of a single point, hence irreducible. To prove irreducibility of N_2 and N_3, we recall that a semi-stable curve with a double twisted cubic component is, up to projectivity, given by Equation (1.2):

$$(ad - bc, ae - c^2 + L^2, be - cd).$$

Similarly, a semi-stable with a double conic component is, up to projectivity, given by Equation (1.3):

$$(ad - bc, ae - c^2 + bL_1 + dL_2, be - cd),$$

Irreducibility of the space of linear forms implies irreducibility of N_2 and N_3.

We proceed to prove the semi-stability of the general point of N_i $(i = 1, \ldots, 4)$ using the Kempf-Morrison criterion [**AFS13**, Proposition 2.4].

PROPOSITION 3.19. *The ideals* I_R, I_{DT}, I_T, I_{DL}, *defined by*

$$\begin{aligned} I_R &:= (ac - b^2, ae - 2bd + c^2, ce - d^2), \\ I_{DT} &:= (ad - b^2, ae - bd + c^2, be - d^2), \\ I_T &:= (ad - bc, ae + bd - c^2, be - cd), \\ I_{DL} &:= (ad, ae + bd - c^2, be), \end{aligned}$$

are semi-stable.

PROOF. Each of the ideals is stabilized by a certain 1-PS acting diagonally with respect to the distinguished basis $\{a, b, c, d, e\}$. Indeed, they are stabilized by $(2, 1, 0, -1, -2)$, $(3, 1, 0, -1, -3)$, $(2, 1, 0, -1, -2)$, and $(2, 1, 0, -1, -2)$, respectively. By the Kempf-Morrison criterion [**AFS13**, Proposition 2.4], it therefore suffices to check that these curves are semi-stable with respect to 1-PS's acting diagonally with respect to this basis. By Theorem 2.1 (see also Remark 2.2), it suffices to check the given finite list of 1-PS's acting diagonally with respect to this basis, and this is an easy exercise. We should remark that semi-stability of the balanced ribbon I_R is a special case of a more general [**AFS13**, Theorem 4.1]. □

Observing that $I_R \in N_1, I_{DT} \in N_2, I_T \in N_3, I_{DL} \in N_4$ finishes the proof of Theorem 3.2. □

REMARK 3.20. We note that $I_T = (ad - bc, ae + bd - c^2, be - cd)$ contains no rank 3 quadric. Thus [**FJ11**, Theorem 2.1] provides an easier, independent proof for the semi-stability of I_T.

REMARK 3.21. We observe that the discriminant quintic of I_T is $2y^3(xz - y^2)$, which is unstable under the natural SL(3)-action on the space of plane quintics.

4. Proofs of the main results

In this section we prove Main Theorems 1–3 from the introduction. Main Theorem 1 follows immediately from Theorems 3.1 and 3.2.

PROOF OF MAIN THEOREM 2. The divisor of singular complete intersections in $\overline{M}^G$ is irreducible. Furthermore, from Main Theorem 1, its general point corresponds to a one-nodal curve on a smooth quartic del Pezzo. It follows that the inverse rational map $f^{-1}\colon \overline{M}^G \dashrightarrow \overline{M}_5$ does not contract divisors and so f is a contraction.

By Theorem [**Has00**, Theorem 6.2 and Theorem 6.3], the general curve in Δ_1 arises as a stable limit of a genus 5 curve with an A_2 singularity and the general curve in Δ_2 arises as a stable limit of the general curve in the $A_5^{\{1\}}$-locus. It follows that Δ_1 and Δ_2 are generically fibered over A_2- and $A_5^{\{1\}}$-loci, respectively.

It remains to prove that the trigonal divisor is contracted to a single point given by Equation (1.4). Recall that a general smooth trigonal curve of genus 5 has a very ample canonical line bundle and its canonical embedding lies on a cubic scroll, whose homogeneous ideal is $(ad - bc, ae - c^2, be - cd)$, up to projectivities. A trigonal curve on the scroll is cut out by two linear independent cubics

$$(aR_1 - bR_2 + cR_3, cR_1 - dR_2 + eR_3),$$

where R_1, R_2, R_3 are quadrics in $\mathbb{C}[a,b,c,d,e]$. In particular, a general trigonal curve is obtained by taking $\{R_i\}_{i=1}^3$ to be general.

PROPOSITION 4.1. *The rational map* $f\colon \overline{M}_5 \dashrightarrow \overline{M}^G$ *contracts the trigonal divisor* Trig_5 *to the point*

$$I_T := (ad-bc, ae+bd-c^2, be-cd) \in \overline{M}^G.$$

PROOF. Since Trig_5 is a divisor, f is defined at the generic point of Trig_5. Hence to show that f contracts Trig_5 to a point, we need to show that a general trigonal curve arises as a stable limit for some deformation of I_T.

Consider the family of nets $\Lambda_t = (Q_1(t), Q_2(t), Q_3(t))$ defined by

$$\begin{aligned}
Q_1(t) &= be-cd+t^2R_1(a,b,c,d,e),\\
Q_2(t) &= ae-c^2+t^2R_2(a,b,c,d,e),\\
Q_3(t) &= ad-bc+t^2R_3(a,b,c,d,e),
\end{aligned}$$

where R_1, R_2, R_3 are general quadrics. Then for $t \neq 0$, Λ_t defines a smooth non-trigonal curve C_t of genus 5, while $\Lambda_0 = (ad-bc, ae-c^2, be-cd)$ defines a cubic scroll in $\mathbb{P}^4$. Let C_0 be the flat limit of $\{C_t\}_{t\neq 0}$ as $t \to 0$. Using linear syzygies among the quadrics containing the scroll, we see that

$$\begin{aligned}
F_1 &:= \frac{1}{t^2}\big(aQ_1(t) - bQ_2(t) + cQ_3(t)\big)|_{t=0} = aR_1 - bR_2 + cR_3,\\
F_2 &:= \frac{1}{t^2}\big(cQ_1(t) - dQ_2(t) + eQ_3(t)\big)|_{t=0} = cR_1 - dR_2 + eR_3
\end{aligned}$$

are cubics in the ideal of C_0. Since R_1, R_2, R_3 were chosen generically, we conclude that C_0 is a general smooth trigonal curve.

It now suffices to show that the limit of $\{\Lambda_t\}_{t\neq 0}$ in $\overline{M}^G$ is the point I_T. Let $\rho_t \in \mathrm{PGL}(5)$ be given by $\rho_t\cdot[a:b:c:d:e] = [a:t^{-1}b:c:t^{-1}d:e]$. Set $\Lambda'_t := \rho_t\cdot\Lambda_t$. Then the flat limit of Λ'_t as $t \to 0$ is $\Lambda'_0 = (ad-bc, ae-c^2+R_2(0,b,0,d,0), be-cd)$. Since R_2 was chosen to be a general quadric, $S(b,d) := R_2(0,b,0,d,0)$ has rank 2.

We claim that, without loss of generality, we may take $R_2(0,b,0,d,0) = bd+\eta d^2$ for some scalar η. This implies that Λ'_0 is semi-stable and that its orbit closure contains I_T, since the limit as $t \to \infty$ of $(ae-bc, ae-c^2+bd+\eta d^2, be-cd)$ under the one-parameter subgroup $(t^2, t, 1, t^{-1}, t^{-2})$ is $I_T = (ad-bc, ae-c^2+bd, be-cd)$.

It remains to show that we may take $R_2(0,b,0,d,0) = bd+\eta d^2$. Let $S(b,d) = L_1(b,d)L_2(b,d)$, where L_1 and L_2 are linearly independent linear forms. Without loss of generality, $L_1(b,d) = d+\mu b$, where $\mu \in C$. Make the following coordinate change:

$$\begin{aligned}
a' &:= a,\\
b' &:= b,\\
c' &:= c+\mu a,\\
d' &:= d+\mu b,\\
e' &:= e+2\mu c+\mu^2 a.
\end{aligned}$$

Let $M(b',d') = \lambda b' + \nu d'$ be the linear form such that $M(b',d') = L_1(b,d)$. Note that $\lambda \neq 0$. After scaling, we can assume that $\lambda = 1$. Then

$$\begin{aligned} a'd' - b'c' &= ad - bc, \\ b'e' - c'd' &= be - cd - \mu(ad - bc), \\ a'e' + M(b',d')d' - (c')^2 &= ae - c^2 + L_1(b,d)L_2(b,d), \end{aligned}$$

as desired. □

Proof of Main Theorem 3. By Main Theorem 2, f is a contraction. We compute $f^*\mathcal{O}(1)$ using two methods.

The most straightforward way is to write down three test families along which f is regular and which are contracted by f. Consider the following families in $\overline{\mathcal{M}}_5$:

(1) A family T_1 of elliptic tails attached to a fixed general pointed genus 4 curve. We have $\lambda \cdot T_1 = 1$, $\delta_0 \cdot T_1 = 12$, $\delta_1 \cdot T_1 = -1$, $\delta_2 \cdot T_1 = 0$. Furthermore, deformations of T_1 cover Δ_1.
(2) A family T_2 of genus 2 tails attached to a fixed general pointed genus 3 curve at a non-Weierstrass point; see [**FS13**, Section 4.4] for a precise description of the construction. We have $\lambda \cdot T_3 = 3$, $\delta_0 \cdot T_3 = 30$, $\delta_2 \cdot T_3 = -1$, $\delta_1 \cdot T_3 = 0$. Furthermore, deformations of T_2 cover Δ_2.
(3) A family T_3 of curves in Trig^0 satisfying $\lambda \cdot T_3 = 4$, $\delta_0 \cdot T_3 = 33$, $\delta_i \cdot T_3 = 0$ for $i = 1, 2$. Such a family exists by [**DP12**], where it is also shown that deformations of T_3 cover Trig_5.

By Main Theorem 2, f contracts each T_i. Namely, $f(T_1)$ is a semi-stable cuspidal curve, $f(T_2)$ is a semi-stable curve in the A_5^1-locus, and $f(T_3) = [I_T]$. Therefore, *assuming that f is regular along each T_i*, we have $f^*\mathcal{O}(1).T_i = 0$ for each $i = 1, 2, 3$. Writing $f^*\mathcal{O}(1) = a\lambda - b\delta_0 - c\delta_1 - d\delta_2$ and intersecting both sides with T_i, we obtain

$$f^*\mathcal{O}(1) \sim 33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2$$

as desired. Unfortunately, proving f is regular along each T_i directly would require several rather subtle stable reduction calculations. Thus, we give an alternative computation of $f^*\mathcal{O}(1)$, which implies the desired regularity *a posteriori*.

Let $\mathcal{D} \subset \overline{M}^G$ be the divisor of nets containing rank 3 quadrics. Note that, $I_R \in \mathcal{D}$ (see Proposition 3.19 for the definition of I_R). In particular, $\mathcal{D}$ is non-empty. $\mathcal{D}$ is irreducible because the divisor of nets in $\mathbb{G}(3, 15)$ containing a rank 3 quadric is irreducible. By Remark 3.20, $I_T \notin \mathcal{D}$, and since I_T lies in the closure of A_2- and $A_5^{\{1\}}$-loci, we conclude that $f^{-1}\mathcal{D}$ does not contain Δ_1, Δ_2, or Trig_5. It follows that $f^{-1}\mathcal{D}$ is the divisor of genus 5 curves with a vanishing theta-null. By [**TiB88**, Proposition 3.1], the class of this divisor is proportional to $4(33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2)$. This proves Part (1) of the theorem.

To prove Part (2), simply observe that

$$K_{\overline{\mathcal{M}}_5} + \frac{14}{33}\delta \sim \big(33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2\big) + 11\delta_1 + 17\delta_2$$

and

$$K_{\overline{\mathcal{M}}_5} + \frac{3}{8}\delta \sim \big(8\lambda - \delta_0 - 4\delta_1 - 6\delta_2\big) + 3\delta_1 + 5\delta_2,$$

where $8\lambda - \delta_0 - 4\delta_1 - 6\delta_2$ is an effective multiple of the divisor class of Trig_5 by Brill-Noether Ray Theorem [**HM98**, Theorem 6.62]. $\square$

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Clay Mathematics Proceedings
Volume **18**, 2013

Restriction of Sections for Families of Abelian Varieties

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This paper is dedicated to Joe Harris on the occasion of his 60th birthday.

Abstract. Given a family of Abelian varieties over a positive-dimensional base, we prove that for a sufficiently general curve in the base, every rational section of the family over the curve is contained in a unique rational section over the entire base.

1. Main results

The starting point for this article is the following theorem.

Theorem 1.1. [**4**] *Let $\pi : X \to B$ be a proper morphism of complex varieties. If π admits a a section when restricted to a very general sufficiently positive curve in B, then there exists a subvariety $Z \subset X$ dominating B whose general fiber is rationally connected.*

We call such a Z a *pseudosection* of π. The relevance of pseudosections is that they will *force* the existence of sections over a generic curve, since families of rationally connected varieties over a curve have sections. Of course, the sections produced by this argument will then be contained in the corresponding Z.

Surprisingly, the proof of this theorem does not establish the following stronger statement.

Conjecture 1.2. *If $\pi : X \to B$ is a morphism of complex varieties, then for a very general, sufficiently positive curve $C \subset B$, every section of the restricted family $X_C = \pi^{-1}(C) \to C$ takes values in a pseudosection.*

This is a birational problem: for any dense, Zariski open $U \subset B$, a positive answer for $\pi : \pi^{-1}(U) \to U$ implies a positive answer for $X \to B$. In particular, we may replace B by its smooth locus. Thus in all that follows, unless explicitly stated otherwise, we assume that B is smooth.

Conjecture 1.2 appeared in [**4**], as well as the special case where X is a family of Abelian varieties over B. In this special case we get a simpler prediction: for a very general, sufficiently positive curve $C \subset B$, the restriction map on sections, $X(B) \to X_C(C)$, is bijective. One special feature is the Weil extension theorem:

2010 *Mathematics Subject Classification.* Primary 14K12, 14C05.

The first author was supported in part by NSF grants DMS-0601074.

The second author was supported in part by NSF Grants DMS-0553921, DMS-0758521, and DMS-0846972.

every rational section extends to a regular section on the entire smooth locus. Because of this, and in order to conform to classical terminology, we shall frequently refer to regular sections as "global rational sections". The purpose of this paper is to prove this prediction, with a stronger than expected bound on the meaning of "sufficiently positive."

To state our result precisely, we need to introduce some notation. Fix a generically finite, generically unramified morphism $f : B \to \mathbb{P}^n$. We then define

- an f-line is a curve of the form $f^{-1}(L)$ for a line $L \subset \mathbb{P}^n$,
- an f-conic is a curve of the form $f^{-1}(C)$ for C a plane conic in $\mathbb{P}^n$.
- an f-line-pair is a curve of the form $f^{-1}(L_1 \cup L_2)$ where the L_i are a pair of incident lines in $\mathbb{P}^n$.
- an f-planar surface is a surface in B of the form $f^{-1}(\Pi)$ for a 2-plane $\Pi \subset \mathbb{P}^n$.

By Bertini's Theorem, for a sufficiently general line L, respectively conic C, plane Π, the inverse image $f^{-1}(L)$, resp. $f^{-1}(C)$, $f^{-1}(\Pi)$, is smooth. We will generally suppress the f in the terminology, but obviously these notions are meaningless in the absence of a choice of f.

Let k be an uncountable algebraically closed field. We remind the reader that a subset of a scheme is *general*, resp. *very general*, if it contains a dense, open subset, resp. the intersection of a countable collection of dense, open subsets. A property of points in a scheme holds at a *general point*, resp. a *very general point*, if the set of points where it holds is general, resp. very general.

Theorem 1.3. *Let B be a integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let A be an Abelian scheme over B. For a very general line-pair C in B, the map*

$$A(B) \to A(C)$$

is a bijection. The theorem also holds with C a very general planar surface in B. If char$(k) = 0$, this also holds with C a very general conic in B.

Remark 1.4. We make a few remarks.

(1) It is perhaps worth remarking that if we take B to be smooth, then $A(B) = A(K(B))$ is the usual Mordell-Weil group of A over the function field of B, and similarly for $A(C)$ if C is a general conic (or any other smooth curve).
(2) If B is a locally closed subset of $\mathbb{P}^N$, then one way to produce a generically finite, generically unramified morphism to $\mathbb{P}^n$ is via generic projection. In this case, a very general f-line with respect to a general projection is just a very general linear section curve. A very general conic will just be a general linear section of the intersection of B with a very general quadric.
(3) There is nothing special about line-pairs, resp. conics. The proof works for any type of curve which is at least as "complex" as line-pairs, resp. conics. In particular, in characteristic zero, we can replace conics with "curves of degree at least 2."
(4) The proof of the characteristic zero portion of this theorem uses a partial compactification of the pair (B, A) to a Néron model $(\widetilde{B}, \widetilde{A})$. A general line-pair, resp. conic, C in $\widetilde{B}$ is projective. Thus, using the Hilbert scheme for instance, the sections of $C \times_{\widetilde{B}} \widetilde{A}$ over C are the k-points of a naturally defined group k-scheme Σ_C. If char$(k) = 0$, this is a reduced group

scheme. But if $\mathrm{char}(k) > 0$, this group scheme is sometimes nonreduced, cf. [**8**, Proposition 3]. This is one reason for the $\mathrm{char}(k) = 0$ hypothesis for conics. To extend our argument to positive characteristic, one would need to prove there exists a homomorphism of group schemes over B

$$\Sigma_C \times_k \widetilde{B} \to \widetilde{A}$$

splitting the restriction map. We do not know whether such a splitting always exists.

(5) In [**3**], N. Fakhruddin has proven a weaker version of the characteristic zero case of this theorem under the hypothesis that $A \to B$ is a proper Abelian scheme and B admits a compactification with boundary in codimension at least 2. It is interesting that it is exactly the divisorial component of the boundary that causes most of the difficulties in our argument, given that our methods are completely different.

Via a standard descent argument, we get as a corollary a strengthening of a theorem from [**4**].

Corollary 1.5. *With A and B as above, for every nonzero element $[\mathcal{T}] \in H^1_{\acute{e}t}(B, A)$, for C a very general line-pair or a very general planar surface, the restriction $[\mathcal{T}|_C] \in H^1_{\acute{e}t}(C, C \times_B A)$ is nonzero. In characteristic zero, we can also take C to be a very general conic.*

Remark 1.6. The group $H^1_{\text{ét}}(B, A)$ is a torsion group. The subgroup

$$H^1_{\text{ét}}(B, A)' \subset H^1_{\text{ét}}(B, A)$$

of elements whose order is not divisible by $\mathrm{char}(k)$ is countable (it is the whole group if $\mathrm{char}(k) = 0$). Therefore, Corollary 1.5 implies that for C a very general line-pair, resp. a very general conic in characteristic zero, the restriction map

$$H^1_{\text{ét}}(B, A)' \to H^1_{\text{ét}}(C, C \times_B A)'$$

is injective.

In the last section, we give some examples related to the theorem and the corollary to indicate limits to possible generalizations.

Proposition 1.7. (i) *There exists a family of elliptic curves $E \to U$ over a dense open $U \subset \mathbb{P}^2$ so that for every line L in $\mathbb{P}^2$, the mapping*

$$E(\mathbb{P}^2) \to E(L \cap U)$$

is not surjective.

(ii) *There exist B and A such that every dense open subset of the parameter space of conics contains a conic C in B for which*

$$A(B) \to A(C)$$

is not surjective.

(iii) *There exist B and A such that for every conic C in B the map*

$$H^1_{\acute{e}t}(B, A) \to H^1_{\acute{e}t}(C, A)$$

is not surjective.

(iv) *If* $char(k) = p$ *is positive, there exist* B *and* A *such that for every line pair* C*, resp. conic* C*, the map*

$$H^1_{ét}(B, A) \to H^1_{ét}(C, A)$$

is not injective. More precisely, there exists an A*-torsor* T *over* B *(depending on* C*) whose order equals* p *and whose restriction* $C \times_B T$ *is a trivial* $C \times_B A$*-torsor over* C*.*

Acknowledgments. We are very grateful to A. J. de Jong who pointed out that a purely inseparable Bertini result as in Corollary 2.2 must exist, who recommended replacing an earlier argument by an application of the Néron extension property, and who explained to us Moret-Bailly's counterexamples from [**8**]. We also thank the anonymous referees for their comments.

2. A Bertini theorem

One consequence of the classical Bertini theorem is that sections of a finite, separable cover of a quasi-projective scheme B of dimension ≥ 2 are detected by the restriction of the cover to a general hyperplane section of B. In this section we recall this and extend the result to covers which may not be separable.

Theorem 2.1. [**5**, Théorème 4.10, 6.10] *Let* B *be an integral scheme and let* $f : B \to \mathbb{P}^N_k$ *be a finite type morphism. If* f *is generically unramified, then for a general hyperplane* H*,* $B \times_{\mathbb{P}^N_k} H$ *is geometrically reduced. If* $dim(f(B)) \geq 2$*, then for a general hyperplane* H*,* $B \times_{\mathbb{P}^N_k} H$ *is geometrically irreducible.*

A straightforward consequence is the separable case of the following result. We also explain the inseparable case.

Corollary 2.2. *Let* B *be an integral scheme of dimension* ≥ 2 *and let* $f : B \to \mathbb{P}^N_k$ *be a generically unramified, finite type morphism. Let* $g : X \to B$ *be a generically finite morphism. For a general hyperplane* $H \subset \mathbb{P}^N_k$*, the restriction map from the set of rational sections of* g *over* B *to the set of rational sections of*

$$g_H : X \times_{\mathbb{P}^N_k} H \to B \times_{\mathbb{P}^N_k} H$$

is a bijection.

Proof. If g has a rational section, its restriction to $B \times_{\mathbb{P}^N_k} H$ is a rational section of g_H for general H. By Noetherian induction, it suffices to consider the case that X is irreducible. If g is not dominant, the result is clear. If g has a rational section, then g is birational and again the result is clear. Thus assume g is dominant and has no rational section. To prove the corollary, we must prove that also g_H has no rational sections for H a general hyperplane.

Because g is dominant, there is an extension of fraction fields

$$g^* : k(B) \to k(X).$$

Because g is generically finite, this is a finite field extension. Because there is no rational section, it is a nontrivial field extension, i.e., it has degree $d > 1$.

Case I. $k(X)/k(B)$ **is separable.** If $k(X)/k(B)$ is separable, then the composition $g \circ f : X \to \mathbb{P}^N_k$ is generically unramified. By Theorem 2.1 applied to $f \circ g$, for a general H, $X \times_{\mathbb{P}^N_k} H$ is integral. By generic flatness, the morphism

$$g_H : X \times_{\mathbb{P}^N_k} H \to B \times_{\mathbb{P}^N_k} H$$

is generically finite and flat of degree d. Thus $k(B \times_{\mathbb{P}^N_k} H) \to k(X \times_{\mathbb{P}^N_k} H)$ is a finite field extension of degree $d > 1$. Since the extension is not degree 1, it has no splitting. Thus g_H has no rational section.

Case II. $k(X)/k(B)$ **is not separable.** In this case, the differential of g is not surjective at a general point of X. It follows that the same is true of g_H for a general H. This precludes the possibility of a section of g_H. □

3. Elementary reductions

The next reduction uses part of the theory of Chow's K/k-trace. We review the part that we need.

DEFINITION 3.1. Let B be an integral, smooth, quasi-projective scheme over an algebraically closed field k. Let A be an Abelian scheme over B. A *Chow B/k-trace of A* is an initial pair $(\mathrm{Tr}_{B/k}(A), u)$ of an Abelian k-scheme $\mathrm{Tr}_{B/k}(A)$ and a morphism of Abelian schemes over B,

$$u : B \times_k \mathrm{Tr}_{B/k}(A) \to A,$$

i.e., for every pair (A_0, v) of an Abelian k-scheme A_0 and a morphism of Abelian schemes over B,

$$v : B \times_k A_0 \to A,$$

there exists a unique morphism of Abelian k-schemes

$$w : A_0 \to \mathrm{Tr}_{B/k}(A)$$

such that $v = u \circ (\mathrm{Id}_B \times w)$.

The basic result concerning the Chow trace is the following.

THEOREM 3.2. [**7**, §VIII.3], [**2**]

(i) *For every integral, smooth, quasi-projective k-scheme B and every Abelian scheme A over B, there exists a Chow B/k-trace of A.*

(ii) *Let E be an integral, smooth, quasi-projective k-scheme and let $E \to B$ be a dominant k-morphism such that $k(B)$ is separably closed in $k(E)$. The induced morphism of Abelian k-schemes*

$$w : Tr_{B/k}(A) \to Tr_{E/k}(E \times_B A)$$

is an isomorphism.

REMARK 3.3. A dense open immersion $U \to B$ satisfies the condition in (ii). Taking the limit over all dense open subsets of B, the Chow trace depends only on the field extension $k(B)/k$ and the Abelian $k(B)$-scheme $A \otimes_{\mathcal{O}_B} k(B)$. Usually the Chow trace is formulated for pairs $(K/k, A_K)$ of a field extension K/k and an Abelian K-scheme. It is more useful for us to formulate it as above.

The Chow trace is closely related to the property of isotriviality of an Abelian scheme.

DEFINITION 3.4. Let B be an integral, smooth, quasi-projective k-scheme. An Abelian scheme A over B is *non-isotrivial*, resp. *strongly non-isotrivial*, if for the geometric generic point of $B \times_k B$,

$$(p, q) : \mathrm{Spec}\ \kappa \to B \times_k B$$

the Abelian κ-schemes

$$A_p := \text{Spec}\ \kappa \times_{p,B} A, \quad A_q := \text{Spec}\ \kappa \times_{q,B} A$$

are not isomorphic, resp. there is no nonzero morphism of Abelian κ-schemes

$$A_p \to A_q.$$

LEMMA 3.5. *Let B be an integral, smooth, quasi-projective k-scheme. An Abelian scheme A over B is strongly non-isotrivial if and only if $Tr_{k(B)^{sep}/k}(A \otimes_{\mathcal{O}_B} k(B)^{sep})$ is zero.*

PROOF. First of all, dual to the morphism of Abelian varieties

$$u : \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k k(B)^{\text{sep}} \to A_{k(B)^{\text{sep}}},$$

there is a morphism

$$v : A_{k(B)^{\text{sep}}} \to \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k k(B)^{\text{sep}}$$

such that $v \circ u$ is multiplication by some positive integer N. Denote by p^*v and q^*u the pullbacks of these morphisms by the morphisms $p, q : \text{Spec}\kappa \to B$. If $\text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}})$ is nonzero, the composition

$$A_p \xrightarrow{p^*v} \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k \kappa \xrightarrow{q^*u} A_q$$

is a nonzero homomorphism.

Denote by $e_p, e_q : k(B)^{\text{sep}} \to \kappa$ the field monomorphism associated to projection p, resp. q. Then, conversely, every nonzero homomorphism

$$A_p \to A_q$$

factors through

$$\text{Tr}_{e_p}(A_q) \otimes_{k(B)^{\text{sep}}, e_p} \kappa \to A_q.$$

By Theorem 3.2, this second map is precisely

$$e_q^* u : \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k \kappa \to A_q.$$

So if there is a nonzero homomorphism $A_p \to A_q$ (or symmetrically $A_q \to A_p$), then $\text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}})$ is nonzero. □

LEMMA 3.6. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 1$. Let Q be an Abelian scheme over B. If $Tr_{B/k}(Q) = 0$, then there are at most countably many sections of Q over B.*

PROOF. Let $\overline{B}$ be a normal, projective scheme containing B as a dense, open subscheme. Let $\overline{Q} \to \overline{B}$ be a projective morphism whose restriction over B equals Q. There is a Chow variety parametrizing cycles in $\overline{Q}$. The Chow variety has countably many irreducible components, for the usual reason (countably many Hilbert polynomials, etc.). The claim is that every cycle $Z \subset \overline{Q}$ for which $Z \to B$ is birational gives an isolated point of the Chow variety.

Towards this claim, let T be an irreducible, quasi-projective curve and let $Z \subset \overline{Q} \times_k T$ be a cycle such that $Z \to \overline{B} \times_k T$ is birational. Replacing T by a dense open subset if necessary, assume T is smooth. Then $Z \cap (B \times_k T)$ is the graph of a B-rational transformation,

$$F : B \times_k T \dashrightarrow Q.$$

By [**6**, Theorem VI.1.9.3], this rational transformation is regular. Fix a point $t_0 \in T$ and denote

$$G : B \times_k T \to Q, \quad G(q,t) = F(q,t) - F(q,t_0).$$

Denote by

$$e : T \to \mathrm{Alb}(T)$$

the Albanese morphism sending t_0 to 0. Then, by the universal property of the Albanese, the morphism G factors through

$$\mathrm{Id}_B \times e : B \times_k T \to B \times_k \mathrm{Alb}(T).$$

Because $\mathrm{Tr}_{B/k}(Q)$ is trivial, by the universal property of the trace, the induced homomorphism of Abelian schemes over B,

$$\widetilde{G} : B \times_k \mathrm{Alb}(T) \to Q$$

is the zero homomorphism. Thus G is the zero map, i.e.,

$$F(q,t) = F(q,t_0)$$

for every $(q,t) \in B \times_k T$. Thus $Z \cap (B \times_k T)$ is independent of $t \in T$. Since Z is the closure of $Z \cap (B \times_k T)$, the same holds for Z, i.e., $Z = Z_0 \times_k T$ for a cycle $Z_0 \subset \overline{Q}$. Therefore every k-morphism from an irreducible, quasi-projective curve T to the Chow variety parametrizing rational transformations Z is constant. In other words, every cycle $Z \subset \overline{Q}$ with $Z \to \overline{B}$ birational gives an isolated point of the Chow variety. □

Corollary 3.7. *Let B be an integral, smooth, quasi-projective k-scheme and let A be an Abelian scheme over B. There exists a dense open subscheme $U \subset B$, a finite, étale, Galois morphism $U' \to U$, an Abelian k-scheme A_0, an Abelian scheme Q over U' and an isogeny of Abelian schemes over U',*

$$u = u_0 \oplus u_Q : (U' \times_k A_0) \times_{U'} Q \to U' \times_B A,$$

with the following properties.

(i) *The pair (A_0, u_0) is a Chow U'/k-trace of $U' \times_B A$.*

(ii) *For every dense open subset V of U and every finite, étale, Galois morphism $V' \to V \times_U U'$, the induced map of Abelian k-schemes*

$$A_0 = Tr_{U'/k}(U' \times_B A) \to Tr_{V'/k}(V' \times_B A)$$

is an isomorphism.

(iii) *The Abelian scheme Q is strongly non-isotrivial.*

(iv) *The quotient of $U' \times_B A$ by $U' \times_k A_0$ is isomorphic to Q in such a way that the composition*

$$Q \xrightarrow{u_Q} U' \times_B A \xrightarrow{quotient} Q$$

is the multiplication by N isogeny for some positive integer N.

4. Proof of the main theorem

Throughout this section we assume that B is a smooth k-variety, and A is an Abelian scheme over B. We fix a generically finite morphism $f : B \to \mathbb{P}^n$ so that we can talk about lines and line-pairs in B. In fact, up to replacing B by a dense open subscheme (which changes none of the results), we may assume that f is a finite, étale morphism onto a dense, Zariski open subset $U \subset \mathbb{P}^n$.

To begin, we prove the first part of our theorem for strongly non-isotrivial families of Abelian varieties. This is essentially the same argument as in [**4**] but it is somewhat easier in this context.

Proposition 4.1. *Let A be a strongly nonisotrivial Abelian scheme over B. Over a very general line-pair X in B, any section of A_X is contained in a unique section of A.*

Proof. Fix a general point $b \in B$ and consider the family D of f-lines through b. The restriction of the universal curve $C_{D_b} \to D$ maps birationally onto B. Over D we have a universal parameter space for sections of A over f-lines. Technically, the scheme structure on this parameter space depends on the choice of projective compactifications of B and A; however, as explained presently, we care only about an associated subset of A that is independent of this choice. The parameter space is a countable union of subschemes, and each subscheme that dominates D is generically finite over D. (This follows easily from the fact that over a generic f-line, the restricted family of Abelian varieties will still be strongly non-isotrivial, and hence have a finitely generated group of rational sections.) Denote the union of the closures of the images of these schemes in A by Ω. Thus Ω is a countable union of closed subvarieties; this set is independent of the choice of projective compactifications of A and B.

If we choose a very general element L_2 of D, and consider the countable set of sections of A over that curve, then each section that is not contained in Ω can meet Ω in at most a countable set of points. We conclude that over a very general point of L_2 any section of A over L_2 that meets Ω is contained in Ω. Thus, if we choose a very general point of L_2, and if we connect it to b via the unique f-line L_1, then we find that any section over the resulting line-pair will have the property that its values over L_1 are contained in Ω (by definition of Ω), and hence its values over L_2 are contained in Ω. But by the Bertini Theorem, Corollary 2.2, any section of Ω over L_2 (which is a very general f-linear curve) extends to a unique section of Ω and hence of A. Moreover, since a point of intersection of L_1 and L_2 is very general in B, it follows that every irreducible component of Ω whose image contains this point is, in fact, étale over that point, and any two distinct components of Ω have disjoint fibers over this point. Hence the section over L_1 extends to the same unique section. □

A very similar argument would establish the analogous lemma for A a trivial family of Abelian varieties over B with line pairs replaced by "triangles." To obtain the desired result seems to require a slightly more elaborate approach. We break this into a series of lemmas.

Lemma 4.2. *Let A_0 be an Abelian variety over k. Let U be a Zariski open subset of $\mathbb{P}^n$ and let $T \to U$ be an A_0 torsor. If T becomes trivial upon restriction to the intersection of U with a general line, then T is trivial.*

PROOF. We work by induction on n with the case of $n = 1$ vacuous. Fix a general hyperplane H, and trivialize T over $H \cap U$. Now fix a general point p not on H. For a general line L through p, there will be a unique section of T over L which takes the value 0 at $L \cap H$ (with respect to our fixed trivialization.) These distinguished sections then sweep out a rational section of T over all of $\mathbb{P}^n$. $\square$

We now introduce some extra notation. As before we have a dense, Zariski open $U \subset \mathbb{P}^n$ and a finite, étale morphism $f : B \to U$. We set $Y = B \times_{\mathbb{P}^n} B = B \times_U B$. Given an Abelian scheme A over $\mathbb{P}^n$ and a rational section s of f^*A, we set ∂s to be the Cech boundary of s, that is to say the rational section of the pullback of A to Y given by the difference of the section pulled back under the two projections to B. In particular, at a general point of Y, which will be of the form (b_1, b_2), the value of ∂s is just $s(b_1) - s(b_2)$.

There is an analogous boundary operator for sections over lines, and formation of the boundary commutes with restriction of sections.

LEMMA 4.3. *Fix an Abelian variety A_0, a connected curve C and a finite map $f : C \to \mathbb{P}^1$. Given a family of morphisms $s_t : C \to A_0$, over a connected, reduced base T, the associated family of boundaries $\partial s_t : C \times_{\mathbb{P}^1} C \to A_0$ is constant.*

PROOF. Because A_0 is separated, it suffices to prove this over a dense open subset of T. By the valuative criterion of properness, for a projective compactification $\overline{C}$ of C, s_t extends to a morphism on $\overline{C}$ for t in a dense open subset of T. Thus, without loss of generality, assume that C is projective.

The tangent space to the space of morphisms from C to A_0 is given by the sections of s^*TA_0. But all of these deformations come from translating the fixed section elements of A_0. Thus each connected component of the Hom scheme is just a copy of A_0 and comes from translations of a fixed morphism. As translation does not affect the boundary, the result follows. $\square$

For the next three results, we let A_0 be a fixed Abelian variety over k and let $A = B \times A_0$ be the corresponding trivial family of Abelian varieties over B.

LEMMA 4.4. *Let C be a very general line pair in B. For every section s of A over C, the boundary ∂s extends to a unique section t of A over Y.*

PROOF. The proof here is identical to the proof of Proposition 4.1. The key point is that by virtue of the previous lemma, the images of boundaries of sections over lines in Y through a fixed y again sweep out a countable union of subschemes of $Y \times A_0$ which are finite over Y. $\square$

LEMMA 4.5. *Let $t : Y \to A$ be a section of A over Y. Assume that for a general line $L \subset \mathbb{P}^n$, there exists a section $s_L : f^{-1}L \to A$ whose boundary ∂s_L equals the restriction t_L of t over $Y \times_{\mathbb{P}^n} L$. Then there exists a section $s : B \to A$ such that $t = \partial s$, and s is unique up to a constant section.*

PROOF. We consider the problem of producing such a section s. Let U be an open subset of $\mathbb{P}^n$ over which f is étale. Fix a general point $p \in U$, and consider the finite set $f^{-1}(p) = \{b_1, \ldots, b_d\}$. If we choose a value $s(b_1)$, then the equation $\partial s = t$ determines the values of the other b_i uniquely. Indeed, they are overdetermined, but the existence of a section s_L over a general line implies the existence of a solution. Given a solution, we can always add a constant from A without affecting ∂s. We conclude that solving the equation $\partial s = t$ is equivalent to trivializing an A_0 torsor over U. By Lemma 4.2 we are done. $\square$

Combining Lemmas 4.4 and 4.5 immediately yields the desired analogue of Proposition 4.1.

PROPOSITION 4.6. *For a trivial Abelian scheme $A = A_0 \times B$ over B, every section over a very general line-pair in B is contained in a unique section over all of B.*

Combining the strongly isotrivial case with the trivial case, we can now prove the main theorem for line pairs.

THEOREM 4.7. *For every Abelian scheme $A \to B$, every section over a very general line pair X is contained in a unique section over all of B.*

PROOF. By Corollary 3.7 we can find an open set $V \subset B$ and a finite étale Galois cover $V' \to V$ such that the pullback of A to V' is isogenous to a product of a strongly nonisotrivial family of Abelian varieties and a trivial family. First we will prove the theorem for the pullback family, $A' \to V'$. Let $u : A' \to Q \times A_0$ be the isogeny. Let s be a section of A'_X. Then $u(s)$ is a section of $Q \times A_0$. By the lemmas above, $u(s)$ is the restriction of a unique section S over all of V'. However, the preimage $u^{-1}(S(V'))$ of $S(V')$ in A' is generically finite over V'. By Corollary 2.2, every section of $u^{-1}(S(V'))$ over X extends to a unique section over all of A'.

We conclude that the theorem holds for A' over V'. Denote by G be the Galois group of the Galois covering $V' \to V$. Then the sections $A(V)$ over V are precisely the G-invariants, $A'(V')^G$, of the sections $A(V')$ over V'. Similarly, $A_X(X)$ equals $A'_{X'}(X')^G$. Since the map $A'(V') \to A'(X')$ is an isomorphism, so is the map on G-invariants. $\square$

COROLLARY 4.8. *For every Abelian scheme $A \to B$, for a very general planar surface $S \subset B$, the restriction map $A(B) \to A(S)$ is an isomorphism.*

PROOF. Applying Theorem 1.3 to both S and B and using the fact that a very general line-pair X in a very general planar surface S is a very general line-pair in B, we find a line pair X such that $A(X) = A(B)$ and $A(X) = A(S)$. $\square$

The hardest part of the argument is to deduce that in characteristic zero, line-pairs can be replaced by smooth conics. The idea of the proof is quite simple: specialize conics to line-pairs. To make effective use of this we will need the existence of Néron models. Much of the rest of this paper can be generalized from Abelian varieties to arbitrary varieties without rational curves. However, without the strong control of degenerations provided by Néron models, we see no way to prove an analogue of the theorem for conics.

First, we remind the reader of the definitions and fundamental existence result regarding Néron models in this setting.

DEFINITION 4.9. Let T be an integral, normal, separated, one dimensional, Noetherian scheme. A smooth, finite type, separated morphism $X \to T$ has the *Néron extension property* if for every triple $(Y \to T, U, s_U)$ of

(i) a smooth morphism $Y \to T$,
(ii) a dense, open subset $U \subset T$,
(iii) and a T-morphism $s_U : Y \times_T U \to X$,

there exists a T-morphism $s : Y \to X$ whose restriction to $Y \times_T U$ equals s_U. If $X \to T$ has the Néron extension property, then it is called a *Néron model*, or a *Néron model of its generic fiber*.

When working with schemes of dimension greater than 1, it is natural to reformulate this as follows.

DEFINITION 4.10. Let T be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. A smooth, finite type, separated morphism $X \to T$ has the *(variant) Néron extension property* if for every triple $(Y \to T, U, s_U)$ of

(i) a smooth morphism $Y \to T$,
(ii) a dense, open subset $U \subset T$,
(iii) and a T-morphism $s_U : Y \times_T U \to X$,

there exists a pair (V, s_V) of

(i) an open subset $V \subset T$ containing U and all codimension 1 points of T
(ii) and a T-morphism $s_V : Y \times_T V \to X$ whose restriction to $Y \times_T U$ equals s_U.

If $X \to T$ has the Néron extension property, then it is called a *(variant) Néron model*, or a *(variant) Néron model of its generic fiber.*

Obviously this definition agrees with the usual definition when T is one dimensional. Also, when they exist, (variant) Néron models are unique in codimension 1. Indeed, let $X_1 \to T$ and $X_2 \to T$ be (variant) Néron models. Let $U \subset T$ be a dense, open subset. Let $U \times_T X_1 \cong U \times_T X_2$ be a T-isomorphism. By the (variant) Néron extension property, there exists an open subset $V \subset T$ containing U and all codimension 1 points and a T-isomorphism $V \times_T X_1 \cong V \times_T X_2$ extending the isomorphism over U. Thus, Néron models are unique in codimension 1.

The basic result about existence of Néron models is the following.

THEOREM 4.11. [**1**, Theorem 3, p.19] *Let T be an integral, normal, separated, one-dimensional, Noetherian scheme. Let $B \subset T$ be a dense open subset. Let $A \to B$ be an Abelian scheme. There exists a Néron model $\widetilde{A}$ over T whose restriction over B equals A.*

Of course this implies the existence of (variant) Néron models in the usual way.

COROLLARY 4.12. *Let W be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. Let B be a dense open subset of W and let A be an Abelian scheme over B. There exists an open subset $\widetilde{B}$ of W containing B and all codimension 1 points and a (variant) Néron model $\widetilde{A}$ over $\widetilde{B}$ whose restriction over B equals A.*

PROOF. The complement $W \setminus B$ is a quasi-compact closed, Noetherian scheme. It has only finitely many generic points. Thus it contains at most finitely many codimension 1 points of W. For every such codimension 1 point η of W, the scheme $T = \mathrm{Spec}\mathcal{O}_{W,\eta}$ is an integral, normal, separated, one-dimensional, Noetherian scheme. By Theorem 4.11, there exists a Néron model $\widetilde{A}_\eta$ of the pullback of A to T. By limit arguments, there exists an open affine $W_\eta \subset W$ containing η and a smooth, finite type, separated morphism $X_\eta \to W_\eta$ whose base change to T is the Néron model and whose base change to $W_\eta \cap B$ equals the restriction of A. Moreover, up to shrinking W_η, we may assume that W_η contains no generic point of $W \setminus B$ other than η. Thus for two distinct codimension 1 points, η_1 and η_2, the intersection $W_{\eta_1} \cap W_{\eta_2}$ is contained in B. Since the morphisms X_{η_1} and X_{η_2} both equal the restriction of A over $W_{\eta_1} \cap W_{\eta_2}$, these morphisms glue to a smooth, finite type, separated morphism $\widetilde{A} \to \widetilde{B}$ over the union $\widetilde{B}$ of B and each of the opens W_η.

Now let $(Y \to \widetilde{B}, U, s_U)$ be a triple as in Definition 4.10. Denote by V the maximal open subscheme of $\widetilde{B}$ over which s_U extends to a W-morphism s_V to $\widetilde{A}$. For every codimension 1 point η of $\widetilde{B} \setminus B$, by the N'eron extension property of $\widetilde{A}_\eta$, the base change of s_U over $\mathrm{Spec}\mathcal{O}_{W,\eta}$ extends to a morphism to $\widetilde{A}$. By limit theorems, there exists an open affine $V_\eta \subset \widetilde{B}$ containing η and an extension of s_U over V_η. Thus V_η is contained in V, i.e., η is contained in V. Therefore V contains every codimension 1 point of $\widetilde{B}$. □

Because of this, we will no longer make any distinction between the original definition of Néron models and the variant definition above. Also, we would like to clarify one point of confusion identified by the referees. It is tempting to imagine that for a Néron model $X \to T$, for every smooth morphism $Y \to T$, also the base change $Y \times_T X \to Y$ is a Néron model; in particular, this would imply that the restriction of X to a "sufficiently general" closed subscheme of T is also a Néron model. Unfortunately, in general there is no such base change property of Néron models. For instance, let T be the spectrum of the DVR, $\mathcal{O}_T = k[t]_{\langle t \rangle}$ with fraction field $K = k(t)$, let $\widetilde{T}$ be the spectrum of the finite, flat extension, $\mathcal{O}_{\widetilde{T}} = \mathrm{Spec}(k[t]_{\langle t \rangle})[y]/\langle y^2 - t\rangle$ with fraction field $L = k(y)$, and let X be $\mathbb{P}^1_L$, considered as a T-scheme. The claim is that X is a Néron model. The point is that for every smooth scheme $Y \to T$, if there exists a morphism $s_K : Y_K \to X_K$, then the composition with projection $X \to \widetilde{T}$ is a T-morphism that extends to an open subscheme U of Y containing every generic point η of the closed fiber Y_k, by the valuative criterion of properness. However, since $\widetilde{T} \to T$ is ramified and factors through $\mathrm{Spec}\mathcal{O}_{Y,\eta} \to T$, also this morphism is ramified. The conclusion is that Y_k is empty, so that the Néron extension property is vacuously true. On the other hand, there are plenty of smooth morphisms $Y \to T$ such that the pullback of X is not a Néron model, e.g., the projective space over K, $Y = \mathbb{P}\mathrm{Hom}_K(K^{\oplus 2}, K^{\oplus 2})$, containing the automorphism group $\mathbf{PGL}_{2,K}$ of $\mathbb{P}^1_K$ as a dense open subscheme. The corresponding universal automorphism of $\mathbb{P}^1_K$ over $\mathbf{PGL}_{2,K}$, or rather the pullback to an automorphism of $\mathbb{P}^1_L$, does not extend over the discriminant divisor in Y. Thus, to make this very clear, we are *not* asserting that Néron models are compatible with smooth base change, nor that the restriction of a Néron model over a "sufficiently general" closed subscheme of T is still a Néron model. Rather, we will apply the Néron extension property to a carefully constructed smooth morphism $Y \to T$.

Just to simplify notation, we will assume from now on that B is a smooth surface (which we are free to do in any case by Corollary 4.8.) We fix a projective completion W of B so that $f : B \to \mathbb{P}^2$ extends to W and choose a Néron model $\widetilde{A} \to \widetilde{B}$ as above. Note that $\widetilde{B}$ is simply the complement of a finite set of points in W.

Now the space of conics in W is $\mathbb{P}^5$ and the space of line-pairs $\mathcal{X} \subset \mathbb{P}^5$ is the discriminant locus. We denote the open locus of conics or line pairs contained in $\widetilde{B}$ by $\mathbb{P}^5_0$ and $\mathcal{X}_0$ respectively. Over $\mathbb{P}^5_0$ we have a universal space of sections over conics. That is to say, there is a scheme H, and a diagram

$$\begin{tikzcd}
& & \widetilde{A} \arrow[d, "\pi"] \\
\mathcal{C}_H \arrow[r] \arrow[d] \arrow[urr] & \mathcal{C} \arrow[r] \arrow[d] & \widetilde{B} \\
H \arrow[r, "\Phi"] & \mathbb{P}^5_0 &
\end{tikzcd}$$

with the left square cartesian, which is universal for the problem of lifting conics from B to A. The scheme H is locally of finite type, but may have infinitely many irreducible components. The key lemma that we need is the following.

LEMMA 4.13. *In characteristic zero, any irreducible component H_0 of H which dominates $\mathbb{P}^5$ also dominates $\mathcal{X}$. That is, the intersection of the image $\Phi(H_0)$ with $\mathcal{X}$ contains a dense open subset of $\mathcal{X}$.*

REMARK 4.14. To put it differently, this lemma is stating that sections over conics specialize to sections over line-pairs. On completely general grounds, given any degeneration of a conic to a line-pair, X, any section over the general conic will specialize to a *stable section* over X, that is, a section over each irreducible component of X, together with a tree of rational curves over each node of X, connecting the corresponding sections, and some collection of trees of π-contracted rational curves attached to the section over other points of X. Moreover, since the nodes of X will be at general points of B, over which there are no rational curves in the fiber of π, we conclude that any section over a general conic will specialize to a section over X with trees of rational curves attached over those points of X which meet the locus in $\widetilde{B}$ over which the fiber fails to be an Abelian variety. The content of the lemma then, is that in fact no such trees occur, at least generically.

PROOF. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & B \\
\mathcal{X} \arrow[r] & \mathbb{P}^5 &
\end{tikzcd}$$

where $\mathcal{C}$ is the universal conic and $\mathcal{C}_{\mathcal{X}}$ is the universal line-pair *with the nodes deleted* so that the leftmost vertical morphism is smooth. Also, the map from $\mathcal{C}$ to B as well as the composition map from $\mathcal{C}_{\mathcal{X}}$ to B are smooth, since they arise by base change from the corresponding maps when $B = \mathbb{P}^2$ where the smoothness is obvious.

We choose a compactification $\overline{H}$ of H_0 such that Φ extends to $\overline{\Phi} : \overline{H} \to \mathbb{P}^5$, and so that $\overline{H}$ is regular at a general point of $\overline{\Phi}^{-1}(\mathcal{X})$. Set $\overline{H}_{\mathcal{X}} = \overline{\Phi}^{-1}(\mathcal{X})_{red}$. Since the characteristic is 0, we have the generic smoothness theorem: there exists a dense open set $V \subset \overline{H}_{\mathcal{X}} \cap \overline{H}^{reg}$ so that the restricted morphism $\overline{\Phi} : \overline{H}_{\mathcal{X}} \to \mathcal{X}$ is smooth on V. Consider the following diagram.

$$\begin{tikzcd}
\mathcal{C}_{\overline{H}_{\mathcal{X}}} \arrow[rr] \arrow[d] \arrow[dr] & & B \\
\overline{H}_{\mathcal{X}} \arrow[dr] & \mathcal{C}_{\mathcal{X}} \arrow[ur] \arrow[d] & \\
& \mathcal{X} &
\end{tikzcd}$$

We see that over V, $\mathcal{C}_{\overline{H}_{\mathcal{X}}} \to B$ is smooth. Then by the regularity of $\overline{H}$, it follows that there is an open set $U \subset \mathcal{C}_{\overline{H}}$ containing $\overline{\Phi}^{-1}(V)$ such that the restriction $U \to B$ is smooth. We include this diagram to clarify the situation.

$$\begin{tikzcd}
& & & A \arrow[d] \\
\mathcal{C}_{\mathcal{X}} \arrow[d] & \mathcal{C}_V \arrow[l] \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] \arrow[ur, dashed, "\rho"] & B \\
\overline{H}_{\mathcal{X}} & V \arrow[l] \arrow[r] & \overline{H} &
\end{tikzcd}$$

Now if we apply the Neron property to the rational map $U \to A$, we find that outside a codimension 2 subset $\zeta \subset B$, it extends to a regular morphism. Since any codimension two subset in B is avoided by a general line-pair, it follows that the rational map ρ which was *a priori* defined over H_0 extends to a regular map over an open set of $\mathcal{C}_{\overline{H}}$ which contains the preimage of a dense open subset of $\mathcal{X}$. Then, since we have already mentioned that we can extend any such rational map over the nodes of the line-pair, the universal property of H implies the lemma. $\square$

Theorem 4.15. *Over a field of characteristic zero, over a very general conic C, every section of A_C over C is the restriction of a unique section of A over B.*

Proof. First, denoting by Y the scheme parameterizing rational sections of A over B, there is a natural rational map $\mathbb{P}^5 \times Y \dashrightarrow H$. We observe that the image of this map is a union of irreducible components of H. This follows immediately from the fact that over a very general point of $\mathcal{X}$, this rational map is regular and in fact an isomorphism (at least on reduced subschemes) by Theorem 4.7.

Now, by way of contradiction, suppose that the theorem is false. Then there exists an irreducible component of H dominating $\mathbb{P}^5$ such that a generic section parameterized by this component is not the restriction of a section. By Lemma 4.13, this component dominates $\mathcal{X}$. But then we see that this component intersects one of the distinguished irreducible components mentioned above. This quickly yields a contradiction, since it implies that the fiber of H over a very general point of $\mathcal{X}$ is nonreduced. As H is a group scheme locally of finite type, this violates Cartier's theorem on reducedness of locally finite type group schemes in characteristic 0. $\square$

5. Examples

In this section we give some examples demonstrating limits to further generalization of Theorem 1.3 and Corollary 1.5.

Example (i). Let B be a smooth quadric in $\mathbb{P}^3_k$ and let f be projection from a general point outside of B. Let A_0 be an ordinary elliptic curve over k and let $A = B \times_k A_0$. Then

$$H^0_{\text{ét}}(B, B \times_k A_0) = A_0(k).$$

Since f conics are curves of type (2,2) on B, every open subset of the space of conics contains curves C isogenous to A_0. For these, the map

$$A_0(k) \to H^0_{\text{ét}}(C, C \times_k A_0)$$

is not surjective. Thus, it is necessary to use *very* general conics in Theorem 1.3 – general conics do not suffice.

Example (ii). Fix a field k and let p be a point of $\mathbb{P}^2_k$. Let B be the space of smooth cubic curves passing through p. Thus B is an open subset of $\mathbb{P}^8$. Let $A \to B$ be the universal smooth cubic curve passing through p, which becomes an Abelian scheme upon declaring p to be the origin. Now it is easy to see that $A(B) = \mathbb{Z}$ but if we fix a general line $L \subset B$, then L corresponds to a pencil of plane cubics with nine base points. These give rise to sections of $A_L \to L$ which are not restrictions of sections over all of B, so we see that Theorem 1.3 would not hold with line pairs or conics replaced with lines. (Note that we can replace $\mathbb{P}^8$ by $\mathbb{P}^2$ in this example by choosing a very general plane in $\mathbb{P}^8$ by virtue of Theorem 1.3 although it is also easy to argue this directly.)

Example (iii). The restriction map

$$H^1_{\text{ét}}(B, A) \to H^1_{\text{ét}}(C, C \times_B A)$$

can fail to be surjective for all line pairs, resp. all conics. For an example, let B be a smooth quadric surface, let f be projection to the plane. and let $A = B \times_k A_0$, where A_0 is a simple Abelian k-variety of dimension $g \geq 2$. Let l be an integer not divisible by char(k). Associated to the multiplication map of group schemes over B

$$0 \longrightarrow B \times_k A_0[l] \longrightarrow B \times_k A_0 \xrightarrow{\text{mult}_l} B \times_k A_0 \longrightarrow 0$$

there is a long exact sequence of cohomology groups part of which is

$$0 \to H^0_{\text{ét}}(B, B \times_k A_0) \otimes \mathbb{Z}/l\mathbb{Z} \to H^0_{\text{ét}}(B, B \times_k A_0[l]) \to H^1_{\text{ét}}(B, B \times_k A_0)[l] \to 0.$$

Because $\text{Alb}(B) = 0$,

$$H^0_{\text{ét}}(B, B \times_k A_0) = A_0(k).$$

Because A_0 is divisible, the exact sequence above gives

$$H^1_{\text{ét}}(B, B \times_k A_0)[l] \cong H^1_{\text{ét}}(B, B \times_k A_0[l]) \cong H^1_{\text{ét}}(B, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

A similar argument applies to conics and line pairs in B. Of course the Albanese variety of a (2,2) curve on a quadric is not zero. But because A_0 is a simple Abelian variety of dimension $g \geq 2$, every morphism from C to A_0 is a constant morphism. Thus

$$H^1_{\text{ét}}(C, C \times_k A_0)[l] \cong H^1_{\text{ét}}(C, C \times_k A_0[l]) \cong H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

Thus the restriction homomorphism on l-torsion is equivalent to

$$H^1_{\text{ét}}(B, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g} \to H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

Now

$$H^1_{\text{ét}}(B, \mathbb{Z}/l\mathbb{Z}) = 0$$

whereas

$$H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z}) = (\mathbb{Z}/l\mathbb{Z})^{\oplus 2}$$

for a (2,2) curve and

$$H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}$$

for a line pair. Thus the restriction map is not surjective.

Example (iv). This example is similar. Let $B = \mathbb{A}^2_k$ and let $f : \mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ be the usual inclusion. Let A_0 be an Abelian k-variety having a nonzero p-torsion k-point,

$$a \in A_0(k)[p] \setminus \{0\},$$

e.g., A_0 is an ordinary elliptic curve over k. Let $A_0 \to A_1$ be the étale isogeny of Abelian varieties whose kernel is generate by a. There is an exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\ a\ } A_0 \longrightarrow A_1 \longrightarrow 0.$$

This gives rise to a long exact sequence. For the same reason as in Example (i), the induced map

$$H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{Z}/p\mathbb{Z}) \to H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{A}^2_k \times_k A_0)[p]$$

is an injection. Thus to produce an A_0-torsor over $\mathbb{A}^2_k$ of order p whose restriction to C is trivial, it suffices to produce a $\mathbb{Z}/p\mathbb{Z}$-torsor T over $\mathbb{A}^2_k$ whose restriction to C is trivial. Let $f \in k[x, y]$ be a polynomial function on $\mathbb{A}^2_k$ vanishing on C and whose degree d is not divisible by p.

Let $T \to \mathbb{A}^2_k$ be the Artin-Schreier cover determined by the ring homomorphism

$$k[x, y] \to k[x, y, t]/\langle t^p - t - f\rangle.$$

Because d is not divisible by p, there is no element $g \in k[x, y]$ such that $g^p - g - f = 0$. Since $k[x, y]$ is normal and $t^p - t - f$ is a monic polynomial in t, also there is no element $g \in k(x, y)$ such that $g^p - g - f = 0$. Thus T is integral and so $T \to \mathbb{A}^2_k$ is a nontrivial torsor. On the other hand, since f is zero on C,

$$C \times_{\mathbb{A}^2_k} T \cong C \times \text{Spec } k[t]/\langle t^p - t\rangle,$$

which is the trivial $\mathbb{Z}/p\mathbb{Z}$-torsor over C. Thus

$$H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{A}^2_k \times_k A_0)[p] \to H^1_{\text{ét}}(C, C \times_k A_0)[p]$$

is not injective.

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Clay Mathematics Proceedings
Volume **18**, 2013

Correspondence and Cycle Spaces: A result comparing their cohomologies

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Outline.

1. Introduction

Let G be a reductive, semi-simple Lie group, $B \subset G$ a Borel subgroup and $X = G/B$ the corresponding *flag manifold*. Let G_0 be a connected real form that contains a compact maximal torus T; this means in particular that the complexified Lie algebra $\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{h}$ is a Cartan subalgebra corresponding to a Cartan subgroup $H \subset G$. By a *flag domain* D we mean an open G_0-orbit $G_0(x_0)$ of G acting on X whose isotropy group is compact. Flag manifolds and flag domains have over the years played a central role in representation theory, both finite and infinite dimensional ([**Sch2**], [**Sch3**], [**BE**], [**FHW**] and the references cited therein). Recently they, together with more general homogeneous complex manifolds G_0/L where $L \supset T$ and L is the compact centralizer of a circle $S^1 \subset T$, have appeared in Hodge theory in the form of Mumford-Tate domains [**GGK1**]. For the case $G_0 = \mathcal{U}(2,1)$, the corresponding Mumford-Tate domains have also appeared in very interesting recent work on arithmetic automorphic representation theory ([**C1**], [**C2**], [**C3**]). In the recent exposition [**GGK2**] of aspects of that work, together with extensions of it, certain constructions concerning the complex geometry of flag domains arose. These constructions play a central role in the use of *Penrose transforms* ([**BE**], [**EGW**], [**C2**]). In the exposition [**GGK2**], in special cases they were used under the term *correspondence spaces*. In that work the general construction and properties of these spaces, together with their relation to the *cycle spaces* [**FHW**] that have been in use since the mid-to-late 1960's ([**Sch1**] and [**GS**]), were discussed. The primary purpose of the present paper is to give the formal definition and some properties of the correspondence space $\mathcal{W}$ and to state and prove a result relating the complex geometry of $\mathcal{W}$ to that of the cycle space $\mathcal{U}$.

To give the informal statement of the result we first comment that both $\mathcal{W}$ and $\mathcal{U}$ are used to relate the cohomology $H^*(D, L_\mu)$ of homogeneous line bundles

2010 *Mathematics Subject Classification.* Primary 20G05, 22E46; Secondary 14M17, 53C56.

$L_\mu \to D$ to global, holomorphic objects. In the case of the correspondence space $\mathcal{W}$ the object is a holomorphic de Rham cohomology group, and it is therefore a quotient of spaces of global holomorphic sections of vector bundles. In this case there is the isomorphism [**EGW**]

$$H^*(D, L_\mu) \cong H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \tag{1.1}$$

In the examples considered so far, there are canonical "harmonic" representatives for classes on the RHS, so that in particular cohomology classes on the LHS can be "evaluated" at points of $\mathcal{W}$. For D a Mumford-Tate domain, $\mathcal{W}$ has an *arithmetic structure* (think of CM points in a Shimura variety), and the main result of [**GGK2**] concerns classes in $H^*(D, L_\mu)$ that take arithmetically defined values at arithmetic points of $\mathcal{W}$.

The "correspondence space" arises from the following consideration: The equivalence classes of homogeneous complex structures on G_0/T are indexed by W/W_K where W, W_K are the Weyl groups of G, G_0 respectively. We denote these by D_w, $w \in W/W_K$. The universality property of $\mathcal{W}$ gives diagrams

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D_w & & D_{w'}. \end{tikzcd} \tag{1.2}$$

Using (1.1) applied to D_w and $D_{w'}$ the existence of certain canonical classes on $\mathcal{W}$ gives multiplication mappings

$$H^q\big(\Gamma(\Omega^\bullet_\pi(L_\mu)); d_\pi\big) \to H^{q'}\big(\Gamma(\Omega^\bullet_{\pi'}(L'_{\mu'})); d_{\pi'}\big)$$

which lead to Penrose transforms

$$H^q(D_w, L_\mu) \to H^{q'}(D_{w'}, L'_{\mu'}).$$

One may think of this as an analogue of the maps on ordinary cohomology in classical algebraic geometry induced by a cycle on $\mathcal{W}$ in a diagram (1.2) where the objects are algebraic varieties.

In the case of the cycle space $\mathcal{U}$ there is always a map

$$H^q(D, L_\mu) \to H^0(\mathcal{U}, F^{0,q}_\mu)$$

where the $F^{p,q}_\mu \to \mathcal{U}$ are holomorphic vector bundles whose rank is $h^q(Z, \Lambda^p N_{Z/D}(L_\mu))$. There are conditions under which this map is injective and a description of its image (cf. [**FHW**], Theorem (3.4) and Corollary (3.5) below).

The main result of this paper is to relate the two global holomorphic objects which realize $H^*(D, L_\mu)$. The result applies only in the case that D is *non-classical*, meaning that it does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain. This is the primary case of interest in [**C1**], [**C2**], [**C3**], [**GGK2**] as it is the situation where new geometric and arithmetic phenomena occur. The result is the

THEOREM 1.1. *In case D is non-classical there is a spectral sequence with*

$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U}, F^{p,q}_\mu) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \end{cases}$$

For D non-classical we have $\mathcal{U} \subset G/K$ and there are maximal compact subvarieties $Z_u \subset D$, $u \in \mathcal{U}$ given by the translates by $g \in G$ of $Z = K/T$ where $gZ \subset D$. Denoting by $N_{Z/D}$ the normal bundle of Z in D, the proof of the theorem will yield the

Corollary 1.2. *If* $H^k(Z, \Lambda^{q-k+1} N_{Z/D}(L_\mu)) = 0$ *for* $0 \leqq k \leqq q-1$*, then*

$$H^q\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big) \cong \ker\{\, H^0(D, F_\mu^{0,q}) \xrightarrow{d_1} H^0(D, F_\mu^{1,q})\,\}.$$

When the isomorphism (1.1) is used on the LHS this corollary is closely related to the result in [**WZ**].

The original motivation for much of the work that this paper is drawing from was concerned not with the $H^q(D, L_\mu)$ but rather with the *automorphic cohomology groups* $H^q(\Gamma\backslash D, L_\mu)$ where $\Gamma \subset G_0$ is a discrete, co-compact and neat subgroup. In section 4.1 we will show that the equation (1.1), and the results (1.1) and (1.2) remain valid as stated when the spaces are factored by Γ. The main new ingredient used here is a result from [**BHH**] which shows that the quotient space $\Gamma\backslash\mathcal{U}$ is Stein.

In section 4.2 we shall show that the de Rham cohomology

$$H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$$

may be written as $\mathfrak{n}$-cohomology $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}$ for a G_0-module $\mathcal{O}G_{\mathcal{W}}$. The spectral sequence (1.1) may then be interpreted as the Hochschild-Serre spectral sequence for $\mathfrak{n}_c \subset \mathfrak{n}$ where $\mathfrak{n}_c = \mathfrak{n} \cap \mathfrak{k}$ for $\mathfrak{k}$ the complexification of Lie algebra of the maximal compact subgroup K_0 of G_0.[1]

The differentials

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

are linear differential operators of degree r, and we shall give a result (Theorem (4.4)) defining and computing their *symbols*. We shall also describe the *characteristic varieties* in our two examples.

In section 4.3 we shall analyze the spectral sequence in the special case of the two examples discussed in [**GGK2**]. This analysis will include determination of the symbol sequence and characteristic varieties for the linear PDE systems whose solutions are the Harish-Chandra module $V_{\mu+\rho}$ with infinitesimal character $\chi_{\mu+\rho}$ where $\mu+\rho$ is in the closure of the anti-dominant Weyl chamber. Of particular interest here are the cases where $\mu+\rho$ is singular, in which case $V_{\mu+\rho}$ is a limit of discrete series. The PDE systems have quite a different character than when $\mu+\rho$ is regular, as is not surprising due to the much greater intricacy of $\mathfrak{n}$-cohomology in these cases.

This paper is a companion work to [**GGK2**], one which completes a definition promised there and which relates the global holomorphic realization of cohomology that was an essential ingredient in [**GGK2**] to the other one that has appeared in the literature. The general context for this work is the relation between representation theory and the geometry of complex homogeneous manifolds. This is a vast and rich subject and we have chosen to refer to the references in some of our primary sources, specifically [**Sch2**], [**Sch3**], [**BE**] and [**FHW**], for excellent expositions of the general theory and for guides to the literature. We also note [**Sch1**], where much of the connection between homogeneous complex manifolds and representation theory had its origin.

[1] The subscript "c" for $\mathfrak{n}_c$ refers to "compact," as $\mathfrak{n}_c$ is the direct sum of the negative root spaces corresponding to the compact roots.

This paper is dedicated to Joe Harris on the occasion of his 60^{th} birthday. The talk that the second author of this paper gave at Joe's 60^{th} conference had the theme that understanding in depth "elementary" examples that have a rich geometry is both interesting in its own right and serves to suggest interesting general structures. The examples presented in that talk were based on [**GGK2**] and are recalled briefly in this paper. We feel that the theme mentioned above is very harmonious with Joe's approach to mathematics.

Notations.

- G is a reductive, semi-simple Lie group with Lie algebra $\mathfrak{g}$;
- $H \subset G$ is a Cartan subgroup with Lie algebra $\mathfrak{h}$;
- $B \subset G$ is a Borel subgroup with associated flag manifold $X = G/B$;
- G_0, with Lie algebra $\mathfrak{g}_0$, is a real form of G;
- we assume that the real form H_0 of H is a compact maximal torus $T \subset G_0$; we shall use the notations H_0 and T interchangeably;
- $K_0 \subset G_0$ is a maximal compact subgroup with $T \subset K_0$ and complexification $K \subset G$;
- $\Phi, \Phi^+, \Phi_c, \Phi_n$ are respectively the roots, positive roots, compact roots and non-compact roots of $(\mathfrak{g}, \mathfrak{h})$;
- W, W_K are the Weyl groups of (G, H), (K_0, T) respectively;
- it is known that the homogeneous complex structures D_w on G_0/T are parametrized by $w \in W/W_K$; they are the open orbits of G_0 acting on X;
- we shall denote by D one of the D_w and by $Z \cong K_0/T \cong K/B_K$, where $B_K = K \cap B$, is a maximal compact subvariety of D;
- the root space decomposition of $\mathfrak{g}$ is denoted
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\underset{\alpha\in\Phi}{\oplus} \mathfrak{g}^\alpha \right)$$
where $\mathfrak{g}^\alpha$ is the α-root space;
- we have
$$\begin{cases} \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \\ \mathfrak{n} = \underset{\alpha\in\Phi^+}{\oplus} \mathfrak{g}^{-\alpha}; \end{cases}$$
- the Cartan decomposition of $\mathfrak{g}$ is
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p};$$
we have
$$\begin{cases} \mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \\ \mathfrak{n} = \mathfrak{n}_c \oplus \mathfrak{p}^- \end{cases}$$
where $\mathfrak{p}^+ = \underset{\alpha\in\Phi^+_{nc}}{\oplus} \mathfrak{g}^\alpha$, $\mathfrak{p}^- = \overline{\mathfrak{p}^+}$ and $\mathfrak{n}_c = \underset{\alpha\in\Phi^+_c}{\oplus} \mathfrak{g}^{-\alpha}$;
- the compact maximal torus is
$$T = \mathfrak{t}/L,$$
where L is a lattice, and we denote by
$$\Gamma \subset i\check{\mathfrak{t}}$$
the weight lattice, which up to a factor of 2π is identified with $\mathrm{Hom}(L, \mathbb{Z})$;
- given a weight μ there is a corresponding character χ_μ of T which induces a homogeneous line bundle $L_\mu \to G_0/T$;

- $L_\mu \to G_0/T$ is made into a holomorphic line bundle over D in the usual way; i.e., by extending it to a holomorphic character $\chi_\mu \colon H \to \mathbb{C}^*$ and then extending it to B via the map $B \to H$.

2. Correspondence and cycle spaces

Cycle spaces and correspondence spaces first arose from special cases of flag domains D, cycle spaces initially in the context of Hodge theory and then representation theory, correspondence spaces in the context of integral geometry and Penrose-type transforms. Both the cycle spaces $\mathcal{U}$ and correspondence spaces $\mathcal{W}$ considered here are open subsets in G-homogeneous projective algebraic varieties, and the basic diagram relating $D, \mathcal{W}$ and $\mathcal{U}$ is an open subset of a diagram of G-homogeneous algebraic varieties. For later reference we now record this diagram:

$$\begin{tikzcd} & G/H \arrow[d] & \\ & G/B_K \arrow[dl] \arrow[dr] & \\ G/B & & G/K. \end{tikzcd} \tag{2.1}$$

The space G/H is sometimes called the *enhanced flag variety.* Double homogeneous space fibrations in the lower part of the diagram are classical in integral geometry [**Ch**]. We note the following general properties:

(2.2a) *The fibres of $G/B_K \to G/B$ and of $G/H \to G/B_K$ are contractible affine algebraic varieties;*

(2.2b) *The fibres of $G/B_K \to G/K$ are projective algebraic varieties.*

Discussion. (see [**FHW**] for detailed proofs): From

$$\mathfrak{b} = \mathfrak{b}_K \oplus \mathfrak{p}^-$$

where $\mathfrak{b}_K = \mathfrak{b} \cap \mathfrak{k}$ one may show that

$$\exp \colon \mathfrak{p}^+ \xrightarrow{\sim} B/B_K$$

is a bi-holomorphic map. A similar argument works for

$$\mathfrak{b}_K = \mathfrak{h} \oplus \mathfrak{n}^+.$$

Finally, K/B_K is the flag variety for K.

The definition of the correspondence space derives from *Matsuki duality* between G_0-orbits $\mathcal{O}_{G_o}$ and K-orbits $\mathcal{O}_K$ in the flag variety X ([**FHW**] and [**Sch3**]). We recall that the pair $(\mathcal{O}_{G_0}, \mathcal{O}_K)$ are *Matsuki dual* if the intersection $\mathcal{O}_{G_0} \cap \mathcal{O}_K$ consists of exactly one K_0-orbit. The relation "contained in the closure of" partially orders the set of K-orbits as well as the set of G_0-orbits, and the duality

$$\{G_0\text{-orbits in } X\} \longleftrightarrow \{K\text{-orbits in } X\}$$

reverses the closure relationships. For $x_0 \in X$ such that $G_0(x_0) = \mathcal{O}_{G_0}$ is open, or equivalently $K(x_0) = \mathcal{O}_K$ is compact, we set

$$\mathcal{W}_G = \{\, g \in G : g\mathcal{O}_K \cap \mathcal{O}_{G_0} \neq \emptyset \text{ and is compact} \,\}^0 / H.$$

Here, $\{\ \}^0$ denotes the connected component of the identity. Since $H \subset K$ this definition makes sense and $\mathcal{W}_G \subset G/H$ is an open set.

Theorem-Definition 2.1. *For an open G_0-orbit $\mathcal{W}_G$ is independent of x_0. We denote it by $\mathcal{W}$ and define it to be the* **correspondence space** *associated to (G_0, H).*

We recall here our blanket assumption that D is non-classical; i.e., it does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain.

The property in the theorem is called *universality*. We will infer it from a similar property for the cycle space $\mathcal{U}$, to which we now turn. Set

$$\mathcal{U}_G = \{\, g \in G : g\mathcal{O}_K \subset \mathcal{O}_{G_0} \,\}^0 / K.$$

We note that the term $\{\quad\}^0$ is the same as for $\mathcal{W}_G$; we have given this description because the cycle spaces associated to $D = G_0/T$ initially arose as

$$\{\, gZ : g \in G \text{ and } gZ \subset D \,\},$$

which is the set of translates gZ of the maximal compact subvariety $Z = K_0/T$ by $g \in G$ such that gZ remains in D (cf. [**Sch1**] and [**GS**]). The universality of $\mathcal{W}$ is a consequence of

Theorem (universality, [**FHW**]). *For open G_0-orbits, $\mathcal{U}_G$ is independent of x_0.*

The proof of this theorem is based on Matsuki duality.

Before turning to the basic diagram and the properties of $\mathcal{W}$ and $\mathcal{U}$, we have previously noted that the open G_0-orbits D_w are indexed by the elements $w \in W/W_K$. Equivalently, a complex structure D_w on G_0/T is given by a choice Φ_w^+ of positive roots, and two such G_0-homogeneous complex structures $D_w, D_{w'}$ are equivalent if $w \equiv w' \bmod W_K$. Each D_w has a distinguished point $x_w \in D_w$ as follows:

If $x_0 \in G/B$ is the identity coset, then

$$x_w = w x_0 w^{-1} \in D_w.$$

It follows that $D_w = G_0(x_w)$ and the compact K-orbit $Z_w \subset D_w$ given by the duality theorem is $Kx_w = wZw^{-1}$ where $Z = K_0/T \subset G_0/T$.

We will now describe the basic diagram for D. By the remark just given there will be a corresponding basic diagram for each D_w. Letting $\{Z_u, u \in \mathcal{U}\}$ be the family of maximal compact subvarieties $Z_u \subset D$ parametrized by $\mathcal{U}$, we define the *incidence correspondence* $\mathcal{I} \subset D \times \mathcal{U}$ by

$$\mathcal{I} = \{(x, u) : x \in Z_u\}.$$

DEFINITION 2.2. The *basic diagram* is

$$
\begin{tikzcd}
 & \mathcal{W} \arrow[d, "\pi_{\mathcal{I}}"] \arrow[ddl, "\pi"'] \arrow[ddr, "\pi'"] & \\
 & \mathcal{I} \arrow[dl, "\pi_D"] \arrow[dr, "\pi_{\mathcal{U}}"'] & \\
D & & \mathcal{U}.
\end{tikzcd}
\tag{2.3}
$$

The maps are those induced by the maps in (2.1), where we note the inclusion

$$\mathcal{I} \subset G/B_K.$$

THEOREM 2.3.

(1) $\mathcal{W}$ *is a Stein manifold;*
(2) *the fibres of* $\mathcal{W} \to D$ *are contractible;*
(3) *the fibres of* $\mathcal{W} \to \mathcal{I}$ *are contractible.*

PROOF. For G_0 of Hermitian type, and recalling our assumption that D is non-classical, this result largely follows from the results in [**FHW**]. Specifically, we have that:

- $\mathcal{U}$ is Stein ([**FHW**], Corollary 6.3.3); and
- the fibres of $\pi' \colon \mathcal{W} \to \mathcal{U}$ are affine algebraic varieties.[2]

The latter statement follows by observing that (2.3) is an open subset of (2.1), and $\mathcal{W} \subset G/H$ is the inverse image of $\mathcal{U} \subset G/K$. A similar argument applies to $\mathcal{W} \to \mathcal{I}$, where a typical fibre is B/B_K. From [**FHW**], (6.23), in case G_0 is of Hermitian type the fibres $\mathcal{I} \to D$ are contractible. This case covers the two examples discussed below. The general case when G_0 is not of Hermitian type is more complex and will be discussed elsewhere. $\square$

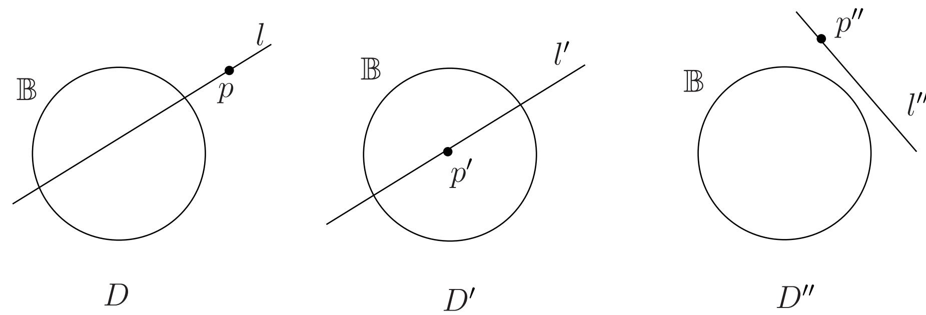
EXAMPLE 2.4. $\mathcal{U}(2,1)$[3] ([**EGW**], [**C1**], [**C2**] and [**GGK2**]).

- $\mathbb{H}$ is the standard Hermitian form on $\mathbb{C}^3$ with matrix $\operatorname{diag}(1,1,-1)$ and
$$\mathcal{U}(2,1) = \{\, g \in \mathrm{GL}(3,\mathbb{C}) : {}^t\bar{g}\mathbb{H}g = \mathbb{H} \,\};$$
- points $p \in \mathbb{P}^2$ are given by homogeneous column vectors $p = {}^t[p_1, p_2, p_3]$ and lines $l \in \check{\mathbb{P}}^2$ by homogeneous row vectors $l = [l_1, l_2, l_3]$;
- the unit ball $\mathbb{B} \subset \mathbb{P}^2$ is given by $\{p : {}^t\overline{p}\mathbb{H}p < 0\}$; $\mathbb{B}^c = \mathbb{P}^2 \backslash \mathcal{C}l(\mathbb{B})$ is the complement of the closed ball;
- the flag variety is described as the standard incidence correspondence $X = \{(p,l) : \langle l, p\rangle = 0\}$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$.

[2]The fibre ${\pi'}^{-1}(u_0) \cong K/H$ is the enhanced flag variety of $Z = Z_{u_0}$. It is a general result [**Bo**] that the quotient by the Cartan subgroup H of the affine variety K is again an affine algebraic variety. We note that K is reductive with center contained in H, so the quotient is the same as one of a semi-simple complex linear group by a Cartan subgroup.

[3]We use $\mathcal{U}(2,1)$ rather than $S\mathcal{U}(2,1)$ because $\mathcal{U}(2,1)$ is the Mumford-Tate group of a generic polarized Hodge structure with Hodge numbers $h^{3,0} = 1$, $h^{2,1} = 2$ and having an action by $\mathbb{Q}(\sqrt{-d})$ (cf. [**GGK1**]).

There are three flag domains given as open orbits of $\mathcal{U}(2,1)$ acting on X and which may be pictured as follows:

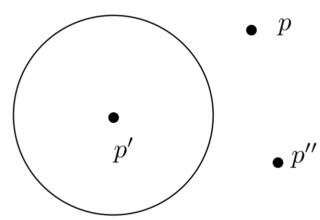


Here, D is non-classical and D', D'' are classical. For example, $(p', l') \to p'$ fibres D' over the ball with $\mathbb{P}^1$ fibres.

The enhanced flag variety $\mathrm{GL}(3, \mathbb{C})/H$ is given by the set of *projective frames*, defined as triples of points $p, p', p'' \in \mathbb{P}^2$ where $p \wedge p' \wedge p'' \neq 0$.

The correspondence space is pictured by

$\mathcal{W} =$



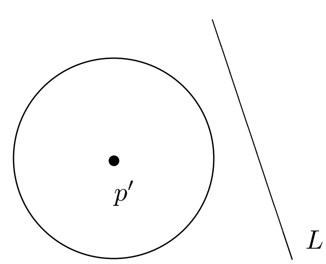
, $\overline{p\,p''} \subset \mathbb{B}^c$

where $\overline{p\,p''}$ is the line joining p and p''. The maps $\mathcal{W} \to D, D', D''$ are given by

$$p, p', p'' \to \begin{cases} (p, \overline{p\,p'}) \in D \\ (p', \overline{p'\,p}) \in D' \\ (p, \overline{p\,p''}) \in D''. \end{cases}$$

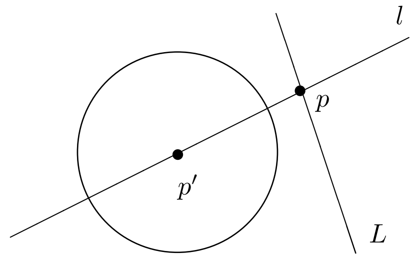
The cycle space $\mathcal{U}$ is pictured by

$\mathcal{U} =$



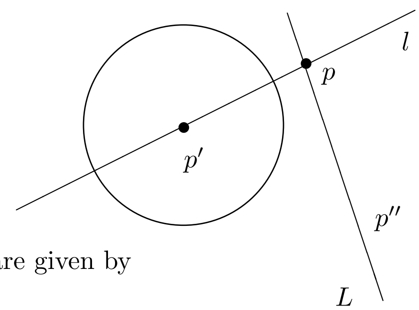
From the picture we see that $\mathcal{U} \cong \mathbb{B} \times \overline{\mathbb{B}}$ where $\overline{\mathbb{B}}$ is the conjugate complex structure on $\mathbb{B}$ and is isomorphic to the set of lines not meeting the closure of $\mathbb{B}$. The corresponding compact subvariety $Z(p', L) \cong \mathbb{P}^1$ is given by $\{(p, l)\} \subset D$ in the picture

$$Z(p', L) =$$

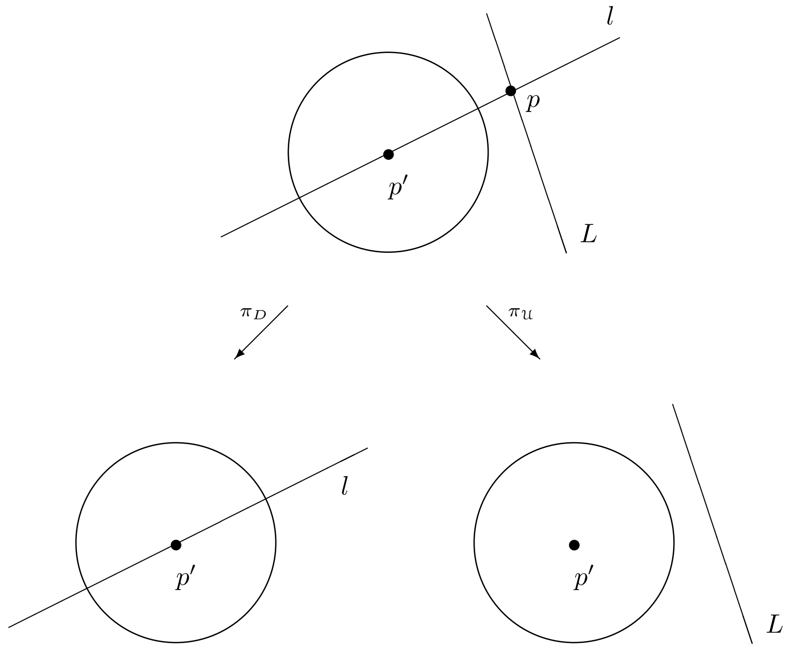


The incidence correspondence $\mathcal{I} = \{(p, l), (p', L)\} \subset D \times \mathcal{U}$ is as pictured in the above figure. The map $\mathcal{W} \to \mathcal{I}$ is given by

$$p, p', p'' \longrightarrow$$



The maps $\mathcal{I} \to D$, $\mathcal{I} \to \mathcal{U}$ are given by



All of the properties in the basic diagram (2.3) may be readily verified from the above pictures. The standard root diagram for $\mathcal{U}(2, 1)$ is where the compact roots are labelled $\boxed{\bullet}$ and the Weyl chambers C, C', C'' correspond to the complex structures D, D', D''. Here, C is non-classical and C', C'' are classical.

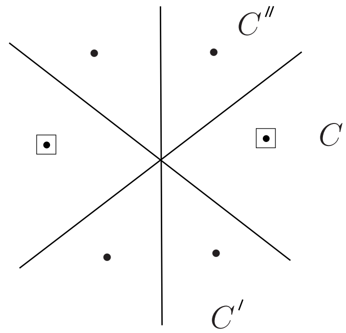
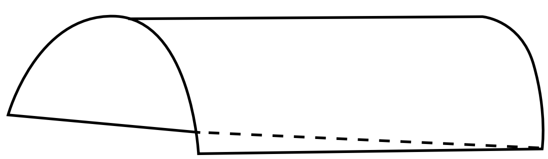


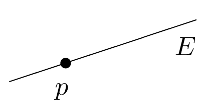
FIGURE 1

EXAMPLE 2.5. Sp(4) ([**GGK2**]).

- Q is the alternating form on $V = \mathbb{C}^4$ with matrix $\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$;
- $\sigma\colon V \to V$ is a conjugation defined in the standard basis v_1, v_2, v_3, v_4 by $\sigma v_1 = iv_4$, $\sigma v_2 = iv_3$ (and then $\sigma v_3 = iv_2$, $\sigma v_4 = iv_1$);
- $\mathbb{H}$ is the Hermitian form defined by $\mathbb{H}(u, v) = iQ(u, \sigma v)$;
- $\mathbb{H}(v, \sigma v) = 0$ defines a real quadric hypersurface $Q_{\mathbb{H}} \subset \mathbb{P}^3$ which we picture as

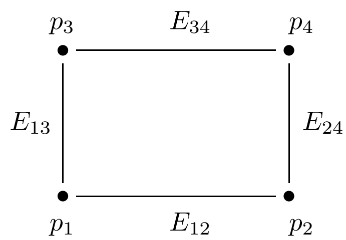

;

- $\mathrm{Sp}(4) = \mathrm{Aut}_\sigma(V, Q)$ is a real form of $\mathrm{Aut}(V, Q)$;
- a *Lagrange flag* is a flag $(0) \subset F_1 \subset F_2 \subset F_3 \subset V$ where $\dim F_j = j$ and with $F_2 = F_2^\perp$, $F_3 = F_1^\perp$, the $\perp$ being with respect to Q;
- a Lagrange flag is given projectively by a pair (p, E) where $p \in \mathbb{P}^3$ and $E \subset \mathbb{P}^3$ is a Lagrange line



  (think of $p = [F_1]$, $E = [F_2]$);
- a *Lagrange frame* is a basis f_1, f_2, f_3, f_4 for V for which $Q(f_i, f_j)$ is the above matrix Q;
- a *Lagrange quadrilateral* is a projective frame p_1, p_2, p_3, p_4 for $\mathbb{P}^3$ where $p_i = [f_i]$ for a Lagrange frame f_1, f_2, f_3, f_4.

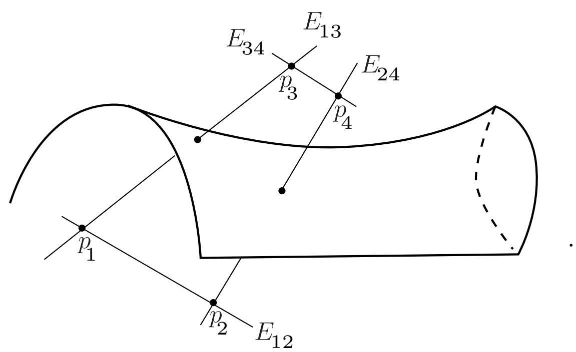
The flag variety X is the set of Lagrange flags. The enhanced flag variety is the set of Lagrange quadrilaterals. We may picture a Lagrange quadrilateral as



where the depicted lines $E_{ij} = \overline{p_i\, p_j}$ are Lagrangian lines in $\mathbb{P}^3$. The diagonal lines are not Lagrangian.

The correspondence space $\mathcal{W}$ is the set of Lagrange quadrilaterals positioned relative to the real hyperquadric $Q_{\mathbb{H}}$ as in the picture

$\mathcal{W} =$

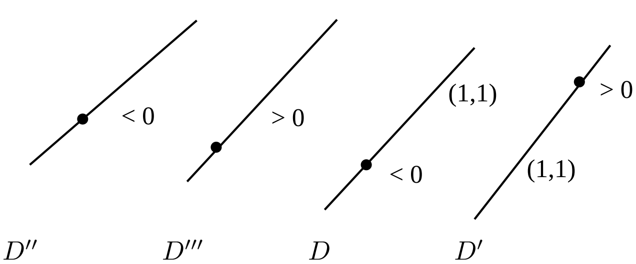


The pictured Lagrangian lines E_{ij} are of three types

- E_{12} lies "inside" $Q_{\mathbb{H}}$, meaning that $\mathbb{H} < 0$ on the corresponding Lagrangian 2-plane $\tilde{E}_{12}$ in V;
- E_{13} meets $Q_{\mathbb{H}}$ in a real circle; as a consequence $\mathbb{H}$ has signature $(1,1)$ on $\tilde{E}_{13}$; E_{24} has a similar property;
- E_{34} lies "outside" $Q_{\mathbb{H}}$, meaning that $\mathbb{H} > 0$ on $\tilde{E}_{34}$.

There are eight orbits of the four Lagrange flags in the above picture; thus we have (p_1, E_{12}) and (p_2, E_{12}) associated to E_{12}. These orbits give eight complex structures on $G_{\mathbb{R}}/T$, of which four pairs are equivalent under the action of W_K. The four types may be pictured as the orbits of

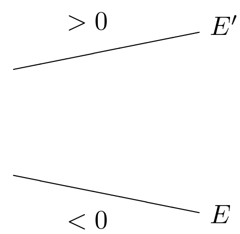
D_w's $=$



The notations mean that $\mathbb{H} < 0$, $\mathbb{H} > 0$ on the first two, $\mathbb{H}$ has signature $(1,1)$ on the second two, and on the two where $\mathbb{H}$ has signature $(1,1)$ we have indicated the sign of $\mathbb{H}$ on the marked points. Here, D and D' are non-classical and D'', D''' are classical.

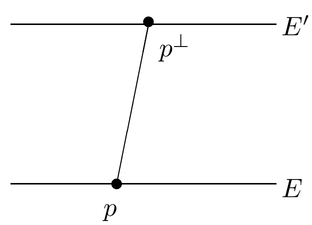
The cycle space is pictured as

$\mathfrak{U} =$



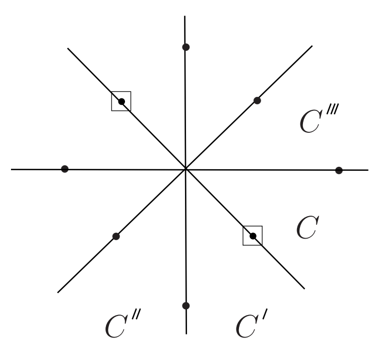
Here E, E' are Lagrangian lines on which $\mathbb{H}\big|_E < 0$, $\mathbb{H}\big|_{E'} > 0$ respectively (think of E as "inside" $Q_{\mathbb{H}}$ and E' as "outside"). The corresponding cycle $Z(E,E')$ in D is $\{(p, \overline{p\,p^{\perp}})\}$

$Z(E,E') =$



where $p \in E$ and $p^{\perp} \in E'$ is the unique point in E' with $Q(p, p^{\perp}) = 0$.

The standard root diagram for $\mathrm{Sp}(4)$ is



where the compact roots are marked $\boxed{\bullet}$ and the Weyl chambers corresponding to the complex structures are as indicated. Here, C and C' are non-classical and C'', C''' are classical.

3. The comparison theorem

Let $D = G_0/T$ be a flag domain and

$$L_\mu \to D$$

a holomorphic line bundle defined by a weight $\mu \in \Lambda$. As will now be explained, over each of the correspondence and cycle spaces there are global holomorphic objects to which the sheaf cohomology groups $H^q(D, L_\mu)$ map. In the case of $\mathcal{W}$ the mapping is an isomorphism and the holomorphic object is a quotient of global holomorphic sections of a holomorphic vector bundle. In the examples of interest to us there will usually be distinguished representatives of equivalence classes in the quotient space. In the case of the cycle space there are conditions under which the mapping is injective and the image can be identified; the global holomorphic object is sections of a bundle. The objective of this section is to relate these two ways of realizing $H^q(D, L_\mu)$.

We begin by recalling the result from [**EGW**]. Let M, N be a complex manifolds and

$$\pi\colon M \to N$$

a holomorphic submersion. We identify holomorphic vector bundles and their sheaves of sections. For $F \to N$ a holomorphic vector bundle we let

- $\pi^{-1}F$ be the pullback to M of the sheaf F;
- $\pi^* F$ be the pullback to M of the bundle F.

We may think of $\pi^{-1}F \subset \pi^* F$ as the sections of $\pi^* F$ that are constant along the fibres of $M \to N$.

Next we let Ω^q_π be the sheaf over M of relative holomorphic q-forms. We have

$$0 \to \pi^*\Omega^1_N \to \Omega^1_M \to \Omega^1_\pi \to 0,$$

and this defines a filtration $F^m\Omega^q_M$ with

$$\Omega^q_\pi \cong \Omega^q_M / F^q\Omega^q_M.$$

In local coordinates (x^i, y^α) on M such that $\pi(x^i, y^\alpha) = (y^\alpha)$, $F^m\Omega^q_M$ are the holomorphic differentials generated over Ω^{q-m}_M by terms $dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_m}$. Thus $F^m\Omega^q_M = \operatorname{image}\{\pi^*\Omega^m_N \otimes \Omega^{q-m}_M \to \Omega^q_M\}$. From this description we see that we have

$$d\colon F^m\Omega^q_M \to F^m\Omega^{q+1}_M,$$

and consequently there is an induced relative differential

$$d_\pi\colon \Omega^q_\pi \to \Omega^{q+1}_\pi.$$

Setting $\Omega^q_\pi(F) = \Omega^q_\pi \otimes_{\mathcal{O}_M} \pi^* F$, since the transition functions of $\pi^* F$ may be taken to involve only the y^α's, we may define

$$d_\pi\colon \Omega^q_\pi(F) \to \Omega^{q+1}_\pi(F)$$

to obtain the complex $(\Omega^\bullet_\pi(F); d_\pi)$. Using the holomorphic Poincaré lemma with holomorphic dependence on parameters one has the resolution

$$0 \to \pi^{-1}F \to \Omega^0_\pi(F) \xrightarrow{d_\pi} \Omega^1_\pi(F) \xrightarrow{d_\pi} \Omega^2_\pi(F) \to \cdots . \tag{3.1}$$

Denoting by $\mathbb{H}^*(M, \Omega^\bullet_\pi(F))$ the hypercohomology of the complex $(\Omega^\bullet_\pi(F), d_\pi)$, from (3.1) we have

$$H^*(M, \pi^{-1}F) \cong \mathbb{H}^*(M, \Omega^\bullet_\pi(F)). \tag{3.2}$$

We denote by

$$H^*\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big)$$

the de Rham cohomology groups arising by taking the global holomorphic sections of the complex $(\Omega^\bullet_\pi(F); d_\pi)$.

THEOREM 3.1. *Assume that M is Stein and the the fibres of $M \to N$ are contractible. Then*
$$H^*(N, F) \cong H^*\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big).$$

DISCUSSION. Using the spectral sequence
$$E_2^{p,q} = H^q_{d_\pi}\big(H^p(M, \Omega^\bullet_\pi(F))\big) \Rightarrow \mathbb{H}^{p+q}\big(M, \Omega^\bullet_\pi(F)\big)$$
and the assumption that M is Stein to have $H^p(M, \Omega^\bullet_\pi(F)) = 0$ for $p > 0$ gives
$$H^*(M, \pi^{-1}F) \cong H^*\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big). \tag{3.3}$$

Next, in the situations with which we shall be concerned, the submersion $M \to N$ will be locally over N a topological product. Then by the contractibility of the fibres the direct image sheaves
$$R^q_\pi(\pi^{-1}F) = 0 \ \text{ for } \ q > 0.$$
The Leray spectral sequence thus gives
$$H^q(N, F) \cong H^q(M, \pi^{-1}F); \tag{3.4}$$
here the LHS is $H^q(N, R^0_\pi(\pi^{-1}F)) = H^q(N, F)$. Combining (3.3) and (3.4) gives the theorem.

NOTE 3.2. The second part of this argument is due to Buchdahl; cf. (14.2.3) in [**FHW**].

Using Theorem (2.3) we now apply this result to $\mathcal{W} \xrightarrow{\pi} D$ and $F = L_\mu$ to have the

COROLLARY 3.3. $H^q(D, L_\mu) \cong H^q\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$.

In this way, the coherent cohomology $H^q(D, L_\mu)$ is realized by global, holomorphic data. As noted above, in examples considered in [**GGK2**] there are canonical "harmonic" representatives of classes in the RHS of the corollary.

To state our main result we first define bundles
$$F^{p,q}_\mu \to \mathcal{U}$$
as follows: For $u \in \mathcal{U}$ let $Z_u \subset D$ be the corresponding maximal compact subvariety. Let $F^{p,q}_\mu = R^q_{\pi_\mathcal{U}}(\Omega^p_{\pi_D}(L_\mu))$. Then the fibre
$$F^{p,q}_{\mu,u} = H^q(Z_u, \Lambda^p N_{Z_u \backslash D}(L_\mu)).$$

THEOREM 3.4. *There exists a spectral sequence with*
$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U}, F^{p,q}_\mu), \quad \textit{and} \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \end{cases}$$

Using (3.3) we have the following result that is implicit in [**WZ**].

COROLLARY 3.5. *There exists a spectral sequence with*
$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U}, F^{p,q}_\mu) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}(D, L_\mu). \end{cases}$$
If $H^0(Z, \Lambda^{q+1} N_{Z/D}(L_\mu)) = \cdots = H^{q-1}(Z, \Lambda^2 N_{Z/D}(L_\mu)) = 0$, then
$$H^q(D, L_\mu) \cong \ker\{\, H^0(\mathcal{U}, F^{0,q}_\mu) \xrightarrow{d_1} H^0(\mathcal{U}, F^{1,q}_\mu)\,\}. \tag{3.5}$$

Thus under the vanishing condition in the corollary, the coherent cohomology $H^q(D, L_\mu)$ is, in a different way from (3.3), realized as a global, holomorphic object.

The differential d_1 is a linear, first order differential operator whose symbol will be identified below following the proof of Theorem (3.4).

Proof of Theorem (3.4). Referring to the basic diagram (3.3) we have on $\mathcal{W}$ the exact sequence of relative differentials

$$0 \to \pi_{\mathcal{I}}^* \Omega^1_{\pi_D} \to \Omega^1_\pi \to \Omega^1_{\pi_{\mathcal{I}}} \to 0. \tag{3.6}$$

This induces a filtration on $\Omega^\bullet_\pi$, and hence one on the complex

$$\Gamma\big(\mathcal{W}, \Omega^\bullet_\pi(L_\mu); d_\pi\big).$$

This filtration then leads to a spectral sequence abutting to

$$H^*(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi).$$

We will identify the E_1-term with that given in the statement of the theorem.

The first observation is that in this spectral sequence we have

$$\begin{cases} E_0^{p,q} \cong \Gamma\big(\mathcal{W}, \Omega^q_{\pi_{\mathcal{I}}} \otimes \pi_{\mathcal{I}}^* \Omega^p_{\pi_D}(L_\mu)\big) \\ d_0 \quad = d_{\pi_{\mathcal{I}}}. \end{cases}$$

Thus

$$\begin{cases} E_1^{p,q} \cong H^q\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi_{\mathcal{I}}} \otimes \pi_{\mathcal{I}}^* \Omega^p_{\pi_D}(L_\mu)); d_{\pi_{\mathcal{I}}}\big) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

By [**EGW**] applied to $\mathcal{W} \xrightarrow{\pi_{\mathcal{I}}} \mathcal{I}$ we have

$$\begin{cases} E_1^{p,q} \cong H^q\big(\mathcal{I}, \Omega^p_{\pi_D}(L_\mu)\big) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

Since $\mathcal{U}$ is Stein, and the sheaves $R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)$ are coherent, the Leray spectral sequence applied to $\mathcal{I} \xrightarrow{\pi_{\mathcal{U}}} \mathcal{U}$ and $\Omega^p_{\pi_D}(L_\mu)$ gives

$$\begin{cases} E_1^{p,q} \cong H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

It remains to establish the identification

$$R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu) \cong F^{p,q}_\mu. \tag{3.7}$$

This will be done by identifying the various tangent spaces at the reference point $(x_0, u_0) \in \mathcal{I}$. For this we will identify locally free sheaves F with vector bundles and denote by F_p the fibre at the point p. We then have the identifications

- $T_{x_0} D = \mathfrak{n}^+$;
- $T_{x_0} Z = \mathfrak{n}^+_c$;
- $N_{Z/D, x_0} = \mathfrak{p}^+$;
- $T_{u_0} \mathcal{U} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$;
- $T_{(x_0,u_0)} \mathcal{I} = \mathfrak{n}^+_c \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$

and $T_{(x_0,u_0)}\mathcal{I}$ maps to $T_{x_0} D = \mathfrak{n}^+ = \mathfrak{n}^+_c \oplus \mathfrak{p}^+$ and $T_{u_0}\mathcal{U} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ by the evident projections.

It follows that

- $\Omega^1_{\pi_D,(x_0,u_0)} = \mathfrak{p}^{-*} \cong \mathfrak{p}^+ = N_{Z/D, x_0}$

where the isomorphism is via the Cartan-Killing form. □

The proof also allows us to identify the symbol $\sigma(d_1)$ of the differential operator d_1, as follows: Recall that

$$\sigma(d_1)\colon F^{0,q}_{u_0} \otimes T^*_{u_0}\mathcal{U} \to F^{1,q}_{\mu,u_0},$$

or using the definition of the $F^{p,q}_{\mu}$

$$\sigma(d_1)\colon H^q(Z, L_\mu) \otimes T^*_{u_0}\mathcal{U} \to H^q(Z, N_{Z/D}(L_\mu)). \tag{3.8}$$

Using the identification $T^*_{u_0}\mathcal{U} \cong \mathfrak{p}^* \cong \mathfrak{p}$ we have the inclusion

$$\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D}) \tag{3.9}$$

given geometrically by considering $X \in \mathfrak{p} \subset \mathfrak{g}$ as a holomorphic vector field along Z and then taking the normal part of X. Combining this with the evident map

$$H^q(Z, L_\mu) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, N_{Z/D}(L_\mu))$$

gives the symbol map (3.8).

This assertion will be proved when we revisit the symbol issue in section 4.2 (cf. Theorem 4.4).

4. Variants and applications

4.1. Quotienting by a discrete group. Let $\Gamma \subset G_0$ be a discrete, co-compact and neat subgroup. A principal motivation for [**GGK2**] was to understand the geometric and arithmetic properties of the *automorphic cohomology groups* $H^q(\Gamma\backslash D, L_\mu)$, objects that had arisen many years ago but whose above mentioned properties had to us remained largely mysterious until the works [**C1**], [**C2**], and [**C3**]. In studying the automorphic cohomology groups it is important to be able to take the quotient of the basic diagram (2.3) by Γ, which is then

$$\begin{tikzcd}
& \Gamma\backslash\mathcal{W} \arrow[d, "\pi_{\mathcal{J}}"] \arrow[ddl, "\pi"'] \arrow[ddr, "\pi'"] & \\
& \Gamma\backslash\mathcal{J} \arrow[dl, "\pi_0"] \arrow[dr, "\pi_{\mathcal{U}}"'] & \\
\Gamma\backslash D & & \Gamma\backslash\mathcal{U}.
\end{tikzcd} \tag{4.1}$$

Here we note that the group G_0 acts equivariantly on the diagram (2.3), and so the above quotient diagram is well-defined. The basic result concerning it is

THEOREM 4.1. *$\Gamma\backslash\mathcal{W}$ is Stein, and the fibres of π, π_D and $\pi_{\mathcal{J}}$ are contractible.*

PROOF. We first note that because Γ is assumed neat, any γ of finite order is the identity. Therefore, no $\gamma \in \Gamma$, $\gamma \neq e$, has a fixed point acting on D or on $\mathcal{U}$. For D this is because the isotropy subgroup of G_0 fixing any point $x \in D$ is compact. For $u \in \mathcal{U}$, if γ fixes u then it maps the compact subvariety $Z_u \subset D$ to itself, so again γ is of finite order. It follows that the fibres in (4.1) are biholomorphic to those in the basic diagram (2.3).

The next, and crucial, step is the result in [**BHH**] (cf. also 6.3.3 in [**FHW**]) that there exist strictly plurisubharmonic functions on $\mathcal{U}$ that are exhaustion functions modulo G_0. As in the proof in loc. cit., this induces a strictly plurisubharmonic exhaustion function of $\Gamma\backslash\mathcal{U}$, which is therefore a Stein manifold. Then $\Gamma\backslash\mathcal{W} \to \Gamma\backslash\mathcal{U}$

is a fibration over a Stein manifold with affine algebraic varieties as fibres, which implies that $\Gamma\backslash\mathcal{W}$ is itself Stein. $\square$

The proof of Theorem (3.4) then applies verbatim to give

$$H^*(\Gamma\backslash D, L_\mu) \cong H^*\big(\Gamma(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \tag{4.2}$$

The double appearance of the notation Γ in the RHS is unfortunate, but we hope that the meaning is clear. We also have a spectral sequence with

$$\begin{cases} E_1^{p,q} = H^0(\Gamma\backslash\mathcal{U}, F_\mu^{p,q}) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}(\Gamma\backslash D, L_\mu). \end{cases} \tag{4.3}$$

4.2. $\mathfrak{n}$-cohomology interpretation. A familiar theme in the study of cohomology of homogeneous spaces and their quotients is to represent that cohomology by Lie algebra cohomology. For flag domains one considers $\mathfrak{n}$-cohomology where $\mathfrak{n}$ is the direct sum of the negative root spaces. Even though $\mathcal{W}$ is not a homogeneous space for G_0, we will show that the global de Rham cohomology groups $H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$ can be realized as $\mathfrak{n}$-cohomology for a certain G_0-module $\mathcal{O}G_\mathcal{W}$. Using this interpretation we will then observe that our spectral sequence is just the familiar Hochschild-Serre spectral sequence.

The definition of $\mathcal{O}G_\mathcal{W}$ is as follows: From the basic diagrams (2.1), (2.3) we obtain

$$\begin{tikzcd} G_\mathcal{W} \arrow[d] & \subset & G \arrow[d, "f"] \\ \mathcal{W} \arrow[d, "\pi_\mathcal{J}"] \arrow[dd, bend right=60, "\pi"'] & \subset & G/H \arrow[d] \\ \mathcal{J} \arrow[d, "\pi_D"] & \subset & G/B_K \arrow[d] \\ D & \subset & G/B. \end{tikzcd} \tag{4.4}$$

Definition 4.2. $G_\mathcal{W} = f^{-1}(\mathcal{W})$ is the open subset of G lying over $\mathcal{W}$ in the diagram (4.4), and

$$\mathcal{O}G_\mathcal{W} = \Gamma(G_\mathcal{W}, \mathcal{O}_{G_\mathcal{W}})$$

is the algebra of holomorphic functions on $G_\mathcal{W}$.

As we shall discuss below, $\mathcal{O}G_\mathcal{W}$ is a somewhat strange object but it is not as intractable as the definition might suggest. Since $G_\mathcal{W} \subset G$ is G_0-invariant, $\mathcal{O}G_\mathcal{W}$ is a G_0-module and therefore $\mathfrak{n}$-cohomology with coefficients in $\mathcal{O}G_\mathcal{W}$ is well-defined.

In fact, since

$$D = G_0(x_0) \subset G/B$$

and

$$\mathcal{W} = \{\, g \in G : gK(x_0) \subseteq D \,\}/H$$

we have

$$G_0\mathcal{W} \subseteq \mathcal{W}, \qquad \mathcal{W}K \subseteq \mathcal{W}$$

where K is acting on $\mathcal{W}$ on the right. Thus, G_0 and K act on $\mathcal{O}G_{\mathcal{W}}$ by

$$\begin{cases} (gh)(w) = h(gw) & g \in G_0, h \in \mathcal{O}G_{\mathcal{W}}, w \in G_{\mathcal{W}} \\ (hk)(w) = h(wk) & k \in K. \end{cases}$$

Because $G_{\mathcal{W}} \subset G$ is an open set; in fact it is $\{g \in G : gK(x_0) \subseteq D\}$, the Lie algebra $\mathfrak{g}$, viewed as left invariant vector fields on G, acts on $\mathcal{O}G_{\mathcal{W}}$ on the left. When $\mathfrak{g}$ is viewed as right invariant vector fields it acts on $\mathcal{O}G_{\mathcal{W}}$ on the right. These two actions commute, and we will use the right action of $\mathfrak{n}$ to define $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$. These groups then have an action of G_0 on the left and an action of H on the right.

THEOREM 4.3.

(1) *There is the natural identification*

$$H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big) \cong H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}.$$

(2) *The Hochschild-Serre spectral sequence associated to the sub-algebra $\mathfrak{n}_c \subset \mathfrak{n}$ coincides with the spectral sequence given in Theorem* (3.4).

PROOF. The notation $(\ )_{-\mu}$ on the RHS of the isomorphism above means the following: The Cartan subgroup H acts on the right on $G_{\mathcal{W}}$ and therefore acts on the complex $(\Lambda^\bullet \mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}, \delta)$ that computes Lie algebra cohomology. Then $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}$ is that part of $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$ that transforms by the character χ_μ^{-1} of H corresponding to the weight $-\mu$. This enters the picture because holomorphic sections of $\pi^* L_\mu \to \mathcal{W}$ are given by holomorphic functions on $G_{\mathcal{W}}$ that transform by χ_μ under the right action of H.

The proof of (1) in the above theorem is basically the observation from the proof of Theorem (3.4), and using the identification (3.6), that we have the natural identification of complexes

$$\Gamma\big(\mathcal{W}, \Omega^\bullet_\pi(L_\mu); d_\pi\big) \cong (\Lambda^\bullet \mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}; \delta)_{-\mu}. \tag{4.5}$$

Here "natural" means that the action of G_0 on the LHS in (4.5) is given by the G_0-module structure of $\mathcal{O}G_{\mathcal{W}}$.

Turning to (2) in the theorem, here the basic observation is that when pulled back to $G_{\mathcal{W}}$, the exact sequence (3.6) is the dual to the restriction to $G_{\mathcal{W}} \subset G$ of the exact sequence of homogeneous vector bundles over G/H given by the exact sequence of H-modules

$$0 \to \mathfrak{n}_c \to \mathfrak{n} \to \mathfrak{p}^- \to 0.$$

From this we may infer (2) in the theorem. □

For later use we note that using the above identifications and $\mathfrak{p}^{-*} \cong \mathfrak{p}^+$ via the Cartan-Killing form,

$$E_1^{p,q} = H^q(\mathfrak{n}_c, \Lambda^p \mathfrak{p}^+ \otimes \mathcal{O}G_{\mathcal{W}})_{-\mu}. \tag{4.6}$$

Using this interpretation we shall now compute the symbol $\sigma(d_1)$ of

$$d_1 \colon H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big) \to H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^{p+1}_{\pi_D}(L_\mu)\big).$$

Following the notation from section 3 and the identification there of the fibre of the vector bundle $F^{p,q}_{\mu,u_0} \to \mathcal{U}$ and tangent space $T_{u_0}\mathcal{U}$ at the reference point, and identifying Z_{u_0} with Z to simplify the notation, the symbol $\sigma(d_1)$ of the 1^{st}-order linear differential operator is a map

$$\sigma(d_1) \colon H^q\big(Z, \Lambda^p N_{Z/D}(L_\mu)\big) \otimes \mathfrak{p}^* \to H^q\big(Z, \Lambda^{p+1} N_{Z/D}(L_\mu)\big).$$

THEOREM 4.4. *With the identifications $\mathfrak{p}^* \cong \mathfrak{p}$ given by the Cartan-Killing form and inclusion $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ the symbol is given by*

$$\sigma(d_1)\varphi \otimes X = \varphi \wedge X.$$

Here, on the LHS we have $X \in \mathfrak{p}$ and $\varphi \in H^q(Z, \Lambda^p N_{Z/D}(L_\mu))$, and on the RHS X is the corresponding normal vector field in $H^0(Z, N_{Z/D})$. The map is $H^q(Z, \Lambda^p N_{Z/D}(L_\mu)) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, \Lambda^{p+1} N_{Z/D}(L_\mu))$ induced by $\Lambda^p N_{Z/D} \otimes N_{Z/D} \to \Lambda^{p+1} N_{Z/D}$.

PROOF. To compute the symbol on $\varphi \otimes X$, we take a section f of $F^{p,q}$ defined near u_0 with $f(u_0) = 0$ and whose linear part is $\varphi \otimes X$. Then by definition

$$\sigma(d_1)\varphi \otimes X = (d_1 f)(u_0).$$

We shall give the computation when $p = 0$, $q = 1$ as this will indicate how the general case goes. Pulled back to $G_{\mathcal{W}}$ we may write

$$f = \sum_{\alpha \in \Phi_c^+} f_\alpha \omega^{-\alpha}$$

where the f_α are holomorphic functions that vanish along the inverse image of Z_{u_0}. Then

$$d_1 f = \sum_{\begin{cases}\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+\end{cases}} (f_\alpha X_{-\beta}) \omega^{-\beta} \wedge \omega^{-\alpha} + \sum_{\alpha \in \Phi_c^+} f_\alpha d_\pi \omega^{-\alpha}.$$

The first term is the right action on f_α by the left invariant vector field $X_{-\beta}$. The second term vanishes along the inverse image of Z_{u_0}. As for the first term, under the pairing

$$\begin{pmatrix}\text{normal vector fields} \\ \text{to } Z_{u_0}\end{pmatrix} \otimes \begin{pmatrix}\text{holomorphic functions} \\ \text{vanishing along } Z_{u_0}\end{pmatrix} \to \mathcal{O}_{Z_0}$$

when evaluated along Z_{μ_0} the first term is the value along Z_{u_0} of

$$\sum_{\begin{cases}\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+\end{cases}} f_\alpha X_{-\beta\alpha} X_\beta \otimes \omega^{-\alpha}$$

where $X_\beta \otimes \omega^{-\alpha} \in \mathfrak{p}^+ \otimes \mathfrak{n}^*$ and $f_\alpha X_{-\beta}\big|_{Z_0} \in \mathcal{O}_{Z_0}$. $\square$

DISCUSSION. The G_0-module $\mathcal{O}G_{\mathcal{W}}$ is certainly not a Harish-Chandra, or HC, module, but it does have an interesting structure, reflecting the fact that $\mathcal{W}$ is a mixed algebro-geometric/complex analytic object, as we now explain.

The fibres of

$$\begin{array}{ccc} \mathcal{W} & \subset & G/H \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow \\ \mathcal{U} & \subset & G/K \end{array}$$

are affine algebraic varieties isomorphic to the enhanced flag variety K/H. We may smoothly and equivariantly compactify G/H so that each fibre $g^{-1}(u)$, $u \in \mathcal{U}$, is the complement of a divisor with normal crossings. Then we may consider the G_0-invariant sub-algebra $\mathcal{O}G_{\mathcal{W}}^{\mathrm{alg}} \subset \mathcal{O}G_{\mathcal{W}}$ of functions that are rational along each fibre, and by truncating Laurent series we may write $\mathcal{O}G_{\mathcal{W}}^{\mathrm{alg}}$ as the union of G_0-submodules that are fibrewise K-finite acting on the right. Thus as a G_0-module over the G_0-module $\mathcal{O}(\mathcal{U}) = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ we see that $\mathcal{O}G_{\mathcal{W}}$ has a reasonable structure.

As for the G_0-module $\mathcal{O}(\mathcal{U})$, from [**FHW**] we see that $\mathcal{U}$ has the function-theoretic characteristics of a bounded domain of holomorphy (contractible, Stein, Kobayashi hyperbolic). In fact, for G_0 of Hermitian type, $\mathcal{U} \cong \Omega \times \overline{\Omega}$ where Ω is an Hermitian symmetric domain and where G_0 acts diagonally. Again, $\mathcal{O}(\mathcal{U})$ is not a HC-module but it seems to be a reasonable object to study. It will be further discussed in a future work. Here we shall illustrate it in the case of $S\mathcal{U}(2,1)$.

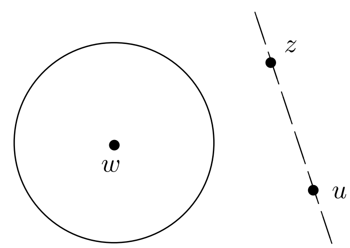
Example 4.5. We represent elements of $G = \mathrm{SL}(3, \mathbb{C})$ as

$$g = \begin{pmatrix} z_1 & w_1 & u_1 \\ z_2 & w_2 & u_2 \\ z_3 & w_3 & u_3 \end{pmatrix} = (z, w, u).$$

Taking as Hermitian form $\mathbb{H} = \mathrm{diag}(1,1,-1)$, $G_{\mathcal{W}} \subset G$ is defined by the conditions

$$\begin{cases} \mathbb{H}(w) < 0 \\ \mathbb{H}(z \wedge w) > 0. \end{cases}$$

The map $G_{\mathcal{W}} \to \mathcal{W}$ is given by

$(z, w, u) \to$ 

the dashed line indicating that the line $\overline{zu}$ lies in $\mathbb{B}^c$. The space $\mathcal{O}G_{\mathcal{W}}$ is spanned by the functions

$$w_1^i w_2^j w_3^k (z_2u_3 - z_3u_2)^l (z_3u_1 - z_1u_3)^m (z_1u_2 - z_2u_1)^n z_1^p z_2^q z_3^r u_1^a u_2^b u_3^c$$

where

$$i, j, i+j+k, l, m, l+m+n, p, q, r, a, b, c \geq 0.$$

There are relations among the generators, such as

$$\left(\frac{z_2u_3 - z_3u_2}{z_1u_2 - z_2u_1}\right)(z_1u_2 - z_2u_1) = z_2u_3 - z_3u_2.$$

4.3. Symbol maps for the two examples. In this section we shall discuss the *symbol sequence* and *characteristic variety* for each of our two examples. Before doing this we shall briefly explain the italicized terms.

In general, over a complex manifold M suppose we are given holomorphic vector bundles $E_i \to M$ and linear, 1st order differential operators $P_i \colon E_i \to E_{i+1}$ that form a complex

$$\begin{cases} E_1 \xrightarrow{P_1} E_2 \xrightarrow{P_2} E_3 \to \cdots \to E_m \to 0, \\ \qquad P_{i+1} \circ P_i = 0. \end{cases} \tag{4.7}$$

This general framework was the object of study of an extensive and rich theory developed by Spencer and his collaborators in the 1960's (cf. [**BCG**^{3}], especially Chapters IX and X). In that theory, one assumes that $E_1 \xrightarrow{P_1} E_2$ is involutive with solution sheaf Θ, and then one seeks to construct the remaining terms in the above sequence that gives an exact sequence of sheaves which then provides a "resolution" of Θ. For each $x \in M$ and $\xi \in T_x^*M$, the pointwise symbol maps $\sigma(P_i)\colon E_{i,x} \otimes T_x^*M \to E_{i+1,x}$ give a complex, called the symbol sequence,

$$E_{1,x} \xrightarrow{\sigma(P_1)(\xi)} E_{2,x} \xrightarrow{\sigma(P_2)\xi} E_{3,x} \to \cdots \to E_{m,x}$$

whose cohomology is an important invariant of the situation (4.7). Also central to the theory is the characteristic variety $\Xi \subset \mathbb{P}T_x^*M$ defined by

$$\Xi = \{\, [\xi] \in \mathbb{P}T^*M : \ker \sigma(P_1)(\xi) \neq 0 \,\}.$$

Roughly speaking one has

- $\Xi = \mathbb{P}T^*M$ means that the PDE system defining Θ is *underdetermined*;
- $\operatorname{codim} \Xi = 1$ means that the PDE system is *determined*;
- $\operatorname{codim} \Xi > 1$ means that it is *overdetermined*;
- $\Xi = \emptyset$ means that the PDE is maximally overdetermined or *holonomic*.

In this latter case the sections of Θ over M are a *finite dimensional* vector space.

We observe that

$$R^q_{\pi_{\mathcal{U}}} \pi_D^* L_\mu \xrightarrow{d_1} R^q_{\pi_{\mathcal{U}}} \Omega^1_{\pi_D}(L_\mu) \xrightarrow{d_1} R^q_{\pi_{\mathcal{U}}} \Omega^2_{\pi_D}(L_\mu) \to \cdots$$

is a complex of the type (4.7) whose symbol sequence and characteristic variety are naturally associated to the spectral sequence (3.4). Although we do not know if the first d_1 is involutive or what the characteristic variety is in general, we shall now discuss the latter for our two examples.

In fact, to a general complex (4.7) there is naturally associated a spectral sequence leading to a definition of "secondary characteristic varieties." We suspect this construction may appear in the literature; we shall give and illustrate it in the situation studied here.

We will omit reference to the character μ and denote by

$$\begin{aligned} \mathcal{F}^{p,q} &= \text{sheaf of holomorphic sections of } F^{p,q} \to \mathcal{U} \\ &= R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L) \end{aligned}$$

with stalks $\mathcal{F}^{p,q}_u$ for $u \in \mathcal{U}$. For $\mathfrak{m}_u \subset \mathcal{O}_{\mathcal{U},u}$ the maximal ideal, we define

$$\operatorname{Gr}^k \mathcal{F}^{p,q}_u = \frac{\mathfrak{m}^k_u \otimes \mathcal{O}_{\mathcal{U},u} \mathcal{F}^{p,q}_u}{\mathfrak{m}^{k+1} \otimes_{\mathcal{O}_{\mathcal{U},u}} F^{p,q}_u}.$$

This is a locally free coherent sheaf over $\mathcal{U}$ whose typical fibre is

$$S^k \mathfrak{p}^* \otimes H^q\bigl(Z, \wedge^p N_{Z/D}(L)\bigr).$$

Combining the above identification $\mathfrak{p}^* \cong \mathfrak{p}$ with the inclusion

$$\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D}),$$

the cup-products give maps

$$S^k \mathfrak{p} \otimes H^q\bigl(Z, \wedge^p_{N_{Z/D}}(L)\bigr) \to S^{k-1}\mathfrak{p} \otimes H^q\bigl(Z, \wedge^{p+1} N_{Z/D}(L)\bigr).$$

The composition of these maps for k followed by $k-1$ is zero, and thus we have for each k a complex of sheaves

$$\mathrm{Gr}^k\mathcal{F}^{0,q} \to \mathrm{Gr}^{k-1}\mathcal{F}^{1,q} \to \cdots \to \mathrm{Gr}^{k-r}\mathcal{F}^{r,q}$$

where $r = \min(k, \mathrm{codim}_D\, Z)$ and where the maps are $\mathcal{O}_{\mathcal{U}}$-linear. For $k = 1$ this is just the symbol map. For $k = 2$ it is

$$\begin{tikzcd}
\mathrm{Gr}^2\mathcal{F}^{0,q} \arrow[r, "d_1"] \arrow[drr, dashed, "d_2"] & \mathrm{Gr}^1\mathcal{F}^{1,q} \arrow[r, "d_1"] & \mathrm{Gr}^0\mathcal{F}^{2,q} \\
\mathrm{Gr}^2\mathcal{F}^{0,q-1} \arrow[r] & \mathrm{Gr}^1\mathcal{F}^{1,q-1} \arrow[r] & \mathrm{Gr}^0\mathcal{F}^{2,q-1}.
\end{tikzcd}$$

Here we continue to denote by $\sigma_1(\xi)$ the natural maps induced by the usual symbol $\sigma_1(\xi)$ when $k = 1$.

The maps in the fibres at u depend only on $\xi \in T_u^*\mathcal{U}$. In a typical fibre we have

$$k=1 \qquad \xi \otimes H^q(Z,L) \xrightarrow{\sigma_1(\xi)} H^q\big(Z, N_{Z/D}(L)\big)$$

$$k=2 \qquad \begin{tikzcd}
\xi^{(2)}H^q(Z,L) \arrow[r, "\sigma_1(\xi)"] \arrow[drr, dashed, "\sigma_2(\xi)"] & \xi \otimes H^q\big(Z, N_{Z/D}(L)\big) \arrow[r, "\sigma_1(\xi)"] & H^q\big(Z, \wedge^2 N_{Z/D}(L)\big) \\
\xi^{(2)} \otimes H^{q-1}(Z,L) \arrow[r] & \xi \otimes H^q\big(Z, N_{Z/D}(L)\big) \arrow[r] & H^{q-1}\big(Z, \wedge^2 N_{Z/D}(L)\big).
\end{tikzcd}$$

Definition 4.6. The *secondary symbol* $\sigma_2(\xi)$ is defined by the dotted arrow above. The *secondary characteristic variety* Ξ_2 is defined by

$$\Xi_2 = \{\xi : \sigma_1(\xi) = 0, \sigma_2(\xi) = 0\}.$$

The definition of $\sigma_2(\xi)$ is clearly related to the differential d_2 in our spectral sequence. Recall that d_2 is a linear differential operator of degree $\leqq 2$ defined on $\ker d_1$. We are not aware of how one may define the symbol of such an operator; the above is one possible construction defined on decomposable elements $\xi^{(2)}$ where $\sigma_1(\xi) = 0$; i.e., $\xi \in \Xi$. In the discussion below of $\mathrm{Sp}(4)$ we shall abuse notation and denote by $\sigma(d_2)$ the above construction extended in the special case (i) there to not necessarily decomposable elements in $\mathfrak{p}^{(2)}$. This discussion is not meant to be rigorous or definitive, but rather our interest is to illustrate interesting behavior of Harish-Chandra modules associated to degenerate, or close to being degenerate in the sense that $\mu + \rho$ is near to a wall, discrete series and limits of such.

$S\mathcal{U}(2,1)$: As complex manifolds, we have $Z = \mathcal{U}(2)/T \cong S\mathcal{U}(2)/T_S$ where $T_S = S\mathcal{U}(2) \cap T$. As homogeneous complex manifolds they are distinct and have different sets of homogeneous vector bundles (cf. section II.A in [**GGK2**] for a discussion and illustration of this point). Here, for simplicity we shall use $Z = S\mathcal{U}(2)/T_S$,[4] and we denote by $W \cong \mathbb{C}^2$ the standard representation of $S\mathcal{U}(2)$ with $W^{(n)}$ being the n^{th} symmetric product. We then have

$$W = H^0(\mathcal{O}_Z(1)).$$

[4]In the case of $\mathrm{Sp}(4)$ discussed below, in order to be able to use weight considerations we shall need to use the homogeneous complex manifold $Z = \mathcal{U}(2)/T$.

From [**GGK2**], (A.IV.F.6) we have the identification of holomorphic vector bundles over Z

$$N_{Z/D} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1).$$

Setting

$$\deg L_\mu\big|_Z = k,$$

we then have the following tables of the fibres of $R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu) \to \mathcal{U}$ (here q is the y-axis and p is the x-axis, and $W^{(j)} = 0$ for $j < 0$):

$k = -l - 2, l > 0$

$W^{(l)^*}$	$\overset{2}{\oplus} W^{(l-1)^*}$	$W^{(l-2)^*}$
0	0	0

$k = -2$

$W^{(0)}$	0	0
0	0	$W^{(0)}$

$k = -1$

0	0	0
0	0	0

$k \geqq 0$

0	0	0
$W^{(k)}$	$\overset{2}{\oplus} W^{(k+1)}$	$W^{(k+2)}$

Using the identification

$$\begin{cases} \mathfrak{p} = H^0(Z, N_{Z/D}) = W \oplus W \\ \mathfrak{p}^* \cong \mathfrak{p}, \text{ as in Theorem (4.4),} \end{cases}$$

we shall analyze the various cases.

$\underline{k = -l - 2}$: The symbols are then

$$\begin{cases} \text{(i)} & W^{(l)^*} \otimes \left(\overset{2}{\oplus} W\right) \to \overset{2}{\oplus} W^{(l-1)^*} \\ \text{(ii)} & \left(\overset{2}{\oplus} W^{(l-1)^*}\right) \otimes \left(\overset{2}{\oplus} W\right) \to W^{(l-2)^*}. \end{cases}$$

These may be identified as follows:

(i) $P \otimes (w \oplus w') \longrightarrow P\lfloor w \oplus P\lfloor w'$ $\qquad P \in W^{(l)^*}; w, w' \in W$

(ii) $(P \oplus P') \otimes (w \oplus w') \longrightarrow P\lfloor w' - P'\lfloor w$ $\qquad P, P' \in W^{(l-1)^*}$.

It follows that (i) is injective unless w, w' are linearly dependent, and by a Koszul-type argument except in this case we have image (i) = kernel (ii). This gives the

Conclusion 4.7. For $k \leqq -3$ the characteristic variety Ξ is a quadric in $\mathbb{P}^3$; hence $\operatorname{codim} \Xi = 1$. For ξ non-characteristic the symbol sequence is exact.

$\underline{k \geqq 0}$: The symbols are then maps

(i) $W^{(k)} \otimes \left(\overset{2}{\oplus} W\right) \to \overset{2}{\oplus} W^{(k+1)}$;

(ii) $\left(\overset{2}{\oplus} W^{(k+1)}\right) \otimes \left(\overset{2}{\oplus} W\right) \to W^{(k+2)}$.

They may be identified as follows:

$$\begin{array}{lll} \text{(i)} & P\otimes(w\oplus w')\to Pw\oplus Pw' & P\in W^{(k)}; w,w'\in W \\ \text{(ii)} & (P\oplus P')\otimes(w\oplus w')\to Pw'-P'w & P,P'\in W^{(k+1)}; w,w'\in W. \end{array}$$

It follows that (i) is injective, unless of course $w=w'=0$, and then the symbol sequence is exact.

CONCLUSION 4.8. For $k\geqq 0$ the characteristic variety Ξ is empty, and the symbol sequence is exact.

We observe that by (3.5) in the cases $k\leqq -2$ and $k\geqq 0$ the maps

$$\begin{array}{rl} H^1(D,L_\mu)\hookrightarrow H^0(\mathcal{U},R^1_{\pi_{\mathcal{U}}}\pi^*_D L_\mu) & \qquad k\leqq -2 \\ -H^0(D,L_\mu)\hookrightarrow H^0(\mathcal{U},R^0_{\pi_{\mathcal{U}}}\pi^*_D L_\mu) & \qquad k\geqq 0 \end{array}$$

are injective. For $k\leqq -3$ and $k\geqq 0$ the image is just

$$\ker\{d_1: H^0(\mathcal{U}R^q_{\pi_{\mathcal{U}}}\pi^*_D L_\mu)\to H^0(\mathcal{U},R^q_{\pi_{\mathcal{U}}}\Omega^1_{\pi_D}(L_\mu))\}.$$

For $k=-2$ a very interesting special circumstance, to be discussed below, arises.

We also note that from the above conclusion we have that when $k\geqq 0$

$$\dim H^0(D,L_\mu)<\infty.$$

REMARK 4.9. For a general $D=G_0/T$ we will have

$$\dim H^q(D,L_\mu)<\infty \qquad 0\leqq q<d=\dim K_0/T \tag{4.8}$$

provided that for $Z=K_0/T$ and non-zero $\xi\in\mathfrak{p}$ the map

$$H^q(Z,L_\mu)\xrightarrow{\xi} H^q(Z,N_{Z/D}(L_\mu)) \tag{4.9}$$

is injective, where we are using the inclusion $\mathfrak{p}\hookrightarrow H^0(Z,N_{Z/D})$. We note that (4.8) is not true for D classical and $q=0$ (take any μ such that $\mu+\rho$ is dominant), and it is not true for D non-classical and $q=d$ (take any μ such that $\mu+\rho$ is anti-dominant). In general, the LHS is known by the Borel-Weil-Bott theorem. For the RHS there is a composition series for $N_{Z/D}$ whose line bundle factors are the L_β where $\beta\in\Phi^+_{nc}$ is a positive non-compact root. Thus, at least in principal, one might hope to analyze the map (4.9).[5] We are not aware of any case of a non-classical D where it fails to be injective for non-zero ξ.

We note finally that

LEMMA 4.10. *If μ is in the anti-dominant Weyl chamber and $N_{Z/D}\to Z$ is ample, then there is a filtration $F^pH^q(D,L_\mu)$ such that for $q<d$, the associated graded has* $\dim \mathrm{Gr}^\bullet H^q(D,L_\mu)<\infty$.

PROOF. The filtration

$$F^pL_\mu=\mathcal{I}^p_Z\underset{\mathcal{O}_Z}{\otimes}L_\mu$$

of L_μ leads to a spectral sequence abutting to $H^*(D,L_\mu)$ and, using that $F^pL_\mu/F^{p+1}L_\mu\cong \mathrm{Sym}^p N^*_{Z/D}(L_\mu)$, with E_1-terms given by

$$E_1^{p,q}=H^{p+q}(Z,\mathrm{Gr}^pL_\mu)=H^{p+q}(Z,\mathrm{Sym}^p N^*_{Z/D}(L_\mu)).$$

[5] As is evident from the works of Schmid (cf. the references in [**Sch2**]) and from part IV of [**FHW**] the combinatorics of the extension data in the composition series are quite intricate.

By the ampleness assumption

$$E_1^{p,q} = 0 \qquad \text{for} \qquad 0 \leqq q < d,\ p \geqq p_0(\mu),$$

which gives the conclusion. □

The condition of ampleness is rare but does occur, especially in low dimensional examples including the two discussed in this paper. We suspect that in fact (4.8) is valid, but to be able to conclude this one needs $\cap_p F^p H^q(D, L_\mu) = (0)$.

Returning to the general discussion of the symbol map for $S\mathcal{U}(2,1)$, in many ways the most interesting is the case $\underline{k = -2}$: Then we have for the symbol a map

$$\sigma(d_2) \cdot W^{(0)} \otimes p^{*(2)} \to W^{(0)}. \tag{4.10}$$

Calculations that are in progress for a separate work indicate that

The symbol map (4.8) *is given by*

$$\sigma(d_2)P = \tfrac{1}{2}\left\langle P, \Omega \right\rangle, \qquad P \in \mathfrak{p}^{*(2)}$$

where $\Omega \in \mathfrak{g}^{(2)}$ *is the Casimir operator.*

Here we are writing $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and thinking of $\mathfrak{p}^{*(2)} \subset \mathfrak{g}^{*(2)}$.

Representation-theoretic interpretation:[6] Referring to the root diagram in Figure 1 where C is the positive Weyl chamber for the non-classical complex structure on $\mathcal{U}(2,1)/T$, for weights μ such that for $\rho = \frac{1}{2}(\sum_{\alpha \in \Phi^*} \alpha)$ we have

$$\mu + \rho \in -C; \tag{4.11}$$

i.e., $\mu + \rho$ is *anti-dominant*, Schmid has shown that $H^1(D, L_\mu)$ is the HC-module $V_{\mu+\rho}$ with infinitesimal character $\chi_{\mu+\rho}$. Since

$$(4.11) \Rightarrow \deg L_\mu \leqq -3$$

from the discussion above and the results of [**Sch2**] we have

Lemma 4.11. *For a weight* μ *satisfying* (4.11)*, the HC-module associated to the discrete series representation with Harish-Chandra character* $\Theta_{\mu+\rho}$ *is realized as the kernel of the linear,* 1st *order differential operator above whose characteristic variety is a quadric in* $\mathbb{P}^3$*.*

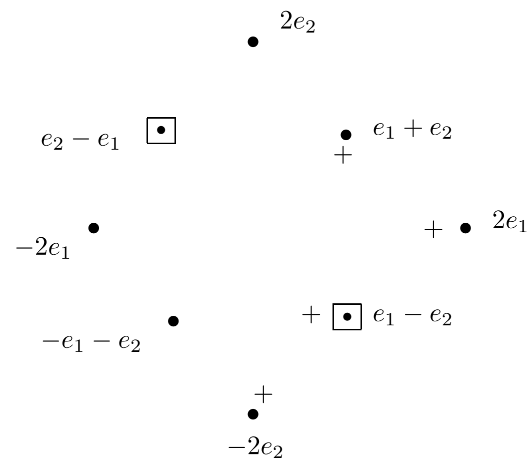
For the weight $\mu = -\rho$, so that $L_\mu\big|_Z = \omega_Z$ is the canonical bundle, $H^1(D, L_{-\rho})$ is the HC-module associated to the *totally degenerate limit of discrete series* (TDLDS) $(0, C)$ with infinitesimal character $\chi_0 = 0$ and corresponding to the non-classical Weyl chamber C. We expect to then have the conclusion

Conclusion 4.12. The HC-module associated to the TDLDS V_0 is the kernel of the scalar, linear 2nd order PDE given above whose symbol is $(1/2)\Omega$ where Ω is the Casimir operator.

$\underline{\mathrm{Sp}(4)}$: In discussing $S\mathcal{U}(2,1)$ we have been treating $Z = \mathcal{U}(2)/T$ as the homogeneous space $S\mathcal{U}(2)/T_S$ where $T_S = S\mathcal{U}(2) \cap T$. For Sp(4) weight considerations

[6]These will be more extensively discussed in joint work in preparation with Matt Kerr which is a sequel to [**GGK2**].

require that we use the full $\mathcal{U}(2)$ symmetry group. In the weight diagram



we have labelled the positive roots for the Weyl chamber C corresponding to our non-classical complex structure D by $+$, and the compact roots by $\boxed{\bullet}$. We denote by $L_{k_1e_1+k_2e_2} \to D$ the $\mathcal{U}(2)$-homogeneous line bundle given by the character of T corresponding to the weight $k_1e_1 + k_2e_2$. We then set

- $W = \mathcal{U}(2)$-module $H^0(Z, L_{e_1})$;
- $\delta = \mathcal{U}(2)$-module $\Lambda^2 W$ given by the character of $\mathcal{U}(2)$ with weight $e_1 + e_2$;
- $W_k^{(n)} = \mathcal{U}(2)$-module $\operatorname{Sym}^n W \otimes \delta^k$.

Then we have as $\mathcal{U}(2)$-modules

$$\text{(4.12)}\qquad \begin{aligned} &\bullet\quad H^0(Z, L_{k_1e_1+k_2e_2}) = W_{k_2}^{(k_1-k_2)} && (= 0 \text{ if } k_1 > k_2);\\ &\bullet\quad H^1(Z, L_{k_1e_1+k_2e_2}) = W_{k_1+1}^{(k_2-k_1-2)} && (= 0 \text{ if } k_2 > k_1 + 2);\\ &\bullet\quad W_k^{(n)^*} = W_{-n-k}^{(n)};\\ &\bullet\quad W_k^{(n)} \otimes W_l^{(m)} = \bigoplus_{i\geqq 0} W_{i+k+l}^{(n+m-2i)} && \text{if } m \le n. \end{aligned}$$

From the root diagram we may infer that for the normal bundle $N_{Z/D} \to Z$ we have as $\mathcal{U}(2)$-homogeneous vector bundles

$$\text{(4.13)}\qquad \begin{cases} N_{Z/D} = L_{-2e_2} \oplus N' \\ 0 \to L_{e_1+e_2} \to N' \to L_{2e_1} \to 0. \end{cases}$$

Using this we see that as $\mathcal{U}(2)$-modules

$$\text{(4.14)}\qquad H^0(Z, N_{Z/D}) = \underbrace{W_0^{(2)} \oplus W_{-2}^{(2)}}_{\mathfrak{p}} \oplus W_1^{(0)}.$$

Here, we have the inclusion $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ given by the terms over the bracket, and there is one "extra" deformation of Z; i.e., not coming from moving Z by G, corresponding to $W_1^{(0)}$. Since $H^1(Z, N_{Z/D}) = 0$, this extra infinitesimal deformation of $Z \subset D$ is unobstructed (cf. Part IV in [**FHW**] for a general discussion of this point).

For the line bundle $L_\mu = L_{k_1 e_1 + k_2 e_2}$ we have

$$\begin{cases} \deg L_\mu = k_1 - k_2 \\ \omega_Z = L_{e_2 - e_1} \qquad (\Rightarrow \deg \omega_Z = -2). \end{cases}$$

The following tables show where the non-zero groups $H^q(Z, \Lambda^p N_{Z/D}(L_\mu))$ occur. The specific $\mathcal{U}(2)$-modules can be identified using (4.12) and (4.13), and this will be done in two cases of particular interest.

$\underline{k \leqq -5}$:

*	*	*	*
0	0	0	0

.

Here, the term in the upper right-hand position is zero if $k = -5$. It may be checked that

$$\mu + \rho \ \text{ anti-dominant } \ \Rightarrow k \leqq -5.$$

Thus, the HC module associated to the discrete series have the above picture. The spectral sequence degenerates at E_2, which is a general phenomenon.

$\underline{k = -4}$:

*	*	*	0
0	0	0	*

$\underline{k = -3}$:

*	*	0	0
0	0	*	*

$\underline{k = -2}$:

*	0	0	0
0	*	*	*

$\underline{k = -1}$:

0	0	0	0
0	*	*	*

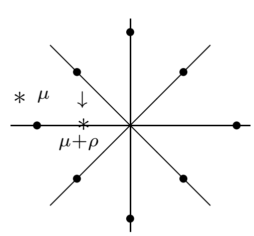
$\underline{k \geq 0}$:

0	0	0	0
*	*	*	*

The cases $k \leqq -5$, which include the discrete series, and $k \geqq 0$ where the characteristic variety $\Xi = \emptyset$ and $\dim H^0(D, L_\mu) < \infty$, are similar to the $S\mathcal{U}(2,1)$ example discussed above. Here we only analyze two particularly interesting cases:

(1) $L_{-\rho} = L_{-2e_1 + e_2}$ corresponding to a TDLDS;
(2) $L_{-3e_1 + e_2}$ corresponding to a non-degenerate limit of discrete series (NDLDS).

In case (2) the picture for $\mu = -3e_1 + e_2$ and $\mu + \rho$ is



The arrow means that the NDLDS is associated to a non-classical, anti-dominant Weyl chamber; that is it is a non-holomorphic NDLDS.

Case (1): The picture here is

$W_{-1}^{(1)}$	$W_0^{(1)}$	0	0
0	0	$W_{-1}^{(1)}$	$W_0^{(1)}$

We have as above

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)} \cong \mathfrak{p}^*.$$

Dualizing the symbol map

$$W_{-1}^{(1)} \otimes \mathfrak{p}^* \to W_0^{(1)}$$

for $d_1\colon E_1^{0,1} \to E_1^{0,2}$ at our reference point gives, using $\mathfrak{p} \cong \mathfrak{p}^*$,

$$W_{-1}^{(1)} \otimes W_{-1}^{(1)} \to \mathfrak{p} = W_{-2}^{(2)} \oplus W_0^{(2)}.$$

By consideration of weights we end up in the $W_{-2}^{(2)}$-factor. This map is thus

$$H^0(Z, \mathcal{O}(1)) \otimes H^0(Z, \mathcal{O}(1)) \to H^0(Z, \mathcal{O}(2)).$$

Unwinding the dualities, we see that the non-zero part of the symbol is a map

$$\begin{array}{ccc} V \otimes S^2V^* & \longrightarrow & V^* \\ \cup & & \cup \\ u \otimes q & \longrightarrow & u \rfloor q \end{array}$$

where V is a 2-dimensional vector space. This map is an isomorphism if, and only if, $q \in S^2V^*$ is a non-singular quadric. This gives the

Conclusion 4.13. The characteristic variety $\Xi \subset \mathbb{P}(W_0^{(2)} \oplus W_{-2}^{(2)})$ is the projectivization of (non-singular quadric in $W_0^{(2)}) \oplus W_{-2}^{(2)}$.

This is a singular quadric in $\mathbb{P}\mathfrak{p}^*$. Since $d_1\colon E_1^{1,0} \to E_1^{1,1}$ is a determined linear PDE system it is consistent that $\operatorname{codim} \Xi = 1$.

Turning to d_2, since for a 1st order determined linear PDE $P\colon E \to E'$ whose characteristic variety is a hypersurface, the solutions that vanish to 2nd order at a point define a linear subspace

$$F_x \subset E_x \otimes S^2T_x^*M$$

that projects *onto* E_x in the sense that the natural map $F_x \otimes S^2T_xM \to E_x$ is surjective as explained above, we may consider the symbol of d_2 as a map

$$\sigma(d_2)\colon W_{-1}^{(1)} \otimes S^2\mathfrak{p} \to W_{-1}^{(1)}.$$

Now by (4.12)

$$S^2\mathfrak{p} = S^2W_0^{(2)} \oplus \left(W_0^{(2)} \otimes W_{-2}^{(2)}\right) \oplus S^2W_{-2}^{(2)}.$$

By weight considerations, only $W_0^{(2)} \otimes W_{-2}^{(2)}$, which contains the map $\mathrm{id}_{W_0^{(2)}}\colon W_0^{(2)} \to W_0^{(2)}$, is going to map $W_{-1}^{(1)}$ to $W_{-1}^{(1)}$. Then by (4.12)

$$\begin{cases} \operatorname{Hom}(W_{-1}^{(1)}, W_{-1}^{(1)}) = W_{-1}^{(2)} \oplus W_0^{(0)} \\ W_{-2}^{(2)} \otimes W_0^{(2)} = W_{-2}^{(4)} \oplus \underbrace{W_{-1}^{(2)} \oplus W_0^{(0)}}. \end{cases} \tag{4.15}$$

Thus the only potentially non-zero piece of $\sigma(d_2)$ arises from the term over the brackets. In fact, since

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)}$$

and

$$q \in W_0^{(2)} \cong S^2 V^*$$

has rank one, taking for example $q = z_1^{*2}$ and $z_2 \in \ker d_1 \subset V$ then

$$q \oplus p \in S^2 V \oplus S^2 V^*$$

maps to $\mathrm{Hom}(W_{-1}^{(1)}, W_{-1}^{(1)})$ by

$$p = az_1^2 + bz_1 z_2 + cz_2^2$$
$$q \otimes p \to 2az_1^* \otimes z_1 + bz_1^* \otimes z_2$$

and $z_2 \to 0$ under $q \otimes p$. Thus $z_2 \in \ker d_2$ so that we are led to the

Conclusion 4.14. Viewing the symbol as a map

$$\mathrm{Hom}\left(W_{-1}^{(1)}, W_{-1}^{(1)}\right) \to S^2\mathfrak{p},$$

from (4.15), the only non-zero part is a map

$$W_{-1}^{(2)} \oplus W_0^{(0)} \to W_{-1}^{(2)} \oplus W_1^{(0)}.$$

This map is a constant c times the identity.

We suspect, but have not proved, that $c \neq 0$; i.e., the characteristic variety of d_2 is non-trivial. The relation, if any, between the symbol $\sigma(d_2)$ and the Casimir operator is not yet understood by the authors. The case where $c = 0$ will be commented on at the end of this section.

Case (2): Here the picture is

$W_{-2}^{(2)}$	$W_{-2}^{(0)} \oplus W_{-1}^{(2)} \oplus W_0^{(0)}$	$W_{-1}^{(0)}$	0
0	0	0	$W_0^{(0)}$

This is derived from (4.12) and (4.13), and for the $E_1^{2,0}$ and $E_1^{2,1}$ terms uses that in the cohomology sequence

$$0 \to L_{-2e_1} \to N' \otimes L_{-2e_2} \otimes L_{-2e_1+e_2} \to L_{-e_1-e_2} \to 0$$

we have

$$H^0(Z, L_{-e_1-e_2}) \xrightarrow{\sim} H^1(Z, L_{-2e_1}).$$

Using $\mathfrak{p} \cong \mathfrak{p}^*$, for the symbol

$$\sigma(d_1)\colon E_1^{0,1} \otimes \mathfrak{p} \to E_1^{1,1} \tag{4.16}$$

we have

$$\begin{aligned} W_{-2}^{(2)} \otimes \mathfrak{p} &= \left(W_{-2}^{(2)} \otimes W_0^{(2)}\right) \oplus \left(W_{-2}^{(2)} \otimes W_{-2}^{(2)}\right) \\ &\cong \left(W_{-2}^{(4)} \oplus \underbrace{W_{-1}^{(2)} \oplus W_0^{(0)}}\right) \oplus \left(W_{-4}^{(4)} \oplus W_{-3}^{(2)} \oplus \underbrace{W_{-2}^{(0)}}\right). \end{aligned}$$

By weight considerations, only the terms over the brackets can map to something non-zero under d_1. Thus the symbol map is

$$\underbrace{\left(W_{-2}^{(2)} \otimes W_0^{(2)}\right)} \oplus \underbrace{\left(W_{-2}^{(2)} \otimes W_{-2}^{(2)}\right)} \to \underbrace{\left(W_{-1}^{(2)} \oplus W_0^{(0)}\right)} \oplus \underbrace{W_{-2}^{(0)}}$$

where the terms over the single and double brackets correspond under the symbol map and may be seen to be surjective. In fact, using from (4.12) that

$$W_0^{(2)} \cong W_{-2}^{(2)^*}$$

the map over the double brackets is just contraction of $Q_1 \in W_{-2}^{(2)}$ with $Q_2 \in W_0^{(2)}$ twisted by δ^2.

The map over the single brackets is of the general form

$$S^2V \otimes S^2V \to (V \otimes V) \otimes (V \otimes V) \to V \otimes \Lambda^2 V \otimes V \to \Lambda^2 V \otimes S^2 V$$

together with

$$S^2V \otimes S^2V \to \Lambda^2 V \otimes S^2 V$$

where V is a 2-dimensional vector space. Together these two maps give

$$S^2V \otimes S^2V \to V \otimes \Lambda^2 V \otimes V \to \Lambda^2 V \otimes V \otimes V.$$

In coordinates and taking the above duality into account, the map is

$$\Big(\sum_{i,j} a_{ij} z_i^* z_j^*\Big) \otimes \Big(\sum_{k,l} b_{kl} z_k z_l\Big) \to \sum_{i,k} \Big(\sum_j a_{ij} b_{jk}\Big) z_i^* z_k.$$

There are three cases depending on the rank of $Q_2 = \sum_{k,l} b_{kl} z_k z_l$.

Rank $Q_2 = 2$: Taking $Q_2 = z_1 z_2$, the above map on $Q_1 = \sum_{i,j} z_i^* z_j^*$ is

$$\begin{cases} z_1^{*2} \to z_1^* \otimes z_2 \\ z_2^{*2} \to z_2^* \otimes z_1 \\ z_1^* z_2^* \to z_1^* \otimes z_2 + z_2^* \otimes z_1. \end{cases}$$

In this case there is no kernel contracting with Q_2.

Rank $Q_2 = 1$: Taking $Q_2 = z_1^2$ we have

$$\begin{cases} z_1^{*2} \to 2z_1^* \otimes z_1 \\ z_1^* z_2^* \to 2z_2^* \otimes z_1 \\ z_2^{*2} \to 0. \end{cases}$$

Now

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)}$$

where $Q_2 \in W_0^{(2)}$ and $Q_1 \in W_{-2}^{(2)}$. Then

- $Q_2 = z_1 z_2$ mapping to the kernel of the $\underbrace{\ }$ part of d_1 is zero;
- $Q_2 = z_1^2$ mapping to the kernel of the $\underbrace{\ }$ part of d_1 is z_2^{*2};
- $Q_2 = 0$ mapping to the kernel of the $\underbrace{\ }$ part of d_1 is all of $W_{-2}^{(2)}$.

If now $Q_2 = z_1^2$, then for $Q_1 = a z_1^{*2} + b z_1^* z_2^* + c z_2^{*2}$ the kernel of the $\underbrace{\underbrace{\ }}$ part of d_1 takes z_2^{*2} to $2cz_2$. So there is one further condition on Q_1, namely $c = 0$, to have a non-trivial $\ker d_1$. If $Q_2 = 0$, then Q_1 contracts to zero with a codimension $\geqq 2$ subspace of $W_{-2}^{(2)}$ for any Q_1. Thus the characteristic variety has codimension 2.

Rank $Q_2 = 0$: Then contracting with Q_1 we always get a rank 2 kernel. But $Q_2 = 0$ is a codimension 3 condition.

CONCLUSION 4.15. The characteristic variety Ξ of the symbol map (4.16) has $\operatorname{codim} \Xi = 2$.

This is consistent with $d_1 = E_1^{0,1} \to E_1^{1,1}$ being an overdetermined PDE system.

Remarks concerning degenerate symbols: We begin with the general

OBSERVATION 4.16. Schmid's result [**Sch2**] that $H^r(D, L_\mu) = 0$ for $r > d = \dim Z$ implies conditions on the differentials in the spectral sequence in Theorem (3.4).

Specifically, no terms in $E_1^{p,q}$ can survive to $E_\infty^{p,q}$ if $p + q > d$.

As we shall now discuss, this has implications for the symbol maps. For this we make the following

CONVENTION. If $P\colon E \to F$ is a differential operator of order $\leqq k$ whose symbol mapping $E \otimes S^k T^* M \to F$ is zero, then P has order $\leqq k-1$. We define *the* symbol of $P\colon E \to F$ to be the first non-zero map $E \otimes S^l T^* M \to F$.

Referring to the discussion below (4.14), if $c = 0$ then d_2 is a differential operator of order $\leqq 1$. If it is truly of order 1, then the symbol is a map

$$W_{-1}^{(2)} \oplus W_0^{(0)} \to W_{-2}^{(2)} \oplus W_0^{(2)},$$

which by weight considerations must be zero. Thus, d_2 is a scalar operator

$$W_{-1}^{(1)} \to W_{-1}^{(1)}$$

which must be a multiple of the identity. We again suspect, but have not proved, that if this situation does occur then the multiple is non-zero.

When we turn to case (ii), from (4.16) the mapping

$$d_2\colon \ker d_1 \cap E_1^{1,1} \to E_1^{3,0}$$

must be an isomorphism. By our convention above, the symbol $\sigma(d_2)$ must be non-zero. The various cases where d_2 is of actual order $2, 1, 0$ can be analyzed using weight considerations, but we shall not do so here.

We conclude with a remark about the case $k \leqq -5$, where the picture is

$*$	$*$	$*$	$*$
0	0	0	0

In this case we have an exact sequence

$$0 \to H^0(D, L_\mu) \to E_1^{0,1} \xrightarrow{d_1} E_1^{1,1} \xrightarrow{d_1} E_1^{2,1} \to E_1^{3,1} \to 0. \tag{4.17}$$

Now $E_1^{p,q} = H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big)$, and since $\mathcal{U}$ is a Stein manifold we believe it follows that setting $\Theta_\mu = \ker\{\, d_1 : R^1_{\pi_{\mathcal{U}}} \pi_D^* L_\mu \to R^1_{\pi_{\mathcal{U}}} \Omega^1(L_\mu)\,\}$ we have over $\mathcal{U}$ the exact *sheaf* sequence

$$\begin{aligned} 0 \to \Theta_\mu \to R^1_{\pi_{\mathcal{U}}} \pi_D^* L_\mu &\xrightarrow{d_1} R^1_{\pi_{\mathcal{U}}} \Omega^1(L_\mu) \\ &\xrightarrow{d_1} R^1_{\pi_{\mathcal{U}}} \Omega^2_{\pi_D}(L_\mu) \xrightarrow{d_1} R^1_{\pi_{\mathcal{U}}} \Omega^3_{\pi_D}(L_\mu) \to 0. \end{aligned}$$

Although we have not tried to analyze this, it seems interesting and reasonable that this should be a Spencer resolution as in [**BCG**^{3}], Chapter X. In fact, this could be a general phenomenon for the discrete series.

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Clay Mathematics Proceedings
Volume **18**, 2013

Geometry of theta divisors — a survey

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Abstract. We survey the geometry of the theta divisor and discuss various loci of principally polarized abelian varieties (ppav) defined by imposing conditions on its singularities. The loci defined in this way include the (generalized) Andreotti-Mayer loci, but also other geometrically interesting cycles such as the locus of intermediate Jacobians of cubic threefolds. We shall discuss questions concerning the dimension of these cycles as well as the computation of their class in the Chow or the cohomology ring. In addition we consider the class of their closure in suitable toroidal compactifications and describe degeneration techniques which have proven useful. For this we include a discussion of the construction of the universal family of ppav with a level structure and its possible extensions to toroidal compactifications. The paper contains numerous open questions and conjectures.

Introduction

Abelian varieties are important objects in algebraic geometry. By the Torelli theorem, the Jacobian of a curve and its theta divisor encode all properties of the curve itself. It is thus a natural idea to study curves through their Jacobians. At the same time, one is led to the question of determining which (principally polarized) abelian varieties are in fact Jacobians, a problem which became known as the Schottky problem. Andreotti and Mayer initiated an approach to the Schottky problem by attempting to characterize Jacobians via the properties of the singular locus of the theta divisor. This in turn led to the introduction of the Andreotti-Mayer loci N_k parameterizing principally polarized abelian varieties whose singular locus has dimension at least k.

In this survey paper we shall systematically discuss loci within the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties (ppav), or within the universal family of ppav $\mathcal{X}_g$, defined by imposing various conditions on the singularities of the theta divisor. Typically these loci are defined by conditions on the dimension of the singular locus or the multiplicity of the singularities or both. A variation is to ask for loci where the theta divisor has singularities of a given multiplicity at a special point, such as a 2-torsion point. We shall discuss the geometric relevance of such

2010 *Mathematics Subject Classification.* Primary 14K10,14K25.

Research of the first author is supported in part by National Science Foundation under grant DMS-10-53313.

Research of the second author is supported in part by DFG grant Hu-337/6-2. He is a member of the Research Training Group "Analysis, Geometry and String Theory" and of the Riemann Center at Leibniz Universität Hannover.

loci, their dimension, and their classes in the Chow ring of either $\mathcal{A}_g$ or a suitable compactification. In some cases we shall also discuss the restriction of these loci to the moduli space $\mathcal{M}_g$ of curves of genus g. Needless to say, we will encounter numerous open problems, as well as some conjectures.

An approach that was successfully applied to the study of these loci in a number of cases is to study these loci by degeneration, i.e. to investigate the intersection of the closure of such a locus with the boundary of a suitable compactification. This requires a good understanding of the universal family $\mathcal{X}_g \to \mathcal{A}_g$, and its possible extension to the boundary, as well as understanding the same for suitable finite Galois covers, called level covers. This is a technically very demanding problem. In this survey we will discuss the construction of the universal family in some detail and we will survey existing results in the literature concerning the extension of the universal family to toroidal compactifications, in particular the second Voronoi compactification. We will also explain how this can be used to compute the classes of the loci discussed above, in the example of intermediate Jacobians of cubic threefolds.

1. Setting the scene

We start by defining the principal object of this paper:

Definition 1.1. A complex *principally polarized abelian variety (ppav)* (A, Θ) is a smooth complex projective variety A with a distinct point (origin) that is an abelian algebraic group, together with the first Chern class $\Theta := c_1(L)$ of an ample bundle L on A which has a unique (up to scalar) section , i.e. $\dim H^0(A, L) = 1$.

We denote $\mathcal{A}_g$ the moduli stack of ppav up to biholomorphisms preserving polarization. This is a fine moduli stack, and we denote $\pi : \mathcal{X}_g \to \mathcal{A}_g$ the universal family of ppav, with the fiber of π over some (A, Θ) being the corresponding ppav. Note that since any ppav has the involution $-1 : z \mapsto -z$, the generic point of $\mathcal{A}_g$ is stacky, and for the universal family to exist, we have to work with stacks. In fact the automorphism group of a generic ppav is exactly $\mathbb{Z}/2\mathbb{Z}$, and thus geometrically as a variety the fiber of the universal family over a generic $[A] \in \mathcal{A}_g$ is $A/\pm 1$, which is called the Kummer variety.

We would now like to define a universal polarization divisor $\Theta_g \subset \mathcal{X}_g$: for this we need to prescribe it (as a line bundle, not just as a Chern class) on each fiber, and to prescribe it on the base, and both issues are complicated. Indeed, notice that translating a line bundle L on an abelian variety A by a point $a \in A$ gives a line bundle $t_a^* L$ on A of the same Chern class as L. In fact for a ppav (A, Θ) the dual abelian variety $\hat{A} := \mathrm{Pic}^0(A)$ is isomorphic to A — the isomorphism is given by the morphism $A \to \hat{A}$, $a \mapsto t_a^* L \otimes L^{-1}$. Thus the Chern class Θ determines the line bundle L uniquely up to translation. It is thus customary to choose a "symmetric" polarization on a ppav, i.e. to require $(-1)^* L = L$. However, for a given $\Theta = c_1(L)$ this still does not determine L uniquely: L can be further translated by any 2-torsion point on A (points of order two on the group A). We denote the group of 2-torsion points on A by $A[2]$.

Thus the set of symmetric polarizations on A forms a torsor over $A[2]$ (i.e. is an affine space over $\mathbb{Z}/2\mathbb{Z}$ of dimension $2g$), and there is in fact no way to universally choose one of them globally over $\mathcal{A}_g$.

Moreover, to define the universal polarization divisor, we would also need to prescribe it along the zero section in order to avoid ambiguities coming from the pullback of a line bundle on the base. The most natural choice would be to require the restriction of the universal polarization to the zero section to be trivial, but this is not always convenient, as one would rather have an ample bundle $\Theta_g \subset \mathcal{X}_g$ (rather than just nef globally, and ample on each fiber of the universal family).

We shall now describe the *analytic* approach to defining the universal theta divisor on $\mathcal{A}_g$, and hence restrict solely to working over the complex numbers. We would, however, like to point out that this is an unnecessary restriction and that there are well developed approaches in any characteristic – the very first example being the Tate curve [**Tat67**] in $g = 1$. In fact $\mathcal{A}_g$ and toroidal compactifications can be defined over $\mathbb{Z}$. We refer the reader to [**FC90**], [**Ale02**] and [**Ols08**]. The basis of these constructions is Mumford's seminal paper [**Mum72**] and the analysis of suitable degeneration data, which take over the role of (limits of) theta functions in characteristic 0. We will later comment on the relationship between moduli varieties in the analytic, respectively the algebraic category and stacks.

Recall that the *Siegel upper half-space* of genus g is defined by

$$\mathcal{H}_g := \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) |\, \tau = \tau^t, \mathrm{Im}(\tau) > 0\}$$

(where the second condition means that the imaginary part of τ is positive definite). To each point $\tau \in \mathcal{H}_g$ we can associate the lattice $\Lambda_\tau \subset \mathbb{C}^g$ (that is, a discrete abelian subgroup) spanned by the columns of the $g \times 2g$ matrix $\Omega_\tau = (\tau, \mathbf{1}_g)$. The torus $A_\tau = \mathbb{C}^g/\Lambda_\tau$ carries a principal polarization by the following construction: the standard symplectic form defines an integral pairing $J : \Lambda_\tau \otimes \Lambda_\tau \to \mathbb{Z}$, whose $\mathbb{R}$-linear extension to $\mathbb{C}^g$ satisfies $J(x, y) = J(ix, iy)$. The form

$$H(x, y) = J(ix, y) + iJ(x, y)$$

then defines a positive definite hermitian form $H > 0$ on $\mathbb{C}^g$ whose imaginary part is the $\mathbb{R}$-linear extension of J. Indeed, using the defining properties $\tau = \tau^t$ and $\mathrm{Im}(\tau) > 0$ of Siegel space, the fact that H is hermitian and positive definite translates into the *Riemann relations*

$$\Omega_\tau^t J^{-1} \Omega_\tau = 0, \quad i\Omega_\tau^t J^{-1} \overline{\Omega}_\tau > 0.$$

By the Lefschetz theorem $H \in H^2(A_\tau, \mathbb{Z}) \cap H^{1,1}(A_\tau, \mathbb{C})$ is the first Chern class of a line bundle L and the fact that $H > 0$ is positive definite translates into L being ample. We refer the reader to [**BL04**] for more detailed discussions.

There are many choices of lattices which define isomorphic ppav. To deal with this problem, one considers the symplectic group $\mathrm{Sp}(g, \mathbb{Z})$ consisting of all integer matrices which preserve the standard symplectic form. This group acts on the Siegel space by the operation

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}$$

where A, B, C and D are $g \times g$ blocks. It is easy to see that $A_\tau \cong A_{\gamma\circ\tau}$, with the isomorphism given by multiplication on the right by $(C\tau + D)^{-1}$. The points of the quotient $\mathrm{Sp}(g, \mathbb{Z})\backslash\mathcal{H}_g$ are in 1-to-1 correspondence with isomorphism classes of ppav of dimension g. This quotient is an analytic variety with finite quotient singularities, and by a well-known result of Satake [**Sat56**] also a quasi-projective variety. Indeed, it is the coarse moduli space associated to the moduli stack of ppav. By misuse of notation we shall also denote it by $\mathcal{A}_g$, and specify which we mean whenever it is not clear from the context. By an abuse of notation we will sometimes write $\tau \in \mathcal{A}_g$, choosing some point $\tau \in \mathcal{H}_g$ mapping to its class $[\tau] \in \mathcal{A}_g$ (otherwise the notation $[\tau]$ is too complicated).

To define principal polarizations explicitly analytically, we consider the Riemann theta function

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \mathbf{e}(n^t \tau n/2 + n^t z),$$

where we denote $\mathbf{e}(x) := \exp(2\pi i x)$ the exponential function. This series converges absolutely and uniformly on compact sets in $\mathcal{H}_g$. For fixed τ the theta function transforms as follows with respect to the lattice Λ_τ: for $m, n \in \mathbb{Z}^g$ we have

$$\theta(\tau, z + n + \tau m) = \mathbf{e}(-\frac{1}{2} m \tau^t m - m^t z)\theta(\tau, z). \tag{1}$$

Moreover, the theta function is even in z:

$$\theta(\tau, -z) = \theta(\tau, z). \tag{2}$$

For fixed $\tau \in \mathcal{H}_g$ the zero locus $\{\theta(\tau, z) = 0\}$ is a divisor in A_τ, and the associated line bundle L defines a principal polarization on A_τ, i.e. the first Chern class $c_1(L) \in H^2(A_\tau, \mathbb{Z}) = \mathrm{Hom}(\Lambda^2 \Lambda_\tau, \mathbb{Z})$ equals the Riemann bilinear form discussed above. Since the theta function is even, the line bundle L is symmetric, i.e. $(-1)^*(L) \cong L$.

The situation becomes more difficult when one works in families. For this, we consider the group $\mathrm{Sp}(g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ where the semi-direct product is given by the natural action of $\mathrm{Sp}(g, \mathbb{Z})$ on vectors of length $2g$. This group acts on $\mathcal{H}_g \times \mathbb{C}^g$ by

$$(\gamma, (m, n)) : (\tau, z) \mapsto (\gamma \circ \tau, (z + \tau m + n)(C\tau + D)^{-1})$$

where $n, m \in \mathbb{Z}^g$. Note that for $\gamma = \mathbf{1}$ this formula simply describes the action of the lattice Λ_τ on $\mathbb{C}^g$. We would like to say that the quotient $\mathcal{X}_g = \mathrm{Sp}(g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g} \backslash \mathcal{H}_g \times \mathbb{C}^g$ is the universal abelian variety over $\mathcal{A}_g$. This is true in the sense of stacks, but not for coarse moduli spaces: note in particular that $(-\mathbf{1}, (0, 0))$ acts on each fiber by the involution $z \mapsto -z$. Further complications arise from points in $\mathcal{H}_g$ with larger stabilizer groups.

We would like to use the Riemann theta function to define a polarization on this "universal family". However, this is not trivial. The transformation formula of $\theta(\tau, z)$ with respect to the group $\mathrm{Sp}(g, \mathbb{Z})$ is difficult and requires the introduction of generalized theta functions which we will discuss below, for details see [**Igu72**, pp. 84,85]. It is, however, still true that the Chern class of the line bundle associated to the divisor $\{\theta(\tau, z) = 0\} \subset A_\tau$ is the same as the Chern class of $\{\theta(\gamma \circ \tau, \gamma \circ z) = 0\} \subset A_{\gamma \circ \tau} \cong A_\tau$, although the line bundles are in general not isomorphic. The point is that a symmetric line bundle representing a polarization is only defined up to translation by a 2-torsion point, and there is no way of making such a choice globally over $\mathcal{A}_g$.

This problem leads to considering generalizations of the classical Riemann theta function. For each $\varepsilon, \delta \in \mathbb{R}^{2g}$ we define the function

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) := \sum_{n \in \mathbb{Z}^g} \mathbf{e}((n + \varepsilon)^t \tau (n + \varepsilon)/2 + (n + \varepsilon)^t (z + \delta)).$$

Note that for $\varepsilon = \delta = 0$ this is just the function $\theta(\tau, z)$ defined before. The pair $m = (\varepsilon, \delta)$ is called the characteristic of the theta function. For the transformation behavior of these functions with respect to translation by Λ_τ we refer the reader to [**Igu72**, pp. 48,49].

This function defines a section of a line bundle $L_{\varepsilon,\delta}$ which differs from the line bundle $L = L_{0,0}$ defined by the standard theta function by translation by the point $\varepsilon\tau + \delta \in A_\tau$. In particular the two line bundles have the same first Chern class and represent the same principal polarization. Of special interest to us is the case where ε, δ are half-integers, of which we think then as $\varepsilon, \delta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$, in which case the line bundle $L_{\varepsilon,\delta}$ is symmetric. We then call the characteristic (ε, δ) *even* or *odd* depending on whether $4\varepsilon \cdot \delta$ is 0 or 1 modulo 2. The function $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, z)$ is even or odd depending on whether the characteristic is even or odd. The value at $z = 0$ of a theta function with characteristic is called *theta constant* with characteristic — thus in particular all theta constants with odd characteristic vanish identically. The action of the symplectic group on the functions $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, z)$ is given by the theta transformation formula, see [**Igu72**, Theorem 5.6] or [**BL04**, Formula 8.6.1], which in particular permutes the characteristics. This too shows that there is no way of choosing a symmetric line bundle universally over $\mathcal{A}_g$.

In order to circumnavigate this problem we shall now consider *(full) level structures*, which lead to Galois covers of $\mathcal{A}_g$. Level structures are a useful tool if one wants to work with varieties rather than with stacks. The advantage is twofold. Firstly, one can thus avoid problems which arise from torsion elements, or more generally, non-neatness of the symplectic group $\mathrm{Sp}(g, \mathbb{Z})$. Secondly, going to suitable level covers allows one to view theta functions and, later on, their gradients, as sections of bundles. One can thus perform certain calculations on level covers of $\mathcal{A}_g$ and, using the Galois group, then interpret them on $\mathcal{A}_g$ itself. We shall next define the full level ℓ covers of $\mathcal{A}_g$ and discuss the construction of universal families over these covers.

Definition 1.2. The full level ℓ subgroup of $\mathrm{Sp}(g, \mathbb{Z})$ is defined by

$$\Gamma_g(\ell) := \{\gamma \in \mathrm{Sp}(g, \mathbb{Z}) |\, \gamma \equiv \mathbf{1}_{2g} \mod \ell\}.$$

Note that this is a normal subgroup since it is the kernel of the projection $\mathrm{Sp}(g, \mathbb{Z}) \to \mathrm{Sp}(g, \mathbb{Z}/\ell\mathbb{Z})$. We call the quotient

$$\mathcal{A}_g(\ell) := \Gamma_g(\ell)\backslash\mathcal{H}_g$$

the level ℓ cover. There is a natural map $\mathcal{A}_g(\ell) \to \mathcal{A}_g$ of varieties which is Galois with Galois group $\mathrm{PSp}(g, \mathbb{Z}/\ell\mathbb{Z})$.

One wonders whether the theta function transforms well under the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$, i.e. if there is a transformation formula similar to (1) relating its values at τ and $\gamma \circ \tau$. To put this in a proper framework, we define

Definition 1.3. A holomorphic function $F : \mathcal{H}_g \to \mathbb{C}$ is called a (Siegel) modular form of weight k with respect to a finite index subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ if

$$F(\gamma \circ \tau) = \det(C\tau + D)^k F(\tau) \qquad \forall\gamma \in \Gamma, \forall\tau \in \mathcal{H}_g$$

(and for $g = 1$, we also have to require suitable regularity near the boundary of $\mathcal{H}_g$).

It then turns out that theta constants with characteristics are modular forms of weight one half with respect to $\Gamma_g(4, 8)$, the finite index normal *theta level subgroup*, such that $\Gamma_g(8) \subset \Gamma_g(4,8) \subset \Gamma_g(4)$, and in fact the eighth powers of theta constants are modular forms of weight 4 with respect to all of $\Gamma_g(2)$, see [**Igu72**] for a proper development of the theory.

The geometric meaning of the variety $\mathcal{A}_g(\ell)$ is the following: it parameterizes ppav with a level ℓ structure. A level ℓ structure is an ordered symplectic basis of the group $A[\ell]$ of ℓ-torsion points of A, where the symplectic form comes form the Weil pairing on $A[\ell]$. Thus a level ℓ structure is equivalent to a choice of a symplectic isomorphism $A[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ where the right-hand side carries the standard symplectic form. We shall also come back to level structures in connection with Heisenberg groups in Section 4.

Indeed, if $\ell \geq 3$, then $\mathcal{A}_g(\ell)$ is a fine moduli space. To see this we define the groups

$$G_g(\ell) := \Gamma_g(\ell) \ltimes (l\mathbb{Z})^{2g} \lhd \mathrm{Sp}(g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}.$$

Note that $\mathrm{Sp}(g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}/G_g(\ell) \cong \mathrm{Sp}(g,\mathbb{Z}/\ell\mathbb{Z}) \ltimes (\mathbb{Z}/\ell\mathbb{Z})^{2g}$.

Definition 1.4. We define the universal family

$$\mathcal{X}_g(\ell) := G_g(\ell)\backslash\mathcal{H}_g \times \mathbb{C}^g.$$

This definition makes sense for all ℓ. Note that, as a variety, $\mathcal{X}_g(1)$ is the universal Kummer family. For $\ell \geq 3$ the group $G_g(\ell)$ acts freely, and $\mathcal{X}_g(\ell)$ is an honest family of abelian varieties over $\mathcal{A}_g(\ell)$. We note that for $\tau \in \mathcal{A}_g(\ell)$ the fiber $\mathcal{X}_g(\ell)_\tau = \mathbb{C}^g/\ell\Lambda_\tau \cong A_\tau = \mathbb{C}^g/\Lambda_\tau$ as ppav.

The family $\mathcal{X}_g(\ell)$ is indeed a universal family for the moduli problem: the points given by Λ_τ in each fiber define disjoint sections of $\mathcal{X}_g(\ell)$ and the sections given by $\tau m+n$ with $m,n \in \{0,1\}^g$ give, properly ordered, a symplectic basis of the ℓ-torsion points and thus a level ℓ structure in each fiber. The group $\mathrm{Sp}(g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}/G_g(\ell) \cong \mathrm{Sp}(g,\mathbb{Z}/\ell\mathbb{Z}) \ltimes (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ acts on $\mathcal{X}_g(\ell)$ with quotient $\mathcal{X}_g$. Under this map each fiber $\mathcal{X}_g(\ell)_\tau$ maps $(2\cdot(\ell)^{2g})$-to-1 to the fiber $(\mathcal{X}_g)_\tau$, the map being given by multiplication by ℓ followed by the Kummer involution.

The next step is to define a universal theta divisor. Provided that the level ℓ is divisible by 8, the theta transformation formula [**Igu72**, Theorem 5.6] shows that the locus $\{\theta(\tau,z) = 0\}$ defines a divisor Θ_g on the universal family $\mathcal{X}_g(\ell)$, which is ℓ^2 times a principal divisor on each fiber. We would like to point out that the condition $8|\ell$ is sufficient to obtain a universal theta divisor, but that we could also have worked with smaller groups, namely the theta groups $\Gamma_g(4\ell,8\ell)$. For a definition we refer the reader to [**Igu72**, Section V]. We would also like to point out that the group $\mathrm{Sp}(g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}/G_g(\ell)$ acts on the theta divisor Θ_g on $\mathcal{X}_g(\ell)$, but does not leave it invariant, in particular the theta divisor does not descend to $\mathcal{X}_g$ in the category of *analytic varieties*.

The previous discussion took place fully in the analytic category but the resulting analytic varieties are in fact quasi-projective varieties (see Section 4). In fact the spaces $\mathcal{A}_g$ and $\mathcal{A}_g(\ell)$ are coarse, and if $\ell \geq 3$, even fine, moduli spaces representing the functors of ppav and ppav with a level ℓ-structure respectively. We can thus also think of $\mathcal{A}_g(\ell) \to \mathcal{A}_g$ as a quotient of stacks, which we want to do from now on.

Unlike in the case of varieties the universal *stack* family $\mathcal{X}_g$ over the *stack* $\mathcal{A}_g$ carries a universal theta divisor. In our notation we will not usually distinguish between the stack and the associated coarse moduli space, but will try to make it clear which picture we use.

At this point it is worth pointing out the connection between level structures and the Heisenberg group, resp. the theta group. To do this we go back to thinking of the fibers $\mathcal{X}_g(\ell)_\tau \cong A_\tau$ of the universal family $\mathcal{X}_g(\ell)$ as ppav. As such they also carry ℓ times a principal polarization and we choose a line bundle L_τ representing this multiple of the principal polarization. Then L_τ is invariant under pullback by translation by ℓ-torsion points, in other words $A_\tau[\ell] = \ker(\lambda_{L_\tau} : A_\tau \to \mathrm{Pic}^0(A_\tau))$. Using the level ℓ structure on A_τ, we can identify $A_\tau[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$. Translations by this group leave the line bundle L_τ invariant, but the action of this group on A_τ does not lift to the total space of L_τ. In order to do this, we must extend the group $(\mathbb{Z}/\ell\mathbb{Z})^{2g}$ to the Heisenberg group H_ℓ of level ℓ. This is a central extension

$$1 \to \mu_\ell \to H_\ell \to (\mathbb{Z}/\ell\mathbb{Z})^{2g} \to 0.$$

Here the commutators of H_ℓ act as homotheties by roots of unity of order ℓ. If we consider the induced action of the Heisenberg group on the line bundle $L_\tau^{\otimes \ell}$, which represents ℓ^2 times the principal polarization, then the commutators act trivially and thus this line bundle descends to $\mathrm{Pic}^0(A_\tau) \cong A_\tau$ where it represents a principal polarization. This is exactly the situation described above. If we choose L_τ to be symmetric, then we can further extend the Heisenberg group by the involution ι. Instead of working with a central extension by μ_ℓ, one can also consider the central extension by $\mathbb{C}^*$. In this way one obtains the (symmetric) theta group. For details we refer the reader to [**BL04**, Section 6].

The smallest group for which we can interpret theta functions as honest sections of line bundles is the group $\Gamma_g(4,8)$. We shall however see later that many computations can in fact be done on the level 2 cover $\mathcal{A}_g(2)$ — see the discussion in Section 2.

The reason that we have taken so much trouble over the definition of the universal family is that it is essential for much of what we will discuss in this paper. It will also be important for us to extend the universal family to (partial) compactifications of $\mathcal{A}_g$. This is indeed a very subtle problem to which we will come back in some detail in Section 4. This will be crucial when we discuss degeneration techniques in Section 6.

Example 1.5. For a smooth algebraic genus g curve C we denote $\mathrm{Jac}(C)$ its Jacobian. The Jacobian can be defined analytically (by choosing a basis for cycles, associated basis for holomorphic differentials, and constructing a lattice in $\mathbb{C}^g$ by integrating one over the other) or algebraically, as $\mathrm{Pic}^{g-1}(C)$ or as $\mathrm{Pic}^0(C)$. We note that $\mathrm{Pic}^0(C)$ is naturally an abelian variety, since adding degree zero divisors on a curve gives a degree zero divisor; however there is no natural choice of an ample divisor on $\mathrm{Pic}^0(C)$ (the polarization class is of course well-defined). On the other hand, $\mathrm{Pic}^{g-1}(C)$ has a natural polarization — the locus of effective divisors of degree $g-1$ — but no natural choice of a group law. Thus to get both polarization and the group structure one needs to identify $\mathrm{Pic}^0(C)$ with $\mathrm{Pic}^{g-1}(C)$, by choosing one point $R \in \mathrm{Pic}^{g-1}(C)$ as the origin — a natural choice for R is the Riemann constant. Alternatively, one could view $\mathrm{Pic}^{g-1}(C)$ as a torsor over the group $\mathrm{Pic}^0(C)$.

We would like to recall that the observation made above is indeed the starting point of Alexeev's work [**Ale02**]. Instead of looking at the usual functor of ppav, Alexeev considered the equivalent functor of pairs (P, Θ) where P is an abelian torsor acted on by the abelian variety $A = \mathrm{Pic}^0(P)$, and Θ is a divisor with $h^0(P,\Theta)=1$. It is the latter functor that Alexeev compactifies, obtaining his moduli space of stable semiabelic varieties, which we will further discuss later on.

For future reference we recall the notion of decomposable and indecomposable ppav. We call a principally polarized abelian variety (A,Θ) *decomposable* if it is a product $(A,\Theta)\cong(A_1,\Theta_1)\times(A_2,\Theta_2)$ of ppav of smaller dimension, otherwise we call it *indecomposable*. We also note that decomposable ppav cannot be Jacobians of smooth projective curves. However, if we consider a nodal curve $C=C_1\cup C_2$ with $C_i, i=1,2$ two smooth curves intersecting in one point, then $\mathrm{Jac}(C)\cong\mathrm{Jac}(C_1)\times\mathrm{Jac}(C_2)$ is a decomposable ppav.

The locus $\mathcal{A}_g^{\mathrm{dec}}$ of decomposable ppav is a closed subvariety of $\mathcal{A}_g$, it is the union of the images of the product maps $\mathcal{A}_i\times\mathcal{A}_{g-i}$ in $\mathcal{A}_g$. Its complement $\mathcal{A}_g^{\mathrm{ind}}=\mathcal{A}_g\backslash\mathcal{A}_g^{\mathrm{dec}}$ is open.

2. Singularity loci of the theta divisor

We are interested in loci of ppav whose theta divisor satisfies certain geometric conditions, in particular we are interested in the loci of ppav with prescribed behavior of the singular locus of the theta divisor. Working over the Siegel upper half-space, we define for a point $\tau\in\mathcal{H}_g$ the set

$$T_a^{(g)}(\tau):=\{z\in A_\tau\mid \mathrm{mult}_z\theta(\tau,z)\geq a\}$$

or more generally

$$T_a^{(g)}\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau):=\{z\in A_\tau\mid \mathrm{mult}_z\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z)\geq a\}.$$

This means that we consider the singularities of the theta divisor *in the z direction*. If we replace τ by a point $\gamma\circ\tau$, $\gamma\in\mathrm{Sp}(g,\mathbb{Z})$, which corresponds to the same ppav in $\mathcal{A}_g$, then the locus $T_a^{(g)}\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\gamma\circ\tau)$ is obtained from $T_a^{(g)}\begin{bmatrix}\varepsilon'\\ \delta'\end{bmatrix}(\tau)$ (where $[\varepsilon,\delta]=\gamma\circ[\varepsilon',\delta']$ is the affine action on characteristics) by applying the linear map $(C\tau+D)^{-1}$, which establishes the isomorphism $A_\tau\to A_{\gamma\circ\tau}$.

Thus the *generalized Andreotti-Mayer locus*

$$F_{a,b}^{(g)}:=\{\tau\in\mathcal{A}_g\mid \dim T_a^{(g)}(\tau)\geq b\}$$

is well-defined (i.e. this condition defines a locus on $\mathcal{H}_g$ invariant under the action of $\mathrm{Sp}(g,\mathbb{Z})$). We will often drop the index (g) if it is clear from the context. Recall that the usual Andreotti-Mayer loci $N_k^{(g)}:=F_{2,k}$ in our notation were defined in [**AM67**] as the loci of ppav for which the theta divisor has at least a k-dimensional singular locus. These were introduced because of their relationship to the Schottky problem, as we will discuss in the next section. The generalized Andreotti-Mayer loci are often denoted $N_k^\ell:=F_{\ell+1,k}$ in our notation, but since we will often need to specify the genus in which we are working, we prefer the F notation.

Varying the point $\tau\in\mathcal{A}_g$, we would also like to define a corresponding cycle $T_a^{(g)}\subset\mathcal{X}_g$ in the universal family. For this we go to the level cover $\mathcal{A}_g(8)$. For each $(\varepsilon,\delta)\in(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ the set

$$T_a^{(g)}\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}=\left\{(\tau,z)\in\mathcal{X}_g(8)\mid(\tau,z)\in T_a^{(g)}\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right\}$$

is well-defined, and since the fiber dimension is upper semi-continuous in the Zariski topology, this is a well-defined subscheme of $\mathcal{X}_g(8)$. The Galois group $\mathrm{Sp}(g, \mathbb{Z}/8\mathbb{Z})$ of the cover $\mathcal{A}_g(8) \to \mathcal{A}_g$ permutes the cycles $T_a^{(g)}\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$, and thus we obtain a cycle $T_a^{(g)} \subset \mathcal{X}_g$ in the universal family over the stack.

Remark 2.1. The Riemann theta singularity theorem describes the singularities of the theta divisor for Jacobians of curves. In particular our locus $T_a^{(g)}$ restricted to the universal family of Jacobians of curves (i.e. the pullback of $\mathcal{X}_g$ to $\mathcal{M}_g$), gives the Brill-Noether locus $\mathcal{W}_{g-1}^{a-1}$. The Brill-Noether loci have been extensively studied, and their projections to $\mathcal{M}_g$ give examples of very interesting geometric subvarieties, see for example [**ACGH85**] for the foundations of the theory, and [**Far01**] for more recent results.

We shall also need the concept of *odd* and *even* 2-torsion points. The 2-torsion points of an abelian variety $A_\tau = \mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g)$ are of the form $\varepsilon\tau + \delta$ where $\varepsilon, \delta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. It is standard to call a 2-torsion point even if $4\varepsilon \cdot \delta = 0$, and odd if this is 1, exactly as in the case of our notion of even or odd characteristics. This can be formulated in a more intrinsic way: if L is a symmetric line bundle representing the principal polarization of an abelian variety A, then the involution $\iota : z \mapsto -z$ can be lifted to an involution on the total space of the line bundle L. A priori there are two such lifts, but we can choose one of them by asking that $\iota^*(s) = s$, where s is the (up to scalar) unique section of L. The even, resp. odd, 2-torsion points are then those where the involution acts by $+1$, resp. -1 on the fiber. The number of even (resp. odd) 2-torsion points is equal to $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$), see [**BL04**, Chapter 4, Proposition 7.5]. Applying this to the Riemann theta function θ and the line bundle defined by it, we obtain our above notion of even and odd 2-torsion points. Replacing θ by $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$ means shifting by the 2-torsion point $\varepsilon\tau + \delta$ (and multiplying by an exponential factor that is not important to us), and thus studying the properties of θ at the point $\varepsilon\tau + \delta$ is equivalent to studying the properties of $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$ at the origin.

We have already pointed out that the non-zero 2-torsion points define an irreducible family over $\mathcal{A}_g$. However, if the level ℓ is even, then the 2-torsion points form sections in $\mathcal{X}_g(\ell)$ and in this case we can talk about even and odd 2-torsion points in families.

We note that the group $\Gamma_g(8)/\Gamma_g(2)$ acts on the functions $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, z)$ by certain signs. This does not affect the vanishing of this theta function, or of its gradient with respect to z, and thus we can often work on $\mathcal{A}_g(2)$, rather than on $\mathcal{A}_g(4, 8)$ or $\mathcal{A}_g(8)$.

Example 2.2. Note that by definition we have $T_1^{(g)} = \Theta$ (more precisely, the union of all the 2^{2g} symmetric theta divisors), as this is the locus of points where θ is zero; thus $F_{1,k} = \mathcal{A}_g$ for any $k \leq g - 1$, and $F_{1,g} = \emptyset$. In general we have $T_{a+1}^{(g)} \subseteq T_a^{(g)}$.

We can think of $T_2^{(g)}$ as the locus of points (τ, z) such that the theta divisor $\Theta_\tau \subset A_\tau$ is singular at the point z; following Mumford's notation, we think of this as the locus $S := \mathrm{Sing}_{vert}\,\Theta := T_2^{(g)}$.

From the *heat equation*

$$\frac{\partial^2\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z)}{\partial z_j\partial z_k} = 2\pi i(1+\delta_{jk})\frac{\partial\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z)}{\partial\tau_{jk}}. \tag{3}$$

we see that the second z-derivatives of $\theta(\tau, z)$ vanish if and only if all the first order τ-derivatives vanish. Thus we have $T_3^{(g)} = \mathrm{Sing}\,\Theta_g$ is the locus of singularities of the global theta divisor, as a subvariety of $\mathcal{X}_g$.

3. Loci in $\mathcal{A}_g$ defined by singularities of the theta divisor

In this section we collect known results and numerous open questions about the properties of the loci of ppav with singular theta divisors defined above. The first result says that the theta divisor of a generic ppav is smooth:

THEOREM 3.1 ([**AM67**]). *For a generic ppav the theta divisor is smooth, i.e.* $N_0 \subsetneq \mathcal{A}_g$.

Thus one is led to ask about the codimension of N_0 in $\mathcal{A}_g$. To this end, note that $S = T_2^{(g)} \subset \mathcal{X}_g$ is the common zero locus of the theta function and its g partial derivatives with respect to z. It follows that $\mathrm{codim}_{\mathcal{X}_g} S \leq g+1$, and in fact the dimension is precisely that:

THEOREM 3.2 ([**Deb92**]). *The locus* $S = T_2^{(g)}$ *is purely of codimension* $g+1$, *and has two irreducible components* $S_{\mathrm{null}} := S \cap \mathcal{X}_g[2]^{\mathrm{even}}$ *(where* $\mathcal{X}_g[2]^{\mathrm{even}} \subset \mathcal{X}_g$ *denotes the universal family of even 2-torsion points), and the "other" component* S'.

Moreover, since the map π from $T_2^{(g)}$ onto its image has fibers of dimension at least k over N_k, this implies that the codimension of (any irreducible component of) N_k within $\mathcal{A}_g$ is at least $k+1$. The $k=1$ case of this result is in fact due to Mumford, who obtained it by an ingenious argument using the heat equation:

THEOREM 3.3 ([**Mum83a**]). $\mathrm{codim}_{\mathcal{A}_g} N_1 \geq 2$.

It thus follows that both components of $S = T_2^{(g)}$ project generically finitely on their image in $\mathcal{A}_g$, and this implies the earlier result of Beauville:

THEOREM 3.4 ([**Bea77**]). *The locus* N_0 *is a divisor in* $\mathcal{A}_g$.

In fact scheme-theoretically we have (this was also first proved in [**Mum83a**])

$$N_0 = \theta_{\mathrm{null}} + 2N_0' = \pi(S_{\mathrm{null}}) \cup \pi(S'),$$

where $\theta_{\mathrm{null}} \subset \mathcal{A}_g$ denotes the theta-null divisor — the locus of ppav for which an even 2-torsion point lies (and thus is a singular point of) the theta divisor, and N_0' denotes the other irreducible component of N_0, i.e. the closure of the locus of ppav whose theta divisor is singular at some point that is not 2-torsion.

We note that unlike $T_2^{(g)}$, which is easily defined by $g+1$ equations in $\mathcal{X}_g$, it is not at all clear how to write defining equations for $N_1 = F_{2,1}$ inside $\mathcal{A}_g$. In particular, note that if the locus $\mathrm{Sing}\,\Theta_\tau$ locally at z has dimension at least one, then the g second derivatives of the theta function of the form $\partial_v\partial_{z_i}\theta(\tau,z)$, where v is a tangent vector to $\operatorname{Sing}\Theta_\tau$ at z, must all vanish — but this is of course not a sufficient condition. Still, one would expect that N_1 has high codimension. However, this, and questions on higher Andreotti-Mayer loci, are exceptionally hard, as there are few techniques available for working with conditions at an arbitrary point on a ppav, as opposed to the origin or a 2-torsion point. Many open questions remain, and are surveyed in detail in [**CM08b**, section 4] and in [**Gru09**, section 7]. We briefly summarize the situation.

The original motivation for Andreotti and Mayer to introduce the loci N_k was their relationship to the Schottky problem.

THEOREM 3.5 ([**AM67**]). *The locus of Jacobians $\mathcal{J}_g$ is an irreducible component of N_{g-4}; the locus of hyperelliptic Jacobians Hyp_g is an irreducible component of N_{g-3}.*

In modern language, this result follows by applying the Riemann-Kempf theta singularity theorem on Jacobians. Generalizing this to singularities of higher multiplicity, we have as a corollary of Martens' theorem

PROPOSITION 3.6. $\operatorname{Hyp}_g = F^{(g)}_{k,g-2k+1}\cap\mathcal{J}_g$, *while* $F^{(g)}_{k,g-2k+2}\cap\mathcal{J}_g=\emptyset$.

One also sees that $N_{g-2}\cap\mathcal{J}_g=\emptyset$, and thus it is natural to ask to describe this locus (note that clearly $N_{g-1}=\emptyset$). We shall give the answer below. A novel aspect was brought to the subject by Kollár who considered the pair (A,Θ) from a new perspective, proving

THEOREM 3.7 (Kollár, [**Kol95**, Theorem 17.3]). *The pair (A,Θ) is log canonical. This implies that the theta function cannot have a point of multiplicity greater than g, i.e. $T^{(g)}_{g+1}=\emptyset=F^{(g)}_{g+1,0}$, and, more generally, $F^{(g)}_{k,g-k+1}=\emptyset$.*

The extreme case $F_{g,0}$ was then considered by Smith and Varley who characterized it as follows:

THEOREM 3.8 (Smith and Varley [**SV96**]). *If the theta divisor has a point of multiplicity g, then the ppav is a product of elliptic curves: $F_{g,0}=\operatorname{Sym}^g(\mathcal{A}_1)$.*

Ein and Lazarsfeld took Kollár's result further and showed:

THEOREM 3.9 (Ein-Lazarsfeld, [**EL97**, Theorem 1]). *If (A,Θ) is an irreducible ppav, then the theta divisor is normal and has rational singularities.*

As an application they obtained:

THEOREM 3.10 (Ein and Lazarsfeld [**EL97**, Corollary 2]). *The locus $F^{(g)}_{k,g-k}$ is equal to the locus of ppav that are products of (at least) k lower-dimensional ppav.*

If $k=g$ then this implies the result by Smith and Varley, if $k=2$, then this gives a conjecture of Arbarello and de Concini from [**ADC87**], namely:

THEOREM 3.11 (Ein-Lazarsfeld [**EL97**]). $N_{g-2}=\mathcal{A}_g^{\mathrm{dec}}$.

In general very little is known about the loci N_k, or even about their dimension. The expectation is as follows:

CONJECTURE 3.12 (Ciliberto-van der Geer [**CvdG00**], [**CvdG08**]). *Any component of the locus N_k whose general point corresponds to a ppav with endomorphism ring $\mathbb{Z}$ (in particular such a ppav is indecomposable) has codimension at least $(k+1)(k+2)/2$ in $\mathcal{A}_g$, and the bound is only achieved for the loci of Jacobians and hyperelliptic Jacobians with $k = g-4$ and $k = g-3$ respectively.*

Ciliberto and van der Geer prove this conjecture in [**CvdG08**] for $k = 1$, and in [**CvdG00**] they obtain a bound of $k+2$ (or $k+3$ for $k > g/3$) for the codimension of N_k, but the full statement remains wide open.

Many results about the Andreotti-Mayer loci are known in low genus; in particular it is known that this approach does not give a complete solution to the Schottky problem: already in genus 4 we have

THEOREM 3.13 ([**Bea77**]). *In genus 4 we have $N_0^{(4)} = \mathcal{J}_4 \cup \theta_{\rm null}$. The locus $N_1^{(4)}$ is irreducible, more precisely $N_1^{(4)} = {\rm Hyp}_4$.*

The situation is also very well understood in genus 5, see [**CM08b**, Table 2]. The varieties $F_{l,k}^{(5)}$ are empty for $l+k > 5$. If $l+k = 5$ then $F_{l,k}$ parameterizes products of k ppav. Moreover we had already seen that $F_{2,0} = N_0 = \theta_{\rm null} + 2N_0'$ is a divisor. To describe the remaining cases, we introduce notation: for $i_1 + \ldots + i_r = g$ we denote by $\mathcal{A}_{i_1,\cdots,i_r} \subset \mathcal{A}_g$ the substack that is the image of the direct product $\mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_r}$. We also denote ${\rm Hyp}_{i_1,\cdots,i_r} := {\rm Hyp}_g \cap \mathcal{A}_{i_1,\cdots,i_r}$.

PROPOSITION 3.14. *In genus 5, the generalized Andreotti-Mayer loci are as follows:*

(i) $F_{0,4}^{(5)} = \mathcal{A}_{1,1} \times {\rm Hyp}_3$, $F_{1,3}^{(5)} = {\rm Hyp}_{1,4} \cup {\rm Hyp}_{2,3} \cup \mathcal{A}_{1,1,3}$, $F_{2,2}^{(5)} = {\rm Hyp}_5 \cup \mathcal{A}_{1,4} \cup \mathcal{A}_{2,3}$.

(ii) $F_{0,3}^{(5)} = \overline{IJ} \cup (\mathcal{A}_1 \times \theta_{\rm null}^4) \cup {\rm Hyp}_{1,4}$, $F_{1,2}^{(5)} = \mathcal{J}_5 \cup \mathcal{A}_{1,4} \cup A \cup B \cup C$, *where $\overline{IJ}$ denotes the closure in $\mathcal{A}_5$ of the locus of intermediate Jacobians of cubic threefolds, the component A has dimension 10, and the components B and C have dimension 9.*

The most interesting cases here are those of $F_{0,3}$, which goes back to Casalaina-Martin and Laza [**CML09**], and $F_{1,2}$, which is due to Donagi [**Don88**, Theorem 4.15] and Debarre [**Deb88**, Proposition 8.2]. The components A,B and C can be described explicitly in terms of Prym varieties. The above results for genus 4 and 5 led to the following folk conjecture, motivated, to the best of our knowledge, purely by the situation in low genus:

CONJECTURE 3.15. (1) *All irreducible components of N_{g-4} except the locus of Jacobians $\mathcal{J}_g$ are contained in the theta-null divisor $\theta_{\rm null}$.*

(2) *The equality $N_{g-3} = {\rm Hyp}_g \cup \mathcal{A}_g^{\rm dec}$ holds (recalling $\mathcal{A}_g^{\rm dec} = N_{g-2}$, this is equivalent to $N_{g-3} \setminus N_{g-2} = {\rm Hyp}_g$).*

Moreover, one sees that in the locus of indecomposable ppav the maximal possible multiplicity of the theta divisor is at most $(g+1)/2$, and we are led to the following folk conjecture:

CONJECTURE 3.16. $T^{(g)}_{\lfloor \frac{g+3}{2} \rfloor} \subset \mathcal{X}_g^{dec}$.

This conjecture is known to hold for $g \le 5$. Indeed, for $g \le 3$ we have $\mathcal{A}_g^{\rm ind} = \mathcal{J}_g$, and by the Riemann theta singularity theorem the statement of the conjecture holds for Jacobians. Similarly, by the Prym-Riemann theta singularity theorem, this also holds for Prym varieties, and can also be shown to hold for their degenerations in $\mathcal{A}_g^{\rm ind}$: see [**CM09**], [**CM08a**], [**CMF05**], [**CM08b**]. Since the Prym map to $\mathcal{A}_g$ is dominant for $g \leq 5$, the conjecture therefore holds for g in this range. A more general question in this spirit was raised by Casalaina-Martin:

QUESTION 3.17 ([**CM08b**, Question 4.7]). *Is it true that $F_{k,g-2k+2} \subset \mathcal{A}_g^{dec}$?*

While studying singularities of theta divisors at arbitrary points appears very hard, geometric properties of the theta divisor at 2-torsion points are often easier to handle: using the heat equation one can translate them into conditions on the ppav itself. Moreover, as we shall see, the resulting loci are of intrinsic geometric interest. We will therefore now concentrate on such questions.

DEFINITION 3.18. We denote by $T_a^{(g)}[2] := T_a^{(g)} \cap \mathcal{X}_g[2]^{\rm odd/even}$ the set of 2-torsion points of multiplicity at least a lying on the theta divisor, where the parity odd/even is chosen to be the parity of a.

This definition is motivated by the fact that the multiplicity of the theta function at a 2-torsion point is odd or even, respectively, depending on the parity of the point. We have already encountered the first non-trivial case, $a = 2$, when $T_2^{(g)}[2]$ is the locus of even 2-torsion points lying on the universal theta divisor, and thus $\pi(T_2^{(g)}[2]) = \theta_{\rm null} \subset \mathcal{A}_g$ is the theta-null divisor discussed above. Already the next case turns out to be much more interesting and difficult, and we now survey what is known about it.

Indeed, we denote $I(g) := \pi(T_3^{(g)}[2])$. Geometrically, this is the locus of ppav where the theta divisor has multiplicity at least three at an odd 2-torsion point; analytically, this is to say that the gradient of the theta function vanishes at an odd 2-torsion point. Beyond being natural to consider in the study of theta functions, this locus has geometric significance. In particular, for low genera we have

$$I^{(3)} = \mathcal{A}_{1,1,1}; \qquad I^{(4)} = {\rm Hyp}_{1,3},$$

which are very natural geometric subvarieties of $\mathcal{A}_3$ and $\mathcal{A}_4$, while for genus 5

$$I^{(5)} = \overline{IJ} \cup \left(\mathcal{A}_1 \times \theta_{\rm null}^{(4)}\right),$$

where $\overline{IJ}$ denotes the closure in $\mathcal{A}_5$ of the locus IJ of intermediate Jacobians of cubic threefolds (this subject originated in the seminal paper of Clemens and Griffiths [**CG72**]; see [**CMF05**], [**CM08a**], [**CML09**], [**GSM09**], [**GH11a**] for further references). In any genus, $\mathcal{A}_1 \times \theta_{\rm null}^{(g-1)}$ is always an irreducible component of $I^{(g)}$ (see [**GSM09**]), but for $g > 4$ the locus $I^{(g)}$ is reducible.

The loci $T_3^{(g)}[2]$ and $T_3^{(g)}$, and the gradients of the theta function at higher torsion points, were studied by Salvati Manni and the first author in [**GSM04**], [**GSM06**], where they showed that the values of all such gradients determine a ppav generically uniquely. Furthermore, in [**GSM09**] Salvati Manni and the first author (motivated by their earlier works [**GSM08**] and [**GSM07**] on double point singularities of the theta divisor at 2-torsion points) studied the geometry of these loci further, and made the following

CONJECTURE 3.19 ([**GSM09**]). *The loci $F_{3,0}^{(g)} = \pi(T_3^{(g)})$ and $I^{(g)} := \pi(T_3^{(g)}[2])$ are purely of codimension g in $\mathcal{A}_g$.*

The motivation for these conjectures comes from the cases $g \leq 5$ discussed above, and also from some degeneration considerations that we will discuss in Section 6. In our joint work [**GH11b**] we proved the above conjecture for $T_3^{(g)}[2]$ for $g \leq 5$ directly, without using the beautiful elaborate geometry of intermediate Jacobians and degenerations of the Prym map that was used in [**CMF05**], [**CM08a**] to previously obtain the proof. Our method was by degeneration: we studied in detail the possible types of semiabelic varieties that can arise in the boundary of the moduli space, and described the closure of the locus $I^{(g)}$ in each such stratum; the details are discussed in Section 6.

4. Compactifications of $\mathcal{A}_g$

Compactifications of $\mathcal{A}_g$ have been investigated extensively, and there is a vast literature on this subject. We will not even attempt to summarize this, but will restrict ourselves to recalling the most important results in so far as they are relevant for our purposes. The first example of such a compactification is the *Satake compactification* of $\mathcal{A}_g$, constructed in [**Sat56**], which was later generalized by Baily and Borel from the Siegel upper half-space to arbitrary bounded symmetric domains [**BB66**]. The idea is simple: theta constants can be used to embed $\mathcal{A}_g$ into some projective space, and the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ is the closure of the image of this embedding. Another way to express this is that the Satake compactification is the Proj of the graded ring of modular forms, or in yet other words, that one uses a sufficiently high multiple of the Hodge line bundle to embed $\mathcal{A}_g$ in a projective space, and then takes the closure. This argument also shows that $\mathcal{A}_g$ is a quasi-projective variety. Set theoretically the Satake compactification is easy to understand:

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \mathcal{A}_{g-2} \sqcup \cdots \sqcup \mathcal{A}_0, \tag{4}$$

but it is non-trivial to equip this set-theoretic union with a good topology and analytic or algebraic structure. The boundary of $\mathcal{A}_g^{\mathrm{Sat}}$ has codimension g, and the compactification is highly singular at the boundary.

To overcome the disadvantages of the Satake compactification, Mumford et al. [**AMRT75**] introduced the concept of toroidal compactifications. Unlike the Satake compactification, the boundary of a toroidal compactification is a divisor in it. There is no canonical choice of a toroidal compactification: in fact toroidal compactifications of $\mathcal{A}_g$ depend on a fan, that is a rational partial polyhedral decomposition of the (rational closure of the) cone of positive definite real symmetric $g \times g$ matrices. There are three known such decompositions (which can be refined by taking subdivisions), namely the *first Voronoi* or *perfect cone decomposition*, the *central cone decomposition*, and the *second Voronoi decomposition*. Each of these leads to a compactification of $\mathcal{A}_g$, namely $\mathcal{A}_g^{\mathrm{Perf}}$, $\mathcal{A}_g^{\mathrm{cent}}$, and $\mathcal{A}_g^{\mathrm{Vor}}$ respectively. The central cone compactification is also denoted $\mathcal{A}_g^{\mathrm{cent}} = \mathcal{A}_g^{\mathrm{Igu}}$ since it coincides with Igusa's blow-up of the Satake compactification along its boundary.

By now the geometric meaning of all these compactifications has been understood. Shepherd-Barron [**SB06**] proved that $\mathcal{A}_g^{\mathrm{Perf}}$ is a canonical model of $\mathcal{A}_g$ in the sense of the minimal model program if $g \geq 12$. Finally Alexeev [**Ale02**] showed that $\mathcal{A}_g^{\mathrm{Vor}}$ has a good modular interpretation, we will comment on this below. The toroidal compactification obtained from a fan with some cones subdivided can be obtained from the original toroidal compactification by a blow-up, and thus by choosing a suitably fine subdivision of the fan one can arrange that all cones are basic. The corresponding toroidal compactification would then have only finite quotient singularities due to non-neatness of the group $\mathrm{Sp}(g,\mathbb{Z})$ — we refer to this as *stack-smooth*. For $g \le 3$ all three compactifications coincide and are stack-smooth. For $g = 4$ the perfect cone and the Igusa compactification coincide: $\mathcal{A}_4^{\mathrm{Igu}} = \mathcal{A}_4^{\mathrm{Perf}}$, but are not stack-smooth, whereas $\mathcal{A}_4^{\mathrm{Vor}}$ is. In genus $g = 4, 5$ the second Voronoi decomposition is a refinement of the perfect cone decomposition, i.e. $\mathcal{A}_g^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_g^{\mathrm{Perf}}$ for $g = 4, 5$, but in general neither is a refinement of the other, and all three fans are different.

We also recall that any toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ admits a natural contracting map to the Satake compactification, $p : \mathcal{A}_g^{\mathrm{tor}} \to \mathcal{A}_g^{\mathrm{Sat}}$. Pulling back the stratification (4) of the Satake compactification thus defines a stratification of any toroidal compactification, and the first two strata of this, $\mathcal{A}'_g = p^{-1}(\mathcal{A}_g \sqcup \mathcal{A}_{g-1})$, are of special interest. Indeed, $\mathcal{A}'_g$ is called *Mumford's partial compactification*, and is the same for all toroidal compactifications. As a stack, it is the disjoint union of $\mathcal{A}_g$ and the universal family $\mathcal{X}_{g-1}$.

In Section 6 we shall discuss degeneration techniques which require the existence of a universal family over a toroidal compactification of $\mathcal{A}_g$. This is a very difficult and delicate problem, which has a long history. The first approach is due to Namikawa [**Nam76a**], [**Nam76b**] who constructed a family over $\mathcal{A}_g^{\mathrm{Vor}}(\ell)$ that carries 2ℓ times a principal polarization. Chai and Faltings [**FC90**, Chapter VI] constructed compactifications of the universal family over stack-smooth projective toroidal compactifications of $\mathcal{A}_g$.

In [**Ale02**] Alexeev introduced a new aspect into the theory, namely the use of log-geometry. He defined the functor of *stable semiabelic pairs* (X, Θ) consisting of a variety X that admits an action of a semiabelian variety (i.e. an extension of an abelian variety of dimension $g - k$ by a torus $(\mathbb{C}^*)^k$) of the same dimension, and an effective ample Cartier divisor $\Theta \subset X$ fulfilling the following conditions: X is seminormal, the group action has only finitely many orbits, and the stabilizer of any point is connected, reduced, and lies in the toric part of the semiabelian variety. The prime example is that of a principally polarized abelian variety acting on itself by translation, together with its theta divisor. This functor is represented by a scheme $\overline{\mathcal{AP}_g}$ which has several components, one of which — the principal component $\overline{\mathcal{P}_g}$ — contains the moduli space $\mathcal{A}_g$ of ppav. It is currently unclear whether $\overline{\mathcal{P}_g}$ is normal, but it is known that its normalization is isomorphic to $\mathcal{A}_g^{\mathrm{Vor}}$. For a discussion of the other components appearing in $\overline{\mathcal{AP}_g}$, also called ET for "extra-type", we refer the reader to [**Ale01**]. In either case the universal family on $\overline{\mathcal{P}_g}$ can be pulled back to give a universal family over $\mathcal{A}_g^{\mathrm{Vor}}$. We would like to point out that Alexeev's construction is in the category of stacks. If we want to work with schemes (and restrict ourselves to ppav and their degenerations), then we obtain the following: for every point in the projective scheme that represents $\overline{\mathcal{P}_g}$ we can find a neighborhood on which, after a finite base change, we can construct a family of stable semiabelic varieties (SSAV). We note that Alexeev's construction can lead to families with non-reduced fibers for $g \ge 5$. Alexeev's theory has been further developed by Olsson [**Ols08**], who modified the functor used by Alexeev in such a way as to single out the principal component through the definition of the functor. Olsson also treats the cases of non-principal polarizations and level structures.

Yet another approach was pursued by Nakamura. In fact Nakamura proposes two different constructions. His first approach uses GIT stability. In [**Nak98**] he defines the functor of *projectively stable quasi-abelian schemes* (PSQAS). For every $\ell \geq 3$ this functor is represented by a projective moduli *scheme* $SQ_g(\ell)$ over which a universal family exists. Nakamura's theory also extends to non-principal polarizations. It should, however, be noted that, as in Alexeev's case, the fibers of this universal family can in general be non-reduced. Also the total space $SQ_g(\ell)$ is known not to be normal, see [**NS06**]. The universal family over $SQ_g(\ell)$ has a polarization which is ℓ times a principal polarization, as well as a universal level ℓ structure (see our discussion in Section 1).

In his second approach Nakamura [**Nak10**] introduces the functor of *torically stable quasi-abelian schemes* (TSQAS). For a given level ℓ this is represented by a projective scheme $SQ_g(\ell)^{\text{toric}}$, which, for $\ell \geq 3$, is a coarse moduli space for families of TSQAS over *reduced* base schemes. There is no global universal family over $SQ_g(\ell)^{\text{toric}}$, but there is locally a universal family after possibly taking a finite base change. The advantage of these families is that all fibers are reduced. By [**Nak10**] there is a natural morphism $SQ_g(\ell)^{\text{toric}} \to SQ_g(\ell)$ which is birational and bijective. Hence both schemes have the same normalization, which is in fact isomorphic to the second Voronoi compactification $\mathcal{A}_g^{\text{Vor}}(\ell)$.

The structure of the semiabelic varieties in question can be deduced from the constructions in [**AN99**] and [**Ale02**]. Indeed, every point in $\mathcal{A}_g^{\text{Vor}}$ lies in a stratum corresponding to some unique Voronoi cone. Let g', for $0 \leq g' \leq g$, be the maximal rank of a matrix in this cone. Then g' is the torus rank of the associated semiabelic variety and the Voronoi cone defines a Delaunay decomposition of the real vector space $\mathbb{R}^{g'}$ which determines the structure of the torus part of the semiabelic variety. This picture also allows one to read off the structure of the polarization on the semiabelic variety, which is in fact given as a limit of the Riemann theta function. In general these constructions turn out to be rather complicated, especially when the rank of the torus part increases. The geometry for most cases of torus rank up to 5 is studied in detail, explicitly, in [**GH11b**].

The construction of universal families is subtle, as one can see already in genus 1. Here one has the well-known universal elliptic curve $S(\ell) \to X(\ell) = \mathcal{A}_1^{\text{Vor}}(\ell)$ over the modular curve of level ℓ. If ℓ is odd, then this coincides with Nakamura's family of PSQAS, if ℓ is even it is different. In the latter case the two families are related by an isogeny, see [**NT01**]. The main technical problems in higher genus, such as non-normality or non-reduced fibers, arise from the difficult combinatorics of the Delaunay polytopes and Delaunay tilings in higher dimensions.

Finally we want to comment on the connection with the moduli space of curves. Torelli's theorem says that the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$, sending a curve C to its Jacobian $Jac(C)$, is injective as a map of coarse moduli schemes (note, however, that as every ppav has an involution -1, and not all curves do, for $g \geq 3$ as a map of stacks the Torelli map is of degree 2, branching along the hyperelliptic locus).

Mumford and Namikawa investigated in the 1970's whether this map extends to a map from $\overline{\mathcal{M}_g}$ to a suitable toroidal compactification, and it was shown in [**Nam76a**], [**Nam76b**] that this is indeed the case for the second Voronoi compactification, i.e. that there exists a morphism $\overline{\mathcal{M}_g} \to \mathcal{A}_g^{\text{Vor}}$. Recently Alexeev and Brunyate [**AB11**] showed that an analogous result also holds for the perfect cone compactification, i.e. there exists a morphism $\overline{\mathcal{M}_g} \to \mathcal{A}_g^{\text{Perf}}$ extending the Torelli map. Moreover, they showed that there is a Zariski open neighborhood of the image of the Torelli map, where $\mathcal{A}_g^{\mathrm{Vor}}$ and $\mathcal{A}_g^{\mathrm{Perf}}$ are isomorphic. Melo and Viviani [**MV11**] further showed that the Voronoi and the perfect cone compactification agree along the so called matroidal locus. Finally, Alexeev et al [**AB11**], [**ALT$^+$11**] proved that the Torelli map can be extended to the Igusa compactification $\mathcal{A}_4^{\mathrm{Igu}}$ if and only if $g \leq 8$. We also recall that the extended Torelli map is no longer injective for $g \geq 3$: the Jacobian of a nodal curve consisting of two irreducible components attached at a node forgets the point of attachment. The fibers of the Torelli map on the boundary of $\overline{\mathcal{M}_g}$ were analyzed in detail by Caporaso and Viviani in [**CV11**].

5. Class computations and intersection theory on $\mathcal{X}_g$

In the cases where one knows the codimension of the loci T_a or $F_{a,b}$, one could then ask to compute their class in the cohomology or Chow rings of $\mathcal{X}_g$ or $\mathcal{A}_g$, respectively.

The cohomology ring $H^*(\mathcal{A}_g)$ and the Chow ring $CH^*(\mathcal{A}_g)$ are not fully known for $g \geq 4$, and are expected to contain various interesting classes (in particular non-algebraic classes in cohomology). One approach to understanding geometrically defined loci within $\mathcal{A}_g$ is by defining a suitable tautological subring of Chow or cohomology, and then arguing that the classes of such loci would lie in this subring.

DEFINITION 5.1. We denote by $\mathbb{E} := \pi_*(\Omega_{\mathcal{X}_g/\mathcal{A}_g})$ the rank g Hodge vector bundle, the fiber of which over a ppav (A, Θ) is the space of holomorphic differentials $H^{1,0}(A, \mathbb{C})$. We then denote by $\lambda_i := c_i(\mathbb{E})$ the Hodge classes, considered as elements of the Chow or cohomology ring of $\mathcal{A}_g$. The tautological ring is then defined to be the subring (of either the Chow or the cohomology ring of $\mathcal{A}_g$) generated by the Chern classes λ_i.

It turns out, see [**Mum77**], [**FC90**] that the Hodge vector bundle extends to any toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ of $\mathcal{A}_g$, and thus one can study the ring generated by λ_i in the cohomology or Chow "ring" of a compactification. We note, however, that an arbitrary toroidal compactification will in general be singular, and thus it is a priori unclear whether there is a ring structure on the Chow groups. We can, however, consider Chern classes of a vector bundle as elements in the operational Chow groups of Fulton and MacPherson. All operations with Chern classes will thus be performed in this operational Chow ring, and the resulting classes will then act on cycles by taking the cup product. Taking the cup product with the fundamental class, we can also associate Chow homology classes to Chern classes and, by abuse of notation, we will not distinguish between a Chern class and its associated Chow homology class. We refer to [**Ful98**, Chapter 17] and the references therein for more details on the operational Chow ring.

It turns out that, unlike the case of the moduli of curves $\mathcal{M}_g$ where the tautological ring is not yet fully known, and there is much ongoing research on Faber's conjectures [**Fab99b**], the tautological rings for $\mathcal{A}_g^{\mathrm{tor}}$ and $\mathcal{A}_g$ can be described completely.

THEOREM 5.2 (van der Geer [**vdG99**] for cohomology, and [**vdG99**] and Esnault and Viehweg [**EV02**] for Chow). *For a suitable toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$, the tautological ring of $\mathcal{A}_g^{\mathrm{tor}}$ is the same in Chow and cohomology, and generated by the classes λ_i subject to one basic relation*

$$(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1.$$

This implies that additive generators for the tautological ring are of the form $\prod \lambda_i^{\varepsilon_i}$ *for* $\varepsilon_i \in \{0,1\}$*, and that all even classes* λ_{2k} *are expressible polynomially in terms of the odd ones.*

Moreover, the tautological ring of the open part $\mathcal{A}_g$ *is also the same in Chow and cohomology, and obtained from the tautological ring of* $\mathcal{A}_g^{\text{tor}}$ *by imposing one additional relation* $\lambda_g = 0$*.*

REMARK 5.3. Notice that from the above theorem it follows that the tautological ring of $\mathcal{A}_{g-1}^{\text{tor}}$ is isomorphic to the tautological ring of $\mathcal{A}_g$. We do not know a geometric explanation for this fact. We also refer to the next section of the text, and in particular to Question 6.1 for further questions on possible structure of suitably enlarged tautological rings of compactifications.

In low genus the entire cohomology and Chow rings are known. Indeed, the Chow rings of $\mathcal{A}_2$ and $\mathcal{A}_2^{\text{tor}}$ (recall that for $g = 2, 3$ all known toroidal compactifications coincide) were computed by Mumford [**Mum83b**] (and are classically known to be equal to the cohomology, see [**HT10**] for a complete proof of this fact), while the cohomology ring of the Satake compactification of $\mathcal{A}_2$ was computed by Hain. The cohomology of $\mathcal{A}_3$ and its Satake compactification was computed by Hain [**Hai02**], the Chow ring of $\mathcal{A}_3$ and its toroidal compactification was computed by van der Geer [**vdG98**], and the second author and Tommasi [**HT10**] computed the cohomology ring of $\mathcal{A}_3^{\text{tor}}$, which turns out to also equal its Chow ring. It turns out that for $g = 2, 3$ the cohomology and Chow rings of $\mathcal{A}_g$ are equal to the tautological rings. Finally, the second author and Tommasi [**HT11**] computed much of the cohomology of the (second Voronoi) toroidal compactification $\mathcal{A}_4^{\text{Vor}}$; in particular they showed that $H^{12}(\mathcal{A}_4)$ contains a non-algebraic class, as does $H^6(\mathcal{A}_3)$, as was shown by Hain [**Hai02**]. We also refer the reader to van der Geer's survey article [**vdG11**]. The methods of computing the cohomology and Chow rings in low genus make extensive use of the explicit geometry, and extending them to higher genus currently appears to be out of reach. However, another natural question, which may possibly give an inductive approach to studying the cohomology by degeneration, is to define a tautological ring for the universal family:

DEFINITION 5.4. We define the tautological rings of $\mathcal{X}_g$ to be the subrings of the Chow and cohomology rings (with rational coefficients) generated by the pullbacks of the Hodge classes $\pi^*\lambda_i$, and the class $[\Theta_g]$ of the universal theta divisor given by the theta function (see Section 1).

Note that $\Theta_g \subset \mathcal{X}_g$ here denotes the (ample) universal theta divisor defined by theta function, but computationally it is often easier to work with the universal theta divisor that is trivial along the zero section — we denote this line bundle by T_g. Since theta constants are modular forms of weight one half we have the relation

$$[T_g] = [\Theta_g] - \pi^*(\frac{\lambda_1}{2}) \tag{5}$$

for the classes of these divisors in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}_g)$.

Then the cohomology tautological ring of $\mathcal{X}_g$ is described simply as follows:

THEOREM 5.5. *The cohomology tautological ring of* $\mathcal{X}_g$ *is generated by the pullback of the tautological ring of* $\mathcal{A}_g$ *and the class of the universal theta divisor trivialized along the zero section, with one relation* $[T_g]^{g+1} = 0$*.*

Proof. Indeed, from the results of Deninger and Murre [**DM91**] it follows (see [**Voi11**, Prop. 0.3] for more discussion) that for the universal family $\pi : \mathcal{X}_g \to \mathcal{A}_g$ there exists a multiplicative decomposition $R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]$. Since T_g is trivialized along the zero section, under the decomposition the class $[T_g]$ only has one term lying in R^2, and by the multiplicativity of the decomposition, $[T_g]^{g+1}$ would then have to lie in R^{2g+2}, which is zero, as the fibers of π have real dimension $2g$. We note also that it follows that the class $[T_g]^g$ is actually equal to $g!$ times the class of the zero section of π, given by choosing the origin on each ppav. By the projection formula it is clear that any class of the form $[T_g]^i\pi^*C$ for C a tautological class on $\mathcal{A}_g$ and $i \leq g$, is non-trivial, and thus there are no further relations. □

It is natural to conjecture that the above description also holds for the tautological Chow ring of $\mathcal{X}_g$.

Of course one cannot expect the tautological ring of $\mathcal{X}_g$ to be equal to the full cohomology or Chow ring, and thus the following question is natural

Question 5.6. *Compute the cohomology and Chow rings of $\mathcal{X}_g$ and their compactifications $\overline{\mathcal{X}_g}$ over $\mathcal{A}_g^{\mathrm{Vor}}$ for small values of g.*

The Chow and cohomology groups of toroidal compactifications of $\mathcal{A}_g$ and those of $\overline{\mathcal{X}_g}$ are closely related: as we have already seen in Section 4, Mumford's partial compactification is the union of $\mathcal{A}_g$ and $\mathcal{X}_{g-1}$, and hence the topology of $\mathcal{X}_{g-1}$ contributes to that of Mumford's partial compactification $\mathcal{A}_g'$. This relationship is an example of a much more general phenomenon. Recall that the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ is stratified as in (4). Taking the preimage of this stratification under the contracting morphism $p : \mathcal{A}_g^{\mathrm{tor}} \to \mathcal{A}_g^{\mathrm{Sat}}$ defines a stratification of any toroidal compactification by taking the strata $p^{-1}(\mathcal{A}_{g-i} \setminus \mathcal{A}_{g-i-1})$. Such a stratum is itself stratified in such a way that each substratum is the quotient of a torus bundle over a $(g-i)$-fold product of the universal family $\mathcal{X}_{g-i} \to \mathcal{A}_{g-i}$ by a finite group. This idea can be used to try to compute the cohomology of $\mathcal{A}_g^{\mathrm{tor}}$ inductively, and this is the approach taken in [**HT11**]. The computation of the cohomology of the various strata then is closely related to computing the cohomology of local systems on $\mathcal{A}_{g-i}$, or rather the invariant part of it under a certain finite group.

While computing the entire cohomology or Chow rings seems out of reach, one could study the stable cohomology: the limit of $H^k(\mathcal{A}_g)$ (or for compactifications) for $g \gg k$. The stable cohomology of $\mathcal{A}_g$ (equivalently, of the symplectic group) was computed by Borel [**Bor74**], and the stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$ was computed by Charney and Lee [**CL83**], using topological methods. There are currently two research projects, [**GS12**] and [**GHT12**], under way, aiming to show the existence of and compute some of the stable cohomology of $\mathcal{A}_g^{\mathrm{Perf}}$. The difference between the classes $[T_g]^g/g!$ and the zero section on the partial compactification of $\mathcal{X}_g$ was explored in [**GZ12**],

We have already pointed out that for $g = 2, 3$ the compactifications $\mathcal{A}_g^{\mathrm{Vor}}$ and $\mathcal{A}_g^{\mathrm{Perf}}$ coincide. It should also be pointed out that $\overline{\mathcal{X}_g}$ is not stack-smooth, already for $g = 2$. The Chow ring of $\mathcal{X}_2$ (and more or less for $\overline{\mathcal{X}_2}$, up to some issues of normalization) is computed by van der Geer [**vdG98**], while the results of the second author and Tommasi [**HT11**] on the cohomology of $\mathcal{A}_4^{\mathrm{Vor}}$ similarly go a long way towards describing the Chow and cohomology of $\overline{\mathcal{X}_3}$.

The above results on the decomposition theorem and the zero section also imply the earlier results of Mumford [**Mum83a**] and van der Geer [**vdG99**] on the pushforwards of the theta divisor: recalling from (5) the class $[T_g]$ of the theta divisor trivialized along the zero section, we have

$$\pi_*([T_g]^g) = g! \cdot [1]; \quad \pi_*([T_g]^{g+a}) = 0 \qquad \forall a \neq 0.$$

Using (5), from the projection formula it then follows that

$$\pi_*([\Theta_g]^g) = g! \cdot [1]; \qquad \pi_*([\Theta_g]^{g-a}) = 0;$$
$$\pi_*([\Theta_g]^{g+a}) = \binom{g+a}{g} 2^{-a} g! \lambda_1^a \qquad \forall a > 0.$$

One can now try to compute the classes of various loci we defined, and in particular ask whether they are tautological on $\mathcal{X}_g$. By definition $T_1^{(g)}$ is the theta divisor, i.e. $T_1^{(g)} = \Theta_g$. We can also compute the class of the locus $T_2^{(g)}$, since it is a complete intersection, defined by the vanishing of the theta function and its z-gradient. The gradient of the theta function is a section of the vector bundle $\mathbb{E} \otimes \Theta_g$: this is to say the gradient of the theta function is a vector-valued modular form for a suitable representation of the symplectic group, see [**GSM04**]. We thus obtain

Proposition 5.7. *The class of $T_2^{(g)}$ can be computed as*

$$[T_2^{(g)}] = c_g(\mathbb{E} \otimes \Theta_g) \cap [\Theta_g] = \sum_{i=0}^{g} \lambda_i [\Theta_g]^{g-i+1} \in CH^{g+1}(\mathcal{X}_g, \mathbb{Q})$$

(recall that Θ_g is not trivialized along the zero section). By pushing this formula to $\mathcal{A}_g$, using the above expressions for pushforwards, we recover the result of Mumford [**Mum83a**]*:*

$$[N_0] = \pi_*[T_2^{(g)}] = \left(\frac{(g+1)!}{2} + g! \right) \lambda_1 \in CH^1(\mathcal{A}_g, \mathbb{Q}).$$

For the locus $T_3^{(g)}$, the situation seems much more complicated, as the codimension is not known, and in particular it is not known to be equidimensional or a locally complete intersection. However, the situation is simpler for $T_3^{(g)}[2]$ — it is given locally in $\mathcal{X}_g$ by $2g$ equations (that the point z is odd 2-torsion, and that the corresponding gradient of the theta function vanishes). If we consider its projection $I^{(g)} \subset \mathcal{A}_g(2)$, it is locally given by the g equations, that the gradient of the theta function vanishes when evaluated at the corresponding 2-torsion point. For future use, we denote

$$\begin{aligned} f_m(\tau) &:= \mathrm{grad}_z \theta(\tau, z)|_{z=m} = \mathrm{grad}_z \theta(\tau, z + \tau\varepsilon + \delta)_{z=0} \\ &= \mathbf{e}(-\varepsilon^t \tau \varepsilon / 2 - \varepsilon^t \delta - \varepsilon^t z) \mathrm{grad}_z \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)|_{z=0} \end{aligned} \tag{6}$$

where $m = \tau\varepsilon + \delta 2$, for $\varepsilon, \delta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$, is an odd 2-torsion point. As discussed above, the gradient of the theta function is a section of $\mathbb{E} \otimes \Theta_g$; the gradient of the theta function evaluated at a 2-torsion point can be thus considered as a restriction of this vector bundle to the zero section of $\mathcal{X}_g \to \mathcal{A}_g$. We thus have $f_m \in H^0(\mathcal{A}_g(4,8), \mathbb{E} \otimes \det \mathbb{E}^{1/2})$. Recall that theta constants are only modular forms for the group $\Gamma_g(4,8)$; however, the action of $\Gamma_g(2)/\Gamma_g(4,8)$ preserves the characteristic and only changes signs; thus the zero locus $\{f_m = 0\}$ is well-defined on $\mathcal{A}_g(2)$.

Therefore, if Conjecture 3.19 holds, the locus $I^{(g)}$ is of codimension g in $\mathcal{A}_g$, and locally a complete intersection, given by the vanishing of a gradient f_m for some m. Summing over all such m yields the following

THEOREM 5.8 ([**GH11a**, Theorem 1.1]). *If Conjecture 3.19 holds in genus g, then the class of the locus $I^{(g)}$ is equal to*

$$[I^{(g)}] = 2^{g-1}(2^g - 1)\sum_{i=0}^{g} \lambda_{g-i}\left(\frac{\lambda_1}{2}\right)^i.$$

We notice that the locus of Jacobians $\mathcal{J}_g$ is very special in $\mathcal{A}_g$ from the point of view of the geometry of the theta divisor. Indeed, the theta divisor of Jacobians has a singular locus of dimension at least $g-4$, and also may have points of high multiplicity. Thus, as is to be expected, the loci $T_a^{(g)}$ do not intersect them transversely.

In particular, only looking at 2-torsion points, and at the loci $T_a^{(g)}[2]$, is equivalent to looking at curves with a theta-characteristic (considered as a line bundle on the curve that is a square root of the canonical line bundle) with a large number of sections. The algebraic study of theta characteristics on algebraic curves is largely due to Mumford [**Mum71**] and Harris [**Har82**]. It is natural to look at the loci $\mathcal{M}_g^k$ of curves of genus g having a theta characteristic with at least $k+1$ sections and the same parity as $k+1$. The connection with what we have just discussed is the (set-theoretic) equality

$$\mathcal{M}_g^k = I_{k+1}^{(g)} \cap \mathcal{M}_g \tag{7}$$

where we define $I_k^{(g)} \subset \mathcal{A}_g$ to be the locus of ppav whose theta divisor has a point of multiplicity k at a 2-torsion point (whose parity is even or odd depending on the parity of k); in particular $I_3^{(g)} = I^{(g)}$ in our notation, and thus $\mathcal{M}_g^2$ is its intersection with $\mathcal{M}_g$. The above equation is a consequence of the Riemann singularity theorem. The following problem was raised by Harris

QUESTION 5.9 (Harris). *Determine the dimension of the loci $\mathcal{M}_g^k$.*

It is known that $\mathcal{M}_g^k$ is non-empty if and only if $k \leq (g-1)/2$; Harris [**Har82**] proved that then the codimension (of any component) of $\mathcal{M}_g^k$ in $\mathcal{M}_g$ is at most $(k+1)/2$, Teixidor i Bigas [**TiB87**] proved an upper bound of $3g-2k+2$ for the dimension of all components of $\mathcal{M}_g^k$, and thus showed in particular that $\mathcal{M}_g^2 = I^{(g)} \cap \mathcal{M}_g$ in $\mathcal{M}_g$ is of pure codimension 3. This also shows that this intersection is highly non-transverse. (Teixidor i Bigas also showed that $\mathcal{M}_{2k+1}^k$ has precisely expected dimension, and that for $g \neq 2k+1$ and $k \geq 3$ a better bound on the codimension can be obtained.)

QUESTION 5.10. *Compute the class of $\mathcal{M}_g^k$, i.e. of the preimage of $I_{k+1}^{(g)}$ in $\mathcal{M}_g$, as well as that of its closure in $\overline{\mathcal{M}_g}$. This question is non-trivial already for $k=2$ and $g \geq 5$, for example the class of the codimension 3 locus $\mathcal{M}_5^2$ is of clear interest.*

In a recent work [**FGSMV11**] of Farkas, Salvati Manni, Verra, and the first author, class computations and geometric descriptions were also given for the loci of ppav within N_0 whose theta divisor has a singularity that is not an ordinary double point.

As we have seen, one of the main problems one encounters is to prove that certain cycles have the expected codimension. One approach to this, which has been used successfully in several situations, is to go to the boundary. Instead of g-dimensional ppavs one can then work with degenerations. Salvati Manni and the first author investigated in [**GSM09**] the boundary of the locus $I^{(g)}$ in the partial compactification $\mathcal{A}'_g$, in particular proving that its intersection with the boundary is codimension g within $\partial\mathcal{A}'_g$ (which is further evidence for Conjecture 3.19). Going further into the boundary of a suitable toroidal compactification, the degenerations can be quite complicated, they are not normal and not necessarily irreducible. However, the normalization of such a substratum has the structure of a fibration, with fibers being toric varieties, over abelian varieties of smaller dimension — and as such is often amenable to concrete calculations. Ciliberto and van der Geer [**CvdG08**] described explicitly the structure of the polarization divisors and their singularities for the locus of semiabelic varieties of torus rank 2, thus proving the $k = 1$ case of their Conjecture 3.12. This is closely related to taking limits of theta functions, resp. to working with the Fourier-Jacobi expansion of these functions. In our work [**GH11b**] we have described completely the geometric structure of all strata of semiabelic varieties that have codimension at most 5 in $\mathcal{A}_g^{\mathrm{Perf}}$, which in principle gives a method to study any locus in $\mathcal{A}_g$, of codimension at most 5, by degeneration. For this reason we shall now discuss degeneration techniques and results.

6. Degeneration: technique and results

In the previous section of the text we discussed the problem of computing the homology (or Chow) classes of various geometrically defined loci in $\mathcal{A}_g$. While computing a divisor class basically amounts to computing one coefficient, that of the generator λ_1 of $\mathrm{Pic}(\mathcal{A}_g)$, for higher codimension loci the problem is much harder, and to the best of our knowledge Proposition 5.8 is the only complete computation of a homology (or Chow) class of a higher-codimension geometric locus for $g \geq 4$. In particular, Proposition 5.8 shows that the class of the locus $I^{(g)}$ (if it is of expected codimension, i.e. if Conjecture 3.19 holds) is tautological. In general this is not clear for the classes of geometric loci in $\mathcal{A}_g$, but one can consider the problem of computing the projection of such a class to the tautological ring. Faber [**Fab99a**] in particular computed the projection of the class of the locus $\mathcal{J}_g$ of Jacobians to the tautological ring, for small genus.

In general a much harder problem still is to consider the classes of closures of various loci in suitable toroidal compactifications $\mathcal{A}_g^{\mathrm{tor}}$. Denoting $\delta \in H^2(\mathcal{A}_g^{\mathrm{tor}})$ the class of the closure of the boundary of the partial compactification, we note that since δ is certainly non-tautological (as it is not proportional to λ_1), it is natural to expect that classes of closures of various loci would not be tautological. The following loosely-phrased problem is thus very natural:

QUESTION 6.1. *Define a suitable extended tautological ring, of either the Chow or cohomology groups, of some toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$, containing the tautological ring, δ, and the classes of various geometrically defined loci (for example of various boundary substrata, see below).*

While we cannot answer the question above, in [**GH11a**], [**GH11b**] we studied the closure of $I^{(g)}$ in $\mathcal{A}_g^{\mathrm{Perf}}$ for $g \le 5$, and also described its projection to the tautological ring. Our result is

THEOREM 6.2 ([**GH11a**]). *For $g \le 5$, we have the following expression for the class of the closure $\overline{I^{(g)}}$ in $CH^g(\mathcal{A}_g^{\mathrm{Perf}})$:*

$$[\overline{I^{(g)}}] = \frac{1}{N} \sum_{m \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}_{\mathrm{odd}}} \sum_{i=0}^{g} p_* \left(\lambda_{g-i} \left(\frac{\lambda_1}{2} - \frac{1}{4} \sum_{n \in Z_m} \delta_n \right)^i \right) \tag{8}$$

where $p : \mathcal{A}_g^{\mathrm{Perf}}(2) \to \mathcal{A}_g^{\mathrm{Perf}}$ is the level cover, $N = |\operatorname{Sp}(g, \mathbb{Z}/2\mathbb{Z})|$ and Z_m is the set of pairs of non-zero vectors $\pm n \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ such that $m + n$ is an even 2-torsion point. We recall that the irreducible components D_n of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(2)$ correspond to non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^{2g} \equiv (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$, and denote their classes $\delta_n := [D_n]$.

When trying to define a suitable extended tautological ring on a toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$, one encounters several problems. The first is that $\mathcal{A}_g^{\mathrm{tor}}$ will in general be singular, also as a stack, as is the case with the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ for $g \ge 5$ and the perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$ for $g \ge 4$, and thus we cannot expect to have a ring structure on Chow classes.

This difficulty could be overcome by taking a suitable refinement to obtain a basic fan, which then leads to a stack-smooth toroidal compactification. The drawback is that no natural examples of basic fans exist for $g \ge 5$. Moreover, such a refinement would introduce numerous new boundary substrata whose geometric meaning is unclear. One possible solution, proposed by Ekedahl and van der Geer [**EvdG05**], is to consider a tautological module, the pushforward of such a ring to the Satake compactification. While natural, this tautological module could not capture all the information of the toroidal compactification, and possible degenerations, which may be more subtle. Independently, it would be of great interest to understand the topology of the Satake compactification itself.

Another approach is to try and define a tautological subring of the operational Chow ring, i.e. to define a suitable collection of geometrically meaningful vector bundles on $\mathcal{A}_g^{\mathrm{tor}}$ and then take the subring of the operational Chow ring which is generated by these classes. Since by [**Mum77**], [**FC90**] the Hodge bundle $\mathbb{E}$ extends to every toroidal compactifications we would naturally always obtain the classes $\lambda_i = c_i(\mathbb{E})$. On $\mathcal{A}_g^{\mathrm{Perf}}$ the boundary δ is an irreducible Cartier divisor, whose first Chern class one would naturally include in an extended tautological ring. Evaluating the elements of the extended tautological ring on the fundamental cycle one would obtain Chow homology classes which could be considered as tautological classes.

A natural class of candidates for such tautological Chow classes arises as follows: let $\mathcal{A}_g^{\mathrm{tor}}(\ell)$ be the level ℓ cover of $\mathcal{A}_g^{\mathrm{tor}}$. Then the boundary is no longer irreducible (not even for the case of $\mathcal{A}_g^{\mathrm{Perf}}$), say $\delta = \sum \delta_n$. It should, however, be noted that although δ is a Cartier divisor this is in general no longer true for the δ_n. Only certain sums of these, e.g. the one appearing in (10) are guaranteed to be Cartier. Nevertheless, we can obtain interesting geometric loci from these. The first example is the sum $\sum_{n \neq m} \delta_n \delta_m$. In the case of the perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$ this is exactly the class of the complement of $\mathcal{A}_g^{\mathrm{Perf}} \setminus \mathcal{A}'_g$ of the partial compactification. One can now go on and study the possible combinatorics of intersections of boundary divisors δ_i, and we investigated this in [**GH11a**].

Indeed, one would like the extended tautological ring to contain all polynomials in the δ_n that are invariant under the action of $\mathrm{Sp}(g,\mathbb{Z}/2\mathbb{Z})$. The action of the symplectic group on tuples of theta characteristics was fully described by Salvati Manni in [**SM94**]: two tuples lie in the same orbit if and only if they can be renumbered $n_1,\ldots,n_k$ and $m_1,\ldots,m_k$ in such a way that: the parity of n_i is the same as the parity of m_i; there exists a linear relation with an even number of terms $n_{i_1}+\ldots+n_{i_{2l}}=0$ if and only if $m_{i_1}+\ldots+m_{i_{2l}}=0$.

It follows that the ring of polynomials in δ_n invariant under the action of the symplectic group is generated by expressions of the form

$$\sum_{|I|\subset(\frac12\mathbb{Z}/\mathbb{Z})^{2g};|I|=i,f_1(I)=\ldots=f_{j_i}(I)=0}\quad\prod_{m\in I}\delta_m^a$$

where each $f_j(I)$ is a sum of an even number of characteristics in I (i.e. a linear relation), and moreover the parities of all the characteristics are prescribed a priori.

We often want to prove results about some toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$. If we want to use degeneration techniques, then we want to be able to compare this to $\mathcal{A}_g^{\mathrm{Vor}}$, over which we have a universal family. The compactification $\mathcal{A}_g^{\mathrm{tor}}$ has various boundary strata, each corresponding to a suitable cone in the fan, and if such a cone is shared with $\mathcal{A}_g^{\mathrm{Vor}}$, we know that the corresponding stratum parameterizes semiabelic varieties of a certain toric type. This is of particular interest in the case of $\mathcal{A}_g^{\mathrm{Perf}}$. For $g\le 3$ we know that $\mathcal{A}_g^{\mathrm{Vor}}$ and $\mathcal{A}_g^{\mathrm{Perf}}$ coincide, while for $g=4,5$ $\mathcal{A}_g^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_g^{\mathrm{Perf}}$. Moreover, by [**AB11**] we know that the two compactifications coincide in an open neighborhood of the locus of Jacobians, and by [**MV11**] their intersection is the matroidal locus, but in general they are different.

Using the relationship between the two compactifications we showed, by explicit computations, that for $g\le 4$ the class $[\overline{I^{(g)}}]$ indeed lies in the ring generated by the λ_i and the classes of the boundary strata. However, for $[\overline{I^{(5)}}]$, an explicit formula for which is given by (8), we could not prove such a result. Studying a subring of the cohomology (or Chow) of $\mathcal{A}_g^{\mathrm{Perf}}$ including the λ_i and the classes of the boundary strata is thus of obvious interest. In particular for $g=5$ it could either turn out that the class of $[\overline{I^{(5)}}]$ lies in this subring — showing this would perhaps involve somehow relating the boundary classes and the Hodge classes — or this could be another class that may need to be added to a suitable extended tautological ring.

We would now like to comment on the proof of Theorem 6.2 by degeneration methods. One could imagine that this is a straightforward extension of Theorem 5.8. Indeed, the components of the preimage of the locus $I^{(g)}$ on the level cover $\mathcal{A}_g(2)$ are given by the vanishing of various gradients of the theta function at 2-torsion points, i.e. by equations $f_m=0$. Thus one could try to investigate the behavior of each f_m at the boundary of $\mathcal{A}_g^{\mathrm{Perf}}$, and determine which bundle it is a section of. However, the geometry of the situation is very subtle. Indeed, on $\mathcal{A}_g(2)$ the locus $I^{(g)}$ can also be defined by the vanishing of the gradients of odd theta functions with characteristics at zero, i.e. by equations

$$F_m:=\mathrm{grad}_z\theta_m(\tau,z)|_{z=0}=0, \tag{9}$$

and it is not a priori clear which of these equations should be used at the boundary.

To deal with this, one first needs to show that f_m (and F_m) extend to sections of some vector bundle on the partial compactification of $\mathcal{A}_g$ — this is reasonably well-known, and can be done by a direct calculation of the Fourier-Jacobi expansion of the theta function (that is, of the Taylor series in the q-coordinate). It turns out that F_m defined by (9) vanishes identically on a boundary component δ_n of $\mathcal{A}_g^{\rm Perf}(2)$ if and only if $n \in Z_m$ (recall that Z_m was defined in Theorem 6.2).

Moreover, in this case the generic vanishing order of F_m on δ_n is $1/4$ in the normal coordinate $q := \exp(2\pi i \tau_{11})$ (where the boundary component is locally given as $\tau_{11} = i\infty$ in the Siegel space). Comparing f_m and F_m by (6) one concludes that the f_m extend to sections

$$\overline{f_m} \in H^0\left(\mathcal{A}_g'(4,8), \mathbb{E} \otimes \det \mathbb{E}^{1/2} \otimes (-\sum_{n \in Z_m} D_n/4)\right) \tag{10}$$

on the partial compactification not vanishing on the generic point of any boundary component. We now use the fact that the codimension of $\mathcal{A}_g^{\rm Perf} \setminus \mathcal{A}_g'$ in $\mathcal{A}_g^{\rm Perf}$ is equal to two (this is not the case for $\mathcal{A}_g^{\rm Vor}$), and thus by Hartogs' theorem $\overline{f_m}$ extend to a section of the above vector bundle on all of $\mathcal{A}_g^{\rm Perf}(4,8)$.

Since on $\mathcal{A}_g(4,8)$ the components of the preimage of $I^{(g)}$ are given by vanishing of the f_m, it follows that on $\mathcal{A}_g^{\rm Perf}(4,8)$ the closure $\overline{I^{(g)}}$ of $I^{(g)}$ is contained in the zero locus of the $\overline{f_m}$. A priori it could happen that the locus $\{\overline{f_m} = 0\}$ on $\mathcal{A}_g^{\rm Perf}(4,8)$ has other irreducible components, but we conjecture that is not so:

CONJECTURE 6.3. *The locus $\{\overline{f_m} = 0\} \subset \mathcal{A}_g^{\rm Perf}(4,8)$ has no irreducible components contained in the boundary.*

If this conjecture holds, it implies that we have $\overline{I^{(g)}} = p(\{\overline{f_m} = 0\})$ (recall that p denotes the level cover). Theorem 6.2 then follows by computing the class of the locus $\{\overline{f_m} = 0\}$, as the zero locus of a section of a vector bundle.

We do not know a general approach to this conjecture, or to similar more general results about the degenerations of the loci defined by vanishing of the gradients of the theta function. The evidence that we have comes from a detailed investigation of the cases of small torus rank. One of the main results of [**GH11b**] is

THEOREM 6.4 ([**GH11b**]). *The conjecture above holds for $g \leq 5$.*

To prove this conjecture, we investigated in detail all strata in the boundary of $\mathcal{A}_g^{\rm Perf}$ that have codimension at most 5. Using the fact that for $g \leq 5$ the Voronoi compactification $\mathcal{A}_g^{\rm Vor}$ admits a morphism to $\mathcal{A}_g^{\rm Perf}$, it follows that all these strata parameterize suitable families of semiabelic varieties of torus rank up to 5, with determined type of the toric bundle and gluing. By examining these strata case by case, describing their semiabelic polarization (or theta) divisors, we could compute the extension $\overline{f_m}$ on each such boundary stratum explicitly. In [**GH11b**] we then proved that $\overline{f_m}$ on such a boundary stratum of $\mathcal{A}_g^{\rm Perf}$ may not have a vanishing locus that is of codimension at most 5 in $\mathcal{A}_g^{\rm Perf}$, thus proving the above theorem. However, to attempt to prove the conjecture for $g \geq 6$, a different approach may be required, especially as it is no longer true that $\mathcal{A}_g^{\rm Vor}$ admits a morphism to $\mathcal{A}_g^{\rm Perf}$, and thus it is not known whether there exists a universal family over $\mathcal{A}_g^{\rm Perf}$, where the computations of the degenerations of the gradients of the theta functions could be carried out. However, our techniques and results of [**GH11b**] are still of independent use, as they provide a way to study any locus in $\mathcal{A}_g$ of codimension at most 5 by taking its closure in $\mathcal{A}_g^{\mathrm{Perf}}$ and considering degenerations.

REMARK 6.5. At this point we would like to take the opportunity to point out and correct an error in the proof of Theorem 6.4. given in [**GH11b**]. There we stated that the only codimension 5 stratum in β_5 is that corresponding to the standard cone $\langle x_1^2, \dots x_5^2 \rangle$, which however turns out not to be true for $g = 5$. Indeed, there are two such cones. Apart from the standard cone, we also have to consider

$$\sigma_1 = \langle x_1^2, \dots x_4^2, (2x_5 - x_1 - x_2 - x_3 - x_4)^2 \rangle.$$

It follows immediately from the definition of the perfect cone decomposition that σ_1 belongs to it. Moreover, it does not lie in the $\mathrm{GL}(5, \mathbb{Z})$ orbit of the standard cone, since its generators do not give a basis of the space of linear forms in 5 variables. This also implies that σ_1 is not in the matroidal locus, see [**MV11**, Section 4]. The fact that, up to $\mathrm{GL}(5, \mathbb{Z})$-equivalence, this is the only other cone of dimension 5 containing rank 5 matrices follows from [**E-VGS10**, p. 7, Table 1] (note that n in this table is the dimension of the cone minus 1), which in turn was confirmed by a computer search, which was performed by M. Dutour Sikiric .

Although the generators of σ_1 do not form a basis of the space of linear forms, the cone σ_1 itself is still a basic cone since its generators are part of a basis of $\mathrm{Sym}^2(\mathbb{Z}^5)$. However, since σ_1 is not contained in the matroidal locus, it is also not contained in the second Voronoi decomposition by [**MV11**, Theorem A] and hence we cannot argue with properties of the theta divisor on semi-abelic varieties as we did for all the other strata in the proof given in [**GH11b**]. Nevertheless, it is possible to give a direct proof that the sections $\overline{f_m}$ do not vanish identically on the stratum corresponding to σ_1. For this we consider the toric variety T_{σ_1}. Since σ_1 is basic of dimension 5 it follows that $T_{\sigma_1} \cong (\mathbb{C}^*)^5 \times \mathbb{C}^{10}$. Let $t_{ij} = e^{2\pi i \tau_{ij}}$. We consider the basis U_{ij} of $\mathrm{Sym}^2(\mathbb{Z})$ where $U_{ii} = x_i^2$ and $U_{ij} = 2x_i x_j$ for $i \neq j$. We denote the dual basis by U_{ij}^*. A straightforward calculation shows that the dual cone $\sigma_1^\vee$ is generated by the following elements:

$$U_{55}^* - 4U_{12}^*,\ U_{25}^* + 2U_{12}^*,\ U_{35}^* + 2U_{12}^*, U_{45}^* + 2U_{12}^*,$$
$$U_{13}^* - U_{12}^*,\ U_{14}^* - U_{12}^*,\ U_{23}^* - U_{12}^*,\ U_{24}^* - U_{12}^*,\ U_{34}^* - U_{12}^*,\ U_{15}^* - U_{25}^*,$$
$$U_{11}^* - U_{12}^*,\ U_{22}^* - U_{12}^*,\ U_{33}^* - U_{12}^*,\ U_{44}^* - U_{12}^*, U_{12}^*,$$

where the first 10 generators are orthogonal to σ_1 and the last 5 generators are orthogonal to 4 of the generators and pair with the remaining generator to 1. Hence we obtain coordinates for $T_{\sigma_1} \cong (\mathbb{C}^*)^{10} \times \mathbb{C}^5$ by setting

$$s_1 = t_{55} t_{12}^{-4},\ s_2 = t_{25} t_{12}^2,\ s_3 = t_{35} t_{12}^2,\ s_4 = t_{45} t_{12}^2,\ s_5 = t_{13} t_{12}^{-1}$$
$$s_6 = t_{14} t_{12}^{-1},\ s_7 = t_{23} t_{12}^{-1},\ s_8 = t_{24} t_{12}^{-1},\ s_9 = t_{34} t_{12}^{-1},\ s_{10} = t_{15} t_{25}^{-1}$$

as coordinates on the torus $(\mathbb{C}^*)^{10}$, and

$$T_1 = t_{11} t_{12}^{-1},\ T_2 = t_{22} t_{12}^{-1},\ T_3 = t_{33} t_{12}^{-1},\ T_4 = t_{44} t_{12}^{-1},\ T_5 = t_{12}$$

as coordinates on the space $\mathbb{C}^5$. From this one can express the t_{ij} in terms of the coordinates s_i, T_j and this in turn enables one to compute the Taylor series expansion of the sections $\overline{f_m}$ in the coordinates s_i, T_j for each of the $16 \cdot 31 = 496$ odd 2-torsion points m. A computer calculation shows they they never vanish identically when restricted to $T_1 = \ldots = T_5 = 0$.

Remark 6.6. Note that one could attempt to follow a similar approach to Conjecture 3.19 for $T_3^{(g)}$, for low genus. However, in this case one needs to study suitable conditions for the singularities of semiabelic theta divisors arising from tangencies of lower-dimensional theta divisors. For the case of torus rank up to 2 this was done by Ciliberto and van der Geer [**CvdG08**], but it is not clear how far this can be extended into the boundary, as it requires a detailed understanding of the possible geometries of intersections of translates of the theta divisor.

Acknowledgements

The first author would like to thank Riccardo Salvati Manni, in collaboration with whom some of the results discussed above were obtained and some of the conjectures made, and from whom we have learned a lot about the geometry of theta functions and abelian varieties. The second author is grateful to Valery Alexeev, Iku Nakamura, and Martin Olsson for discussions on degenerations of abelian varieties and universal level families. We are grateful to Achil Schürmann for pointing out a reference on the perfect cone decomposition and to Mathieu Dutour Sikiric for double checking the number of cones in the perfect cone decomposition in genus 5.

We thank the referee for reading the text very carefully and for numerous suggestions and corrections.

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Clay Mathematics Proceedings
Volume **18**, 2013

Singular curves and their compactified Jacobians

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Abstract. We survey the theory of the compactified Jacobian associated to a singular curve. We focus on describing low genus examples using the Abel map.

In this article we study how to assign a degenerate Jacobian, called a compactified Jacobian, to a singular curve. The title of this article is intended to recall Mumford's book "Curves and their Jacobians" [**Mum75**]. That book contains a beautiful introduction to the geometric theory of the Jacobian variety associated to a smooth curve, and the present article is intended to be a survey of the analogous theory for a singular curve, written in a similar spirit. The focus is on providing examples and indicating various technical issues. We omit many proofs and instead direct the interested reader to the literature.

The main goal of this article is to show how the Abel map can be used to describe the compactified Jacobian $\bar{J}^d_X$ associated to a singular curve X. One novel feature of this article is that we link the theory of the Abel map to the theory of linear systems of generalized divisors. Such a link is certainly well-known to experts (see [**Har86**, Rmk. 1.6.5]), but we develop the relation in greater depth than has previously been done in the literature. We also discuss several examples that have not appeared in the literature before. The most interesting example is the compactified Jacobian of a genus 2 non-Gorenstein curve, which is studied throughout the paper but especially in Example 5.0.11.

Most of the results in this paper are not due to the author. Many mathematicians have contributed to the body of work discussed, but the author owes a particularly large mathematical debt to Allen Altman, Steve Kleiman, and Robin Hartshorne. The theory of generalized divisors was developed by Hartshorne in [**Har86**], [**Har94**], and [**Har07**], and the Abel map that we study here was constructed by Altman–Kleiman in [**AK80**]. Kleiman's article [**Kle05**] was also very helpful in writing Section 1.

The reader who has looked below at the "Conventions" section may have noticed that in this paper the term "curve" always refers to an irreducible and reduced curve. It is possible to assign a compactified Jacobian $\bar{J}^d_X$ to, say, a reducible curve X, but both the definition of $\bar{J}^d_X$ and the associated Abel map then become more complicated. One major barrier to constructing an Abel map for a reducible curve is that the theory of linear systems on a reducible curve has undesirable properties,

2010 *Mathematics Subject Classification.* Primary 14K30; Secondary 14H20, 14C20.

which are discussed in [**Har07**, Rmk. 2.9]. There is, however, a growing body of work (e.g. [**CE07, CCE08, CP10**]) on constructing an Abel map for a reducible but reduced curve. The papers just cited also provide a guide to the literature on compactified Jacobians of reducible curves. Much less is known about assigning a compactified Jacobian to a non-reduced curve, and in particular, there seems to be no literature on constructing an Abel map for such a curve.

0.1. Organization. This paper is organized as follows. In Section 1 we recall some basic facts about the Jacobian variety associated to a smooth curve. The material developed there is our model for the theory of the compactified Jacobian associated to a singular curve. We begin discussing singular curves in Section 2, where we define the generalized Jacobian of a singular curve. This variety typically fails to be proper, and so we are naturally led to compactify the generalized Jacobian. This is done in Section 3. At the end of that section we show by example that the most naive approach to constructing an Abel map into the compactified Jacobian fails. The rest of the article is devoted to constructing a suitable Abel map. We recall the theory of generalized divisors in Section 4 and then we apply that theory to construct the Abel map in Section 5. Finally, Section 6 is an appendix that contains some facts about the dualizing sheaf that are used in this article.

0.2. Personal Remarks. A few personal words about this article. I wrote this article for the proceedings for the conference "A Celebration of Algebraic Geometry," held in honor of Joe Harris' 60th birthday. Joe was my adviser in graduate school, and I hope that this article demonstrates Joe's influence on me as a mathematician.

During my last year in graduate school, William Fulton told me about working with Joe when Joe first moved to Brown University. Fall semester that year Joe taught a topics course on Brill–Noether theory. After reviewing the necessary definitions, Joe begin by working out the theory of special divisors on a genus 2 curve. The next day of class was spent on genus 3 curves, which took up one lecture, and Joe then moved on to genus 4 curves, which took a bit more class time. Fulton said that he expected Joe to soon run out of examples and then state and prove the general Brill–Noether Theorem.

Joe continued doing examples until the Thanksgiving break.

Right before the break, he was discussing curves whose genera was in the double digits. When he returned from break, he apologized and explained that he had run out of examples. It was only then that he stated and proved the Brill–Noether Theorem. Fulton cited this as an example of Joe's excellent mathematical taste: explicitly working through such a large class of curves provided incredible insight into Brill–Noether theory, insight that is not conveyed by just proving general theorems.

In the present article, we will certainly not get to singular curves with double digit genera, but I hope the choice of material shows the influence of Joe's good taste. We will work out examples of compactified Jacobians associated to curves of low genus, and the selection of examples was influenced by [**ACGH85**], a book that Joe co-authored. Indeed, the examples in Section 1 are all answers to exercises in [**ACGH85**], and the examples studied later are chosen as examples analogous to the ones from Section 1.

Conventions

- The letter k denotes an algebraically closed field.
- If V is a k-vector space, then $\mathbf{P}V$ is the Grassmannian of 1-dimensional quotients of V.
- If T is a k-scheme, then we write X_T for $X \times_k T$.
- If T and X are k-schemes, then a **T-valued point** of X is a k-morphism $T \to X$.
- A **variety** is a finite type, separated, and integral k-scheme.
- A **curve** is a finite type, separated, integral, and projective k-scheme of pure dimension 1.
- The **genus** g of a curve X is $g := 1 - \chi(\mathcal{O}_X)$.
- The symbol $k(X)$ denotes the **field of fractions** (or field of rational functions) of a curve X.
- The symbol $\mathcal{K}$ denotes the locally constant sheaf associated to $k(X)$.
- The symbol $\mathcal{K}_\omega$ denotes the locally constant sheaf of rational 1-forms.
- If X is a k-scheme and F, G are two $\mathcal{O}_X$-modules, then we write $\underline{\mathrm{Hom}}(F, G)$ for the sheaf of homomorphisms from F to G.

1. The Jacobian

Here we discuss the Jacobian variety J^d_X of a smooth curve X. We begin by discussing three different approaches to constructing J^d_X. Of these approaches, one uses the Abel map, and we also review the properites of this map. After recalling the definition, we conclude this section by describing the Abel map of a curve of genus at most 4. In the subsequent sections of this article we will work to extend results of this section from smooth curves to singular curves.

The most succinct definition of the Jacobian J^0_X of a smooth curve X requires us to assume that the ground field k is the field of complex numbers $k = \mathbf{C}$. If X is a smooth projective complex curve, then the associated Jacobian is the complex torus

$$J^0_X := H^1(X_{\mathrm{an}}, \mathcal{O}_X)/H^1(X_{\mathrm{an}}, \underline{\mathbf{Z}}).$$

We write X_{an} for the space X with the analytic or classical topology (rather than the Zariski topology). The group $H^1(X_{\mathrm{an}}, \underline{\mathbf{Z}})$ is considered as a subgroup of $H^1(X_{\mathrm{an}}, \mathcal{O}_X)$ using the map induced by the natural inclusion $\underline{\mathbf{Z}} \xrightarrow{2\pi i} \mathcal{O}_X$. This inclusion is part of a short exact sequence

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0, \tag{1}$$

which provides us with an alternative description of J^0_X. An inspection of the associated long exact sequence shows that there is a canonical isomorphism

$$J^0_X \cong \ker(H^1(X_{\mathrm{an}}, \mathcal{O}^*_X) \xrightarrow{c_1} H^2(X_{\mathrm{an}}, \underline{\mathbf{Z}})), \tag{2}$$

where c_1 is the 1st Chern class map.

The group $H^1(X_{\mathrm{an}}, \mathcal{O}^*_X)$ is canonically isomorphic to the set of isomorphism classes of line bundles, and under this isomorphism, the 1st Chern class map corresponds to the degree map, so the points of J^0_X are in natural bijection with the degree 0 line bundles on X.

This description suggests a way of defining J^0_X over an arbitrary ground field: J^0_X is the moduli space of degree 0 line bundles. In slightly more generality, if d

is any integer, then we define the degree d Jacobian, or the moduli space of line bundles of degree d, as follows.

DEFINITION 1.0.1. The **Jacobian functor** $J_X^{d,\sharp}\colon$ k-Sch $\to$ Set of degree d is the étale sheaf associated to the functor that assigns to a k-scheme T the set of isomorphism classes of line bundles L on X_T that have the property that the restriction of L to every fiber of $X_T \to T$ has degree d. The **Jacobian variety** J_X^d of degree d is the k-scheme J_X that represents $J_X^{d,\sharp}$.

Our definition of $J_X^{d,\sharp}$ is somewhat unsatisfactory. Because $J_X^{d,\sharp}$ is the sheafification of the functor parameterizing isomorphism classes of line bundles and not the functor itself, it is not immediately clear what the functor $J_X^{d,\sharp}$ parametrizes. For example, a line bundle L on X_T defines a T-valued point of $J_X^{d,\sharp}$ (for a suitable d, provided the degree of the restriction of L to the fiber of $X_T \to T$ over $t \in T$ is constant as a function of t, but it is not immediate from the definition that every T-valued point of $J_X^{d,\sharp}$ is defined by some L. Similarly, it is unclear when two line bundles L and M on X_T define the same T-valued point.

A very careful discussion of this topic can be found in [**Kle05**]. In this article, the distinction between the functor parameterizing line bundles on X and its associated sheaf will be largely irrelevant for the following reason: when T equals $\mathrm{Spec}(K)$ for K an algebraically closed field, $J_X^{d,\sharp}(T)$ equals the set of isomorphisms classes of degree d line bundles on X_T, which is what one should naively expect. (This is [**Kle05**, Ex. 2.3].)

In an arithmetic context, however, it is important to distinguish between the functor parameterizing line bundles and its sheafification because when K fails to be algebraically closed there may be K-valued points of $J_X^{d,\sharp}$ that cannot be represented by line bundles. (See [**Kle05**, Ex. 2.4] for an example.)

In any case, to make use of Definition 1.0.1, we need to prove that $J_X^{d,\sharp}$ can be represented by a k-scheme. In general, a standard approach to proving representability of a functor is to use a theorem of Artin. In [**Art74, Art69**], Artin gave a criteria for a functor to be representable by an algebraic space, and one approach to proving that a functor can be represented by a scheme is to first prove representability by an algebraic space by verifying Artin's criteria and then to prove that the resulting algebraic space is actually a scheme by some other means.

In the specific case of $J_X^{d,\sharp}$, this strategy was carried out by Artin in [**Art69**]. He first proved that $J_X^{d,\sharp}$ is representable by an algebraic space by verifying that the functor satisfies Artin's criteria ([**Art69**, Thm. 7.3, p. 67]; see also [**Art74**, p.186-187]). Artin then proved that this algebraic space is a k-scheme by proving more generally that any torsor for a locally finite type group space over k is actually a scheme [**Art69**, Lem. 4.2, p. 43].

A second approach to proving that $J_X^{d,\sharp}$ is representable is to make use of the Abel map. While this approach to proving representability is not as general as the first approach, it has the advantage of providing more insight into the geometry of J_X^d. Let us review this construction, beginning with the definition of the symmetric power.

DEFINITION 1.0.2. The **symmetric product** $X^{(d)}$ is defined to be the quotient of the d-fold self-product $X \times \cdots \times X$ of X by the action of the symmetric group Sym_d given by permuting factors.

Recall that the quotient of a quasi-projective variety V by a finite group always exists as a quasi-projective variety (by, say, [**Mum70**, p. 66]). Hence the symmetric power $X^{(d)}$ of a smooth curve is a projective variety of dimension d. This symmetric power is also smooth, but this is more difficult to establish. A proof of smoothness can be found in e.g. [**FG05**, Thm. 7.2.3]. The symmetric power of a smooth curve also has a moduli-theoretic interpretation.

Definition 1.0.3. The **Hilbert functor** $\mathrm{Hilb}_X^{d,\sharp}$ of degree d is defined by setting $\mathrm{Hilb}^{d,\sharp}(T)$ equal to the set of T-flat closed subschemes $Z \subset X_T$ with the property that every fiber of $Z \to T$ has degree d.

Lemma 1.0.4. *The k-scheme $X^{(d)}$ represents* $\mathrm{Hilb}_X^{d,\sharp}$.

Proof. We will prove this lemma under the additional assumption that $\mathrm{Hilb}_X^{d,\sharp}$ can be represented by some k-scheme Hilb_X^d. To begin, let us construct a natural transformation $X^{(d)} \to \mathrm{Hilb}_X^{d,\sharp}$. It is enough to construct a Sym_d-invariant transformation $X \times \cdots \times X \to \mathrm{Hilb}_X^{d,\sharp}$, and we construct this second transformation by exhibiting the corresponding closed subscheme $Z \subset (X \times \cdots \times X) \times X$.

We define Z to be the multi-diagonal that consists of tuples $(p_1, \ldots, p_d, q)$ with $p_i = q$ for some i. Working on the level of local rings, one can show that $Z \to X \times \cdots \times X$ is flat and finite of degree d. Because Z is visibly Sym_d-invariant, this subscheme induces the desired transformation $X^{(d)} \to \mathrm{Hilb}_X^{d,\sharp}$. To complete the proof, we need to show that this transformation is an isomorphism.

First, observe that $X^{(d)}(K) \to \mathrm{Hilb}_X^{d,\sharp}(K)$ is injective for any algebraically closed field K. Indeed, this observation is just the statement that no two fibers of $X \times X^{(d)} \supset Z \to X^{(d)}$ are equal as subschemes, a statement that can be verified by working affine locally and writing out the ideal of Z in terms of symmetric polynomials. (Show that the fiber of $Z \to X^{(d)}$ over the point that is the image of $(p_1, \ldots, p_d)$ is a closed subscheme supported on $\{p_1, \ldots, p_d\}$ and then compute the length of $\mathcal{O}_{Z,p_i}$.)

Now a similar explicit computation shows that $X^{(d)} \to \mathrm{Hilb}_X^d$ is an isomorphism over the locus in Hilb_X^d parameterizing reduced subschemes, so the morphism is birational. The morphism is also quasi-finite because we have just shown that it is injective on K-points. Thus $X^{(d)} \to \mathrm{Hilb}_X^d$ is a quasi-finite, birational morphism. Furthermore, both the target and the source of the morphism are smooth and proper. Indeed, we made this observation about $X^{(d)}$ just after defining the variety, and one can verify that Hilb_X^d is smooth and proper by using the functorial characterization of these properties. We can conclude by Zariski's Main Theorem that $X^{(d)} \to \mathrm{Hilb}_X^d$ is an isomorphism. $\square$

By the lemma, we can identify $X^{(d)}$ with $\mathrm{Hilb}_X^{d,\sharp}$ when X is smooth. The symmetric power is still defined when X is singular, but then $X^{(d)}$ may not represent the Hilbert functor. In Section 5 we will describe how the symmetric power of a singular curve is related to the Hilbert functor.

The closed subschemes $Z \subset X_T$ that correspond to elements of $\mathrm{Hilb}_X^d(T)$ are all effective relative Cartier divisors. That is, locally the ideal of $Z \subset X_T$ is generated by a single element. Indeed, this is a consequence of the proof of Lemma 1.0.4, but the fact can also be proven directly (see [**Kle05**, Lem. 9.3.4]). Linear equivalence classes of Cartier divisors are in natural bijection with isomorphism classes of line bundles, and the Abel map $A\colon X^{(d)} \to J_X^d$ is a manifestation of this relation.

Before defining the Abel map, let us review the definition of a Cartier divisor from [**Har77**, Chap. 2, Sect. 5]. We define a **Cartier divisor** D to be a global section of the quotient sheaf $\mathcal{K}^*/\mathcal{O}_X^*$, where $\mathcal{K}^*$ is the locally constant sheaf of nonzero rational functions. Because Cartier divisors are sections of an abelian sheaf, we can form the **minus** $-D$ of a Cartier divisor D and the **sum** $D+E$ of D and a second Cartier divisor E. As a section of a quotient sheaf, D can be represented by a collection $(f_i, U_i)_{i \in I}$ consisting of open subsets $\{U_i\}_{i \in I}$ that cover X and rational functions $\{f_i\}_{i \in I}$ that have the property that $f_i f_j^{-1}$ is regular on $U_i \cap U_j$. To this data we can associate a nonzero coherent subsheaf

$$I_D \subset \mathcal{K},$$

namely the subsheaf generated by f_i on U_i.

The reader may verify that the correspondence $D \mapsto I_D$ defines a bijection between the set of global sections of $\mathcal{K}^*/\mathcal{O}_X^*$ and the set of nonzero coherent subsheaves $I_D \subset \mathcal{K}$ that are locally principal. (In fact, nonzero subsheaves of $\mathcal{K}$ are always locally principal; this can be proven using, say, the classification of f.g. modules over a DVR.) Thus we can think of Cartier divisors as nonzero subsheaves of $\mathcal{K}$ rather than as global sections of $\mathcal{K}^*/\mathcal{O}_X^*$. We will work with Cartier divisors as subsheaves in this article.

A Cartier divisor D is said to be **effective** if we have $I_D \subset \mathcal{O}_X$. In other words, an effective Cartier divisor is a 0-dimensional closed subscheme. A rational function f defines a Cartier divisor, the Cartier divisor $\mathcal{O}_X \cdot f \subset \mathcal{K}$, and we denote this divisor by $\operatorname{div}(f)$. We say that two Cartier divisors D and E are **linearly equivalent** if $D = E + \operatorname{div}(f)$ for some rational function f. The set of all effective divisors linearly equivalent to a given divisor D is denoted by $|D|$ and called the **complete linear system** associated to D.

The divisors of the form $\operatorname{div}(f)$ naturally form a subgroup of the group of all Cartier divisors. In fact, the rule $D \mapsto I_D$ defines a surjective group homomorphism from the group of Cartier divisors to the group of isomorphism classes of line bundles, and the kernel of this map is precisely the subgroup consisting of divisors of the form $\operatorname{div}(f)$. Thus a linear equivalence class of Cartier divisors is the same thing as an isomorphism class of line bundles. In slightly different form, this is [**Har77**, Prop. 6.13].

The complete linear system $|D|$ associated to a Cartier divisor naturally has the structure of a projective space. Indeed, consider the $\mathcal{O}_X$-linear dual $\mathcal{L}(D) := \underline{\operatorname{Hom}}(I_D, \mathcal{O}_X)$ of I_D. The natural map $H^0(X, \mathcal{L}(D)) \to \operatorname{Hom}(I_D, \mathcal{O}_X)$ is an isomorphism, and the rule that sends a nonzero global section of $\mathcal{L}(D)$ to the image of the corresponding homomorphism $I_D \to \mathcal{O}_X$ defines a bijection $\mathbf{P}H^0(X, \mathcal{L}(D)) \cong |D|$. We use $\mathcal{L}(D)$ to define the Abel map.

Definition 1.0.5. If $D \in \operatorname{Hilb}_X^{d,\sharp}(T)$ (for some k-scheme T), then we define $\mathcal{L}(D) := \underline{\operatorname{Hom}}(I_D, \mathcal{O}_{X_T})$. The d-th **Abel map** $A\colon X^{(d)} \to J_X^d$ is defined by the rule $D \mapsto \mathcal{L}(D)$.

Because I_D is a line bundle, the formation of $\mathcal{L}(D)$ commutes with base change, so the Abel map is well-defined.

By definition, the fiber of A over $[L] \in J_X^d$ is the set of divisors D satisfying $\mathcal{L}(D) \cong L$. In other words, $A^{-1}([L]) = \mathbf{P}H^0(X, L)$. This equality holds on the level of schemes, though this is perhaps not clear from the work we have done so far. A proof of scheme-theoretic equality and other technical results describing the structure of A can be found in [**Kle05**, Sect. 9.3] (esp. Thm. 9.3.13). In any case, it follows from those structural results that A fibers $X^{(d)}$ over J_X^d by projective spaces of possibly varying dimension.

What dimensions can these projective spaces have? The dimension $\dim |D|$ is controlled by the Riemann–Roch Formula. Define a **canonical divisor** K to be a divisor with the property that $\mathcal{L}(K) = \omega$, the dualizing sheaf of X. Given a Cartier divisor D, an **adjoint divisor** $\operatorname{adj} D$ is defined to be a Cartier divisor satisfying $\mathcal{L}(\operatorname{adj} D) = \underline{\operatorname{Hom}}(\mathcal{L}(D), \omega)$ or equivalently a divisor linearly equivalent to $K - D$. The Riemann–Roch Formula relates $\dim |\operatorname{adj} D|$ to $\dim |D|$. The formula states

$$\dim |D| - \dim |\operatorname{adj} D| = d + 1 - g,$$

where d is the degree of D.

To use this formula, we need information about $\operatorname{adj} D = K - D$. The canonical divisor K has degree $2g - 2$, so if D has degree $d > 2g - 2$, then $\operatorname{adj} D$ has negative degree, which forces $\dim |\operatorname{adj} D| = -1$. To study $|\operatorname{adj} D|$ for $d \leq 2g-2$, we introduce the canonical map.

Assume $g \geq 1$. (The case $g = 0$ can be dealt with separately.) The canonical divisor K of a curve of genus $g \geq 1$ is base-point free (by [**Har77**, Prop. 5.1]), and so K determines a morphism $X \to \mathbf{P}H^0(X, \mathcal{L}(K))^\vee$ to projective space that we call the **canonical map**. The image is a curve when $g \geq 2$, and we call this curve the **canonical curve**. This curve is a curve in $\mathbf{P}^{g-1}$ as $\dim H^0(X, \mathcal{L}(K)) = g$. In terms of the canonical map, the adjoint linear system $|\operatorname{adj} D|$ associated to an effective divisor D can be described as the set of hyperplanes in $\mathbf{P}^{g-1}$ whose preimage contains D. Using this description and the Riemann–Roch Formula, one can prove that

$$\dim |D| = \begin{cases} -1 & \text{if } d < g \\ d - g & \text{if } d \geq g \end{cases} \tag{3}$$

for a general effective divisor D of degree d and for every divisor when $d \geq 2g - 1$. We give the proof later in Section 4, where we prove a more general statement as Corollary 4.1.16.

For the purpose of constructing J_X^d, the most important fact about Cartier divisors is that $\dim |D| = d-g$ for every divisor of degree $d > 2g-2$. In other words, for such a degree, the fibers of $A\colon X^{(d)} \to J_X^{d,\sharp}$ are all $\mathbf{P}^{d-g}$'s. Given the existence of the Jacobian J_X^d, the stronger technical results about A in [**Kle05**, Sect. 9.3], which were mentioned earlier, show that $A\colon X^{(d)} \to J_X^d$ is a $\mathbf{P}^{d-g}$-bundle.

If we do not assume that J_X^d exists, then we can use the above fact about Cartier divisors to construct the scheme. It is enough to construct J_X^d for $d >> 0$, and for such a d, we can define J_X^d to be the quotient of $X^{(d)}$ by the relation of linear equivalence. For this to be a valid definition, however, we must show that the quotient of $X^{(d)}$ exists as a k-scheme.

In general, the quotient of a k-scheme by an equivalence relation may not exist as a k-scheme, but linear equivalence is a particularly nice equivalence relation: the equivalence classes are smooth and projective subschemes of $X^{(d)}$. More formally, the relation is smooth and projective in the sense that the graph $R \subset X^{(d)} \times X^{(d)}$ of the equivalence relation has the property that the two projection morphisms $R \to X^{(d)}$ are smooth and projective. The quotient of a quasi-projective k-scheme by such an equivalence relation always exists, as can be shown using an elementary argument (that realizes the quotient as a subscheme of a suitable Hilbert scheme). We direct the reader to [**Kle05**, Lem. 4.9] for details about the construction of the quotient by a smooth and projective relation and to [**Kle05**, Thm. 4.8] for the proof that the quotient of $X^{(d)}$ is J_X^d. In any case, this gives a second construction of the Jacobian.

The analytic construction from the beginning of this section showed that not only does the Jacobian exist, but it has the structure of a g-dimensional complex torus. We can now show that J_X^0 is the algebro-geometric analogue of a g-dimensional complex torus: J_X^0 is a g-dimensional abelian variety. Recall this means J_X^0 is a smooth proper variety of dimension g that has group scheme structure. The group structure on J_X^0 comes from the tensor product, and one can deduce the remaining properties from the fact that $X^{(d)}$ is a smooth projective variety of dimension d by using $A\colon X^{(d)} \to J_X^d$ for $d >> 0$.

So far we have just made use of the Abel maps $A\colon X^{(d)} \to J_X^d$ for d sufficiently large, but the Abel maps for small values of d are also of interest. The Abel map of degree $d = g$ is particularly interesting because $X^{(g)} \to J_X^g$ is birational. We can thus describe the Jacobian J_X^g in terms of the symmetric power $X^{(g)}$ and the exceptional locus of the Abel map $A\colon X^{(g)} \to J_X^g$.

The Abel map of degree $d = g - 1$ is also distinguished.

Definition 1.0.6. The image of $A\colon X^{(g-1)} \to J_X^{g-1}$ is the **theta divisor** Θ.

The map $A\colon X^{(g-1)} \to J_X^{g-1}$ is birational onto its image, so $\Theta \subset J_X^{g-1}$ is a subscheme of codimension 1, hence is a locally principal divisor that can be shown to be ample. Historically, the geometric study of Jacobian varieties has been dominated by the study of their theta divisors. We conclude this section by showing how the geometry of Θ encodes the geometry of J_X^d by describing the theta divisor for curves of genus at most 4.

Somewhat more generally, we will describe the Abel maps

$$A\colon X^{(g)} \to J_X^g \text{ and}$$
$$A\colon X^{(g-1)} \to \Theta.$$

The locus $C_d^1 \subset X^{(d)}$ of points $[D]$ satisfying $\dim |D| \geq 1$ is closed, and the restriction of the Abel map to the complement $X^{(d)} \setminus C_d^1$ is an isomorphism onto its image. We will describe the structure of the Abel map by describing the subset $C_d^1 \subset J_X^d$ and the contraction $C_d^1 \to J_X^d$.

We derive such a description using the Riemann–Roch Formula. By that formula, an effective divisor D of degree $d = g-1$ satisfies $\dim |D| \geq 1$ (i.e. $[D] \in C_{g-1}^1$) if and only if $\dim |\operatorname{adj} D| \geq 1$ or equivalently there are two distinct canonical divisors that contain D. Similarly, an effective divisor D of degree $d = g$ satisfies $\dim |D| \geq 1$ if and only if $|\operatorname{adj} D| \geq 0$ or equivalently some canonical divisor contains D.

These observations demonstrate the key role played by the canonical divisor K in describing C_d^1. The canonical divisor of a low genus curve is well-understood. Indeed, effective canonical divisors are the preimages of hyperplanes under the canonical map, and the canonical map of a curve of genus at most 4 is computed in [**Har77**, Chap. 4, Sect. 6]. We now describe the Jacobian of a low genus curve.

Example 1.0.7 (Genus 1). The canonical divisor of a genus 1 curve X is trivial. In particular, $C_g^1 = \emptyset$, and so the Abel map $A\colon \operatorname{Hilb}_X^1 = X \to J_X^1$ is an isomorphism. Now J_X^1 does not have natural group structure, but J_X^0 does, so if we fix a point $p_0 \in X$, then we can identify $X = J_X^1$ with J_X^0 by tensoring with $\mathcal{O}_X(-p_0)$, and this identification makes X into a group with identity p_0. In a different form, this group law is often introduced in a first course in algebraic geometry. The map associated to the complete linear system $|3p_0|$ embeds X as a cubic curve in the plane, and the group law coming from the isomorphism $X \cong J_X^0$ is the group law that is defined using the tangent-chord construction (as in, say, [**Har77**, p. 321]).

The theta divisor $\Theta \subset J_X^0$ is not very interesting. The only effective degree 0 divisor is the empty divisor, so we have $\Theta = \{[\mathcal{O}_X]\}$.

Example 1.0.8 (Genus 2). Every genus 2 curve is hyperelliptic, and the effective canonical divisors K are the fibers of the degree 2 map $f\colon X \to \mathbb{P}^1$ to the line. These fibers $K = f^{-1}(t)$ are exactly the effective divisors of degree $g = 2$ that move in a positive dimensional linear system. Indeed, if $[D] \in C_g^1$, then, as we observed earlier, D is contained in a canonical divisor, and hence equal to a canonical divisor by degree considerations. This classification shows that C_g^1 is a rational curve

$$\mathbf{P}^1 = C_g^1 \subset X^{(g)},$$

and this curve is contracted to a point by the Abel map.

What about $\Theta \subset J_X^{g-1}$? No degree $1 = g-1$ divisor D can satisfy $\dim |D| \geq 1$. Indeed, again using the observations we made after stating the Riemann–Roch Formula, we see that any such divisor would be contained in two distinct fibers of f, which is absurd. We can conclude that the Abel map $A\colon X^{(g-1)} = X \to J_X^{g-1}$ is injective, so the theta divisor is

$$X = \Theta \subset J_X^{g-1},$$

an embedded copy of the curve.

Example 1.0.9 (Genus 3). The genus 3 curves fall into two classes: the hyperelliptic curves and the non-hyperelliptic curves. We will first consider the case of hyperelliptic curves, which are similar to the genus 2 curves that we just discussed.

Let X be a genus 3 hyperelliptic curve with degree 2 map $f\colon X \to \mathbb{P}^1$ to the line. The effective canonical divisors of X are the divisors of the form $K = f^{-1}(t_1) + f^{-1}(t_2)$ for $t_1, t_2 \in \mathbb{P}^1$. From this description, we can conclude that the elements $[D]$ of C_g^1 are the divisors of the form $f^{-1}(t_0) + p_0$ for $t_0 \in \mathbf{P}^1$ and $p_0 \in X$. Furthermore, these divisors satisfy

$$\dim |f^{-1}(t_0) + p_0| = 1$$

by the Riemann–Roch Formula (as there is a unique canonical divisor containing $f^{-1}(t_0) + p_0$). Indeed, the divisors $f^{-1}(t) + p_0$ with $t \in \mathbb{P}^1$ exhaust the effective divisors linearly equivalent to $f^{-1}(t_0) + p_0$.

How does this classification of divisors translate into a description of $A\colon X^{(g)} \to J_X^g$? The exceptional locus C_g^1 is isomorphic to the surface $X \times \mathbf{P}^1$, and the Abel map collapses this surface to the curve X by projection onto the first factor.

What about the theta divisor $\Theta \subset J_X^{g-1}$? The exceptional locus C_{g-1}^1 of $A\colon X^{(2)} \to \Theta$ is the locus of effective degree $g-1 = 2$ divisors D that are contained in two distinct canonical divisors. From our description of K, we see that these are exactly the divisors of the form $f^{-1}(t)$, $t \in \mathbf{P}^1$, and all these divisors are linearly equivalent. In other words, the exceptional locus is a rational curve

$$\mathbf{P}^1 = C^1_{g-1} \subset X^{(g-1)}$$

that is contracted to a point. Set-theoretically, this is the same as the description of the degree g Abel map $X^{(g)} \to J^g_X$ of a genus 2 curve. There is, however, an important difference: when X is a genus 2 curve, the image of the rational curve $\mathbb{P}^1 \subset X^{(2)}$ is a smooth point, but when X is a genus 3 hyperelliptic curve, the image is a singularity of Θ. This is a consequence of a general result — the Riemann Singularity Theorem — that computes the multiplicity of Θ as

$$\operatorname{mult}_{[L]} \Theta = h^0(X, L).$$

Two proofs of this result can be found in [**ACGH85**, Chap. 6].

What about non-hyperelliptic curves? The canonical map of a non-hyperelliptic curve X of genus 3 embeds X as a degree 4 plane curve $X \subset \mathbb{P}^2$, and the effective canonical divisors are just the divisors that are the intersection of X with a line $\ell \subset \mathbb{P}^2$. From this description of the canonical divisors, we see that $[D] \in C^1_g$ if and only if D lies on a line ℓ (which is necessarily unique). We can define a map $\pi\colon C^1_g \to X$ as follows. Given $[D] \in C^1_g$, there is a unique line ℓ containing D, and we can write $\ell \cap X = D + q_0$ for some point q_0 of X. We set $\pi([D]) = q_0$.

The fiber $\pi^{-1}(q_0)$ of π over a point is a $\mathbf{P}^1$, the projective line parameterizing lines $\ell \subset \mathbf{P}^2$ containing q_0. In particular, we see that C^1_g is a surface. The Abel map $A\colon X^{(g)} \to J^g_X$ contracts the fibers of π, so the image of C^1_g is a curve:

$$X = A(C^1_g) \subset J^g_X.$$

We now turn our attention to the theta divisor. We have $C^1_{g-1} = \emptyset$. Indeed, no effective degree $g-1 = 2$ divisor is contained in two distinct canonical divisors because two distinct lines meet in a single point. In particular, the theta divisor is a smooth projective surface:

$$X^{(g-1)} = \Theta \subset J^{g-1}_X.$$

This is a special case of Marten's Theorem. That theorem states that if X is a curve of genus g, $g \geq 3$, then we have

$$\dim \Theta_{\text{sing}} = \begin{cases} g-3 & \text{if } X \text{ is hyperelliptic;} \\ g-4 & \text{otherwise.} \end{cases}$$

This statement, along with various generalizations, is proven in [**ACGH85**, Chap. 4, Sect. 5].

Example 1.0.10 (Genus 4). The structure of the Abel map of a genus 4 hyperelliptic curve is similar to the structure of the Abel map of a genus 3 hyperelliptic curve, so we will only discuss the non-hyperelliptic case. The canonical map of a non-hyperelliptic genus 4 curve X embeds the curve in space $X \subset \mathbb{P}^3$. As a space curve, X is the complete intersection of a (non-unique) cubic hypersurface and a (unique) quadric surface, which we denote by Q. The quadric surface Q is either smooth or a cone over a plane curve of degree 2. The shape of the quadric influences the structure of the Abel map, so we consider these two cases separately.

Suppose first that Q is smooth. Then the quadric Q must be isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ embedded by the complete linear system $|\mathcal{O}(1,1)|$. The description of C^1_g is similar to the description we gave for a non-hyperelliptic genus 3 curve. By the Riemann–Roch Formula, the exceptional locus $C^1_g \subset X^{(g)}$ of $A\colon X^{(g)} \to J^g_X$ is the locus of effective degree 4 divisors that are contained in a hyperplane $h \subset \mathbf{P}^3$, which is necessarily unique (for otherwise D would be contained in a line that lies on Q by degree considerations, but the intersection of a line lying on Q with X is a divisor of degree 3). We describe C^1_g by constructing a map $\pi\colon C^1_g \to X^{(2)}$. Given $[D] \in C^1_g$, let $h \subset \mathbf{P}^3$ be the unique hyperplane containing D. The intersection $h \cap X$ is a degree 6 effective divisor that we can write as $h \cap X = D + E$ for some effective divisor E of degree 2. We set $\pi([D]) = [E]$. The fibers of the resulting map $\pi\colon C^1_g \to X^{(2)}$ are $\mathbf{P}^1$'s, so C^1_g is a smooth 3-fold. The Abel map contracts C^1_g to the surface

$$X^{(2)} = A(C^1_g) \subset J^g_X$$

by contracting each fiber $\pi^{-1}([E])$ to a point.

What about the degree $3 = g-1$ Abel map $A\colon X^{(g-1)} \to J^{g-1}_X$? If $[D] \in X^{(g-1)}$ satisfies $\dim |D| \geq 1$, then D lies on two distinct hyperplane sections, and hence lies on their intersection which is a line $\ell \subset \mathbf{P}^3$. We can construct divisors satisfying this condition by using the geometry of the quadric surface Q. Given a line ℓ on the quadric surface, the intersection $D := \ell \cap X$ is a degree 3 effective divisor, so $[D] \in C^1_{g-1}$. In fact, these points exhaust C^1_{g-1}. If $[D] \in C^1_{g-1}$, then the unique line ℓ containing D must be contained in Q because otherwise $Q \cap \ell$ would be a degree 2 closed subscheme that contains the degree 3 effective divisor D, which is absurd. The lines on Q consist of two 1-dimensional linear systems (the lines $\{t\} \times \mathbf{P}^1$ and the lines $\mathbf{P}^1 \times \{t\}$). We thus have

$$\mathbf{P}^1 \cup \mathbf{P}^1 = C^1_{g-1} \subset X^{(3)}.$$

Each curve is contracted to a point of Θ which is a singularity. This shows that Θ is singular at exactly two points.

What about the case where the quadric Q containing X is singular? The structure of the degree g Abel map $A\colon X^{(g)} \to J^g_X$ is as before; the map contracts a threefold $C^1_g \subset X^{(g)}$ that is a $\mathbf{P}^1$-bundle over $X^{(2)}$. The structure of the degree $g-1$ Abel map $A\colon X^{(g-1)} \to \Theta$, however, is different. The argument used in the previous case remains valid except now Q contains only one 1-dimensional linear system of lines. Recall that Q is the cone over a smooth plane quadric $Y \subset \mathbf{P}^3$. The lines ℓ on Q are the lines that join a point on Y to the vertex of the cone. Thus

$$\mathbf{P}^1 = C^1_g \subset X^{(g-1)}.$$

This curve is contracted to the unique singularity of Θ. (Warning: We have only defined C^1_g as a set, and the locus naturally has non-reduced scheme structure.)

This completes our discussion of Jacobians of smooth curves. The remainder of the article is devoted to extending this theory of the Jacobian to singular curves.

2. Generalized Jacobians of Singular Curves

How should the Jacobian of a singular curve be defined? One approach is to simply repeat Definition 1.0.1, which is the definition of the Jacobian of a smooth curve.

DEFINITION 2.0.11. Given a curve X, the **generalized Jacobian functor** $J^{d,\sharp}_X$ of degree d is defined to be the étale sheaf associated to the functor that assigns to a k-scheme T the set of isomorphism classes of lines bundles L on X_T that have the property the restriction of L to any fiber of $X_T \to T$ has degree d. The **generalized Jacobian variety** is the k-scheme that represents $J_X^{d,\sharp}$.

This generalized Jacobian can be constructed by, for example, using Artin's Criteria. However, the generalized Jacobian has a major deficiency: it is not proper.

Consider the generalized Jacobian of a genus 1 curve X with a node $p_0 \in X$. We observed in Example 1.0.7 that the degree 1 line bundles on a smooth genus 1 curve are all of the form $\mathcal{L}(p)$ for $p \in X$ a point, and this fact remains valid for X provided we require that p lies in the smooth locus. Thus we have

$$\begin{aligned} J_X^1 =& X \setminus \{\text{node}\} \\ \cong& \mathbf{P}^1 \setminus \{0, \infty\}. \end{aligned}$$

In particular, J_X^1 is not proper. This suggests a question: how to compactify J_X^1 to a proper k-scheme?

We answer this question in Section 3. Before studying compactifications of J_X^d, let's first examine the structure of the scheme. The generalized Jacobian J_X^0 is a smooth connected g-dimensional quasi-projective variety that admits a group scheme structure coming from the tensor product. This group structure can be described in terms of the singularities of X.

When $k = \mathbf{C}$, we described the group of k-points $J_X^0(k)$ of the Jacobian cohomologically as a kernel in Eq. (2). In that displayed equation, the terms in the exact sequence were cohomology groups that we interpreted as sheaf cohomology computed with respect to the analytic (or classical) topology and the curve X was a smooth curve. However, our computation remains valid if we let X be a singular curve over an arbitrary k provided we replace the analytic topology with the étale topology. Let us explain this.

In what follows, we only use the very basic properties of the étale topology, and the reader unfamiliar with this formalism is directed to [**BLR90**, Sect. 8.1]. We write $T_{\text{ét}}$ for the étale site of a scheme. Given a smooth curve X over $k = \mathbf{C}$, the isomorphism Eq. (2) is constructed by first constructing an isomorphism between the group of line bundles of arbitrary degree and the group $H^1(X_{\text{an}}, \mathcal{O}_X^*)$ and then observing that this isomorphism identifies the degree map on line bundles with the Chern class map $c_1 \colon H^1(X_{\text{an}}, \mathcal{O}_X^*) \to H^2(X_{\text{an}}, \mathbf{Z})$. These facts remain valid when X has singularities and $k = \mathbf{C}$. If k is an arbitrary field, then X_{an} does not make sense, but there is an isomorphism between the group of line bundles and the étale cohomology group $H^1(X_{\text{ét}}, \mathcal{O}_X^*)$ that identifies the degree map with a map valued in $H^2(X_{\text{ét}}, \mathbf{Z}_\ell(1))$ (for ℓ a prime distinct from the characteristic). More generally, if we define $\operatorname{Pic}^\sharp_{X/k}$ to be the disjoint union of all the $J_X^{d,\sharp}$'s, then this functor is isomorphic to the 1st higher direct image of $\mathcal{O}_X^*$ under the structure morphism $X_{\text{ét}} \to k_{\text{ét}}$. This result is due to Grothendieck, and a recent discussion of the identification can be found in [**Kle05**, Sect. 2].

The reason for introducing this cohomological formalism is that it makes it easy to relate the generalized Jacobian J_X^0 of X to the Jacobian $J_{X^\nu}^0$ of the normalization X^ν. Let $\nu \colon X^\nu \to X$ be the normalization map. The homomorphism $\nu^{-1}\mathcal{O}_X^* \to \mathcal{O}_{X^\nu}$ given by pulling back by ν is adjoint to a homomorphism $\mathcal{O}_X^* \to \nu_*(\mathcal{O}_{X^\nu}^*)$ that is injective and an isomorphism away from the singular locus. The cokernel $\mathcal{F}$ is supported on X_{sing} and fits into a short exact sequence

$$0 \to \mathcal{O}_X^* \to \nu_*(\mathcal{O}_{X^\nu}^*) \to \mathcal{F} \to 0. \tag{4}$$

Consider the associated long exact sequence relating the higher direct images of these sheaves under $X_{\text{ét}} \to k_{\text{ét}}$. We just explained that the 1st direct image of $\mathcal{O}_X^*$ is $\mathrm{Pic}^\sharp_{X/k}$. Similarly, the 1st direct image of $\nu_*\mathcal{O}_X^*$ is $\mathrm{Pic}^\sharp_{X^\nu/k}$ as ν is finite. The natural map $\mathrm{Pic}^\sharp_{X/k} \to \mathrm{Pic}^\sharp_{X^\nu/k}$ is surjective as the cokernel injects into the 1st direct image of $\mathcal{F}$, which is zero as $\mathcal{F}$ has 0-dimensional support. For the same reason, the direct image of $\mathcal{F}$ is the locally constant sheaf F associated to the group $H^0(X_{\text{ét}}, \mathcal{F})$. The connecting homomorphism $F \to \mathrm{Pic}^\sharp_{X/k}$ is injective as $\mathcal{O}_X^* \to \nu_*\mathcal{O}_{X^\nu}^*$ is easily seen to induce an isomorphism on direct images (the only global functions on X, X^ν are constants). To summarize, we can extract from the long exact sequence on higher direct images a short exact sequence

$$0 \to F \to \mathrm{Pic}^\sharp(X/k) \to \mathrm{Pic}^\sharp(X^\nu/k) \to 0. \tag{5}$$

As $\mathrm{Pic}^\sharp(X/k) \to \mathrm{Pic}^\sharp(X^\nu/k)$ respects degree maps, we can also write

$$0 \to F \to J_X^0 \to J_{X^\nu}^0 \to 0. \tag{6}$$

This is the desired description of the generalized Jacobian J_X^0: it is an extension of the abelian variety $J_{X^\nu}^0$ by the commutative group variety F Let us examine the structure of F.

Label the singularities of X as $q_1, \dots, q_n$ and then label the points of the fiber $\nu^{-1}(q_i)$ as $p_{i,j}$, $j = 1, \dots, m_i$. We defined F to be the locally constant étale sheaf associated to $H^0(X_{\text{ét}}, \mathcal{F})$, and this group of sections is isomorphic to the direct sum of the stalks of $\mathcal{F}$:

$$H^0(X_{\text{ét}}, \mathcal{F}) = \bigoplus (\mathcal{O}_{X,p_{i,1}}^* \oplus \cdots \oplus \mathcal{O}_{X,p_{i,m_i}}^*)/\mathcal{O}_{X,q_i}^*. \tag{7}$$

Here the stalks are taken with respect to the étale topology, and the group $\mathcal{O}_{X,q_i}^*$ is embedded diagonally.

The quotients appearing in Eq. (7) are unchanged if we replace the unit groups $\mathcal{O}_{X,q_i}^*$ and $\mathcal{O}_{X^\nu,p_{i,j}}^*$ with the unit groups $\widehat{\mathcal{O}}_{X,q_i}^*$ and $\widehat{\mathcal{O}}_{X^\nu,p_{i,j}}^*$ in the appropriate completions. In particular, the structure of F depends only on the analytic type of the singularities of X. Let's now compute F for some specific singularities.

Example 2.0.12 (The Node). The node $\mathcal{O} = k[[x,y]]/(xy)$ has normalization isomorphic to the product $\widetilde{\mathcal{O}} = k[t]] \times k[[t]]$. An isomorphism

$$\widetilde{\mathcal{O}}^*/\mathcal{O}^* \cong k^*$$

is given by the rule $(f,g) \in \widetilde{\mathcal{O}}^* \mapsto f(0)/g(0)$. In other words, the associated algebraic group is the multiplicative group $\mathbf{G}_m$. More generally, if X is a nodal curve, then $F = \ker(J_X^0 \to J_{X^\nu}^0)$ is a multiplicative torus $\mathbf{G}_m^\delta$ of dimension equal to the number of nodes δ.

Example 2.0.13 (The Cusp). If $\mathcal{O} = k[[x,y]]/(y^2 - x^3)$ is the cusp, then the quotient $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is isomorphic to the additive group k^+. In particular, if X is a curve with only cusps as singularities, then F is an additive torus $\mathbf{G}_a^\delta$ of dimension equal to the number of cusps δ.

Example 2.0.14 (The Tacnode). Like the node, the normalization of the tacnode $\mathcal{O} = k[[x,y]]/(y^2 - x^4)$ is the product $\widetilde{\mathcal{O}} = k[[t]] \times k[[t]]$ of two power series rings. An isomorphism $\widetilde{\mathcal{O}}^*/\mathcal{O}^* \cong k^* \oplus k^+$ is given by the rule $(f,g) \mapsto (f(0)/g(0), f'(0)/f(0) - g'(0)/g(0))$. Here $f'(t)$ and $g'(t)$ are the formal derivatives. In particular, each tacnode of X contributes a factor of $\mathbf{G}_m \times \mathbf{G}_a$ to F.

This concludes our discussion of examples. A more detailed description of F can be found in [**BLR90**, Sect. 9.1]. Rather than discussing that description, we turn our attention to the problem of compactifying the generalized Jacobian.

3. Compactified Jacobians of Singular Curves

In the special case of the genus 1 nodal curve X from the beginning Section 2, it is clear how to compactify the associated generalized Jacobian J^1_X. We described that generalized Jacobian as

$$J^1_X \cong X \setminus \{\text{node}\}, \tag{8}$$

and we can construct a compactification $\bar{J}^1_X \supset J^1_X$ by defining

$$\bar{J}^1_X \cong X. \tag{9}$$

In this section, we will describe how to interpret this compactification $\bar{J}^1_X$ in terms of moduli and then how to generalize that interpretation to arbitrary curves.

We can give a moduli-theoretic interpretation of the compactification (9) as follows. Let $p_0 \in X$ be the node. In the isomorphism (8), a point $p \in X \setminus \{p_0\}$ corresponds to $[\mathcal{L}(p)] \in J^1_X$, where $\mathcal{L}(p)$ is the $\mathcal{O}_X$-linear dual of the ideal I_p of p. When $p = p_0$, the ideal I_{p_0} is not a line bundle, but we can still define $\mathcal{L}(p_0)$ by $\mathcal{L}(p_0) := \underline{\mathrm{Hom}}(I_{p_0}, \mathcal{O}_X)$. The compactification $\bar{J}^1_X$ parameterizes degree 1 line bundles on X and the sheaf $\mathcal{L}(p_0)$. Both the line bundles $\mathcal{L}(p)$ and the sheaf $\mathcal{L}(p_0)$ are examples of rank 1, torsion-free sheaves, a class of sheaves that we now define.

Definition 3.0.15. A coherent sheaf I on a curve X is **torsion-free** if for every local section f of $\mathcal{O}_X$ and every local section s of I satisfying $f \cdot s = 0$, we have $f = 0$ or $s = 0$. The sheaf I is said to be **rank** 1 if there exists a dense open subset $U \subset X$ such that $I|_U$ is isomorphic to $\mathcal{O}_U$.

The degree of a rank 1, torsion-free sheaf is defined as follows.

Definition 3.0.16. If I is a rank 1, torsion-free sheaf, then the **degree** $\deg(I)$ of I is defined by

$$\deg(I) := \chi(I) - \chi(\mathcal{O}_X).$$

The compactified Jacobian is defined in the expected manner.

Definition 3.0.17. The degree d **compactified Jacobian functor** $\bar{J}^{d,\sharp}_X$ is defined to be the étale sheaf associated to the functor that assigns to a k-scheme T the set of isomorphism classes of $\mathcal{O}_T$-flat, finitely presented $\mathcal{O}_{X_T}$-modules I on X_T that have the property that the restriction of I to any fiber of $X_T \to T$ is a rank 1, torsion-free sheaf of degree d. The degree d **compactified Jacobian** is the k-scheme that represents $\bar{J}^{d,\sharp}_X$.

By [**AK80**, Thm. 8.1], the compactified Jacobian exists as a projective k-scheme. Since every line bundle is a rank 1, torsion-free sheaf, we have $J^d_X \subset \bar{J}^d_X$, and so $\bar{J}^d_X$ compactifies the generalized Jacobian J^d_X. This compactification is particularly well behaved when X has only planar singularities. For such a curve, Altman–Iarrobino–Kleiman have proven that the compactified Jacobian $\bar{J}^d_X$ is an irreducible variety that contains the generalized Jacobian as a dense open subset [**AIK77**].

The compactified Jacobian has undesirable properties when X has a non-planar singularity. In this case, Kleiman–Kleppe [**KK81**] have proven that $\bar{J}^d_X$ has at least two irreducible components. The generalized Jacobian is contained in a single irreducible component, so the compactified Jacobian of a curve with a non-planar singularity has the undesirable property of having extra components.

The compactified Jacobian of a singular curve can be constructed using techniques similar to those used to construct the Jacobian of a smooth curve. In Section 1, we sketched two different proofs that the Jacobian of a non-singular curve exists. The first proof used Artin's Criteria to construct the Jacobian. The author expects this proof can be modified to prove the existence of the compactified Jacobian of a singular curve (but additional care must be taken in showing that the relevant algebraic space is actually a scheme — the compactified Jacobian is not a group scheme).

In any case, we are more interested in generalizing the second construction of the Jacobian. In the second construction, given a smooth curve X, we fixed a sufficiently large integer d and then constructed J^d_X as the quotient of the symmetric power $X^{(d)}$ by the relation of linear equivalence. The quotient scheme exists essentially because the relevant equivalence classes are $\mathbf{P}^{d-g}$'s, and the quotient map $X^{(d)} \to J^d_X$ is then the Abel map.

In [**AK80**], Altmann–Kleiman construct the compactified Jacobian by extending the theory of linear equivalence and of the Abel map to singular curves. This extension is, however, non-trivial. As we now demonstrate, the most naive approach to constructing the Abel map of a singular curve fails.

Given a singular curve X, let $X^{(d)}_{\mathrm{sm}}$ denote the d-th symmetric power of the smooth locus of X. Applied to $X^{(d)}_{\mathrm{sm}}$, the construction from the proof of Lemma 1.0.4 produces a regular map

$$A\colon X^{(d)}_{\mathrm{sm}} \to J^d_X$$

or equivalently a rational map

$$A\colon X^{(d)} \dashrightarrow \bar{J}^d_X.$$

However, this second rational map may not be a regular map; the locus of indeterminacy may be non-empty. We show this by example.

EXAMPLE 3.0.18. We exhibit a singular curve X with the property that the map $A\colon X^{(2)} \dashrightarrow \bar{J}^2_X$ is not a regular map. Let X be a general genus 2 curve that has a single node $p_0 \in X$. That is, let X be the projective curve that contains $U := \mathrm{Spec}(k[x,y]/(y^2 - x^2(x-a_1)\dots(x-a_4))$ (for some distinct, nonzero constants $a_1, \dots, a_4 \in k$) and is smooth away from U. The maximal ideal $(x,y) \subset \mathcal{O}_U$ corresponds to the node $p_0 \in X$. The curve X admits a degree 2 morphism $\pi\colon X \to \mathbf{P}^1$ that extends the ring map $k[t] \to \mathcal{O}_U$ defined by $t \mapsto x$. We will show that $A\colon X^{(2)} \dashrightarrow \bar{J}^2_X$ is undefined at the point $[2p_0] \in X^{(2)}$.

Our strategy is to construct two maps $f_1, f_2\colon \mathrm{Spec}(k[[t]]) \to X^{(2)}$ out of the formal disc with the property that the closed point $0 \in \mathrm{Spec}(k[[t]])$ is sent to $[2p_0]$ and the generic point does not map into the locus of indeterminacy of A. Since A is well-defined at the image of the generic point, the composition $A \circ f_i\colon \mathrm{Spec}(\mathrm{Frac}\, k[[t]]) \to \bar{J}^2_X$ is defined and hence extends to a morphism $\mathrm{Spec}(k[[t]]) \to \bar{J}^2_X$ by the valuative criteria of properness. Write $\widetilde{g}_i$ for this extension. We will show directly that $\widetilde{g}_1(0) \neq \widetilde{g}_2(0)$. However, if A was a regular map, then we would have $\widetilde{g}_1(0) = A([2p_0])$ and $\widetilde{g}_2(0) = A([2p_0])$, which is absurd.

We now construct the maps $f_1, f_2\colon \mathrm{Spec}(k[[t]]) \to X^{(2)}$ by exhibiting corresponding algebra maps $(\mathcal{O}_U \otimes \mathcal{O}_U)^{\mathrm{Sym}_2} \to k[[t]]$. Set $\sqrt{(t-a_1)\dots(t-a_4)}$ equal to the power series that is the usual Taylor series expression for the square root of $(t-a_1)\dots(t-a_4)$. (Choose $\sqrt{a_i}$ for $i=1,\dots,4$ arbitrarily.)

The map f_1 is defined to be the regular map that corresponds to the algebra homomorphism $(\mathcal{O}_U \otimes \mathcal{O}_U)^{\mathrm{Sym}_2} \to k[[t]]$ that is the restriction of the map $\mathcal{O}_U \otimes \mathcal{O}_U \to k[[t]]$ defined by

$$\begin{aligned} x \otimes 1 &\mapsto 0, \\ y \otimes 1 &\mapsto 0, \\ 1 \otimes x &\mapsto t, \\ 1 \otimes y &\mapsto t\sqrt{(t-a_1)\dots(t-a_4)}. \end{aligned}$$

Intuitively, this is the formal arc that sends a parameter t to an unordered pair of points that consists of p_0 and a point in X that tends to p_0 as t tends to 0.

Similarly, we define f_2 to be the regular map that corresponds to the algebra homomorphism defined by

$$\begin{aligned} x \otimes 1 &\mapsto t, \\ y \otimes 1 &\mapsto t\sqrt{(t-a_1)\dots(t-a_4)}, \\ 1 \otimes x &\mapsto t, \\ 1 \otimes y &\mapsto -t\sqrt{(t-a_1)\dots(t-a_4)}. \end{aligned}$$

Intuitively, this is the formal arc that sends the parameter value t to the two points in $\pi^{-1}(t)$.

Both maps have the property that 0 maps to $[2p_2]$ and the generic point maps to a point where $X^{(d)} \dashrightarrow \bar{J}^2_X$ is regular. To complete this example, we need to show that $\widetilde{g}_1(0) \neq \widetilde{g}_2(0)$.

By construction, the composition $\operatorname{Spec}(\operatorname{Frac} k[[t]]) \xrightarrow{f_1} X^{(2)} \overset{A}{\dashrightarrow} \bar{J}^2_X$ corresponds to the line bundle on $X \otimes \operatorname{Frac} k[[t]]$ that is the pullback of $\mathcal{O}(1)$ under $\pi \otimes 1\colon X \otimes \operatorname{Frac} k[[t]] \to \mathbf{P}^1 \otimes \operatorname{Frac} k[[t]]$. This line bundle extends to the pullback of $\mathcal{O}(1)$ under $\pi \otimes 1\colon X \otimes k[[t]] \to \mathbf{P}^1 \otimes k[[t]]$, and so the extension $\widetilde{g}_1$ of $A \circ f_1$ must satisfy $\widetilde{g}_1(0) = [\pi^*(\mathcal{O}(1))]$. The composition $A \circ f_2\colon \operatorname{Spec}(\operatorname{Frac} k[[t]]) \to \bar{J}^2_X$ corresponds to a sheaf that fails to be locally free. This property persists under specialization, so if $\widetilde{g}_2(0) = [I]$, then I must fail to be locally free. In particular, $\widetilde{g}_2(0) \neq \widetilde{g}_1(0)$. We can conclude that $A\colon X^{(2)} \to \bar{J}^2_X$ is not a regular map.

The above example shows that, to define a suitable Abel map for a singular curve, we must replace $X^{(d)}$ with a blow-up that resolves the indeterminacy of $X^{(d)} \dashrightarrow \bar{J}^d_X$. In Section 5, we resolve this indeterminacy by a blow-up that is a moduli space that parameterizes generalized divisors, objects we discuss in the next section.

4. Generalized Divisors

Here we develop the theory of generalized divisors in analogy with the theory of Cartier divisors. Recall that on a smooth curve a line bundle is equivalent to a linear equivalence class of Cartier divisors (see Sect. 1). Our goal is to define more general divisors on a singular curve so that a rank 1, torsion-free sheaf is equivalent to a linear equivalence class of these more general divisors. We define two types of divisors that generalize Cartier divisors: generalized divisors and generalized ω-divisors. On a Gorenstein curve, generalized divisors are essentially equivalent to generalized ω-divisors, but the two divisors are fundamentally different on a non-Gorenstein curve, and it is only generalized ω-divisors that behave as one expects by analogy with Cartier divisors. The material in this section is derived from Hartshorne's papers [**Har86**], [**Har94**], and [**Har07**].

4.1. Generalized Divisors. Let X be an integral curve. We begin by defining a generalized divisor on X.

Definition 4.1.1. Let $\mathcal{K}$ denote the sheaf of total quotient rings of X. That is, $\mathcal{K}$ is the field of rational functions, considered as a locally constant sheaf. A **generalized divisor** D on X is a nonzero subsheaf $I_D \subset \mathcal{K}$ that is a coherent $\mathcal{O}_X$-module. If additionally I_D is a line bundle, then we say that D is a **Cartier divisor**. We say that a generalized divisor D is **effective** if $I_D \subset \mathcal{O}_X$.

An effective generalized divisor on X is just a 0-dimensional closed subscheme $Z \subset X$. In particular, every closed point $p \in X$ defines a generalized divisor that, by abuse of notation, we denote by p. A second source of generalized divisors is rational functions. A rational function f generates a subsheaf $\mathcal{O}_X \cdot f \subset \mathcal{K}$, and we write $\text{div}(f)$ for the corresponding generalized divisor. The generalized divisor $\text{div}(1)$ is written 0.

Many of the familiar operations on Cartier divisors extend to generalized divisors.

Definition 4.1.2. The **sum** $D+E$ of two generalized divisors D and E is the subsheaf $I_{D+E} \subset \mathcal{K}$ generated by local sections of the form fg with $f \in I_D$ and $g \in I_E$. The **minus** $-D$ of a generalized divisor D is the subsheaf $I_{-D} \subset \mathcal{K}$ whose local sections are elements $f \in \mathcal{K}$ satisfying $f \cdot I_D \subset \mathcal{O}_X$.

Remark 4.1.3. Our definitions of sum and minus are different from the definitions found on [**Har07**, p. 88]. There Hartshorne defines I_{D+E} to be the ω-reflexive hull of the module generated by the elements fg rather than the module itself and similarly with I_{-D}. His definition is, however, equivalent to the one just given because the module generated by the fg's is ω-reflexive by Lemma 4.2.7 below.

The sum operation is easily seen to make the set of generalized divisors into a commutative monoid with identity 0. More precisely, properties of sum and minus are summarized by the lemma below.

Lemma 4.1.4. *The sum and minus operations have the following properties:*

(a) *sum is associative and commutative;*
(b) $D+0=D$*;*
(c) $D+-D=0$ *provided D is Cartier;*
(d) $-(D+E)=-D+-E$ *provided E is Cartier.*

Proof. This is [**Har07**, Prop. 2.2]. □

In the last two parts of Lemma 4.1.4, it is necessary to assume that one of the divisors is Cartier. The sum operation is not well-behaved when applied to non-Cartier divisors, as is shown by the examples in [**Har94**, Sect. 3].

We now define linear equivalence.

Definition 4.1.5. Two generalized divisors D and E are **linearly equivalent** if there exists a rational function f with $D = E + \mathrm{div}(f)$. Given D, the associated **complete linear system** $|D|$ is the set of all effective generalized divisors linearly equivalent to D.

Linear equivalence is immediately seen to define an equivalence relation on the set of generalized divisors. The lemma below provides an alternative characterization of linear equivalence.

Lemma 4.1.6. *The following relations between generalized divisors and rank* 1*, torsion-free sheaves hold:*

(a) *D is linearly equivalent to E if and only if I_D is isomorphic to I_E;*
(b) *every rank* 1*, torsion-free sheaf I is isomorphic to I_D for some generalized divsior D;*
(c) *$I_D \otimes I_E \cong I_{D+E}$ provided E is Cartier;*
(d) *$I_{-D} \cong \underline{\mathrm{Hom}}(I_D, \mathcal{O}_X)$.*

Proof. Conditions (a) and (b) are [**Har07**, Prop. 2.4], and the reader may find a proof of Condition (d) in [**Har94**, Lem. 2.2]. To prove Condition (c), it is enough to prove that the natural surjection $I_D \otimes I_E \to I_{D+E}$ is injective. Injectivity can be checked locally, so we may assume I_E is principal, in which case injectivity is immediate. □

To study $|D|$, we make the following definition.

Definition 4.1.7. Write $I^\vee := \underline{\mathrm{Hom}}(I, \mathcal{O}_X)$ for the **$\mathcal{O}_X$-linear dual** of a coherent $\mathcal{O}_X$-module I. The sheaf $\mathcal{L}(D)$ associated to a generalized divisor D is defined by $\mathcal{L}(D) := I_D^\vee$. An **adjoint generalized divisor** $\mathrm{adj}\, D$ of the generalized divisor D is a generalized divisor satisfying $\mathcal{L}(\mathrm{adj}\, D) = \underline{\mathrm{Hom}}(\mathcal{L}(D), \omega)$. A **canonical generalized divisor** K is an adjoint divisor of 0 (so $\mathcal{L}(K) = \omega$).

A canonical divisor exists when ω is reflexive (e.g. when X is Gorenstein). Indeed, by Lemma 4.1.6 we can write $I_K = \omega^\vee$ for some generalized divisor K. We then have $\mathcal{L}(K) = (\omega^\vee)^\vee$, which equals ω by reflexivity. Thus K is a canonical divisor.

When X is Gorenstein, ω is not only reflexive but in fact locally free, so K exists and is Cartier. The canonical divisor K is base-point free when $g \geq 1$ by [**Har86**, Thm. 1.6], and so it determines a **canonical map** $X \to \mathbf{P}H^0(X, \mathcal{L}(K))^\vee = \mathbf{P}^{g-1}$. The image of this morphism is a curve provided $g \geq 2$, and in this case, we define the image to be the **canonical curve**. There is a definition of the canonical curve and the canonical map of a non-Gorenstein curve (see e.g. [**KM09**]), but the definitions are more complicated, and we do not study them here.

The assumption that X is Gorenstein also implies that the adjoint divisor of a given divisor D exists as we can take $\mathrm{adj}\, D := K + -D$. Just as with a smooth curve, the elements of $|\mathrm{adj}\, D|$ are the preimages of hyperplanes in canonical space $\mathbf{P}^{g-1}$ that contain D.

When X is non-Gorenstein, an adjoint divisor $\mathrm{adj}\, D$ of a divisor D may not exist, and when it exists, it may not be unique even up to linear equivalence because we cannot recover $I_{\mathrm{adj}\, D}$ from its dual $\mathcal{L}(\mathrm{adj}\, D)$. We can, however, recover $I_{\mathrm{adj}\, D}$ when this sheaf is reflexive because then $I_{\mathrm{adj}\, D} \cong \mathcal{L}(\mathrm{adj}\, D)^\vee$. The following lemma shows that $I_{\mathrm{adj}\, D}$ is always reflexive when X is Gorenstein.

LEMMA 4.1.8. *If X is Gorensten, then every rank 1, torsion-free sheaf is reflexive. That is, the natural map $I \to (I^{\vee})^{\vee}$ is an isomorphism. In fact, we have*

$$D = -(-D)$$

for all generalized divisors D.

PROOF. This is [**Har86**, Lem. 1.1]. □

The lemma is false if we omit the hypothesis that X is Gorenstein.

EXAMPLE 4.1.9. We provide an example of a curve whose dualizing sheaf is not reflexive. Take X to be the curve in Example 6.0.15. This is a genus 2 curve with a unique singularity p_0 that is unibranched and non-Gorenstein.

We compute $\omega^{\vee}$ and $(\omega^{\vee})^{\vee}$. Let $\mathcal{K}_{\omega}$ denote the locally constant sheaf of rational 1-forms. Write ∂_t for the functional $(\mathcal{K}_{\omega})|_{X_1} \to \mathcal{K}|_{X_1}$ that sends dt to 1 and similarly for $\partial_s \colon (\mathcal{K}_{\omega})|_{X_2} \to \mathcal{K}|_{X_2}$. A computation shows that the image of any functional $\omega|_{X_1} \to \mathcal{O}_{X_1}$ is contained in the subsheaf of regular functions that vanish at the singularity p_0, so $\omega^{\vee}$ can be described as the sheaf generated by $t^6\partial_t, t^7\partial_t, t^8\partial_t$ on X_1 and by ∂_s on X_2. The same reasoning shows that $(\omega^{\vee})^{\vee}$ is generated by $dt/t^3, dt/t^2, dt/t$ on X_1 and by ds on X_2. In particular, $\omega \to (\omega^{\vee})^{\vee}$ is not an isomorphism because dt/t is not in the image.

The curve from the above example does not admit a canonical divisor. Indeed, we have seen that the reflexivity of ω is a sufficient condition for the existence of K, but it is also a necessary condition because every dual module is reflexive. As the dualizing module of the curve in Example 4.1.9 is not reflexive, this curve does not admit a canonical divisor.

We now develop the properties of complete linear systems. The sheaf $\mathcal{L}(D) = \underline{\mathrm{Hom}}(I_D, \mathcal{O}_X)$ has the property that the natural map

$$H^0(X, \underline{\mathrm{Hom}}(I_D, \mathcal{O}_X)) \to \mathrm{Hom}(I_D, \mathcal{O}_X)$$

is an isomorphism. (Indeed, this holds quite generally when I_D is replaced by an arbitrary coherent sheaf.) We use this observation in the following lemma to describe $|D|$ in terms of a cohomology group.

LEMMA 4.1.10. *Let D be a generalized divisor. Then the rule that sends a nonzero global section of $\mathcal{L}(D)$ to the image of the corresponding homomorphism $I_D \to \mathcal{O}_X$ defines a bijection*

$$\mathbf{P}H^0(X, \mathcal{L}(D)) \cong |D|. \tag{10}$$

PROOF. Under the hypothesis that X is Gorenstein, this is stated on [**Har86**, p. 378, top], but the fact remains valid when X is non-Gorenstein. To show that the map is well-defined, we need to show that any nonzero homomorphism $\phi \colon I_D \to \mathcal{O}_X$ is injective. Such a ϕ must be generically nonzero because $\mathcal{O}_X$ is torsion-free. Because both I_D and $\mathcal{O}_X$ are rank 1, we can conclude that ϕ is in fact generically an isomorphism. In particular, the kernel is supported on a proper closed subset of X. But the only such subsheaf of I_D is the zero subsheaf as I_D is torsion-free, so ϕ is injective.

Given that the map (10) is well-defined, it is immediate that the map is surjective. To show injectivity, we argue as follows. Suppose that we are given two nonzero global sections σ_1 and σ_2 that correspond to two homomorphisms

$\phi_1, \phi_2 \colon I_D \to \mathcal{O}_X$ with the same image. We have just shown that ϕ_2 is injective, so this homomorphism is an isomorphism onto its image. If we write $\phi_2^{-1} \colon \operatorname{im}(\phi_1) = \operatorname{im}(\phi_2) \to I_D$ for the inverse homomorphism, then $\alpha := \phi_2^{-1} \circ \phi_1$ satisfies $\phi_1 = \phi_2 \circ \alpha$. The only automorphisms of I_D are the maps given by multiplication by nonzero scalars by [**AK80**, Cor. 5.3, Lem. 5.4]. If α is given by multiplication by $c \in k^*$, then $\phi_1 = c \cdot \phi_2$ or equivalently $\sigma_1 = c \cdot \sigma_2$. In other words, σ_1 and σ_2 define the same point of $\mathbf{P}H^0(X, \mathcal{L}(D))$, showing injectivity. This completes the proof. □

Motivated by the previous lemma, we now study the cohomology of $\mathcal{L}(D)$. The most important numerical invariant controlling the cohomology is the degree.

DEFINITION 4.1.11. The **degree** $\deg(D)$ of a generalized divisor D is defined by $\deg(D) := -\deg(I_D)$.

LEMMA 4.1.12. *The degree function has the following properties:*

(a) *if D is an effective generalized divisor, then* $\deg(D)$ *is the length of* $\mathcal{O}_D$*;*
(b) $\deg(D + E) = \deg(D) + \deg(E)$ *provided E is Cartier;*
(c) $\deg(-D) = -\deg(D)$ *provided X is Gorenstein;*
(d) *if D is linearly equivalent to E, then* $\deg(D) = \deg(E)$.

PROOF. Property (a) follows from the additivity of the Euler characteristic χ, and Property (d) is immediate from Lemma 4.1.6(a). Property (b) follows from the more general identity $\chi(I_D \otimes I_E^{\otimes n}) = n \deg(I_E) + \chi(I_D)$ which is e.g. p. 295 and Corollary 2, p. 298 of [**Kle66**, Chap. 1]. Property (b) and coherent duality imply Property (c). Indeed, if X is Gorenstein, then $\underline{\operatorname{Hom}}(I_D, \omega) = I_D^{\vee} \otimes \omega$. By (b), $\deg(\omega) + \chi(I_D^{\vee}) = \chi(I_D^{\vee} \otimes \omega)$, and $\chi(\underline{\operatorname{Hom}}(I_D, \omega)) = -\chi(I_D)$ by coherent duality. We now deduce (c) by elementary algebra.

□

REMARK 4.1.13. When X is Gorenstein, Hartshorne defines the degree differently on [**Har86**, p. 2], but the two definitions coincide by [**Har94**, Prop. 2.16].

One consequence of Lemma 4.1.12(c) is that

$$\deg(\mathcal{L}(D)) = \deg(D)$$

when X is Gorenstein (as $I_{-D} = \mathcal{L}(D)$). When X is non-Gorenstein, this equality can fail, as the example below shows.

EXAMPLE 4.1.14. We give an example where $\deg(\mathcal{L}(D)) \neq \deg(D)$ or equivalently $\deg(-D) \neq -\deg(D)$. Consider the non-Gorenstein genus 2 curve X from Example 6.0.15 and the generalized divisor $D = p_0$ that is the singularity. We have $\deg(p_0) = 1$, but we claim $\deg(-p_0) = -2$. The sheaf I_{p_0} is the ideal generated by (t^3, t^4, t^5) on X_1 and by (1) on X_2. A direct computation shows that I_{-p_0} is the subsheaf of $\mathcal{K}$ generated by $(1, t, t^2)$ on X_1 and by (1) on X. The degree of $\mathcal{L}(p_0)$ is $2 = 1 + \deg(p_0)$, not $\deg(p_0)$. (To compute the degree, observe that $\mathcal{O}_X \subset I_{-D}$ has colength 2.)

While we always have $|D| = \mathbf{P}H^0(X, \mathcal{L}(D))$, the dimension of $|D|$ is only well-behaved when X is Gorenstein. First, we have the following form of the Riemann–Roch Formula.

PROPOSITION 4.1.15. *If X is Gorenstein and D is a generalized divisor, then we have*
$$\dim|D| - \dim|\operatorname{adj} D| = \deg(D) + 1 - g.$$

PROOF. This is [**Har86**, Thm. 1.3, 1.4], and with our definition of degree, the equation is a consequence of Eq. (15). □

We now use the Riemann–Roch Formula to compute $\dim|D|$.

COROLLARY 4.1.16. *Assume X is Gorenstein. Then the equation*
$$\dim|D| = \begin{cases} -1 & \text{if } d < g; \\ d-g & \text{if } d \geq g. \end{cases} \tag{11}$$
holds for every generalized divisor D of degree $d > 2g-2$.

Furthermore, if D is a divisor of degree $d \leq 2g-2$, then there exists a degree 0 Cartier divisor E such that $D+E$ satisfies Eq. (11).

REMARK 4.1.17. In Section 5, we will introduce the moduli space of effective generalized divisors of degree d, called the Hilbert scheme Hilb_X^d. The present corollary implies that the locus of divisors satisfying Eq. (11) is dense in Hilb_X^d (i.e. Eq. (11) is satisfied by a general effective D of degree d).

PROOF. Suppose first that D is a generalized divisor of degree $d > 2g-2$. Then the degree of $\operatorname{adj} D$ is $2g-2-d < 0$, and it follows from degree considerations that the only homomorphism $I_{\operatorname{adj} D} \to \mathcal{O}_X$ is the zero homomorphism. In other words, $|\operatorname{adj} D| = \emptyset$, and the claim immediately follows from the Riemann–Roch Formula.

Now suppose that D is a given generalized divisor of degree $d \leq 2g-2$. Choose an integer e large enough so that $d+e > 2g-2$ and then choose a Cartier divisor A that is the sum of e distinct points lying in the smooth locus X^{sm}. We claim that there exist distinct points $p_1, \ldots, p_e \in X^{\mathrm{sm}}$ such that $D + A - p_1 - \cdots - p_e$ satisfies Eq. (11).

We construct the p_i's by induction. It is enough to show that if B is a generalized divisor with $\dim|B| > 1$, then there exists a point $p \in X^{\mathrm{sm}}$ with $h^0(X, \mathcal{L}(B-p)) = h^0(X, \mathcal{L}(B)) - 1$. Fix a nonzero section $\sigma \in H^0(X, \mathcal{L}(B))$. Then the image $\sigma(p) \in k(p) \otimes \mathcal{L}(B)$ of σ in the fiber $\mathcal{L}(B)$ at p is zero for only finitely many points p. Pick a point p in the smooth locus of X such that $\sigma(p) \neq 0$. Then $h^0(X, \mathcal{L}(B-p)) = h^0(X, \mathcal{L}(B)) - 1$ because $H^0(X, \mathcal{L}(F-p))$ is the kernel of the nonzero homomorphism $H^0(X, \mathcal{L}(F)) \to k(p) \otimes \mathcal{L}(F)$. This completes the proof. □

The corollary is false without the Gorenstein assumption.

EXAMPLE 4.1.18. Let X be the non-Gorenstein genus 2 curve from Example 6.0.15. If $p_0 \in X$ is the singularity, then for any 4 general points $p_1, p_2, q_1, q_2 \in X$, the divisors $p_0 + p_1 + p_2$ and $p_0 + q_1 + q_2$ are linearly equivalent. Indeed, assume the points all lie in X_1 and are respectively given by the ideals $(t-a_1), (t-a_2), (t-b_1), (t-b_2)$. Then
$$p_0 + p_1 + p_2 = p_0 + q_1 + q_2 + \operatorname{div}(f) \text{ for } f = \frac{(t-a_1)(t-a_2)}{(t-b_1)(t-b_2)}.$$
This shows that $\dim|p_0 + p_1 + p_2| \geq 2 > 1 = d - g$ for $d = \deg(p_0 + p_1 + p_2)$ even though $d > 2g-2$.

A more satisfactory theory of linear systems on a non-Gorenstein curve can be constructed by changing the definition of divisor.

4.2. Generalized ω-divisors. We now define generalized ω-divisors and develop their properties in analogy with generalized divisors. The definition is as follows.

DEFINITION 4.2.1. Let $\omega_{\mathcal{K}}$ be the sheaf of rational 1-forms. A **generalized ω-divisor** D_ω on X is a nonzero subsheaf $I_{D_\omega} \subset \omega_{\mathcal{K}}$ that is a coherent $\mathcal{O}_X$-module. We say that a generalized ω-divisor D_ω is **Cartier** if I_{D_ω} is a line bundle. A generalized ω-divisor is **effective** if $I_{D_\omega} \subset \omega$.

(We denote generalized ω-divisors with a subscript ω to avoid confusing them with generalized divisors.)

When X is Gorenstein, the definition of a generalized ω-divisor is essentially equivalent to the definition of a generalized divisor. Indeed, for such an X, tensoring with the dualizing sheaf ω defines a bijection between the set of generalized ω-divisors and generalized divisors that preserves properties such as effectiveness. No such bijection exists when X is non-Gorenstein, and the remainder of this section is devoted to demonstrating to the reader that generalized ω-divisors behave better than generalized divisors.

Unlike effective generalized divisors, effective generalized ω-divisors do not correspond to closed subschemes of X. However, given a point $p \in X$, the subsheaf $I_p \cdot \omega \subset \omega$ of 1-forms that vanish at p defines an effective ω-divisor that we denote by p_ω. (Later in this section we will define the degree of an ω-divisor, and the reader is warned that p_ω may not have degree 1 when p is a singularity.) The submodule $\omega \subset \mathcal{K}_\omega$ defines an effective generalized ω-divisor that we denote 0_ω. The ω-divisor $\operatorname{div}(\eta)$ associated to a rational 1-form $\eta \in \mathcal{K}_\omega$ is defined by setting $\operatorname{div}(\eta) := \mathcal{O}_X \cdot \eta \subset \mathcal{K}_\omega$.

We can define basic operations on generalized ω-divisors in analogy with the operations we defined on generalized divisors, although there are a few complications.

DEFINITION 4.2.2. We define the **sum** $D + E_\omega$ of a generalized divisor D and a generalized ω-divisor E_ω by setting $I_{D+E_\omega} \subset \mathcal{K}_\omega$ equal to the subsheaf generated by elements of the form $f\eta$ with f a local section of I_D and η a local section of I_{E_ω}. (The sum of two generalized ω-divisors is not well-defined.) The **negation** $n(D_\omega)$ of a generalized ω-divisor is the generalized divisor defined by setting $I_{n(D_\omega)} \subset \mathcal{K}$ equal to the subsheaf generated by elements f that have the property that $f \cdot I_{D_\omega} \subset \omega$. Swapping the roles of $\mathcal{K}$ and $\mathcal{K}_\omega$, we get the definition of the negation $n(D)$ of a generalized divisor.

LEMMA 4.2.3. *The sum and negation operations have the following properties:*

(a) *sum is associative (i.e. $(D+E)+F_\omega = D+(E+F_\omega)$;*
(b) $0 + D_\omega = D_\omega$*;*
(c) $D + n(D) = 0_\omega$ *for every Cartier divisor D and* $n(D_\omega) + D_\omega = 0_\omega$ *for every Cartier ω-divisor D_ω;*
(d) $n(D+E_\omega) = n(E_\omega) + n(D)$ *provided D or E_ω is Cartier.*

PROOF. The proof is analogous to the proof of Lemma 4.1.4. We leave the details to the interested reader. $\square$

DEFINITION 4.2.4. Two generalized ω-divisors D_ω and E_ω are said to be **linearly equivalent** if there exists a rational function f such that $D_\omega = \operatorname{div}(f) + E_\omega$. The set of all effective generalized ω-divisors linearly equivalent to a given generalized ω-divisor is written $|D_\omega|$ and called the associated **complete linear system**.

LEMMA 4.2.5. *The following relations between generalized ω-divisors and rank 1, torsion-free sheaves hold:*

(a) *two generalized ω-divisors D_ω and E_ω are linearly equivalent if and only if $I_{D_\omega} \cong I_{E_\omega}$;*
(b) *every rank 1, torsion-free sheaf is isomorphic to I_{D_ω} for some generalized ω-divisor D_ω;*
(c) *$I_{D+E_\omega} \cong I_D \otimes I_{E_\omega}$ provided either D or E_ω is Cartier;*
(d) *$I_{n(D)} = \underline{\operatorname{Hom}}(I_D, \omega)$ and $I_{n(E_\omega)} = \underline{\operatorname{Hom}}(I_{E_\omega}, \omega)$.*

PROOF. The proof is entirely analogous to the proof of Lemma 4.1.6. □

Proceeding as we did with generalized divisors, we now define the canonical and adjoint divisors.

DEFINITION 4.2.6. Write $I^* := \underline{\operatorname{Hom}}(I, \omega)$ for the **ω-linear dual** of a coherent $\mathcal{O}_X$-module I. The sheaf $\mathcal{M}(D_\omega)$ associated to a generalized ω-divisor is defined by $\mathcal{M}(D_\omega) := I^*_{D_\omega}$. An **adjoint generalized ω-divisor** $\operatorname{adj} D_\omega$ of a generalized ω-divisor D_ω is a generalized ω-divisor satisfying $\mathcal{M}(\operatorname{adj} D_\omega) = \underline{\operatorname{Hom}}(\mathcal{M}(D_\omega), \omega)$. A **canonical generalized ω-divisor** K_ω is an adjoint ω-divisor of 0_ω (so $\mathcal{M}(K_\omega) = \omega$).

The lemma below implies that an adjoint ω-divisor of a generalized ω-divisor exists and is unique up to linear equivalence. In particular, every curve admits a canonical ω-divisor K_ω. We do not study the relation between K_ω and the canonical map here.

LEMMA 4.2.7. *A rank 1, torsion-free sheaf I is ω-reflexive. That is, the natural map*

$$I \to (I^*)^*$$

is an isomorphism. More generally, $n(n(D_\omega)) = D_\omega$ for every generalized ω-divisor D_ω and $n(n(D)) = D$ for every generalized divisor D.

PROOF. This is [**Har07**, Lem. 1.4]. □

We now develop the theory of the complete linear system associated to a ω-divisor.

LEMMA 4.2.8. *Let D_ω be a generalized ω-divisor. Then the rule that sends a nonzero global section of $\mathcal{M}(D_\omega)$ to the image of the corresponding homomorphism $I_{D_\omega} \to \omega$ defines a bijection*

$$\mathbf{P}H^0(X, \mathcal{M}(D_\omega)) \cong |D_\omega|.$$

PROOF. The proof is analogous to the proof of Lemma 4.1.10. The details are left to the interested reader. □

DEFINITION 4.2.9. The **degree** of a generalized ω-divisor D_ω is defined by $\deg(D_\omega) := \deg(\omega) - \deg(I_{D_\omega})$.

LEMMA 4.2.10. *The degree function has the following properties:*

(a) *if D_ω is an effective generalized ω-divisor, then* $\deg(D_\omega)$ *equals the length of the quotient module* ω/I_{D_ω}*;*
(b) $\deg(D+E_\omega) = \deg(D) + \deg(E_\omega)$ *provided D or E_ω is Cartier;*
(c) $\deg(n(D)) = -\deg(D)$ *and* $\deg(n(D_\omega)) = -\deg(D_\omega)$*;*
(d) *if D_ω is linearly equivalent to E_ω, then* $\deg(D_\omega) = \deg(E_\omega)$.

Proof. The proof of this lemma is similar to the proof of Lemma 4.1.12. □

We can use the lemma to conclude that

$$\deg(\mathcal{M}(D_\omega)) = \deg(D_\omega).$$

For generalized divisors, the analogous equality only holds when X is Gorenstein.

We now state the Riemann–Roch Formula.

Proposition 4.2.11. *We have*

$$\dim |D_\omega| - \dim |\operatorname{adj} D_\omega| = \deg(D_\omega) + 1 - g.$$

Proof. This is a consequence of coherent duality (14).

□

Having established a version of the Riemann–Roch Formula for generalized ω-divisors, we can now describe $\dim |D_\omega|$ just as we did for divisors on a smooth curve.

Corollary 4.2.12. *The equation*

$$\dim |D_\omega| = \begin{cases} -1 & \text{if } d < g; \\ d-g & \text{if } d \geq g. \end{cases} \tag{12}$$

holds for every generalized ω-divisor D_ω of degree $d > 2g-2$

Furthermore, if D_ω is a ω-divisor of degree $d \leq 2g-2$, then there exists a degree 0 *Cartier divisor E such that $E + D_\omega$ satisfies Eq.* (12).

Remark 4.2.13. As with Corollary 4.1.16, the second part of the present corollary implies that the general effective ω-divisor satisfies Eq. (12). To be precise, in Section 5 we will introduce the Quot scheme $\operatorname{Quot}^d_\omega$ which is a projective k-scheme that parameterizes effective generalized ω-divisors of degree d, and the present corollary implies that the locus of ω-divisors satisfying Eq. (12) is dense in Quot^d_X.

Proof. Replace generalized divisors with ω-divisors in the proof of Corollar 4.1.16. □

4.3. Examples. We now study generalized divisors on a singular curve of low genus. Our goal is to provide examples similar to those given at the end of Section 1 in order to illustrate the differences and similarities between divisors on non-singular curves and divisors on singular curves. We focus on describing divisors of degree $d = g-1$ and $d = g$.

Example 4.3.1 (Genus 1). A genus 1 curve X is always Gorenstein with trivial canonical divisor $K = 0$. The only effective generalized divisor of degree $0 = g-1$ divisor is the empty divisor 0, and a degree $1 = g$ effective divisor is a point $p \in X$. No divisor of degree $g-1$ or g moves in a positive dimensional linear system.

Example 4.3.2 (Genus 2). Here we first encounter non-Gorenstein curves. Let us first dispense with the Gorenstein case, which is analogous to the smooth case. If X is a Gorenstein curve of genus 2, then X admits a degree 2 morphism $f\colon X \to \mathbf{P}^1$ whose fibers $f^{-1}(t)$ are exactly the effective canonical divisors K ([**KM09**, Prop. 3.2]). Arguing as in Example 1.0.8, we see that no degree $1 = g-1$ divisor moves in a positive dimensional linear system, and the effective degree $2 = g$ divisors that satisfy $\dim |D| \geq 1$ are exactly the effective canonical divisors.

What about the non-Gorenstein curves? We will just consider the curve X from Example 6.0.15 (which is a representative example). Recall that X is a rational curve with a unique singularity $p_0 \in X$ that is unibranched and non-Gorenstein. We saw in Example 4.1.18 that generalized divisors on X do not behave as they do on a Gorenstein curve: there exists a degree $d = 2g-1$ generalized divisor D with $\dim |D| > d - g$, and this cannot happen on a Gorenstein curve. Let us examine generalized divisors of degree $1 = g-1$ and $2 = g$.

A degree $1 = g - 1$ divisor $D = p$ must satisfy $\dim |D| = 0$. This is [**AK80**, Thm. 8.8]. If $\dim |p| > 0$, then p would necessarily be linearly equivalent to a point q lying in the smooth locus and any non-constant function with at worst a pole at q would define an isomorphism $X \cong \mathbf{P}^1$, which is impossible.

There are degree $2 = g$ divisors that move in a positive dimensional linear system. If $p, q \in X$ are points distinct from the singularity p_0, then a modification of the construction from Example 4.1.18 shows that $p_0 + p$ is linearly equivalent to $p_0 + q$, so $\dim |p_0 + p| \geq 1$. In fact, $\dim |p_0 + p| = 1$. We can prove this as follows. Take p_1 to be the point in X_2 defined by the ideal (s). Then $I_{p_0+p_1}$ is the ideal generated by (t^3, t^4, t^5) on X_1 and by (s) on X_2. A computation shows that $\mathcal{L}(p_0 + p_1)$ is the subsheaf of $\mathcal{K}$ generated by $(1, t, t^2)$ on X_1 and by (s^{-1}) on X_2, and this rank 1, torsion-free sheaf is isomorphic to the direct image $\nu_* \mathcal{O}(1)$ of the line bundle $\mathcal{O}(1)$ under the normalization map $\nu\colon \mathbf{P}^1 = X^\nu \to X$. In particular, $h^0(X, \mathcal{L}(p_0 + p_1)) = 2$, so $\dim |p_0 + p_1| = 1$.

What are all the elements of $|p_0 + p_1|$? In addition to the divisors $p_0 + p$, the linear system $|p_0 + p_1|$ contains a non-reduced divisor: the non-reduced divisor D_0 whose ideal I_{D_0} is

$$I_{D_0} = \begin{cases} (t^4, t^5, t^6) & \text{on } X_1; \\ (1) & \text{on } X_2. \end{cases}$$

Indeed, D_0 lies in $|p_0 + p_1|$ as $D_0 = p_0 + p_1 + \operatorname{div}(t^{-1})$. The generalized divisors that we have constructed are all the effective degree 2 divisors that move in a positive dimensional linear system. Let us prove this statement.

There are no degree 2 Cartier divisors that move in a positive dimensional linear system because the existence of such a divisor would imply the existence of a non-constant degree 2 morphism $X \to \mathbf{P}^1$, forcing X to be Gorenstein ([**KM09**, Prop. 2.6]). To handle non-Cartier divisors, we need to do more work.

We claim that if E is an effective generalized divisor of degree 2 that does not lie in $|p_0 + p_1|$ and is not Cartier, then $\mathcal{L}(E)$ is the subsheaf of $\mathcal{K}$ generated by $1, t, t^2$ on X_1 and by 1 on X_2. This subsheaf is isomorphic to the direct image $\nu_* \mathcal{O}$ of the structure sheaf of the normalization, so any such E satisfies $h^0(\mathcal{L}(E)) = 1$ or equivalently $\dim |E| = 0$. Thus the claim implies that the elements of $|p_0 + p_1|$ are exactly the degree 2 divisors that move in a positive dimensional linear system as we wished to show.

We prove the claim by direct computation. If E does not lie in $|p_0 + p_1|$ and is not Cartier, then E must be supported at the singularity p_0, so we can pass from X to the open affine X_1. By considering the dimension of $\mathcal{O}_E = \mathcal{O}_{X_1}/I_E$, we see that the ideal I_E of E must contain the square of the maximal ideal (t^3, t^4, t^5) and a 2-dimensional subspace W of $k \cdot t^3 + k \cdot t^4 + k \cdot t^5$. Certainly the elements $1, t, t^2$ are all contained in $\mathcal{L}(E)$, so $\mathcal{L}(E)$ contains $\mathcal{O}^{\nu}_{X_1} = k[t]$. To complete the proof of the claim, we need to show that $\mathcal{L}(E)$ is no larger.

Thus suppose $f \in \mathcal{L}(E)$. Write f as a Laurent series in t. Because $D_0 \neq E$, the vector space W must contain an element of the form $t^3 + at^4 + bt^5$. If multiplication by $f \in k(X)$ maps $t^3 + at^4 + bt^5$ into $\mathcal{O}_{X_1}$, then the Laurent series of f must be of the form $c_{-3}t^{-3} + c_0 + c_1 t^1 + \dots$. However, W must also contain an element of the form $ct^4 + dt^5$ with c and d not both zero. By examining the coefficients of $(ct^4 + dt^5)f$, we see that $c_{-3} = 0$. This proves the claim, completing our discussion of generalized divisors on X.

Generalized ω-divisors on X are easier to analyze because we can use the Riemann–Roch Formula. Effective canonical ω-divisors K_ω are ω-divisors of degree $2 = g$ that move in a positive dimensional linear system. Indeed, K_ω has degree 2 and satisfies

$$
\begin{aligned}
\dim |K_\omega| &= \dim H^0(X, \mathcal{M}(K_\omega)) - 1 \\
&= \dim H^0(X, \omega) - 1 \\
&= 1.
\end{aligned}
$$

The canonical ω-divisors are the only ω-divisors of degree g that move in a positive dimensional linear system. If $\deg(D_\omega) = g$ and $\dim |D_\omega| > 0$, then by the Riemann–Roch Formula, $\operatorname{adj} D_\omega$ is a ω-divisor of degree 0 that satisfies $|\operatorname{adj} D_\omega| \neq \emptyset$. This is only possible if $\operatorname{adj} D_\omega = 0$ (as degree considerations show that any nonzero global section of $I_{\operatorname{adj} D_\omega}$ defines an isomorphism with $\mathcal{O}_X$). We can conclude that $D_\omega = K_\omega$.

What about the generalized ω-divisors of degree $1 = g - 1$? A point $p \in X$ distinct from p_0 defines an effective ω-divisor p_ω of degree 1. There are additional effective ω-divisors of degree 1 that correspond to quotients supported at the singularity p_0. In Example 6.0.15 of the Appendix we compute a presentation of ω, and from that presentation, we see that the stalk $\omega/I_{p_0} \cdot \omega$ of ω at p_0 is a 2-dimensional k-vector space. There is thus a 1-dimensional family of quotients of ω supported at p_0, corresponding to the surjections $\omega/I_p \cdot \omega \to k$.

None of these ω-divisors moves in a positive dimensional linear system. Indeed, fix a point $q \in X$ distinct from the singularity p_0. If D_ω is a ω-divisor of degree $1 = g-1$ with $\dim |D_\omega| > 0$, then $-q + D_\omega$ is a degree 0 ω-divisor that is effective, so $-q + D_\omega$ is linearly equivalent to 0_ω. We can conclude that D_ω is linearly equivalent to q_ω. In particular, D_ω must be Cartier. As in the case of generalized divisors, if we fix two linearly independent global sections of $\mathcal{M}(D_\omega)$, then these sections define an isomorphism $X \cong \mathbf{P}^1$, which is impossible. This completes our study of genus 2 curves.

Example 4.3.3 (Genus 3). The classification of non-Gorenstein curves of genus g becomes complicated once $g \geq 3$, so we now focus on the Gorenstein case. As in the smooth case, if X is a Gorenstein curve of genus 3, then the canonical map $X \to \mathbf{P}^2$ is either an embedding of X as a plane curve of degree 4 or a degree

2 map onto a plane quadric curve, in which case X is hyperelliptic (by [**KM09**, Thm. 4.13]). We begin by analyzing the hyperelliptic case.

If X is hyperelliptic with degree 2 map to the line $f\colon X \to \mathbf{P}^1$, then the effective canonical divisors are the divisors of the form $K = f^{-1}(t_1)+f^{-1}(t_2)$ for $t_1, t_2 \in \mathbf{P}^1$. Arguing as in Example 1.0.9, we see that the degree $2 = g-1$ effective divisors that are contained in a positive dimensional linear system are the divisors $f^{-1}(t)$ with $t \in \mathbf{P}^1$, and the degree $3 = g$ effective divisors with this property are the divisors $p + f^{-1}(t)$ for $p \in X$. In particular, every degree $g-1$ effective generalized divisor that moves in a positive dimensional linear system is Cartier, but this is not true for degree g divisors. If $p_0 \in X$ is a singularity, then $p_0 + f^{-1}(t)$ for $t \in \mathbf{P}^1$ is not Cartier but $\dim |p_0 + f^{-1}(t)| = 1$.

What about when X is non-hyperelliptic? The curve is then a degree 4 plane curve $X \subset \mathbf{P}^2$, and the effective canonical divisors are restrictions of lines $K = \ell \cap X$. As in the hyperelliptic case, the analysis we gave for non-singular curves at the end of Section 1 extends to the present case. No effective generalized divisor of degree $g-1$ moves in a positive dimensional linear system, but an effective generalized divisor of degree g moves in a positive dimensional linear system when it is contained in a line, so e.g. three points p_0, p_1, p_2 of X that lie on a line satisfy $\dim |p_0 + p_1 + p_2| = 1$. In particular, if p_0 is a node of X, then $p_0 + p_1 + p_2$ is an effective generalized divisor of degree g that is not Cartier but moves in a positive dimensional linear system.

Example 4.3.4 (Genus 4). Here we only study a few specific examples of genus 4 curves. On a curve of genus $g < 4$, every effective generalized divisor D of degree $g-1$ that moves in a positive dimensional linear system is Cartier. This remains true on the general curve of genus $g = 4$, but not on certain special genus 4 curves. We focus on describing divisors on these special curves.

A singular hyperelliptic curve of genus 4 is an example of a special curve. If $f\colon X \to \mathbf{P}^1$ is the degree 2 map to $\mathbf{P}^1$, then the effective canonical divisors are the divisors of the form $K = f^{-1}(t_1) + f^{-1}(t_2) + f^{-1}(t_3)$. We can conclude that the degree $g-1$ divisors that move in a positive dimensional linear system are the divisors of the form $p_0 + f^{-1}(t)$ for $p_0 \in X$ and $t \in \mathbf{P}^1$. In particular, the divisor $p_0 + f^{-1}(t)$ is not Cartier when $p_0 \in X$ is a singularity.

What about the non-hyperelliptic case? If X is a non-hyperelliptic Gorenstein curve of genus 4, then the canonical map realizes X as a degree 6 space curve $X \subset \mathbf{P}^3$. Just as in the smooth case, X is the complete intersection of a (non-unique) cubic hypersurface and a (unique) quadric hypersurface Q. If the two hypersurfaces are general, then X is smooth, but X can have singularities when the hypersurfaces are special. Consider the case where Q is the cone over a plane quadric curve Y with vertex $p_0 \in \mathbf{P}^3$ and the cubic hypersurface contains p_0 but is otherwise general. The curve X then has a unique node that is the vertex p_0 of Q.

The effective canonical divisors of X are the divisors of the form $K = h \cap X$ for $h \subset \mathbf{P}^3$ a hyperplane. Our analysis in Example 1.0.10 shows that an effective degree $3 = g-1$ generalized divisor D moves in a positive dimensional linear system precisely when D is contained in a line $\ell \subset Q$ that lies on the quadric surface. Every such line meets X in 3 points, one of which is the vertex p_0. The lines on Q are exactly the lines joining a point of the plane curve Y to the vertex p_0. If the (set-theoretic) intersection of a line and X consists of the points p_0, p_1, p_2, then $p_0 + p_1 + p_2$ is a degree $g-1$ effective generalized divisor that moves in a positive dimensional linear system, and the general divisor with this property is of the form $p_0+p_1+p_2$ for a suitable line. Because $p_0 \in X$ is a node, $p_0+p_1+p_2$ is not Cartier. Thus every degree $g-1$ generalized divisor that moves in a positive dimensional linear system fails to be Cartier.

5. The Abel map

At the end of Section 3, we posed the problem of defining an Abel map associated to a singular curve. There are two motivations. First, a theory of the Abel map provides us with a tool for constructing and describing the compactified Jacobian $\bar{J}^d_X$. Second, a theory of the Abel map allows us to define and study the theta divisor $\Theta \subset \bar{J}^{g-1}_X$ associated to a singular curve.

Example 3.0.18 demonstrated that the most naive approach to constructing an Abel map fails: If $X^{(d)}$ is the d-th symmetric power of a curve, then the rule that assign to d general points $p_1, \dots, p_d$ the line bundle $\mathcal{O}_X(p_1 + \cdots + p_d)$ defines a rational map

$$A\colon X^{(d)} \dashrightarrow \bar{J}^d_X,$$

but this map may not be regular. In the example, the 2nd Abel map $A\colon X^{(2)} \dashrightarrow \bar{J}^2_X$ is undefined at a unique point.

Lemma 1.0.4 suggests an alternative approach to constructing the Abel map. For a smooth curve X, the symmetric power $X^{(d)}$ can be interpreted as the moduli space of effective Cartier divisors of degree d, and the Abel map $A\colon X^{(d)} \to J^d_X$ can be interpreted as the map that sends a divisor D to its associated line bundle $\mathcal{L}(D)$. When X is singular, the symmetric power no longer has such a moduli theoretic interpretation. Here we resolve the indeterminacy of $A\colon X^{(d)} \dashrightarrow \bar{J}^d_X$ by constructing an Abel map out of a moduli space that maps to $X^{(d)}$.

Our discussion from Section 4 suggests two possibilities for the moduli space: the moduli space of effective generalized divisors and the moduli space of effective generalized ω-divisors. We will see that there are two different Abel maps corresponding to the two different types of divisors. The two maps are essentially equivalent when X is Gorenstein but have different properties in general. We begin by defining the relevant moduli spaces.

Effective generalized divisors are parameterized by the **Hilbert scheme** Hilb^d_X which is defined to be the k-scheme that represents the Hilbert functor $\mathrm{Hilb}^{d,\sharp}_X$ defined in Definition 1.0.3. (There X was assumed to be smooth, but the definition remains valid when X is singular.) Generalized ω-divisors are parameterized by a Quot scheme that we now define.

Definition 5.0.5. The **Quot functor** $\mathrm{Quot}^{d,\sharp}_\omega$ of degree d is defined by setting $\mathrm{Quot}^{d,\sharp}_\omega(T)$ equal to the set of isomorphism classes of T-flat quotients $q\colon \omega_T \twoheadrightarrow Q$ of the dualizing sheaf with the property that the restriction Q to any fiber of $X_T \to T$ has length d. The **Quot scheme** Quot^d_ω of degree d is the k-scheme that represents $\mathrm{Quot}^{d,\sharp}_\omega$.

A quotient map $q\colon \omega \to Q$ is equivalent to the inclusion $\ker(q) \hookrightarrow \omega$ which defines a generalized ω-divisor, so Quot^d_ω is the moduli space of effective generalized ω-divisors of degree d. For the remainder of the section, we will write D_ω rather than $[q]$ for an element of Quot^d_ω to emphasize the connection with generalized ω-divisors.

The moduli spaces Hilb^d_X and Quot^d_ω both exist as projective k-schemes by a theorem of Grothendieck ([**AK80**, Thm. 2.6]). The construction realizes both of these moduli spaces as suitable subschemes of a Grassmannian variety.

The Hilbert scheme and the Quot scheme are related to the symmetric power by the Hilbert–Chow morphism. Suppose that $D \subset X$ is a degree d closed subscheme. Given $p \in X$, write $\ell_p(D)$ for the length of $\mathcal{O}_{D,p}$. With this definition, we can associate to D the point in the symmetric power

$$[\sum \ell_p(D) \cdot p] \in X^{(d)}. \tag{13}$$

We would like to assert that there is a morphism

$$\rho\colon \mathrm{Hilb}^d_X \to X^{(d)}$$

with the property that ρ is defined by Eq. (13) as a set-map. We also like to assert the existence of an analogous morphism $\rho\colon \mathrm{Quot}^d_\omega \to X^{(d)}$ out of the Quot scheme.

These assertions are surprisingly difficult to prove. The difficulty is that our construction of $X^{(d)}$ does not provide direct access to the scheme's functor of points. One approach to constructing ρ is to describe the functor of points of $X^{(d)}$ in a way that makes it possible to interpret Eq. (13) as the definition of a transformation of functors. A detailed discussion of this approach can be found in the thesis of David Rydh ([**Ryd08a**] and esp. [**Ryd08b**, Sect. 6]).

Rydh attributes this construction of ρ, which he calls the Grothendieck–Deligne norm map, to Grothendieck and Deligne ([**SGA73**, Exposé XVII, 6.3.4]). In this approach, one first identifies $X^{(d)}$ with the space of divided powers. The space of divided powers has a moduli-theoretic interpretation in terms of multiplicative polynomial laws, and ρ is then constructed as the morphism that sends a closed subscheme to the multiplicative law given by a determinant construction. A similar construction produces a morphism $\rho\colon \mathrm{Quot}^d_\omega \to X^{(d)}$ out of the Quot scheme. There are other approaches to constructing ρ, using e.g. projective geometry, and we direct the interested reader to [**Ryd08b**, Sect. 6] for a review of the relevant literature.

We will try to resolve the indeterminacy of the Abel map $A\colon X^{(d)} \dashrightarrow J^d_X$ by constructing morphisms $\mathrm{Hilb}^d_X \to \bar{J}^d_X$ and $\mathrm{Quot}^d_\omega \to \bar{J}^d_X$ that factor as $\mathrm{Hilb}^d_X \xrightarrow{\rho} X^{(d)} \overset{A}{\dashrightarrow} \bar{J}^d_X$ and $\mathrm{Quot}^d_X \xrightarrow{\rho} X^{(d)} \overset{A}{\dashrightarrow} \bar{J}^d_X$. We will always be able to construct a suitable morphism out of Quot^d_ω, but we will only be able to construct a morphism out of Hilb^d_X when X is Gorenstein.

Before constructing the morphisms, we warn the reader that ρ is not always a resolution of indeterminacy in the usual sense because ρ is not always birational. The Hilbert–Chow morphism is an isomorphism over the locus parameterizing d distinct points ([**Ive70**, Thm. II.3.4]). When the singularities of X are planar, the locus of distinct points in Hilb^d_X is dense, so ρ is a birational map. The Hilbert–Chow morphism is not birational when X has a singularity of embedding dimension 3 or more. Indeed, when X has such a singularity, Hilb^d_X is reducible, and the locus of d distinct points is contained in a single component. The other components are collapsed by ρ, so ρ is not birational. The issue with Quot^d_ω is similar.

In any case, we now define Abel maps associated to singular curves.

Definition 5.0.6. The **Abel map**

$$A_h\colon \mathrm{Hilb}^d_X \to \bar{J}^{-d}_X$$

out of the Hilbert scheme is defined by the rule

$$[D] \mapsto [I_D],$$

where I_D is the ideal of the closed subscheme $D \subset X_T$. The **Abel map**

$$A_q\colon\ \mathrm{Quot}^d_\omega \to \bar{J}^{2g-2-d}_X$$

out of the Quot scheme is defined by the rule

$$[D_\omega] \mapsto [I_{D_\omega}],$$

where I_{D_ω} is the kernel of the quotient map corresponding to D_ω.

These are the Abel maps that are constructed in [**AK80**], and they enjoy many of the properties that we would like the Abel map to have. For example, if D is an effective generalized divisor, then the fiber $A^{-1}([I_D])$ containing $[D]$ is equal to the complete linear system $|D|$, and the same remains true if we let D be a generalized ω-divisor and replace Hilb^d_X with Quot^d_ω.

From this description of A, we can conclude from Corollary 4.1.16 that if $d \geq 2g-1$ and X is Gorenstein, then the fibers of $A\colon\ \mathrm{Hilb}^d_X \to \bar{J}^{-d}_X$ are all $\mathbf{P}^{d-g}$'s, and in fact, this Abel map is smooth of relative dimension $d-g$ ([**AK80**, Thm. 8.6]). Example 4.1.18 shows that this is not true when X is non-Gorenstein, but if we replace Hilb^d_X with Quot^d_X, then we recover the property that $A\colon\ \mathrm{Quot}^d_X \to \bar{J}^{-d}_X$ is smooth with $\mathbf{P}^{d-g}$ fibers ([**AK80**, Thm. 8.4]), as is suggested by Corollary 4.2.12.

With this property of the Abel map, we can now construct the compactified Jacobian of a singular curve from the Quot scheme in the same manner that we constructed the Jacobian of a smooth curve from the symmetric power. If we fix $d \geq 2g-1$, then linear equivalence partitions Quot^d_ω into equivalence classes that are all isomorphic to $\mathbf{P}^{d-g}$. In fact, linear equivalence is a smooth and projective equivalence relation on Quot^d_ω. We can conclude that the quotient exists as a scheme, and this quotient is the compactified Jacobian $\bar{J}^d_X$. This argument is roughly the construction of $\bar{J}^d_X$ given in [**AK80**, Thm. 8.5].

The Abel map in Definition 5.0.6 however suffers from one deficiency: it does not quite resolve the indeterminacy of $A\colon\ X^{(d)} \dashrightarrow \bar{J}^d_X$. The Abel map $A_h\colon\ \mathrm{Hilb}^d_X \to \bar{J}^{-d}_X$ sends a closed subscheme D consisting of d general points to the ideal sheaf I_D, but the composition $\mathrm{Hilb}^d_X \xrightarrow{\rho} X^{(d)} \overset{A}{\dashrightarrow} \bar{J}^d_X$ sends D to $\mathcal{L}(D)$.

This is not just an issue of convention. In Example 4.1.14, we saw an example of a generalized divisor D of degree d with the property that the degree of $\mathcal{L}(D)$ is $d+1$. Since the degree of a rank 1, torsion-free sheaf is locally constant in flat families, we can conclude that the rule $D \mapsto \mathcal{L}(D)$ does not define a morphism $\mathrm{Hilb}^d_X \to \bar{J}^d_X$ when X is the curve from the example.

That curve is non-Gorenstein, and there are no problems provided we restrict our attention to Gorenstein curves. Given $[D] \in \mathrm{Hilb}^d_X(T)$ (for some T), form the sheaf

$$\mathcal{L}(D) := \underline{\mathrm{Hom}}(I_D, \mathcal{O}_{X_T}),$$

If X is Gorenstein, then for any rank 1, torsion-free sheaf I, $\underline{\mathrm{Ext}}^1(I, \mathcal{O}_X)$ vanishes by [**HK71**, 6.1]. In particular, this group vanishes when I is the restriction of I_D to a fiber of $X_T \to T$, so we can conclude by [**AK80**, Thm. 1.10] that $\mathcal{L}(D)$ is T-flat and its formation commutes with base-change. (This is essentially [**EGK00**, 2.2].) We can thus make the following definition.

DEFINITION 5.0.7. Assume X is Gorenstein. Given a k-scheme T and $[D] \in \mathrm{Hilb}^d_X(T)$, set $\mathcal{L}(D) := \underline{\mathrm{Hom}}(I_D, \mathcal{O}_{X_T})$ where I_D is the ideal of D. The **modified Abel map** $A^\vee_h \colon \mathrm{Hilb}^d_X \to \bar{J}^d_X$ is defined by the rule $D \mapsto \mathcal{L}(D)$.

The modified Abel map is well-defined by the preceding discussion. Furthermore, unlike A_h, the modified Abel map $A^\vee_h$ resolves the indeterminacy of $A\colon X^{(d)} \dashrightarrow \bar{J}^d_X$ in the sense described earlier.

When X is non-Gorenstein, our discussion of generalized divisors from the previous section suggests that we should try to define a modified Abel map out of Quot^d_ω rather than Hilb^d_X. We can always do this. By [**EGK00**, 2.2] the sheaf

$$\mathcal{M}(D_\omega) := \underline{\mathrm{Hom}}(I_{D_\omega}, \omega_T)$$

associated to $D_\omega \in \mathrm{Quot}^d_\omega(T)$ is always T-flat and its formation commutes with base-change. We can therefore make the following definition.

DEFINITION 5.0.8. Given a k-scheme T and $D_\omega \in \mathrm{Quot}^d_\omega(T)$, set $\mathcal{M}(D_\omega) := \underline{\mathrm{Hom}}(I_{D_\omega}, \omega_T)$. The **modified Abel map** $A^*_q \colon \mathrm{Quot}^d_\omega \to \bar{J}^d_X$ out of the Quot scheme is defined by the rule

$$D_\omega \mapsto \mathcal{M}(D_\omega)$$

As with the modified Abel map out of the Hilbert scheme, this modification of the Abel map resolves the indeterminacy of $A\colon X^{(d)} \dashrightarrow \bar{J}^d_X$.

Having constructed Abel maps, we can now use these maps to define and study the theta divisor. Imitating the construction for smooth curves, we make the following definition.

DEFINITION 5.0.9. The **theta divisor** $\Theta \subset \bar{J}^{g-1}_X$ of a curve X is defined to be the image of $A^*_q \colon \mathrm{Quot}^{g-1}_\omega \to \bar{J}^{g-1}_X$ with the reduced scheme structure.

We are abusing language in calling Θ a divisor because it is not known in general whether Θ is always a divisor. The subscheme Θ is known to be a Cartier divisor when the singularities of X are planar. For such an X, Soucaris [**Sou94**] and Esteves [**Est97**] have proven that Θ is not only a Cartier divisor, but in fact Θ is an ample Cartier divisor. They prove this statement by constructing and studying Θ using the formalism of the determinant of cohomology.

Soucaris uses the same formalism to construct a subscheme in $\bar{J}^{g-1}_X$ when X is a curve with arbitrary singularities, but then it is not known whether his subscheme coincides with Θ, and it is not known whether his subscheme is a Cartier divisor. By construction, Soucaris' subscheme is locally defined by a single equation, and Soucaris proves that the subscheme does not contain an irreducible component of $\bar{J}^{g-1}_X$ [**Sou94**, Thm. 8]. (His argument is essentially our Corollary 4.2.12.) This does not quite show that Soucaris' subscheme is a Cartier divisor. To show that, it is necessary to also prove that the subscheme does not contain any embedded component of $\bar{J}^{g-1}_X$, a result that is unknown. More generally, it is not known whether $\bar{J}^{g-1}_X$ can have embedded components or even whether $\bar{J}^{g-1}_X$ can be non-reduced.

It is also not known whether Soucaris' subscheme equals Θ when X has non-planar singularities. The two subschemes are supported on the same subset of $\bar{J}^{g-1}_X$, so to show equality, it is enough to prove that Soucaris' subscheme is reduced.

There is a third subscheme of $\bar{J}^{g-1}_X$ that we can define: the image of the Abel map $A_h \colon \mathrm{Hilb}^{g-1}_X \to \bar{J}^{-(g-1)}_X$. This subscheme can be identified with Θ when X

is Gorenstein, but we will see in Example 5.0.11 that the two subschemes can be different in general.

We conclude by describing the Abel map in some low genus examples. With the theory of the Abel map that we have developed, this will be a straightforward application of the results from the end of Section 4. As in Section 1, we let $C^1_d \subset \mathrm{Hilb}^d_X$ denote the subset of effective generalized divisors D with $\dim |D| \geq 1$ and similarly with $C^1_d \subset \mathrm{Quot}^d_\omega$.

Example 5.0.10 (Genus 1). If X is a genus 1 curve, then the singularities of X are planar and hence Gorenstein. In Example 4.3.1, we showed that $C^1_g = \emptyset$, so the Abel map $A^\vee_h \colon \mathrm{Hilb}^1_X = X \to \bar{J}^1_X$ is an isomorphism. When X is a nodal curve, this recovers the moduli-theoretic interpretation of the compactification of the generalized Jacobian from the beginning of Section 3. The theta divisor $\Theta = \{[\mathcal{O}_X]\} \subset \bar{J}^0_X$ consists of a single point that lies in the smooth locus.

Example 5.0.11 (Genus 2). In describing the structure of the Abel map, we need to distinguish between Gorenstein curves and non-Gorenstein curves. When X is a Gorenstein curve, we showed in Example 4.3.2 that C^1_g is the rational curve $\mathbf{P}^1 = C^1_g \subset \mathrm{Hilb}^g_X$ that consists of effective canonical divisors K. The Abel map $A^\vee_h$ contracts this curve to the point $\{[\omega]\} \subset J^g_X$ of the generalized Jacobian. The locus C^1_{g-1} is empty, so the theta divisor $\Theta \subset \bar{J}^{g-1}_X$ is an embedded copy of Hilb^{g-1}_X, which is just X itself. Thus the structure of the Abel map is the same as for a smooth genus 2 curve except the geometry of both the source and the target are more complicated.

What happens when X is non-Gorenstein? We will only consider the non-Gorenstein curve X with a unique unibranched singularity $p_0 \in X$ from Example 6.0.15. For this curve, we can consider both the Abel map A_h out of the Hilbert scheme and the Abel map A^*_q out of the Quot scheme. We begin by describing A_h. The locus $C^1_g \subset \mathrm{Hilb}^g_X$ is the rational curve that parameterizes elements of $|p + p_0|$ (where $p \neq p_0$). This curve is contracted by the Abel map $A_h \colon \mathrm{Hilb}^g_X \to \bar{J}^{-g}_X$, and away from this curve, A_h is an isomorphism. The locus $C^1_{g-1} \subset \mathrm{Hilb}^1_X$ is empty, so $X = \mathrm{Hilb}^1_X \to \bar{J}^{-(g-1)}_X$ is an embedding.

Let us now turn our attention to the Abel map A^*_q out of the Quot scheme. This Abel map contracts the rational curve $\mathbf{P}^1 = C^g_1 \subset \mathrm{Quot}^g_\omega$ parameterizing effective canonical ω-divisors to the point $[\omega] \in \bar{J}^2_X$ and is an isomorphism away from C^1_g. The locus $C^1_{g-1} \subset \mathrm{Quot}^{g-1}_\omega$ is empty, so $A^*_q \colon \mathrm{Quot}^{g-1}_X \to \bar{J}^1_X$ is an embedding. How does A^*_q compare with A_h?

In degree g, both Abel maps contract a rational curve $\mathbf{P}^1$, but the contractions are different. The image of $\mathbf{P}^1$ under A_h is $[I_{p+p_0}] \in \bar{J}^{-g}_X$ which is a singularity, but the image of $\mathbf{P}^1$ under A^*_q is the smooth point $[\omega]$ of $\bar{J}^g_X$.

To see that $[I_{p+p_0}]$ is a singularity, we estimate the tangent space dimension. The tangent space $\mathrm{T}_{[p+p_0]}(\mathrm{Hilb}^2_X)$ is the sum of the tangent spaces $\mathrm{T}_p(X)$ and $\mathrm{T}_{p_0}(X)$, so $\dim \mathrm{T}_{[p+p_0]}(\mathrm{Hilb}^2_X) = 4$. We can conclude that $\dim \mathrm{T}_{[I_{p+p_0}]}(\bar{J}^{-g}_X) \geq 3$ because the kernel of

$$\mathrm{T}(A_h) \colon \mathrm{T}_{[p+p_0]}(\mathrm{Hilb}^2_X) \to \mathrm{T}_{[I_{p+p_0}]}(\bar{J}^{-g}_X)$$

is the 1-dimensional space

$$\mathrm{Hom}(I_{p+p_0}, \mathcal{O}_X) / \mathrm{Hom}(I_{p+p_0}, I_{p+p_0}) = H^0(X, \mathcal{L}(p + p_0)) / H^0(X, \mathcal{O}_X).$$

Now the point $[I_{p+p_0}]$ lies in the closure of the line bundle locus, so either $[I_{p+p_0}]$ lies on the intersection of two irreducible components or the local dimension of $\bar{J}_X^{-g}$ at $[I_{p+p_0}]$ is $2 < 3 \leq \dim \mathrm{T}_{[I_{p+p_0}]}(\bar{J}_X^{-g})$. In either case, we can conclude that $[I_{p+p_0}] \in \bar{J}_X^{-g}$ is a singularity.

Tangent space techniques can also be used to prove that $[\omega] \in \bar{J}_X^{g}$ is a smooth point, and the reader is directed to the proof of [**Kas12**, Thm. 2.7] for the computation. In [**Kas12**], the author also enumerates the irreducible components of $\bar{J}_X^{-g}$. There are two.

The Abel maps in degree $g-1$ are also different. Both A_h and A_q^* are embeddings of curves but of different curves. The morphism A_h embeds Hilb_X^{g-1} which is isomorphic to the irreducible curve X. The morphism A_q^*, on the other hand, embeds Quot_X^{g-1}, and this scheme is not isomorphic to X. As $g-1=1$, the Quot scheme $\mathrm{Quot}_\omega^{g-1}$ is isomorphic to $\mathbf{P}\omega$, the projectivization of ω. From the presentation of ω in Example 6.0.15 of Section 6, we see that $\mathrm{Quot}_\omega^{g-1} = \mathbf{P}\omega$ is a curve with two irreducible components, one whose general element corresponds to a quotient supported at the singularity p_0 and one whose general element corresponds to a quotient supported on the smooth locus X_{sm}. The image $A_q^*(\mathrm{Quot}_\omega^{g-1})$ is the theta divisor Θ, so this example shows that Θ and $A_h(\mathrm{Hilb}_X^{g-1})$ may not be isomorphic as schemes.

Example 5.0.12 (Genus 3). We only consider Gorenstein curves, and we consider the hyperelliptic curves and the non-hyperelliptic curves separately. If X is a singular hyperelliptic curve of genus 3, then our description of complete linear systems on X in Example 4.3.3 shows that C_g^1 is isomorphic to $X \times \mathbf{P}^1$, and the Abel map $A_h^\vee \colon C_g^1 \to \bar{J}_X^g$ collapses the $\mathbf{P}^1$ factor. In particular, the image $A_h^\vee(C_g^1)$ is a copy of X.

The locus $C_{g-1}^1 \subset \mathrm{Hilb}_X^{g-1}$ is the rational curve that parameterizes effective canonical divisors. As in the smooth case, this curve is blown down to a point $\{[\omega]\} \subset \Theta$ that is a singularity by [**CMK12**, Prop. 6.1]. The point $[\omega]$ is the only point on the theta divisor with positive dimensional preimage under $A_h^\vee$, but it is not the only singularity. The theta divisor is also singular at points corresponding to sheaves that fail to be locally free. This is another consequence of [**CMK12**, Prop. 6.1].

What if X is non-hyperelliptic? The canonical map then embeds X as a degree 4 plane curve, and we showed in Example 4.3.3 that the elements of C_g^1 are effective generalized divisors of degree 3 that are contained in a line. Arguing as in the smooth case (Example 1.0.9), we can construct a map $C_g^1 \to X$ with $\mathbf{P}^1$-fibers that are collapsed by $A_h^\vee \colon \mathrm{Hilb}_X^g \to \bar{J}_X^g$. We can conclude that $A_h^\vee(C_g^1)$ is a copy of X. The locus C_{g-1}^1 is empty, so Θ is an embedded copy of Hilb_X^2.

Example 5.0.13 (Genus 4). Here we just describe the degree $g-1$ Abel map associated to the special singular curves that we studied in Example 4.3.4. Consider first the special non-hyperelliptic curve X that lies on a singular quadric $Q \subset \mathbf{P}^3$. From our description in Example 4.3.4, we see that $C_{g-1}^1 \subset \mathrm{Hilb}_X^{g-1}$ is the rational curve that parameterizes effective degree $g-1$ generalized divisors that lie on a ruling of the quadric surface $Q \subset \mathbf{P}^3$. This curve is contracted to a point by $A_h^\vee$. Recall that every element of C_{g-1}^1 is a divisor that is not Cartier. This is a new phenomenon: this is the first example of a point on the theta divisor that is not a line bundle and has positive dimensional preimage under $A_h^\vee$.

The theta divisor of a singular hyperelliptic curve of genus 4 contains similar points. For such a curve, the work we did in Example 4.3.4 shows that C^1_{g-1} is isomorphic to $X \times \mathbf{P}^1$, and the Abel map $A_h^\vee$ collapses the second $\mathbf{P}^1$ factor. The image of $A_h^\vee$ is thus a copy of $X \subset \Theta$, and the singularities of X correspond to points on the theta divisor that are not line bundles and have positive dimensional preimage.

In [**CMK12**], the author and Casalaina-Martin computed the multiplicity of the theta divisor of a nodal curve at a point. In general, if X is a nodal curve and $[I] \in \Theta$, then the main theorem of that paper asserts

$$\mathrm{mult}_I\,\Theta = 2^n \cdot h^0(X, I),$$

where n is number of nodes at which I fails to be locally free. When X is the special genus 4 non-hyperelliptic curve and $[I] \in \Theta$ is the image of C^1_{g-1}, the theorem states that $[I] \in \Theta$ is a multiplicity 4 point.

6. Appendix: The dualizing sheaf and coherent duality

The dualizing sheaf ω of a curve plays an important role in the study of compactified Jacobians and generalized divisors. Here we recall the definition and the basic properties of ω. Two references for this material are [**AK70**] and [**Har77**, Chap. III, Sect. 7].

The dualizing sheaf ω of a curve X is defined as follows. Given a point $p \in X^\nu$ of the normalization of X, fix a uniformizer t of X^ν at p. We define the **residue** $\mathrm{Res}_p(\eta)$ of a rational 1-form η at p as follows. In the local ring $\mathcal{O}_{X^\nu,p}$, we can write η as

$$\eta = (b(t) + a_{-1}t^{-1} + a_{-2}t^{-2} + \dots)dt$$

for $b(t) \in \mathcal{O}_{X^\nu,p}$ and $a_n \in k$. We define

$$\mathrm{Res}_p(\eta) := a_{-1}.$$

The functional Res_p is independent of the choice of t (though the proof is non-trivial when $\mathrm{char}(k) > 0$). The dualizing sheaf can be defined in terms of residues.

Definition 6.0.14. The **dualizing sheaf** ω of the curve X is defined to be the sheaf whose sections $\eta \in H^0(U, \omega)$ over an open subset $U \subset X$ are rational 1-forms η with the property that

$$\sum_{\nu(p)=q} \mathrm{Res}_p(f\eta) = 0$$

for all regular functions $f \in H^0(U, \mathcal{O}_X)$ and all points $q \in U$.

The dualizing module admits a distinguished functional $t\colon H^1(X, \omega) \to k$ (whose definition we omit) that induces an isomorphism

$$\mathrm{Ext}^n(F, \omega) \cong H^{1-n}(X, F)^\vee$$

for every coherent sheaf F and every integer n. This statement is the Coherent Duality Theorem (see [**AK70**, Chap. IV, Sect. 5] or [**Har77**, Thm. 7.6]). By general formalism, the pair (ω, t) is unique up to a unique isomorphism.

We can say even more when F is a rank 1, torsion-free sheaf. If $F = I$ is a rank 1, torsion-free sheaf, then the natural map $H^n(X, \underline{\mathrm{Hom}}(I, \omega)) \to \mathrm{Ext}^n(I, \omega)$ is an isomorphism for all n because the higher cohomology sheaves $\underline{\mathrm{Ext}}^n(I, \omega)$, $n > 0$, vanish [**HK71**, 6.1]. In particular, if $I = \mathcal{M}(D_\omega)$ is the rank 1, torsion-free sheaf associated to a generalized ω-divisor, then coherent duality takes the form

$$H^n(X, \mathcal{M}(\operatorname{adj} D_\omega)) \cong H^{1-n}(X, \mathcal{M}(D_\omega))^\vee \tag{14}$$

Similarly, if X is Gorenstein (so the adjoint of a generalized divisor exists), then for every generalized divisor D, we have

$$H^n(X, \mathcal{L}(\operatorname{adj} D)) \cong H^{1-n}(X, \mathcal{L}(D))^\vee. \tag{15}$$

We conclude this section by computing the dualizing sheaf of a specific non-Gorenstein curve.

Example 6.0.15. Define X to be the curve constructed by gluing the affine curves

$$X_1 := \operatorname{Spec}(k[t^3, t^4, t^5]),$$
$$X_2 := \operatorname{Spec}(k[s])$$

by the isomorphism $t = s^{-1}$. This is a rational curve of genus 2 with a unique singularity that is non-Gorenstein and unibranched.

Using Definition 6.0.14, we see that the dualizing sheaf ω is the subsheaf of the sheaf $\omega_{\mathcal{K}}$ of rational 1-forms generated by

$$dt/t^3, dt/t^2 \text{ on } X_1; \tag{16}$$
$$ds \text{ on } X_2.$$

Acknowledgments

The author thanks the anonymous referee and Maxim Arap for helpful comments on earlier drafts of this article. The author also thanks David Rydh for a helpful email exchange about the Hilbert–Chow morphism.

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Clay Mathematics Proceedings
Volume **18**, 2013

On the Göttsche Threshold

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Abstract. For a line bundle $\mathcal{L}$ on a smooth surface S, it is now known that the degree of the Severi variety of cogenus-δ curves is given by a universal polynomial in the Chern classes of $\mathcal{L}$ and S if $\mathcal{L}$ is δ-very ample. For S rational, we relax the latter condition substantially: it suffices that three key loci be of codimension more than δ. As corollaries, we prove that the condition conjectured by Göttsche suffices if S is $\mathbb{P}^2$ or S is any Hirzebruch surface, and that a similar condition suffices if S is any classical del Pezzo surface.

1. Introduction

Fix $\delta \geq 0$. Fix a smooth irreducible projective complex surface S, and a line bundle $\mathcal{L}$. Let $|\mathcal{L}|$ be the complete linear system, and $|\mathcal{L}|^\delta \subset |\mathcal{L}|$ the Severi variety, the locus of reduced curves C of *cogenus* δ; so δ is the genus drop, $\delta := p_aC - p_gC$, or $\delta = \chi(\mathcal{O}_{\widetilde{C}}) - \chi(\mathcal{O}_C)$ where $\widetilde{C}$ is the normalization. Let $|\mathcal{L}|^\delta_+ \subset |\mathcal{L}|^\delta$ be the sublocus of δ-nodal curves. Often enough when S is rational, $|\mathcal{L}|^\delta_+$ is open and dense in $|\mathcal{L}|^\delta$, so that $\deg|\mathcal{L}|^\delta_+ = \deg|\mathcal{L}|^\delta$; see Proposition 1.2 below.

The degree $\deg|\mathcal{L}|^\delta_+$ can be found recursively if S is the plane [**28**, Theorem 3C.1], [**7**, Theorem 1.1], if S is any Hirzebruch (rational ruled) surface [**35**, §8], or if S is any classical del Pezzo surface (that is, its anticanonical bundle is very ample) [**35**, §9]. If δ and S are arbitrary, but $\mathcal{L}$ is sufficiently ample, then by [**23, 24**], by [**34**], or by [**21**], there's a universal polynomial $G_\delta(S, L)$ in the Chern classes of S and $\mathcal{L}$ with

$$(+) \qquad \deg|\mathcal{L}|^\delta_+ = G_\delta(S, \mathcal{L}).$$

Further, set $r := \dim|\mathcal{L}|$. In those cases, $\deg|\mathcal{L}|^\delta_+$ is the number of δ-nodal curves through $r - \delta$ general points, and each curve is counted with multiplicity 1 by [**19**, Lemma (4.7)]. See [**20**] for a brief survey of related work and open problems.

Given δ and S, for precisely which $\mathcal{L}$ does (+) *hold?* It is known [**21**, Theorem 4.1] that (+) holds if $\mathcal{L}$ is δ-very ample, that is if, for any subscheme $Z \subset S$ of length $\delta+1$, the restriction map $\mathrm{H}^0(\mathcal{L}) \to \mathrm{H}^0(\mathcal{L}|_Z)$ is surjective. In particular, (+)

2010 *Mathematics Subject Classification.* Primary 14N10; Secondary 14C20, 14M20.
Key words and phrases. Göttsche Threshold, Severi variety, Enumeration of nodal curves.
Vivek V. Shende gratefully acknowledges the support of the Simons Foundation.
Ilya Tyomkin gratefully acknowledges the support of the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement 248826.

holds for $S = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}(d)$ if $d \ge \delta$. Previously, this bound had been confirmed by F. Block [**6**, Theorem 1.3], who also coined the term *Göttsche threshold* for the value of d at which (+) begins to hold. However, as conjectured by Göttsche [**13**, Conjecture 4.1, Remark 4.4] and proved by Block [**6**, Theorem 1.4] for $\delta = 3, \dots, 14$, in fact the threshold appears to be $\lceil \delta/2 \rceil + 1$ if $\delta \ge 3$; whereas, it is 1 if $\delta = 0, 1, 2$. Göttsche [**13**, Remark 4.3, 4.4] also conjectured a value for the threshold if S is any Hirzebruch surface.

Here we prove Göttsche's conjectured value is at least an upper bound on the threshold if S is $\mathbb{P}^2$ or if S is any Hirzebruch surface, and we prove a similar bound if S is any classical del Pezzo surface; see Corollaries 1.3, 1.4, 1.6 and Remark 1.5 stated just below. Although we cannot say exactly when the bound is tight, in Remark 1.5 we show it isn't if S is the first Hirzebruch surface, the blowup of $\mathbb{P}^2$ at a point. We derive those results directly from our main results, Theorem 1.1 and Proposition 1.2, stated next.

Note that the term immersed is used here in the sense of differential geometry; specifically, we call an embedded curve $D \subset S$ *immersed* if D is reduced and the tangent map $T_{\widetilde{D}} \to T_S$ is injective, where $\widetilde{D}$ is the normalization.

THEOREM 1.1. *Assume S is rational with canonical class K. Let V be a closed subset of $|\mathcal{L}|$ that contains every $D \in |\mathcal{L}|$ such that either*

(1) *D is nonreduced, or*
(2) *D has a component D_1 with $-K \cdot D_1 \le 0$, or*
(3) *D has a nonimmersed component D_1 with $-K \cdot D_1 = 1$.*

Then the closure of $|\mathcal{L}|^\delta - V$ has codimension δ at all its points (if any), and its sublocus of immersed curves is open and dense, and is smooth off V. Further, if $\operatorname{codim} V > \delta$, then $|\mathcal{L}|^\delta$ has codimension δ at all its points, and $\deg |\mathcal{L}|^\delta = G_\delta(S, \mathcal{L})$.

PROPOSITION 1.2. *Under the conditions of Theorem 1.1, assume $D \in V$ also if either*

(4) *D has a component D_1 with a point of multiplicity at least 3 and with $-K \cdot D_1 \le 3$, or*
(5) *D has two components D_1, D_2 with a common point that is double on D_1 and with $-K \cdot D_1 = 1$ or $-K \cdot D_2 = 1$, or*
(6) *D has two components D_1, D_2 with a common point that is double on D_1 and on D_2 and with $-K \cdot D_1 = 2$ and $-K \cdot D_2 = 2$, or*
(7) *D has two components D_1, D_2 with a common point that is double on D_1 and simple on D_2 and with $-K \cdot D_1 = 2$, or*
(8) *D has three components D_1, D_2, D_3 with a common point that is simple on each and with $-K \cdot D_1 = 1$, or*
(9) *D has two components D_1, D_2 with a common point that is simple on each and at which they are tangent and with $-K \cdot D_1 = 1$, or*
(10) *D has a component D_1 with a nonnodal double point and with $-K \cdot D_1 \le 2$.*

Then in the closure of $|\mathcal{L}|^\delta - V$, its sublocus of nodal curves is open and dense. Further, if $\operatorname{codim} V > \delta$, then $|\mathcal{L}|^\delta_+$ is open and dense in $|\mathcal{L}|^\delta$, and (+) holds.

COROLLARY 1.3. *Assume $S = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}(d)$. If $d \ge \lceil \delta/2 \rceil + 1$, then (+) holds.*

COROLLARY 1.4. *Assume S is the Hirzebruch surface with section E of self-intersection $-e$ with $e \ge 0$. Assume these subloci of $|\mathcal{L}|$ have codimension more* *than δ: (1) the nonreduced curves, (2) if $e \geq 1$, the curves with E as a component. Then (+) holds.*

REMARK 1.5. Göttsche [**13**, Remark 4.3, 4.4] stated without proof that the codimension condition of Corollary 1.4 is equivalent to essentially this condition: say $\mathcal{L} = \mathcal{O}(nF + mE)$ where F is a ruling, and set $p := n - em$; then either $m = 0$, $p = 1$, and $\delta = 1$ or

$$(1.5.1) \qquad m + p \geq 1 \quad \text{and} \quad \delta \leq \begin{cases} \min(2m,\, p) & \text{if } e \geq 1, \\ \min(2m,\, 2p) & \text{if } e = 0. \end{cases}$$

In fact, more is true; the proof of this equivalence plus the main results yield the following statements. Assume $e \geq 1$ and $m \geq 2$ and $p \geq 0$. Assume the nonreduced $D \in |\mathcal{L}|$ appear in codimension more than δ, or equivalently,

$$(1.5.2) \qquad \delta \leq \min(2m,\, 2p + e + 1).$$

Assume $\delta \geq p + e$ too. Then there are curves in $|\mathcal{L}|^\delta$ with E as a component, and they form a component of $|\mathcal{L}|^\delta$ of codimension $\delta - e + 1$; the other components are of codimension δ. Lastly, if $e = 1$, then $\deg |\mathcal{L}|^\delta = G_\delta(S, \mathcal{L})$; further, (+) holds at least if $\delta = p + 1$ too.

COROLLARY 1.6. *Assume S is a classical del Pezzo surface. Assume these subloci of $|\mathcal{L}|$ have codimension more than δ: (1) the nonreduced curves, (2) the curves with a -1-curve as a component. Then (+) holds.*

Section 2 derives the three corollaries from the theorem and the proposition. It also proves the remark. Section 3 proves four lemmas about the Severi variety and the relative Hilbert scheme. Section 4 uses those lemmas to prove the theorem and the proposition, which are the main results.

Throughout, δ, S, $\mathcal{L}$, K, and so forth continue to be as above. In particular, C denotes a reduced member of $|\mathcal{L}|$, and D an arbitrary member. In addition, Γ denotes an arbitrary reduced curve on S, usually integral, but not always.

As some loci may be empty, we adopt the convention that the empty set has dimension -1, and so codimension 1 more than the dimension of the ambient space. Thus, in the theorem and the proposition, the hypothesis $\operatorname{codim} V > \delta \geq 0$ implies that $\dim |\mathcal{L}| \geq 0$; in particular, $\mathcal{L}$ is nontrivial.

2. Proof of the corollaries and the remark

Before addressing the corollaries and the remark, we prove the following lemma, which we use to handle the bounds in Corollary 1.3 and Remark 1.5.

LEMMA 2.1. *Assume that S is rational and that $D \in |\mathcal{L}|$. Then $\mathrm{H}^2(S, \mathcal{L}) = 0$ and $\dim |\mathcal{L}| \geq D \cdot (D - K)/2$. Equality holds and $\mathrm{H}^1(S, \mathcal{L}) = 0$ if this condition obtains: every component Γ of D satisfies $-K \cdot \Gamma \geq 1$, and every Γ that is a -1-curve appears with multiplicity 1.*

PROOF. Since S is integral, $\mathrm{H}^0(S, \mathcal{O}_S) = 1$. Since S is rational, $\mathrm{H}^q(S, \mathcal{O}_S) = 0$ for $q = 1, 2$. Hence the Riemann–Roch theorem yields

$$(2.1.1) \qquad \dim |\mathcal{L}| = D \cdot (D - K)/2 + \dim \mathrm{H}^1(S, \mathcal{L}) - \dim \mathrm{H}^2(S, \mathcal{L}).$$

Thus it suffices to study the vanishing of $H^1(S, \mathcal{L})$ and $H^2(S, \mathcal{L})$.

Given a component Γ of D, let m_Γ denote its multiplicity of appearance. Set $m := \sum m_\Gamma$, and proceed by induction on m. Suppose $m = 0$. Then $D = 0$. So $\mathcal{L} = \mathcal{O}_S$. Hence in this case, both groups vanish.

Suppose $m \geq 1$. Fix a component Γ, and set $\mathcal{L}' := \mathcal{L}(-\Gamma)$. Form the standard sequence $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}|\Gamma \to 0$, and take cohomology to get this sequence:

$$\mathrm{H}^q(S, \mathcal{L}') \to \mathrm{H}^q(S, \mathcal{L}) \to \mathrm{H}^q(S, \mathcal{L}|\Gamma) \quad \text{for} \quad q = 1, 2.$$

By induction, $\mathrm{H}^2(S, \mathcal{L}') = 0$. As Γ is a curve, $\mathrm{H}^2(\Gamma, \mathcal{L}|\Gamma) = 0$. Thus $\mathrm{H}^2(S, \mathcal{L}) = 0$, as desired.

Assume the stated condition obtains. Then by induction, $\mathrm{H}^1(S, \mathcal{L}') = 0$. Thus, it suffices to show $\mathrm{H}^1(\Gamma, \mathcal{L}|\Gamma) = 0$.

Let K_Γ be the canonical class. By adjunction, $\mathcal{O}_\Gamma(K_\Gamma) = \mathcal{O}_\Gamma(\Gamma + K)$. So

$$\mathrm{H}^1(\Gamma, \mathcal{L}|\Gamma) = \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(D - \Gamma + K_\Gamma - K)).$$

The latter group is dual to $\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma(-D + \Gamma + K))$, which vanishes as desired, since Γ is integral and since, as shown next, $(-D + \Gamma + K) \cdot \Gamma < 0$.

First, by hypothesis, $K \cdot \Gamma < 0$. Second, if $m_\Gamma = 1$, then $D - \Gamma$ does not contain Γ, and so $(-D + \Gamma) \cdot \Gamma \leq 0$. Finally, suppose $m_\Gamma \geq 2$. Then, by hypothesis, Γ is not a -1-curve; so $\Gamma^2 \neq -1$ if $K \cdot \Gamma = -1$. But $(\Gamma + K) \cdot \Gamma = \deg K_\Gamma \geq -2$. So $\Gamma^2 \geq -K \cdot \Gamma - 2 \geq -1$. Hence $\Gamma^2 \geq 0$. Thus again $(-D + \Gamma) \cdot \Gamma \leq 0$, as desired. □

Note in passing that, if $\mathcal{L} = \mathcal{O}_S(m\Gamma)$ where Γ is a -1-curve and $m \geq 1$, then (2.1.1) yields $\dim \mathrm{H}^1(S, \mathcal{L}) = m(m-1)/2$.

Proof of Corollary 1.3. Note $\deg K = -3$; so $-K \cdot \Gamma \geq 3$ for every integral curve Γ on S, and $-K \cdot \Gamma \geq 9$ if Γ is singular. So no $D \in |\mathcal{L}|$ satisfies any of (2)–(10) of Theorem 1.1 and Proposition 1.2. Thus it remains to consider (1).

The nonreduced $D \in |\mathcal{L}|$ are of the form $D = A + 2B$ with A, B effective. Set $b := \deg B$. Fix $b \geq 1$. Then these D form a locus of dimension $\dim |A| + \dim |B|$, so of codimension $b(4d - 5b + 3)/2$ owing to Lemma 2.1. But $d \geq 2b$. So

$$\begin{aligned} b(4d - 5b + 3)/2 - (2d - 1) &= (b - 1)(4d - 5b - 2)/2 \\ &\geq (b - 1)(3b - 2)/2 \geq 0. \end{aligned}$$

Therefore, when $b = 1$, the codimension achieves its minimum value, namely, $2d - 1$. This value is more than δ, as desired. □

Proof of Corollary 1.4. For the following basic properties of Hirzebruch surfaces, see [**15**, Chapter V, §2]. Let F be a ruling. Then every curve Γ is equivalent to $nF + mE$ with $n, m \geq 0$. Suppose Γ is integral and $\Gamma \neq E$. Then $n > 0$ and $n - me \geq 0$. Further, $-K = (e + 2)F + 2E$. Finally, $F^2 = 0$ and $F \cdot E = 1$.

Hence $-K \cdot \Gamma = n + (n - me) + 2m$. Suppose $-K \cdot \Gamma \leq 3$. Then either $n = 1$ and $m = 0$, or $n, m, e = 1$. In first case, $-K \cdot \Gamma = 2$; further, $\Gamma = F$, so Γ is smooth. In the second case, $-K \cdot \Gamma = 3$; further, $\Gamma \cdot F = 1$, whence Γ is smooth. On the other hand, E is smooth, and $-K \cdot E = 2 - e$. So if $-K \cdot E \leq 1$, then $e \geq 1$.

In $|\mathcal{L}|$ consider the locus of D with a component Γ such that $-K \cdot \Gamma \leq k$. By the above, if $k = 1$, then $\Gamma = E$ and $e \geq 1$. So by hypothesis, the locus has codimension more than δ. Further, if $k = 3$, then Γ is smooth. Thus all the hypotheses of Theorem 1.1 and Proposition 1.2 obtain; whence, (+) holds, as asserted. □

Proof of Remark 1.5. Fix a section G of S complementary to E. Then G is equivalent to $eF+E$, so that $\mathcal{L}=\mathcal{O}(pF+mG)$. Let's see that, if there's a $D\in|\mathcal{L}|$, then $m\ge 0$; further, $p\ge 0$ if also either $e=0$ or $e\ge 1$ and D doesn't contain E. Indeed, as $|F|$ has no base points, $m=D\cdot F\ge 0$. If $e=0$, then $S=\mathbb{P}^1\times\mathbb{P}^1$; whence by symmetry, $p\ge 0$. If $e\ge 1$, then $p=D\cdot E\ge 0$.

Note that, if the nonreduced $D\in|\mathcal{L}|$ form a locus of codimension more than δ, then $\dim|\mathcal{L}|\ge 0$; in particular, $\mathcal{L}$ is nontrivial. Then $m\ge 0$. Further, if some $D\in|\mathcal{L}|$ doesn't contain E, then $p=D\cdot E\ge 0$. In particular, if the codimension condition of Corollary 1.4 obtains, then $m,\,p\ge 0$. On the other hand, if (1.5.1) obtains, then $m,\,p\ge\delta\ge 0$. Thus to prove the remark, we may assume $m,\,p\ge 0$ and $m+p\ge 1$.

If $m=0$ and $p=1$, then $\dim|\mathcal{L}|=1$, no $D\in|\mathcal{L}|$ contains E, and every D is reduced; whence, then the codimension condition of Corollary 1.4 obtains if and only if $\delta\le 1$, if and only if either $\delta=1$ or (1.5.1) obtains. If $m=0$ and $p\ge 2$, then $\dim|\mathcal{L}|\ge 2$, no $D\in|\mathcal{L}|$ contains E, and the nonreduced D form a locus of codimension 1. Hence, then the codimension condition of Corollary 1.4 obtains if and only if $\delta=0$, if and only if (1.5.1) obtains. Thus, to complete the proof, we may assume $m\ge 1$; further, if $e=0$, then by symmetry, we may assume $p\ge 1$ too.

The proof of Corollary 1.4 yields $-K\cdot F=2$ and $-K\cdot G=e+2$. Also $\mathcal{L}=\mathcal{O}(pF+mG)$ and $m,\,p\ge 0$. So Lemma 2.1 yields this formula:

$$\dim|\mathcal{L}|=pm+p+m+me(1+m)/2.$$

The $D\in|\mathcal{L}|$ containing E are of the form $D=A+E$ with A effective. Set

$$\mathcal{L}':=\mathcal{O}_S\big((p+e)F+(m-1)G\big). \tag{2.1.2}$$

Then $A\in|\mathcal{L}'|$. But we now assume $p\ge 0$ and $m\ge 1$. So Lemma 2.1 yields

$$\dim|\mathcal{L}'|=pm-1+m+me(1+m)/2\ge 1.$$

If $e\ge 1$, then $\dim|E|=0$ as $E^2=-e$ (whereas if $e=0$, then $\dim|E|=1$); so the $D\in|\mathcal{L}|$ containing E form a nonempty locus of codimension exactly $p=1$:

$$\dim|\mathcal{L}|-\dim|\mathcal{L}'|=p+1.$$

Thus, if $e\ge 1$, then the $D\in|\mathcal{L}|$ containing E appear in codimension more than δ if and only if $\delta\le p$.

By the same token, if $e\ge 1$ and if $m\ge 2$, then the $A\in|\mathcal{L}'|$ containing E appear in codimension $p+e+1$. Conversely, if $e\ge 1$ and if there exists such an A, then $m-2=(A-E)\cdot F\ge 0$. Thus if $e\ge 1$, then there exists a $D\in|\mathcal{L}|$ containing $2E$ if and only if $m\ge 2$; if so, then these D form a locus of codimension $2p+e+2$.

Given a nonreduced $D\in|\mathcal{L}|$, say $D=A+2B$ with A,B effective and $B\ne 0$. Say B is equivalent to $aF+bG$. Then A is equivalent to $(p-2a)F+(m-2b)G$. Since A and B are effective, $m-2b\ge 0$ and $b\ge 0$. If $e\ge 1$, assume D does not contain E. Then $p-2a\ge 0$ and $a\ge 0$ for any e. Hence, for fixed a and b, these D form a locus of dimension $\dim|A|+\dim|B|$; so Lemma 2.1 yields its codimension to be

$$\epsilon(a,b):=2pb+2am-5ab+a+b+(1+4m-5b)be/2.$$

The above analysis assumed given some D and A and B. However, given $a,\,b\ge 0$ such that $p-2a\ge 0$ and $m-2b\ge 0$, set

$$A:=(p-2a)F+(m-2b)G,\quad B:=aF+bG,\quad D:=A+2B.$$

Then A and B are effective. Also, $D \in |\mathcal{L}|$, and D does not contain E. Further, $B \neq 0$ if $a+b \geq 1$. So the above analysis yields a locus of nonreduced members of $|\mathcal{L}|$ of codimension $\epsilon(a,b)$.

Note $\epsilon(0,1) = 2p+1+2e(m-1)$. But $p \geq 2a$ and $m \geq 2b$. So if $b \geq 1$, then

$$\begin{aligned}\epsilon(a,b) - \epsilon(0,1) &= (2p+1)(b-1) + a(2m-5b+1) \\ &\qquad + (4m-5b-4)(b-1)e/2 \\ &\geq (3a+1+(3b-4)e/2)(b-1).\end{aligned}$$

The latter term is nonnegative. Further,

$$\epsilon(a,0) = a(2m+1) \geq \epsilon(1,0) = 2m+1.$$

Thus $\min \epsilon(a,b) = \min\bigl(\epsilon(1,0),\, \epsilon(0,1)\bigr) = \min\bigl(2m+1,\, 2p+1+2e(m-1)\bigr)$.

Suppose $e = 0$. Then we are assuming $m, p \geq 1$. Hence the nonreduced $D \in |\mathcal{L}|$ form a nonempty locus of codimension exactly $\min\bigl(2m+1,\, 2p+1\bigr)$. Thus the codimension condition of Corollary 1.4 obtains if and only if (1.5.1) obtains, as asserted.

Suppose $e \geq 1$ and the codimension condition of Corollary 1.4 obtains. In this case, we assume $m \geq 1$ and $p \geq 0$. Then, as proved above, $\delta \leq p$. So if $p \leq 1$, then $\delta \leq 2m$. If $p \geq 2$, take $a := 1$ and $b := 0$; then the codimension condition yields $\epsilon(1,0) > \delta$. But $\epsilon(1,0) = 2m+1$. Thus (1.5.1) obtains, as asserted.

Conversely, suppose $e \geq 1$ and (1.5.1) obtains. Then, as proved above, the $D \in |\mathcal{L}|$ containing E appear in codimension more than δ. Also, the nonreduced $D \in |\mathcal{L}|$ not containing E appear in codimension $\min\bigl(2m+1,\, 2p+1+2e(m-1)\bigr)$. But we assume $m-1 \geq 0$. Thus the codimension condition of Corollary 1.4 obtains, as asserted.

Finally, assume $e \geq 1$ and $m \geq 2$. Then $2e(m-1) \geq e+1$. Let W be the locus of all nonreduced curves. Then $\operatorname{codim} W = \min(2m+1,\, 2p+e+2)$. Thus $\operatorname{codim} W > \delta$ if and only if (1.5.2) obtains, as asserted. Assume (1.5.2) does obtain.

Assume $\delta \geq p+e$ too. Set $\delta' := \delta - p - e$. Then $\delta' \leq p+1$ as $\delta \leq 2p+e+1$; so $\delta' \leq p+e$ as $e \geq 1$. Further, $\delta \leq 2m$, so $\delta' \leq 2m-p-e$. Hence $\delta' \leq 2m-2$, except possibly if $p=0$; but then, $\delta' \leq 1$, so after all $\delta' \leq 2m-2$ as $m \geq 2$.

Consider the $\mathcal{L}'$ of (2.1.2). By the above analysis, the Severi variety $|\mathcal{L}'|^{\delta'}$ is nonempty and everywhere of codimension δ' in $|\mathcal{L}'|$, so of codimension $\delta - e + 1$ in $|\mathcal{L}|$. Further, $|\mathcal{L}'|^{\delta'}$ contains a dense open subset of curves A not containing E. Set $D := A+E$. Then $D \in |\mathcal{L}|^{\delta}$ as $p_a D = p_a A + p_a E + A \cdot E - 1$ and $p_g D = p_g A + p_g E - 1$ by general principles. Conversely, given a $D \in |\mathcal{L}|^{\delta}$ containing E, set $A := D - E$; then, plainly, $A \in |\mathcal{L}'|^{\delta'}$, and A does not contain E.

Recall that $\operatorname{codim} W > \delta$; further, if Γ is an integral curve with $-K \cdot \Gamma \leq 1$, then $\Gamma = E$. Let V be the union of W and the locus of $D \in |\mathcal{L}|$ containing E. Then by Theorem 1.1, the closure of $|\mathcal{L}|^{\delta} - V$ has codimension δ everywhere. Consequently, there are $D \in |\mathcal{L}|^{\delta}$ containing E, and they form a component of $|\mathcal{L}|^{\delta}$ of codimension $\delta - e + 1$; the other components of $|\mathcal{L}|^{\delta}$ are of codimension δ, as asserted.

Lastly, assume $e = 1$ in addition. Then $-K \cdot E = 1$ and E is immersed. Thus Theorem 1.1 yields $\deg |\mathcal{L}|^{\delta} = G_{\delta}(S, \mathcal{L})$, as asserted.

Further, by Proposition 1.2, the nodal curves form an open and dense subset of $|\mathcal{L}|^{\delta} - V$. Assume $\delta = p+1$ also. Then $\delta' = 0$. So the $D \in |\mathcal{L}|^{\delta}$ containing E are the $D \in |\mathcal{L}|$ of the form $A+E$ where $A \in |\mathcal{L}'| - V$. The A that meet E transversally form a dense open sublocus, because the restriction map $\mathrm{H}^0(S,\, \mathcal{L}) \to \mathrm{H}^0(S,\, \mathcal{L}|E)$

is surjective as $\mathrm{H}^1(S,\mathcal{L})=0$ by Lemma 2.1. Hence the nodal locus is open and dense in $|\mathcal{L}|^\delta$. Thus (+) holds, as asserted. □

PROOF OF COROLLARY 1.6. Since S is a classical del Pezzo surface, we may regard S as embedded in a projective space with $-K$ as the hyperplane class. Let $\Gamma\subset S$ be an integral curve. Suppose $-K\cdot\Gamma=1$. Then Γ is a line. So adjunction yields $\Gamma^2=-1$. Hence Γ is a -1-curve. In $|\mathcal{L}|$ consider the locus of D with a component D_1 such that $-K\cdot D_1=1$; by hypothesis, this locus therefore has codimension more than δ. If $-K\cdot\Gamma=2$, then Γ is an integral plane conic, so smooth. Finally, if $-K\cdot\Gamma=3$, then Γ is either a twisted cubic, so smooth, or else an integral plane cubic, so has no point of multiplicity at least 3. Thus all the hypotheses of Theorem 1.1 and Proposition 1.2 obtain; whence, (+) holds, as asserted. □

3. Four lemmas

We now set the stage to prove Theorem 1.1 and Proposition 1.2. First off, we recall some basic deformation theory from [**8**] and [**14**].

Fix the reduced curve $C\in|\mathcal{L}|$. There exist a smooth (analytic or étale) germ

$$(\Lambda,0):=(\mathrm{Def}_{\mathrm{loc}}(C),\,0)$$

and a family $\mathcal{C}_\Lambda/\Lambda$ realizing a miniversal deformation of the singularities of C; that is, given any family $\mathcal{C}_B/B$ and point $b\in B$ such that the fiber $\mathcal{C}_b$ is a multigerm of C along its singular locus Σ, there exists a map of germs $(B,b)\to(\Lambda,0)$ such that the multigerm $(\mathcal{C}_B,\Sigma)$ is the pullback of the multigerm $(\mathcal{C}_\Lambda,\Sigma)$. The tangent map $T_bB\to T_0\Lambda$ is canonical. Further, there is an identification

$$T_0\Lambda=\mathrm{H}^0(C,\,\mathcal{O}_C/\mathcal{J}) \tag{3.0.1}$$

where $\mathcal{J}$ is the *Jacobian* ideal of C, the first Fitting ideal of its Kähler differentials.

Denote the cogenus of C by $\delta(C)$ and the *normalization* map by

$$n\colon\widetilde{C}\to C.$$

So $\delta(C)=\dim\mathrm{H}^0(n_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$. Denote the locus of $a\in\Lambda$ with $\delta(\mathcal{C}_a)=\delta(C)$ by $\Lambda^{\delta(C)}$. It is called the *equigeneric locus* or *δ-constant stratum.* Its codimension is $\delta(C)$. Its reduced tangent cone $(\mathbf{C}_0\Lambda^\delta)_{\mathrm{red}}$ is a vector space; namely,

$$(\mathbf{C}_0\Lambda^\delta)_{\mathrm{red}}=\mathrm{H}^0(C,\,\mathcal{A}/\mathcal{J}) \tag{3.0.2}$$

under the identification (3.0.1). Here $\mathcal{A}$ denotes the *conductor* ideal sheaf; namely,

$$\mathcal{A}:=\mathcal{H}om(n_*\mathcal{O}_{\widetilde{C}},\,\mathcal{O}_C).$$

The following lemma regarding $\mathcal{A}$ is fundamental. It is more or less well known.

LEMMA 3.1. *Denote by $K_{\widetilde{C}}$ the canonical class of $\widetilde{C}$. Then*

$$\mathcal{A}\cdot n^*\mathcal{O}_S(C)=O_{\widetilde{C}}(K_{\widetilde{C}}-n^*K) \tag{3.1.1}$$

where, doing double duty, n also denotes the composition $n\colon\widetilde{C}\to C\hookrightarrow S$.

Let $\widetilde{M}$ be a line bundle on $\widetilde{C}$, and $\widetilde{C}_1,\dots,\widetilde{C}_h$ be the components of $\widetilde{C}$. Then

$$\dim\mathrm{H}^1\big(C,\ \mathcal{A}\cdot n_*\widetilde{M}\otimes\mathcal{O}_S(C)\big)\le\textstyle\sum_{i=1}^h\max\big(0,\ 1+\deg\big(\widetilde{M}^{-1}(n^*K)\big|\widetilde{C}_i\big)\big). \tag{3.1.2}$$

PROOF. By adjunction, $\mathcal{O}_C(K_C) = \mathcal{O}_C \otimes \mathcal{O}_S(C+K)$. And relative duality yields

$$n_*\mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}}) = \mathcal{H}om(n_*\mathcal{O}_{\widetilde{C}},\, \mathcal{O}_C(K_C)) = \mathcal{A} \otimes \mathcal{O}_C(K_C).$$

Hence $\mathcal{A} \otimes \mathcal{O}_S(C) = n_*\mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}}) \otimes \mathcal{O}_S(-K)$. But n is finite, and that equation is just the image under n_* of (3.1.1). Thus (3.1.1) holds.

By the same token, $\mathrm{H}^1(C,\, \mathcal{A} \cdot n_*\widetilde{M} \otimes \mathcal{O}_S(C)) = \mathrm{H}^1(\widetilde{C},\, \widetilde{M}(K_{\widetilde{C}} - n^*K))$. By duality, the right side is just $\mathrm{H}^0(\widetilde{C},\, \widetilde{M}^{-1}(n^*K))^\vee$; whence, (3.1.2) holds. □

Since $C \in |\mathcal{L}|$, the tangent map $T_C|\mathcal{L}| \to T_0\Lambda$ is just this restriction map:

$$\mathrm{H}^0(S,\, \mathcal{L}) \big/ \operatorname{Im} \mathrm{H}^0(S, \mathcal{O}_S) \to \mathrm{H}^0(C,\, \mathcal{O}_C/\mathcal{J}). \tag{3.1.3}$$

Consequently, using Lemma 3.1, we can prove the following results about the Severi variety and the Hilbert scheme. The results about the Severi variety are already known in various forms, see [**4**, (10.1), p. 845], [**7**, Proposition 2.21 p. 355], [**32**, Theorem 2.8, p. 8], [**35**, Theorem 3.1, p. 59], and [**38**, Theorem 1, p. 215; Theorem 2, p. 220]. However, our particular approach and results appear to be new.

LEMMA 3.2. *Assume* $C \in |\mathcal{L}|^\delta$. *Set* $\lambda := \dim \operatorname{Ker}(\mathrm{H}^1(S,\, \mathcal{O}_S) \to \mathrm{H}^1(S,\, \mathcal{L}))$ *and* $\alpha := \dim \operatorname{Ker}(\mathrm{H}^1(C,\, \mathcal{A} \cdot \mathcal{O}_C(C)) \to \mathrm{H}^1(C, \mathcal{O}_C(C)))$. *Then*

$$\delta - \alpha - \lambda \le \dim_C |\mathcal{L}| - \dim_C |\mathcal{L}|^\delta \le \delta \quad \textit{and} \tag{3.2.1}$$

$$(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}} \subset \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)). \tag{3.2.2}$$

In addition, assume $\lambda = 0$ *and* $\alpha = 0$. *Then*

$$(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}} = \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)). \tag{3.2.3}$$

Finally, assume C *is immersed too. Then* $|\mathcal{L}|^\delta$ *is smooth at* C.

PROOF. Plainly, $|\mathcal{O}_S(C)|^\delta$ is, locally at C, the preimage of the equigeneric locus Λ^δ in $\Lambda := \mathrm{Def}_{\mathrm{loc}}(C)$. As codimension cannot increase on taking a preimage from a smooth ambient target, the right-hand bound holds in (3.2.1).

In general, let $f\colon X \to Y$ be a map of schemes, $x \in X$ a point, $y := f(x) \in Y$ the image. Plainly, f induces maps of tangent spaces $\mathbf{T}_f\colon \mathbf{T}_x(X) \to \mathbf{T}_y(Y)$ and tangent cones $\mathbf{C}_x(X) \to \mathbf{C}_y(Y)$, so a map of reductions $\mathbf{C}_x(X)_{\mathrm{red}} \to \mathbf{C}_y(Y)_{\mathrm{red}}$. Thus $\mathbf{C}_x(X)_{\mathrm{red}} \subset \mathbf{T}_f^{-1}(\mathbf{C}_y(Y)_{\mathrm{red}})$. Now, take $|\mathcal{L}|^\delta \to \Lambda^\delta$ for f, and take C for x. Therefore, $(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}}$ lies in the preimage of $(\mathbf{C}_0\Lambda^\delta)_{\mathrm{red}}$ in $\mathbf{T}_C|\mathcal{L}|^\delta$. However, $\mathbf{T}_C|\mathcal{L}|^\delta \subset \mathbf{T}_C|\mathcal{L}|$. Thus $(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}}$ lies in the preimage of $(\mathbf{C}_0\Lambda^\delta)_{\mathrm{red}}$ in $\mathbf{T}_C|\mathcal{L}|$.

Further, the tangent map $T_C|\mathcal{L}| \to T_C\Lambda$ is given by this composition:

$$\theta\colon\ \mathrm{H}^0(S,\, \mathcal{L}) \big/ \operatorname{Im} \mathrm{H}^0(S, \mathcal{O}_S) \xhookrightarrow{\eta} \mathrm{H}^0(C,\, \mathcal{O}_C(C)) \xrightarrow{\nu} \mathrm{H}^0(C,\, \mathcal{O}_C/\mathcal{J}). \tag{3.2.4}$$

Therefore, (3.0.2) and the injectivity of η yield

$$(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}} \subset \theta^{-1}\, \mathrm{H}^0(C,\, \mathcal{A}/\mathcal{J}) \subset \nu^{-1}\, \mathrm{H}^0(C,\, \mathcal{A}/\mathcal{J}). \tag{3.2.5}$$

Consider the following composition:

$$\xi\colon\ \mathrm{H}^0(C,\, \mathcal{O}_C(C)) \xrightarrow{\nu} \mathrm{H}^0(C,\, \mathcal{O}_C/\mathcal{J}) \xrightarrow{\rho} \mathrm{H}^0(C,\, \mathcal{O}_C/\mathcal{A}). \tag{3.2.6}$$

The left-exactness of H^0 yields $\mathrm{H}^0(C,\, \mathcal{A}/\mathcal{J}) = \operatorname{Ker}\rho$ and $\mathrm{H}^0(C,\, \mathcal{A} \cdot \mathcal{O}_C(C)) = \operatorname{Ker}\xi$. But $\nu^{-1}\operatorname{Ker}\rho = \operatorname{Ker}\xi$. Hence $\nu^{-1}\,\mathrm{H}^0(C,\, \mathcal{A}/\mathcal{J}) = \mathrm{H}^0(C,\, \mathcal{A} \cdot \mathcal{O}_C(C))$. But (3.1.1) implies $\mathrm{H}^0(C,\, \mathcal{A} \cdot \mathcal{O}_C(C)) = \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K))$. Thus (3.2.2) holds.

The above considerations also yield $\nu^{-1}\,\mathrm{H}^0(C,\, \mathcal{A}/\mathcal{J}) = \operatorname{Ker}\xi$. So (3.2.5) yields

$$\dim_C |\mathcal{L}|^\delta = \dim(\mathbf{C}_C|\mathcal{L}|^\delta)_{\mathrm{red}} \le \dim \operatorname{Ker}\xi. \tag{3.2.7}$$

On the other hand, the long exact cohomology sequences involving η and ξ yield

$$-\dim|\mathcal{L}| + \dim \mathrm{H}^0(C, \mathcal{O}_C(C)) - \lambda = 0 \tag{3.2.8}$$

$$\dim \operatorname{Ker}\xi - \dim \mathrm{H}^0(C, \mathcal{O}_C(C)) + \dim \mathrm{H}^0(C, \mathcal{O}_C/\mathcal{A}) - \alpha = 0. \tag{3.2.9}$$

But $\dim \mathrm{H}^0(C, \mathcal{O}_C/\mathcal{A}) = \delta$. Thus, combined, (3.2.7) and (3.2.8) and (3.2.9) yield the left-hand bound in (3.2.1).

In addition, assume $\lambda = 0$ and $\alpha = 0$. To prove (3.2.3), let's show both sides of (3.2.2) are of the same dimension. The left-hand side is of dimension $\dim|\mathcal{L}| - \delta$ by (3.2.1). On the other hand, (3.2.8) and (3.2.9) yield $\dim \operatorname{Ker}\xi = \dim|\mathcal{L}| - \delta$, and the considerations after (3.2.6) show $\operatorname{Ker}\xi$ is equal to the right-hand side, as desired.

Finally, assume C is immersed too. Then Λ^δ is smooth at C by Theorem 2.59(1)(c) of [**14**, p. 355]. So $\mathbf{T}_0\Lambda^\delta = H^0(C, \mathcal{A}/\mathcal{J})$ by (3.0.2). Always, $\mathbf{T}_C|\mathcal{L}|^\delta$ maps into $\mathbf{T}_0\Lambda^\delta$; so $\mathbf{T}_C|\mathcal{L}|^\delta$ lies in the preimage $\mathbf{T}$ of $\mathbf{T}_0\Lambda^\delta$ in $\mathbf{T}_C|\mathcal{L}|$. But $\mathbf{T}$ is a vector space of codimension δ owing to the above analysis; indeed, $\mathbf{T} = \theta^{-1}\,\mathrm{H}^0(C, \mathcal{A}/\mathcal{J})$, and in (3.2.5), the two extreme terms are of codimension δ. But $\operatorname{codim} \mathbf{T}_C|\mathcal{L}|^\delta \le \delta$ by (3.2.1). Thus $\dim \mathbf{T}_C|\mathcal{L}|^\delta = \dim_C |\mathcal{L}|^\delta$. Thus $|\mathcal{L}|^\delta$ is smooth at C. $\square$

In the remaining two lemmas, we assume S is *regular*; that is, $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. As a consequence, in Theorem 1.1 and Proposition 1.2, instead of assuming S is rational, we may assume S is regular. But the "generalization is illusory," as noted in [**30**, (v), p. 116] in a similar situation. Indeed, assume $\dim|\mathcal{L}| \ge 1$, else $\mathcal{L}$ holds little interest. Assume $C \in |\mathcal{L}| - V$. Let C' be its variable part, so that $|C'|$ has no fixed components. Then C' is nonzero and nef. Hence $\mathrm{H}^0(S, mK) = 0$ for all $m \ge 1$; else, $K \cdot C' \ge 0$, but $-K \cdot \Gamma \ge 1$ for every component Γ of C' as $C \notin V$. Since $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, Castelnuovo's Criterion implies S is rational.

The first lemma below addresses the immersedness of a general member of $|\mathcal{L}|^\delta$. The discussion involves another invariant of the reduced curve C on S, namely, the (total) multiplicity of its Jacobian ideal $\mathcal{J}$, or what is the same, the colength of its extension $\mathcal{J}\mathcal{O}_{\widetilde{C}}$ to the normalization of C. This invariant was introduced by Teissier [**31**, II.6′, p. 139] in order to generalize Plücker's formula for the class (the degree of the dual) of a plane curve.

This invariant was denoted $\kappa(C)$ by Diaz and Harris [**8**, (3.2), p. 441], but they defined it by the formula

$$\kappa(C) = 2\delta(C) + m(C)$$

where $m(C)$ denotes the (total) ramification degree of $\widetilde{C}/C$. The two definitions are equivalent owing to the following formula, due to Piene [**27**, p. 261]:

$$\mathcal{J}\mathcal{O}_{\widetilde{C}} = \mathcal{A} \cdot \mathcal{R} \tag{3.2.10}$$

where $\mathcal{R}$ is the ramification ideal.

The invariant $\kappa(C)$ is upper semicontinuous in C; see [**31**, p. 139] or [**8**, bot., p. 450]. So $|\mathcal{L}|^\delta$ always contains a dense open subset $|\mathcal{L}|^\delta_\kappa$ on which $\kappa(C)$ is locally constant, termed an *equiclassical locus* in [**8**].

By definition, C is immersed if and only if $m(C) = 0$. Thus if $C \in |\mathcal{L}|^\delta_\kappa$, then $\kappa(C) \ge 2\delta$, and C is immersed if and only if $\kappa(C) = 2\delta$. Further, if so, then every curve D in every component of $|\mathcal{L}|^\delta_\kappa$ containing C is immersed.

LEMMA 3.3. *Assume S regular, and $C \in |\mathcal{L}|^\delta_\kappa$. Assume $-K \cdot C_1 \ge 1$ for every component C_1 of C. If some C_1 is not immersed, then $-K \cdot C_1 = 1$.*

Proof. Fix a C_1. Assume C_1 is not immersed, but $-K \cdot C_1 \geq 2$. Then there's a point $\widetilde{P}$ in the normalization of C_1 at which n ramifies. Set $\mathcal{A}' := \mathcal{A} \cdot n_* \mathcal{O}_{\widetilde{C}}(-\widetilde{P})$. Then owing to Lemma 3.1, the restriction map

$$\mathrm{H}^0(C, \mathcal{O}_C(C)) \to \mathrm{H}^0(C, \mathcal{O}_C/\mathcal{A}')$$

is surjective. Since S is regular, the following restriction map too is surjective:

$$\mathrm{H}^0(S, \mathcal{L}) \to \mathrm{H}^0(C, \mathcal{O}_C(C)).$$

Set $\mathcal{H} := n_*(\mathcal{J}\mathcal{O}_{\widetilde{C}})$. Then $\mathcal{A}' \supset \mathcal{H}$ owing to Piene's Formula (3.2.10). But $\mathcal{H} \supset \mathcal{J}$. Set $\Lambda := \mathrm{Def}_{\mathrm{loc}}(C)$. It follows, as in the proof of Lemma 3.2, that the image of $T_C|\mathcal{L}|$ in $T_0\Lambda$ is transverse to $\mathcal{A}'/\mathcal{J}$. Thus the image of $|\mathcal{L}|$ in Λ contains a 1-parameter equigeneric family whose tangent space at 0 is transverse to $\mathcal{A}'/\mathcal{J}$ inside $\mathcal{A}/\mathcal{J}$.

Diaz and Harris [**8**, (5.5), p. 459] proved that $\mathcal{H}/\mathcal{J}$ is the reduced tangent cone to the locus of equiclassical deformations. Thus the above 1-parameter family exits $|\mathcal{L}|^\delta_\kappa$ while remaining in $|\mathcal{L}|^\delta$, contrary to the openness of $|\mathcal{L}|^\delta_\kappa$ in $|\mathcal{L}|^\delta$. □

Finally, we consider the smoothness over $\mathbb{C}$ of the relative Hilbert scheme of a family. To be precise, given a family of curves with parameter space B and total space $\mathcal{C}_B$, denote by $\mathcal{C}_B^{[n]}$ the relative Hilbert scheme of n points. Further, if $B \subset |\mathcal{L}|$, take $\mathcal{C}_B$ to be the total space of the tautological family.

Lemma 3.4. *Assume S regular, and $-K \cdot C_1 \geq 1$ for every component C_1 of C. Fix $n \geq 0$. Then the relative Hilbert scheme $\mathcal{C}^{[n]}_{|\mathcal{L}|}$ is smooth over $\mathbb{C}$ along the Hilbert scheme $C^{[n]}$ of C over $\mathbb{C}$.*

Proof. The proof has three steps: (1) show that $\mathcal{C}^{[n]}_\Lambda$ is smooth over $\mathbb{C}$ along $C^{[n]}$; (2) show that, for any point $z \in C^{[n]}$, the image in $T_0\Lambda$ of the tangent space $T_z\mathcal{C}^{[n]}_\Lambda$ contains $\mathcal{A}/\mathcal{J}$ in $T_0\Lambda$; and (3) show that $\mathcal{C}^{[n]}_{|\mathcal{L}|}$ is smooth over $\mathbb{C}$ along $C^{[n]}$. The hypothesis that S is regular and $-K \cdot C_1 \geq 1$ is not used in the first two steps.

Step (1) was done in [**29**, Proposition 17]. Here's the idea. First, embed $\mathcal{C}^{[n]}_\Lambda$ in $S^{[n]} \times \Lambda$, where $S^{[n]}$ is the Hilbert scheme. The latter is smooth by Fogarty's theorem. Form the tangent bundle-normal bundle sequence (constructed barehandedly as (6) in [**29**]); it's the dual of the Second Exact Sequence of Kähler differentials [**15**, Proposition 8.12, p. 176] . It shows the question is local analytic about the singularities of C, as the smoothness in question is equivalent to the surjectivity of the right-hand map owing to [**11**, (17.12.1)]. So we may replace C by an affine plane curve $\{f = 0\}$.

Take a vector space $\mathbb{V}$ of polynomials containing f and also every polynomial of degree at most n. Form the tautological family $\mathcal{C}_\mathbb{V}/\mathbb{V}$. Its relative Hilbert scheme $\mathcal{C}^{[n]}_\mathbb{V}$ is smooth over $\mathbb{C}$ along $C^{[n]}$ owing to the analogous tangent bundle-normal bundle sequence; its right-hand map is surjective by choice of $\mathbb{V}$. Finally, as Λ is versal, there's a map of germs $\lambda\colon (\mathbb{V}, 0) \to (\Lambda, 0)$ such that $\mathcal{C}^{[n]}_\mathbb{V}$ is the pullback of $\mathcal{C}^{[n]}_\Lambda$. It's smooth as the map on tangent spaces is surjective. Thus $\mathcal{C}^{[n]}_\Lambda$ is smooth over $\mathbb{C}$ along $C^{[n]}$, as desired.

To do Step (2), we may assume that z represents a subscheme Z of C supported on its singular locus Σ, because the map of tangent spaces (essentially the map on the left in [**29**, (6)]) is the product of the corresponding maps at the various points p in the support of Z, and these maps are clearly surjective at the p where C is smooth. Set $\mathcal{O} := \mathcal{O}_{C,\Sigma}$, and let $I \subset \mathcal{O}$ be the ideal of Z. Then $T_z\mathcal{C}^{[n]}_\Lambda$ is the set of first-order deformations of the inclusion map $I \hookrightarrow \mathcal{O}$. Further, the map $T_z\mathcal{C}^{[n]}_\Lambda \to T_0\Lambda$ forgets the inclusion, and just keeps the deformation of $\mathcal{O}$.

Let J be the Jacobian ideal of $\mathcal{O}$, the ideal of Σ. Then (3.0.1) yields $T_0\Lambda = \mathcal{O}/J$. Further, let A be the conductor ideal of $\mathcal{O}$.

The map $T_z\mathcal{C}^{[n]}_\Lambda \to T_0\Lambda$ factors through the set $D(\mathcal{O}, I)$ of first-order deformations of the pair $(\mathcal{O}, I)$ with I viewed as an abstract $\mathcal{O}$-module. The map $D(\mathcal{O}, I) \to T_0\Lambda$ was studied by Fantechi, Göttsche and van Straten in [**10**, Sec. C]; they showed that, in $\mathcal{O}/J$, the image of this map contains A/J.

It remains to show $T_z\mathcal{C}^{[n]}_\Lambda \to D(\mathcal{O}, I)$ is surjective. So take $(\mathcal{O}', I') \in D(\mathcal{O}, I)$. As $\mathcal{O}$ is Gorenstein, $\mathrm{Ext}^1_{\mathcal{O}}(I,\, \mathcal{O}) = 0$. Hence, since deformations are flat, the Property of Exchange [**1**, Theorem (1.10)] implies this natural map is bijective:

$$\mathrm{Hom}_{\mathcal{O}}(I',\, \mathcal{O}') \otimes_{\mathcal{O}'} \mathcal{O} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(I,\, \mathcal{O}).$$

So the inclusion map $I \hookrightarrow \mathcal{O}$ lifts to a map $I' \to \mathcal{O}'$. The latter is injective and its cokernel is flat owing to the Local Criterion of Flatness, as $\mathcal{O}'$ is flat and $I' \to \mathcal{O}'$ reduces to an injection with flat cokernel, namely, $I \hookrightarrow \mathcal{O}$.

Finally, consider Step (3). Since Λ is versal, there exists a map of germs $(|\mathcal{L}|, C) \to (\Lambda, 0)$ such that the germ $(\mathcal{C}^{[n]}_{|\mathcal{L}|}, z)$ is the pullback of the germ $(\mathcal{C}^{[n]}_\Lambda, z)$, which is smooth over $\mathbb{C}$ by Step (1). Since $(|\mathcal{L}|, C)$ and $(\Lambda, 0)$ are smooth over $\mathbb{C}$, the pullback $(\mathcal{C}^{[n]}_{|\mathcal{L}|}, z)$ is therefore smooth over $\mathbb{C}$ by general principles, if the images in $T_0\Lambda$ of the tangent spaces $T_C|\mathcal{L}|$ and $T_z\mathcal{C}^{[n]}_\Lambda$ sum to $T_0\Lambda$.

Owing to (3.1.3) and to Step (2), the latter holds if this composition is surjective:

$$\mathrm{H}^0(S,\, \mathcal{L}) \to \mathrm{H}^0(C,\, \mathcal{O}_C(C)) \to \mathrm{H}^0(C,\, \mathcal{O}_C/\mathcal{A}).$$

However, the first map is surjective as S is regular, and the second map is surjective by Lemma 3.1 with $\widetilde{M} = \mathcal{O}_{\widetilde{C}}$ owing to the hypothesis $-K \cdot C_1 \geq 1$. □

4. Proof of the main results

Theorem 1.1 can now be proved by revisiting the construction in [**21**] of the universal polynomial $G_\delta(S, \mathcal{L})$ and making use of the lemmas in the preceding section.

Proof of Theorem 1.1. First, (3.2.1) gives $\mathrm{codim}_C\, |\mathcal{L}|^\delta \leq \delta$ for all $C \in |\mathcal{L}|^\delta$. Also, $\mathrm{H}^1(S,\, \mathcal{O}_S) = 0$ as S is rational, and by (3.1.2), if $C \in (|\mathcal{L}|^\delta - V)$, then $\mathrm{H}^1(C,\, \mathcal{A}\cdot\mathcal{O}_C(C)) = 0$; hence, if $C \in (|\mathcal{L}|^\delta - V)$, then (3.2.1) yields $\mathrm{codim}_C\, |\mathcal{L}|^\delta \geq \delta$. Therefore, if $\mathrm{codim}\, V > \delta$, then $\mathrm{codim}_C\, |\mathcal{L}|^\delta = \delta$ for all C in the closure $\big(|\mathcal{L}|^\delta - V\big)^-$, and then $\big(|\mathcal{L}|^\delta - V\big)^- = |\mathcal{L}|^\delta$.

Note that Lemma 3.3 and the discussion before it imply that, if $C \in \big(|\mathcal{L}|^\delta - V\big)^-$, then $C \in |\mathcal{L}|^\delta_\kappa$ if and only if C is immersed, and that $|\mathcal{L}|^\delta_\kappa$ is open and dense in $|\mathcal{L}|^\delta$. Further, the last assertion of Lemma 3.2 now implies $|\mathcal{L}|^\delta_\kappa$ is smooth at C if $C \notin V$.

It remains to compute $\deg |\mathcal{L}|^\delta$ assuming $\mathrm{codim}\, V > \delta$. Denote by g the common arithmetic genus $p_a D$ of the $D \in |\mathcal{L}|$. Bertini's theorem [**15**, Corollary 10.9, p. 274] yields a δ-plane $\mathbb{P} \subset |\mathcal{L}|$ avoiding $V \bigcup \big(|\mathcal{L}| - |\mathcal{L}|^\delta_\kappa\big)$ and such that $\mathcal{C}^{[n]}_{\mathbb{P}}$ is smooth over $\mathbb{C}$ for $n \leq g$. But $\mathcal{C}^{[n]}_{\mathbb{P}}$ is, by [**2**, Theorem 5, p. 5], cut out of $\mathbb{P} \times S^{[n]}$, where $S^{[n]}$ is the Hilbert scheme, by a transversally regular section of the rank-n bundle $\mathcal{L}^{[n]}$ that is obtained by pulling $\mathcal{L}$ back to the universal family and then pushing it down. Hence the topological Euler characteristic $\chi(\mathcal{C}_{\mathbb{P}}^{[n]})$ can be computed by integrating polynomials in the Chern classes of $\mathcal{L}^{[n]}$ and $S^{[n]}$. But, as Ellingsrud, Göttsche, and Lehn [**9**] show, such integrals admit universal polynomial expressions in the Chern classes of S and $\mathcal{L}$.

Following [**18**], define $n_h(\mathbb{P})$ by this relation:

$$\textstyle\sum_{n=0}^{\infty} q^n \chi(\mathcal{C}_{\mathbb{P}}^{[n]}) = \sum_{h=-\infty}^{g} n_h(\mathbb{P}) q^{g-h}(1-q)^{2h-2}.$$

For $D \in |\mathcal{L}|$, define $n_h(D)$ similarly. By additivity of the Euler characteristic, these definitions are compatible: $\chi(\mathbb{P}, n_h) = n_h(\mathbb{P})$ where $n_h\colon \mathbb{P} \to \mathbb{Z}$ is the constructible function $b \mapsto n_h(\mathcal{C}_b)$. By [**26**, Appendix B.1], if D is reduced of geometric genus $\widetilde{g}$, then $n_h(D) = 0$ for $h < \widetilde{g}(D)$. Thus the $n_h(\mathbb{P})$ admit universal polynomial expressions.

For each ϵ, Lemma 3.2 implies $|\mathcal{L}|^\epsilon$ is of codimension ϵ at every $D \in (|\mathcal{L}|^\epsilon - V)$. So $|\mathcal{L}|^\epsilon - V$ is empty if $\epsilon > \dim |\mathcal{L}|$. Further, replacing $\mathbb{P}$ by a more general δ-plane if neccessary, we may assume $\mathbb{P} \bigcap |\mathcal{L}|^\epsilon$ is empty if $\delta < \epsilon \le \dim |\mathcal{L}|$. Then there are only finitely many $D \in \mathbb{P}$ of cogenus δ, and none of greater cogenus. Thus

$$n_{g-\delta}(\mathbb{P}) = \textstyle\sum_{D \in \mathbb{P} \cap |\mathcal{L}|^\delta} n_{\widetilde{g}}(D).$$

Alternatively, instead of using (3.2.1) to bound the $\operatorname{codim} |\mathcal{L}|^\epsilon$, we could use [**25**, Corollary 9], which asserts that, given any family of locally planar curves whose nth relative Hilbert scheme is smooth over $\mathcal{C}$ and any $\epsilon \le n$, the curves of cogenus ϵ form a locus of codimension at least ϵ in the base.

Finally, as each $D \in \mathbb{P} \cap |\mathcal{L}|^\delta$ is immersed, $n_{\widetilde{g}}(D) = 1$ by [**29**, Eqn. 5] plus [**5**, Proposition 3.3]. Alternatively, this statement follows from [**29**, Theorem A], because $|\mathcal{L}|^\delta$ is smooth at D. Thus $n_{g-\delta}(\mathbb{P}) = \deg |\mathcal{L}|^\delta$. □

Lastly, we prove Proposition 1.2, which provides conditions under which the nodal curves in the Severi variety $|\mathcal{L}|^\delta$ form a dense open subset $|\mathcal{L}|_+^\delta$. It is well known that $|\mathcal{L}|_+^\delta$ is open and dense if S is the plane; see [**38**, Theorem 2, p. 220] and [**4**, (10.7), p. 847] and [**7**, Proposition 2.2, p. 355]. Similar arguments work if S is a Hirzebruch surface; see [**35**, Proposition 8.1, p. 74]. The broadest statement is given in [**32**, Theorem 2.8, p. 8].

However, even that statement is not broad enough to cover our needs. Moreover, our approach appears to be new in places. In addition, the appendix develops the ideas in [**32**] further, so as to provide another proof of Proposition 1.2 and the codimension statement in Theorem 1.1.

Proof of Proposition 1.2. Clearly, $\deg |\mathcal{L}|_+^\delta = \deg |\mathcal{L}|^\delta$ if $|\mathcal{L}|_+^\delta$ is open and dense in $|\mathcal{L}|$. Thus Theorem 1.1 and the first assertion of Proposition 1.2 yield the second.

To prove the first assertion, assume $C \in |\mathcal{L}|^\delta$, fix $P \in C$, and consider the local Milnor number $\mu(C, P)$, which vanishes if C is smooth at P. It is, by [**14**, Theorem 2.6(2), p. 114], upper semicontinuous in this sense: there is an (analytic or étale) neighborhood B of the point in $|\mathcal{L}|^\delta$ representing C and a neighborhood U of P in the tautological total space $\mathcal{C}_B$ such that, for each $b \in B$,

$$\mu(C, P) \ge \textstyle\sum_{Q \in \mathcal{C}_b \cap U} \mu(\mathcal{C}_b, Q). \tag{4.0.1}$$

So the total Milnor number $\mu(C) := \sum_z \mu(C, z)$ too is upper semicontinuous in C.

Therefore, $|\mathcal{L}|^\delta$ always contains a dense open subset $|\mathcal{L}|_\mu^\delta$ on which $\mu(C)$ is locally constant. So fix $C \in |\mathcal{L}|_\mu^\delta$. Then after B is shrunk, equality holds in (4.0.1).

Therefore, there is a section $B \to \mathcal{C}_B$ along which the family is equisingular by work of Zariski's [**36, 37**], of Lê and Ramanujam's [**22**] and of Teissier's — see both [**14**, Proposition 2.62, p. 359] and [**31**, Theorem 5.3.1, p. 123], as well as the historical note [**31**, 5.3.10, p. 129].

Consider Milnor's Formula $\mu(C) = 2\delta - \sum_{Q\in C}(r(Q)-1)$ where $r(Q)$ is the number of branches of C at Q; see [**14**, Proposition 3.35, p. 208]. It implies $\mu(C) \ge \delta$, with equality if and only if C is δ-nodal. So the nodal locus $|\mathcal{L}|^\delta_+$ is always a union of components of $|\mathcal{L}|^\delta_\mu$. Thus to complete the proof of Proposition 1.2, it suffices to show $|\mathcal{L}|^\delta_\mu - V$ consists of nodal curves. So assume $C \in |\mathcal{L}|^\delta_\mu - V$.

First of all, C is immersed by Lemma 3.3. So Lemma 3.2 implies $|\mathcal{L}|^\delta_\kappa$ is smooth at C with tangent space equal to $\mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K))$.

Form the composition $B \to \mathcal{C}_B \to S$ of the above equisingular section and of the projection. Denote the preimage of $P \in S$ by B'. Evidently, $\dim_C B - \dim_C B' \le 2$.

By equisingularity, P has the same multiplicity m on every $D \in B'$. Denote by S' the blowup of S at P, by E the exceptional divisor, by C' the strict transform of C. Set $\mathcal{L}' := \mathcal{O}_{S'}(C')$ and $\delta' := \delta - m(m-1)/2$. Taking strict transforms gives a map $B' \to |\mathcal{L}'|^{\delta'}$. It is injective as taking images gives an inverse.

Denote by $n'\colon \widetilde{C} \to C'$ the normalization map, by K' the canonical class of S'. Then (3.2.2) yields $(\mathbf{C}_{C'}|\mathcal{L}'|^{\delta'})_{\mathrm{red}} \subset \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n'^*K'))$. Therefore,

$$\dim \mathrm{H}^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)) - \dim \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n'^*K')) \tag{4.0.2}$$
$$\le \dim_C |\mathcal{L}|^\delta - \dim_{C'} |\mathcal{L}'|^{\delta'} = \dim_C B - \dim_C B' \le 2. \tag{4.0.3}$$

The groups in (4.0.2) belong to the long exact cohomology sequence arising from

$$0 \to \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n'^*K') \to \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K) \to \mathcal{O}_{n'^*E} \to 0.$$

Further, $\mathrm{H}^1(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)) = 0$ by (3.1.2) as $C \notin V$. Hence (4.0.2) is equal to

$$\dim \mathrm{H}^0(\mathcal{O}_{n'^*E}) - \dim \mathrm{H}^1(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n'^*K')). \tag{4.0.4}$$

But $\deg(n'^*E) = m$; so $\dim \mathrm{H}^0(\mathcal{O}_{n'^*E}) = m$.

Denote by $C_1, \ldots, C_h$ the components of C, by $\widetilde{C}_i$ the normalization of C_i. Set

$$k_i := -K \cdot C_i \quad \text{and} \quad m_i := \mathrm{mult}(P,\, C_i) = \deg(n'^*E|\widetilde{C}_i) \ge 0.$$

Now, $n'^*K' = n^*K + n'^*E$. Therefore, (3.1.1) and (3.1.2) yield

$$\dim \mathrm{H}^1(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n'^*K')) \le \textstyle\sum_{i=1}^h \max(0,\, 1 - k_i + m_i).$$

Note $m = \sum_{i=1}^h m_i$. Consequently, (4.0.4) and (4.0.3) yield

$$\textstyle\sum_{i=1}^h s_i \le 2 \quad \text{where} \quad s_i := m_i - \max(0,\, 1 - k_i + m_i). \tag{4.0.5}$$

Note $m_i \ge s_i \ge 0$ for all i, as $0 \le \max(0,\, 1 - k_i + m_i) \le m_i$ since $k_i \ge 1$ owing to (2) of Theorem 1.1. Also, $s_i = 0$ if $k_i = 1$ for any i and any m_i; conversely, if $s_i = 0$ and $m_i \ge 1$, then $k_i = 1$. Further, $m_i = s_i$ if and only if $k_i \ge m_i + 1$, as both conditions are obviously equivalent to $\max(0,\, 1 - k_i + m_i) = 0$. Clearly, $k_i \le m_i + 1$ if and only if $s_i = k_i - 1$.

Using (4.0.3), let's now rule out $m \ge 3$. Aiming for a contradiction, assume

$$m_1 \ge \cdots \ge m_h \quad \text{and} \quad m_1 + \cdots + m_h = m \ge 3. \tag{4.0.6}$$

Now, (4.0.5) yields $s_1 \leq 2$ as $s_i \geq 0$ for all i. So if $m_1 \geq 3$, then $m_1 - 2 \leq 1 - k_1 + m_1$; whence, $k_1 \leq 3$, contrary to (4) of Proposition 1.2. Thus (4.0.6) yields $2 \geq m_1 \geq m_2 \geq 1$.

Suppose $m_1 = 2$. Then (5) of Proposition 1.2 rules out $k_1 = 1$ and $k_2 = 1$. So $k_1 \geq 2$ and $k_2 \geq 2$. Suppose $k_1 = 2$. Then $s_1 = 1$. So $s_2 \leq 1$. If $m_2 = 2$, then $k_2 = 2$, contrary to (6) of Proposition 1.2. If $m_2 = 1$, then already $k_1 = 2$ is contrary to (7) of Proposition 1.2. Suppose $k_1 \geq 3$. Then $s_1 = 2$. So $s_2 = 0$. So $k_2 = 1$. But this case was already ruled out. Thus the case $m_1 = 2$ is ruled out completely.

Lastly, suppose $m_1 = 1$. Then (4.0.6) yields $m_2 = 1$ and $m_3 = 1$ too. So (8) of Proposition 1.2 yields $k_i \geq 2$ for $i = 1, 2, 3$. Hence $s_i = 1$ for $i = 1, 2, 3$, contradicting (4.0.5). Thus $m = 2$, as claimed.

Finally, given $m = 2$, let's show P is a simple node. Since C is immersed at P, it is locally analytically given by an equation of the form $y^2 = x^{2k}$ for some $k \geq 1$. Denote by $\widetilde{P}, \widetilde{Q} \in \widetilde{C}$ the points above P on the branches with equations $y = x^k$ and $y = -x^k$. Then (3.1.1) and (3.1.2) imply

$$\dim \mathrm{H}^1(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K - \widetilde{P} - \widetilde{Q})) = 0,$$

because $k_i \geq 1$ for all i owing to (2) of Theorem 1.1 and because either $k_i \geq 2$ for $i = 1, 2$ if $\widetilde{P} \in \widetilde{C}_1$ and $\widetilde{Q} \in \widetilde{C}_2$ owing to (9) of Proposition 1.2 or $k_1 \geq 3$ if $\widetilde{P}, \widetilde{Q} \in \widetilde{C}_1$ owing to (10) of Proposition 1.2. Hence the following restriction map is surjective:

$$\mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)) \twoheadrightarrow \mathrm{H}^0(\widetilde{C},\, \mathcal{O}_{\widetilde{P}+\widetilde{Q}}).$$

Therefore, there's a section of $\mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}} - n^*K)$ that doesn't vanish at $\widetilde{P}$, but does at $\widetilde{Q}$. Correspondingly, there's a first-order deformation of C. Say it's given locally by $y^2 - x^{2k} + \epsilon g(x, y)$. Then $g(t, t^k)$ is of degree k, but $g(t, -t^k)$ is of degree $k + 1$. Clearly, any such g is, up to scalar multiple, necessarily of the form $g(x, y) = x^k + y + O(x^{k+1}, y^2)$. However, the Jacobian ideal of the singularity is $\langle y,\, x^{2k-1}\rangle$. This ideal must contain $g(x, y)$ as the deformation under consideration is equisingular and as the Jacobian ideal is equal to the equisingular ideal by [**14**, Lemma 2.16, p. 287]. Hence $k = 1$. Thus P is an simple node of C, as desired. □

Appendix A. An alternative proof by Ilya Tyomkin

Our goal is to use the deformation theory of maps to provide an alternative proof of Proposition 1.2 and the codimension statement in Theorem 1.1. The general idea goes back to Arbarello and Cornalba [**3**], but the proof contains new ingredients, most of which were introduced in [**32**].

A.1. Notation. Let δ, S, $\mathcal{L}$, K, $|\mathcal{L}|^\delta$, $|\mathcal{L}|^\delta_+$ be as in the Introduction. Again, we work over the complex numbers $\mathbb{C}$, but as is standard, we denote the residue field at a point p by $k(p)$. Moreover, as our treatment is purely algebraic, all the statements and proofs are valid over an arbitrary algebraically closed field of characteristic 0.

Given a morphism $f\colon X \to Y$, and $p^1, \ldots, p^r \in X$ points where X is smooth, $\mathrm{Def}(X, f;\, \underline{p})$ denotes the functor of deformations of $(X, f;\, p^1, \ldots, p^r)$; i.e., if $(T, 0)$ is a local Artinian $\mathbb{C}$-scheme, then $\mathrm{Def}(X, f;\, \underline{p})(T, 0)$ is the set of isomorphism classes of this data: $(X_T, f_T; p^1_T, \ldots, p^r_T; \iota)$ where X_T is T-flat, each $p^i_T\colon T \to X_T$

is a section, $f_T\colon X_T \to Y \times T$ is a T-morphism, and ι is an isomorphism

$$\iota\colon (X_0, f_0; p_0^1, \dots, p_0^r) \xrightarrow{\sim} (X, f; p^1, \dots, p^r).$$

Let $\mathrm{Def}^1(X, f; \underline{p})$ denote the set of first-order deformations $\mathrm{Def}(X, f; \underline{p})(T, 0)$ where $T := \mathrm{Spec}(\mathbb{C}[\epsilon])$ and $\mathbb{C}[\epsilon]$ is the ring of dual numbers.

If X and Y are smooth, set $\mathcal{N}_f := \mathrm{Coker}(T_X \to f^*T_Y)$; it's the *normal sheaf.*

A.2. Three Lemmas.

Lemma A.1. *Let $(C; p^1, \dots, p^r)$ be a smooth curve with marked points, and $f\colon C \to S$ a map that does not contract components of C. Then there is a natural exact sequence*

$$0 \to \textstyle\bigoplus_{i=1}^r \mathbf{T}_{p^i}(C) \to \mathrm{Def}^1(C, f; \underline{p}) \to H^0(C, \mathcal{N}_f) \to 0.$$

Proof. Consider the forgetful map $\phi\colon \mathrm{Def}^1(C, f; \underline{p}) \to \mathrm{Def}^1(C, f)$. It is surjective by the infinitesimal lifting property, since C is smooth at all the p^i. Its kernel is canonically isomorphic to $\bigoplus_{i=1}^r \mathrm{Def}^1(p^i \to C)$, so to $\bigoplus_{i=1}^r \mathbf{T}_{p^i}C$. Finally, since $T_C \to f^*T_S$ is injective, $\mathrm{Def}^1(C, f) = \mathrm{Ext}^1(\mathbf{L}_{C/S}, \mathcal{O}_C) = H^0(C, \mathcal{N}_f)$, where $\mathbf{L}_{C/S}$ is the cotangent complex of $f\colon C \to S$; see [**17**, (2.1.5.6), p. 138; Proposition 3.1.2, p. 203; Theorem 2.1.7, p. 192] or [**16**, pp. 374–376]. □

Lemma A.2. *Let C be a smooth curve, $f\colon C \to S$ a map, $D \subset S$ a closed curve. Set $Z := D \times_S C$, and assume Z is reduced and zero-dimensional. Let $g\colon Z \to D$ be the inclusion, and set $T := \mathrm{Spec}(\mathbb{C}[\epsilon])$ and $(Z_T, g_T) := (C_T, f_T) \times_{S \times T} (D \times T)$. Then sending (C_T, f_T) to (Z_T, g_T) defines a map $d\psi\colon \mathrm{Def}^1(C, f) \to \mathrm{Def}^1(Z, g)$. Furthermore, $d\psi(H^0(C, \mathcal{N}_f^{\mathrm{tor}})) = 0$.*

Proof. To prove $d\psi$ is well defined, it suffices to show that Z_T is T-flat. Let $0 \in T$ be the closed point, $q \in Z \subset Z_T$ a preimage of 0, and $h = 0$ a local equation of D at $f(q)$. Then there exists an exact sequence $0 \to \mathcal{O}_{C_T,q} \to \mathcal{O}_{C_T,q} \to \mathcal{O}_{Z_T,q} \to 0$ where the first map m_h is the multiplication by $f_T^*(h)$. Also, $m_h \otimes k(0)\colon \mathcal{O}_{C,q} \to \mathcal{O}_{C,q}$ is injective, since the locus of zeroes of $f^*(h)$ in C is of codimension 1, and so $f^*(h) \in \mathcal{O}_{C,q}$ is not a zero-divisor. Thus, $\mathcal{O}_{Z_T,q}$ is flat by the local criterion of flatness [**12**, Corollary 5.7]. Thus $d\psi$ is well defined.

As Z is reduced, $Z \bigcap \mathrm{Supp}(\mathcal{N}_f^{\mathrm{tor}}) = \emptyset$. Set $U := C \backslash \mathrm{Supp}(\mathcal{N}_f^{\mathrm{tor}})$. Then $d\psi$ factors through $\mathrm{Def}^1(U, f|_U) = \mathcal{N}_f(U) = \big(\mathcal{N}_f/\mathcal{N}_f^{\mathrm{tor}}\big)(U)$. Thus $d\psi(H^0(C, \mathcal{N}_f^{\mathrm{tor}})) = 0$. □

Lemma A.3. *Let W be an algebraic variety, $C_W \to W$ a flat family of reduced curves, $\widetilde{C}_W \to C_W$ the normalization, and $Z_W \subset \widetilde{C}_W$ a reduced closed subvariety quasi-finite over W. Then there exists an étale morphism $U \to W$ and sections $s_i\colon U \to \widetilde{C}_U$ such that the following two conditions hold:* (1) *$C_U \to U$ is equinormalizable, i.e., $\widetilde{C}_U \to U$ is flat and $\widetilde{C}_u \to C_u$ is the normalization for any $u \in U$; and* (2) *$Z_U \to U$ is étale and $Z_U = \cup_{i=1}^r s_i(U)$.*

Proof. The generic fiber $\widetilde{C}_\eta$ is normal since normalization commutes with arbitrary localizations. Then it is geometrically normal, since the characteristic is zero; and hence $\widetilde{C}_\eta \to \eta$ is smooth by flat descent. Then $\widetilde{C}_W \to W$ is generically smooth by generic flatness theorem, i.e., there exists an open dense subset $U_0 \subset W$ such that $\widetilde{C}_{U_0} \to U_0$ is smooth. In particular, $\widetilde{C}_{U_0} \to U_0$ is flat and has normal fibers. But, $\widetilde{C}_u \to C_u$ is finite for any $u \in U_0$, and hence the normalization.

Furthermore, for any étale map $U \to U_0$, the family $C_U \to U$ is equinormalizable since normalization commutes with étale base changes.

The morphism $Z_W \to W$ is finite, and Z_W is reduced. Thus, $Z_\eta \to \eta$ is finite and étale since the characteristic is zero. Hence, after shrinking U_0, we may assume that $Z_{U_0} \to U_0$ is finite and étale. Then there exists an étale morphism $U \to U_0$ such that Z_U is the disjoint union of $\deg(Z_\eta \to \eta)$ copies of U and the map $Z_U \to U$ is the natural projection. Hence U is as needed. $\square$

A.3. The results.

Proposition A.4. *Let $W \subseteq |\mathcal{L}|^\delta$ be an irreducible subvariety, $C_W \to W$ the tautological family of curves, $\widetilde{C}_W \to C_W$ the normalization, $f_W\colon \widetilde{C}_W \to S$ the natural morphism, and $0 \in W$ a* general *closed point. Assume that C_0 is reduced.*

(1) *Then there exists a natural embedding $\mathbf{T}_0(W) \hookrightarrow H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}})$.*

(2) *If $-K.C \geq 1$ for any irreducible component $C \subseteq C_0$, then*

$$\dim(W) \leq h^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) \leq -K.C_0 + p_g(C_0) - 1. \tag{A.4.1}$$

(3) *If (A.4.1) is an equality and $-K.C > 1$ for an irreducible component C of C_0, then C is immersed.*

(4) *If (A.4.1) is an equality and $-K.C > 1$ for any irreducible component C of C_0, then $\mathcal{N}_{f_0}$ is invertible and $\mathbf{T}_0(W) \to H^0(\widetilde{C}_0, \mathcal{N}_{f_0})$ is an isomorphism.*

Proof. Pick a smooth irreducible closed curve $D \subset S$ in a very ample linear system such that $h^0(S, \mathcal{L}(-D)) = 0$. Then $D \cap C_w$ is finite for any $w \in W$, and is reduced for almost all $w \in W$ by Bertini's theorem. In particular, $D \cap C_0$ is reduced since $0 \in W$ is general. Hence the projection $Z_W := \widetilde{C}_W \times_S D \to W$ is finite, since it is a projective morphism with finite fibers. Let $g_0\colon Z_0 \to D$ be the closed immersion. Then, by Lemma A.2 and Lemma A.3, there exists a commutative diagram

$$\begin{tikzcd}
\mathbf{T}_0(W) \arrow[r] \arrow[d, hook] & \mathrm{Def}^1(\widetilde{C}_0, f_0) \arrow[r] \arrow[d] & H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) \arrow[d] \\
\mathbf{T}_{Z_0}(|\mathcal{L} \otimes \mathcal{O}_D|) \arrow[r, hook] & \mathrm{Def}^1(Z_0, g_0) \arrow[r, equal] & H^0(Z_0, \mathcal{N}_{g_0})
\end{tikzcd} \tag{A.4.2}$$

where $\mathbf{T}_0(W) \to \mathbf{T}_{Z_0}(|\mathcal{L} \otimes \mathcal{O}_D|)$ is injective since $W \subseteq |\mathcal{L}| \subseteq |\mathcal{L} \otimes \mathcal{O}_D|$ by the choice of D; and $\mathbf{T}_{Z_0}(|\mathcal{L} \otimes \mathcal{O}_D|) \to \mathrm{Def}^1(Z_0, g_0)$ is injective since $\mathbf{T}_{Z_0}(|\mathcal{L} \otimes \mathcal{O}_D|)$ is a subspace of the space of first-order embedded deformations of $g_0(Z_0) \subset D$, and the latter is canonically isomorphic to $\bigoplus_{p \in g_0(Z_0)} \mathbf{T}_p(D) = \mathrm{Def}^1(Z_0, g_0)$. Thus, the composition $\mathbf{T}_0(W) \to H^0(Z_0, \mathcal{N}_{g_0})$ is injective, and hence so is $\mathbf{T}_0(W) \to H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}})$ as asserted by (1).

(2) The first inequality in (A.4.1) follows from (1). Since both sides of the second inequality in (A.4.1) are additive with respect to unions, we may assume that C_0 is irreducible. Let $0 \to \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}} \to \mathcal{F}$ be an invertible extension such that $c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0})$. By the assumption, $c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0}) = c_1(\omega_{\widetilde{C}_0}) - K.C_0 > c_1(\omega_{\widetilde{C}_0})$. Thus, $h^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) \leq h^0(\widetilde{C}_0, \mathcal{F}) = c_1(\mathcal{F}) + 1 - p_g(C_0) = -K.C_0 + p_g(C_0) - 1$ by Riemann–Roch theorem, since $h^0(\widetilde{C}_0, \mathcal{F}^\vee \otimes \omega_{\widetilde{C}_0}) = 0$; and hence (A.4.1) holds.

(3) Once again, we may assume that C_0 is irreducible. To prove that C_0 is immersed, it is sufficient to show that $\mathcal{N}_{f_0}^{\mathrm{tor}} = 0$. Assume to the contrary that

$\mathcal{N}_{f_0}^{\mathrm{tor}} \neq 0$. Pick an invertible extension $0 \to \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}} \to \mathcal{F}$ with $c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0}) - 1$. By the assumption, $c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0}) - 1 = c_1(\omega_{\widetilde{C}_0}) - K.C_0 - 1 > c_1(\omega_{\widetilde{C}_0})$. Thus, $h^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) \leq h^0(\widetilde{C}_0, \mathcal{F}) = c_1(\mathcal{F}) + 1 - p_g(C_0) = -K.C_0 + p_g(C_0) - 2$ by Riemann–Roch theorem, which is a contradiction.

(4) Note that by (3) we have: $\mathcal{N}_{f_0}^{\mathrm{tor}} = 0$, and hence $\mathcal{N}_{f_0}$ is invertible. Then by (2), $\dim(\mathbf{T}_0(W)) = h^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) = h^0(\widetilde{C}_0, \mathcal{N}_{f_0})$. Thus, (1) implies that $\mathbf{T}_0(W) \hookrightarrow H^0(\widetilde{C}_0, \mathcal{N}_{f_0}) = H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}})$ is an isomorphism. $\square$

Remark A.5. By definition, $\delta := p_a(C_0) - p_g(C_0)$. Hence, if S is rational and $-K.C_0 \geq 1$, then the adjunction formula and Lemma 2.1 yield

$$-K.C_0 + p_g(C_0) - 1 = C_0.(C_0 - K)/2 - \delta = \dim|\mathcal{L}| - \delta.$$

Proposition A.6. *Fix a point $q \in S$, and a curve $E \subset S$. Let $W \subseteq |\mathcal{L}|^\delta$ be an irreducible subvariety, $C_W \to W$ the tautological family of curves, $\widetilde{C}_W \to C_W$ the normalization, $f_W \colon \widetilde{C}_W \to S$ the natural morphism, and $0 \in W$ a* general *closed point. Assume that C_0 is reduced and immersed,* $\dim(W) = -K.C_0 + p_g(C_0) - 1$, *and $-K.C \geq 1$ for any irreducible component C of C_0.*

(1) *If $-K.C > 1$ for any irreducible component C of C_0, then $q \notin C_0$.*

(2) *Let $q_0 \in C_0$ be a point of multiplicity at least three, and $p^1, p^2, p^3 \in \widetilde{C}_0$ three distinct preimages of q_0. Then there exists an irreducible component $C \subseteq C_0$ such that $-K.C \leq \left|\widetilde{C} \cap \{p^1, p^2, p^3\}\right|$.*

(3) *Let $q_0 \in C_0$ be a singular point with at least two tangent branches, and $p^1, p^2 \in \widetilde{C}_0$ the preimages of q_0 on these branches. Then there exists an irreducible component $C \subseteq C_0$ such that $-K.C \leq \left|\widetilde{C} \cap \{p^1, p^2\}\right|$.*

(4) *If $-K.C > 1$ for any irreducible component C of C_0, then any branch of C_0 intersects E transversally. Furthermore, if $C_0^{\mathrm{sing}} \cap E \neq \emptyset$, then there exists an irreducible component $C \subseteq C_0$ such that $C^{\mathrm{sing}} \cap E \neq \emptyset$ and $-K.C = 2$.*

Proof. First, note that $\mathcal{N}_{f_0}$ is invertible since C_0 is immersed. Thus, the embedding $\mathbf{T}_0(W) \hookrightarrow H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/\mathcal{N}_{f_0}^{\mathrm{tor}}) = H^0(\widetilde{C}_0, \mathcal{N}_{f_0})$ of Proposition A.4 (1) is an isomorphism by Proposition A.4 (2). Then $h^0(\widetilde{C}_0, \mathcal{N}_{f_0}) = -K.C_0 + p_g(C_0) - 1 = \chi(\mathcal{N}_{f_0})$, and hence $h^1(\widetilde{C}_0, \mathcal{N}_{f_0}) = 0$. Let $A_W \subset C_W$ be the locus of singular points of the fibers $C_W \to W$. Set $Z_W := \nu^{-1}(A_W) \cup f_W^{-1}(q \cup E) \subset \widetilde{C}_W$, where $\nu \colon \widetilde{C}_W \to C_W$ is the normalization. Then $Z_W \subset \widetilde{C}_W$ is locally closed, and $Z_W \to W$ has finite fibers. Thus, by Lemma A.3, there exists an étale neighborhood U of 0 and disjoint sections $s_i \colon U \to \widetilde{C}_U$ such that $Z_U = \cup_{i=1}^r s_i(U)$. Set $p^i := s_i(0)$. Then the isomorphism $\mathbf{T}_0(W) \to H^0(\widetilde{C}_0, \mathcal{N}_{f_0}) = \mathrm{Def}^1(\widetilde{C}_0, f_0)$ factors through $\mathrm{Def}^1(\widetilde{C}_0, f_0; \underline{p})$ for any $1 \leq i_1 < \cdots < i_m \leq r$, where $\underline{p} = (p^{i_1}, \dots, p^{i_m})$.

Consider the exact sequence of Lemma A.1

$$0 \to \oplus_{j=1}^m \left(T_{\widetilde{C}_0} \otimes k(p^{i_j})\right) \to \mathrm{Def}^1(\widetilde{C}_0, f_0; \underline{p}) \to H^0(\widetilde{C}_0, \mathcal{N}_{f_0}) \to 0,$$

the restriction map $\gamma \colon H^0(\widetilde{C}_0, \mathcal{N}_{f_0}) \to \oplus_{j=1}^m \left(\mathcal{N}_{f_0} \otimes k(p^{i_j})\right)$, and the forgetful map

$$\beta \colon \mathrm{Def}^1(\widetilde{C}_0, f_0; \underline{p}) \to \oplus_{j=1}^m \mathrm{Def}^1(p^{i_j}, f_0|_{p^{i_j}}) = \oplus_{j=1}^m \left(f_0^* T_S \otimes k(p^{i_j})\right).$$

Then the following diagram is commutative:

$$\text{(A.6.1)}\qquad \begin{tikzcd} \mathbf{T}_0(W) \arrow[r] \arrow[d, equal] & \mathrm{Def}^1(\widetilde{C}_0, f_0; \underline{p}) \arrow[r, "\beta"] \arrow[d] & \bigoplus_{j=1}^m \left(f_0^* T_S \otimes k(p^{i_j})\right) \arrow[d, "\pi"] \\ \mathbf{T}_0(W) \arrow[r, "\sim"] & H^0(\widetilde{C}_0, \mathcal{N}_{f_0}) \arrow[r, "\gamma"] & \bigoplus_{j=1}^m \left(\mathcal{N}_{f_0} \otimes k(p^{i_j})\right) \end{tikzcd}$$

From the long exact sequence of cohomology associated to the short exact sequence of sheaves $0 \to \mathcal{N}_{f_0}(-\sum_{j=1}^m p^{i_j}) \to \mathcal{N}_{f_0} \to \oplus_{j=1}^m \mathcal{N}_{f_0} \otimes k(p^{i_j}) \to 0$ we obtain: $\mathrm{Ker}(\gamma) = H^0(\widetilde{C}_0, \mathcal{N}_{f_0}(-\sum_{j=1}^m p^{i_j}))$, and $\mathrm{Coker}(\gamma) \subseteq H^1(\widetilde{C}_0, \mathcal{N}_{f_0}(-\sum_{j=1}^m p^{i_j}))$. Thus, the map γ is not surjective if and only if $h^1(\widetilde{C}_0, \mathcal{N}_{f_0}(-\sum_{j=1}^m p^{i_j})) \neq 0$, since $h^1(\widetilde{C}_0, \mathcal{N}_{f_0}) = 0$. In particular, if γ is not surjective then there exists an irreducible component $C \subseteq C_0$ such that $c_1(\mathcal{N}_{f_0}(-\sum_{j=1}^m p^{i_j})|_{\widetilde{C}}) \leq c_1(\omega_{\widetilde{C}})$, or, equivalently, $-K.C \leq |\widetilde{C} \cap \{p^{i_j}\}_{j=1}^m|$.

Let $q_0 \in C_0$ be either a singular point, or a point of intersection $C_0 \cap E$, or $q_0 = q$. Assume that $\nu(s_i(0)) = q_0$ for $1 \leq i \leq m$. Since $0 \in W$ is general, $\nu \circ s_i = \nu \circ s_j$ for all $1 \leq i \leq j \leq m$. Set $i_j := j$, and consider diagram (A.6.1). For any $1 \leq j \leq m$, the tensor product $f_0^* T_S \otimes k(p^j)$ is canonically isomorphic to $\mathbf{T}_{q_0}(S) = \mathrm{Def}^1(q_0 \to S)$, and β factors through the diagonal map $\Delta\colon \mathbf{T}_{q_0}(S) \to \bigoplus_{j=1}^m \left(f_0^* T_S \otimes k(p^j)\right)$. Hence $\mathrm{Im}(\gamma) \subseteq \mathrm{Im}(\pi \circ \Delta)$.

(1) Assume to the contrary that $q \in C_0$, and set $q_0 := q$. Without loss of generality, $\nu(s_1(0)) = q_0$. Set $m := 1$, and consider diagram (A.6.1). Then the image of $\mathbf{T}_0(W)$ in $\mathrm{Def}^1(p^1, f_0|_{p^1})$ is trivial since q is fixed. Thus, γ is the zero map, and hence there exists an irreducible component $C \subseteq C_0$ such that $-K.C \leq 1$, which is a contradiction.

(2) Assume that $q_0 \in C_0$ is a singular point of multiplicity at least three. Without loss of generality, $s_1(0), s_2(0), s_3(0)$ are preimages of q_0. Set $m := 3$, and consider diagram (A.6.1). Then $\dim(\mathrm{Im}(\gamma)) \leq \dim(\mathrm{Im}(\pi \circ \Delta)) = 2 < 3$, and hence γ is not surjective. Thus, there exists an irreducible component as asserted.

(3) Assume that C_0 has at least two tangent branches at q_0. Without loss of generality, $s_1(0), s_2(0)$ are the preimages of q_0 on the tangent branches. Set $m := 2$, and consider diagram (A.6.1). Then $\dim(\mathrm{Im}(\gamma)) \leq \dim(\mathrm{Im}(\pi \circ \Delta)) = 1 < 2$, and hence γ is not surjective. Thus, there exists an irreducible component as asserted.

(4) Assume that $q_0 \in C_0 \cap E$. Then $q_0 \notin E^{\mathrm{sing}}$ by (1). Without loss of generality, $s_1(0)$ is a preimage of q_0. Assume to the contrary that $df_0(\mathbf{T}_{s_1(0)}(\widetilde{C}_0)) = \mathbf{T}_{q_0}(E)$. Set $m := 1$, and consider diagram (A.6.1). The image of γ belongs to the image of $\mathrm{Def}^1(q_0 \to E) = \mathbf{T}_{q_0}(E) \to \mathcal{N}_{f_0} \otimes k(p^1)$, which is zero. Thus, there exists an irreducible component $C \subset C_0$ such that $-K.C \leq 1$, which is a contradiction. Hence no branch of C_0 is tangent to E. Assume now that $q_0 \in C_0^{\mathrm{sing}}$. Without loss of generality, $s_1(0)$ and $s_2(0)$ are preimages of q_0. Set $m := 2$, and consider diagram (A.6.1). The image of γ belongs to the image of $\mathbf{T}_{q_0}(E) \to \oplus_{i=1}^2 \left(\mathcal{N}_{f_0} \otimes k(p^i)\right)$, which is at most one-dimensional. Thus, γ is not surjective, and hence there exists an irreducible component $C \subseteq C_0$ such that $-K.C \leq |\widetilde{C} \cap \{p^1, p^2\}| \leq 2$. However, $-K.C \geq 2$ by the assumption. Hence $p^1, p^2 \in \widetilde{C}$, $q_0 \in C^{\mathrm{sing}}$, and $-K.C = 2$. □

A.4. Conclusions and final remarks. First, let us prove the assertion about the codimension in Theorem 1.1: The upper bound follows easily from the fact that the locus of equigeneric deformations in the space of all deformations has codimension δ, as explained at the very beginning of the proof of Lemma 3.2. The lower bound follows from Proposition A.4 (2) and Remark A.5 applied to every irreducible component $W \subseteq |\mathcal{L}|^\delta \setminus V$.

Second, let us prove the most difficult part of Proposition 1.2, namely the nodality of a general curve in $|\mathcal{L}|^\delta \setminus V$: Pick an irreducible component $W \subseteq |\mathcal{L}|^\delta \setminus V$, and let $0 \in W$ be a general closed point. Then $\dim(W) = -K.C_0 + p_g(C_0) - 1$ by Theorem 1.1 and Remark A.5. Furthermore, C_0 is immersed by Proposition A.4 (3) and assumption (3) of Theorem 1.1. By Proposition A.6 (2), if C_0 has a point of multiplicity at least three, then we get a contradiction to assumption (4), or (5), or (6), or (7), or (8) of Proposition 1.2. Similarly, by Proposition A.6 (3), if C_0 has a singular point with at least two tangent branches, then we get a contradiction to assumption (9) or (10) of Proposition 1.2. Thus, C_0 is nodal.

Third, note that Proposition A.4 and Proposition A.6 imply few previously known results about families of curves on algebraic surfaces such as [**38**, Theorem 2, p. 220], [**3**, (3.1), p. 95], [**4**, (10.7), p. 847], [**7**, Proposition 2.2, p. 355], [**35**, Proposition 8.1, p. 74], and [**32**, Theorem 2.8, p. 8].

Finally, let us mention that in positive characteristic Proposition A.4 and Proposition A.6 are no longer true. It was shown in [**33**] that there exist $S, \mathcal{L}, W$ as in the Propositions such that: (a) for any étale morphism $U \to W$ the family C_U is not equinormalizable, (b) $\dim(W) = -K.C_0 + p_g(C_0) - 1$, and (c) all curves C_w are non-immersed, have tangent branches, and intersect each other non-transversally. However, at least for toric surfaces S, it was shown that the bound $\dim(W) \leq -K.C_0 + p_g(C_0) - 1$ holds true in arbitrary characteristic.

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Clay Mathematics Proceedings
Volume **18**, 2013

Curve Counting à la Göttsche

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Abstract. Summary of a Problem Session held August 25, 2011, updated September 13, 2012.

Let n_δ be the number of δ-nodal curves lying in a suitably ample complete linear system $|L|$ and passing through appropriately many points on a smooth projective complex algebraic surface. Often n_δ is referred to as a *Severi degree*. A major problem is to understand the behavior of n_δ, specifically to finish off Lothar Göttsche's mostly proved 1997 conjectures [**17**] and then go on to treat the new refinements by Göttsche and Vivek Shende [**18**], [**19**].

The general area has been very active for over fifteen years, and is now busier and more exciting than ever before. Among many other people involved have been Joe Harris himself, some of his students, and some of theirs. The area is unusually broad — embracing ideas from physics, symplectic differential geometry, complex analytic geometry, algebraic geometry, tropical geometry, and combinatorics.

Problem number one is to find the two power series

$$B_1(q),\ B_2(q) \in \mathbb{Z}[[q]]$$

appearing in Göttsche's remarkable formula for the generating function of the n_δ. The formula expresses the function, so the n_δ, in terms of the four basic numerical invariants of the system and the surface. In fact, n_δ is a polynomial in the four. See (1) and (4) and (5) below.

Göttsche [**17**, Rmk. 2.5(2)] computed the coefficients of $B_1(q)$ and $B_2(q)$ up to degree 28 on the basis of the recursive formula for the n_δ of the plane due to Lucia Caporaso and Harris [**8**, Thm. 1.1]. (A different recursion had been given earlier by Ziv Ran [**34**, Thm. 3C.1].) Göttsche checked the result against much of what was known, including Ravi Vakil's enumeration [**40**] for the Hirzebruch surfaces (that is, the rational ruled surfaces).

The **problem** is to find a closed form for each $B_i(q)$, or a functional equation.

Second, given δ, how ample is suitable, so that n_δ has the predicted value? After all, for any system, the polynomial yields a number, but it isn't always n_δ. For example, consider plane curves of degree d. If $d = 1$, then n_3 is the number of

2010 *Mathematics Subject Classification.* 14N10 (Primary); 14C20, 14H20 (Secondary).

The author offers his heartfelt thanks to Florian Block, to Eduardo Esteves, to Ragni Piene, to Nikolay Qviller, and to Vivek Shende for reading previous drafts and making apt comments.

3-nodal lines, namely 0, but the polynomial yields 75. Considering the geometry, Göttsche [**17**, Cnj. 4.1, Rmk. 4.4] conjectured the polynomial works if $d \geq \delta/2 + 1$.

The latter conjecture was proved for $\delta \leq 8$ by Ragni Piene and the author [**25**, Thm. 3.1] using algebraic methods, then for $\delta \leq 14$ by Florian Block [**4**, Prp. 1.4]. He built on ideas of Sergey Fomin and Grigory Mikhalkin [**15**, Thm. 5.1], who used tropical methods to set up the enumeration from scratch and to validate its predictions for $d \geq 2\delta$. In principle, the problem is purely combinatorial: to show formally the Caporaso–Harris recursion yields a polynomial in d for $d \geq \delta/2 + 1$.

On any surface, Martijn Kool, Shende, and Richard Thomas [**27**, Prp. 2.1] proved it suffices for L to be δ-very ample. Piene and the author [**24**, Thm. 1.1] proved it suffices for L to be of the form $M^{\otimes m} \otimes N$ where M is very ample, $m \geq 3\delta$, and N is spanned, provided $\delta \leq 8$. Both results were inspired by Göttsche's [**17**, Prp. 5.2]; in turn, Göttsche had been inspired by Harris and Rahul Pandharipande's paper [**20**], which treats the case $\delta \leq 3$ in the plane.

In fact, Göttsche conjectured the polynomial works for plane curves of degree d *iff* $d \geq \delta/2+1$. And Block proved, for $3 \leq \delta \leq 14$, that $\lceil \delta/2 \rceil + 1$ is, indeed, a *threshold*, as he called it; namely, it is the least integer d^* such that the polynomial works for $d \geq d^*$. Further, Göttsche conjectured a similar statement for the Hirzebruch surfaces. Shende and the author [**26**] proved that, above Göttsche's conjectured threshold, the polynomials work for the plane and for the Hirzebruch surfaces and that a similar statement holds for the classical del Pezzo surfaces; moreover, there's at least one case where the polynomial works below the conjectured threshold too.

Sometimes, the curves are required to belong to a general linear subsystem of $|L|$ rather than to pass through appropriately many points. However, the latter condition does yield a general subsystem by Piene and the author's [**24**, Lem. (4.7)].

The **problem** is to determine just when the polynomial yields n_δ.

Third, what about nonlinear systems? After all, Gromov–Witten theory fixes not the linear equivalence class, but the homology class, and this class determines the four basic invariants, (1) below. Jim Bryan and Naichung Conan Leung [**5**, Thm. 1.1] handled primitive complete nonlinear systems on generic Abelian surfaces for all δ. They used symplectic methods. Piene and the author [**25**, § 5] obtained similar results algebraically, but for $\delta \leq 8$.

Israel Vainsencher [**39**, § 6.2] treated a remarkable system. His parameter space was the Grassmannian of $\mathbb{P}^2$ in $\mathbb{P}^4$. His surface was $\mathbb{P}^2$, but moving in $\mathbb{P}^4$. His curves arose by intersecting the moving $\mathbb{P}^2$ with a fixed general quintic 3-fold X. Thus he found X contains 17,601,000 irreducible 6-nodal quintic plane curves. Piene and the author [**25**, Thm. 4.3] validated the number. Pandharipande [**11**, (7.54)] noted each curve has six double covers previously unconsidered in mirror symmetry.

Given any suitably general algebraic system of curves on surfaces, Piene and the author [**25**, Thm. 2.5 and Rmk. 2.7] found on the parameter space the class of the curves with δ nodes for $\delta \leq 8$ and conjectured the formula generalizes to any δ.

The **problem** is to generalize the formula for n_δ, in (4), to algebraic systems.

Fourth, what about higher singularities? This question is related to the previous one, about algebraic systems. For example, given a system, consider those curves with a triple point and δ double points. Their number can be viewed as the number of curves with δ double points in the following system: take the subsystem of curves with a triple point, and resolve the locus of triple points. This example was treated for $0 \leq \delta \leq 3$ by Vainsencher and by Piene and the author [**24**, Thm. 1.2]. A substantial amount of work has been done; see Maxim Kazarian's paper [**21**], Dmitry Kerner's papers [**22**], [**23**], Jun Li and Yu-Jong Tzeng's paper [**28**], Jørgen Rennemo's paper [**35**] and their references.

The **problem** is to enumerate the curves of fixed global equisingularity type lying in a given algebraic system — that is, to find on the parameter space the class of these curves.

Fifth, what about positive characteristic? Sometimes an enumeration is more tractable modulo a prime. Thus Göttsche [**16**, Thm. 0.1] found the Betti numbers of the Hilbert schemes of points on a smooth surface. (In [**13**, pp. 175–178], he and Barbara Fantechi discuss other proofs and refinements of the result.) This result, combined with others, led to the celebrated formula of Shing-Tung Yau and Eric Zaslow [**41**, p. 5] enumerating rational curves on a K3 surface. They developed ideas of Cumrun Vafa et al.: see [**37**, p. 438] for a similar formula; see [**38**, p. 44] for the use of Göttsche's result; see [**3**, p. 437] for the use of varying Jacobians. In turn, Arnaud Beauville [**1**] and Fantechi, Göttsche, and Duco van Straten [**14**] developed the ideas in [**41**] further, and Xi Chen [**9**, Thm. 1] proved the curves are nodal.

The Yau–Zaslow formula too inspired Göttsche to develop his conjectures. For K3 surfaces and Abelian surfaces, $B_1(q)$ and $B_2(q)$ disappear, leaving explicit formulas in any geometric genus. These formulas were proved for primitive classes on generic such surfaces by Bryan and Leung; see [**6**] for a lovely survey.

The **problem** is to determine just when Göttsche's conjectures hold in positive characteristic.

To define the $B_i(q)$, denote the surface by S, and its canonical bundle by K. The four basic invariants are these numbers:

$$x := L^2, \quad y := L \cdot K, \quad z := K^2, \quad t := c_2(S). \tag{1}$$

For $\delta \le 6$, Vainsencher [**39**, § 5] worked out formulas for the n_δ, getting enormous polynomials in x, y, z, t. Afterwards, it was natural to conjecture this statement:

(2) *The number n_δ is given by a universal polynomial of degree δ in $\mathbb{Q}[x, y, z, t]$.*

For plane curves of degree d, we have $(x, y, z, t) = (d^2, -3d, 9, 3)$. So Philippe Di Francesco and Claude Itzykson [**12**, p. 85] conjectured n_δ is given by a polynomial in d of a certain shape for $\binom{d-1}{2} \ge \delta$. Youngook Choi [**10**, p. 12] established their conjecture for $d \ge \delta$ on the basis of Ran's work [**34**]. Göttsche [**17**, § 4] refined the conjecture. Given (2) in the form of (4) below, Nikolay Qviller [**33**, § 4] established most of Göttsche's refinements concerning the shape.

In full generality, (2) was given a symplectic proof and an algebraic proof by Ai-ko Liu [**29**], [**30**]. It was recently given new proofs by Tzeng [**36**, Thm. 1.1] and Kool, Shende, and Thomas [**27**, Thm. 4.1]; the former is purely algebraic, whereas the latter also relies on topology. These new proofs have caused quite a stir!

Göttsche [**17**, Cnj. 2.1] did conjecture (2) in full generality, but his elaboration is far more important. First, he proved (2) is equivalent to this statement:

$$\sum n_\delta u^\delta = A_1^x A_2^y A_3^z A_4^t \quad \textit{for some } A_i \in \mathbb{Q}[[u]]. \tag{3}$$

The A_i are the exponentials of their logarithms. Hence (3) is equivalent to this:

$$n_\delta = P_\delta(a_1, \ldots, a_\delta)/\delta! \quad \textit{where } \textstyle\sum_{\delta \ge 0} P_\delta u^\delta/\delta! = \exp\bigl(\sum_{\kappa \ge 1} a_\kappa u^\kappa/\kappa!\bigr) \tag{4}$$

for some *linear* forms $a_\kappa(x, y, z, t)$. The polynomials P_δ were studied extensively in 1934 by Eric Temple Bell [**2**]. Piene and the author [**24**, p. 210] determined a_κ

for $\kappa \le 8$, and found its coefficients to be integers. Recently, Qviller [**33**, Thm. 2.4] (see [**32**, § 6] too) proved the coefficients are always integers.

The $B_i(q)$ appear in the next formula, the *Göttsche–Yau–Zaslow Formula*:

$$\sum n_\delta u(q)^\delta = B_1(q)^z B_2(q)^y B_3(q)^\chi B_4(q)^{-\nu/2} \tag{5}$$

where $u(q)$, $B_3(q)$, $B_4(q) \in \mathbb{Z}[[q]]$ are explicit quasi-modular forms and where

$$\chi := \chi(L) = (x-y)/2 + \nu \quad \text{and} \quad \nu := \chi(\mathcal{O}_S) = (z+t)/12.$$

Göttsche [**17**, Cnj. 2.4] conjectured (5). He [**17**, Rmks. 2.5(1), 3.1] noted (5) implies (2) and generalizes the Yau–Zaslow Formula. Tzeng [**36**, Thm. 1.2] derived (5) from (3) via Bryan and Leung's work on K3 surfaces [**7**, Thm. 1.1] and via Piene and the author's [**25**, Lem. 5.3]; the latter provides enough suitably ample primitive classes.

Finally, Göttsche and Shende were inspired by Kool, Shende, and Thomas's work to conjecture, *among many other statements,* refinements [**19**, Cnj. 75] of the Caporaso–Harris and Vakil recursions. Further, Göttsche and Shende [**19**, Cnj. 5, 7] refine (5) with this conjecture: there should be polynomials $n_\delta(v) \in \mathbb{Z}[v]$ and power series with polynomial coefficients $u(v,q)$, $B_i(v,q) \in \mathbb{Q}[v, v^{-1}][[q]]$ such that

$$\sum n_\delta(v)\, u(v,q)^\delta = B_1(v,q)^z B_2(v,q)^y B_3(v,q)^\chi B_4(v,q)^{-\nu/2}$$

and such that putting $v = 1$ recovers (5). Again $u(v,q)$ and $B_3(v,q)$ and $B_4(v,q)$ are known; however, it is an **open problem** to find the geometric meaning of $n_\delta(v)$.

If S is a real toric variety, then $n_\delta(-1)$ is conjectured in [**19**, Cnj. 90] to be the tropical Welschinger invariant — the number of real δ-nodal curves lying in a suitably ample real complete linear system and passing through a subtropical set of appropriately many real points, each curve counted with an appropriate sign. The notion of subtropical set was introduced and studied by Mikhalkin in [**31**]. This conjecture is also stated by Block and Göttsche in a paper currently being written; further, there the conjecture is proved for $\delta \le 8$ using methods like those in [**4**]

The **refined problem number one** is to find $B_1(v,q)$ and $B_2(v,q)$.

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Clay Mathematics Proceedings
Volume **18**, 2013

Mnëv-Sturmfels universality for schemes

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Abstract. We prove a scheme-theoretic version of Mnëv-Sturmfels Universality, suitable to be used in the proof of Murphy's Law in Algebraic Geometry [**Va**, Main Thm. 1.1]. Somewhat more precisely, we show that any singularity type of finite type over $\mathbb{Z}$ appears on some incidence scheme of points and lines, subject to some particular further constraints.

This paper is dedicated to Joe Harris on the occasion of his birthday, with warmth and gratitude.

1. Introduction

Define an equivalence relation $\sim$ on pointed schemes generated by the following: if $(X, P) \to (Y, Q)$ is a smooth morphism of pointed schemes ($P \in X$, $Q \in Y$) — i.e. a smooth morphism $\pi : X \to Y$ with $\pi(P) = Q$ — then $(X, P) \sim (Y, Q)$. We call equivalence classes *singularity types*, and we call pointed schemes *singularities*. We say that *Murphy's Law* holds for a (moduli) scheme M if every singularity type appearing on a finite type scheme over $\mathbb{Z}$ also appears on M. (This use of the phrase "Murphy's Law" is from [**Va**, §1], and earlier appeared informally in [**HM**, p. 18]. Folklore ascribes it to Mumford.)

Definition 1.1. Define an *incidence scheme* of points and lines in $\mathbb{P}^2_{\mathbb{Z}}$ as a locally closed subscheme of $(\mathbb{P}^2_{\mathbb{Z}})^M \times (\mathbb{P}^{2\vee}_{\mathbb{Z}})^N = \{p_1, \ldots, p_M, l_1, \ldots, l_N\}$ parametrizing M labeled points and N labeled lines, satisfying the following conditions.

(i) $p_1 = [0, 0, 1]$, $p_2 = [0, 1, 0]$, $p_3 = [1, 0, 0]$, $p_4 = [1, 1, 1]$.
(ii) We are given some specified incidences: for each pair (p_i, l_j), either p_i is required to lie on l_j, or p_i is required not to lie on l_j.
(iii) The marked points are required to be distinct, and the marked lines are required to be distinct.
(iv) Given any two marked lines, there is a marked point required to be on both of them (necessarily unique, given (iii)).
(v) Each marked line contains at least three marked points.

Note that even though our definition over $\mathbb{Z}$, these conditions may force us into positive characteristic. For instance, the Fano plane point-line configuration would force us into characteristic 2.

The goal of this paper is to establish the following.

2010 *Mathematics Subject Classification.* Primary 14J10; Secondary 14N20, 14B12.

The first author was supported by Samsung Scholarship. The second author was supported by NSF grant DMS-1100771.

Theorem 1.2 (Mnëv-Sturmfels Universality Theorem for Schemes). The disjoint union of all incidence schemes (over all possible M, N, and data of the form described in Definition 1.1) satisfies Murphy's Law.

Theorem 1.2 appeared as [**Va**, Thm. 3.1], as an essential step in proving [**Va**, Main Thm. 1.1], which stated that many important moduli spaces satisfy Murphy's Law. A number of readers of [**Va**] have pointed out to the second author that the references given in [**Va**] and elsewhere in the literature do not establish the precise statement of Theorem 1.2, and that it is not clear how to execute the glib parenthetical assertion ("The only subtlety ...," [**Va**, p. 577, l. 3-5]) to extend Lafforgue's argument [**L**, Thm. I.14] to obtain the desired result. (In particular, our problem is that Lafforgue does not require (iv) in his moduli spaces, as it is not needed for his purposes. If we then add marked points to all pairwise intersections of lines in Lafforgue's construction, then it is not clear that (iii) holds in the configurations he constructs, and we suspect it does not always hold.)

This paper was written in order to fill a possible gap in [**Va**], or at least to clarify details of an important construction. We hope this paper will be of use to those studying the singularities of moduli spaces not covered by [**Va**] (the moduli space of vector bundles, [**P**], or the Hilbert scheme of points, [**E**], say). Although no one familiar with this area would doubt that Theorem 1.2 holds, or how the general idea should go, we will see that some care is needed to rigorously establish it. In particular, our argument is characteristic-dependent.

1.1. Key features in the argument. Given any polynomials $f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n]$, our goal is to build a smooth cover of

$$\operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

by (open subsets of) incidence schemes, by encoding the variables and relations in incidence relations. We build the relations by combining "atomic" calculations encoding equality, negation, addition, and multiplication. We point out new features of the argument we use, in order to ensure 1.1(iii) in particular. We perform each "atomic" calculation on a separate line of the plane, to avoid having too many important points on a single line, because points on a line must be shown to not overlap. We need various cases to deal with when the "variable" in question is "near" 0 or 1 (i.e. has value 0 or 1 at a given geometric point Q of $\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$, but is not required to have that value "near" Q). Furthermore, the "usual" construction of addition and multiplication runs into problems in characteristic 2 due to unintended coincidences of points, so some care is required in this case (see §4.6).

1.2. Algebro-geometric history. Vershik's "universality" philosophy (e.g. [**Ve**, Sect. 7]) has led to a number of important constructions in many parts of mathematics. One of the most famous is Mnëv's Universality Theorem [**M1, M2**]. It was independently proved by Bokowski and Sturmfels [**BS, S1, S2**]. We follow Belkale and Brosnan [**BB**, §10] in naming the result after both Mnëv and Sturmfels. (The idea is more ancient; von Staudt's "algebra of throws" goes back at least to [**Ma**], see also [**Ku**].)

Lafforgue outlined a proof of a scheme-theoretic version in [**L**, Thm. I.14]. Keel and Tevelev used this construction in [**KT**] (see §1.8 and Theorem 3.13 of that article). Another algebro-geometric application of Mnëv's theorem (this time in its manifestation in the representation problem of matroids) was Belkale and

Brosnan's surprising counterexample to a conjecture of Kontsevich, [**BB**]. More recently Payne applied this construction in [**P**] to toric vector bundles, and Erman applied it in [**E**] to the Hilbert scheme of points. (These examples are representative but not exhaustive.)

Acknowledgments. We thank the two referees for their thoughtful suggestions. The second author thanks A. J. de Jong, M. Roth, and of course J. Harris.

2. Structure of the construction

2.1. Strategy. Fix a singularity (Y, Q) of finite type over $\operatorname{Spec}\mathbb{Z}$. We will show that there exists a point P of some incidence scheme X (i.e. some configuration of points and lines, as described in Definition 1.1), along with a smooth morphism $\pi : (X, P) \to (Y, Q)$ of pointed schemes. Because smooth morphisms are open, it suffices to deal with the case where Q is a *closed* point of Y. Then the residue field $\kappa(Q)$ has finite characteristic p. (The reduction to characteristic p is not important; it is done to allow us to construct a configuration over a fixed infinite field. Those interested only in the characteristic 0 version of this result will readily figure out how to replace $\overline{\mathbb{F}}_p$ with $\mathbb{Q}$ or $\mathbb{C}$.)

By replacing Y with an affine neighborhood of Q, we may assume Y is affine, say $Y = \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. The morphism

$$\operatorname{Spec}\overline{\mathbb{F}}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \to \operatorname{Spec}\mathbb{F}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

is surjective by the Lying Over Theorem. Choose a pre-image

$$\overline{Q} \in \operatorname{Spec}\overline{\mathbb{F}}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

of Q — say the (closed) point $(x_1, \ldots, x_n) = (q_1, \ldots, q_n)$, where $\vec{q} \in \overline{\mathbb{F}}_p^n$.

We make the following constructions.

(a) We describe a configuration of points and lines over $\overline{\mathbb{F}}_p$, which is thus an $\overline{\mathbb{F}}_p$-valued point $\overline{P}$ of an incidence scheme X.

(b) The incidence scheme X will be an open subscheme of an affine scheme X', and we construct (a finite number of) coordinates on X', which we name X_1, …, X_n, Y_1, …, Y_s subject only to the relations $f_i(X_1, \ldots, X_n) = 0$ $(1 \leq i \leq r)$. We thus have a smooth morphism $\pi : X \to Y$ given by $X_i \mapsto x_i$. Letting $X_K = X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} K$ for $K = \mathbb{F}_p$ and $\overline{\mathbb{F}}_p$, and similarly for Y_K and π_K, we have a diagram:

$$\begin{tikzcd}
X_{\overline{\mathbb{F}}_p} \arrow[r] \arrow[d, "\pi_{\overline{\mathbb{F}}_p}"'] & X_{\mathbb{F}_p} \arrow[r] \arrow[d, "\pi_{\mathbb{F}_p}"'] & X \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"'] & X' \arrow[r, equal] & Y \times_{\mathbb{Z}} \mathbb{A}^s_{\mathbb{Z}} \arrow[dll] \\
Y_{\overline{\mathbb{F}}_p} \arrow[r] & Y_{\mathbb{F}_p} \arrow[r] & Y & &
\end{tikzcd}$$

In the course of the construction, we will not explicitly name the variables Y_j, but whenever a free choice is made this corresponds to adding a new variable Y_j.

(c) We will have $\pi_{\overline{\mathbb{F}}_p}(\overline{P}) = \overline{Q}$. Thus the image of π includes Q.

2.2. Notation and variables for the incidence scheme. The traditional (and only reasonable) approach is to construct a configuration of points and lines encoding this singularity, by encoding the "atomic" operations of equality, negation, addition, and multiplication. The most difficult desideratum is 1.1(iii).

Our incidence scheme will parametrize points and lines of the following form. In the course of this description we give names to the relevant types of points and lines, and give our chosen coordinates. We will later describe our particular point $\overline{P} \in X_{\overline{\mathbb{F}}_p}$.

The first type of points are p_1 through p_4 (see 1.1(i)). We call these *anchor points*. We interpret $\mathbb{P}^2$ in the usual way: p_2p_3 is the line at infinity, and p_1 is the origin; lines through p_3 are called horizontal. The first type of lines in our incidence scheme are the lines p_ip_j. We call these *anchor lines*.

The next type of line, which we call *variable-bearing lines*, will be required to pass through $p_3 = [1, 0, 0]$ (they are "horizontal"), and not through p_1, p_2, or p_4. Each variable-bearing line is parametrized by where it meets the y-axis (which is finite, as the lines do not pass through $p_2 = [0, 1, 0]$). Thus for each variable-bearing line l_i, we have a coordinate y_i. In order to satisfy 1.1(iii), we will always arrange that the y_i are distinct and not 0 or 1. These y_i will be among the Y_j of 2.1(b) above.

Each variable-bearing line l_i has a *framing-type* Fr_i, which is a size two subset of $\{-1, 0, 1\}$ if $p > 2$, and of $\{0, 1, j\}$ where j is a chosen solution of $j^2 + j - 1 = 0$ (see §4.6.3 for more) if $p = 2$. Each variable-bearing line l_i contains (in addition to p_3) the following three distinct marked points:

- two *framing points* $P_{i,s}$, where $s \in \mathrm{Fr}_i$; and
- one *variable-bearing point* V_i.

The point $P_{i,s}$ we parametrize by its x-coordinate, which we confusingly name $y_{i,s}$ (because it will be one of the "free" variables Y_i of 2.1(b)). The variable-bearing point V_i we parametrize using the isomorphism $l_i \to \mathbb{P}^1$ obtained by sending p_3 to ∞, and $P_{i,s}$ to s for $s \in \mathrm{Fr}_i$. We denote this coordinate x_i. (In our construction, these coordinates will be either among those X_i of 2.1(b) above, or will be determined by the other variables.) A variable-bearing line over $\overline{\mathbb{F}}_p$ of framing-type Fr_i, whose variable-bearing point carries the variable $x_i = q \in \overline{\mathbb{F}}_p$, we will call a (Fr_i, q)-line or a (Fr_i, x_i)-line.

We have a number of additional configurations of points and lines, called *connecting configurations*, which are required to contain a specified subset of the above-named points, and required to *not* contain the rest. These will add additional free variables (which, in keeping with 2.1(b) above, we call Y_j for an appropriate j), and will (scheme-theoretically) impose a single constraint upon the x-variables:

- $x_a = x_b$ (an *equality configuration*)
- $x_a = -x_b$ (a *negation configuration*),
- $x_a + x_b = x_c$ (an *addition configuration*), or
- $x_ax_b = x_c$ (a *multiplication configuration*).

Finally, for each pair of above-named lines that do not have an above-named point contained in both, we have an additional marked point at their intersection (in order that 1.1(iv) holds), which we call *bystander points*. In our construction, we never have more than two lines meeting at a point except at the previously-named points, and the only lines passing through the previously-named points are the ones specified above. The name "bystander points" reflects the fact that they play no further role, and no additional variables are needed to parametrize them.

2.3. Reduction to four problems. We reduce Theorem 1.2 to four atomic problems.

We construct the expression for each f_i sequentially, starting with the variables x_i and the constant 1 (which we name x_0 to simplify notation later), and using negation of one term or addition or multiplication of two terms at each step. Somewhat more precisely, we make a finite sequence of intermediate expressions, where each expression is the negation, sum, or product of earlier (one or two) expressions. We assign new variables $x_{n+1}, x_{n+2}, \dots$ for each new intermediate expression. Those additional variables x_k will come along with a single equation — negation, addition, or multiplication — describing how x_k is obtained from its redecessor(s). In case of sum and product, we additionally require two predecessors to be different, even in case of adding or multiplying same expression. Finally, for the variable x_a representing the final expression f_i (one for each f_i), we add the equation $x_a = 0$. These simple equations (which we call g_i) are equivalent to our original equations $f_i = 0$, so we have (canonically)

$$\mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r) \cong \mathbb{Z}[x_1, \dots, x_n, x_{n+1}, \dots, x_m]/(g_1, g_2, \dots, g_{r'}) \tag{2.1}$$

where each g_j is of the form $x_a - x_b$, $x_a + x_b$, $x_a + x_b - x_c$, $x_a x_b - x_c$, or x_a. As one example of this procedure:

$$\frac{\mathbb{Z}[x_1, x_2, x_3]}{(x_1 x_2 + x_3^2 - 2)} \cong \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]/I$$

where $I = (x_4 - x_1 x_2, x_5 - x_3, x_6 - x_3 x_5, x_7 - x_4 - x_6, x_8 - 1, x_9 - 1, x_{10} - x_8 - x_9, x_7 - x_{10})$.

We now construct our configuration over $\overline{\mathbb{F}}_p$. Via (2.1), we interpret $\overline{\mathbb{Q}}$ as a geometric point of $\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_m]/(g_1, g_2, \dots, g_{r'})$, and we let $q_i \in \overline{\mathbb{F}}_p$ be coordinates of x_i for all i. For each $i \in \{1, \dots, m\}$, we choose two distinct q's in $\{-1, 0, 1\}$ or $\{0, 1, j\}$ (according to whether $p > 2$ or $p = 2$) distinct from q_i; this will be the framing-type Fr_i of x_i. We place (generally chosen) variable-bearing lines l_i, one for each x_i for $i \in \{1, \dots, m\}$, with framing points (corresponding to the framing-type Fr_i) chosen generally on l_i, then with variable-bearing points chosen so that the coordinate of the variable point for the line l_i is q_i.

We then sequentially do the following for each simple equation g_j. For each g_j involving variables x_a, x_b (and possibly x_c), we place a corresponding configuration joining variable-bearing points for those variables and enforcing (scheme-theoretically) the equation g_j. We will do this in such a way that the connecting configuration passes through no points or lines it is not supposed to. We will of course do this by a general position argument.

We are thus reduced to four problems, which we describe below (with italicized titles) after setting the stage for them. Suppose we are given a configuration of points and lines in the plane, including the anchor points p_i, and the anchor lines $p_i p_j$ ($1 \le i < j \le 4$) (and hence implicitly a point of some incidence scheme). Note that this incidence scheme is quasiaffine, say $U \subset \operatorname{Spec} A$:

- the non-vertical lines (those non-anchor lines not containing $p_2 = [0, 1, 0]$) $y = mx + b$ are parametrized by m and b;
- vertical lines (those non-anchor lines passing through p_2) $x = a$ are parametrized by a;
- those points $(x, y) = (a, b)$ not on the line at infinity are parametrized by a and b;
- and those non-anchor points $[1, c, 0]$ on the line at ∞ are parametrized by c.

The conditions of §1.1 are clearly locally closed.

Here now are the four problems.

Equality problem. If we have two variable-bearing lines l_a and l_b with coordinates x_a and x_b, we must show that we may superimpose an equality configuration (i.e. add more points and lines), where except for the framing and variable- earing points on these two lines l_a and l_b, no point of the additional configuration lies on any pre-existing lines, and no line in the additional configuration passes through any pre-existing points (including pre-existing bystander points — pairwise intersections of pre-existing lines). Furthermore, the addition of this configuration must add only open conditions for added free variables and x_a, x_b, and enforce exactly (scheme-theoretically) the equation $x_a = x_b$. More precisely, we desire that the morphism from the new incidence scheme to the old one is of the following form:

$$\begin{tikzcd}
U' \arrow[r, hook, "\text{open}"] \arrow[d] & \operatorname{Spec} A[y_1, \ldots, y_N]/(x_a - x_b) \arrow[d] \\
U \arrow[r, hook, "\text{open}"] & \operatorname{Spec} A
\end{tikzcd}$$

(for some value of N).

Negation problem: the same problem, except with $x_a = -x_b$ replacing $x_a = x_b$.

Addition problem: the analogous problem, except $x_c = x_a + x_b$ $(a \neq b)$.

Multiplication problem: the analogous problem, except $x_c = x_a x_b$ $(a \neq b)$.

3. The configurations

We now describe the configurations needed to make this work.

3.1. Building blocks for the building blocks: five configurations. The building blocks we use are shown in Figures 2–5. The figures follow certain conventions. (See Figure 1 for a legend.) Lines that appear horizontal are indeed so — they are required to pass through $p_3 = [1, 0, 0]$. The horizontal lines often have (at least) three labeled points, which suggest an isomorphism with $\mathbb{P}^1$. The dashed lines (and marked points thereon) are those that are in the configuration before we begin. The points and lines marked with a box are added next, and involve free choices (two coordinates for each boxed point, one for the each boxed horizontal line). The remaining points and lines are then determined. The triangle indicates the "goal" of the construction, if interpreted as constructing midpoint, addition, multiplication, and so forth (which is admittedly not our point of view).

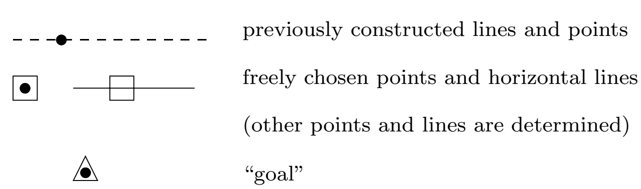

FIGURE 1. Legend for Figures 2–5

The first building block, *parallel shift*, gives the projection from a point X of three points (P_1, V, P_2) on the horizontal line l onto (P_1', V', P_2') on the horizontal line l'. Invariance of cross-ratio under projection gives

$$(P_1, V; P_2, p_3) = (P'_1, V'; P'_2, p_3)$$

(where $(\cdot,\cdot;\cdot,\cdot)$ throughout the paper means cross-ratio, or moduli point in $\mathcal{M}_{0,4}$), so if l and l' are lines of the same framing-type, with (P_1, P_2) and (P'_1, P'_2) the corresponding framing points and V and V' the variable-bearing points, then the coordinates of V and V' are the same (scheme-theoretically). Note that we are adding three free variables (two for the point, one for the line), plus an open condition to ensure no unintended incidences with preexisting points and lines.

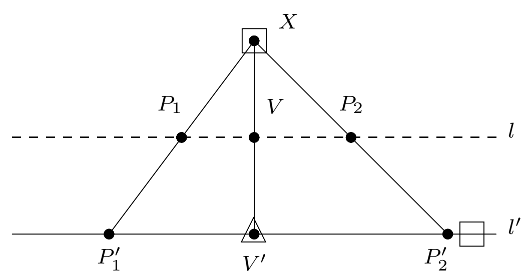

FIGURE 2. Parallel shift

The second building block, *midpoint* (Figure 3), will be used for constructing the midpoint M of two distinct points A and B on line l (where p_3 is considered as usual to be infinity). This construction will be used outside characteristic 2. (In $p = 2$, the diagram is misleading: XY passes through p_3, resulting in $M = p_3$.) We have equality of cross-ratios

$$\begin{aligned}(A, M; B, p_3) &= (A', M'; B', p_3) \quad \text{(projection from } X\text{)}\\ &= (B, M; A, p_3) \quad \text{(projection from } Y\text{)}\\ &= 1/(A, M; B, p_3) \quad \text{(property of cross-ratio)}\end{aligned}$$

so $(A, M; B, p_3)$ is either 1 or -1. For $p \neq 2$ and $A \neq B$, it is straightforward to verify that $M \neq p_3$ so $(A, M; B, p_3) \neq 1$. Thus $(A, M; B, p_3) = -1$, so M is the "midpoint" of AB. (More precisely: given any isomorphism of l with $\mathbb{P}^1$ identifying p_3 with ∞, the coordinate of M is the average of the coordinates of A and B. In classical language, M is the harmonic conjugate of p_3 with respect to A and B.)

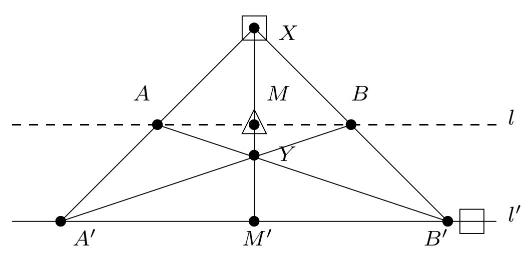

FIGURE 3. Midpoint

The *generic addition* configuration (Figure 4) deals with addition $x_a + x_b$ in the "generic" case where x_a, x_b, and $x_a + x_b$ are distinct from 0 and 1, and the framing-type of their lines are all $\{0,1\}$. Given two lines l_a and l_b with variables x_a, x_b, with framing points $(P_{a,0}, P_{a,1})$ and $(P_{b,0}, P_{b,1})$ on l_a and l_b respectively, we choose a general horizontal line l' and a general point X, and superimpose the construction shown in Figure 4. If the line l' is given the framing-type $\mathrm{Fr} = \{0,1\}$ with framing points P'_0 and P'_1, the reader will readily verify that the coordinate of V' is $x' = x_a + x_b$, and that this equation is precisely what is (scheme-theoretically) enforced by the configuration.

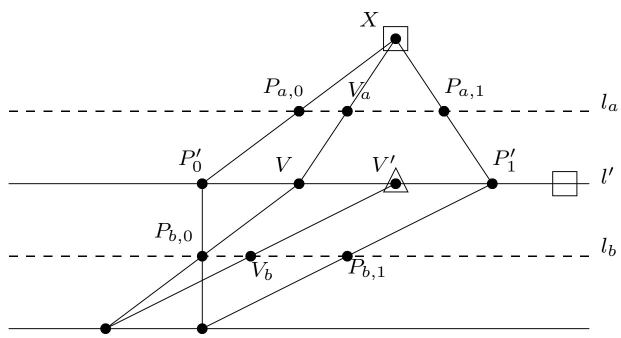

FIGURE 4. Generic addition

The *generic multiplication* configuration (Figure 5) constructs/enforces multiplication $x_c = x_a x_b$ in the "generic" case where x_a, x_b, and $x_a x_b$ are distinct from 0 and 1, and the framing-type of their lines are all $\{0,1\}$. As with the "generic addition" case, given two lines l_a and l_b with variables x_a, x_b, with framing points $(P_{a,0}, P_{a,1})$ and $(P_{b,0}, P_{b,1})$ on l_a and l_b respectively, we choose a general horizontal line l' and a general point X, and superimpose the construction shown in Figure 5. If the line l' is given the framing-type $\mathrm{Fr} = \{0,1\}$ with framing points P'_0 and P'_1, the reader will readily verify that the coordinate x' of V' is $x_a x_b$, and that this equation is precisely what is (scheme-theoretically) enforced by the configuration. The main part of the argument is that

$$x_a = (P_{a,0}, V_a; P_{a,1}, p_3) = (P'_0, V; P'_1, p_3)$$

and

$$(P'_0, V'; V, p_3) = (P_{b,0}, V_b; P_{b,1}, p_3) = x_b$$

yield

$$x' = (P'_0, V'; P'_1, p_3) = (P'_0, V'; V, p_3)(P'_0, V; P'_1, p_3) = x_a x_b.$$

We remark that we are parallel-shifting the point V_b from l_b to l' to avoid accidental overlaps of points in our later argument.

4. Putting everything together

We now put the atomic configurations together in various ways in order to solve the four problems of §2.3. We begin with the case $p \neq 2$, leaving the case $p = 2$ until §4.6.

4.1. Relabeling. Before we start, we note that it will be convenient to use the same framing points but a different framing-type to change the value of the variable "carried" by the line. For example, a $(\{0,1\}, q)$-line may be interpreted as a $(\{0,-1\}, -q)$-line (as $(0,1;q,\infty) = (0,-1;-q,\infty)$).

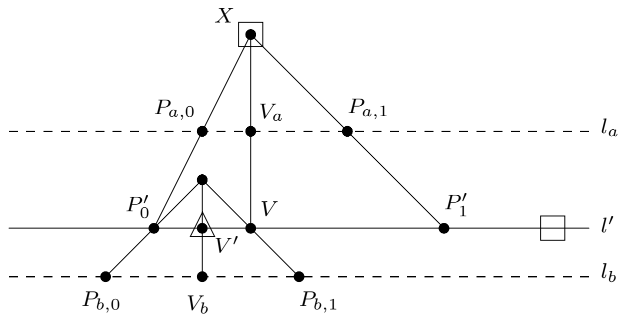

FIGURE 5. Generic multiplication

4.1.1. *Initial framing.* Before we start, we "construct -1 on the x-axis". More precisely, on the x-axis, we have identified the points $0 := [0, 0, 1] = p_1$ and $1 := [1, 0, 1] = \overline{p_2 p_4} \cap \overline{p_1 p_3}$. We use the midpoint construction (Figure 3) to construct $-1 := [-1, 0, 1]$ as well (using $M = 0$, $B = 1$, $A = -1$).

We now construct equality, negation, addition, and multiplication.

4.2. Equality: enforcing $x_a = x_b$. We enforce equality $x_a = x_b$ as follows.

4.2.1. *First case: same framing-type.* Suppose first that two variables x_a and x_b are of same framing-type $\{s_1, s_2\}$. Then after a general choice of horizontal line l', we parallel shift (Figure 2) $(P_{a,s_1}, V_a, P_{a,s_2})$ onto (P'_1, V', P'_2) on l', using a generally chosen projection point X. Then for $X' = P'_1 P_{b,s_1} \cap P'_2 P_{b,s_2}$, we shift $(P_{b,s_1}, V_b, P_{b,s_2})$ onto (P'_1, V'', P'_2) on l', using X' as projection point. The reader will verify that if we impose the codimension 1 condition that $V' = V''$, we enforce the equality $x_a = x_b$. The reader will verify that with the general choice of projection point X and line l', the newly constructed points will miss any finite number of previously constructed points and lines (except for those in the Figure); and the newly constructed lines will miss any finite number of previously constructed points (except for those in the Figure, and of course p_3) — we will have no "unintended coincidences". This can be readily checked in all later constructions (an essential point in the entire strategy!), but for concision's sake we will not constantly repeat this.

4.2.2. *Second case: different framing-type.* Next, suppose that x_a and x_b have different framing-type, say $\{s_{a,1}, s_{a,2}\}$ and $\{s_{b,1}, s_{b,2}\}$ respectively (two distinct subsets of $\{-1, 0, 1\}$). Then $q_a = q_b$ is not in $\{-1, 0, 1\}$. We apply parallel shift (Figure 2) to move x_a to a generally chosen horizontal line l'. We then parallel shift the points -1, 0, and 1 on the x-axis to l', so we have marked points on l' that can be identified (with the obvious isomorphism to $\mathbb{P}^1$) with $\infty = p_3$, -1, 0, 1, and $q_a = q_b$. Then (using the subset $\{s_{b,1}, s_{b,2}\}$ of the marked points on l') l' and l_b have same framing-type and we can apply previous construction.

We remark that in this and later constructions, we can take x_a or x_b (or, later, x_c) to be the constants 0 or 1, by treating the x-axis as a variable-bearing line. For example, to take $x_a \equiv 1$, treat the x-axis as a $(\{-1, 0\}, 1)$-line.

4.2.3. *Remark: choosing framing-type freely.* The argument of §4.2.2 shows that given a variable "carried by" a variable-bearing line, we can change the framing-type of the line it "lives on", at the cost of moving it to another generally chosen horizontal line (so long as the value of the variable does not lie in the new framing-type of course). From now on, given a variable, we freely choose a framing-type to suit our purposes at the time.

4.3. Negation: enforcing $x_b = -x_a$. We now explain how to enforce $x_b = -x_a$. Suppose x_a is carried on a line with framing-type $\{s_1, s_2\}$, and x_b is carried on a line with framing-type $\{-s_1, -s_2\}$ (possible as $q_a = -q_b$ — here we use Remark 4.2.3). We enforce $x_b = -x_a$ by adding the equality configuration (first case, §4.2.1), except interpreting line l_b as an $(\{s_1, s_2, \}, -q_b)$-line (the relabel construction, §4.1).

4.4. Addition: enforcing $x_a + x_b = x_c$.

4.4.1. *First case: "(general) + (general) = (general)".* Suppose q_a, q_b, q_c are all distinct from 0 and 1. We apply parallel shifts to move the three relevant variable-bearing lines onto generally chosen lines, and then superimpose the "generic addition" configuration of Figure 4. (The parallel shifts are to guarantee no unintended coincidences.)

4.4.2. *Second case: "1 + (general) = (general)".* Suppose next that $q_a = 1$, and q_b and q_b are neither 0 nor 1. Then $q_c \neq -1$ (or else q_b would be 0). The equation we wish to enforce may be rewritten as $-x_c + x_b = -x_a$, and $-x_c$, x_b, and $-x_a$ are all distinct from 0 and 1. (Here we use $p \neq 2$, as we require $-1 \neq 1$.) We thus accomplish our goal by applying the negation configuration to x_a and x_c, then applying the first case of the addition construction, §4.4.1.

4.4.3. *Third case: "0 + (general) = (general)".* Suppose that $q_a = 0$, and q_b and q_c are not in $\{-1, 0, 1, 2\}$. We take the framing-sets on l_a and l_c to be $\{-1, 1\}$ (using Remark 4.2.3). As in §4.1, we interpret/relabel the $(\{-1, 1\}, x_a)$-line l_a as a $(\{0, 1\}, x'_a)$-line (where $x'_a = (x_a + 1)/2$) and the $(\{-1, 1\}, x_c)$-line as a $(\{0, 1\}, x'_c)$-line (where $x'_c = (x_c + 1)/2$). We take the framing-set $\{0, 1\}$ on l_b. We parallel shift x_b onto a general horizontal line l'_b, then use the midpoint construction (Figure 3) to construct the midpoint of V_b and $P_{b,0}$ on l'_b, so we have constructed the variable $x_b/2$, which we name x'_b. The equation we wish to enforce, $x_a + x_b = x_c$, is algebraically equivalent to $x'_a + x'_b = x'_c$, and the values of x'_a, x'_b, and x'_c are all distinct from 0 and 1, so we can apply the construction of the first case of addition, §4.4.1.

4.4.4. *Fourth case: everything else.* We begin by adding two extra free variables s and t on two generally chosen horizontal lines. More precisely, for s, we pick a generally chosen horizontal line l_i, and three generally chosen points $P_{i,0}$, $P_{i,1}$, and V_i on it, and define $s = (P_{i,0}, P_{i,1}; V_i, p_3)$, so l_i is a $(\{0, 1\}, s)$-line. We do the same for t. Using the previous cases of addition, we successively construct $x_a + s$, $x_b + t$, $(x_a + s) + (x_b + t)$, $s + t$, and $x_c + (s + t)$. (Because s and t were generally chosen, one of the three previous cases can always be used.) Then we impose the equation

$$(x_a + s) + (x_b + t) = x_c + (s + t)$$

(using the third case of addition, §4.4.3, twice). Thus we have scheme-theoretically enforced $x_a + x_b = x_c$ as desired.

4.5. Multiplication: enforcing $x_a x_b = x_c$. As with addition, we deal with a "sufficiently general" case first, and then deal with arbitrary cases by translating by a general value.

4.5.1. *First case: "(general)$\times$(general) = (general)".* Suppose $q_a, q_b, q_c \neq 0, 1$. We parallel shift all variables x_a, x_b, x_c to generally chosen lines l'_a, l'_b, and l'_c (to avoid later unintended incidences), and then superimpose the generic multiplication configuration to impose $x_c = x_a x_b$ (where l'_a, l'_b, and l'_c here correspond to l_a, l_b, and l' in Figure 5).

4.5.2. *Second case: everything else.* To enforce $x_a x_b = x_c$, we proceed as follows. We add two extra free variables u and v as in §4.4.4. We then use the addition constructions of §4.4.1–4.4.4 to construct $x_a + u$ and $x_b + v$ (on generally chosen horizontal lines). We use the construction of §4.5.1 to construct $(x_a+u)(x_b+v)$, uv, $(x_a+u)v$, and $(x_b+v)u$ (each on generally chosen lines). Finally, we use the addition constructions (several times) to enforce

$$(x_a+u)(x_b+v) + uv = x_c + (x_a+u)v + (x_b+v)u.$$

The result then follows from the algebraic identity

$$(a+c)(b+d) + cd = ab + (a+c)d + (b+d)c.$$

4.6. Characteristic 2. As the above constructions at several points use $-1 \neq 1$, the case $p = 2$ requires a variant strategy.

4.6.1. *Addition and multiplication: general cases (*§4.4.1, §4.5.1*).* We begin by noting that the general cases of addition and multiplication, given in §4.4.1 and §4.5.1 respectively, work as before (where q_a, q_b, and q_c are all distinct from $\{0,1\}$, and the framing-type is taken to be $\{0,1\}$ in all cases).

4.6.2. *Relabeling (*§4.1*), and the first case of equality (*§4.2.1*).* Relabeling (§4.1) works as before. Equality in the case of same framing-type (§4.2.1) does as well.

4.6.3. *Initial framing.* In analogy with the initial framing of §4.1.1, before we begin the construction, we construct j and $1-j = j^2$ on the x-axis as follows. More precisely, we will add whose points on the x-axis which we label j and k, as well as configurations forcing the coordinates to satisfy $j^2 + j - 1 = 0$, and $k = j^2$. (We then hereafter call the point k by the name j^2.) It is important to note that this construction of j is étale over $\operatorname{Spec}\mathbb{Z}$ away from $[(5)]$, and in particular at 2; thus this choice will not affect the singularity type.

We construct these points as follows. Choose $j \in \mathbb{F}_4 \setminus \mathbb{F}_2$, and place a marked point at j on the x-axis. Construct the product of j with j by parallel shifting j separately onto two generally chosen horizontal lines, and then using the construction of §4.6.1, i.e. §4.5.1 (possible as j and j^2 are distinct from 0 and 1). Then construct $1-j$ using the relabeling trick of §4.1 (§4.6.2): parallel shift 0, 1, and j to a generally chosen line, then reinterpret the $(\{0,1\}, j)$-line as a $(\{1,0\}, 1-j)$-line, and parallel-shift it back to the x-axis. Finally, we use the equality configuration (the "same framing-type" case, §4.6.2 = §4.2.1) to enforce $j^2 = 1-j$.

4.6.4. *Equality in general (*§4.2.2*), and freely choosing framing-type (Remark* 4.2.3*).* Now that we have constructed j, the second case of the equality construction works (with $\{-1,0,1\}$ replaced by $\{0,1,j\}$), and we may choose framing-type freely on lines as observed in Remark 4.2.3.

4.6.5. *Addition, second case: "*1 *+ (general) = (general)", cf.* §4.4.2. Suppose $q_a = 1$, and q_b and q_c are not in $\{0,1,j\}$. Then use the general case of multiplication (§4.6.1, i.e. §4.5.1) to construct (on separate general horizontal lines) $q'_b = q_b/j$ and $q'_c = q_c/j$. By considering the $(\{0,j\}, x_a)$-line as a $(\{0,1\}, x_a/j)$-line (§4.6.2, i.e. §4.1), construct (on a general horizontal line, using parallel shift) $x'_a = x_a/j$. Then impose $x'_a + x'_b = x'_c$ using the general case of addition (§4.6.1)

4.6.6. *Addition, third case: "0 + (general) = (general)", cf.* §4.4.3. Suppose $q_a = 0$, $q_b \notin \{0, 1, j, j^2\}$, and $q_c \notin \{1, j\}$. Then construct $x'_a = (x_a - 1)/j^2$ (on a general horizontal line of framing-type $\{0, 1\}$) by considering the $(\{1, j\}, x_a)$-line (carrying the variable x_a) as a $(\{0, 1\}, (x_a - 1)/j^2)$-line (as $j - 1 = j^2$). Similarly, construct $x'_c = (x_c - 1)/j^2$. Using the general multiplication construction (§4.6.1, i.e. §4.5.1) twice, construct $x'_b = x_b/j^2$ (by way of the intermediate value of x_b/j). Then impose $x'_a + x'_b = x'_c$ (using the general addition construction of §4.6.1, i.e. §4.4.1), and note that this is algebraically equivalent to $x_a + x_b = x_c$.

4.6.7. *Addition and multiplication, final cases: everything else (*§4.4.4, §4.5.2*).* These now work as before.

4.6.8. *Negation (*§4.3*).* Finally, negation can be imposed by constructing the configuration imposing $x_a + x_b = 0$ (using the final case of addition).

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Clay Mathematics Proceedings
Volume **18**, 2013

Gromov-Witten theory and Noether-Lefschetz theory

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Dedicated to J. Harris on the occasion of his 60th birthday

Abstract. Noether-Lefschetz divisors in moduli spaces of $K3$ surfaces are the loci corresponding to Picard rank at least 2. We relate the degrees of the Noether-Lefschetz divisors in 1-parameter families of $K3$ surfaces to the Gromov-Witten theory of the 3-fold total space. The reduced $K3$ theory and the Yau-Zaslow formula play an important role. We use results of Borcherds and Kudla-Millson for $O(2,19)$ lattices to determine the Noether-Lefschetz degrees in classical families of $K3$ surfaces of degrees 2, 4, 6 and 8. For the quartic $K3$ surfaces, the Noether-Lefschetz degrees are proven to be the Fourier coefficients of an explicitly computed modular form of weight 21/2 and level 8. The interplay with mirror symmetry is discussed. We close with a conjecture on the Picard ranks of moduli spaces of $K3$ surfaces.

Contents

0. Introduction

0.1. K3 families. Let C be a nonsingular complete curve, and let

$$\pi : X \to C$$

be a 1-parameter family of nonsingular quasi-polarized $K3$ surfaces. Let $L \in \mathrm{Pic}(X)$ denote the quasi-polarization of degree

$$\int_{K3} L^2 = l \in 2\mathbb{Z}^{>0}.$$

The family π yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_l,$$

to the 19 dimensional moduli space of quasi-polarized $K3$ surfaces of degree l. A review of the definitions can be found in Section 1.

0.2. Noether-Lefschetz numbers. Noether-Lefschetz numbers are defined by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_l$. Noether-Lefschetz divisors can be described via Picard lattices or Picard classes. We briefly review the two approaches.

Let $(\mathbb{L}, v)$ be a rank 2 integral lattice with an even symmetric bilinear form

$$\langle , \rangle : \mathbb{L} \times \mathbb{L} \to \mathbb{Z}$$

and a distinguished primitive vector $v \in \mathbb{L}$ satisfying

$$\langle v, v \rangle = l.$$

The invariants of $(\mathbb{L}, v)$ are the discriminant $\triangle \in \mathbb{Z}$ and the coset

$$\delta \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm .$$

If the data are presented as

$$\mathbb{L}_{h,d} = \begin{pmatrix} l & d \\ d & 2h-2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the discriminant is

$$\triangle_l(h,d) = -\det \begin{vmatrix} l & d \\ d & 2h-2 \end{vmatrix} = d^2 - 2lh + 2l$$

and the coset is

$$\delta = d \bmod l \quad \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm .$$

Two lattices $(\mathbb{L}_{h,d}, v)$ and $(\mathbb{L}_{h',d'}, v')$ are equivalent if and only if

$$\triangle_l(h,d) = \triangle_l(h',d') \text{ and } \delta_{h,d} = \delta_{h',d'}.$$

However, not all pairs $(\triangle, \delta)$ are realized.

The first type of Noether-Lefschetz divisor is defined by specifying a Picard lattice. Let

$$P_{\triangle,\delta} \subset \mathcal{M}_l$$

be the closure of the locus of quasi-polarized $K3$ surfaces (S, L) of degree l for which $(\mathrm{Pic}(S), L)$ is of rank 2 with discriminant $\triangle$ and coset δ. By the Hodge index theorem, $P_{\triangle,\delta}$ is empty unless $\triangle > 0$.

The second type of Noether-Lefschetz divisor is defined by specifying a Picard class. In case $\triangle_l(h,d) > 0$, let

$$D_{h,d} \subset \mathcal{M}_l$$

have support on the locus of quasi-polarized $K3$ surfaces (S, L) for which there exists a class $\beta \in \mathrm{Pic}(S)$ satisfying

$$\int_S \beta^2 = 2h-2 \text{ and } \int_S \beta \cdot L = d.$$

More precisely, $D_{h,d}$ is the weighted sum

$$D_{h,d} = \sum_{\triangle,\delta} \mu(h, d \,|\triangle, \delta) \cdot [P_{\triangle,\delta}] \tag{1}$$

where the multiplicity

$$\mu(h, d \,|\triangle, \delta) \in \{0, 1, 2\}$$

is defined to be the number of elements β of the lattice $(\mathbb{L}, v)$ associated to $(\triangle, \delta)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \;\text{ and }\; \langle \beta, v \rangle = d. \tag{2}$$

If no lattice corresponds to $(\triangle, \delta)$, the multiplicity $\mu(h, d \,|\triangle, \delta)$ vanishes and $P_{\triangle,\delta}$ is empty. If the multiplicity is nonzero, then

$$\triangle | \triangle_l (h, d).$$

Hence, the sum on the right of (1) has only finitely many terms.

As relation (1) is easily seen to be triangular, the divisors $P_{\triangle,\delta}$ and $D_{h,d}$ are essentially equivalent. However, the divisors $D_{h,d}$ will be seen to have better formal properties.

A natural approach to studying the divisors $D_{h,d}$ is via intersections with test curves. In case $\triangle_l(h, d) > 0$, the Noether-Lefschetz number $NL^{\pi}_{h,d}$ is the classical intersection product

$$NL^{\pi}_{h,d} = \int_C \iota^*_{\pi}[D_{h,d}]. \tag{3}$$

If $\triangle_l(h, d) < 0$, the divisor $D_{h,d}$ vanishes by the Hodge index theorem. A definition of $NL^{\pi}_{h,d}$ for all values $\triangle_l(h, d) \geq 0$ is given by classical intersection theory in the period domain for $K3$ surfaces in Section 1.

The divisibility of a nonzero element β of a lattice is the maximal positive integer m dividing β. Refined divisors $D_{m,h,d}$ are defined by

$$D_{m,h,d} = \sum_{\triangle,\delta} \mu(m, h, d \,|\triangle, \delta) \cdot [P_{\triangle,\delta}]$$

where the multiplicity

$$\mu(m, h, d \,|\triangle, \delta) \in \{0, 1, 2\}$$

is the number of elements β of divisibility m of the lattice $(\mathbb{L}, v)$ associated to $(\triangle, \delta)$ satisfying (2). Refined Noether-Lefschetz number are defined by

$$NL^{\pi}_{m,h,d} = \int_C \iota^*_{\pi}[D_{m,h,d}].$$

0.3. Invariants. We will study three types of invariants associated to a 1-parameter family π of quasi-polarized $K3$ surfaces in case the total space X is nonsingular:

(i) the Noether-Lefschetz numbers of π,
(ii) the Gromov-Witten invariants of X,
(iii) the reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin.

The Gromov-Witten invariants (ii) of the 3-fold X and the reduced Gromov-Witten invariants (iii) of a $K3$ surface are defined via integration against virtual classes of moduli spaces of stable maps. We view both of these Gromov-Witten theories in terms of the associated BPS state counts defined by Gopakumar and Vafa [**19, 20**].

Let $n^X_{g,d}$ denote the Gopakumar-Vafa invariant of X of genus g for π-vertical curve classes of degree d with respect to L. Let $r_{g,m,h}$ denote the Gopakumar-Vafa reduced $K3$ invariant of genus g and curve class $\beta \in H_2(K3,\mathbb{Z})$ of divisibility m satisfying

$$\int_{K3} \beta^2 = 2h-2.$$

A review of these quantum invariants is presented in Section 2.

A geometric result intertwining the invariants (i)-(iii) is derived in Section 3 by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π.

THEOREM 1. For $d > 0$,

$$n^X_{g,d} = \sum_h \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^\pi_{m,h,d}.$$

Theorem 1 is the main geometric result of the paper. The proof is given in Section 3.

0.4. Applications. Since Theorem 1 relates three distinct geometric invariants, the result can be effectively used in several directions.

An application for studying reduced invariants of $K3$ surfaces is given in [**27**]. A central conjecture discussed in Section 2.3 is the *independence*[1] of $r_{g,m,h}$ on m. In genus 0, the independence is the non-primitive Yau-Zaslow conjecture proven in [**27**] as a consequence of Theorem 1.

The approach taken there is the following. For a specific 1-parameter family of $K3$ surfaces, known in the physics literature as the STU model, the BPS states $n^{STU}_{0,d}$ are known by proven mirror transformations and the Noether-Lefschetz numbers $NL^{STU}_{m,h,d}$ can by exactly determined. Theorem 1 is then used in [**27**] to solve for $r_{0,m,h}$:

$$r_{0,m,h} = r_{0,1,h}, \qquad \sum_{h\geq 0} r_{0,1,h} = \prod_{n\geq 1} \frac{1}{(1-q^n)^{24}} \ .$$

The genus 1 results

$$r_{1,m,h} = r_{1,1,h} = -\frac{h}{12}\ r_{0,1,h}$$

are an easy consequence, see Section 2.3. We write $r_{g,m,h} = r_{g,h}$ independent of m for $g = 0, 1$.

Using [**27**], the genus 0 and 1 specialization takes a much simpler form.

COROLLARY 1. For $g \leq 1$ and $d > 0$,

$$n^X_{g,d} = \sum_{h=g}^{\infty} r_{g,h} \cdot NL^\pi_{h,d}.$$

By Corollary 1, the Gromov-Witten invariants $n^X_{g,d}$ are completely determined by the Noether-Lefschetz numbers of π for any 1-parameter family of quasi-polarized $K3$ surfaces. The result may be viewed as giving a fully classical interpretation of the Gromov-Witten invariants of X in π-vertical classes.

[1] If m^2 does not divide $2h-2$, then $r_{g,m,h} = 0$. The independence is conjectured only when m^2 divides $2h-2$. When we write $r_{g,m,h}$, the divisibility condition is understood to hold.

Theorem 1 can also be used to constrain the Noether-Lefschetz degrees themselves. An important approach to the Noether-Lefschetz numbers (already used in the STU calculation) is via results of Borcherds [**7**] and Kudla-Millson [**29**]. The Noether-Lefschetz numbers of π are proven to be the Fourier coefficients of a vector-valued modular form.[2] For several classical families of $K3$ surfaces, Corollary 1 in genus 0 provides an alternative method of calculating the Noether-Lefschetz numbers via the invariants $n^X_{0,d}$. Together, we obtain a remarkable sequence of identities intertwining hypergeometric series from mirror transformations (calculating $n^X_{0,d}$) and modular forms. The Harvey-Moore identity [**22**] for the STU model is a special case.

As a basic example, we provide a complete calculation of the Noether-Lefschetz numbers for the family of $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^3$. The required mirror symmetry calculations (iii) for the quartic pencil have long been established rigorously [**17, 18**]. We give the derivation of the Noether-Lefschetz numbers via Gromov-Witten calculations in Section 5. The resulting hypergeometric-modular identity follows immediately in Section 5.5. A second approach to calculating Noether-Lefschetz numbers directly via more sophisticated modular form techniques is explained for quartics and several other classical families in Section 6.

Once the Noether-Lefschetz numbers are calculated for the 1-parameter family π, Corollary 1 yields the genus 1 Gromov-Witten invariants of X in π-vertical classes. There are very few methods for the exact calculation of genus 1 invariants in Calabi-Yau geometries.[3] Corollary 1 provides a new class of complete solutions.

0.5. Heterotic duality. In rather different terms, approach (i)-(iii) was pursued in the string theoretic work of Klemm, Kreuzer, Riegler, and Scheidegger [**26**] with the goal of calculating the BPS counts $n^X_{g,d}$ from the genus 0 values $n^X_{0,d}$. Heterotic duality was used in [**26**] for (i) since the connection to the intersection theory of the Noether-Lefschetz divisors

$$D_{h,d} \subset \mathcal{M}_l$$

and the work of Borcherds was not made. The perspective of [**26**] can be turned upside down by using Gromov-Witten theory to calculate the Noether-Lefschetz numbers. On the other hand, modularity allows the calculations of [**26**] to be pursued in much greater generality.

In fact, the back and forth here between heterotic duality and mathematical results is older. Borcherds' paper on automorphic functions [**6**] which underlies [**7**] was motivated in part by the work of Harvey and Moore [**22, 23**] on heterotic duality. The first higher genus results for $K3$ fibrations were by Mariño and Moore [**38**].

Finally, we mention the circle of ideas here can be considered for interesting isotrivial families of $K3$ surfaces with double Enriques fibers [**28, 39**]. While heterotic duality arguments apply there, Borcherds' result does not directly apply.

[2]While the paper [**7, 29**] have considerable overlap, we will follow the point of view of Borcherds.

[3]See [**54**] for a different mathematical approach to genus 1 invariants for complete intersections.

0.6. Modular forms. Let A and B be modular forms of weight 1/2 and level 8,

$$A = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{8}}, \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}.$$

Let Θ be the modular form of weight 21/2 and level 8 defined by

$$\begin{aligned} 2^{22}\Theta \quad = \quad & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & -20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & -621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & -346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & -361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & -4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \end{aligned}$$

We can expand Θ as a series in $q^{\frac{1}{8}}$,

$$\Theta = -1 + 108q + 320q^{\frac{9}{8}} + 50016q^{\frac{3}{2}} + 76950q^2 \dots.$$

The modular form Θ first appeared in calculations of [**26**].

Let π be the family of quasi-polarized $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^4$. Let $\Theta[m]$ denote the coefficient of q^m in Θ.

Theorem 2. *The Noether-Lefschetz numbers of the quartic pencil π are coefficients of Θ,*

$$NL^{\pi}_{h,d} = \Theta\left[\frac{\triangle_4(h,d)}{8}\right].$$

0.7. Classical quartic geometry. Let V be a 4-dimensional $\mathbb{C}$-vector space. A quartic hypersurface in $\mathbb{P}(V)$ is determined by an element of $\mathbb{P}(\mathrm{Sym}^4V^*)$. Let

$$\mathcal{U} \subset \mathbb{P}(\mathrm{Sym}^4V^*)$$

be the Zariski open set of nonsingular quartic hypersurfaces. Since $[S] \in \mathcal{U}$ corresponds to a polarized $K3$ surface of degree 4, we obtain a canonical morphism

$$\phi : \mathcal{U} \to \mathcal{M}_4.$$

If $\triangle_4(h,d) > 0$, the pull-back

$$\mathcal{D}_{h,d} = \phi^{-1}(D_{h,d}) \subset \mathcal{U}$$

is a closed subvariety of pure codimension 1. As a Corollary of Theorem 2, we obtain a complete calculation of the degrees of the hypersurfaces

$$\overline{\mathcal{D}}_{h,d} \subset \mathbb{P}(\mathrm{Sym}^4V^*).$$

Corollary 2. *If $\triangle_4(h,d) > 0$, the degree of $\overline{\mathcal{D}}_{h,d}$ is*

$$\deg(\overline{\mathcal{D}}_{h,d}) = \Theta\left[\frac{\triangle_4(h,d)}{8}\right] - \Psi\left[\frac{\triangle_4(h,d)}{8}\right]$$

where the correction term is

$$\Psi = 108\sum_{n>0} q^{n^2}.$$

The correction term, obtained from the contribution of the nodal quartics, is explained in Section 5.6. Formulas for the degrees of

$$\overline{\phi^{-1}(P_{\triangle,\delta})} \subset \mathbb{P}(\mathrm{Sym}^4 V^*)$$

are easily obtained from (1) and a parallel nodal analysis. While Corollary 2 answers a classical question about the Hodge theory of quartic $K3$ surfaces, the method of proof is modern.

0.8. Outline. In Section 1, we give a precise definition of Noether-Lefschetz numbers and establish several elementary properties. The definitions of BPS invariants for 3-folds and reduced Gromov-Witten invariants of $K3$ surfaces are recalled in Section 2. Two central conjectures about the reduced theory of $K3$ surfaces are stated in Section 2.3. The proof of Theorem 1 is presented in Section 3.

We review of the work of Borcherds on Heegner divisors and explain the application to families of $K3$ surfaces in Section 4. The results are applied with Theorem 1 to prove Theorem 2 via mirror symmetry calculations in Section 5. A direct approach to Noether-Lefschetz degrees for classical familes of $K3$ surfaces of degrees $2, 4, 6$, and 8 is given in Section 6 via a deeper study of vector-valued modular forms. Finally, in Section 7, we state a conjecture regarding Picard ranks of moduli spaces of $K3$ surfaces of degree l .

0.9. Acknowledgments. Discussions with A. Klemm about the calculations in [**26**] played a crucial role. We are grateful to D. Huybrechts for a careful reading of the paper.

We thank R. Borcherds, J. Bruinier, J. Bryan, B. Conrad, I. Dolgachev, S. Grushevsky, E. Looijenga, G. Moore, K. Ranestad, P. Sarnak, E. Scheidegger, C. Skinner, A. Snowden, W. Stein, G. Tian, I. Vainsencher, and W. Zhang for conversations about Noether-Lefschetz divisors, reduced invariants of $K3$ surfaces, and modular forms.

D. M. was partially supported by an NSF graduate fellowship. R.P. was partially support by NSF grant DMS-0500187 and a Packard foundation fellowship. The paper was written in the spring of 2007 and revised in 2009.

1. Noether-Lefschetz numbers

1.1. Picard lattice. Let S be a $K3$ surface. The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S, \mathbb{Z}) \stackrel{\sim}{=} U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \tag{4}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (4) is even.

The *divisibility* of $\beta \in H^2(S,\mathbb{Z})$ is the maximal positive integer dividing β. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm are equivalent up to orthogonal transformation of $H^2(S,\mathbb{Z})$, see [**51**].

The Hodge decomposition of the second cohomology of S

$$H^2(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{2,0}(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^{0,2}(S,\mathbb{C})$$

has dimensions $(1,20,1)$. The *Picard lattice* of S is

$$\mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C}).$$

1.2. Quasi-polarization. A *quasi-polarization* on S is a line bundle L with primitive Chern class $c_1(L) \in H^2(S,\mathbb{Z})$ satisfying

$$\int_S L^2 > 0 \ \text{ and } \ \int_S L \cdot [C] \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasi-polarization is base point free and determines a birational morphism

$$S \to \widetilde{S}$$

contracting A-D-E configurations of (-2)-curves on S [**47**]. Hence, every quasi-polarized $K3$ surface (S,L) is algebraic.

Let X be a compact 3-dimensional complex manifold equipped with a holomorphic line bundle L and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve. The triple (X,L,π) is a *family of quasi-polarized $K3$ surfaces of degree l* if the fibers (X_ξ, L_ξ) are quasi-polarized $K3$ surfaces satisfying

$$\int_{X_\xi} L_\xi^2 = l$$

for every $\xi \in C$. The family (X,L,π) yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_l,$$

to the moduli space of quasi-polarized $K3$ surfaces of degree l.

We will often refer to the triple (X,L,π) just by π. Associated to π is the projective variety $\widetilde{X}$ obtained from the relative quasi-polarization,

$$X \to \widetilde{X} \subset \mathbb{P}(R^0\pi_*(L^n)^*) \to C,$$

for sufficiently large n. The complex manifold X may be a non-projective small resolution of the singular projective variety $\widetilde{X}$.

1.3. Period domain. Let V be a rank 22 integer lattice with intersection form $\langle , \rangle$ obtained from the second homology of a $K3$ surface,

$$V \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

A 1-dimensional subspace $\mathbb{C}\cdot\omega \in V \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying

$$\langle \omega, \omega \rangle = 0 \ \text{ and } \ \langle \omega, \overline{\omega} \rangle > 0 \tag{5}$$

determines a Hodge structure of type $(1,20,1)$ on V,

$$V \otimes_{\mathbb{Z}} \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2} = \mathbb{C}\cdot\omega \ \oplus \ (\mathbb{C}\cdot\omega \oplus \mathbb{C}\cdot\overline{\omega})^{\perp} \ \oplus \ \mathbb{C}\cdot\overline{\omega}.$$

Conversely, a Hodge structure of type $(1, 20, 1)$ determines a 1-dimensional subspace $\mathbb{C} \cdot \omega$ satisfying (5).

The moduli space M^V of Hodge structures of type $(1, 20, 1)$ on V is therefore an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M^V \subset Q \subset \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{C}).$$

The moduli space M^V is the *period domain.*

For nonzero $\beta \in V$, let $D^V_\beta \subset M^V$ denote the locus of Hodge structures for which $\beta \in V^{1,1}$. Certainly,

$$D^V_\beta = M^V \cap \beta^\perp \subset \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{C})$$

where $\beta^\perp$ is the linear space orthogonal to β. Hence, D^V_β is simply a 19-dimensional hyperplane section of M^V.

1.4. Local systems. Let (X, L, π) be a quasi-polarized family of $K3$ surfaces over a nonsingular curve C. Let

$$\mathcal{V} = R^2\pi_*(\mathbb{Z}) \to C$$

denote the rank 22 local system determined by the middle cohomology of the fibration

$$\pi : X \to C.$$

The local system $\mathcal{V}$ is equipped with the fiberwise intersection form $\langle , \rangle$.

Let $\mathcal{M}^{\mathcal{V}}$ be the π-relative moduli space of Hodge structures

$$\mu : \mathcal{M}^{\mathcal{V}} \to C$$

with fiber

$$\mu^{-1}(\xi) = M^{\mathcal{V}_\xi}.$$

The moduli space $\mathcal{M}^{\mathcal{V}}$ is a complex manifold, and μ is a locally trivial fibration in the analytic topology.

Duality and homological push-forward yield a canonical map

$$\epsilon : \mathcal{V} \to H_2(X, \mathbb{Z})$$

where the right side can be viewed as a trivial local system. Let $H_2(X, \mathbb{Z})^\pi$ denote the kernel of the projection map

$$\pi_* : H_2(X, \mathbb{Z}) \to H_2(C, \mathbb{Z}).$$

For $h \in \mathbb{Z}$ and $\gamma \in H_2(X, \mathbb{Z})^\pi$, we will define a Noether-Lefschetz number $NL^\pi_{h,\gamma}$ for the $K3$ fibration π.

Informally, $NL^\pi_{h,\gamma}$ counts the number of points $\xi \in C$ for which there exists an integral class $\beta \in V_\xi$ of type $(1, 1)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma.$$

The formal definition is given in Section 1.5.

1.5. Classical intersection. Define the relative divisor

$$\mathcal{D}^{\mathcal{V}}_{h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$$

by the set of Hodge structures which contain a class $\beta \in \mathcal{V}_\xi$ of type $(1,1)$ satisfying

$$\langle \beta, \beta \rangle = 2h-2 \ \text{ and } \ \epsilon(\beta) = \gamma.$$

When $\mathcal{M}^{\mathcal{V}}$ is trivialized[4] over a Euclidean open set $U \subset C$,

$$\mathcal{M}^{\mathcal{V}_U} = M^V \times U,$$

the subset $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$ restricts to

$$\mathcal{D}^{\mathcal{V}_U}_{h,\gamma} = \cup_\beta \ D^V_\beta \times U$$

where the union is over all $\beta \in V$ satisfying

$$\langle \beta, \beta \rangle = 2h-2 \ \text{ and } \ \epsilon(\beta) = \gamma.$$

Hence, $\mathcal{D}^{\mathcal{V}}_{h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$ is a countable union of divisors.

The Noether-Lefschetz number is defined by a tautological intersection product. The family π determines a canonical section

$$\sigma : C \to \mathcal{M}^{\mathcal{V}}.$$

where

$$\sigma(\xi) = [H^{2,0}(X_\xi, \mathbb{C})] \in \mathcal{M}^{\mathcal{V}_\xi}$$

is the Hodge structure determined by the $K3$ surface X_ξ. Let

$$NL^\pi_{h,\gamma} = \int_C \sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}]. \tag{6}$$

The divisor $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$ may have infinitely many components. However, by the finiteness result of Proposition 1, $NL^\pi_{h,\gamma}$ is well-defined.

While $NL^\pi_{h,\gamma}$ is a classical intersection number, an excess calculation is required in case $\sigma(C) \subset \mathcal{D}^{\mathcal{V}}_{h,\gamma}$. The informal counting interpretation is not always well-defined.

Proposition 1. *$NL^\pi_{h,\gamma}$ is finite.*

Proof. Let L be the quasi-polarization on X. If there exists a point $\xi \in C$ for which L_ξ is ample, then L is π-relatively ample over an open set of C. If L_ξ is never ample, then the morphism

$$X \to \widetilde{X} \subset \mathbb{P}(R^0\pi_*(L^n))$$

for sufficiently large n contracts divisors on X which intersect the generic fiber X_ξ in (-2)-curves. After modification[5] of L by these contracted divisors, a new quasi-polarization L' of X may be obtained which is π-relatively ample over a nonempty open set of C.

We assume now (after possible modification) the quasi-polarization L is π-relatively ample over a nonempty open set $U \subset C$. Let

$$d = \int_\gamma L$$

[4]We take trivializations obtained from trivializing $R^2\pi_*(\mathbb{Z})$ compatibly with ϵ .

[5]A base change of $\pi : X \to C$ is not required since the modification can be averaged over the symmetries of the (-2)-curve configuration.

be the degree of γ. Let

$$l = \int_{X_\xi} L_\xi^2 > 0$$

be the degree of the $K3$ fibers of π.

Let $\beta \in \mathcal{V}_\xi$ of type $(1,1)$ satisfy

$$\langle \beta, \beta \rangle = 2h - 2 \ \text{ and } \ \epsilon(\beta) = \gamma.$$

We will prove

$$\sigma(C) \subset \mathcal{M}^{\mathcal{V}}$$

intersects only finitely many components of $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$.

Let k be an integer satisfying

$$d + lk > 0 \ \text{ and } \ lk^2 + 2dk + 2h - 2 > -4.$$

The first step is to show

$$\tilde{\beta} = \beta + kc_1(L_\xi)$$

is an effective curve class on X_ξ by Riemann-Roch.

Let $L_{\tilde{\beta}}$ denote the unique line bundle on X_ξ with

$$c_1(L_{\tilde{\beta}}) = \tilde{\beta}.$$

By Serre duality,

$$H^2(X_\xi, L_{\tilde{\beta}}) = H^0(X_\xi, L^*_{\tilde{\beta}})^*$$

Since

$$\langle c_1(L^*_{\tilde{\beta}}), L_\xi \rangle \leq -d - lk < 0,$$

$h^0(X_\xi, L^*_{\tilde{\beta}})$ vanishes. Then, by Riemann-Roch,

$$\begin{aligned} h^0(X_\xi, L_{\tilde{\beta}}) &\geq \chi(X_\xi, L_{\tilde{\beta}}) - h^2(X_\xi, L_{\tilde{\beta}}) \\ &= \chi(X_\xi, L_{\tilde{\beta}}) \\ &= \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta} \rangle + 2 \\ &> 0. \end{aligned}$$

Hence, $\tilde{\beta}$ is an effective curve class on X_ξ.

Consider first the open set $U \subset C$ over which L is π-relatively ample. Let

$$\mathcal{H} \to U$$

be the π-relative Hilbert scheme parameterizing of curves in $X_{\xi \in U}$ of degree

$$\langle \tilde{\beta}, c_1(L_\xi) \rangle = d + lk$$

and Euler characteristic

$$\chi(X_\xi, \mathcal{O}_{X_\xi}) - \chi(X_\xi, L^*_{\tilde{\beta}}) = -\frac{1}{2}\langle \tilde{\beta}, \tilde{\beta} \rangle = -\frac{1}{2}(lk^2 + 2dk + 2h - 2).$$

The scheme $\mathcal{H}$ is projective over U and of finite type.

An irreducible component $\mathcal{H}_{irr} \subset \mathcal{H}$ either dominates U or maps to a point $\xi \in U$. In the former case, the classes of curves represented by $\mathcal{H}_{irr}$ yield a *finite* monodromy invariant subset of $\mathcal{V}$. In the latter case, the curves represented by $\mathcal{H}_{irr}$ yield a single element of $\mathcal{V}_\xi$.

After shifting the finiteness statements back by $kc_1(L_\xi)$, we obtain the finiteness of the intersection geometry

$$\sigma(C) \cap \mathcal{D}^{\mathcal{V}}_{h,\gamma} \tag{7}$$

over $U \subset C$. Indeed, the dominant components $\mathcal{H}_{irr}$ correspond to finitely many excess intersections and the non-dominant components correspond to finitely many true intersections.

Finally consider the complement $U^c \subset C$. The complement is a finite set. For each $\xi^c \in U^c$, let $L^c_{\xi^c}$ be an ample line bundle. The above arugment using the ample bundles $L^c_{\xi^c}$ for the fibers X_{ξ^c} shows there are finitely many intersections in (7) over $U^c \subset C$ as well.

We conclude the intersection geometry is finite over all of C and the product

$$NL^{\pi}_{h,\gamma} = \int_C \sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}]$$

is well-defined. □

Let γ_L denote the push-forward of the ample class on the fibers,

$$\gamma_L = c_1(L) \cap [X_\xi] \in H_2(X,\mathbb{Z})^{\pi}.$$

By an elementary comparison,

$$\sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}] = \sigma^*[\mathcal{D}^{\mathcal{V}}_{h+d+\frac{l}{2},\gamma+\gamma_L}].$$

We obtain the following result.

Proposition 2. $NL^{\pi}_{h,\gamma} = NL^{\pi}_{h+d+\frac{l}{2},\gamma+\gamma_L}$.

The proof of Proposition 1 show the vanishing of the Noether-Lefschetz number for high h.

Proposition 3. *For fixed γ, the numbers $NL^{\pi}_{h,\gamma}$ vanish for sufficiently high h.*

The Noether-Lefschetz numbers $NL_{h,\gamma}(\pi)$ have non-trivial dependence on γ despite the linear equivalence

$$D^V_\beta \cong D^V_{\beta'}$$

on M^V. The Noether-Lefschetz numbers involve also the twisting of the local system $\mathcal{V}$ over C.

1.6. Refinements. The Noether-Lefschetz numbers $NL^{\pi}_{h,d}$ defined in Section 0.3 are obtained from the relation

$$NL^{\pi}_{h,d} = \sum_{\int_\gamma L = d} NL^{\pi}_{h,\gamma}. \tag{8}$$

The finiteness of the sum on the right is a consequence of the negative definiteness of the intersection matrix of divisors in X_ξ contracted by L_ξ. The invariants $NL^{\pi}_{h,\gamma}$ may be viewed as a refinement of $NL^{\pi}_{h,d}$ with the nonvanishing discriminant hypothesis lifted.

Further refined Noether-Lefschetz numbers may be defined with respect to any additional monodromy invariant data. For example, the divisibility m of an element $\beta \in \mathcal{V}_\xi$ is a monodromy invariant. Let

$$\mathcal{D}^{\mathcal{V}}_{m,h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$$

be the divisor of Hodge structures which contain a class $\beta \in \mathcal{V}_\xi$ of type $(1,1)$ of divisibility m satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma.$$

We define

$$NL^\pi_{m,h,\gamma} = \int_C \sigma^*[\mathcal{D}_{m,h,\gamma}].$$

The relation

$$NL^\pi_{h,\gamma} = \sum_{m \geq 1} NL^\pi_{m,h,\gamma} \tag{9}$$

certainly holds.

1.7. Intersection theory of $\mathcal{M}_l$. Let $v \in V$ be a vector of norm l, and let

$$\mathcal{M}^V_v = v^\perp \cap \mathcal{M}^V.$$

Let Γ denote the group of orthogonal transformations of the lattice V, and let

$$\Gamma_v \subset \Gamma$$

be the subgroup fixing v. The moduli space of quasi-polarized $K3$ surfaces of degree l is the quotient

$$\mathcal{M}_l = \mathcal{M}^V_v / \Gamma_v.$$

The moduli space is a nonsingular orbifold. We refer the reader to [**14**] for a more detailed discussion.

In case $\triangle_l(h,d) \neq 0$, the above construction of $\mathcal{M}_l$ shows the definitions of the Noether-Lefschetz number by (3) and (8) agree.

2. Gromov-Witten theory

2.1. BPS states for 3-folds. Let (X, L, π) be a quasi-polarized family of $K3$ surfaces. While X may not be a projective variety, X carries a $(1,1)$-form ω_K which is Kähler on the $K3$ fibers of π. The existence of a fiberwise Kähler form is sufficient to define Gromov-Witten theory for vertical classes

$$0 \neq \gamma \in H_2(X, \mathbb{Z})^\pi.$$

The fiberwise Kähler form ω_K is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.[6]

Let $\overline{M}_g(X,\gamma)$ be the moduli space of stable maps from connected genus g curves to X. Gromov-Witten theory is defined by integration against the virtual class,

$$N^X_{g,\gamma} = \int_{[\overline{M}_g(X,\gamma)]^{vir}} 1. \tag{10}$$

The expected dimension of the moduli space is 0.

The Gromov-Witten potential $F^X(\lambda, v)$ for nonzero vertical classes is the series

$$F^X = \sum_{g \geq 0} \sum_{0 \neq \gamma \in H_2(X,\mathbb{Z})^\pi} N^X_{g,\gamma} \, \lambda^{2g-2} v^\gamma$$

[6]See [**30, 36**] for treatments of Gromov-Witten invariants for fiberwise Kähler geometry.

where λ and v are the genus and curve class variables. The BPS counts $n^X_{g,\gamma}$ of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^X = \sum_{g\geq 0} \sum_{0\neq\gamma\in H_2(X,\mathbb{Z})^\pi} n^X_{g,\gamma}\ \lambda^{2g-2} \sum_{d>0} \frac{1}{d}\left(\frac{\sin(d\lambda/2)}{\lambda/2}\right)^{2g-2} v^{d\gamma}.$$

Conjecturally, the invariants $n^X_{g,\gamma}$ are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on X.

2.2. Reduced theory. Let C be a connected, nodal, genus g curve. Let S be a $K3$ surface, and let $\beta \in \mathrm{Pic}(S)$ be a nonzero class. The moduli space $M_C(S,\beta)$ parameterizes maps from C to S of class β. Let

$$\nu : C \times M_C(S,\beta) \to M_C(S,\beta)$$

denote the projection, and let

$$f : C \times M_C(S,\beta) \to S$$

denote the universal map. The canonical morphism

$$R^\bullet\nu_*(f^*S)^\vee \to L^\bullet_{M_C} \tag{11}$$

determines a perfect obstruction theory on $M_C(S,\beta)$, see [**2, 3, 34**]. Here, $L^\bullet_{M_C}$ denotes the cotangent complex of $M_C(S,\beta)$.

Let Ω_S denote the cotangent bundle of S. Let Ω_ν and ω_ν denote respectively the sheaf of relative differentials of ν and the relative dualizing sheaf of ν. There are canonical maps

$$f^*(\Omega_S) \to \Omega_\nu \to \omega_\nu \tag{12}$$

The sections of the canonical bundle $H^0(S,K_S)$ determine a 1-dimensional space of holomorphic symplectic forms. Hence, there is a canonical isomorphism

$$T_S \otimes H^0(S,K_S) \stackrel{\sim}{=} \Omega_S$$

where T_S is the tangent bundle. We obtain a map

$$f^*(T_S) \to \omega_\nu \otimes (H^0(S,K_S))^\vee$$

and a map

$$R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S) \to R^\bullet\nu_*(f^*T_S)^\vee. \tag{13}$$

From (13), we obtain the cut-off map

$$\iota : \tau_{\leq -1}R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S) \to R^\bullet\nu_*(f^*T_S)^\vee.$$

The complex $\tau_{\leq -1}R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S)$ is represented by a trivial bundle of rank 1 tensored with $H^0(S,K_S)$ in degree -1. Consider the mapping cone $C(\iota)$ of ι. Certainly $R^\bullet\pi_*(f^*T_S)^\vee$ is represented by a two term complex. An elementary argument using nonvanishing $\beta \neq 0$ shows the complex $C(\iota)$ is also two term.

By Ran's results[7] on deformation theory and the semiregularity map, there is a canonical map

$$C(\iota) \to L^\bullet_{M_C} \tag{14}$$

[7]The required deformation theory can also be found in a recent paper by M. Manetti [**37**]. A different approach to the construction of the reduced virtual class is available in [**48**].

induced by (11), see [**46**]. Ran proves the obstructions to deforming maps from C to a holomorphic symplectic manifold lie in the kernel of the semiregularity map. After dualizing, Ran's result precisely shows (11) factors through the cone $C(\iota)$.

The map (14) defines a *new* perfect obstruction theory on $M_C(S,\beta)$. The conditions of cohomology isomorphism in degree 0 and the cohomology surjectivity in degree -1 are both induced from the perfect obstruction theory (11). We view (11) as the *standard* obstruction theory and (14) as the *reduced* obstruction theory.

Following [**2, 3**], the morphism (14) is an obstruction theory of maps to S relative to the Artin stack $\mathfrak{M}_g$ of genus g curves. A reduced absolute obstruction theory

$$E^\bullet \to L^\bullet_{\overline{M}_g(S,\beta)} \tag{15}$$

is obtained via a distinguished triangle in the usual way, see [**2, 3, 34**]. The obstruction theory (15) yields a reduced virtual class

$$[\overline{M}_g(S,\beta)]^{red} \in A_g(\overline{M}_g(S,\beta),\mathbb{Q})$$

of dimension g.

2.3. BPS for $K3$ surfaces. Let (S,ω_K) be a $K3$ surface with a Kähler form ω_K. Let $\beta \in \mathrm{Pic}(S)$ be a nonzero class of positive degree

$$\int_\beta \omega_K > 0.$$

We are interested in the following reduced Gromov-Witten integrals,

$$R_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g. \tag{16}$$

Here, the integrand λ_g is the top Chern class of the Hodge bundle

$$\mathbb{E}_g \to \overline{M}_g(S,\beta)$$

with fiber $H^0(C,\omega_C)$ over moduli point

$$[f : C \to S] \in \overline{M}_g(S,\beta).$$

See [**15, 21**] for a discussion of Hodge classes in Gromov-Witten theory.

The definition of the BPS counts associated to the Hodge integrals (16) is straightforward. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive class of positive degree with respect to ω_K. The Gromov-Witten potential $F_\alpha(\lambda,v)$ for classes proportional to α is

$$F_\alpha = \sum_{g\geq 0}\sum_{m>0} R_{g,m\alpha}\ \lambda^{2g-2} v^{m\alpha}.$$

The BPS counts $r_{g,m\alpha}$ are uniquely defined by the following equation:

$$F_\alpha = \quad \sum_{g\geq 0}\sum_{m>0} r_{g,m\alpha}\ \lambda^{2g-2}\sum_{d>0}\frac{1}{d}\left(\frac{\sin(d\lambda/2)}{\lambda/2}\right)^{2g-2} v^{dm\alpha}.$$

We have defined BPS counts for both primitive and divisible classes.

The string theoretic calculations of Katz, Klemm and Vafa [**24**] via heterotic duality yield two conjectures.

Conjecture 1. *The BPS count $r_{g,\beta}$ depends upon β only through the square $\int_S \beta^2$.*

Assuming the validity of Conjecture 1, let $r_{g,h}$ denote the BPS count associated to a class β satisfying

$$\int_S \beta^2 = 2h-2.$$

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory. By deformation arguments, the invariants $R_{g,\beta}$ depend upon both the divisibility m of β and $\int_S \beta^2$. Hence, BPS counts $r_{g,m,h}$ depending upon both the divisibility and the norm are well-defined unconditionally.

Conjecture 2. *The BPS counts $r_{g,h}$ are uniquely determined by the following equation:*

$$\sum_{g\geq 0}\sum_{h\geq 0}(-1)^g r_{g,h}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{2g}q^h = \prod_{n\geq 1}\frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

As a consequence of Conjecture 2, $r_{g,h}$ vanishes if $g>h$ and

$$r_{g,g} = (-1)^g(g+1).$$

The first values are tabulated below:

$r_{g,h}$	$h=0$	1	2	3	4
$g=0$	1	24	324	3200	25650
1		-2	-54	-800	-8550
2			3	88	1401
3				-4	-126
4					5

The right side Conjecture 2 is related to the generating series of Hodge numbers of the Hilbert schemes of points $\mathrm{Hilb}(S,n)$. The genus 0 specialization of Conjecture 2 recovers the Yau-Zaslow formula

$$\sum_{h\geq 0} r_{0,h}q^h = \prod_{n\geq 1}\frac{1}{(1-q^n)^{24}}$$

related to the Euler characteristics of $\mathrm{Hilb}(S,n)$.

The Conjectures are proven in very few cases. A mathematical approach to the genus 0 Yau-Zaslow formula following [**52**] can be found in [**4, 12, 16**]. The Yau-Zaslow formula is proven for primitive classes β by Bryan and Leung [**10**]. If β has divisibility 2, the Yau-Zaslow formula is proven by Lee and Leung in [**31**]. Using Theorem 1, a complete proof of the Yau-Zaslow formula for all divisibilities is given in [**27**]. Since

$$R_{1,\beta} = \int_{[\overline{M}_1(S,\beta)]^{red}} -\lambda_1 = -\frac{\langle\beta,\beta\rangle}{24}R_{0,\beta},$$

we obtain

$$r_{1,h} = -\frac{h}{12}\ r_{0,h}$$

and Conjectures 1 and 2 for genus 1 from the genus 0 results.

Conjecture 2 for primitive classes β is connected to Euler characteristics of the moduli spaces of stable pairs on $K3$ by the correspondence of [**44, 45**]. A proof of Conjecture 2 for primitive classes is given in [**40**].

3. Theorem 1

3.1. Result. Consider a quasi-polarized family of $K3$ surfaces of degree l as in Section 1.2,

$$\pi : X \to C \ .$$

We restate Theorem 1 in terms of $\gamma \in H_2(X, \mathbb{Z})^\pi$ following the notation of Section 1.4.

THEOREM 1. *For $\gamma \neq 0$,*

$$n^X_{g,\gamma} = \sum_h \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^\pi_{m,h,\gamma}.$$

3.2. Proof. Since the formulas relating the BPS counts to Gromov-Witten invariants are the same for X and the $K3$ surface, Theorem 1 is equivalent to the analogous Gromov-Witten statement:

$$N^X_{g,\gamma} = \sum_h \sum_{m=1}^{\infty} R_{g,m,h} \cdot NL^\pi_{m,h,\gamma} \tag{17}$$

for $\gamma \neq 0$.

Following the notation of Section 1.5, let σ denote the section

$$\sigma : C \to \mathcal{M}^{\mathcal{V}}$$

determined by the Hodge structure of the $K3$ fibers

$$\sigma(\xi) = [H^0(X, K_{X_\xi})] \in \mathcal{M}^{\mathcal{V}_\xi}.$$

For each $\xi \in C$, let

$$\mathcal{V}_\xi(m, h, \gamma) \subset \mathcal{V}_\xi$$

be the set of classes with divisibility m, square $2h-2$, and push-forward γ. Let

$$B_\xi(m, h, \gamma) = \{ \ \beta \in \mathcal{V}_\xi(m, h, \gamma) \mid \sigma(\xi) \in \beta^\perp \ \}.$$

By Proposition 1, the set $B_\xi(m, h, \gamma)$ is finite.

Equation (17) is proven by showing the contributions of the classes $B_\xi(m, h, \gamma)$ to both sides are the same. The set

$$B(m, h, \gamma) = \bigcup B_\xi(m, h, \gamma) \subset \mathcal{V}$$

can be divided into two disjoint subsets

$$B(m, h, \gamma) = B_{\text{iso}}(m, h, \gamma) \cup B_\infty(m, h, \gamma).$$

The elements of $B_{\text{iso}}(m, h, \gamma)$ are isolated while the elements of $B_\infty(m, h, \gamma)$ form a finite local system over C,

$$\epsilon : B_\infty(m, h, \gamma) \to C. \tag{18}$$

We address the contributions of the isolated issues and the local system separately.

Consider first the local system (18). The contribution of ϵ to the Gromov-Witten invariant $N^X_{g,\gamma}$ is the integral

$$N^X_{g,\epsilon} = \int_{[\overline{M}_g(X,\epsilon)]^{vir}} 1$$

where $\overline{M}_g(X,\epsilon) \subset \overline{M}_g(X,\gamma)$ is the connected component[8] of the moduli space of stable maps which represent curve classes in ϵ. Alternatively,

$$N^X_{g,\epsilon} = \int_{[\overline{M}_g(\pi,\epsilon)]^{vir}} c_g(\mathbb{E}^*_g \otimes T_C) \tag{19}$$

where $\overline{M}_g(\pi,\epsilon) \subset \overline{M}_g(\pi,\gamma)$ is a connected component of the relative moduli space of maps. By standard arguments [**15**], the difference in the absolute and relative obstruction theories is $\mathbb{E}^*_g \otimes T_C$ and hence yields the Hodge integrand in (19).

The family π determines a canonical line bundle

$$K \to C$$

with fiber $H^0(X_\xi, K_{X_\xi})$ over $\xi \in C$. By the construction of the reduced class in Section 2.2,

$$[\overline{M}_g(\pi,\epsilon)]^{vir} = c_1(K^*) \cap [\overline{M}_g(\pi,\epsilon)]^{red}$$

where, on the right side, the reduced virtual class for the relative moduli space of maps appears. Expanding (19) yields

$$\begin{aligned} N^X_{g,\epsilon} &= \int_{[\overline{M}_g(\pi,\epsilon)]^{red}} c_g(\mathbb{E}^*_g \otimes T_C) \cdot c_1(K^*) \\ &= \int_{[\overline{M}_g(K3,m\alpha)]^{red}} (-1)^g \lambda_g \cdot \int_{B_\infty(m,h,\gamma)} c_1(K^*) \\ &= R_{g,m,h} \cdot \int_{B_\infty(m,h,\gamma)} c_1(K^*). \end{aligned}$$

In the second equality, α is primitive and satisfies

$$\langle m\alpha, m\alpha \rangle = 2h - 2.$$

The contribution of the local system ϵ to the Noether-Lefschetz number $NL^\pi_{m,h,\gamma}$ is much easier to calculate. The local system represents an excess intersection contribution

$$\int_{B_\infty(m,h,\gamma)} c_1(\mathrm{Norm})$$

where Norm is the line bundle with fiber

$$\mathrm{Hom}(H^0(X_\xi, K_{X_\xi}), \mathbb{C} \cdot \beta)$$

at $\beta \in B_\infty(m,h,\gamma)$ lying over $\xi \in C$. Over $B_\infty(m,h,\gamma)$, the fibration $\mathbb{C} \cdot \beta$ is a trivial line bundle. Hence, the excess contribution of $B_\infty(m,h,\gamma)$ to $NL^\pi_{m,h,\gamma}$ is

$$\int_{B_\infty(m,h,\gamma)} c_1(K^*).$$

We conclude the contributions of $B_\infty(m,h,\gamma)$ to the left and right sides of equation (17) exactly match.

We consider now the contributions of the isolated classes $B_{iso}(m,h,\gamma)$ to the two sides of (17). Let

$$\beta \in B_{\text{iso}}(m,h,\gamma)$$

[8]By connected component, we mean both open and closed. Formally, the condition is usually stated as a union of connected components.

be an isolated class lying over $\xi \in C$. We trivialize $\mathcal{M}^{\mathcal{V}}$ over a Euclidean open set $U \subset C$ as in Section 1.5. The local intersection of the section σ with the divisor

$$D_\beta^{V_\xi} \times U \subset M^{V_\xi} \times U$$

has an isolated point corresponding to (β, ξ). The local intersection multiplicity may not be 1. However, by deformation equivalence of the Gromov-Witten contributions on the left side of (17) and the intersection products on the right side of (17), we may assume *the local intersection multiplicity is 1* after local holomorphic perturbation of the section σ. Then, the contribution of the isolated class β to $NL^\pi_{m,h,\gamma}$ is certainly 1.

The final step is to show the contribution of the isolated class β with intersection multiplicity 1 to $N^X_{g,\gamma}$ is simply $R_{g,m,h}$. The result is obtained by a comparison of obstruction theories.

By the multiplicity 1 hypothesis, a connected component of the moduli space of stable maps to X coincides with the moduli stable of stable maps to fiber X_ξ,

$$\overline{M}_g(X_\xi, \beta) \subset \overline{M}_g(X, \gamma). \tag{20}$$

At the level of points, the assertion is obvious. The multiplicity 1 conditions prohibits any infinitesimal deformations of maps away from the fiber X_ξ and implies the scheme theoretic assertion.

From the fibration π, we obtain an exact sequence

$$0 \to T_{X_\xi} \to T_X|_{X_\xi} \to T_{C,\xi} \to 0, \tag{21}$$

and an induced map

$$\widetilde{\iota} : R^\bullet \nu_*(f^* T_{X_\xi})^\vee \to T^*_{C,\xi}$$

where the second complex is a trivial bundle in degree -1. Following the notation of Section 2.2, we have a canonical map

$$\iota : H^0(X_\xi, K_{X_\xi}) \to R^\bullet \nu_*(f^* T_{X_\xi})^\vee$$

where the first complex is a trivial bundle with fiber $H^0(X_\xi, K_{X_\xi})$ in degree -1. By Lemma 1 below, the composition

$$\widetilde{\iota} \circ \iota : H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi}$$

is an isomorphism. Hence, by sequence (21), the obstruction theories $R^\bullet \nu_*(f^* T_X)^\vee$ and $C(\iota)$ differ by only by the Hodge bundle $\mathbb{E}_g \otimes T^*_{C,\xi}$. We conclude

$$[\overline{M}_g(X_\xi, \beta)]^{virx} = (-1)^g \lambda_g \cap [\overline{M}_g(X_\xi, \beta)]^{red}$$

where the virtual class on the left is obtained from the obstruction theory of maps to X via (20). The contribution of the isolated class β to $N^X_{g,\gamma}$ is thus $R_{g,h,m}$.

Since the contributions of $B_{\text{iso}}(m, h, \gamma)$ to the left and right sides of equation (17) also match, the proof of Theorem 1 is complete. □

Lemma 1. *The composition*

$$\widetilde{\iota} \circ \iota : H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi}$$

is an isomorphism.

Proof. Consider the differential of the period map at ξ,

$$T_{C,\xi} \to H^1(T_{X_\xi}) \to \mathrm{Hom}(H^0(K_{X_\xi}), H^1(\Omega_{X_\xi})).$$

The multiplicity 1 condition implies that the image of this map is not contained in the tangent space to the hyperplane $\beta^\perp = 0$. More explicitly, if we apply the cup-product pairing of $H^1(\Omega_{X_\xi})$ with the class $\beta \in H^2(X_\xi, \mathbb{Z})$, the composition

$$T_{C,\xi} \to H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) \xrightarrow{\beta\cup} H^0(K_{X_\xi})^* \otimes \mathbb{C}$$

is nonzero. This sequence can be included in the diagram

$$\begin{tikzcd}
T_{C_\xi} \arrow[r] \arrow[d, equal] & H^1(T_{X_\xi}) \arrow[r] \arrow[d] & H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) \arrow[r, "\beta\cup"] \arrow[d] & H^0(K_{X_\xi})^* \arrow[d, equal] \\
T_{C_\xi} \arrow[r] & R^\bullet\nu_*(f^*T_{X_\xi}) \arrow[r] & H^0(K_{X_\xi})^* \otimes R^\bullet\nu_*(f^*\Omega_{X_\xi}) \arrow[r] & H^0(K_{X_\xi})^*
\end{tikzcd}$$

where the vertical maps are given by base-change morphisms and the bottom row is the map $(\widetilde{\iota}\circ\iota)^*$. Standard comparison results imply that this diagram commutes. Since the top row is nonvanishing, so is the bottom row. $\square$

3.3. Conjectures 1 and 2 revisited. The proof of Conjectures 1 and 2 in the following case allows us to bound from below the h summation in Theorem 1.

Lemma 2. *If $\int_{K3} \beta^2 < 0$, then $r_{g,\beta} = 1$ if*

$$g = 0 \quad \textit{and} \quad \int_{K3} \beta^2 = -2$$

and $r_{g,\beta} = 0$ otherwise.

Proof. Let S be a $K3$ surface, and let $\beta \in \mathrm{Pic}(S)$ be primitive with

$$\int_S \beta^2 = -2.$$

We may assume β is represented by an isolated -2 curve $P \subset S$. Let

$$\pi : X \to \triangle_0$$

be a 1-parameter deformation of S over the disk $\triangle_0$ for which β fails (even infinitesimally) to remain algebraic. By the proof of Theorem 1, the reduced invariants $r_{g,m,\beta}$ are obtained[9] from the contribution of P to the BPS state counts of X. Since P is a rigid $(-1,-1)$ curve, P contributes a single BPS state [**15**]. We conclude

$$r_{g,m,\beta} = 1$$

if $(g,m) = (0,1)$ and $r_{g,m,\beta} = 0$ otherwise.

If $\beta \in \mathrm{Pic}(S)$ is primitive with square $2h-2$ strictly less than -2, then all reduced invariants $r_{g,m,\beta}$ vanish. The proof is obtained by considering elliptically fibered $K3$ surfaces $S \to \mathbb{P}^1$. Let

$$[s], [f] \in \mathrm{Pic}(S)$$

be the classes of a section and a fiber respectively. Then,

$$[s] + h[f],\ -[s] - h[f] \in \mathrm{Pic}(S)$$

[9]The local NL intersection number here is 1.

are both primitive with square $2h-2$. Since the moduli spaces

$$\overline{M}_g\left(S, m([s]+h[f])\right),\ \overline{M}_g\left(S, m(-[s]-h[f])\right)$$

are easily seen to be empty, all reduced invariants $r_{g,m,\beta}$ vanish. □

By Lemma 2, the integrals $r_{g,m,h<0}$ all vanish. Hence, Theorem 1 may be written as

$$n^X_{g,\gamma} = \sum_{h\geq 0}\sum_{m=1}^{\infty} r_{g,m,h}\cdot NL^{\pi}_{m,h,\gamma}.$$

If Conjecture 1 and the vanishing $r_{g,h}$ for $g>h$ of Conjecture 2 hold, then

$$r_{g,h} = r_{g,m,h}$$

and Theorem 1 implies the following result. by relation (9).

THEOREM 1*. *For* $\gamma \neq 0$,

$$n^X_{g,\gamma} = \sum_{h\geq g} r_{g,h}\cdot NL^{\pi}_{h,\gamma}\ .$$

The asterisk here indicates the dependence of Theorem 1* upon Conjectures 1 and 2.

3.4. Invertibility. Theorem 1* and Conjecture 2 imply the BPS states $n^X_{g,\gamma}$ of the total space contain exactly the same information as the Noether-Lefschetz numbers $NL^{\pi}_{h,\gamma}$.

PROPOSITION 4*. *For classes* $\gamma\in H_2(X,\mathbb{Z})^{\pi}$ *of positive degree, the invariants* $\{n^X_{g,\gamma}\}_{g\geq 0}$ *determine the Noether-Lefschetz numbers* $\{NL^{\pi}_{h,\gamma}\}_{h\geq 0}$ *in terms of the invariants* $\{r_{g,h}\}_{g,h\geq 0}$.

PROOF. Fix $\gamma\in H_2(X,\mathbb{Z})^{\pi}$. By Proposition 2, the numbers $NL^{\pi}_{h,\gamma}$ vanish for $h>h_{top}$. So we need only determine

$$NL^{\pi}_{0,\gamma},\dots NL^{\pi}_{h_{top},\gamma}.$$

The equations

$$n^X_{g,\gamma} = \sum_{h=g}^{h_{top}} r_{g,h}\cdot NL^{\pi}_{h,\gamma}$$

for $g=0,\dots,h_{top}$ of Theorem 1* are triangular and invertible by Conjecture 2. □

4. Modular forms

4.1. Overview. We explain here work of Borcherds [**7**] relating Noether-Lefschetz numbers to Fourier coefficients of modular forms.[10] His results apply in great generality to arithmetic quotients of symmetric spaces associated to the orthogonal group $O(2,n)$ for any n. While we are mainly interested in the case of $O(2,19)$, we will first explain the statement in full generality. Other values of n play a role, for example, in studying 1-parameter families of $K3$ surfaces with generic Picard rank at least 2.

[10]Borcherds' original result is modular only up to a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. The strengthening of [**7**] by the more recent rationality result of [**41**] removes the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ issue.

4.2. Vector-valued modular forms of half-integral weight. We first summarize standard facts and notation regarding modular forms of half-integral weight. In order to make sense of the modular transformation law with half-integer exponents, a double cover of the standard modular group $SL_2(\mathbb{Z})$ is required.

The metaplectic group $Mp_2(\mathbb{R})$ is the unique connected double cover of $SL_2(\mathbb{R})$. The elements of $Mp_2(\mathbb{R})$ can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau) = \pm\sqrt{c\tau + d} \right)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi(\tau)$ is a choice of square root of the function $c\tau+d$ on the upper-half plane $\mathcal{H}$. The group structure is defined by the product

$$(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1A_2, \phi_1(A_2\tau)\phi_2(\tau)).$$

Here, we write $A\tau$ for the usual action of $SL_2(\mathbb{R})$ on $\tau \in \mathcal{H}$.

The group $Mp_2(\mathbb{Z})$ is the preimage of $SL_2(\mathbb{Z})$ under the projection map

$$\pi : Mp_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

It is generated by the two elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

where $\sqrt{\tau}$ denotes the choice of square root with positive real part.

Suppose we are given a representation ρ of $Mp_2(\mathbb{Z})$ on a finite-dimensional complex vector space V with the property that ρ factors through a finite quotient. Given $k \in \frac{1}{2}\mathbb{Z}$, we define a modular form of weight k and type ρ to be a holomorphic function

$$f : \mathcal{H} \to V$$

such that, for all $g = (A, \phi(\tau)) \in Mp_2(\mathbb{Z})$, we have

$$f(A\tau) = \phi(\tau)^{2k} \cdot \rho(g)(f(\tau)).$$

For $k \in \mathbb{Z}$ and ρ trivial, this reduces to the usual transformation rule.

If we fix an eigenbasis $\{v_\gamma\}$ for V with respect to T, we can take the Fourier expansion of each component of f at the cusp at infinity. That is, we write

$$f(\tau) = \sum_\gamma \sum_{k \in \mathbb{Z}} c_{k,\gamma} q^{k/R} v_\gamma \in V$$

where

$$q = e^{2\pi i \tau}$$

and R is the smallest positive integer for which $T^R \in \mathrm{Ker}(\rho)$. The function f is holomorphic at infinity if $c_{k,r} = 0$ for $k < 0$. The space $\mathrm{Mod}(Mp_2(\mathbb{Z}), k, \rho)$ of holomorphic modular forms of weight k and type ρ is finite-dimensional.

Given an integral lattice M with an even bilinear form $\langle , \rangle$ with signature $(2, n)$, we associate to M the following unitary representation of $Mp_2(\mathbb{Z})$. Let

$$M^\vee \subset M \otimes \mathbb{Q}$$

denote the dual lattice and $M^\vee/M$ the finite quotient. The pairing $\langle , \rangle$ extends linearly to a $\mathbb{Q}$-valued pairing on $M^\vee$. The functions $\frac{1}{2}\langle \gamma, \gamma \rangle$ and $\langle \gamma, \delta \rangle$ descend to $\mathbb{Q}/\mathbb{Z}$-valued functions on $M^\vee/M$.

We construct a representation ρ_M of $Mp_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[M^\vee/M]$. It suffices to define ρ_M in terms of the action of the generators T and S with respect to the standard basis v_γ for $\gamma \in M^\vee/M$,

$$\rho_M(T)v_\gamma = e^{2\pi i \frac{\langle \gamma,\gamma\rangle}{2}} v_\gamma \ ,$$
$$\rho_M(S)v_\gamma = \frac{\sqrt{i}^{n-2}}{\sqrt{|M^\vee/M|}} \sum_\delta e^{-2\pi i \langle \gamma,\delta\rangle} v_\delta \ .$$

Let N denote the smallest positive integer for which $N\langle\gamma,\gamma\rangle/2 \in \mathbb{Z}$ for all $\gamma \in M^\vee$. The representation factors through a double cover of $SL_2(\mathbb{Z}/N\mathbb{Z})$. We will be primarily interested in the dual representation ρ_M^* of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[M^\vee/M]$. We have given the action of ρ_M to match Borcherds' notation.

4.3. Heegner divisors. Given the lattice M of type $(2,n)$ as before, consider the Hermitian symmetric domain

$$\mathcal{D} = \{\omega \in \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C}) \mid \langle \omega,\omega\rangle = 0, \langle \omega,\bar{\omega}\rangle > 0\}$$

naturally associated to M. We will study the quotient

$$\mathcal{X}_M = \mathcal{D}/\Gamma_M \tag{22}$$

of $\mathcal{D}$ by the arithmetic subgroup of $O(2,n)$

$$\Gamma_M = \{g \in \mathrm{Aut}(M) \mid g \text{ acts trivially on } M^\vee/M\}\,.$$

The quotient (22) is a quasi-projective algebraic variety.

For every $n \in \mathbb{Q}^{<0}$ and $\gamma \in M^\vee/M$, we associate a divisor class $y_{n,\gamma} \in \mathrm{Pic}(\mathcal{X}_M)$ as follows. Given an element $v \in M^\vee$, there is an associated hyperplane

$$v^\perp = \{\omega \in \mathcal{D} \mid \langle \omega, v\rangle = 0\}\,.$$

Both $\langle v,v\rangle$ and the residue class $v \bmod M$ are invariant under the action of Γ_M. Therefore, if we fix $n \in \mathbb{Q}$ and $\gamma \in M^\vee/M$, the set of $v \in M^\vee$ with

$$\frac{1}{2}\langle v,v\rangle = n, \quad v \equiv \gamma \bmod M$$

is also Γ_M-invariant. The union over the set of the associated hyperplanes

$$\sum_{\substack{\frac{1}{2}\langle v,v\rangle = n \\ v \equiv \gamma \bmod M}} v^\perp$$

is Γ_M-invariant and descends to an algebraic divisor

$$y_{n,\gamma} = \left(\sum_{\frac{1}{2}\langle v,v\rangle = n,\ v\equiv\gamma \bmod M} v^\perp \right) /\Gamma_M .$$

The $y_{n,\gamma}$ are the *Heegner divisors* of $\mathcal{X}_M$. Because of the symmetry $v^\perp = (-v)^\perp$, there is a redundancy

$$y_{n,\gamma} = y_{n,-\gamma}$$

in our notation, and $y_{n,\gamma}$ is multiplicity 2 everywhere if $2\gamma \equiv 0 \bmod M$.

In the degenerate case where $n = 0$, we have the following prescription. The line bundle $\mathcal{O}(-1)$ on $\mathcal{D} \subset \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C})$ admits a natural Γ_M action and therefore descends to a line bundle K on $\mathcal{X}_M$. If $n = 0$ and $\gamma = 0$, we set

$$y_{0,0} = K^*.$$

If $n = 0$ and $\gamma \neq 0$, we set $y_{n,\gamma} = 0$.

We place the Heegner divisors in a formal power series $\Phi_M(q)$ with coefficients in $\mathrm{Pic}(\mathcal{X}_M) \otimes \mathbb{C}[M^\vee/M]$. More precisely, we consider the generating function

$$\Phi(q) = \sum_{n \in \mathbb{Q}^{\geq 0}} \sum_{\gamma \in M^\vee/M} y_{-n,\gamma} q^n v_\gamma \in \mathrm{Pic}(\mathcal{X}_M)[[q^{1/N}]] \otimes_{\mathbb{Z}} \mathbb{C}[M^\vee/M].$$

The main result of [**7**] together with the refinement of [**41**] yield the following Theorem.

Theorem ([**7**],[**41**]). *Let M have signature $(2, n)$. The generating function $\Phi(q)$ is an element of*

$$\mathrm{Pic}(\mathcal{X}_M) \otimes_{\mathbb{Z}} \mathrm{Mod}(Mp_2(\mathbb{Z}), 1 + \frac{n}{2}, \rho_M^*).$$

As a consequence, given any linear functional

$$\lambda : \mathrm{Pic}(\mathcal{X}_M) \otimes \mathbb{C} \to \mathbb{C},$$

the contraction $\lambda(\Phi_M(q))$ is the Fourier expansion of a vector-valued modular form of weight $1 + \frac{n}{2}$ and type ρ_M^*.

Borcherds' proof uses the singular theta lift of [**6**] to construct automorphic forms on $\mathcal{X}_M$ starting from vector-valued meromorphic modular forms on the upper half-plane. The zeroes and poles of these automorphic forms lie precisely along the Heegner divisors with multiplicity determined by the singular part of the initial modular form. Each such lifting gives a relation in $\mathrm{Pic}(\mathcal{X}_M)$. The total collection of relations arising in this way are encoded in the modularity statement.

In [**6**], Borcherds only shows that $\Phi_M(q)$ lies in a certain Galois closure of the space of modular forms. For the representations ρ arising in [**6**], MacGraw proves in [**41**] that $\mathrm{Mod}(Mp_2(\mathbb{Z}), k, \rho)$ admits a basis with rational coefficients. Therefore, the Galois closure does not enlarge the space.

4.4. Application to K3 surfaces. Let V be the rank 22 lattice obtained from the second cohomology of a $K3$ surface with fixed polarization L of norm l. In order to apply Borcherds' results to the moduli spaces $\mathcal{M}_l$, we consider the lattice of signature $(2, 19)$

$$M = L^\perp = \{v \in V \mid \langle L, v\rangle = 0\}.$$

A direct check yields

$$M \cong \mathbb{Z}w \oplus U^2 \oplus E_8(-1)^2$$

where $\langle w, w\rangle = -l$. Therefore

$$M^\vee/M = \mathbb{Z}/l\mathbb{Z}$$

and is generated by $\frac{1}{l}w$. Here, we will write ρ_l for the representation ρ_M.

From the definitions, we find $\mathrm{Aut}(V, L) = \Gamma_M$, so we have the identification

$$\mathcal{M}_l = \mathcal{X}_M.$$

We claim the Heegner divisors correspond precisely to our Noether-Lefschetz divisors.

LEMMA 3. *We have* $D_{h,d} = y_{n,\gamma}$, *where*

$$n = -\frac{\Delta_l(h,d)}{2l} \quad \textit{and} \quad \gamma \equiv d(\frac{1}{l}w) \bmod M.$$

PROOF. The Noether-Lefschetz divisor $D_{h,d}$ is the quotient by Γ_M of the union of hyperplanes

$$\sum_{\substack{\langle \beta,\beta\rangle = 2h-2 \\ \langle L,\beta\rangle = d}} \beta^\perp.$$

It therefore suffices to establish a bijection between the two sets of hyperplanes. Given an element $\beta \in V$ satisfying

$$\langle \beta,\beta\rangle = 2h-2, \quad \langle \beta, L\rangle = d,$$

let $v = \beta - \frac{d}{l}L \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the projection of β to $M = L^\perp$. A direct calculation shows

$$\frac{1}{2}\langle v,v\rangle = h - 1 - \frac{d^2}{2l} = -\frac{\triangle_l(h,d)}{2l}\ ,$$

$$v \equiv d\cdot(\frac{1}{l}w) \bmod M\ .$$

Conversely, given $v \in M^\vee$ satisfying the above conditions,

$$\beta = v + \frac{d}{l}L$$

gives the inverse construction. Since $\beta^\perp = v^\perp$, we obtain the result. □

It is important for our applications that the constant term $y_{0,0}$ of $\Phi_M(q)$ matches with the line bundle K^* from our excess calculation in the proof of Theorem 1. This occurs because automorphic forms can be viewed as sections of powers of K^* on $\mathcal{M}_l$.

Let π be a 1-parameter family of quasi-polarized $K3$ surfaces of degree l, and let ι be the associated morphism to moduli space:

$$\pi : X \to C,$$

$$\iota : C \to \mathcal{M}_l.$$

We can apply Borcherds' theorem to the functional on $\mathrm{Pic}(\mathcal{M}_l)$ given by

$$D \mapsto \int_C \iota^* D.$$

COROLLARY 3. *There is a vector-valued modular form of weight* $21/2$ *and type* ρ_l^*,

$$\Phi^\pi(q) = \sum_{r=0}^{l-1} \Phi_r^\pi(q) v_r \in \mathbb{C}[[q^{1/2l}]] \otimes \mathbb{C}[\mathbb{Z}/l\mathbb{Z}],$$

with nonzero coefficients determined by the equality

$$NL^\pi_{h,d} = \Phi_r^\pi\left[\frac{\triangle_l(h,d)}{2l}\right]$$

where $r \equiv d \bmod l$.

4.5. Quartic $K3$ surfaces. We now apply Borcherds' modularity to the study of $K3$ surfaces of degree 4. If $l = 4$, the isomorphism class of a rank two lattice $(\mathbb{L}, v)$ with primitive polarization $\langle v, v\rangle = l$ is determined only by the discriminant $\triangle$.

Given a 1-parameter family $\pi : X \to C$ of quasi-polarized $K3$ surfaces of degree 4, we have the generating function

$$\Phi^\pi(q) = \Phi^\pi_0(q)v_0 + \Phi^\pi_1(q)v_1 + \Phi^\pi_2(q)v_2 + \Phi^\pi_3(q)v_3$$

which is a modular form of weight $21/2$ and type ρ_4^* by Corollary 3.

Consider the scalar-valued power series

$$\phi^\pi(q) = \Phi^\pi_0(q) + \frac{1}{2}\Phi^\pi_1(q) + \Phi^\pi_2(q) + \frac{1}{2}\Phi^\pi_3(q).$$

By chasing definitions, we see $\phi^\pi(q)$ has the following property:

$$NL^\pi_{h,d} = \phi^\pi\left[\frac{\triangle_4(h,d)}{8}\right]. \tag{23}$$

The factor of $1/2$ is included to correct for the redundancy

$$\Phi^\pi_1(q) = \Phi^\pi_3(q).$$

PROPOSITION 5. *The function $\phi^\pi(q)$ is a homogeneous polynomial of degree* 21 *in*

$$A = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{8}} \quad \textit{and} \quad B = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{8}}.$$

PROOF. While the vector $\Phi^\pi(q)$ is modular with respect to the full metaplectic group, $\phi^\pi(q)$ is a priori only modular with respect to the subgroup $\widetilde{\Gamma}(8) = \mathrm{Ker}(\rho_4^*)$. However, we can write $\phi^\pi(q)$ as a sum

$$\phi^\pi(q) = \frac{3}{4}\phi_+(q) + \frac{1}{4}\phi_-(q)$$

where

$$\phi_+(q) = \Phi^\pi_0(q) + \Phi^\pi_1(q) + \Phi^\pi_2(q) + \Phi^\pi_3(q),$$
$$\phi_-(q) = \Phi^\pi_0(q) - \Phi^\pi_1(q) + \Phi^\pi_2(q) - \Phi^\pi_3(q).$$

Consider the congruence subgroup of $SL_2(\mathbb{Z})$

$$\Gamma^0(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \bmod 8 \right\}.$$

A direct calculation of the representation ρ_4^* shows that $\phi_+(q)$ and $\phi_-(q)$ are modular forms of weight $21/2$ with respect to

$$\widetilde{\Gamma}^0(8) = \left\{(A,\phi) \in Mp_2(\mathbb{Z}) \mid A \in \Gamma^0(8)\right\}$$

and distinct characters

$$\chi_+, \chi_- : \widetilde{\Gamma}^0(8) \to \mathbb{C}^*.$$

Moreover, A and B are modular forms of weight $1/2$ with respect to $\widetilde{\Gamma}^0(8)$ and the same characters χ_+ and χ_- respectively.

We will not describe $\chi_\pm$ explicitly. While they are distinct, their squares are equal and $\chi = \chi_+^2 = \chi_-^2$ descends to a character

$$\chi : \Gamma^0(8) \to \mathbb{C}^*.$$

The character χ is specified completely by the following evaluations:

$$\chi(\Gamma^1(8)) = 1,\ \chi\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1,\ \chi\begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix} = -1$$

where

$$\Gamma^1(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \bmod 8, a \equiv d \equiv 1 \bmod 8 \right\}.$$

Consider the space $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$ of holomorphic modular forms of weight 11 and type χ. The space $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$ is 12-dimensional space with basis

$$A^{22}, A^{20}B^2, \cdots, A^2B^{20}, B^{22}.$$

Both $\phi_+(q) \cdot A$ and $\phi_-(q) \cdot B$ lie in $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$. Since A^{22}/B and B^{22}/A are not holomorphic at the boundary, we conclude $\phi_\pm(q)$ are each homogeneous polynomials of degree 21 in A and B and therefore so is $\phi^\pi(q)$. □

5. Lefschetz pencil of quartics

5.1. Quartics. A general Lefschetz pencil of quartics can be viewed as a hypersurface of type $(4, 1)$,

$$\pi : X_{4,1} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1 \tag{24}$$

where the last projection is onto the second factor. Unfortunately, π contains 108 nodal fibers, so the family (24) does not fit the specifications of Section 1.2.

A family of quasi-polarized $K3$ surfaces of degree 4 can be obtained from the Lefschetz pencil π by the following construction. Let

$$\epsilon : C_{53} \xrightarrow{2-1} \mathbb{P}^1 \tag{25}$$

be the genus 53 hyperelliptic curve branched over the 108 points of $\mathbb{P}^1$ corresponding to the nodal fibers of π. The family

$$\epsilon^*(X_{4,1}) \to C_{53}$$

has 3-fold double point singularities over the 108 nodes of the fibers of the original family π. Let

$$\widetilde{\pi} : \widetilde{X} \to C_{53}$$

be obtained from a small resolution

$$\widetilde{X} \to \epsilon^*(X_{4,1}).$$

Then, $\widetilde{\pi}$ is easily seen to be a family of quasi-polarized $K3$ surfaces of degree 4. The quasi-polarization is the pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$.

5.2. Invariants. The Noether-Lefschetz numbers are defined in Section 1 only for the family $\widetilde{\pi}$. However, for convenience, we define

$$NL^{\pi}_{g,d} = \frac{1}{2} NL^{\widetilde{\pi}}_{g,d}\ .$$

Instead of a curve class γ, the degree d against the polarization is taken as the second subscript.

The family $\widetilde{\pi}$ may be viewed as twice the Lefschetz pencil of quartics. Let

$$\pi_{4,2} : X_{4,2} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1$$

be the family obtained from a nonsingular Calabi-Yau hypersurface. The family $\pi_{4,2}$ may also be viewed as twice the Lefschetz pencil.

LEMMA 4. $n^{\widetilde{X}}_{g,d} = n^{X_{4,2}}_{g,d}$.

PROOF. It suffices to prove the analogous statement for Gromov-Witten invariants. Consider the degeneration of $X_{4,2}$ to the union

$$X_{4,1} \cup_{K3} X_{4,1}$$

of two $(4,1)$ hypersurfaces along a smooth $K3$ surface. The degeneration formula of [**32, 33**] implies

$$N^{X_{4,2}}_{g,d} = 2N^{X_{4,1}/K3}_{g,d}$$

where the latter term denotes the Gromov-Witten theory of $X_{4,1}$ relative to the $K3$ fiber. Since the Gromov-Witten theory of $K3 \times \mathbb{P}^1$ vanishes, the trivial degeneration

$$X_{4,1} \cup_{K3} (K3 \times \mathbb{P}^1)$$

yields the equality of relative and absolute invariants

$$N^{X_{4,1}}_{g,d} = N^{X_{4,1}/K3}_{g,d}.$$

To study the small resolution $\widetilde{\pi}$, consider the family of double covers

$$\epsilon_t : C_t \mapsto \mathbb{P}^1$$

ramified at 108 generic points which specializes to our particular double cover (25) as $t \to 0$. The behavior of Gromov-Witten theory in the conifold transition from

$$X_t = \epsilon_t^*(X_{4,1})$$

to $\tilde{X}$ has been calculated by Li and Ruan [**32**]:

$$N^{\widetilde{X}}_{g,d} = N^{X_t}_{g,d}.$$

By degenerating the base C_t to two copies of $\mathbb{P}^1$, we have a degeneration of X_t to two copies of $X_{4,1}$ attached at 54 smooth $K3$ fibers. As before, we apply the degeneration formula and the identification of relative and absolute invariants to obtain the equality

$$N^{\widetilde{X}}_{g,d} = N^{X_t}_{g,d} = 2N^{X_{4,1}}_{g,d} = N^{X_{4,2}}_{g,d}.$$

□

Instead of studying the Gromov-Witten invariants of $\widetilde{X}$, we may study the Gromov-Witten invariants of $X_{4,2}$.

5.3. Mirror symmetry.

5.3.1. *Overview.* The genus 0 invariants of $X_{4,2}$ are determined from hypergeometric series by the mirror transformation. The mirror formulas of Candelas, de la Ossa, Green, and Parkes [**11**] have been proven mathematically in many settings [**17, 18, 35**]. In particular, the case of $X_{4,2}$ is understood rigorously. We follow the notation of [**43**].

5.3.2. *Potential.* Let the variables T_1, T_2 correspond to the hyperplane classes

$$H_1 \subset \mathbb{P}^3, \quad H_2 \subset \mathbb{P}^1$$

respectively. The genus 0 potential of $X_{4,2}$ for classes restricted from $\mathbb{P}^3 \times \mathbb{P}^1$ is

$$\mathcal{F}(T_1, T_2) = \frac{1}{3}T_1^3 + 2T_1^2 T_2 + \sum_{d_1, d_2 \geq 0,\ (d_1,d_2) \neq (0,0)} N^{X_{4,2}}_{0,(d_1,d_2)}\ e^{d_1 T_1} e^{d_2 T_2}$$

where we follow the Gromov-Witten notation of Section 2. The curve class (d_1, d_2) is not a fiber class for $\pi^{4.2}$ if $d_2 > 0$.

5.3.3. *Hypergeometric series.* Let t_1, t_2 be new variables. Define the hypergeometric series $I_{i,j}(t_1,t_2)$ by

$$\sum_{i=0}^{3}\sum_{j=0}^{1} I_{i,j}(t_1,t_2)H_1^iH_2^j = \sum_{d_1,d_2\geq 0} e^{(H_1+d_1)t_1}e^{(H_2+d_2)t_2}\frac{\Pi_{r=0}^{4d_1+2d_2}(4H_1+2H_2+r)}{\Pi_{r=1}^{d_1}(H_1+r)^4\ \Pi_{r=1}^{d_2}(H_2+r)^2}.$$

The right side, taken mod H_1^4 and H_2^2, is valued in $H^*(\mathbb{P}^3\times\mathbb{P}^1,\mathbb{Q})$. Formally,

$$I_{i,j}(t_1,t_2)\in\mathbb{Q}[[t_1,e^{t_1},t_2,e^{t_2}]].$$

The functions $I_{i,j}(t)$ form a solution of the Picard-Fuchs differential equation associated to the mirror geometry.

5.3.4. *Mirror transformation.* The mirror transformation is defined using two auxiliary functions. Let

$$F(e^{t_1},e^{t_2}) = \sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty} e^{d_1t_1}e^{d_2t_2}\frac{(4d_1+2d_2)!}{(d_1!)^4(d_2!)^2},$$

and let

$$G_{a,b}(e^{t_1},e^{t_2}) = \sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty} e^{d_1t_1}e^{d_2t_2}\frac{(4d_1+2d_2)!}{(d_1!)^4(d_2!)^2}\Big(\sum_{r=1}^{ad_1+bd_2}\frac{1}{r}\Big)$$

for $a,b\geq 0$.

The mirror transformation relating the variables T_i and t_i is determined by the following equations:

$$T_1 = t_1 + \frac{4(G_{4,2}(e^{t_1},e^{t_2}) - G_{1,0}(e^{t_1},e^{t_2}))}{F(e^{t_1},e^{t_2})},$$

$$T_2 = t_2 + \frac{2(G_{4,2}(e^{t_1},e^{t_2}) - G_{0,1}(e^{t_1},e^{t_2}))}{F(e^{t_1},e^{t_2})}.$$

Exponentiation yields

$$e^{T_1} = e^{t_1}\cdot\exp\left(\frac{4(G_{4,2}(e^{t_1},e^{t_2}) - G_{1,0}(e^{t_1},e^{t_2}))}{F(e^{t_1},e^{t_2})}\right),$$

$$e^{T_2} = e^{t_2}\cdot\exp\left(\frac{2(G_{4,2}(e^{t_1},e^{t_2}) - G_{0,1}(e^{t_1},e^{t_2}))}{F(e^{t_1},e^{t_2})}\right).$$

Together, the above four equations define a change of variables from formal series in $T_1, e^{T_1}, T_2, e^{T_2}$ to formal series in $t_1, e^{t_1}, t_2, e^{t_2}$. The mirror transformation is easily seen to be invertible.

5.3.5. *Genus 0 invariants.* The genus 0 potential $\mathcal{F}$ is determined by mirror symmetry,

$$\mathcal{F}(T_1(t_1,t_2),T_2(t_1,t_2)) = \left(\frac{2I_{1,1}-I_{2,0}}{I_{1,0}}\right)\left(\frac{I_{3,0}}{I_{1,0}}\right) + 2\left(\frac{I_{2,0}}{I_{1,0}}\right)\left(\frac{I_{2,1}}{I_{1,0}}\right) - 2\left(\frac{I_{3,1}}{I_{1,0}}\right).$$

The arguments of the functions on the right side are understood to be t_1 and t_2. The genus 0 BPS states $n_{0,d}^{X_{4,2}}$ are determined by $\mathcal{F}$.

5.4. Proof of Theorem 2. Consider twice the Lefschetz pencil of quartics

$$\widetilde{\pi} : \widetilde{X} \to C_{53}.$$

Corollary 1 in genus 0 is

$$n_{0,d}^{\widetilde{X}} = \sum_{h=0}^{\infty} r_{0,h} \cdot NL_{h,d}^{\widetilde{\pi}} \ . \tag{26}$$

We now solve for the Noether-Lefschetz numbers of $\tilde{\pi}$. By (23),

$$NL_{h,d}^{\widetilde{\pi}} = \phi^{\widetilde{\pi}} \left[\frac{\triangle_4(h,d)}{8} \right]$$

where $\phi^{\widetilde{\pi}}(q)$ is a homogeneous polynomial of degree 21 in A and B. We need only 22 equations to determine $\phi^{\widetilde{\pi}}(q)$. Using the mirror symmetry calculation of $n_{0,d}^{\widetilde{X}}$, equation (26) provides infinitely many relations. In particular, $\phi^{\widetilde{\pi}}(q)$ is easily determined by linear algebra.

The precise formula for $\phi^{\widetilde{\pi}}$ is 2Θ where Θ is given in Section 0.6 since $\widetilde{\pi}$ is twice the Lefschetz pencil of quartics. The modular form Θ was first computed in [**26**].

5.5. Modular identity. Equation (26) may be viewed as a rather intricate relation between hypergeometric functions (after mirror transformation) on the left and modular forms on the right. Let

$$\mathcal{G}(q) = -\frac{2}{q} + 168 + \sum_{d \geq 1} n_{0,d}^{X_{4,2}} q^{\frac{d^2}{8}}$$

be the generating function determined by the property

$$\sum_{d=1}^{\infty} \sum_{k=1}^{\infty} n_{0,d}^{X_{4,2}} \frac{1}{k^3} e^{dkT_1} = \left(\mathcal{F}(T_1, T_2) - \frac{1}{3} T_1^3 - 2T_1^2 T_2 \right) |_{e^{T_2}=0}$$

where $\mathcal{F}$ is determined as above.

Corollary 4. *We have the equality*

$$\mathcal{G}(q) = 2 \frac{\Theta(q)}{\Delta(q)} \ ,$$

where $\Theta(q)$ *is given in Section* 0.6 *and*

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \ .$$

Such relations are produced by Theorem 1 for many classical examples. For any 1-parameter family of $K3$ surfaces obtained via a toric complete intersection, there is an associated identity of special functions. The relation obtained from the STU model studied in [**27**] is the Harvey-Moore identity. In fact, the Harvey-Moore identity is the *only* one for which a direct proof (avoiding Theorem 1) is known. The proof is due to Zagier and can be found in [**27**].

5.6. Proof of Corollary 2. Let π be the Lefschetz pencil of quartic $K3$ surfaces. The difference between $NL^{\pi}_{h,d}$ and the degree of

$$\overline{\mathcal{D}}_{h,d} \subset \mathbb{P}(\mathrm{Sym}^4(V^*))$$

is simply the contribution of the nodal quartics. The nodal quartics contribute to $NL^{\pi}_{h,d}$ but not the hypersurface $\overline{\mathcal{D}}_{h,d}$.

Using the relation $NL^{\pi}_{h,d} = \frac{1}{2} NL^{\widetilde{\pi}}_{h,d}$, we can study instead the doubled family. The Picard lattice of each of the 108 fibers of $\widetilde{\pi}$ corresponding to the original nodal fibers of π is

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}. \tag{27}$$

We use here the genericity of the Lefschetz pencil π.

The equation $\langle \beta, L \rangle = d$ is solvable in the lattice (27) if and only if d is divisible by 4. Then, $\langle \beta, \beta \rangle = 2h - 2$ is solvable if and only if

$$4(\frac{d}{4})^2 - 2n^2 = 2h - 2$$

in which case there are two solutions. In the solvable cases,

$$\triangle_4(h, d) = 8n^2.$$

Hence, the contribution of the nodal fiber to the Noether-Lefschetz numbers of $\widetilde{\pi}$ is

$$\Psi(q) = 108 \cdot 2 \sum_{n>0} q^{n^2}.$$

The Corollary follows by halving. □

6. Direct Noether-Lefschetz calculations

6.1. Overview. We apply Corollary 3 to directly study $K3$ surfaces of low degree via a more sophisticated approach to modular forms. The key idea is to construct a basis of the space of vector-valued modular forms of Corollary 3 instead of working with the much larger space of scalar-valued modular forms as in Section 4.5. For many classical families, the dimensions of the associated spaces of vector-valued modular forms are very small. The Noether-Lefschetz numbers can often be specified by a few classical calculations. In particular, we see another derivation of Theorem 2.

6.2. Rankin-Cohen brackets. Since each component of a vector-valued modular form is a half-weight modular form of level $2l$, we can use a basis of the latter to construct all vector-valued modular forms. In practice, however, the method is tedius since the dimensions of the spaces of scalar-valued modular forms are much larger. We will instead apply the following shortcut for low degree $K3$ surfaces.

Let $f(q)$ and $g(q)$ be scalar-valued level N modular forms on the upper-half plane $\mathcal{H}$ of weights k_1 and k_2 respectively. For each integer $n \geq 0$, the n-th Rankin-Cohen bracket is a bilinear differential operator defined by the expression

$$[f(q), g(q)]_n = \sum_{r=0}^{n} (-1)^r \binom{n+k_1-1}{n-r} \binom{n+k_2-1}{r} f^{(r)}(q) \cdot g^{(n-r)}(q),$$

where $f^{(r)}$ denote r applications of the differential operator

$$\frac{d}{d\tau} = q\frac{d}{dq} \ .$$

For $n = 0$, the 0-th bracket is just multiplication.

The key feature of Rankin-Cohen brackets is the preservation of modularity. Suppose we are given a representation ρ of $Mp_2(\mathbb{Z})$ on V, a modular form $f \in \mathrm{Mod}(Mp_2(\mathbb{Z}), k_1, \rho)$ of weight k_1 and type ρ, and a scalar-valued modular form $g \in \mathrm{Mod}(SL_2(\mathbb{Z}), k_2)$ of weight k_2 and level 1. Let

$$f(q) = \sum_\gamma f_\gamma(q) v_\gamma \in V$$

denote the decomposition of f into components with respect to some basis of V. For each integer $n \geq 0$, the Rankin-Cohen bracket is a holomorphic function on $\mathcal{H}$ with values in V defined by

$$[f, g]_n(q) = \sum_\gamma [f_\gamma(q), g(q)]_n v_\gamma.$$

We then have the following result.

Lemma 5. *$[f, g]_n(q) \in \mathrm{Mod}(Mp_2(\mathbb{Z}), k_1 + k_2 + 2n, \rho)$.*

Proof. For scalar-valued modular forms, a proof is given in [**53**]. Since g is scalar-valued and level 1, the same argument translates to the vector-valued context without change. □

6.3. Bases of modular forms. Following the notation of Corollary 3, we now look for modular forms of weight 21/2 and type ρ_l^* for even

$$l = 2, 4, 6, 8 \ .$$

From the dimension formula given in Section 7 below,

$$\dim(\mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)) = 2, 3, 4, 5$$

for $l = 2, 4, 6, 8$ respectively. We are only interested[11] in the subspace

$$\mathrm{Mod}_0(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$$

of forms $\sum f_i(q) v_i$ where $f_r(q)$ is a cusp form for $r \neq 0$. In the $l = 8$ case, we have a 4-dimensional subspace.

We can use Rankin-Cohen brackets to construct explicit bases. Indeed, for each l, there is a canonical weight 1/2 modular form given by the Siegel theta function (see [**6**], Section 4),

$$\theta^{(l)}(q) = \sum_{i=0}^{l-1} \sum_{s \in \mathbb{Z}} q^{\frac{(ls+i)^2}{2l}} v_i \in \mathrm{Mod}(Mp_2(\mathbb{Z}), 1/2, \rho_l^*).$$

Therefore, for $n = 0, 1, 2, 3$, Lemma 5 gives us a modular form,

$$F_n^l(q) = [\theta^{(l)}(q), E_{10-2n}(q)]_n \in \mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*),$$

of weight 21/2 where $E_{2k}(q)$ denotes Eisenstein series of weight $2k$.

Using the explicit formula for Rankin-Cohen brackets and the dimension formula, the following Lemma is obtained by calculating the initial Taylor coefficients.

[11]The cusp condition is obtained from Borcherds' results and was omitted in the statement of Corollary 3 for simplicity.

Lemma 6. *For $l = 2, 4, 6$, the modular forms*

$$F_n^l(q) = [\theta^{(l)}(q), E_{10-2n}(q)]_n, n = 0, \dots, l/2$$

form a basis of $\mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$. *For $l = 8$, the modular forms for $n = 0, \dots, 3$ form a basis of the subspace* $\mathrm{Mod}_0(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$.

6.4. Classical families of $K3$ surfaces. A general $K3$ surface of degree $l = 2, 4, 6, 8$ is either a branched cover of $\mathbb{P}^2$ (for $l = 2$) or a complete intersection in projective space. We obtain 1-parameter families of quasi-polarized $K3$ surfaces of degree l by taking a generic Lefschetz pencil of these constructions (and resolving singularities as discussed in Section 5.1). Because the space of vector-valued forms is of low dimension, we only need a few classical constraints to completely determine the associated modular form. In fact, we will use only the following constraints:

(i) the degree of the Hodge bundle $R^2\pi_*\mathcal{O}$ (the coefficient of $q^0 v_0$),
(ii) the number of nodal fibers (the coefficient of $q^1 v_0$),
(iii) vanishing obtained from Castelnuovo's bound in Lemma 7 below.

The following result is a special case of Castelnuovo's bound for projective curves [**1**].

Lemma 7. *Given a $K3$ surface with very ample bundle L and an primitive curve class β, we have the inequality*

$$\langle \beta, \beta \rangle \leq 2\binom{L \cdot \beta - 1}{2} - 2 \ .$$

We now apply these constraints for 1-parameter families of $K3$ given by Lefschetz pencils for $l = 2, 4, 6, 8$.

• *Degree* 2 *$K3$ surfaces*

A generic $K3$ surface of degree 2 is a double cover of $\mathbb{P}^2$ branched along a nonsingular sextic plane curve. Consider a family

$$R \subset \mathbb{P}^1 \times \mathbb{P}^2$$

of sextics defined by a generic hypersurface of type $(2, 6)$. Let X be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified over R. Since all the singular fibers of

$$R \to \mathbb{P}^1$$

are irreducible and nodal, the associated family

$$\pi : X \to \mathbb{P}^1$$

of $K3$ surfaces is smooth except for finitely many fibers with nodal singularities.

The degree of the Hodge bundle is -1 by a Riemann-Roch calculation. The number of nodal fibers of π is 150, twice the degree of the discriminant locus of sextics. Since we have a 2-dimensional space of forms, the generating series of Noether-Lefschetz numbers is the vector-valued modular form

$$\overrightarrow{\Theta}(q) = -F_0^{(2)}(q) - \frac{1}{2}F_1^{(2)}(q).$$

In the case of $l = 2$, the discriminant Δ of a rank 2 lattice with degree 2 polarization determines the coset class δ by $\delta = \Delta \mod 2$. So there is no loss of information if we replace $\overrightarrow{\Theta}(q)$ by the sum of the components $\Theta(q) = \overrightarrow{\Theta}_0 + \overrightarrow{\Theta}_1$.

If we consider the theta functions

$$U = \sum_{n\in\mathbb{Z}} q^{n^2/4}, \quad V = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/4},$$

we can express Θ as a polynomial function of U and V:

$$\begin{aligned}\Theta(q) &= \frac{1}{1024}(U^{21} - 12U^{17}V^4 - 402U^{13}V^8 - 572U^9V^{12} - 39U^5V^{16}) \\ &= -1 + 150q + 1248q^{5/4} + 108600q^2 + 332800q^{9/4} + 5113200q^3 \cdots .\end{aligned}$$

To see equivalence of the two expressions, we observe both are modular forms of weight $21/2$ with respect to $\Gamma(4)$ and check the agreement of sufficiently many coefficients.

• *Degree* 4 $K3$ *surfaces*

A generic $K3$ surface of degree 4 is a quartic hypersurface in $\mathbb{P}^3$. If we take a generic Lefschetz pencil of such quartics, the degree of the Hodge bundle is -1. Using Lemma 7, the Noether-Lefschetz degrees associated to the lattices

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$

both vanish. Indeed, by choosing a generic pencil, we can assume all fibers containing these Picard lattices have very ample quasi-polarization. The coefficients of $q^0v_0, q^{1/8}v_1$, and $q^{1/2}v_2$ determine

$$\overrightarrow{\Theta}(q) = -F_0^{(4)}(q) - \frac{5}{4}F_1^{(4)}(q) - \frac{16}{21}F_2^{(4)}(q).$$

Again, as in the degree 2 case, we can recover all Noether-Lefschetz degrees from

$$\Theta(q) = \overrightarrow{\Theta}_0(q) + \overrightarrow{\Theta}_1(q) + \overrightarrow{\Theta}_2(q).$$

In terms of

$$A = \sum_{n\in\mathbb{Z}} q^{n^2/8}, \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/8},$$

we recover the expression for $\Theta(q)$ given in Section 0.6 since both are modular forms of weight $21/2$ and level 8 which agree on initial terms.

• *Degree* 6 $K3$ *surfaces*

A generic $K3$ surface of degree 6 is the intersection of a quadric and cubic hypersurface in $\mathbb{P}^4$. We have two basic families. We can fix a quadric and take a Lefschetz pencil of cubics or vice versa. In each case, we have vanishings associated to the lattices

$$\begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix}$$

from the Castelnuovo bound. Along with the Hodge bundle degree and the number of nodal fibers, we completely determine the Noether-Lefschetz series.

For the first family, the Hodge and nodal degrees are -1 and 98 respectively. We obtain the series

$$\overrightarrow{\Theta}(q) = -F_0^{(6)}(q) - \frac{49}{24}F_1^{(6)}(q) - \frac{8}{3}F_2^{(6)}(q) - \frac{12}{5}F_3^{(6)}(q).$$

For the second family, the Hodge and nodal degrees are -1 and 7. We obtain the series

$$\overrightarrow{\Theta}(q) = -F_0^{(6)}(q) - \frac{17}{8}F_1^{(6)}(q) - \frac{22}{7}F_2^{(6)}(q) - \frac{18}{5}F_3^{(6)}(q).$$

One can read off other classical calculations from our results. For example, the number of surfaces containing elliptic plane curves or containing lines are the Noether-Lefschetz degrees associated to the lattices

$$\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 6 & 1 \\ 1 & -2 \end{pmatrix}$$

respectively. In the first family, the degrees are 0 and 168 respectively. In the second family, the degrees are 10 and 198. In both cases, the numbers agree with earlier enumerative calculations.

- *Degree* 8 $K3$ *surfaces*

A generic $K3$ surface of degree 8 is the intersection of three quadric hypersurfaces in $\mathbb{P}^5$. The basic family comes from fixing two quadrics and allowing the third to vary in a Lefschetz pencil. Again, the series is determined by the Hodge term, the nodal term, and the two Castelnuovo vanishings from Lemma 7. The Hodge term is given by -1, and the number of nodal fibers is 80. We find

$$\overrightarrow{\Theta}(q) = -F_0^{(8)}(q) - \frac{49}{18}F_1^{(8)}(q) - \frac{128}{27}F_2^{(8)}(q) - \frac{256}{45}F_3^{(8)}(q).$$

Again, we can read off that the number of fibers containing a line is 128, agreeing with the classical calculation.

For all the classical examples discussed above, the mirror symmetry calculation of the genus 0 Gromov-Witten invariants is solvable in terms of hypergeometric functions. In each case, Theorem 1 yields a remarkable identity with hypergeometric functions (after mirror transformation) on the left and modular forms on the right, as in Section 5.5.

The lower Noether-Lefschetz degrees in the above classical examples can often be pursued by alternative methods. In particular, matches with our modular form calculations have been found in [**5, 13**].

7. Picard rank of $\mathcal{M}_l$

The Picard ranks of the moduli spaces of quasi-polarized $K3$ surfaces $\mathcal{M}_l$ are unknown. By an argument of O'Grady, the ranks can grow arbitrarily large [**42**]. Let

$$\mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \subset \mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{Q} \tag{28}$$

denote the span of the Noether-Lefschetz divisors $D_{h,d}$. We make the following conjecture.

Conjecture 3. *The inclusion is an isomorphism,*

$$\mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \cong \mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{Q}.$$

Bruinier has calculated the dimension of the space $\mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q}$ in equations (6-7) of [**8**]. If Conjecture 3 holds, we obtain a formula for the Picard rank of $\mathcal{M}_l$.

We now recount Bruinier's formula for the span of the Noether-Lefschetz divisors. By Borcherds' work, we have a map

$$\mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)^* \to \mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{C}. \tag{29}$$

Let $\mathrm{Cusp}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$ denote the subspace of cusp forms — modular forms for which the Fourier coefficients $c_{0,\gamma}$ vanish for all γ. The map (29) induces a map

$$\mathrm{Cusp}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)^* \to (\mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{C})/\mathbb{C}K, \tag{30}$$

where K is the Hodge bundle on $\mathcal{M}_l$. Bruinier shows the map (30) is injective [**8**]. Specifically, if L is a $(2, n)$ lattice containing two copies of U as direct summands, Bruinier shows that every relation among Heegner divisors is obtained from Borcherds' theta lifting. Therefore,

$$\dim\ \mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 1 + \dim\ \mathrm{Cusp}(Mp_2(\mathbb{Z}), \rho_l^*, 21/2).$$

A direct calculation of the dimension of the space of cusp forms via Riemann-Roch yields the following evaluation [**8**]:

$$\begin{aligned}\dim\ \mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \quad = \quad & 1 + \frac{31}{24} + \frac{31}{48}l - \frac{1}{8\sqrt{l}}\mathrm{Re}(G(2, 2l)) \\ & - \frac{1}{6\sqrt{3l}}\mathrm{Re}(e^{-2\pi i \frac{19}{24}}(G(1, 2l) + G(-3, 2l))) \\ & - \sum_{k=0}^{l/2} \left\{ \frac{k^2}{2l} \right\} - C,\end{aligned}$$

where $G(a, b)$ denotes the quadratic Gauss sum

$$G(a, b) = \sum_{k=0}^{b-1} e^{-2\pi i \frac{ak^2}{b}},$$

the braces $\{,\}$ denote fractional part, and C is the cardinality of the set

$$\left\{ k \ \mid \ 0 \le k \le \frac{l}{2}, \frac{k^2}{2l} \in \mathbb{Z} \right\}.$$

For $l = 2, 4, 6$, the formula yields

$$\dim\ \mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 2, 3, 4$$

respectively. For $l = 2$ and 4, we have agreement with the Picard ranks of $\mathcal{M}_l$ calculated in [**25, 49, 50**]. Hence, the inclusion (28) is an isomorphism in at least the first two cases.

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Clay Mathematics Proceedings
Volume **18**, 2013

Numerical Macaulification

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Dedicated to Joe Harris on the occasion of his 60th birthday

Abstract. An unpublished example due to Joe Harris from 1983 (or earlier) gave two smooth space curves with the same Hilbert function, but one of the curves was arithmetically Cohen-Macaulay (ACM) and the other was not. Starting with an arbitrary homogeneous ideal in any number of variables, we give two constructions, each of which produces, using a finite number of basic double links, an ideal with the Hilbert function of a codimension two ACM subscheme. We call the subscheme associated to such an ideal "numerically ACM." We study the connections between these two constructions, and in particular show that they produce ideals with the same Hilbert function. We call the resulting ideal from either construction a "numerical Macaulification" of the original ideal. Specializing to the case where the ideals are unmixed of codimension two, we show that (a) every even liaison class, $\mathcal{L}$, contains numerically ACM subschemes, (b) the subset, $\mathcal{M}$, of numerically ACM subschemes in $\mathcal{L}$ has, by itself, a Lazarsfeld-Rao structure, and (c) the numerical Macaulification of a minimal element of $\mathcal{L}$ is a minimal element of $\mathcal{M}$. Finally, if we further restrict to curves in $\mathbb{P}^3$, we show that the even liaison class of curves with Hartshorne-Rao module concentrated in one degree and having dimension n contains smooth, numerically ACM curves, for all $n \geq 1$. The first (and smallest) such example is that of Harris. A consequence of our results is that the knowledge of the Hilbert function of an integral curve alone is not enough to decide whether it contains zero-dimensional arithmetically Gorenstein subschemes of arbitrarily large degree.

1. Introduction

A natural, and very old, problem is to determine what information can be obtained about an algebraic variety, X, based on knowledge of its Hilbert function, possibly with some reasonable additional assumptions on X. There is a vast literature on this subject (see, e.g., [**3**], [**13**], [**18**], [**24**]). It is sometimes the case that one can determine whether or not X is arithmetically Cohen-Macaulay (ACM), based on the Hilbert function (e.g., when X is a line). An old example due to Joe Harris [**12**] shows a limitation to this, by exhibiting two smooth curves in $\mathbb{P}^3$ with the

2010 *Mathematics Subject Classification.* Primary 14M06; Secondary 13C40, 14M07, 14M05.

The first author is very grateful to Joe Harris, not only for the letter in 1983 explaining Joe's example, but even more for the preceding five years in which he patiently explained so much more! The authors thank the referee for helpful comments.

The first author was partially supported by grant #208579 from the Simons Foundation.

The second author was partially supported by grant #208869 from the Simons Foundation.

same Hilbert function, but one ACM and the other not. Harris's example consists of two curves of degree 10 and genus 11, where the non-ACM one is in the liaison class of a set of two skew lines. In Section 2 we recall this example.

The ACM property provides an important distinction between two curves, even irreducible ones with the same Hilbert function. For instance, if X is a reduced, irreducible ACM curve then it has arithmetically Gorenstein subsets of arbitrarily large degree [**14**]. However, if X is irreducible but not ACM then we will see that there is a bound on the degree of an arithmetically Gorenstein zero-dimensional subscheme that it contains (Proposition 6.8). Thus, the question of whether the number of degrees of arithmetically Gorenstein subschemes of an integral curve is finite or infinite cannot be solved (in general) only from knowing the Hilbert function of the curve.

In this note we will develop a different approach to this problem than that of Harris. In Lemma 3.6 we give several equivalent conditions for a standard graded algebra to have the same Hilbert function as that of a Cohen-Macaulay ideal of codimension two. Some are from the perspective of the graded Betti numbers, and some from the perspective of the Hilbert function, especially the h-vector. These lead to Algorithm 4.3 in Section 4 that starts with an arbitrary homogeneous ideal and produces, after a finite number of repetitions of a construction called *basic double linkage*, an ideal with the Hilbert function of an ACM codimension two subscheme. We call such an ideal *numerically ACM*. For a later application we also provide an alternative algorithm (cf. Algorithm 4.6) that also takes an arbitrary ideal to one with the Hilbert function of an ACM codimension two subscheme. In fact, we show that its resulting ideal has the same Hilbert function as the ideal obtained from the original ideal via Algorithm 4.3 (see Proposition 4.9) though it typically has more minimal generators than the latter. We give examples in Section 7.

In the event that our original ideal is the ideal of an unmixed codimension two subscheme of $\mathbb{P}^n$, we show something more: if $\mathcal{L}$ is an even liaison class of codimension two subschemes of $\mathbb{P}^n$ then the subset, $\mathcal{M}$, of numerically ACM schemes in $\mathcal{L}$ satisfies a Lazarsfeld-Rao property much like that satisfied by $\mathcal{L}$ itself. This is Theorem 5.11. To achieve this, we combine results about the Lazarsfeld-Rao property in $\mathcal{L}$ with an analysis of the change of the Hilbert function when carrying out Algorithm 4.6.

In Section 6 we generalize the example of Harris, following our approach, and show in Theorem 6.3 that in any liaison class of space curves corresponding to a Hartshorne-Rao module of diameter one and dimension n, there are smooth, maximal rank, numerically ACM curves. The example of Harris is the first case, $n = 1$.

2. Harris's original example

With his permission, we first give the example, due to Joe Harris, of two smooth curves C, C' with the same Hilbert function, one ACM and the other not. We quote directly from [**12**].

The point is, to say the Hilbert function is the same for C and C' is to say the ranks of the maps

$$\rho_n : H^0(\mathbb{P}^3, \mathcal{O}(n)) \to H^0(C, \mathcal{O}(n)) \quad \text{and} \quad \rho'_n : H^0(\mathbb{P}^3, \mathcal{O}(n)) \to H^0(C', \mathcal{O}(n))$$

are the same for all n (of course the degree and genus of C and C' must therefore be the same); but to say that C is ACM and C' is not means that ρ_n is surjective for all n, while ρ'_n is not, for some n. Thus we must have $h^0(C, \mathcal{O}(n)) \neq h^0(C', \mathcal{O}(n))$ for some n; since it would be simplest if C, C' were linearly normal, we might as well take $n = 2$ here. Finally (just as a matter of personal preference) let's arrange for C' to be semicanonical — i.e. $K_{C'} = \mathcal{O}_{C'}(2)$, so that $h^0(\mathcal{O}_{C'}(2)) = g$ — while C is not, so $h^0(\mathcal{O}_C(2)) = g - 1$. If we at the same time assume that C, C' do not lie on quadric surfaces, this then says that $g = 11, d = 10$.

For C', take S a quartic containing two skew lines L_1, L_2, and let C' be a general member of the linear system $C' \in |\mathcal{O}_S(2)(L_1 + L_2)|$. Since $L_i^2 = -2$ on S, $C' \cdot L_i = 0$, — i.e. C' is disjoint from L_i — so that, inasmuch as $K_S = O_S$,

$$K_{C'} = \mathcal{O}_{C'}(C') = \mathcal{O}_{C'}(2)$$

so C' is semicanonical. Note that C' lies on no cubic surfaces T: if it did, we could write $S \cdot T = C' + D$ and then on S we would have $\mathcal{O}_S(D + L_1 + L_2) = \mathcal{O}_S(1)$; but L_1 and L_2 do not lie in a hyperplane. Thus the Hilbert function of C' is

$$\begin{array}{rcll} h(1) & = & 4 & \\ h(2) & = & 10 & \\ h(3) & = & 20 & \\ h(4) & = & 30 & \\ & \cdots & & \\ h(n) & = & 10n - 10 & (= \text{Hilbert polynomial}) \end{array}$$

You may also recognize C' as being residual, in the intersection of two quartic surfaces S and T, to a curve of type $(2, 4)$ on a quadric; the Hilbert function and semicanonicality can be deduced from this.

As for C, let now S be a quartic surface containing a smooth, non-hyperelliptic curve D of degree 6 and genus 3 in $\mathbb{P}^3$, and let C be a member of the linear series $C \in |\mathcal{O}_S(1)(D)|$. Since D does lie on a cubic surface U, and we can write $U \cap S = D + E$ where E is again a septic of genus 3, C can be described as the curve residual to a non-hyperelliptic sextic of genus 3 in a complete intersection of quartics — i.e. $C \in |\mathcal{O}_S(4)(-E)|$. C is thus linked to a twisted cubic, and so is ACM; likewise, since D lies on no quadrics we see that the Hilbert function of C is equal to that of C'.

3. Background

We will consider homogeneous ideals in the polynomial ring $R = K[x_0, \ldots, x_n]$ or projective subschemes $X \subset \mathbb{P}^n := \mathbb{P}^n$ whose codimension is at least two, where K is an infinite field. Thus, if $I \subset R$ is an ideal of codimension c then the projective dimension of R/I satisfies

$$c \leq \mathrm{pd}(R/I) \leq n + 1$$

and so $\mathrm{pd}(I) \leq n$.

The following result will be a key tool throughout this note (cf., e.g, [**18**] for a proof).

Lemma 3.1. *Let $I \subset R$ be any non-zero ideal. Let $F \in I$ be any non-zero element of degree d, and let $G \in R$ be a general form of degree a. Consider the ideal $J = G \cdot I + (F)$. Then*

(a) *J has codimension two.*

(b) *If I has a minimal free resolution*

$$0 \to \mathbb{F}_n \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_1 \to I \to 0$$

then J has a free resolution

$$0 \to \mathbb{F}_n(-a) \to \mathbb{F}_{n-1}(-a) \to \cdots \to \mathbb{F}_3(-a) \to \begin{array}{c} \mathbb{F}_2(-a) \\ \oplus \\ R(-d-a) \end{array} \to \begin{array}{c} \mathbb{F}_1(-a) \\ \oplus \\ R(-d) \end{array} \to J \to 0.$$

The only possible cancellation is a copy of $R(-a-d)$ between the first and second free modules, which occurs if and only if there is a minimal generating set for I that includes F.

(c) *Let $\mathfrak{a} = \langle F, G \rangle$. We have the formula for the Hilbert functions:*

$$h_{R/J}(t) = h_{R/I}(t-a) + h_{R/\mathfrak{a}}(t).$$

(d) *There are graded isomorphisms between local cohomology modules with support in the maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$*

$$H^i_{\mathfrak{m}}(R/J) \cong H^i_{\mathfrak{m}}(R/I)(-a)$$

whenever $i \leq n-2$.

(e) *If we have $I = I_X$, the saturated ideal of a subscheme $X \subset \mathbb{P}^n$, then J is also a saturated ideal, defining a codimension two subscheme X_1.*

(f) *If X is unmixed of codimension two, then the degree of X_1 is $ad + \deg X$, and, as sets, X_1 is the union of X and the complete intersection of F and G. Furthermore, in this case X is linked to X_1 in two steps.*

Definition 3.2. The ideal J produced in Lemma 3.1 will be called a *basic double link of I of type (d,a)*. We also sometimes call a the *height* of the basic double link.

Notice that if X is not unmixed of codimension two then the construction given in Lemma 3.1 is not actually related to linkage, but we retain the terminology since it is the standard one. We also note that basic double linkage has been generalized, in the context of Gorenstein liaison, to a construction called *basic double G-linkage* – cf. [**14**].

Consider a free resolution

$$0 \to \mathbb{F}_n \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0.$$

Let D be the associated Betti diagram. It is well-known that the Hilbert function of R/I can be computed from D, and that any free summand $R(-a)$ that occurs in consecutive $\mathbb{F}_i$ does not contribute to the Hilbert function computation, and so can be canceled from the numerical computation, even if removing it produces a diagram that is not the Betti diagram of any module. Several papers have used the idea of formally canceling free summands in this way, in order to obtain useful numerical information (e.g. [**8**], [**23**], [**7**], [**17**]). We make a small extension of this idea by applying it to a diagram that may or may not be a Betti diagram.

DEFINITION 3.3. Consider a finite diagram, $D = (d_{i,j})$, with $0 \le j \le n$, $0 \le i$, $d_{0,0} = 1$, and $d_{i,0} = 0$ for $i \ge 2$. A *numerical reduction* of D is a diagram obtained as follows. Whenever $d_{i,j} > 0$ and $d_{i-1,j+1} > 0$, let $d = \min\{d_{i,j}, d_{i-1,j+1}\}$. Then replace $d_{i,j}$ by $d_{i,j} - d$ and replace $d_{i-1,j+1}$ by $d_{i-1,j+1} - d$. Two ideals, I_1 and I_2, are *numerically equivalent* if their Betti diagrams admit the same numerical reduction.

Clearly two numerically equivalent ideals have the same Hilbert function. Note that a given diagram may have more than one numerical reduction.

REMARK 3.4. Fix integers

$$s_1 \ge s_2 \ge \cdots \ge s_{\nu-1} > 0, \qquad r_1 \ge r_2 \ge \cdots \ge r_\nu > 0$$

such that $\sum_{j=1}^{\nu} r_j = \sum_{i=1}^{\nu-1} s_i$. Consider the $(\nu-1) \times \nu$ integer matrix

$$A = (s_i - r_j).$$

Notice that the columns are non-decreasing from bottom to top, and the rows are non-decreasing from left to right. Sauer [**24**] remarks without proof (page 84) that there is an ACM curve $Y \subset \mathbb{P}^3$ with free resolution

$$0 \to \bigoplus_{i=1}^{\nu-1} R(-s_i) \to \bigoplus_{j=1}^{\nu} R(-r_j) \to I_Y \to 0 \tag{3.1}$$

if and only if the entries of the main diagonal of A are non-negative. We make some additional remarks.

(1) If we restrict to minimal free resolutions, then we have the stronger condition that the entries of the main diagonal are strictly positive.

(2) If we do not restrict to minimal free resolutions, then the criterion of Sauer is not correct. For instance, choosing $s_1 = 4$ and $r_1 = r_2 = 2$ clearly gives the Koszul resolution for a complete intersection of type $(2, 2)$, but adding a trivial summand $R(-1)$ to both free modules produces a non-minimal free resolution with a negative entry on the main diagonal.

We will adjust Sauer's criterion. It works equally well for ACM codimension two subvarieties of $\mathbb{P}^n$. Before stating the result, let us introduce some notation. We denote the Hilbert function of R/I by $h_{R/I}(j) = \dim_K[R/I]_j$. A sequence $(h_0, h_1, \ldots)$ of non-negative integers is called an *O-sequence* if it is the Hilbert function of some standard graded K-algebra A, that is, $h_j = h_A(j)$ for all j. O-sequences can also be characterized by an explicit numerical growth condition (see, e.g., [**6**, Theorem 4.2.10]).

If I has codimension two, then the Hilbert series of R/I can be written as

$$\sum_{j \ge 0} h_{R/I}(j) z^j = \frac{h_0 + h_1 z + \cdots + h_e z^e}{(1-z)^{n-1}},$$

where $h_e \neq 0$. Then $h = (h_0, \ldots, h_e)$ is called the h-vector of R/I. Equivalently, the h-vector is the list of non-zero values of the $(n-1)$st difference of the Hilbert function, $\Delta^{n-1} h_{R/I}$, where $\Delta h_{R/I}(j) = h_{R/I}(j) - h_{R/I}(j-1)$ and $\Delta^i h_{R/I} = \Delta(\Delta^{i-1} h_{R/I})$ if $i \ge 1$. Abusing terminology, in this note the following convention is helpful.

DEFINITION 3.5. If $I \subset R$ is a homogenous ideal whose codimension is at least two, then we define the *h-vector* of R/I to be the non-zero values of $\Delta^{n-1} h_{R/I}$.

Then we have:

LEMMA 3.6. *Let $I \subset R$ be a homogeneous ideal whose codimension is at least two. Consider a (not necessarily minimal) free resolution of R/I:*

$$0 \to \bigoplus_{k\geq 1}(-c_{n,k}) \to \bigoplus_{k\geq 1}(-c_{n-1,k}) \to \cdots \to \bigoplus_{k\geq 1}(-c_{2,k}) \to \bigoplus_{k\geq 1}(-c_{1,k}) \to R \to R/I \to 0.$$

Let

$$\underline{s} = \{s_i\} = \{c_{p,q} \mid p \text{ is even}\}, \quad \underline{r} = \{r_j\} = \{c_{p,q} \mid p \text{ is odd}\},$$

and assume that the sets $\underline{s}$ and $\underline{r}$ are ordered so that

$$s_1 \geq s_2 \geq \cdots \geq s_{\nu-1}, \qquad r_1 \geq r_2 \geq \cdots \geq r_\nu.$$

Note that

$$\sum_{j=1}^{\nu} r_j = 1 + \sum_{i=1}^{\nu-1} s_i.$$

Assume that $s_{\nu-1} > r_\nu \geq 1$. Then the following conditions are equivalent:

(a) *I has the same Hilbert function as that of a CM ideal of codimension two.*
(b) *The h-vector of R/I is an O-sequence.*
(c) *For every integer $k \geq r_\nu$,*

$$\#\{r_i \mid r_i \leq k\} > \#\{s_j \mid s_j \leq k\}.$$

(d) *For all $i = 1, \ldots, \nu-1$, $\quad s_i \geq r_i$.*
(e) *There is a (CM) ideal $J \subset R$ of codimension two having a free resolution of the form*

$$0 \to \bigoplus_{i=1}^{\nu-1} R(-s_i) \to \bigoplus_{j=1}^{\nu} R(-r_j) \to J \to 0 \tag{3.2}$$

As preparation for its proof we need:

LEMMA 3.7. *Using the notation of the preceding lemma, the h-vector of R/I can be computed using, for each k, the formula*

$$h(k) = k + 1 - \sum_{r_i \leq k}(k - r_i + 1) + \sum_{s_j \leq k}(k - s_j + 1). \tag{3.3}$$

It follows that

$$h(k+1) = h(k) + 1 - \#\{r_i \mid r_i \leq k+1\} + \#\{s_j \mid s_j \leq k+1\}. \tag{3.4}$$

PROOF. The additivity of vector space dimension along exact sequences provides for each integer k

$$h_{R/I}(k) = h_R(k) - \sum_i h_R(k - r_i) + \sum_j h_R(k - s_j).$$

The claim follows by passing to the $(n-1)$st differences as $\Delta^{n-1}h_R(k) = k + 1$ if $k \geq 0$. $\square$

Now we are ready to come back to Lemma 3.6.

Proof of Lemma 3.6. This is probably known to specialists. However, for the convenience of the reader we provide a brief argument.

If R/J is Cohen-Macaulay of dimension $n-1$, then its h-vector is the Hilbert function of $R/(I, \ell_1, \dots, \ell_{n-1})$, where $\ell_1, \dots, \ell_{n-1} \in R$ are general linear forms. Thus, (b) is a consequence of (a).

Lemma 3.7 shows that (c) follows from (b) because the h-vector is weakly decreasing after it stopped to increase strictly.

It is elementary to check that (c) provides (d).

Assume Condition (d) is true. Then let $A = (a_{i,j})$ be the $(\nu - 1) \times \nu$ matrix such that $a_{i,i} = x_0^{s_i - r_i}$, $a_{i,i+1} = x_1^{s_i - r_{i+1}}$, and $a_{i,j} = 0$ if $j \neq i, i+1$. Then the ideal J generated by the maximal minors of A has codimension two and a free resolution of the form (3.2), establishing Condition (e).

Condition (a) follows from (e) as I and J have the same Hilbert function. $\square$

Remark 3.8. In Lemma 3.6 we used the h-vector of R/I, which, by abuse of notation, we defined to be the $(n-1)$st difference of the Hilbert function. Equivalently, we could have used the γ-character, which is the nth difference of the Hilbert function. The conditions in Lemma 3.6 are also equivalent to the positivity of the γ-character as defined in [**16**, Definition V.1.1]. Moreover, using γ-characters, the equivalence of conditions (a) and (b) in Lemma 3.6 is shown in [**16**, Theorem V.1.3].

We need the following consequence of Lemma 3.6.

Corollary 3.9. (a) *Fix integers $s_1 \geq s_2 \geq \cdots \geq s_{\nu-1} > 0$ and $r_1 \geq r_2 \geq \cdots \geq r_\nu > 0$ such that $\sum_{j=1}^{\nu} r_j = \sum_{i=1}^{\nu-1} s_i$. Consider the $(\nu-1) \times \nu$ integer matrix $A = (s_i - r_j)$. Then there is a codimension two subscheme $Y \subset \mathbb{P}^n$ with minimal free resolution* (3.1) *if and only if the entries of the main diagonal of A are strictly positive.*

(b) *Let $\underline{s}$ and $\underline{r}$ be sets satisfying the conditions in (a). Suppose that equal entries are added to both sets, all $\geq r_\nu$ (corresponding to the addition of trivial summands $R(\ )$ to both free modules in the resolution). Let $f(t) = \#\{i \mid s_i \leq t\}$ and $g(t) = \#\{j \mid r_j \leq t\}$. Then, for all $t \geq r_\nu$, we have $g(t) > f(t)$.*

4. Numerical Macaulification: Two Algorithms

In this section we will introduce some terminology used throughout the paper. The main goal of the section, though, is to give two algorithms to produce, from an arbitrary ideal I of height ≥ 2, a numerically ACM ideal, using only basic double linkage. The Hilbert functions of the two resulting ideals turn out to be equal. Thus we will call the end result of these constructions the *numerical Macaulification* of I. The result is numerically unique but not unique as an ideal, since there are two algorithms, and even within one algorithm there are several choices of polynomials (of fixed degree).

Definition 4.1. A homogeneous ideal $J \subset R$ is *numerically r-ACM* if R/J has the Hilbert function of some codimension r ACM subscheme of $\mathbb{P}^n$. When $r = 2$ we will simply say that J is *numerically ACM.*

The main result of this section is the following.

Theorem 4.2. *If I is an ideal whose codimension is at least two, then there is a finite sequence of basic double links, starting from I, that results in an ideal that is numerically ACM.*

To achieve this, we give two algorithms. Then we will compare the algorithms to see that they result in ideals with the same Hilbert function.

Algorithm 4.3. Let $I \subset R$ be a homogeneous ideal of height ≥ 2. Consider a minimal free resolution of R/I:

$$0 \to \bigoplus_{k\geq 1}(-c_{n,k}) \to \bigoplus_{k\geq 1}(-c_{n-1,k}) \to \cdots \to \bigoplus_{k\geq 1}(-c_{2,k}) \to \bigoplus_{k\geq 1}(-c_{1,k}) \to R \to R/I \to 0.$$

(1) Let

$$\underline{s} = \{s_i\} = \{c_{p,q} \mid p \text{ is even}\}, \quad \underline{r} = \{r_j\} = \{c_{p,q} \mid p \text{ is odd}\},$$

and assume that the sets $\underline{s}$ and $\underline{r}$ are ordered so that the entries are non-increasing. Note that $1 + \sum s_i = \sum r_j$.

(2) Remove equal elements r_j and s_i pairwise one at a time. For convenience of notation, we will still call the sets $\underline{s}$ and $\underline{r}$. So now we may assume that $\underline{s}$ and $\underline{r}$ are disjoint sets.

(3) Form the matrix $A = (s_i - r_j)$. Let $\{-d_1, \ldots, -d_\ell\}$ be the negative entries of A on the main diagonal. Assume for convenience that they are ordered according to *non-decreasing* values of r_j. (That is, we are taking the negative entries of the main diagonal beginning from the bottom right and moving up and left, regardless of the values of the d_k.)

(4) **(Main step)** Say that $-d_1 = s_{i_1} - r_{i_1} < 0$ (since it is on the main diagonal). Using general polynomials, let J be the ideal obtained from I by a basic double link of type (r_{i_1}, d_1).

(5) Repeat steps (1) – (4) for J. Continue repeating until there are no longer negative entries on the main diagonal.

Proposition 4.4. *This algorithm terminates, and the result is an ideal that is numerically ACM. Thus, we define the resulting ideal to be a* numerical Macaulification *of I.*

Proof. Observe first that thanks to Lemma 3.1, the Betti numbers (up to one possible cancellation) and hence the Hilbert function of the resulting ideal depend only on the degrees of the polynomials used. Let I be the original ideal, and J the result of performing steps (1) to (4). As a result of step (2), associated to I are the sets $\underline{s} = \{s_i\}$ and $\underline{r} = \{r_j\}$, with no common entries. Then J has a (not necessarily minimal) free resolution with Betti numbers giving new lists

$$\underline{s}' = \{s_i + d_1\} \ \cup \ \{r_{i_1} + d_1\}, \quad \underline{r}' = \{r_j + d_1\} \ \cup \ \{r_{i_1}\}.$$

(Note that $\{s_i + d_1\}$ and $\{r_j + d_1\}$ contain, in general, more than one element, while $\{r_{i_1} + d_1\}$ and $\{r_{i_1}\}$ are sets with one single element.) Thus $\underline{s}'$ contains at least one $s_{i_1} + d_1 = r_{i_1}$ and at least one $r_{i_1} + d_1$, and $\underline{r}'$ contains at least one $r_{i_1} + d_1$ and at least one r_{i_1}. In performing step (2) for J, we remove these two entries from both lists. One checks that as the result of this removal, the new matrix A is obtained from the original one by removing row i_1 and column i_1. Thus the new matrix has the same entries on the main diagonal as the original one, except that one negative entry (namely $-d_1$) has been removed. Thus the algorithm terminates. But one result of the algorithm are two lists, $\underline{s}'$ and $\underline{r}'$, satisfying the conditions of Lemma 3.6. (Notice that the construction guarantees that A will have no entries that are equal to 0, in particular on the main diagonal.) Since the Hilbert function of R/J (where J is the result of the completion of the algorithm) can be computed from $\underline{r}'$ and $\underline{s}'$ and seen to be the same as that of the ACM subscheme determined by $\underline{r}'$ and $\underline{s}'$, the result follows from Lemma 3.6. $\square$

EXAMPLE 4.5. Let $I = (w^3, x^3) \cap (y^3, z^3) \subset k[w, x, y, z]$. This curve has Betti diagram

```
           0    1    2    3
-------------------------
 0:        1    -    -    -
 1:        -    -    -    -
 2:        -    -    -    -
 3:        -    -    -    -
 4:        -    -    -    -
 5:        -    4    -    -
 6:        -    -    -    -
 7:        -    -    4    -
 8:        -    -    -    -
 9:        -    -    -    1
-------------------------
Tot:       1    4    4    1
```

and h-vector $(1, 2, 3, 4, 5, 6, 3, 0, -3, -2, -1)$.

The two lists $\underline{s}$ and $\underline{r}$ are

$$\underline{s} = \{9, 9, 9, 9\}, \quad \underline{r} = \{12, 6, 6, 6, 6\}$$

so the matrix has the form

$$A = \begin{bmatrix} \boxed{-3} & 3 & 3 & 3 & 3 \\ -3 & 3 & 3 & 3 & 3 \\ -3 & 3 & 3 & 3 & 3 \\ -3 & 3 & 3 & 3 & 3 \end{bmatrix} \tag{4.1}$$

with only one negative entry in the main diagonal. Thus we perform only one basic double link, using $\deg G = 3$ and $\deg F = 12$. The h-vector of the resulting ideal, J, is computed using Lemma 3.1:

$$\begin{array}{cccccccccccccc} & & & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 0 & -3 & -2 & -1 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 6 & 3 & & & \end{array}$$

and Betti diagram

```
           0    1    2    3
-------------------------
 0:        1    -    -    -
 1:        -    -    -    -
           ...
 7:        -    -    -    -
```

8:	-	4	-	-
9:	-	-	-	-
10:	-	-	4	-
11:	-	1	-	-
12:	-	-	-	1
13:	-	-	1	-
Tot:	1	5	5	1

ALGORITHM 4.6. Let $I \subset R = K[x_0, \dots, x_n]$ be a homogeneous ideal of height ≥ 2. Assume for convenience that I contains no linear forms (otherwise use a smaller R). Let $h_{R/I}$ be the Hilbert function of R/I, and consider the $(n-1)$-nd difference

$$\underline{a} = \Delta^{n-1} h_{R/I} = (1, a_1, \dots, a_e)$$

where a_e is the last non-zero value and $a_1 \leq 2$. Note that this is a finite sequence, and if I has codimension two then this is the h-vector of R/I. Repeat the following step until $\underline{a}$ becomes is an O-sequence:

(∗) *If $\underline{a}$ is not an O-sequence then set $a_{e+1} = 0$ and let i be the smallest index such that $a_i \leq i$ and $a_i < a_{i+1}$. Let $F \in I$ be a form of degree $i+2$, and let $J = L \cdot I + (F)$, with L a general linear form. Set $\underline{b}$ to be the $(n-1)$-nd difference of $h_{R/J}$.*

Note that again this algorithm uses basic double links, but this time always of type $(d, 1)$ for different d. We illustrate its idea.

EXAMPLE 4.7. Let C be the general union of a line C_1, a plane cubic C_2, and a curve C_3 that is linked to a line in a complete intersection of type $(4, 8)$. The Betti diagram for R/I_C has the form

	0	1	2	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	-	-	-
5:	-	2	1	-
6:	-	-	-	-
7:	-	2	3	1
8:	-	-	-	-
9:	-	2	1	-
10:	-	1	-	-
11:	-	1	5	2
12:	-	1	3	2
Tot:	1	9	13	5

The two lists $\underline{s}$ and $\underline{r}$ are

$$\underline{s} = \{14, 14, 14, 13, 13, 13, 13, 13, 11, 9, 9, 9, 7\},$$

$$\underline{r} = \{15, 15, 14, 14, 13, 12, 11, 10, 10, 10, 8, 8, 6, 6\}.$$

Thus, the h-vector of the original curve is

```
[1, 2, 3, 4, 5, 6, 5, 5, 3, 4, 2, 0, -3, -2]
```

Following Algorithm 4.6, we perform four basic double links, using, successively, forms $F_1, \dots, F_4$ of degree $\deg F_1 = 10, \deg F_2 = 14+1 = 15, \deg F_3 = 15+1+1 = 17$, and $\deg F_4 = 15+1+1+1 = 18$. The h-vectors of the successive basic double links are

```
[1, 2, 3, 4, 5, 6, 7, 6, 6, 4, 4, 2, 0, -3, -2]
------------------------------
[1, 2, 3, 4, 5, 6, 7, 8, 7, 7, 5, 5, 3, 1, -2, -2]
------------------------------
[1, 2, 3, 4, 5, 6, 7, 8, 9, 8, 8, 6, 6, 4, 2, -1, -1]
------------------------------
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 9, 9, 7, 7, 5, 3]
```

whereby we recognize that only the last has the Hilbert function of an ACM curve. We remark that each of these h-vectors is obtained by shifting the previous h-vector by one and adding a vector consisting of ($\deg F_i$) 1's. For example, the first h-vector above is obtained by

$$\begin{array}{ccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 5 & 3 & 4 & 2 & 0 & -3 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 & 6 & 4 & 4 & 2 & 0 & -3 & -2 \end{array}$$

The effect of this first basic double link is to "fix" the growth from 3 to 4 (from degree 8 to degree 9) in the original h-vector, which violates maximal growth of the Hilbert function. This illustrates the idea of the proof of the next result; indeed, a basic double link using a form of smaller degree would be "wasted," since it would not serve to change any part of the h-vector that fails to be an O-sequence, and any basic double link using a form of larger degree would forever eliminate our ability to "fix" the impossible growth from 3 to 4, since subsequent basic double links use forms of strictly larger degrees. Thus in order to obtain a numerically ACM curve in the fewest possible steps, the first basic double link *must* be using a form of degree 10. The other three basic double links are similarly forced. For purposes of comparing with Algorithm 4.3, in Example 7.4 we give the Betti table of the numerically ACM curve thus obtained.

PROPOSITION 4.8. *Algorithm* 4.6 *terminates, and the result is an ideal J' that is numerically ACM. Furthermore, the degrees of the forms F used in the repeated applications of step (*) are strictly increasing.*

PROOF. Observe that if we apply step (*), $\underline{b}$ is obtained by the computation

$$\begin{array}{cccccccccc} & 1 & 2 & a_2 & a_3 & \dots & a_i & a_{i+1} & \dots & a_e \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 & & & \\ \hline 1 & 2 & 3 & a_2+1 & a_3+1 & \dots & a_i+1 & a_{i+1} & \dots & a_e \end{array}$$

By assumption, this is an O-sequence up to and including degree $i+1$ (where the value is $b_{i+1} = a_i + 1$), and the possible failure to be an O-sequence from (now) degree $i+1$ to degree $i+2$ has decreased by one. If it is still not an O-sequence in this degree (because $b_{i+1} = a_i + 1 < a_{i+1} = b_{i+2}$), for the next step we choose F of degree $(i+1)+2 = i+3$, so the degrees of the forms are strictly increasing. If $a_i + 1 = a_{i+1}$ then $\deg F$ increases even more in the next application of step (*).

As long as $a_{i+1} < i+2$, we shall define the *deficit in degree* i to be $\delta_i = \max\{0, a_{i+1} - a_i\}$ and the *deficit* to be $\delta = \sum \delta_i$. An algebra is numerically ACM if and only if its deficit is zero. Clearly each application of step (*) reduces the deficit by one, hence the result follows. □

We now compare the results of Algorithms 4.3 and 4.6 when they are applied to the same ideal.

PROPOSITION 4.9. *Let I be a homogeneous ideal that is not Cohen-Macaulay and whose codimension is at least two. Then the numerical Macaulification J produced from I by Algorithm* 4.3 *has the same Hilbert function as the ideal J' produced from I by Algorithm* 4.6. *Furthermore, $\sum d_i = \delta$, where δ is the number of basic double links applied in that algorithm and*

$$H^i_{\mathfrak{m}}(R/J) \cong H^i_{\mathfrak{m}}(R/J')$$

whenever $i \leq n-2$.

PROOF. In order to show the claim about the Hilbert function it is enough to prove that the two ideals have the same $(n-1)$-nd differences of their Hilbert functions. Since all ideals involved except possibly I have codimension two, we will refer to these $(n-1)$-nd differences as h-vectors. Let $\underline{s} = \{s_1 \geq \cdots \geq s_{\nu-1}\}$ and $\underline{r} = \{r_1 \geq \cdots \geq r_\nu\}$ be the sequences of integers obtained after applying Steps (1) and (2) of Algorithm 4.3. In the proof of Proposition 4.8 we have seen that the smallest number of basic double links of type $(t,1)$ that can be used to obtain a numerically Cohen-Macaulay ideal starting with I is the deficit

$$\delta = \sum_{i \geq r_\nu} \max\{0, h_{i+1} - h_i\}$$

and that Algorithm 4.6 uses exactly δ such basic double links. Note that r_ν is the least index i such $h_i \leq i$. Moreover, we know the h-vector of the ideal J' obtained by Algorithm 4.6. Indeed, denote by $(1, h_1, \ldots, h_e)$ the h-vector of R/I, and let m be the smallest index i such that $i \geq r_\nu$ and $h_i < h_{i+1}$. Then the first entries of the h-vector $(1, h'_1, \ldots, h'_{e+\delta})$ of R/J' are given by

$$h'_i = \begin{cases} i+1 & \text{if } i < \delta \\ h_{i-\delta} + \delta & \text{if } \delta \leq i \leq m + \delta \end{cases}$$

Suppose that the original h-vector failed to be an O-sequence in another degree. Denote by m' the smallest index i such that $i > m$ and $h_i < h_{i+1}$. Then the next entries of the h-vector of R/J' are given by

$$h'_i = h_{i-d} + \delta' \quad \text{whenever} \quad m + \delta < i \leq m' + \delta,$$

where

$$\delta' = \sum_{i > m} \max\{0, h_{i+1} - h_i\}.$$

Continuing in this fashion we get the h-vector of R/J'.

We now analyze Algorithm 4.3. To simplify notation, let k be the largest index such that $r_k > s_k$, and set $d = d_1 = r_k - s_k$. Then the main step of Algorithm 4.3 says that we should perform a basic double link of type (r_k, d). Denote the resulting ideal by $\widetilde{J}$. We want to compare the h-vectors of R/I and $R/\widetilde{J}$.

By the choice of k, we know that $r_k > s_k \geq s_{k-1} > r_{k-1}$. This implies that whenever $s_k \leq j \leq r_k - 1$

$$\#\{r_i \mid r_i \leq j\} = \nu - k$$

and

$$\#\{s_i \mid s_i \leq j\} \geq \nu - k.$$

Hence Lemma 3.7 provides

$$h_{s_k-1} < h_{s_k} < \cdots < h_{r_k-1} \tag{4.2}$$

and that $m = s_k - 1$ is the smallest index i such that $i \geq r_1$ and $h_i < h_{i+1}$.

Since $d = r_k - s_k < r_k$, a complete intersection of type (r_k, d) has h-vector

$$(1, 2, \ldots, d, d, \ldots, d, d-1, \ldots, 2, 1),$$

where the last entry is in degree $d + r_k - 2$. Hence, by Lemma 3.1 the h-vector $(1, a_1, \ldots, a_s)$ of $R/\widetilde{J}$ satisfies

$$a_{r_k-1+j} = h_{s_k-1+j} + \max\{0, d-j\} \tag{4.3}$$

whenever $j \geq 0$. This means, in particular, that h_{s_k-1} is increased by d, h_{s_k} is increased by $d-1$, $\ldots$, h_{r_k-2} is increased by 1. Comparing with Equation (4.2), it follows that the deficit of the original ideal I is decreased, in one step, by a total of $d = d_1 = r_k - s_k$. Repeating the argument we see that the basic double links used in Algorithm 4.3 reduce the deficit by $\sum d_i$. Since the result of this algorithm is numerically Cohen-Macaulay by Proposition 4.4, i.e., the deficit for the h-vector of R/J is zero, we conclude that

$$\delta = \sum_{r_i > s_i} (r_i - s_i) = \sum d_i.$$

Applying Lemma 3.1, the result about the local cohomology modules follows.

It remains to show that R/J and R/J' have the same h-vector. Indeed, we have seen above that the first basic double link used in Algorithm 4.3 reduces the deficit by one in each of d consecutive degrees beginning with the leftmost possible degree. If needed, the second basic double link similarly reduces the deficit beginning with the then leftmost possible degree. Comparing with the above description for obtaining the h-vector of J' from the one of I, it follows that the numerical Macaulification J computed by Algorithm 4.3 has the same h-vector as J'. This concludes our argument. $\square$

Example 4.10. If we apply Algorithm 4.6 to the curve in Example 4.5, we make the following computation to get the h-vector of the resulting numerically ACM curve:

$$\begin{array}{cccccccccccccc}
 & & & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 0 & -3 & -2 & -1 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 6 & 3 & & &
\end{array}$$

which agrees with the one given in Algorithm 4.3.

Although the results of Algorithms 4.3 and 4.6 are numerically equivalent, the numerical Macaulification produced by the former algorithm will typically have fewer minimal generators than the result of the latter algorithm.

5. The Lazarsfeld-Rao property

The goal of this section is to understand the role played by the numerically ACM schemes within a fixed even liaison class of codimension two subschemes of $\mathbb{P}^n$. It has been shown in a sequence of papers including [**15**], [**1**], [**16**], [**19**] and [**21**] that any such even liaison class satisfies the so-called *Lazarsfeld-Rao property (LR-property)*, recalled below. In this section we show that within an even liaison class, the subclass of ideals that are numerically ACM itself satisfies a Lazarsfeld-Rao property. Throughout this section, we consider unmixed codimension two ideals.

We first recall the Lazarsfeld-Rao property, summarized as follows.

Theorem 5.1 (Lazarsfeld-Rao (LR) Property). *Let $\mathcal{L}$ be an even liaison class of unmixed codimension two subschemes of $\mathbb{P}^n$. Assume that the elements of $\mathcal{L}$ are not ACM, so that, for all $X \in \mathcal{L}$, we have $M^i := \bigoplus_t H^i(\mathbb{P}^n, \mathcal{I}_X(t)) \neq 0$ for at least one i, $1 \leq i \leq n-2$. Then we have a partition $\mathcal{L} = \mathcal{L}^0 \cup \mathcal{L}^1 \cup \mathcal{L}^2 \cup \dots$, where $\mathcal{L}^0$ is the set of those $X \in \mathcal{L}$ for which M^i has the leftmost possible shift, and $\mathcal{L}^h$ is the set of those elements of $\mathcal{L}$ for which M^i is shifted h places to the right (this partition does not depend on the choice of i). Furthermore, we have*

(a) *If $X_1, X_2 \in \mathcal{L}^0$ then there is a flat deformation from one to the other through subschemes all in $\mathcal{L}^0$, which furthermore preserves the Hilbert function.*
(b) *Given $X_0 \in \mathcal{L}^0$ and $X \in \mathcal{L}^h$ $(h \geq 1)$, there exists a sequence of basic double link schemes (see Lemma 3.1 and Definition 3.2) $X_0, X_1, \dots, X_t$ such that for all j, $1 \leq j \leq t$, X_j is a basic double link of X_{j-1}, and X is a deformation of X_t through subschemes all in $\mathcal{L}^h$, all of which furthermore have the same Hilbert function.*

Remark 5.2. In the literature, it is sometimes stated that the deformations are through subschemes "with the same cohomology" (referring to the cohomology of the ideal sheaf). This is equivalent to what we have stated, since the fact that they lie in the same $\mathcal{L}^h$ fixes the i-th cohomology for $1 \leq i \leq \dim X$, the preservation of the Hilbert function fixes the zeroth cohomology, and then the equality of the Hilbert polynomials fixes the $(\dim X + 1)$-st cohomology (and the rest are determined by $\mathbb{P}^n$).

We begin by recalling some technical results that will be important in this section. They were not originally formulated in this generality, but the same proofs work.

Lemma 5.3 ([**5**], Proposition 3.1). *Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$, and let $X, Y \in \mathcal{L}^h$ be elements such that X and Y have the same Hilbert function. Then there exists an irreducible flat family $\{X_s\}_{s \in S}$ of codimension two subschemes of $\mathbb{P}^n$ to which both X and Y below. Moreover, S can be chosen so that for all $s \in S$, $X_s \in \mathcal{L}^h$ and X_s has the same Hilbert function as X and Y.*

We remark that the conclusion that $X_s \in \mathcal{L}^h$ is a very strong one: it means that all the elements of the flat family are in the same even liaison class, and their modules have the same shift.

Lemma 5.4 ([**5**] Corollary 3.4). *If X, Y are codimension two subschemes of $\mathbb{P}^n$, both in $\mathcal{L}^h$, and if the general hyperplane section of X has the same Hilbert* *function as the general hyperplane section of Y, then X and Y belong to the same flat family, with the properties described in Lemma* 5.3.

Lemma 5.5 ([**5**], Corollary 3.9 (b)). *Let I be an ideal (not necessarily of codimension two). Then the ideal obtained from I by a basic double link of type (d, a) is numerically equivalent to the ideal obtained from I by a sequence of a basic double links of type $(d, 1)$. In particular, these ideals have the same Hilbert function.*

Lemma 5.6 ([**5**], in proof of Lemma 5.2). *Let I be an ideal. Define $a_0 = \min\{t \mid [I]_t \neq 0\}$. Let $a, b \geq a_0$ be integers. Assume that $b \neq a_0$. Let J_1 be the ideal obtained from I by a sequence of two basic double links, first of type $(a, 1)$ and then of type $(b, 1)$ (assume that this is possible). Then it is also possible to do a sequence of basic double links of types $(b-1, 1)$ and $(a+1, 1)$, resulting in an ideal J_2 that is numerically equivalent to J_1.*

Lemma 5.7 ([**5**], Section 5). *Let $\mathcal{L}$ be an even liaison class of codimension two subschemes. Let X_0 be an arbitrary minimal element of $\mathcal{L}$. Let $X \in \mathcal{L}$ be an arbitrary element. Assume that $X \in \mathcal{L}^h$. Let a be the initial degree of I_{X_0}. Then associated to X is a uniquely determined sequence of integers $(b, g_2, g_3, \ldots, g_r)$ such that*

1. *$b \geq 0$;*
2. *$a < g_2 < \cdots < g_r$;*
3. *$b + r - 1 = h$;*
4. *X is numerically equivalent to the scheme obtained from X_0 by a sequence of basic double links of types*
$$(a, b), (g_2, 1), (g_3, 1), \ldots, (g_r, 1).$$

Lemma 5.7 implies that up to numerical equivalence (i.e. in this case, up to flat deformation preserving the cohomology of the ideal sheaf, and in particular preserving the Hilbert function), we only have to consider sequences of basic double links as in (4), i.e. beginning with a minimal element and satisfying conditions (1), (2), (3).

The following observation is a consequence of Lemma 3.1.

Lemma 5.8. *If I is numerically ACM and J is the result of applying any basic double link to I, then J is again numerically ACM.*

The next result gives a class of curves, all of which fail to be numerically ACM.

Lemma 5.9. *Let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$. Then the minimal elements of $\mathcal{L}$ are not numerically ACM.*

Proof. Let C be a minimal curve. We know that all minimal curves have the same minimal free resolution. In fact, let $M(C)$ denote the Hartshorne-Rao module of C. Combining a result of Rao ([**22**] Theorem 2.5) with a result of Martin-Deschamps and Perrin ([**16**] Proposition 4.4), we see that there are free modules $\mathbb{F}_4, \mathbb{F}_3, \mathbb{F}_2, \mathbb{F}_1, \mathbb{F}_0$ and $\mathbb{F}$ fitting into the following minimal free resolutions:

$$0 \to \mathbb{F}_4 \to \mathbb{F}_3 \to \mathbb{F}_2 \to \mathbb{F}_1 \to F_0 \to M(C) \to 0$$

$$0 \to \mathbb{F}_4 \to \mathbb{F}_3 \to \mathbb{F} \to I_C \to 0$$

(See also [**19**].) Let

$$\mathbb{F}_4 = \bigoplus R(-c_i), \quad \mathbb{F}_3 = \bigoplus R(-b_i) \quad \mathbb{F} = \bigoplus R(-a_i).$$

Since $M(C)$ is Cohen-Macaulay, we have $\max\{c_i\} > \max\{b_i\}$. As I_C is a height two ideal, we have $\max\{b_i\} > \max\{a_i\}$. It follows that in the matrix A constructed in Algorithm 4.3, the $(1,1)$ entry of the matrix A is < 0. Thus C is not numerically ACM. □

We believe that Lemma 5.9 continues to hold for equidimensional codimension two subschemes, but we do not have a proof. Thus, we propose:

CONJECTURE 5.10. Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$ $(n \geq 4)$. Then the minimal elements of $\mathcal{L}$ are not numerically ACM.

We will prove a Lazarsfeld-Rao structure theorem for the numerically ACM elements of an even liaison class $\mathcal{L}$. We illustrate the idea with an example.

THEOREM 5.11. *Let $\mathcal{L}$ be a codimension two even liaison class of subschemes of $\mathbb{P}^n$. Let $\mathcal{M}$ be the subset of $\mathcal{L}$ consisting of numerically ACM subschemes. Then $\mathcal{M}$ also satisfies the LR property. That is, there are minimal elements of $\mathcal{M}$, unique up to flat deformation preserving the cohomology of the ideal sheaf (hence also the Hilbert function), and every element of $\mathcal{M}$ can be produced from a minimal one by a sequence of basic double links followed by a flat deformation preserving the cohomology.*

Furthermore, the numerical Macaulification of any minimal element in $\mathcal{L}$ is a minimal element in $\mathcal{M}$.

PROOF. In this proof we will heavily use the fact that $\mathcal{L}$ has the LR property. We also use the fact that without loss of generality we can assume that our sequence of basic double links is performed with $\deg G = 1$ and $\deg F_i$ *strictly increasing.* We also use the fact that if C is a minimal element with h-vector $(1, 2, h_2, \dots, h_k)$ (with h_i not necessarily non-negative) and a basic double link of type $(t, 1)$ is performed, then the h-vector of the resulting curve is obtained by shifting that of C by one, and adding a vector of $(\deg F)$ ones as in the previous example. Finally, recall that C is numerically ACM if and only if its h-vector is an O-sequence.

First we produce the minimal elements of $\mathcal{M}$ using Algorithm 4.6. Let C_0 be a minimal element of $\mathcal{L}$, and let $\underline{h} = (1, 2, h_2, h_3, \dots, h_e)$ be its h-vector (recall that minimal elements of $\mathcal{L}$ all have the same h-vector). If all the minimal elements of $\mathcal{L}$ are numerically ACM then it follows from Lemma 5.8 and the LR-property for $\mathcal{L}$ that $\mathcal{M} = \mathcal{L}$, hence $\mathcal{M}$ has the LR-property. (In the case of curves in $\mathbb{P}^3$, we have seen in Lemma 5.9 that minimal elements are never numerically ACM.)

So assume that the minimal elements are not numerically ACM. By Lemma 5.9, $\underline{h}$ is not an O-sequence. Thus there is a least degree i such that $h_i < h_{i+1}$ (possibly $h_i < 0$ as well). Each application of the step $(*)$ of Algorithm 4.6 reduces the deficit by 1 in the leftmost possible degree, and we ultimately obtain an O-sequence, hence a numerically ACM subscheme. This subscheme has deficiency modules (which are not all zero) shifted δ places to the right of that of C_0, where δ is the number of basic double links applied.

To prove the LR-property for $\mathcal{M}$, we must show that

(a) no sequence of fewer than δ basic double links (always with $\deg G = 1$) achieves a numerically ACM subscheme;
(b) the (strictly increasing) degrees of the forms obtaining a numerically ACM subscheme in δ steps is uniquely determined;

(c) any sequence of basic double links using forms of strictly increasing degrees and ending with a numerically ACM subscheme is numerically equivalent to another sequence whose first δ steps are the ones from above (but now the remaining steps are not necessarily using forms of strictly increasing degree).

Parts (a) and (b) give us that the minimal shift of $\mathcal{M}$ can be obtained using this procedure, and the elements lie in a flat family since they have the same Hilbert function and cohomology. Part (c) says that any element of $\mathcal{M}$ can be obtained from a minimal element using basic double linkage. We do not claim that any element of $\mathcal{M}$ can be obtained from a minimal element using basic double links of strictly increasing degree!

We make two observations at this point. First, if instead of applying step $(*)$ of Algorithm 4.6, we make a choice of F of degree $> i + 2$, it results in a situation where no subsequent sequence of basic double links using forms of strictly increasing degree can be numerically ACM, since there will always be a step in the h-vector where the deficit is positive. Second, if we choose F to be of degree $< i + 2$, the number of steps remaining to obtain a numerically ACM subscheme does not drop. These observations together imply (a) and (b).

For (c), suppose that C is a numerically ACM subscheme in $\mathcal{L}$ that is obtained from a minimal element by some sequence of basic double links using forms of strictly increasing degree (after possibly a number of basic double links using forms of least degree), and then deforming. Let δ be the deficit of the minimal element. By the above observations, there are precisely δ of the basic double links that reduce the deficit by one; the others do not change the deficit. Consider the first deficit-reducing basic double link. If it is actually the first basic double link, leave it alone and consider the second deficit-reducing basic double link. If all of the deficit-reducing basic double links come at the beginning of the sequence of basic double links, there is nothing to prove. Otherwise, suppose that the i-th deficit-reducing basic double link is the first one that is not at the i-th step, and suppose that it comes at the $(i + k)$-th step. Taking the $(i + k - 1)$-st and $(i + k)$-th basic double links, we can apply Lemma 5.6, resulting in a new sequence where the first $(i + k - 2)$ basic double links are unchanged, and the $(i + k - 1)$-st one is deficit-reducing. Continuing in this way, we produce a new sequence where the first i basic double links are all deficit-reducing. Then we move to the next deficit-reducing basic double link, and in this way ultimately we produce the desired result of all δ basic double links coming at the beginning of the sequence (but possibly losing the property of being strictly increasing). $\square$

6. A class of smooth numerically ACM curves

In this section we give a class of smooth numerically ACM space curves that contains Harris's example as a special case. We will find such an example in the liaison class $\mathbb{L}_n$ consisting of curves whose Hartshorne-Rao module is K^n, i.e. is n-dimensional, concentrated in one degree. The example of Joe Harris is the case $n = 1$. It will first be necessary to recall some facts. Except as indicated, all of these results can be found in (or deduced from) [**3**].

Our main tool is a result from [**3**] that is stated in the language of *numerical characters*. The numerical character of a curve was introduced by Gruson and

Peskine [**11**]. Rather than give the definition, we give an equivalent formulation from [**9**] that is more useful to us in our setting.

Let C be a curve in $\mathbb{P}^3$ and let H be a general hyperplane. Let $\underline{h}$ be the h-vector of the hyperplane section $Z = C \cap H$. Suppose that σ is the initial degree of I_Z in the coordinate ring of $H = \mathbb{P}^2$. Then $\underline{h}$ reaches a maximum value of σ in degree $\sigma - 1$ (and possibly beyond). The numerical character of C is a σ-tuple of integers defined as follows: First, if $\Delta\underline{h}$ takes a negative value $-k$ in degree t then there are k occurrences of t in the numerical character. These account for all the entries of the numerical character. However, we list the entries of the numerical character in non-increasing order. For example, if Z has h-vector

$$(1, 2, 3, 4, 5, 6, 7, 8, 8, 6, 5, 5, 3, 2, 1)$$

then $\sigma = 8$ and the numerical character is $(15, 14, 13, 12, 12, 10, 9, 9)$. Notice that there is a gap (there is no 11), corresponding to the fact that 5 occurs twice in the h-vector.

Since K^n is self-dual, we may view $\mathbb{L}_n$ as an even liaison class. The minimal elements of $\mathbb{L}_n$ form a flat family (thanks to the LR-property) and the general element is smooth. All minimal elements have degree $2n^2$ and arithmetic genus $\frac{1}{3}(2n-3)(2n-1)(2n+1)$. The Hartshorne-Rao module of such a curve occurs in degree $2n-2$. The ideal is generated by forms of degree $2n$, and there are $3n+1$ such generators. The h-vector of a minimal curve is

$$(1, 2, 3, 4, \dots, 2n-1, 2n, -n). \tag{6.1}$$

Since the Hartshorne-Rao module occurs in degree $2n-2$ and the initial degree of the ideal is $2n$, any minimal curve has maximal rank.

The h-vector of the general hyperplane section of such a curve is

$$(1, 2, 3, 4, \dots, 2n-2, 2n-1, n); \tag{6.2}$$

in particular, the initial degree is $\sigma = 2n-1$ and there are n generators of that degree. The numerical character of the curve is

$$(\underbrace{2n, \dots, 2n}_{n}, \underbrace{2n-1, \dots, 2n-1}_{n-1}).$$

Given a curve C and its numerical character $(n_0, n_1, \dots, n_{\sigma-1})$ with $n_0 \geq n_1 \geq \dots \geq n_{\sigma-1}$, we define for any integer i

$$A_i = \#\{j | n_j = i\}.$$

One of the main results of [**3**] is the following:

THEOREM 6.1 ([**3**], Theorem 5.3). *Let $N = (n_0, n_1, \dots, n_{\sigma-1})$ be a sequence of integers without gaps satisfying*

- $n_0 \geq n_1 \geq \dots \geq n_{\sigma-1} \geq \sigma$;
- $\sigma \geq 2n-1$;
- $A_\sigma \geq n-1$;
- $A_{\sigma+1} \geq n$, *and if* $A_{\sigma+1} = n$ *then* $A_t = 0$ *for all* $t > \sigma+1$.

Then there exists a smooth maximal rank curve $C \in \mathbb{L}_n$ with numerical character N.

Another result proved there, which will be useful to us, is the following. Note that it was shown much earlier by Gruson and Peskine [**11**] that the general hyperplane section of an integral curve in $\mathbb{P}^3$ has a numerical character that is without gaps. This turns out to be equivalent to the observation by Joe Harris [**13**] that the h-vector of the general hyperplane section of an irreducible curve in $\mathbb{P}^3$ is of *decreasing type*.

THEOREM 6.2 ([**3**], Corollary 3.6, Notation 3.7). *Let $Y \in \mathbb{L}_n$ be a smooth maximal rank curve, and let $N(Y) = (n_0, n_1, \dots, n_{\sigma-1})$ be its numerical character. Then the sufficient conditions listed in Theorem* 6.1 *are also necessary.*

We first produce a minimal numerically ACM curve in $\mathbb{L}_n$ (which will not be smoothable in $\mathbb{L}_n$). According to the algorithm in Theorem 5.11, since we begin with the h-vector (6.1) for the minimal curve, we must perform a series of n basic double links, using forms in the ideal of degrees $2n+2, 2n+3, \dots, 3n+1$. The resulting curve, Y, is numerically ACM. Its Hartshorne-Rao module occurs in degree $(2n-2)+n = 3n-2$ and the initial degree of I_Y is $(2n)+n = 3n$. Thus Y also has maximal rank.

To compute the h-vector of the general hyperplane section of Y, we begin with the h-vector of the general hyperplane section of the minimal curve in $\mathbb{L}_n$, namely (6.2), and perform the same sequence of basic double link calculations using forms of degrees $2n+2, 2n+3, \dots, 3n+1$. We obtain that the general hyperplane section of Y has h-vector

$$(1, 2, 3, \dots, 3n-2, 3n-1, 2n, n)$$

where the $2n$ occurs in degree $3n-1$. This translates to a numerical character

$$(\underbrace{3n+1, \dots, 3n+1}_{n}, \underbrace{3n, \dots, 3n}_{n}, \underbrace{3n-1, \dots, 3n-1}_{n-1})$$

with $\sigma = 3n-1$. Notice that this does not satisfy the last condition of Theorem 6.1, hence by Theorem 6.2 we cannot find a smooth Y with these numerical properties. However, if we perform one more basic double link, using a form in I_Y of degree $3n+1$, we obtain a curve C' with Hartshorne-Rao module in degree $3n-1$, initial degree of $I_{C'}$ equal to $3n+1$ (hence C' has maximal rank). Its general hyperplane section has h-vector

$$(1, 2, 3, \dots, 3n, 2n+1, n),$$

so the numerical character of C' is

$$(\underbrace{3n+2, \dots, 3n+2}_{n}, \underbrace{3n+1, \dots, 3n+1}_{n+1}, \underbrace{3n, \dots, 3n}_{n-1}). \tag{6.3}$$

Now $\sigma = 3n$, and we observe that this numerical character does satisfy the conditions of Theorem 6.1. The degree of C' can be obtained by adding the entries in the h-vector:

$$\deg C' = \frac{9n^2 + 9n + 2}{2}.$$

At this point we have a curve $C' \in \mathbb{L}_n$ that is numerically ACM and also satisfies the conditions of Theorems 6.1 and 6.2. This is not quite enough to guarantee that C' lies in an irreducible flat family of numerically ACM elements of $\mathbb{L}_n$, the general one of which is smooth. A priori, it is conceivable that there are several families, all of whose general *hyperplane sections* have the described Hilbert function, but some of which have elements that are numerically ACM and others of which have general elements that are smooth, but none having both. (This could happen if different shifts of the Hartshorne-Rao module are involved.) Our next observation is a uniqueness result that eliminates this danger.

Observe that starting with the h-vector of the general hyperplane section of the minimal curve, given in (6.2), if we perform basic double links using forms of strictly increasing degrees, the *only* way to obtain the numerical character (6.3) is via the given sequence of $n+1$ basic double links. Thus in $\mathbb{L}_n$, there is only one flat family of curves with this numerical character, namely the one containing C', which is numerically ACM. But the property of being numerically ACM is preserved in the flat family, and by Theorem 6.1 this flat family contains a smooth curve, C.

We have thus shown:

THEOREM 6.3. *The liaison class $\mathbb{L}_n$ contains a smooth, maximal rank, numerically ACM curve C of degree $\frac{9n^2+9n+2}{2}$. C is the smooth curve of least degree in $\mathbb{L}_n$ that is numerically ACM, but it does not have least degree simply among the numerically ACM curves in $\mathbb{L}_n$.*

Notice that when $n = 1$, we obtain Harris's curve of degree 10.

REMARK 6.4. Consider a pair of skew lines $C \subset \mathbb{P}^3$. It is a minimal curve in $\mathbb{L}_1$, and it is not an ACM curve. A numerical Macaulification of C is obtained by performing a basic double link of type $(4, 1)$. By Theorem 5.11, the resulting curve D is a minimal numerical ACM curve in $\mathbb{L}_1$. Nollet showed in [**20**] that D cannot be flatly deformed within its liaison class to an integral curve.

QUESTION 6.5. Does Harris's curve have the smallest possible degree among integral numerically ACM (but not ACM) curves in $\mathbb{P}^3$?

QUESTION 6.6. We believe that at least for curves in $\mathbb{P}^3$, every even liaison class contains smooth numerically ACM elements. Does every even liaison class of locally Cohen-Macaulay, codimension two subschemes of $\mathbb{P}^n$ contain smooth numerically ACM elements?

REMARK 6.7. We wonder if the smooth numerically ACM curves in $\mathbb{L}_n$, or perhaps in any even liaison class of curves in $\mathbb{P}^3$, satisfy some sort of Lazarsfeld-Rao property, similar to what was studied in [**20**] (without regard to the numerically ACM property). Since our main tool here involves only maximal rank curves in $\mathbb{L}^n$, we do not know the answer to this question. We remark that thanks to the results in [**4**], a result similar to Theorem 6.3 is probably possible for smooth, maximal rank arithmetically Buchsbaum curves whose Hartshorne-Rao module has diameter two.

As mentioned in the introduction, smooth curves, even having the same Hilbert function, can behave very differently depending on whether they are ACM or not. The following gives an interesting illustration. On an integral ACM curve, there are Gorenstein sets of points with arbitrarily large degree (see [**14**]). This in no longer true on non-ACM curves, even if we just ask for zero-dimensional schemes.

PROPOSITION 6.8. *Let C be an integral non-ACM curve in $\mathbb{P}^n$. Then there is an integer N, depending only on the regularity of I_C, such that C contains no arithmetically Gorenstein zero-dimensional scheme of degree $> N$. In fact, $N = \deg C \cdot \operatorname{reg} I_C$ has this property.*

Proof. Let $Z \subset C$ be a zero-dimensional scheme. Let $\mathbb{F}_\bullet$ be the minimal free resolution of I_C, and let $\mathbb{G}_\bullet$ be the minimal free resolution of I_Z. The length of both $\mathbb{F}_\bullet$ and $\mathbb{G}_\bullet$ is $n-1$ (the former since C is not ACM).

Let $d = \operatorname{reg} I_C$. In particular, all the minimal generators of I_C have degree $\le d$. In fact, in the Betti diagram for $\mathbb{F}_\bullet$, the last non-zero row is the d-th one.

Now assume that $|Z| > d \cdot \deg C$. Since C is integral, any hypersurface of degree $\le d$ that contains Z also contains C, so any minimal generator of I_Z that is not in I_C has degree $> d$. Consequently, the first d rows of the Betti diagram for I_Z are precisely the Betti diagram for I_C. It also follows that the largest twist of $\mathbb{G}_{n-1}$ is strictly larger than the largest twist of $\mathbb{F}_{n-1}$. But the summands of $\mathbb{F}_{n-1}$ are also summands of $\mathbb{G}_{n-1}$. Thus $\mathbb{G}_{n-1}$ has at least two summands, so Z cannot be arithmetically Gorenstein. $\square$

7. Examples

We illustrate our algorithms and results by a few more examples, and we raise some questions that, we believe, deserve further consideration.

Example 7.1. Let C_1 be the scheme in $\mathbb{P}^3$ defined by the cube of the ideal of a general line. Let C_2 and C_3 be general complete intersections of types $(1,2)$ and $(4,8)$ respectively. Let $C = C_1 \cup C_2 \cup C_3$. The Betti diagram for R/I_C is

```
        0    1    2    3
-------------------------
 0:     1    -    -    -
 1:     -    -    -    -
 2:     -    -    -    -
 3:     -    -    -    -
 4:     -    -    -    -
 5:     -    -    -    -
 6:     -    -    -    -
 7:     -    4    3    -
 8:     -    4    7    3
 9:     -    -    -    -
10:     -    -    -    -
11:     -    5    4    -
12:     -    4    8    4
13:     -    -    4    3
-------------------------
Tot:    1   17   26   10
```

and the h-vector of R/I_C is $(1,2,3,4,5,6,7,8,5,1,4,4,-1,-6,-3)$. We illustrate the two algorithms, and why they produce numerically equivalent results. We begin with the first algorithm.

Collecting our lists of $\{r_i\}$ and $\{s_i\}$, we obtain

$\{s_i\}$	15	15	15	15	14	14	14	14	14	14	14	14	13	13	13	13
$\{r_i\}$	16	16	16	15	15	15	15	13	13	13	13	12	12	12	12	12
	10	10	10	10	10	10	10	9	9	9						
	11	11	11	9	9	9	9	8	8	8	8					

However, we first remove duplicates and re-align the lists:

$\{s_i\}$	14	14	14	14	14	14	14	14	10	10	10	10	10	10	10	
$\{r_i\}$	16	16	16	12	12	12	12	12	11	11	11	9	8	8	8	8

The negative entries of the main diagonal of the matrix are precisely those integers for which $s_i < r_i$. We thus immediately see that we will need three basic double links of height 2 and three of height 1, a fact that was not at all evident before removing the duplicates. More careful analysis shows that in fact the sequence of basic double links consists of types

$$(11, 1), (12, 1), (13, 1), (19, 2), (21, 2), \text{ and } (23, 2).$$

Notice that the sum of the heights of the basic double links is $1+1+1+2+2+2 = 9$.

As for the second algorithm, we begin with the h-vector

$$(\ 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8,\quad 5,\quad 1,\quad 4,\quad 4,\quad -1,\quad -6,\quad -3.\ )$$

We look for places where the value in one degree is smaller than that in the next degree. The total deficit is $3 + 3 + 3 = 9$, which is equal to the sum of the heights, i.e. the sum of the absolute values of the negative entries on the main diagonal of the matrix. In fact, Algorithm 4.6 gives that we must use a sequence of basic double links of type $(d_i, 1)$ where d_i takes the values 11, 12, 13, 18, 19, 20, 22, 23, 24.

The resulting curve from the first approach in the same flat family of the even liaison class as the curve resulting from the second approach because the two curves have the same Hilbert function by Proposition 4.9. Alternatively, we can see this by replacing the basic double links used in the first algorithm by numerically equivalent basic double links. First, using Lemma 5.5 three times we see that we have a sequence of basic double links of type $(d_i, 1)$ with d_i taking the values 11, 12, 13, 19, 19, 21, 21, 23, 23. Applying Lemma 5.6 six times to this 9-tuple gives sequentially the 9-tuples

11	12	13	19	19	21	21	23	23
11	12	13	18	20	21	21	23	23
11	12	13	18	20	20	22	23	23
11	12	13	18	19	21	22	23	23
11	12	13	18	19	21	22	22	24
11	12	13	18	19	21	21	23	24
11	12	13	18	19	20	22	23	24

where the last row represents the same sequence of basic double links prescribed (in the end) by the first approach.

We note that in general it seems rather complicated to show directly that the sequences of basic double links used in Algorithms 4.3 and 4.6 are numerically equivalent.

Example 7.2. Let $I \subset k[w, x, y, z]$ be the ideal of a set Z of 11 general points in $\mathbb{P}^3$. The Betti diagram for R/I is

```
            0     1     2     3
------------------------------
  0:        1     -     -     -
  1:        -     -     -     -
  2:        -     9    12     3
```

```
 3:     -    -    -    1
------------------------
Tot:    1    9   12    4
```

Even though I has codimension three, we can still apply Algorithm 4.3 to I. We perform a sequence of basic double links of type $(5, 1)$, $(6, 1)$, $(7, 1)$ and $(9, 2)$, obtaining an ideal with Betti diagram

```
        0    1    2    3
------------------------
 0:     1    -    -    -
 1:     -    -    -    -
 2:     -    -    -    -
 3:     -    -    -    -
 4:     -    -    -    -
 5:     -    -    -    -
 6:     -    -    -    -
 7:     -    9   12    3
 8:     -    4    3    1
 9:     -    -    1    -
------------------------
Tot:    1   13   16    4
```

and h-vector $[1, 2, 3, 4, 5, 6, 7, 8]$.

Example 7.3. Let Z be a set of 11 general points in $\mathbb{P}^3$ as in Example 7.2, and let I be the ideal generated by a general set of four forms in I_Z of degree 4. I defines Z scheme-theoretically, but is not saturated. The Betti diagram for R/I is

```
        0    1    2    3    4
-----------------------------
 0:     1    -    -    -    -
 1:     -    -    -    -    -
 2:     -    -    -    -    -
 3:     -    4    -    -    -
 4:     -    -    -    -    -
 5:     -    -    -    -    -
 6:     -    -    6    -    -
 7:     -    -    -    -    -
 8:     -    -    1    -    -
 9:     -    -    3   16    9
-----------------------------
Tot:    1    4   10   16    9
```

Applying Algorithm 4.3 involves a sequence of seven basic double links, resulting in a Betti diagram

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	-	-	-	-
			...		
19:	-	-	-	-	-
20:	-	4	-	-	-
21:	-	-	-	-	-
22:	-	-	-	-	-
23:	-	-	6	-	-
24:	-	3	-	-	-
25:	-	-	1	-	-
26:	-	1	3	16	9
27:	-	3	7	-	-
Tot:	1	11	17	16	9

and an h-vector

$$[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,18,15,12,9,9,9,9,9].$$

Example 7.4. The Betti diagram of the curve studied in Example 4.7 is

	0	1	2	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	-	-	-
5:	-	-	-	-
6:	-	-	-	-
7:	-	-	-	-
8:	-	-	-	-
9:	-	2	1	-
10:	-	-	-	-
11:	-	2	3	1
12:	-	1	-	-
13:	-	1	1	-
14:	-	1	-	-
15:	-	1	5	2
16:	-	2	4	2
17:	-	2	2	-
Tot:	1	12	16	5

One can verify that this gives a matrix with only positive entries in the main diagonal (after removing redundant terms), and so the curve is numerically ACM.

Remark 7.5. Using the methods of this paper, many ACM Hilbert functions of curves can be obtained starting from non-ACM curves. It would be interesting to know which ACM Hilbert functions do *not* occur in this way, i.e. which force the curve to be ACM (a trivial example is if it is a plane curve). Is it a finite list?

How does the question change if we restrict to smooth or integral curves? This question has been studied from the point of view of the Hilbert function of the general hyperplane section (see e.g. [**10**]) but this is a different question!

Remark 7.6. It will be noted that all of our numerically ACM subschemes have codimension two. It would be interesting to find a construction that produces numerically ACM subschemes of higher codimension. In this paper we have heavily used methods and results that apply only to codimension two, so it is unlikely that results as complete as those given here will be obtained for higher codimension. Still, it is an interesting problem.

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Clay Mathematics Proceedings
Volume **18**, 2013

The non-nef locus in positive characteristic

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Dedicated to Joe Harris on the occasion of his sixtieth birthday

Abstract. We give an analogue in positive characteristic of the description of the non-nef locus from [**ELMNP**]. In this case, the role of the asymptotic multiplier ideals is played by the asymptotic test ideals. The key ingredient is provided by a uniform global generation statement involving twists by such ideals.

1. Introduction

Let X be a smooth, projective variety over an algebraically closed field k. If D is a divisor on X, then for every positive integer m we may consider the closed subset $\mathrm{Bs}(mD)$, the base-locus of the linear system $|mD|$. The intersection $\bigcap_{m\geq 1}\mathrm{Bs}(mD)$ is equal to $\mathrm{Bs}(\ell D)$ for ℓ divisible enough; this is the *stable base locus* $\mathbf{B}(D)$ of D. By definition, we have $\mathbf{B}(D)=\mathbf{B}(rD)$ for every positive integer r, and using this one extends in the obvious way the definition of $\mathbf{B}(D)$ to the case when D is a $\mathbf{Q}$-divisor.

The stable base locus is a very interesting invariant, but it is quite subtle: for example, two numerically equivalent divisors can have different stable base loci. A related subset is the *non-nef locus*, defined as follows. If D is an $\mathbf{R}$-divisor on X, then

$$\mathbf{B}_{-}(D):=\bigcup_{A}\mathbf{B}(D+A),$$

where the union is over all ample $\mathbf{R}$-divisors A such that $D+A$ is a $\mathbf{Q}$-divisor. It follows from the definition that $\mathbf{B}_{-}(D)$ only depends on the numerical equivalence class of D, and $\mathbf{B}_{-}(D)$ is empty if and only if D is nef.

This locus was studied in [**ELMNP**] over a ground field of characteristic zero. The key tool in this study is the asymptotic multiplier ideal and a certain uniform global generation result for twists by such ideals. In that context, the global generation statement is a consequence of vanishing theorems of Kodaira-type and of Castelnuovo-Mumford regularity. The main point of the present paper is that a similar uniform global generation result also holds in positive characteristic, if one replaces the asymptotic multiplier ideal by the asymptotic test ideal (despite the

2010 *Mathematics Subject Classification.* Primary 14A10; Secondary 13A35.
Key words and phrases. Non-nef locus, test ideals, multiplier ideals, big line bundles.
The author was partially supported by NSF grant DMS-1068190 and a Packard Fellowship.

fact that in positive characteristic Kodaira's vanishing theorem and its generalizations may fail).

We recall that test ideals give an analogue in positive characteristic of multiplier ideals in characteristic zero. They were introduced by Hara and Yoshida in [**HY**] using a generalization of tight closure theory, and it was noticed from the beginning that they satisfy similar formal properties with those of multiplier ideals in characteristic zero. Furthermore, there are some very interesting results and open problems concerning the connection between multiplier ideals and test ideals via reduction mod p. We refer to §3 for the definition of test ideals and to [**ST**] for a more comprehensive overview.

As it is the case for multiplier ideals in characteristic zero, for a divisor D on a variety X in positive characteristic with $\mathcal{O}_X(D)$ of non-negative Iitaka dimension, one can use an asymptotic construction to obtain *asymptotic test ideals* $\tau(\lambda\cdot \parallel D \parallel)$ for every $\lambda \in \mathbf{R}_{\geq 0}$. The following is our main technical result (see Theorem 4.1 below).

Theorem A. Let X be a smooth projective variety over an algebraically closed field of positive characteristic and let H be an ample divisor on X, with $\mathcal{O}_X(H)$ globally generated. If D and E are divisors on X such that $\mathcal{O}_X(D)$ has non-negative Iitaka dimension, and $\lambda \in \mathbf{Q}_{\geq 0}$ is such that $E - \lambda D$ is nef, then the sheaf

$$\tau(\lambda\cdot \parallel D \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + E + dH)$$

is globally generated for every $d \geq \dim(X) + 1$.

Here K_X denotes a canonical divisor on X. For the corresponding result in characteristic zero, in which $\tau(\lambda\cdot \parallel D \parallel)$ is replaced by the asymptotic multiplier ideal $\mathcal{J}(\lambda\cdot \parallel D \parallel)$, see [**Laz**, Corollary 11.2.13]. It was Schwede who first noticed in [**Sch**] that one can use an argument due to Keeler [**Kee**] and Hara (unpublished) to obtain global generation statements involving test ideals. The idea is to use Castelnuovo-Mumford regularity and the fact that by pushing-forward via the Frobenius morphism one can reduce the desired vanishings to Serre's asymptotic vanishing. Our argument follows the one in [**Sch**], with some modifications coming from the fact that we need to consider test ideals of not necessarily locally principal ideals, and we have the extra nef divisor $E - \lambda D$ to deal with (in order to do this, we use Fujita's vanishing theorem instead of Serre's asymptotic vanishing).

Once we have the above uniform global generation statement and its corollaries, the basic results describing $B_{-}(D)$ for a big divisor D follow as in [**ELMNP**]. Recall that given a closed point $x \in X$[1] and a big divisor D on X, one defines the *asymptotic order of vanishing* $\mathrm{ord}_x(\parallel D \parallel)$ by

$$\mathrm{ord}_x(\parallel D \parallel) := \inf_{m\geq 1} \frac{\mathrm{ord}_x |mD|}{m} = \lim_{m\to\infty} \frac{\mathrm{ord}_x |mD|}{m},$$

where $\mathrm{ord}_x |mD|$ is the order of vanishing at x of a general element in $|mD|$. The following are the main properties of this function (see Theorem 6.1 below).

Theorem B. Let X be a smooth projective variety over an algebraically closed field of positive characteristic, and x a closed point on X.

i) For every big divisor D, the asymptotic order of vanishing $\mathrm{ord}_x(\parallel D \parallel)$ only depends on the numerical class of D.

[1] In the main body of the paper we will deal with an arbitrary irreducible proper closed subset $Z \subset X$, not just with points.

ii) The function $D \to \mathrm{ord}_x(\| D \|)$ extends as a continuous function to the cone of big divisor classes $\mathrm{Big}(X)_{\mathbf{R}}$.

In characteristic zero, this was also proved by Nakayama in [**Nak**]. As a consequence of Theorems A and B, we obtain the following description of the non-nef locus (see Theorem 6.2 below).

Theorem C. Let X be a smooth projective variety over an algebraically closed field of positive characteristic and x a closed point on X. For a big divisor D, the following are equivalent:

i) x does not lie in the non-nef locus $\mathbf{B}_-(D)$.
ii) There is a divisor G on X such that x does not lie in the base locus of $|mD+G|$ for every $m \geq 1$.
iii) There is a real number M such that $\mathrm{ord}_x |mD| \leq M$ for every m with $|mD|$ non-empty.
iv) $\mathrm{ord}_x(\| D \|) = 0$.
v) For every $m \geq 1$, the ideal $\tau(\| mD \|)$ does not vanish at x.

At the boundary of the big cone, the situation is more complicated. If D is a pseudo-effective $\mathbf{R}$-divisor, then $D + A$ is big for every ample $\mathbf{R}$-divisor A. As in [**Nak**], we define $\sigma_x(D) := \sup_A \mathrm{ord}_x(\| D + A \|)$, where A varies over all ample $\mathbf{R}$-divisors. This is tautologically a lower semi-continuous function on the pseudo-effective cone, but it is not continuous in general. Following an idea of Hacon, we show that for every $\lambda \in \mathbf{R}_{\geq 0}$, there is a unique minimal element in the set of ideals $\tau(\lambda \cdot \| D + A \|)$, where A is as above. We denote this ideal by $\tau_+(\lambda \cdot \| D \|)$. The following theorem gives the description of the non-nef locus for pseudo-effective divisors.

Theorem D. Let X be a smooth projective variety over an algebraically closed field of positive characteristic and x a closed point on X. If D is a pseudo-effective $\mathbf{R}$-divisor on X, then the following are equivalent:

i) x does not lie in the non-nef locus $\mathbf{B}_-(D)$.
ii) $\sigma_x(D) = 0$.
iii) For every $m \geq 1$, the ideal $\tau_+(m \cdot \| D \|)$ does not vanish at x.

The paper is organized as follows. In §2 we review, following [**ELMNP**], the definition and elementary properties of the asymptotic order function and of the non-nef locus. In §3 we recall the definition of test ideals and of its asymptotic version. We prove here that given an arbitrary graded sequence of ideals, its asymptotic order of vanishing along a subvariety can be computed from the orders of vanishing of the corresponding sequence of asymptotic test ideals. Section 4 contains the proof of our key technical result, Theorem A. Some applications to asymptotic test ideals and their F- jumping numbers are given in the following section. In §6 we deduce the results stated in Theorems B and C above, while the last section of the paper contains the description of the non-nef locus for pseudo-effective $\mathbf{R}$-divisors.

Acknowledgment. I am indebted to Karl Schwede for several inspiring conversations on the results of his preprint [**Sch**] and for comments on a preliminary version of this note. I would also like to thank Rob Lazarsfeld for many discussions over the years on multiplier ideals and asymptotic invariants. Last but not least, I am grateful to the anonymous referees for their comments.

2. Non-nef loci and asymptotic orders of vanishing

In this section we review, following [**ELMNP**], the definition of the non-nef locus and of the asymptotic order of vanishing of a divisor along a subvariety. Let X be a smooth variety[2] over an algebraically closed field k (in this section we make no restriction on the characteristic).

Recall that a *graded sequence of ideals* on X consists of a sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\geq 1}$ of ideals of $\mathcal{O}_X$ (all ideals are assumed to be coherent) that satisfies

$$\mathfrak{a}_{m_1}\cdot\mathfrak{a}_{m_2}\subseteq\mathfrak{a}_{m_1+m_2} \tag{2.1}$$

for all $m_1, m_2 \geq 1$. We assume that all our graded sequences are *nonzero*, that is, $\mathfrak{a}_m \neq 0$ for some $m \geq 1$.

The most interesting examples of graded sequences arise as follows. Suppose that X is complete, and that D is a divisor on X such that $\mathcal{O}_X(D)$ has non-negative Iitaka dimension. Let $\mathfrak{a}_{|mD|}$ be the ideal defining the base locus of $\mathcal{O}_X(mD)$, that is, evaluation of sections induces a surjective map

$$H^0(X,\mathcal{O}_X(mD))\otimes\mathcal{O}_X\to\mathfrak{a}_{|mD|}\cdot\mathcal{O}_X(mD).$$

In this case $\mathfrak{a}^D_\bullet = (\mathfrak{a}_{|mD|})_{m\geq 1}$ is a graded sequence of ideals (note that some $\mathfrak{a}_{|mD|}$ is nonzero by the assumption on the Iitaka dimension of $\mathcal{O}_X(D)$).

Suppose now that X is not necessarily complete and Z is an irreducible proper closed subset of X. For a nonzero ideal $\mathfrak{a}$ on Z we denote by $\mathrm{ord}_Z(\mathfrak{a})$ the order of vanishing of $\mathfrak{a}$ along Z; in other words, if $\mathcal{O}_{X,Z}$ is the local ring of X at the generic point of Z, having maximal ideal $\mathfrak{m}_Z$, then $\mathrm{ord}_Z(\mathfrak{a})$ is the largest r such that $\mathfrak{a}\cdot\mathcal{O}_{X,Z}\subseteq\mathfrak{m}_Z^r$. By convention, we put $\mathrm{ord}_Z(0)=\infty$. It is clear that given two ideals $\mathfrak{a}$ and $\mathfrak{a}'$ on X, we have

$$\mathrm{ord}_Z(\mathfrak{a}\cdot\mathfrak{a}') = \mathrm{ord}_Z(\mathfrak{a}) + \mathrm{ord}_Z(\mathfrak{a}'). \tag{2.2}$$

If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then the *asymptotic order of vanishing* of $\mathfrak{a}_\bullet$ along Z is

$$\mathrm{ord}_Z(\mathfrak{a}_\bullet) := \inf_{m\geq 1}\frac{\mathrm{ord}_Z(\mathfrak{a}_m)}{m}.$$

It is easy to deduce from properties (2.1) and (2.2) that $\mathrm{ord}_Z(\mathfrak{a}_\bullet) = \lim_{m\to\infty}\frac{\mathrm{ord}_Z(\mathfrak{a}_m)}{m}$, where the limit is over those m such that $\mathfrak{a}_m$ is nonzero (see for example [**JM**, Lemma 2.3]).

If X is complete and D is a divisor on X such that $\mathcal{O}_X(D)$ has non-negative Iitaka dimension, then we consider the graded sequence $\mathfrak{a}^D_\bullet = (\mathfrak{a}_{|mD|})_{m\geq 1}$. The asymptotic order of vanishing $\mathrm{ord}_Z(\mathfrak{a}^D_\bullet)$ is denoted by $\mathrm{ord}_Z(\parallel D\parallel)$. Since $\mathrm{ord}_Z(\parallel D\parallel)$ is the limit of the corresponding normalized orders of vanishing, we have the equality $\mathrm{ord}_Z(\parallel mD\parallel) = m\cdot\mathrm{ord}_Z(\parallel D\parallel)$ for every positive integer m. Given a $\mathbf{Q}$-divisor D with $h^0(X,\mathcal{O}_X(mD))\neq 0$ for some positive integer m such that mD has integer coefficients, we can therefore define $\mathrm{ord}_Z(\parallel D\parallel) := \frac{1}{m}\mathrm{ord}_Z(\parallel mD\parallel)$. It is clear that this is well-defined and $\mathrm{ord}_Z(\parallel\lambda D\parallel) = \lambda\cdot\mathrm{ord}_Z(\parallel D\parallel)$ for every $\lambda\in\mathbf{Q}_{\geq 0}$.

Remark 2.1. If D and E are divisors on X such that both $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ have non-negative Iitaka dimension, we have $\mathfrak{a}_{|mD|}\cdot\mathfrak{a}_{|mE|}\subseteq\mathfrak{a}_{|m(D+E)|}$ for every m.

[2] We assume that all varieties are irreducible and reduced.

This easily implies

$$\mathrm{ord}_Z(\| D+E \|) \leq \mathrm{ord}_Z(\| D \|) + \mathrm{ord}_Z(\| E \|).$$

We now turn to the definition of the stable base locus and of the non-nef locus. Suppose that X is a smooth projective variety. We denote by $\mathrm{N}^1(X)_{\mathbf{R}}$ the finite-dimensional real vector space of numerical equivalence classes of $\mathbf{R}$-divisors on X, and by $\mathrm{Big}(X)_{\mathbf{R}}$ the *big cone*, that is, the open cone of big $\mathbf{R}$-divisor classes. The closure of the big cone is the cone of *pseudo-effective* divisor classes.

For a divisor D on X, we denote by $\mathrm{Bs}(D)$ the base-locus of $|D|$ (with the reduced scheme structure). It is clear that for every positive integers m and r, we have $\mathrm{Bs}(rmD) \subseteq \mathrm{Bs}(mD)$, hence the Noetherian property implies that the intersection $\bigcap_{m\geq 1} \mathrm{Bs}(mD)$ is equal to $\mathrm{Bs}(\ell D)$ if ℓ is divisible enough. This is the *stable base locus* of D, denoted by $\mathbf{B}(D)$. It is clear that $\mathbf{B}(D) = \mathbf{B}(rD)$ for every positive integer r. Therefore we can define $\mathbf{B}(D)$ for a $\mathbf{Q}$-divisor D as $\mathbf{B}(rD)$, where r is any positive integer such that rD has integer coefficients.

Suppose now that D is an $\mathbf{R}$-divisor on X. The *non-nef locus* of D (called *restricted base locus* in [**ELMNP**]) is the union

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}(D+A),$$

where the union is over all ample $\mathbf{R}$-divisors A such that $D+A$ is a $\mathbf{Q}$-divisor. The properties in the following proposition are simple consequences of the definition, see [**ELMNP**, §1].

Proposition 2.2. Let D_1 and D_2 be $\mathbf{R}$-divisors on X.

i) $\mathbf{B}_-(D_1) = \mathbf{B}_-(\lambda D_1)$ for every $\lambda > 0$.
ii) If D_1 and D_2 are numerically equivalent, then $\mathbf{B}_-(D_1) = \mathbf{B}_-(D_2)$.
iii) The non-nef locus $\mathbf{B}_-(D_1)$ is empty if and only if D_1 is nef.
iv) If D_1 is a $\mathbf{Q}$-divisor, then $\mathbf{B}_-(D_1) \subseteq \mathbf{B}(D_1)$.
v) We have $\mathbf{B}_-(D_1 + D_2) \subseteq \mathbf{B}_-(D_1) \cup \mathbf{B}_-(D_2)$.
vi) If $(A_m)_{m\geq 1}$ is a sequence of ample $\mathbf{R}$-divisors with each $D + A_m$ having rational coefficients, and such that the classes of the A_m converge to zero in $\mathrm{N}^1(X)_{\mathbf{R}}$, then $\mathbf{B}_-(D) = \bigcup_{m\geq 1} \mathbf{B}(D + A_m) = \bigcup_{m\geq 1} \mathbf{B}_-(D + A_m)$.

It is not known whether $\mathbf{B}_-(D)$ is always a Zariski closed subset of X, though property vi) above shows that it is a countable union of closed subsets. This property also implies that if the ground field k is uncountable, then $\mathbf{B}_-(D) = X$ if and only if D is not pseudo-effective.

3. Asymptotic test ideals

We start by reviewing the definition of test ideals. These ideals have been introduced and studied in [**HY**]. Since we only deal with smooth varieties, we use an alternative definition from [**BMS**], which is more suitable for our applications (this description goes back to [**HT**, Lemma 2.1]). Suppose that X is a smooth n-dimensional variety over an algebraically closed field k of characteristic $p > 0$ (in fact, for what follows it is enough to assume k perfect). Let ω_X denote the sheaf of n-forms on X. We denote by $F\colon X \to X$ the Frobenius morphism, that is given by the identity on the topological space, and by taking the p-power on regular functions.

The key object is the *trace* map $\mathrm{Tr} = \mathrm{Tr}_X : F_*(\omega_X) \to \omega_X$. This is a surjective map that can be either defined as a trace map for duality with respect to F, or as coming from the Cartier isomorphism. Given algebraic coordinates $x_1, \ldots, x_n$ on an open subset U of X, the trace map is characterized by

$$\mathrm{Tr}(x_1^{i_1} \cdots x_n^{i_n} dx_1 \wedge \cdots \wedge dx_n) = x_1^{\frac{i_1-p+1}{p}} \cdots x_n^{\frac{i_n-p+1}{p}} dx_1 \wedge \cdots \wedge dx_n,$$

where the monomial on the right-hand side is understood to be zero if one of the exponents is not an integer. Iterating this map e times we obtain a surjective map $\mathrm{Tr}^e : F_*^e(\omega_X) \to \omega_X$.

Given an ideal $\mathfrak{b}$ in $\mathcal{O}_X$ and $e \geq 1$, the image $\mathrm{Tr}^e(\mathfrak{b} \cdot \omega_X)$ can be written as $\mathfrak{b}^{[1/p^e]} \cdot \omega_X$ for some ideal $\mathfrak{b}^{[1/p^e]}$ in $\mathcal{O}_X$. This is not the definition in [**BMS**], but it can be easily seen to be equivalent to the definition therein via [**BMS**, Proposition 2.5]. For example, when $\mathfrak{a}$ is the ideal defining a smooth divisor E on X, we have $(\mathfrak{a}^m)^{[1/p^e]} = \mathcal{O}_X(-\lfloor m/p^e \rfloor E)$, where $\lfloor u \rfloor$ denotes the largest integer $\leq u$.

Given a nonzero ideal $\mathfrak{a}$ on X and $\lambda \in \mathbf{R}_{\geq 0}$, one shows that

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}\right)^{[1/p^{e+1}]}$$

for every $e \geq 1$. Here we put $\lceil u \rceil$ for the smallest integer $\geq u$. By the Noetherian property, there is an ideal $\tau(\mathfrak{a}^\lambda)$, the *test ideal* of $\mathfrak{a}$ of exponent λ, that is equal to $\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$ for $e \gg 0$. One can show that if r is a positive integer, then $\tau(\mathfrak{a}^{r\lambda}) = \tau((\mathfrak{a}^r)^\lambda)$. Furthermore, we have $\mathfrak{a} \subseteq \tau(\mathfrak{a})$.

Test ideals share many of the properties of the multiplier ideals. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$ for every λ. We have $\tau(\mathfrak{a}^\lambda) = \mathcal{O}_X$ if $0 \leq \lambda \ll 1$. For $\lambda > \mu$ we have $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{a}^\mu)$. Given $\lambda \geq 0$, there is $\varepsilon > 0$ such that $\tau(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^\mu)$ for μ with $\lambda \leq \mu \leq \lambda + \varepsilon$. One says that $\lambda > 0$ is an *F-jumping number* of $\mathfrak{a}$ if $\tau(\mathfrak{a}^\lambda) \neq \tau(\mathfrak{a}^{\lambda'})$ for every $\lambda' < \lambda$. It is known that the set of F-jumping numbers of $\mathfrak{a}$ is a discrete set of rational numbers. For the proof of all these properties, we refer to [**BMS**].

A nice feature of the theory is that the Subadditivity Theorem for multiplier ideals (see [**Laz**, Theorem 9.5.20]) has an analogue in this setting. This says that for every nonzero ideals $\mathfrak{a}$ and $\mathfrak{b}$ and every $\lambda \geq 0$, we have

$$\tau((\mathfrak{a} \cdot \mathfrak{b})^\lambda) \subseteq \tau(\mathfrak{a}^\lambda) \cdot \tau(\mathfrak{b}^\lambda). \tag{3.1}$$

In particular, we have $\tau(\mathfrak{a}^{m\lambda}) \subseteq \tau(\mathfrak{a}^\lambda)^m$ for every positive integer m. For a proof see [**BMS**, Proposition 2.11].

One can similarly define a mixed test ideal: given nonzero ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{O}_X$ and $\lambda, \mu \in \mathbf{R}_{\geq 0}$, there is an ideal $\tau(\mathfrak{a}^\lambda \mathfrak{b}^\mu)$ that is equal to $(\mathfrak{a}^{\lceil \lambda p^e \rceil} \mathfrak{b}^{\lceil \mu p^e \rceil})^{[1/p^e]}$ for $e \gg 0$. One can show that for every λ and μ, there is $\varepsilon > 0$ such that $\tau(\mathfrak{a}^\lambda \mathfrak{b}^\mu) = \tau(\mathfrak{a}^{\lambda'} \mathfrak{b}^{\mu'})$ if $\lambda \leq \lambda' \leq \lambda + \varepsilon$ and $\mu \leq \mu' \leq \mu + \varepsilon$ (this follows by adapting the argument in the non-mixed case, see the proof of [**BMS**, Proposition 2.14]). The notion of mixed test ideals will only come up in the proof of Proposition 3.1 iv) below.

It is straightforward to define an asymptotic version of test ideals, proceeding in the same way as in the case of multiplier ideals (this has been noticed already in [**Ha**], where the construction was used to compare symbolic powers with usual powers of ideals in positive characteristic). Suppose that $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X (recall that we always assume that some $\mathfrak{a}_m$ is nonzero) and $\lambda \in \mathbf{R}_{\geq 0}$.

If m and r are positive integers such that $\mathfrak{a}_m$ is nonzero, then

$$\tau(\mathfrak{a}_m^{\lambda/m}) = \tau((\mathfrak{a}_m^r)^{\lambda/mr}) \subseteq \tau(\mathfrak{a}_{mr}^{\lambda/mr}),$$

where the inclusion follows from $\mathfrak{a}_m^r \subseteq \mathfrak{a}_{mr}$. It follows from the Noetherian property that there is a unique ideal $\tau(\mathfrak{a}_\bullet^\lambda)$ such that $\tau(\mathfrak{a}_m^{\lambda/m}) \subseteq \tau(\mathfrak{a}_\bullet^\lambda)$ for all m (such that $\mathfrak{a}_m$ is nonzero), with equality if m is divisible enough. This is the *asymptotic test ideal* of $\mathfrak{a}_\bullet$ of exponent λ. We collect in the next proposition some easy properties of asymptotic test ideals. The proof follows the case of multiplier ideals (see [**Laz**, §11.2]).

Proposition 3.1. Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two graded sequences of ideals on X.

i) We have $\tau(\mathfrak{a}_\bullet^\lambda) \subseteq \tau(\mathfrak{a}_\bullet^\mu)$ for every $\lambda \geq \mu$.
ii) We have $\tau(\mathfrak{a}_\bullet^{m\lambda}) \subseteq \tau(\mathfrak{a}_\bullet^\lambda)^m$ for all positive integers m.
iii) For every $\lambda \in \mathbf{R}_{\geq 0}$, there is $\varepsilon > 0$ such that $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_\bullet^\mu)$ for all μ with $\lambda \leq \mu \leq \lambda + \varepsilon$.
iv) If there is a nonzero ideal $\mathfrak{c}$ on X such that $\mathfrak{c} \cdot \mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all $m \gg 0$, then $\tau(\mathfrak{a}_\bullet^\lambda) \subseteq \tau(\mathfrak{b}_\bullet^\lambda)$ for all $\lambda \in \mathbf{R}_{\geq 0}$.

Proof. The assertions in i) and ii) follow from the definition of asymptotic test ideals, using the corresponding properties in the case of test ideals. For iii), let m be such that $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_m^{\lambda/m})$. There is $\varepsilon > 0$ such that $\tau(\mathfrak{a}_m^{\lambda/m}) = \tau(\mathfrak{a}_m^{(\lambda+\varepsilon)/m}) \subseteq \tau(\mathfrak{a}_\bullet^{\lambda+\varepsilon})$, which proves iii).

For iv), given $\lambda \in \mathbf{R}_{\geq 0}$ let us choose m such that $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_m^{\lambda/m})$. If $\ell \gg 0$, then

$$\begin{aligned} \tau(\mathfrak{a}_\bullet^\lambda) &= \tau(\mathfrak{a}_m^{\lambda/m}) = \tau(\mathfrak{c}^{\lambda/m\ell}\mathfrak{a}_m^{\lambda/m}) \\ &= \tau((\mathfrak{c}\mathfrak{a}_m^\ell)^{\lambda/m\ell}) \subseteq \tau((\mathfrak{c}\mathfrak{a}_{m\ell})^{\lambda/m\ell}) \subseteq \tau(\mathfrak{b}_{m\ell}^{\lambda/m\ell}) \subseteq \tau(\mathfrak{b}_\bullet^\lambda). \end{aligned}$$

□

We say that $\lambda > 0$ is an F-jumping number of $\mathfrak{a}_\bullet$ if $\tau(\mathfrak{a}_\bullet^\lambda) \neq \tau(\mathfrak{a}_\bullet^\mu)$ for every $\mu < \lambda$. We will see in §5 that if $\mathfrak{a}_\bullet$ is associated to a divisor on a projective variety, then the set of F-jumping numbers of $\mathfrak{a}_\bullet$ is discrete.

If $\mathfrak{a}_\bullet$ is a graded sequence as above, we also consider $\mathfrak{b}_\bullet = (\mathfrak{b}_m)_{m\geq 1}$, where $\mathfrak{b}_m = \tau(\mathfrak{a}_\bullet^m)$. Note that $\mathfrak{a}_m \subseteq \tau(\mathfrak{a}_m) \subseteq \mathfrak{b}_m$ for every m. The sequence $\mathfrak{b}_\bullet$ is not a graded sequence, but Proposition 3.1 ii) implies that $\mathfrak{b}_{mr} \subseteq \mathfrak{b}_m^r$ for every $m, r \geq 1$. Furthermore, we have $\mathfrak{b}_{m_1} \subseteq \mathfrak{b}_{m_2}$ for $m_1 > m_2$. It is easy to deduce that if Z is an irreducible proper closed subset of X, then

$$\mathrm{ord}_Z(\mathfrak{b}_\bullet) := \sup_m \frac{\mathrm{ord}_Z(\mathfrak{b}_m)}{m} = \lim_{m\to\infty} \frac{\mathrm{ord}_Z(\mathfrak{b}_m)}{m}$$

(see [**JM**, Lemma 2.6]).

Proposition 3.2. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X and $\mathfrak{b}_\bullet$ is the corresponding sequence of asymptotic test ideals, then for every irreducible proper closed subset Z of X, we have $\mathrm{ord}_Z(\mathfrak{a}_\bullet) = \mathrm{ord}_Z(\mathfrak{b}_\bullet)$.

Proof. Since $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for every m, it is clear that we have $\mathrm{ord}_Z(\mathfrak{a}_m) \geq \mathrm{ord}_Z(\mathfrak{b}_m)$ for all m, hence $\mathrm{ord}_Z(\mathfrak{a}_\bullet) \geq \mathrm{ord}_Z(\mathfrak{b}_\bullet)$. In order to prove the reverse inequality, given $m \geq 1$, let us choose r such that $\mathfrak{b}_m = \tau(\mathfrak{a}_{mr}^{1/r})$. It follows from Proposition 3.3 below that

$$\mathrm{ord}_Z(\mathfrak{b}_\bullet) \geq \frac{\mathrm{ord}_Z(\mathfrak{b}_m)}{m} > \frac{\mathrm{ord}_Z(\mathfrak{a}_{mr})}{mr} - \frac{\mathrm{codim}(Z,X)}{m} \geq \mathrm{ord}_Z(\mathfrak{a}_\bullet) - \frac{\mathrm{codim(Z,X)}}{m}.$$

Since this holds for every $m \geq 1$, we deduce that $\mathrm{ord}_Z(\mathfrak{b}_\bullet) \geq \mathrm{ord}_Z(\mathfrak{a}_\bullet)$, which completes the proof of the proposition. $\square$

The next proposition is an instance of the fact that "the test ideal is contained in the multiplier ideal", which goes back to [**HY**, Theorem 3.4]. We give a direct proof, since the argument is particularly transparent in our setting.

Proposition 3.3. If $\mathfrak{a}$ is a nonzero ideal on X and Z is an irreducible proper closed subset of X, then for every $\lambda \in \mathbf{R}_{\geq 0}$ we have

$$\mathrm{ord}_Z(\tau(\mathfrak{a}^\lambda)) > \lambda \cdot \mathrm{ord}_Z(\mathfrak{a}) - \mathrm{codim}(Z, X).$$

Proof. Since construction of test ideals commutes with restriction to an open subset, after replacing X by a suitable open neighborhood of the generic point of Z, we may assume that Z is smooth. Let $\pi\colon Y \to X$ be the blow-up of X along Z, with exceptional divisor E. If $c = \mathrm{codim}(Z, X)$, then the relative canonical divisor $K_{Y/X}$ is equal to $(c-1)E$.

We have a commutative diagram

$$\begin{array}{ccc} F_*^e\pi_*(\omega_Y) & \xrightarrow{\pi_*(\mathrm{Tr}_Y^e)} & \pi_*(\omega_Y) \\ {\scriptstyle F_*^e(\rho)}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ F_*^e(\omega_X) & \xrightarrow{\mathrm{Tr}_X^e} & \omega_X \end{array} \tag{3.2}$$

in which the vertical maps are isomorphisms. Note that if J is an ideal in $\mathcal{O}_Y$, then $\rho(\pi_*(J \cdot \omega_Y)) = \pi_*(J \cdot \mathcal{O}_Y(K_{Y/X})) \cdot \omega_X$.

Given the nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_X$, we put $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_Y$, and consider $M := F_*^e(\mathfrak{a}^m \cdot \omega_X)$. Since $K_{Y/X}$ is effective, we have $F_*^e(\rho)^{-1}(M) \subseteq F_*^e\pi_*(\mathfrak{b}^m \cdot \omega_Y)$, and using the commutativity of the above diagram to compute $\mathrm{Tr}_X^e(M)$ gives

$$(\mathfrak{a}^m)^{[1/p^e]} \subseteq \pi_*(\mathcal{O}_Y(K_{Y/X}) \cdot (\mathfrak{b}^m)^{[1/p^e]}).$$

If $s = \mathrm{ord}_Z(\mathfrak{a})$, then $\mathfrak{b} \subseteq \mathcal{O}_Y(-sE)$, and since E is nonsingular we have

$$\mathcal{O}_Y(K_{Y/X}) \cdot (\mathfrak{b}^m)^{[1/p^e]} \subseteq \mathcal{O}_Y((c - 1 - \lfloor ms/p^e \rfloor)E).$$

For a fixed $\lambda \in \mathbf{R}_{\geq 0}$, let us take $m = \lceil \lambda p^e \rceil$ with $e \gg 0$, so that $\lfloor ms/p^e \rfloor = \lfloor \lambda s \rfloor$, and we conclude

$$\mathrm{ord}_Z(\tau(\mathfrak{a}^\lambda)) = \mathrm{ord}_Z((\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}) \geq \lfloor \lambda s \rfloor - c + 1,$$

which is equivalent with the inequality in the proposition. $\square$

In the following sections we will be interested in the case when X is projective and D is a divisor on X such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m. We then denote by $\tau(\lambda \cdot \parallel D \parallel)$ the asymptotic test ideal of exponent λ associated to the graded sequence $(\mathfrak{a}_{|mD|})_{m \geq 1}$. If D is a $\mathbf{Q}$-divisor such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some m such that mD is integral, then we put $\tau(\lambda \cdot \parallel D \parallel) := \tau(\lambda/r \cdot \parallel rD \parallel)$ for every r such that rD has integer coefficients. If $\lambda = 1$, then we simply write $\tau(\parallel D \parallel)$. It is clear from definition that if $\lambda \in \mathbf{Q}_{\geq 0}$, then $\tau(\lambda \cdot \parallel D \parallel) = \tau(\parallel \lambda D \parallel)$ (note that when $\lambda = 0$, both sides are trivially equal to $\mathcal{O}_X$). In particular, we see using Proposition 3.1 i) that if D is as above and $\lambda_1 \leq \lambda_2$ are in $\mathbf{Q}_{\geq 0}$, then $\tau(\parallel \lambda_2 D \parallel) \subseteq \tau(\parallel \lambda_1 D \parallel)$.

4. A uniform global generation result

In this section we prove the main technical result of the paper. Let X be a smooth projective variety over an algebraically closed field k of positive characteristic. We denote by K_X a canonical divisor (that is, we have $\mathcal{O}_X(K_X) \simeq \omega_X$). We put $n = \dim(X)$, and consider an ample divisor H on X, such that $\mathcal{O}_X(H)$ is globally generated.

THEOREM 4.1. *With the above notation, let D and E be divisors on X, and $\lambda \in \mathbf{Q}_{\geq 0}$. If $\mathcal{O}_X(D)$ has non-negative Iitaka dimension, and $E - \lambda D$ is nef, then the sheaf*

$$\tau(\lambda\cdot \parallel D \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + E + dH)$$

is globally generated for every $d \geq n+1$.

The proof that we give below follows the proof of [**Sch**, Theorem 4.3], which in turn makes use of an argument of Keeler [**Kee**] and Hara (unpublished). In our proof we also use of the following theorem of Fujita [**Fuj**]. If $\mathcal{F}$ is a coherent sheaf on X and A is an ample divisor, then there is ℓ_0 such that $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell A + P)) = 0$ for every $i \geq 1$, every $\ell \geq \ell_0$, and every nef divisor P.

PROOF OF THEOREM 4.1. By definition of the asymptotic test ideal, we can find m such that if $\mathfrak{a}_m$ is the ideal defining the base-locus of $|mD|$, then we have $\tau(\lambda\cdot \parallel D \parallel) = \tau(\mathfrak{a}_m^{\lambda/m})$. Let us fix such m. For $e \gg 0$ we have $\tau(\mathfrak{a}_m^{\lambda/m}) = (\mathfrak{a}_m^{\lceil \lambda p^e/m \rceil})^{[1/p^e]}$, and therefore there is a surjective map

$$F^e_*(\mathfrak{a}_m^{\lceil \lambda p^e/m \rceil} \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X)) \to \tau(\lambda\cdot \parallel D \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X). \tag{4.1}$$

By tensoring with $\mathcal{O}_X(E + dH)$ and using the projection formula, we obtain the surjective map

(4.2)

$$F^e_*(\mathfrak{a}_m^{\lceil \lambda p^e/m \rceil} \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + p^e(E + dH))) \to \tau(\lambda\cdot \parallel D \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + E + dH).$$

On the other hand, we have by definition a surjective map

$$H^0(X, \mathcal{O}_X(mD)) \otimes_k \mathcal{O}_X(-mD) \to \mathfrak{a}_m,$$

hence a surjective map

$$W \otimes_k \mathcal{O}_X(-m\lceil \lambda p^e/m \rceil D) \to \mathfrak{a}_m^{\lceil \lambda p^e/m \rceil}, \tag{4.3}$$

where $W = \mathrm{Sym}^{\lceil \lambda p^e/m \rceil} H^0(X, \mathcal{O}_X(mD))$. Tensoring (4.3) with the line bundle $\mathcal{O}_X(K_X + p^e(E + dH))$ and pushing forward by F^e (note that F^e_* is exact since the Frobenius morphism is affine), we obtain a surjective map

(4.4)

$$W \otimes_k F^e_* \mathcal{O}_X(K_X + p^e(E+dH) - m\lceil \lambda p^e/m \rceil D) \to F^e_*(\mathfrak{a}_m^{\lceil \lambda p^e/m \rceil} \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + p^e(E+dH))).$$

It follows from the surjective maps (4.2) and (4.4) that in order to complete the proof of the theorem, it is enough to show that for $e \gg 0$, the sheaf

$$F^e_* \mathcal{O}_X(K_X + p^e(E + dH) - m\lceil \lambda p^e/m \rceil D)$$

is globally generated. In fact, it is enough to show that this sheaf is 0-regular in the sense of Castelnuovo-Mumford regularity with respect to the ample globally generated line bundle $\mathcal{O}_X(H)$ (we refer to [**Laz**, §1.8] for basic facts about

Castelnuovo-Mumford regularity). Therefore, it is enough to show that if $e \gg 0$, then

$$H^i(X, \mathcal{O}_X(-iH) \otimes_{\mathcal{O}_X} F^e_*\mathcal{O}_X(K_X + p^e(E + dH) - m\lceil \lambda p^e/m \rceil D)) = 0 \tag{4.5}$$

for all i with $1 \leq i \leq n$.

Using again the projection formula and the fact that F^e is affine, we obtain

$$\begin{aligned} &H^i(X, \mathcal{O}_X(-iH) \otimes_{\mathcal{O}_X} F^e_*\mathcal{O}_X(K_X + p^e(E + dH) - m\lceil \lambda p^e/m \rceil D)) \\ &\simeq H^i(X, F^e_*\mathcal{O}_X(K_X + p^e(E + (d-i)H) - m\lceil \lambda p^e/m \rceil D)) \\ &\simeq H^i(X, \mathcal{O}_X(K_X + p^e(E + (d-i)H) - m\lceil \lambda p^e/m \rceil D)). \end{aligned}$$

Note that by assumption $d - i \geq 1$ for $i \leq n$.

Claim. We can find finitely many divisors $T_1, \ldots, T_r$ on X that satisfy the following property: for every e, there is j such that the difference $p^eE - m\lceil \lambda p^e/m \rceil D - T_j$ is nef. If this is the case, then by applying Fujita's vanishing theorem to each of the sheaves $\mathcal{F}_j = \mathcal{O}_X(K_X + T_j)$ and to the ample divisor $(d - i)H$ we obtain

$$H^i(X, \mathcal{O}_X(K_X + p^e(E + (d-i)H) - m\lceil \lambda p^e/m \rceil D)) = 0$$

for all i with $1 \leq i \leq n$ and all $e \gg 0$. Therefore in order to finish the proof, it is enough to check the assertion in the claim.

Let us write $\lambda = \frac{a}{b}$, for non-negative integers a and b, with b nonzero. For every $e \geq 1$, we write $p^e = mbs + t$, for non-negative integers s and t, with $t < mb$. In this case $\lceil \lambda p^e/m \rceil = as + \lceil \frac{at}{bm} \rceil$, hence

$$p^eE - m\lceil \lambda p^e/m \rceil D = ms(bE - aD) + \left(tE - m\left\lceil \frac{at}{bm} \right\rceil D\right),$$

and the claim follows since $bE - aD$ is nef by assumption, and t can only take finitely many values. This completes the proof of the theorem. □

REMARK 4.2. In Theorem 4.1, we may allow D to be a **Q**-divisor: in this case we may simply replace D by mD and λ by λ/m, with m divisible enough.

5. Applications to asymptotic test ideals of divisors

In this section we give some consequences of Theorem 4.1 to general properties of asymptotic test ideals. From now on, we always assume that X is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. Our first result says that the asymptotic test ideals of a big **Q**-divisor only depend on the numerical equivalence class of the divisor.

PROPOSITION 5.1. *If D and E are numerically equivalent big **Q**-divisors on X, then*

$$\tau(\lambda \cdot \parallel D \parallel) = \tau(\lambda \cdot \parallel E \parallel)$$

for every $\lambda \in \mathbf{R}_{\geq 0}$.

PROOF. The proof follows as in the case of multiplier ideals in characteristic zero, see [**Laz**, Example 11.3.12]. After replacing D and E by multiples, we may clearly assume that both D and E have integer coefficients. Let H be a very ample divisor and $n = \dim(X)$. Since D is big, there is a positive integer ℓ such that $\ell D - (K_X + (n+1)H)$ is linearly equivalent with an effective divisor G. It follows from Theorem 4.1 that $\tau(\| (m-\ell)D \|) \otimes_{\mathcal{O}_X} \mathcal{O}_X(mE-G)$ is globally generated for every $m \geq \ell$, hence $\tau(\| (m-\ell)D \|)$ is contained in the ideal $\mathfrak{a}_{|mE-G|}$. Therefore

$$\mathcal{O}_X(-G) \cdot \tau(\| (m-\ell)D \|) \subseteq \mathfrak{a}_{|G|} \cdot \mathfrak{a}_{|mE-G|} \subseteq \mathfrak{a}_{|mE|}$$

and we deduce

$$\mathcal{O}_X(-G) \cdot \mathfrak{a}_{|mD|} \subseteq \mathcal{O}_X(-G) \cdot \tau(\| mD \|) \subseteq \mathcal{O}_X(-G) \cdot \tau(\| (m-\ell)D \|) \subseteq \mathfrak{a}_{|mE|}$$

for every $m \geq \ell$. Proposition 3.1 iv) implies $\tau(\lambda \cdot \| D \|) \subseteq \tau(\lambda \cdot \| E \|)$ for every $\lambda \in \mathbf{R}_{\geq 0}$, and the reverse inclusion follows by symmetry. $\square$

Proposition 5.2. If D is a divisor on X such that $\mathcal{O}_X(D)$ has non-negative Iitaka dimension, then the set of F-jumping numbers of the graded sequence of ideals $(\mathfrak{a}_{|mD|})_{m \geq 1}$ is discrete.

Proof. It is enough to show that for every $\lambda_0 > 0$, there are only finitely many different values for $\tau(\lambda \cdot \| D \|)$, with $\lambda < \lambda_0$. Furthermore, it follows from Proposition 3.1 iii) that it is enough to only consider $\lambda \in \mathbf{Q}_{\geq 0}$.

Let H be a very ample divisor on X and let $\dim(X) = n$. We also fix a divisor A such that both A and $A - \lambda_0 D$ are nef (for example, A could be a large multiple of an ample divisor). In this case $A - \lambda D$ is nef for every λ with $0 \leq \lambda \leq \lambda_0$. For every such λ which is rational, Theorem 4.1 implies that

$$\tau(\lambda \cdot \| D \|) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + A + (n+1)H)$$

is globally generated. In particular, its space of global sections V_λ, which is a linear subspace of $V := H^0(X, \mathcal{O}_X(K_X + A + (n+1)H))$, determines $\tau(\lambda \cdot \| D \|)$. Furthermore, if $\lambda_r < \ldots < \lambda_1 < \lambda_0$, then $V_{\lambda_1} \subseteq \ldots \subseteq V_{\lambda_r}$. Since V is finite-dimensional, this clearly bounds the number of distinct values for $\tau(\lambda \cdot \| D \|)$ with $\lambda < \lambda_0$. $\square$

In characteristic zero, Hacon used global generation results to attach a type of asymptotic multiplier ideal to a pseudo-effective divisor. The analogous construction works also in positive characteristic, as follows. Suppose that D is a pseudo-effective $\mathbf{R}$-divisor on X. For every ample $\mathbf{R}$-divisor A, the sum $D + A$ is big. In particular, if $D + A$ is a $\mathbf{Q}$-divisor, then we may consider $\tau(\lambda \cdot \| D + A \|)$ for every $\lambda \in \mathbf{R}_{\geq 0}$.

Proposition 5.3. For every pseudo-effective $\mathbf{R}$-divisor D and all $\lambda \in \mathbf{R}_{\geq 0}$, there is a unique minimal element, that we denote by $\tau_+(\lambda \cdot \| D \|)$, among all ideals of the form $\tau(\lambda \cdot \| D + A \|)$, where A varies over the ample $\mathbf{R}$-divisors such that $D + A$ is a $\mathbf{Q}$-divisor. Furthermore, there is an open neighborhood $\mathcal{U}$ of the origin in $\mathrm{N}^1(X)_{\mathbf{R}}$ such that

$$\tau_+(\lambda \cdot \| D \|) = \tau(\lambda \cdot \| D + A \|)$$

for every ample divisor A with $D + A$ a $\mathbf{Q}$-divisor and such that the class of A lies in $\mathcal{U}$.

Proof. Note first that if A_1 and A_2 are ample divisors with both $D + A_1$ and $D + A_2$ having $\mathbf{Q}$-coefficients and such that $A_1 - A_2$ is ample, then $\mathfrak{a}_{|m(D+A_2)|} \subseteq \mathfrak{a}_{|m(D+A_1)|}$ for all $m \gg 0$. This implies

$$\tau(\lambda \cdot \| D + A_2 \|) \subseteq \tau(\lambda \cdot \| D + A_1 \|).$$

Choose a very ample divisor H on X and put $n = \dim(X)$. Suppose now that B is a fixed ample divisor such that $B - \lambda D$ is ample. If A is an ample $\mathbf{R}$-divisor such that $D + A$ is a $\mathbf{Q}$-divisor and $B - \lambda(D + A)$ is ample, then Theorem 4.1 implies that

$$\tau(\lambda\cdot \parallel D + A \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + (n+1)H + B)$$

is globally generated (if λ is not rational, then we apply the theorem to some rational $\lambda' > \lambda$ such that $B - \lambda'(D+A)$ is still ample and $\tau(\lambda\cdot \parallel D+A \parallel) = \tau(\lambda'\cdot \parallel D+A \parallel)$). In particular, we see that $\tau(\lambda\cdot \parallel D + A \parallel)$ is determined by the subspace

$$\begin{aligned} W_A &:= H^0(X, \tau(\lambda\cdot \parallel D + A \parallel) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + (n+1)H + B)) \\ &\subseteq W := H^0(X, \mathcal{O}_X(K_X + (n+1)H + B)). \end{aligned}$$

Since W is finite dimensional, we can find some A as above such that W_A is minimal among all such subspaces. Given any ample A_1 such that $D + A_1$ is a $\mathbf{Q}$-divisor, we may choose an ample A_2 such that both $A - A_2$ and $A_1 - A_2$ are ample $\mathbf{Q}$-divisors. As we have seen, this implies

$$\tau(\lambda\cdot \parallel D+A_2 \parallel) \subseteq \tau(\lambda\cdot \parallel D+A_1 \parallel) \text{ and } \tau(\lambda\cdot \parallel D+A_2 \parallel) \subseteq \tau(\lambda\cdot \parallel D+A \parallel). \tag{5.1}$$

Note that $B - \lambda(D + A_2)$ is ample, and the second inclusion in (5.1) implies in particular that $W_{A_2} \subseteq W_A$. By the minimality in the choice of A we have, in fact, $W_A = W_{A_2}$, and therefore

$$\tau(\lambda\cdot \parallel D + A_2 \parallel) = \tau(\lambda\cdot \parallel D + A \parallel) \subseteq \tau(\lambda\cdot \parallel D + A_1 \parallel).$$

This shows that $\tau(\lambda\cdot \parallel D + A \parallel)$ satisfies the minimality requirement in the proposition.

Suppose now that $\mathcal{U}$ consists of the classes of those E such that $A - E$ is ample. In this case $\mathcal{U}$ is an open neighborhood of the origin in $\mathrm{N}^1(X)_{\mathbf{R}}$ which satisfies the last assertion in the proposition. Indeed, if A' is ample such that $D + A'$ is a $\mathbf{Q}$-divisor and the class of A' lies in $\mathcal{U}$, then the argument at the beginning of the proof gives the inclusion $\tau(\lambda\cdot \parallel D + A' \parallel) \subseteq \tau(\lambda\cdot \parallel D + A \parallel)$, while the reverse inclusion follows from the minimality of $\tau(\lambda\cdot \parallel D + A \parallel)$, which we have proved. □

In the next proposition we list several properties of this new version of asymptotic test ideals.

Proposition 5.4. Let D be a pseudo-effective $\mathbf{R}$-divisor on X and let $\lambda \in \mathbf{R}_{\geq 0}$.

i) If E is a pseudo-effective $\mathbf{R}$-divisor on X, numerically equivalent to D, then
$$\tau_+(\lambda\cdot \parallel D \parallel) = \tau_+(\lambda\cdot \parallel E \parallel).$$

ii) If $\mu \geq \lambda$, then
$$\tau_+(\mu\cdot \parallel D \parallel) \subseteq \tau_+(\lambda\cdot \parallel D \parallel).$$

iii) If B is a nef $\mathbf{R}$-divisor, then
$$\tau_+(\lambda\cdot \parallel D \parallel) \subseteq \tau_+(\lambda\cdot \parallel D + B \parallel).$$

iv) We have $\tau_+(\lambda\cdot \parallel D \parallel) = \tau_+(\parallel \lambda D \parallel)$.

Proof. The assertion in i) follows from definition, once we note that if A is ample, then we can write $D + A = E + (A + D - E)$ and $A + D - E$ is ample. The inclusion in ii) follows from definition and the fact that for every ample $\mathbf{R}$-divisor A such that $D + A$ has rational coefficients, we have

$$\tau(\mu\cdot \parallel D + A \parallel) \subseteq \tau(\lambda\cdot \parallel D + A \parallel).$$

In order to prove iii), let A be ample such that $D+B+A$ is a $\mathbf{Q}$-divisor, and the class of A in $\mathrm{N}^1(X)_{\mathbf{R}}$ is in a small enough neighborhood of the origin, so that

$$\tau_+(\lambda\cdot \parallel D+B \parallel) = \tau(\lambda\cdot \parallel D+B+A \parallel).$$

Since B is nef, $A+B$ is ample, hence we can find an ample divisor A' such that $A+B-A'$ is ample and $D+A'$ is a $\mathbf{Q}$-divisor. In this case

$$\tau(\lambda\cdot \parallel D+B+A \parallel) \supseteq \tau(\lambda\cdot \parallel D+A' \parallel) \supseteq \tau_+(\lambda\cdot \parallel D \parallel),$$

where the first inclusion follows from the fact that $\mathfrak{a}_{|m(D+A+B)|} \supseteq \mathfrak{a}_{|m(D+A')|}$ for all m divisible enough.

Let us now prove iv). Suppose that A is an ample divisor such that $D+A$ has rational coefficients and the class of A in $\mathrm{N}^1(X)_{\mathbf{R}}$ lies in a sufficiently small neighborhood of the origin. If $\lambda' > \lambda$ is rational and close enough to λ (depending on A), then

$$\tau_+(\lambda\cdot \parallel D \parallel) = \tau(\lambda\cdot \parallel D+A \parallel) = \tau(\lambda'\cdot \parallel D+A \parallel) = \tau(\parallel \lambda'(D+A) \parallel).$$

On the other hand, the difference $\lambda'(D+A) - \lambda D = (\lambda'-\lambda)D + \lambda' A$ is ample if $\lambda' - \lambda$ is small enough, hence

$$\tau_+(\parallel \lambda D \parallel) \subseteq \tau(\parallel \lambda'(D+A) \parallel) = \tau_+(\lambda\cdot \parallel D \parallel).$$

In order to prove the reverse inclusion, let us choose an ample $\mathbf{R}$-divisor B such that $\lambda D + B$ is a $\mathbf{Q}$-divisor and $\tau_+(\parallel \lambda D \parallel) = \tau(\parallel \lambda D + B \parallel)$. Since B is ample, we can choose an ample $\mathbf{R}$-divisor A' such that $B - \lambda A'$ is ample and $D + A'$ is a $\mathbf{Q}$-divisor. We can choose now $\mu > \lambda$ such that $\mu \in \mathbf{Q}$ and $\mu - \lambda$ is small enough, so that

$$(\lambda D + B) - \mu(D + A') = (\lambda - \mu)D + (B - \mu A')$$

is ample. Furthermore, since $\mu - \lambda \ll 1$, we have

$$\tau(\lambda\cdot \parallel D+A' \parallel) = \tau(\mu\cdot \parallel D+A' \parallel) = \tau(\parallel \mu(D+A') \parallel) \subseteq \tau(\parallel \lambda D+B \parallel) = \tau_+(\parallel \lambda D \parallel),$$

hence by definition we obtain $\tau_+(\lambda\cdot \parallel D \parallel) \subseteq \tau_+(\parallel \lambda D \parallel)$. This completes the proof of iv). □

REMARK 5.5. In general, even for a big $\mathbf{Q}$-divisor D, the two ideals $\tau_+(\lambda\cdot \parallel D \parallel)$ and $\tau(\lambda\cdot \parallel D \parallel)$ might be different. Suppose, for example, that $\pi\colon X \to W$ is the blow-up of a smooth projective variety W of dimension ≥ 2 at a point, and E is the exceptional divisor. Let H be a very ample divisor on W such that $\pi^*(H) - E$ is ample. Note that for every non-negative integers r and s, the ideal $\mathfrak{a}_{|r\pi^*(H)+sE|}$ is equal to $\mathcal{O}_X(-sE)$. Using this, it is easy to see that if $D = \pi^*(H) + E$, then for every positive integer m we have

$$\tau(m\cdot \parallel D \parallel) = \mathcal{O}_X(-mE) \text{ and } \tau_+(m\cdot \parallel D \parallel) = \mathcal{O}_X(-(m-1)E).$$

6. The non-nef locus of big divisors

In this section we prove Theorems B and C stated in the Introduction (in a more general version, in which we sometimes do not need to restrict to closed points). As in the previous sections, we assume that X is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. Let $n = \dim(X)$.

THEOREM 6.1. *Let Z be an irreducible proper closed subset of X. For a big* $\mathbf{Q}$*-divisor D, the value* $\mathrm{ord}_Z(\| D \|)$ *only depends on the numerical equivalence class of D. Furthermore, the function* $D \to \mathrm{ord}_Z(\| D \|)$ *extends as a continuous function on* $\mathrm{Big}(X)_{\mathbf{R}}$*, also denoted by* $\mathrm{ord}_Z(\| - \|)$.

PROOF. For the first assertion, by homogeneity we may assume that D has integer coefficients. If $\mathfrak{b}_m = \tau(m \cdot \| D \|)$, then Proposition 3.2 implies

$$\mathrm{ord}_Z(\| D \|) = \sup_{m \geq 1} \frac{\mathrm{ord}_Z(\mathfrak{b}_m)}{m}.$$

Since the ideals $\mathfrak{b}_m$ only depend on the numerical equivalence class of D by Proposition 5.1, we obtain the first assertion in the theorem. The second assertion now follows as in [**ELMNP**, §3], where the argument is characteristic-free. □

THEOREM 6.2. *Let Z be an irreducible proper closed subset of X and assume that either the ground field k is uncountable, or Z consists of a point. If D is a big divisor on X, then the following are equivalent:*

i) *Z is not contained in* $\mathbf{B}_-(D)$.
ii) *There is a divisor G on X such that Z is not contained in the base locus of $|mD + G|$ for every $m \geq 1$.*
iii) *There is a real number M such that* $\mathrm{ord}_Z(\mathfrak{a}_{|mD|}) \leq M$ *for all m with $|mD|$ non-empty.*
iv) $\mathrm{ord}_Z(\| D \|) = 0$.
v) *For every $m \geq 1$, the ideal $\tau(\| mD \|)$ does not vanish along Z.*

PROOF. The proof is similar to the one in characteristic zero (see [**ELMNP**, §2]). With the notation in the proof of Theorem 6.1, we see that $\mathrm{ord}_Z(\| D \|) = 0$ if and only if $\mathrm{ord}_Z(\mathfrak{b}_m) = 0$ for every $m \geq 1$. This proves the equivalence iv)⇔v).

We now prove the implications v)⇒ii)⇒i)⇒iv). Let us show v)⇒ii). Let H be a very ample divisor on X and let $G = K_X + (n+1)H$. It follows from Theorem 4.1 that the sheaf $\tau(\| mD \|) \otimes_{\mathcal{O}_X} \mathcal{O}_X(mD + G)$ is globally generated. If v) holds, this implies that for every $m \geq 1$ there is a divisor in $|mD + G|$ that does not vanish at the generic point of Z, hence ii).

For the implication ii)⇒i) we make use of the hypothesis on Z and k. This implies that if Z is contained in a countable union of closed subsets, then it is contained in one of these sets. Therefore if $Z \subseteq \mathbf{B}_-(D)$, then there is an ample $\mathbf{Q}$-divisor A such that $Z \subseteq \mathbf{B}(D + A)$. However, if G is as in ii), then for $m \gg 0$ the divisor $mA - G$ is ample, hence $Z \subseteq \mathbf{B}(D + A) = \mathbf{B}(mD + mA) \subseteq \mathbf{B}(mD + G)$, a contradiction.

We now show i)⇒iv). Let $(A_i)_{i \geq 1}$ be a sequence of ample $\mathbf{Q}$-divisors whose classes in $\mathrm{N}^1(X)_{\mathbf{R}}$ converge to zero. It follows from i) that $Z \not\subseteq \mathbf{B}(D + A_i)$ for any i, which in turn implies that $\mathrm{ord}_Z(\mathfrak{a}_{|r(D+A_i)|}) = 0$ for r divisible enough. We thus deduce that $\mathrm{ord}_Z(\| D + A_i \|) = 0$ for all i, and by continuity of $\mathrm{ord}_Z(\| - \|)$ we conclude that $\mathrm{ord}_Z(\| D \|) = 0$.

In order to complete the proof of the theorem it is enough to also show the implications ii)⇒iii)⇒iv). Suppose first that ii) holds. Since D is big, there is a positive integer m_0 and an effective divisor T linearly equivalent to $m_0 D - G$. In this case $T + mD + G$ is linearly equivalent to $(m + m_0)D$, and the assumption in ii) implies $\mathrm{ord}_Z(\mathfrak{a}_{|(m+m_0)D|}) \leq \mathrm{ord}_Z \mathcal{O}_X(-T)$. This gives iii), by taking M to be the maximum of $\mathrm{ord}_Z \mathcal{O}_X(-T)$ and of those $\mathrm{ord}_Z(\mathfrak{a}_{|mD|})$, with $m \leq m_0$ and

$|mD|$ non-empty. Since iii) clearly implies iv), this completes the proof of the theorem. $\square$

REMARK 6.3. If in Theorem 6.2 we allow D to have rational coefficients, the equivalence between i), iv), and v) still holds. Indeed, it is enough to apply the theorem to rD, where r is a positive integer such that rD has integer coefficients.

COROLLARY 6.4. If D is a big $\mathbf{Q}$-divisor on X, then D is nef if and only if $\tau(\| mD \|) = \mathcal{O}_X$ for every $m \geq 1$.

PROOF. Note that D is nef if and only if $x \notin \mathbf{B}_-(D)$ for every $x \in X$. By Theorem 6.2, this is equivalent with the fact that $\tau(\| mD \|)$ does not vanish at x, for every $x \in X$ and every $m \geq 1$. $\square$

COROLLARY 6.5. If Z and k satisfy the condition in Theorem 6.2, then for every big $\mathbf{R}$-divisor D on X, we have $Z \not\subseteq \mathbf{B}_-(D)$ if and only if $\mathrm{ord}_Z(\| D \|) = 0$.

PROOF. If Z is not contained in $\mathbf{B}_-(D)$, then we obtain $\mathrm{ord}_Z(\| D \|) = 0$ arguing as in the proof of the implication i)$\Rightarrow$iv) in Theorem 6.2. Conversely, suppose that we have $\mathrm{ord}_Z(\| D \|) = 0$. Let us consider a sequence of ample $\mathbf{R}$-divisors $(A_m)_{m\geq 1}$ whose classes in $\mathrm{N}^1(X)_{\mathbf{R}}$ converge to zero and such that all $D + A_m$ are $\mathbf{Q}$-divisors. It is easy to see that $\mathrm{ord}_Z(\| D + A_m \|) \leq \mathrm{ord}_Z(\| D \|) = 0$, hence applying Theorem 6.2 (see also Remark 6.3) we get $Z \not\subseteq \mathbf{B}_-(D + A_m)$ for every m. Under our assumptions on Z and k this implies that Z is not contained in $\mathbf{B}_-(D)$ (see Proposition 2.2 vi)). $\square$

7. The case of pseudo-effective divisors

The picture at the boundary of the pseudo-effective cone is more complicated. In particular, the function $\mathrm{ord}_Z(\| - \|)$ might not admit a continuous extension to the pseudo-effective cone, see [**Nak**, IV.2.8]. In this section we explain, following the approach in [**Nak**], how the results in the previous section need to be modified in this context.

If D is a preudo-effective $\mathbf{R}$-divisor on X, then for every ample $\mathbf{R}$-divisor A, we know that $D + A$ is big. If Z is an irreducible proper closed subset of X, then we put

$$\sigma_Z(D) := \sup_A \mathrm{ord}_Z(\| D + A \|) \in \mathbf{R}_{\geq 0} \cup \{\infty\},$$

where the supremum is over all ample $\mathbf{R}$-divisors A. Note that if A_1 and A_2 are ample and $A_1 - A_2$ is ample, then $\mathrm{ord}_Z(\| D + A_1 \|) \leq \mathrm{ord}_Z(\| D + A_2 \|)$. It is then easy to deduce that if $(A_m)_{m\geq 1}$ is a sequence of ample divisors whose classes in $\mathrm{N}^1(X)_{\mathbf{R}}$ converge to zero, then $\sigma_Z(D) = \lim_{m\to\infty} \mathrm{ord}_Z(\| D + A_m \|)$. Using the continuity of $\mathrm{ord}_Z(\| - \|)$ on the big cone, we see that $\sigma_Z(D) = \mathrm{ord}_Z(\| D \|)$ if D is big.

It is straightforward to see from definition that $\sigma_Z(D)$ only depends on the equivalence class of D. Therefore we may and will consider σ_Z as a function on the pseudo-effective cone of X.

PROPOSITION 7.1. The function σ_Z is lower semi-continuous on the pseudo-effective cone.

PROOF. Note that by Theorem 6.1, each function φ_A given by

$$\varphi_A(D) = \mathrm{ord}_Z(\| D + A \|)$$

is continuous on the pseudo-effective cone (here A is an arbitrary ample $\mathbf{R}$-divisor). Since $\sigma_Z = \sup_A \varphi_A$, it follows that σ_Z is lower semi-continuous. $\square$

THEOREM 7.2. *Let Z be an irreducible proper closed subset of X and assume that either the ground field k is uncountable, or Z consists of a point. If D is a pseudo-effective divisor on X, then the following are equivalent:*

i) *Z is not contained in $\mathbf{B}_-(D)$.*
ii) *$\sigma_Z(D) = 0$.*
iii) *The ideal $\tau_+(\| mD \|)$ does not vanish along Z for any $m \geq 1$.*

PROOF. Let us fix a sequence of ample $\mathbf{R}$-divisors $(A_i)_{i\geq 1}$ whose classes in $\mathrm{N}^1(X)_{\mathbf{R}}$ converge to zero, and such that all $D + A_i$ have rational coefficients. By definition, we have $\sigma_Z(D) = 0$ if and only if $\mathrm{ord}_Z(\| D + A_i \|) = 0$ for all i. On the other hand, Proposition 2.2 vi) gives $\mathbf{B}_-(D) = \bigcup_i \mathbf{B}_-(D + A_i)$, hence our hypothesis on Z and k implies that $Z \not\subseteq \mathbf{B}_-(D)$ if and only if for every i we have $Z \not\subseteq \mathbf{B}_-(D + A_i)$. Therefore the equivalence of i) and ii) follows from the equivalence of i) and iv) in Theorem 6.2 (see Remark 6.3).

Suppose now that ii) holds, hence $\mathrm{ord}(\| D + A_i \|) = 0$ for all i. It follows from Theorem 6.2 (see also Remark 6.3) that for every $m \geq 1$, the ideal $\tau(\| m(D+A_i) \|)$ does not vanish along Z. Since $\tau_+(\| mD \|) = \tau(\| m(D + A_i) \|)$ for $i \gg 0$, we get the assertion in iii).

Suppose now that iii) holds. If ii) fails, then there is i with $\mathrm{ord}_Z(\| D+A_i \|) > 0$, hence for some $m \geq 1$, the ideal $\tau(\| m(D + A_i) \|)$ vanishes along Z. Since we have $\tau_+(\| mD \|) \subseteq \tau(\| m(D+A_i) \|)$, we obtain a contradiction with iii). This completes the proof of the theorem. $\square$

REMARK 7.3. It is shown in [**Nak**, Proposition II.1.10] that if D is a pseudo-effective divisor on X and $E_1, \ldots, E_r$ are prime divisors such that $\sigma_{E_i}(D) > 0$ for all i, then for every $\alpha_1, \ldots, \alpha_r \in \mathbf{R}_{\geq 0}$ and every i, one has

$$\sigma_{E_i}(\alpha_1 E_1 + \ldots + \alpha_r E_r) = \alpha_i$$

(note that the proof therein is characteristic-free). In particular, this implies that the classes of $E_1, \ldots, E_r$ in $\mathrm{N}^1(X)_{\mathbf{R}}$ are linearly independent, hence r is bounded above by the Picard number $\rho(X)$ of X. If we assume that the ground field is uncountable, we deduce using Theorem 7.2 that the number of irreducible codimension one subsets of $\mathbf{B}_-(D)$ is bounded above by $\rho(X)$.

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Clay Mathematics Proceedings
Volume **18**, 2013

Pairwise incident planes and Hyperkähler four-folds

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Abstract. We address the following question: what are the cardinalities of maximal finite families of pairwise incident planes in a complex projective space? One proves easily that the span of the planes has dimension 5 or 6. Up to projectivities there is one such family spanning a 6-dimensional projective space - this is an elementary result. Maximal finite families of pairwise incident planes in a 5-dimensional projective space are considerably more mysterious: they are linked to certain special (EPW) sextic hypersurfaces which have a non-trivial double cover, generically a hyperkaehler 4-fold. We prove that the cardinality of such a set cannot exceed 20. We also show that there exist such families of cardinality 16 - in fact we conjecture that 16 is the maximum.

1. Introduction

A family of pairwise incident lines in a projective space consists of lines through a point or lines contained in a plane. Is there an analogous characterization of families of pairwise incident planes in a complex projective space? A beautiful theorem of Ugo Morin [**6**] states that an algebraic **irreducible** family of pairwise incident planes is contained in one of the following families:

(1) Planes containing a fixed point.
(2) Planes whose intersection with a fixed plane has dimension at least 1.
(3) Planes contained in a fixed 4-dimensional projective space.
(4) One of the two irreducible components of the set of planes contained in a fixed smooth 4-dimensional quadric.
(5) The planes tangent to a fixed Veronese surface (image of $\mathbb{P}^2 \to |\mathcal{I}_{\mathbb{P}^2}(2)|^\vee$).
(6) The planes intersecting a fixed Veronese surface along a conic.

In the present paper we will address the following question: what are the cardinalities of finite families of pairwise incident planes? As stated the question is not interesting because the families of pairwise incident planes listed above contain sets of arbitrary finite cardinality. In order to formulate a meaningful question we recall the following definition of Morin: a family of pairwise incident planes is *complete* if there exists no plane outside the family which is incident to all planes in the family - in other words if the family is maximal. We ask the following question: what are the cardinalities of finite complete family of pairwise incident planes? Before stating our main result we will describe a finite complete family of pairwise incident

2010 *Mathematics Subject Classification.* Primary 14N10, 14J40.
Supported by PRIN 2010.

planes in $\mathbb{P}^6$. Let $\{v_0,\ldots,v_6\}$ be a basis of $\mathbb{C}^7$. Identify the set $\{[v_0],\ldots,[v_6]\}$ and $\mathbb{P}^2_{\mathbb{F}_2}$ (the projective plane on the field with 2 elements) as follows:

$$[v_0]\mapsto[010],\quad [v_1]\mapsto[011],\quad [v_2]\mapsto[001],\quad [v_3]\mapsto[101],\quad [v_4]\mapsto[100],\quad [v_5]\mapsto[110],\quad [v_6]\mapsto[111].$$

Given the above identification we let $\Lambda_1,\ldots,\Lambda_7\in\mathrm{Gr}(2,\mathbb{P}^6)$ be the planes spanned by the points on a line in $\mathbb{P}^2_{\mathbb{F}_2}$. Explicitly

$$\begin{aligned}(1.1)\quad &\Lambda_1=\mathbb{P}\langle v_0,v_1,v_2\rangle,\quad \Lambda_2=\mathbb{P}\langle v_2,v_3,v_4\rangle,\quad \Lambda_3=\mathbb{P}\langle v_0,v_4,v_5\rangle,\quad \Lambda_4=\mathbb{P}\langle v_1,v_3,v_5,\rangle,\\ &\Lambda_5=\mathbb{P}\langle v_0,v_3,v_6\rangle,\quad \Lambda_6=\mathbb{P}\langle v_1,v_4,v_6\rangle,\quad \Lambda_7=\mathbb{P}\langle v_2,v_5,v_6\rangle.\end{aligned}$$

As is easily checked the planes $\Lambda_1,\ldots,\Lambda_7$ are pairwise incident: we will show (see Claim 2.1) that they form a complete family.

THEOREM 1.1. *Let $T\subset\mathrm{Gr}(2,\mathbb{P}^N)$ be a finite complete family of pairwise incident planes. The planes in T span a projective space of dimension* 5 *or* 6*. If the span has dimension* 6 *then T is projectively equivalent to the family $\{\Lambda_1,\ldots,\Lambda_7\}$ described above. If the span has dimension* 5 *then T has at most* 20 *elements. For any $10\le k\le 16$ there exists a complete family of k pairwise incident planes: in fact it has at least $(20-k)$ moduli.*

In §2 we will study finite complete families of pairwise incident planes which span a projective space of dimension greater than 5: the proofs are of an elementary nature. In §3 we will make the connection between our question and the geometry of certain Hyperkähler 4-folds which are double covers of special sextic hypersurfaces in $\mathbb{P}^5$ named EPW-sextics. Then we will apply results of Ferretti [**4**] on degenerations of double EPW-sextics in order to show that there exist finite complete families of pairwise incident planes in $\mathbb{P}^5$ of cardinality between 10 and 16; we will also get the lower bound on the number of moduli given in Theorem 1.1. In §4 we will prove that a finite complete family of pairwise incident planes has cardinality at most 20.

A few comments. I suspect that 16 is the maximum cardinality of a finite complete family of pairwise incident planes. Our (we might say Ferretti's) proof that there exist complete families of pairwise incident planes of cardinality between 10 and 16 is a purely existential proof: it does not give explicit families. One may ask for explicit examples. The paper [**2**] of Dolgachev and Markushevich provides a general framework for the study of this problem. In particular the authors associate to a generic Fano model of an Enriques surface (plus a suitable choice of 10 elliptic curves on the surface) a finite collection of complete families of 10 pairwise incident planes in $\mathbb{P}^5$ - they also study the problem of classifying the irreducible components (there are several such) of the locus parametrizing ordered 10-tuples of pairwise incident planes in $\mathbb{P}^5$. In the same paper Dolgachev and Markushevich give explicit constructions of complete families of 13 pairwise incident planes.

Notation and conventions. We work throughout over $\mathbb{C}$. Let $T\subset\mathrm{Gr}(2,\mathbb{P}^N)$ be a family of planes: the *span* of T is the span of the union of the planes parametrized by T.

2. Families of pairwise incident planes in $\mathbb{P}^N$ for $N>5$

Let $T\subset\mathrm{Gr}(2,\mathbb{P}^N)$ be a finite complete family of pairwise incident planes. If the span of T is contained in a projective space M of dimension at most 4 then T is contained in the infinite family of pairwise incident planes $\mathrm{Gr}(2,M)$, that is a contradiction. Hence the span of T has dimension at least 5. In the present section we will classify finite complete family of pairwise incident planes whose span has dimension greater than 5. We will start by showing that the planes $\Lambda_1, \ldots, \Lambda_7 \subset \mathbb{P}^7$ defined by (1.1) form a complete family of pairwise incident planes. Let $v_0, \ldots, v_6$ be as in §1; we let

$$\mathbb{P}^5 := \mathbb{P}\langle v_0, \ldots, v_5\rangle. \tag{2.1}$$

The set of lines in $\mathbb{P}^5$ meeting $\Lambda_1, \Lambda_2, \Lambda_3$ has 4 irreducible components, each isomorphic to $\mathbb{P}^2$; more precisely

$$\begin{aligned}\{L\in \mathrm{Gr}(1,\mathbb{P}^5) \mid L\cap\Lambda_i\neq\emptyset,\ i=1,2,3\} = \mathrm{Gr}(1,\mathbb{P}\langle v_0,v_2,v_4\rangle)\cup \\ \cup\{\mathbb{P}\langle v_0,u\rangle \mid 0\neq u\in\langle v_2,v_3,v_4\rangle\}\cup\{\mathbb{P}\langle v_2,u\rangle \mid 0\neq u\in\langle v_0,v_4,v_5\rangle\}\cup\{\mathbb{P}\langle v_4,u\rangle \mid 0\neq u\in\langle v_0,v_1,v_2\rangle\}.\end{aligned} \tag{2.2}$$

In fact suppose that a line L intersects $\Lambda_1, \Lambda_2, \Lambda_3$ and does not belong to $\Lambda_0 := \mathbb{P}\langle v_0, v_2, v_4\rangle$. It suffices to show that one of $[v_0], [v_2], [v_4]$ belongs to L. Suppose the contrary and let $L\cap\mathbb{P}\langle v_0, v_2, v_4\rangle = \{p\}$. Then there exist at least two planes among $\Lambda_1, \Lambda_2, \Lambda_3$ which do not contain p, call them Λ_i, Λ_j. It follows that L belongs to the intersection $\mathbb{P}\langle\Lambda_0, \Lambda_i\rangle\cap\mathbb{P}\langle\Lambda_0, \Lambda_j\rangle$. The latter is equal to Λ_0, that is a contradiction. Equation (2.2) gives that there are exactly 3 lines in $\mathbb{P}^5$ meeting $\Lambda_1, \ldots, \Lambda_4$. More precisely let

$$L_5 := \mathbb{P}\langle v_0,v_3\rangle = \Lambda_5\cap\mathbb{P}^5, \quad L_6 := \mathbb{P}\langle v_1,v_4\rangle = \Lambda_6\cap\mathbb{P}^5, \quad L_7 := \mathbb{P}\langle v_2,v_5\rangle = \Lambda_7\cap\mathbb{P}^5. \tag{2.3}$$

Then

$$\{L \in \mathrm{Gr}(1,\mathbb{P}^5) \mid L\cap\Lambda_i \neq \emptyset,\ i = 1, 2, 3, 4\} = \{L_5, L_6, L_7\}. \tag{2.4}$$

CLAIM 2.1. *The collection of planes $\Lambda_1, \ldots, \Lambda_7 \subset \mathbb{P}^6$ defined by* (1.1) *is a complete family of pairwise incident planes.*

PROOF. We need to show that the family is complete. First we notice that the span of $\Lambda_1, \ldots, \Lambda_4$ is equal to $\mathbb{P}^5$, notation as in (2.1). Now let $\Lambda \subset \mathbb{P}^6$ be a plane intersecting $\Lambda_1, \ldots, \Lambda_7$. Since the intersection of $\Lambda_1, \ldots, \Lambda_4$ is empty one of the following holds:

(1) $\Lambda \subset \mathbb{P}^5$,
(2) $\dim(\Lambda\cap\mathbb{P}^5) = 1$.

Suppose that (1) holds. Then Λ meets each of the lines L_5, L_6, L_7 given by (2.3). Since L_5, L_6, L_7 generate $\mathbb{P}^5$ it follows that Λ intersects L_i in a single point p_i and that Λ is spanned by p_5, p_6, p_7. Imposing the condition that $\langle p_5, p_6, p_7\rangle$ (for $p_i \in L_i$) meet each of $\Lambda_1, \ldots, \Lambda_4$ we get that $\langle p_5, p_6, p_7\rangle$ is one of $\Lambda_1, \ldots, \Lambda_4$. This proves that if (1) holds then $\Lambda \in \{\Lambda_1, \ldots, \Lambda_4\}$. Next suppose that (2) holds and let $L = \Lambda\cap\mathbb{P}^5$. Then L meets each of $\Lambda_1, \ldots, \Lambda_4$. By (2.4) it follows that L equals one of L_5, L_6, L_7. Suppose that $L = L_5$. Then Λ meets Λ_6 and Λ_7 in points outside $\mathbb{P}^5$. Now notice that the span of $\Lambda, \Lambda_6, \Lambda_7$ is all of $\mathbb{P}^6$: it follows that $\Lambda, \Lambda_6, \Lambda_7$ meet in a single point, which is necessarily $[v_6]$. Thus $\Lambda = \Lambda_5$. If L equals one of L_6 or L_7 a similar argument shows that $\Lambda = \Lambda_6$ or $\Lambda = \Lambda_7$ respectively. □

Our next goal is to prove that if T is a finite complete family of pairwise incident planes spanning a projective space of dimension greater than 5 then T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$ where the planes $\Lambda_1, \ldots, \Lambda_7$ are defined by (1.1). First we make the following observation.

Proposition 2.2. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a family of pairwise incident planes. Suppose that there exist $\Lambda, \Lambda' \in T$ such that their intersection is a line. Then T is contained in an infinite family of pairwise incident planes.*

Proof. Let $L := \Lambda \cap \Lambda'$ and $M := \langle \Lambda, \Lambda' \rangle$. Thus L is a line and M is a 3-dimensional projective space. Let $\Lambda'' \in T$: since Λ'' intersects both Λ and Λ' one of the following holds:

(1) $\dim(\Lambda'' \cap M) \geq 1$,
(2) $\Lambda'' \cap L \neq \emptyset$.

Now let $\Lambda_0 \subset M$ be a plane containing L. If (1) holds then Λ_0 intersects $(\Lambda'' \cap M)$, if (2) holds then Λ_0 contains the non-empty intersection $(\Lambda'' \cap L)$: in both cases we get that Λ_0 intersects Λ''. Hence the union of T and the set of planes in M containing L is an infinite family of pairwise incident planes containing T. $\square$

The result below follows immediately from Proposition 2.2.

Corollary 2.3. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a finite complete family of pairwise incident planes. If $\Lambda, \Lambda' \in T$ are distinct their intersection is a single point.*

Proposition 2.4. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a finite complete family of pairwise incident planes. Suppose that the span of T has dimension greater than* 5. *Then T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$ where $\Lambda_1, \ldots, \Lambda_7$ are as in* (1.1).

Proof. Let $\Lambda_1, \Lambda_2 \in T$ be distinct: by Corollary 2.3 they intersect in a single point p and hence they span a 4-dimensional projective space M. We claim that there does exist $\Lambda_3 \in T$ which is not contained in M and which intersects Λ_1, Λ_2 in distinct points. In fact suppose the contrary. Then we get an infinite family of pairwise incident planes by adding to T the planes $\Lambda \in \mathrm{Gr}(2, M)$ containing p: that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. Since the planes $\Lambda_1, \Lambda_2, \Lambda_3$ have distinct pairwise intersections and they span a 5-dimensional projective space there exists linearly independent $v_0, \ldots, v_5 \in \mathbb{C}^6$ such that $\Lambda_1, \Lambda_2, \Lambda_3$ are as in (1.1). We claim that there exists $\Lambda_4 \in T$ which is contained in $\mathbb{P}^5$ (notation as in (2.1)) and does not intersect $\mathbb{P}\langle v_0, v_2, v_4 \rangle$. In fact if no such Λ_4 exists then the plane $\mathbb{P}\langle v_0, v_2, v_4 \rangle$ is incident to all planes in T and intersects each of $\Lambda_1, \Lambda_2, \Lambda_3$ along a line: that is a contradiction because of Proposition 2.2. We may rename v_1, v_3, v_5 so that Λ_4 is as in (1.1). Now let $\Lambda \in T$: since Λ intersects $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ one of the following holds:

(1) $\Lambda \subset \mathbb{P}^5$.
(2) $\dim(\Lambda \cap \mathbb{P}^5) = 1$ and the line $\Lambda \cap \mathbb{P}^5$ is one of L_5, L_6, L_7, see (2.3).

Since the span of T has dimension greater than 5 there does exist $\Lambda \in T$ such that Item (2) holds. By Corollary 2.3 we have an injection

$$
\begin{array}{ccc}
T \setminus \mathrm{Gr}(2, \mathbb{P}^5) & \hookrightarrow & \{L_5, L_6, L_7\} \\
\Lambda & \mapsto & \Lambda \cap \mathbb{P}^5
\end{array}
\tag{2.5}
$$

We claim that Map (2.5) is surjective. In fact suppose that the image consists of a single line L_i: then every plane containing L_i is incident to every plane in T, that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. Now suppose that the image consists of 2 lines: without loss of generality we may assume that they are L_5, L_6. A straightforward computation gives that

(2.6)
$$\{\Lambda\in\mathrm{Gr}(2,\mathbb{P}^5)|\Lambda \text{ is incident to } L_5,\ L_6,\ \Lambda_1,\ \Lambda_2,\ \Lambda_3 \text{ and } \Lambda_4\}=\mathbb{P}\langle v_0,v_1,v_3,v_4\rangle\cup\{\mathbb{P}\langle v_0,v_1,av_2+bv_3+cv_4\rangle\}\cup$$
$$\cup\{\mathbb{P}\langle v_1,v_3,av_0+bv_4+cv_5\rangle\}\cup\{\mathbb{P}\langle v_3,v_4,av_0+bv_1+cv_2\rangle\}\cup\{\mathbb{P}\langle v_0,v_4,av_1+bv_3+cv_5\rangle\}.$$

Now notice that the right-hand side of (2.6) is an infinite family of pairwise incident planes: that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. We have proved that Map (2.5) is surjective. Now let $\Lambda \in T$ be such that Item (1) holds: then Λ is incident to $\Lambda_1, \ldots, \Lambda_4$ and to L_1, L_2, L_3: it follows that $\Lambda \in \{\Lambda_1, \ldots, \Lambda_4\}$ - see the proof of Claim 2.1. The set of $\Lambda \in T$ such that Item (2) holds consists of 3 elements, say $\{\Lambda_5, \Lambda_6, \Lambda_7\}$ where $\Lambda_i \cap \mathbb{P}^5 = L_i$. Since L_5, L_6, L_7 span $\mathbb{P}^5$ the planes $\Lambda_5, \Lambda_6, \Lambda_7$ intersect in a single point which lies outside $\mathbb{P}^5$: thus we may complete $v_0, \ldots, v_5$ to a basis of $\mathbb{C}^7$ by adding a vector v_6 such that $\Lambda_5 \cap \Lambda_6 \cap \Lambda_7 = \{[v_6]\}$. Then it is clear that T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$. □

3. Complete finite families of pairwise incident planes in $\mathbb{P}^5$

In the present section we will associate to a finite complete family of pairwise incident planes in $\mathbb{P}^5$ an EPW-sextic - a special sextic hypersurface in $\mathbb{P}^5$ which comes equipped with a double cover. The double cover of a generic EPW-sextic is a Hyperkähler 4-fold deformation equivalent to the Hilbert square of a $K3$. There is a divisor Σ in the space of EPW-sextics whose generic point corresponds to a double cover X whose singular locus is a $K3$-surface of degree 2: it is obtained from a HK 4-fold $\widetilde{X}$ by contracting a divisor E which is a conic bundle over the $K3$, see [**12**]. Let Y be the EPW-sextic corresponding to X: the covering map $X \to Y$ takes the singular locus of X to a plane. There are more special EPW-sextics parametrized by points of Σ which correspond to a HK 4-fold $\widetilde{X}$ containing more than one of the divisors E: the images of these divisors under the composition $\widetilde{X} \to X \to Y$ are pairwise incident planes. We will show that certain of these EPW-sextics (introduced by Ferretti [**4**]) provide examples of complete families of k pairwise incident planes in $\mathbb{P}^5$ for $10 \leq k \leq 16$. Choose a volume-form $\mathrm{vol}\colon \bigwedge^6 \mathbb{C}^6 \xrightarrow{\sim} \mathbb{C}$ and equip $\bigwedge^3 \mathbb{C}^6$ with the symplectic form

$$(\alpha, \beta) := \mathrm{vol}(\alpha \wedge \beta). \tag{3.1}$$

Let $A \subset \bigwedge^3 \mathbb{C}^6$ be a subspace: we let

$$\Theta_A := \{W \in \mathrm{Gr}(3, \mathbb{C}^6) \mid \bigwedge^3 W \subset A\}, \tag{3.2}$$
$$\mathbf{\Theta}_A := \{\Lambda \in \mathrm{Gr}(2, \mathbb{P}^5) \mid \Lambda = \mathbb{P}(W) \text{ where } W \in \Theta_A\}. \tag{3.3}$$

The following simple observation will be our starting point.

REMARK 3.1. Let $A \subset \bigwedge^3 \mathbb{C}^6$ be **isotropic** for the symplectic form $(,)$. Then $\mathbf{\Theta}_A$ is a family of pairwise incident planes. Conversely let $T \subset \mathrm{Gr}(2, \mathbb{P}^5)$ be a family of pairwise incident planes and $B \subset \bigwedge^3 \mathbb{C}^6$ be the subspace spanned by the vectors $\bigwedge^3 W$ for $W \in \mathrm{Gr}(3, \mathbb{C}^6)$ such that $\mathbb{P}(W) \in T$: then B is isotropic for $(,)$.

Let $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\bigwedge^3 \mathbb{C}^6$ - of course $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ does not depend on the choice of volume-form. Notice that $\dim \bigwedge^3 \mathbb{C}^6 = 20$ and hence elements of $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ have dimension 10.

Claim 3.2. *Let $T \subset \mathrm{Gr}(2,\mathbb{P}^5)$ be a* **complete** *family of pairwise incident planes. Then there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that*

$$\boldsymbol{\Theta}_A = T. \tag{3.4}$$

Conversely suppose that $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ is **spanned** *by Θ_A (embedded in $\bigwedge^3 \mathbb{C}^6$ by Plücker). Then $\boldsymbol{\Theta}_A$ is a complete family of pairwise incident planes.*

Proof. Let $B \subset \bigwedge^3 \mathbb{C}^6$ be the subspace spanned by the vectors $\bigwedge^3 W$ for $W \in \mathrm{Gr}(3,\mathbb{C}^6)$ such that $\mathbb{P}(W) \in T$: then B is $(,)$-isotropic, see Remark 3.1. Thus there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ containing B. Then $\boldsymbol{\Theta}_A$ is a family of pairwise incident planes, see Remark 3.1, and it contains T. Since T is complete we get that (3.4) holds. Now suppose that $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ is spanned by $\bigwedge^3 W_1, \ldots, \bigwedge^3 W_{10}$ where $W_1, \ldots, W_{10} \in \Theta_A$. Suppose that $\mathbb{P}(W_*) \in \mathrm{Gr}(2,\mathbb{P}^5)$ is incident to all $\Lambda \in \boldsymbol{\Theta}_A$. Then $\bigwedge^3 W_*$ is orthogonal to $\bigwedge^3 W_1, \ldots, \bigwedge^3 W_{10}$ and hence to all of A. Since A is lagrangian we get that $\mathbb{P}(W_*) \in \boldsymbol{\Theta}_A$. This proves that $\boldsymbol{\Theta}_A$ is a complete family of pairwise incident planes. $\square$

Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$: according to Eisenbud-Popescu-Walter (see the appendix of [**3**] or [**8**]) one associates to A a subset of $\mathbb{P}^5$ as follows. Given a non-zero $v \in \mathbb{C}^6$ we let

$$F_v := \{\alpha \in \bigwedge^3 \mathbb{C}^6 \mid v \wedge \alpha = 0\}. \tag{3.5}$$

Notice that $F_v \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$. We let

$$Y_A = \{[v] \in \mathbb{P}^5 \mid F_v \cap A \neq \{0\}\}. \tag{3.6}$$

The lagrangians F_v are the fibers of a vector-bundle F on $\mathbb{P}^5$ with $\det F \cong \mathcal{O}_{\mathbb{P}^5}(-6)$: it follows that Y_A is the zero-locus of a section of $\mathcal{O}_{\mathbb{P}^5}(6)$. Thus either $Y_A = \mathbb{P}^5$ (this happens for "degenerate" choices of A, for example $A = F_w$) or else Y_A is a sextic hypersurface - an *EPW-sextic*. We emphasize that EPW-sextics are very special hypersurfaces, in particular their singular locus has dimension at least 2. An EPW-sextic Y_A comes equipped with a finite map [**10**]

$$f_A \colon X_A \to Y_A. \tag{3.7}$$

X_A is the *double EPW-sextic* associated to A. The following result [**8**] motivates the adjective "double". Suppose that

$$\Theta_A = \emptyset \text{ and } \dim(F_v \cap A) \leq 2 \text{ for all } [v] \in \mathbb{P}^5. \tag{3.8}$$

(A dimension count shows that (3.8) holds for generic $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$.) Then $Y_A \neq \mathbb{P}^5$ and X_A is a Hyperkähler variety deformation equivalent to the Hilbert square of a $K3$ surface[1], moreover (3.7) is identified with the quotient map of an anti-symplectic involution on X_A. What if one of the conditions of (3.8) are violated? If Θ_A is empty but there do exist $[v] \in \mathbb{P}^5$ such that $\dim(F_v \cap A) > 2$ then

[1]Notice that if A is general then X_A is not isomorphic nor birational to the Hilbert square of a $K3$.

necessarily $\dim(F_v \cap A) = 3$ and X_A is obtained from a holomorphic symplectic 4-fold by contracting certain copies of $\mathbb{P}^2$ (one for each point violating the second condition of (3.8)): thus X_A is almost as good as a HK variety. On the other hand suppose that $\Lambda \in \mathbf{\Theta}_A$: then $\Lambda \in Y_A$ and Y_A and X_A (assuming that $Y_A \neq \mathbb{P}^5$) may be quite singular along Λ. The following result will be handy.

Proposition 3.3 (Cor. 2.5 of [**9**] and Prop. 1.11, Claim 1.12 of [**10**]). *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $[v] \in \mathbb{P}^5$. Then the following hold:*

(1) *If no $\Lambda \in \mathbf{\Theta}_A$ contains $[v]$ then $Y_A \neq \mathbb{P}^5$, $\operatorname{mult}_{[v_0]} Y_A = \dim(A \cap F_{v_0})$ and*
  - (1a) *if $\dim(F_v \cap A) \leq 2$ then X_A is smooth at $f_A^{-1}([v])$,*
  - (1b) *if $\dim(F_v \cap A) > 2$ then the analytic germ of X_A at $f_A^{-1}([v])$ (a single point) is isomorphic to the cone over $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$.*

(2) *If there exists $\Lambda \in \mathbf{\Theta}_A$ containing $[v]$ then either $Y_A = \mathbb{P}^5$ or else X_A is singular at $f_A^{-1}([v])$.*

Next we will define an $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that Y_A is a triple quadric: the example will be a key element in the construction of complete families of pairwise incident planes of cardinality between 10 and 16. Choose an isomorphism $\mathbb{C}^6 = \bigwedge^2 U$ where U is a complex vector-space of dimension 4. Thus $\operatorname{Gr}(2, U) \subset \mathbb{P}(\mathbb{C}^6)$ is a smooth quadric hypersurface: we let

$$Q(U) := \operatorname{Gr}(2, U). \tag{3.9}$$

We have an embedding

$$\begin{array}{ccc} \mathbb{P}(U) & \overset{i_+}{\hookrightarrow} & \operatorname{Gr}(2, \mathbb{P}^5) \\ [u_0] & \mapsto & \mathbb{P}\{u_0 \wedge u \mid u \in U\} \end{array} \tag{3.10}$$

Definition 3.4. Let $A_+(U) \subset \bigwedge^3(\mathbb{C}^6)$ be the subspace spanned by the cone over $\operatorname{Im}(i_+)$ - here we view $\operatorname{Gr}(2, \mathbb{P}^5)$ as embedded in $\mathbb{P}(\wedge^3 \mathbb{C}^6)$ by the Plücker map.

Let $\mathcal{L}$ be Plücker line-bundle on $\operatorname{Gr}(2, \mathbb{P}^5)$. Then $i_+^* \mathcal{L} \cong \mathcal{O}_{\mathbb{P}(U)}(2)$ and the induced map on global sections is surjective: thus $\dim A_+(U) = 10$. On the other hand any two planes in the image of i_+ are incident: thus $A_+(U) \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$, see Remark 3.1. One has (see Claim 2.14 of [**9**])

$$Y_{A_+(U)} = 3Q(U). \tag{3.11}$$

Let $\mathbf{K} \subset \mathbb{P}(U)$ be a Kummer quartic surface and let $\mathbf{p}_1, \ldots, \mathbf{p}_{16}$ be its nodes. Choose k nodes $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$. There exist arbitrarily small deformations of $\mathbf{K}$ which contain exactly k nodes which are small deformations of $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ and are smooth elsewhere (it suffices to deform the minimal desingularization of $\mathbf{K}$ keeping the rational curves lying over $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ of type $(1,1)$ and not keeping of type $(1,1)$ the rational curves lying over the remaining nodes). Let S_0 be such a small deformation of $\mathbf{K}$ and $p_1, \ldots, p_k$ be its nodes. Let $\widetilde{S}_0 \to S_0$ be the minimal desingularization: thus $\widetilde{S}_0$ is a $K3$ surface containing k smooth rational curves $R_1, \ldots, R_k$ mapping to $p_1, \ldots, p_k$ respectively. The HK 4-fold $\widetilde{S}_0^{[2]}$ contains k disjoint copies of $\mathbb{P}^2$ namely $R_1^{(2)}, \ldots, R_k^{(2)}$. We have a regular map

$$\begin{array}{ccc} \widetilde{S}_0^{[2]} \setminus \bigcup_{i=1}^k R_i^{(2)} & \longrightarrow & Q(U) \\ Z & \mapsto & \langle Z \rangle \end{array} \tag{3.12}$$

where $\langle Z\rangle$ is the unique line containing the scheme Z. One cannot extend the above map to a regular map over $R_i^{(2)}$. Let $\widetilde{S}_0^{[2]} \dashrightarrow X$ be the flop of $R_1^{(2)},\ldots,R_k^{(2)}$ i.e. the blow-up of each $R_i^{(2)}\cong\mathbb{P}^2$ followed by contraction of the exceptional fiber E_i (which is isomorphic to the incidence variety in $\mathbb{P}^2\times(\mathbb{P}^2)^\vee$) along the projection $E_i\to(\mathbb{P}^2)^\vee$. Map (3.12) extends [**4**] to a regular degree-6 map

$$X \longrightarrow Q(U). \tag{3.13}$$

The following result is due to Ferretti:

Proposition 3.5 (Ferretti, Prop. 4.3 of [**4**]). *Keep notation as above. There exist a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "G"] \arrow[dr, "\pi"'] & & \mathcal{U}\times\mathbb{P}^5 \arrow[dl] \\ & \mathcal{U} & \end{tikzcd} \tag{3.14}$$

and maps

$$\mathcal{U}\xrightarrow{A}\mathbb{LG}(\bigwedge^3\mathbb{C}^6),\qquad \mathcal{U}\xrightarrow{\Lambda_i}\mathrm{Gr}(2,\mathbb{P}^5),\quad i=1,\ldots,k$$

such that the following hold:

1. *$\mathcal{U}$ is a connected contractible manifold of dimension $(20-k)$.*
2. *π is a proper map and a submersion of complex manifolds: for $t\in\mathcal{U}$ we let $X_t:=\pi^{-1}(t)$ and $g_t\colon X_t\to\mathbb{P}^5$ be the regular map induced by G.*
3. *There exists $0\in\mathcal{U}$ and an isomorphism $X_0\cong X$ such that g_0 gets identified with Map (3.13). Moreover $A(0)=A_+(U)$ and $\Lambda_i(0)=i_+(p_i)$.*
4. *There exist a regular map $c_t\colon X_t\to X_{A(t)}$ and prime divisors $E_i(t)$ on X_t for $i=1,\ldots,k$ such that the following hold for all t belonging to an open dense $\mathcal{U}^0\subset\mathcal{U}$:*
   - (4a) *$g_t=f_{A(t)}\circ c_t$.*
   - (4b) *$g_t(E_i(t))=\Lambda_i(t)$ for $i=1,\ldots,k$.*
   - (4c) *c_t contracts each $E_i(t)$ to a $K3$ surface $S_i(t)\subset X_{A(t)}$ and is an isomorphism of the complement of $\cup_{i=1}^k E_i(t)$ onto its image.*
5. *The period map $\mathcal{U}\to\mathbb{P}(H^2(X_0;\mathbb{C}))$ is an immersion i.e. the family of deformations of X_0 parametrized by $\mathcal{U}$ has $(20-k)$ moduli.*

Given Proposition 3.5 it is easy to show that there exist complete families of pairwise incident planes of cardinality k for $10\le k\le 16$. Before stating the relevant result we recall that the $K3$ surface $\widetilde{S}_0$ depends on the choice of nodes $\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}$ and hence so does the variety X.

Proposition 3.6. *Keep notation as in Proposition 3.5 and let $10\le k\le 16$. Let $t\in\mathcal{U}^0$ be close to 0. One can choose the nodes $\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}$ of $\mathbf{K}$ so that $\mathbf{\Theta}_{A(t)}$ is a complete family of pairwise incident planes of cardinality k.*

Proof. The map i_+ is identified with the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}(U)}(2)|$. It is well-know that no quadric in $\mathbb{P}(U)$ contains $\mathbf{p}_1,\ldots,\mathbf{p}_{16}$[2].

[2]Suppose that the quadric Q_0 contains $p_1,\ldots,p_{16}$. There exist 16 planes $L_1,\ldots,L_{16}\subset\mathbb{P}(U)$ such that each L_j contains 6 of the nodes of $\mathbf{K}$ and moreover $L_j\cdot\mathbf{K}=2C_j$ where C_j is a smooth conic - see for example Exercise VIII.5 of [**1**]. It follows that Q_0 contains $C_1,\ldots,C_{16}$ and hence $Q_0\cap\mathbf{K}$ has degree at least 32: that contradicts Bézout.

Thus $i_+(\mathbf{p}_1), \ldots, i_+(\mathbf{p}_{16})$ span a 10-dimensional subspace of $\mathbb{P}(\bigwedge^3 \mathbb{C}^6)$. Since $10 \le k \le 16$ we may choose $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ such that no quadric in $\mathbb{P}(U)$ contains them. It follows that for small enough $t \in \mathcal{U}^0$ the planes $\Lambda_1(t), \ldots, \Lambda_k(y)$ span $\mathbb{P}(A(t))$. By Claim 3.2 it remains to prove that no other plane is contained in $\mathbf{\Theta}_{A(t)}$. Suppose that $\Lambda \in \mathbf{\Theta}_{A(t)}$ and that $\Lambda \notin \{\Lambda_1(t), \ldots, \Lambda_k(t)\}$. By Item (2) of Proposition 3.3 we get that $X_{A(t)}$ is singular along $f^{-1}_{A(t)}(\Lambda)$: since $\Lambda \notin \{\Lambda_1(t), \ldots, \Lambda_k(t)\}$ that contradicts Item (4c) of Proposition 3.5. □

4. Upper bound

We will prove that a finite complete family of pairwise incident planes in $\mathbb{P}^5$ has at most 20 elements. The key element in the proof is the following construction from [**11**]: given $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $W \in \Theta_A$ we consider the locus

$$C_{W,A} := \{[v] \in \mathbb{P}(W) \mid \dim(F_v \cap A) \ge 2\}. \tag{4.1}$$

(Notice that $\dim(F_v \cap A) \ge 1$ for $[v] \in \mathbb{P}(W)$ because $\bigwedge^3 W \subset (F_v \cap A)$.) One describes $C_{W,A}$ as the degeneracy locus of a map between vector-bundles of rank 9: the fiber over $[v]$ of the domain is equal to $F_v / \bigwedge^3 W$, the codomain is the trivial vector-bundle with fiber $\bigwedge^3 W^{\perp} / \bigwedge^3 W$ - see [**11**] for details. It follows that either $C_{W,A} = \mathbb{P}(W)$ or else $C_{W,A}$ is a sextic curve. The link with our problem is the following. Suppose that $C_{W,A} \ne \mathbb{P}(W)$ and that $W' \in \Theta_A$ is distinct from W: then $\mathbb{P}(W \cap W')$ is contained in the singular locus of $C_{W,A}$. In order to state the relevant results from [**11**] we give a couple of definitions. Let $W \subset V$ be a subspace: we let

$$S_W := (\bigwedge^2 W) \wedge \mathbb{C}^6. \tag{4.2}$$

Definition 4.1. Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. We let $\mathcal{B}(W, A) \subset \mathbb{P}(W)$ be the set of $[v]$ such that one of the following holds:

(1) There exists $W' \in (\Theta_A \setminus \{W\})$ such that $[v] \in \mathbb{P}(W')$.
(2) $\dim(A \cap F_v \cap S_W) \ge 2$.

One checks easily that $\mathcal{B}(W, A)$ is closed subset of $\mathbb{P}(W)$.

Proposition 4.2. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. Then $C_{W,A} = \mathbb{P}(W)$ if and only if $\mathcal{B}(W, A) = \mathbb{P}(W)$. Suppose that $C_{W,A} \ne \mathbb{P}(W)$: then every non-reduced component of $C_{W,A}$ is contained in $\mathcal{B}(W, A)$.*

Proof. The first statement follows from Corollary 3.2.7 of [**11**]. The second statement follows from Proposition 3.2.6 of [**11**]. □

Lemma 4.3. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$. Suppose that Θ_A is finite of cardinality at least* 15. *Then there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced curve.*

Proof. By contradiction. Assume that for every $W \in \Theta_A$ one of the following holds:

(1) $C_{W,A} = \mathbb{P}(W)$.
(2) $C_{W,A}$ is a non-reduced curve.

By Proposition 4.2 we get that $\dim \mathcal{B}(W, A) \ge 1$. Let $W' \in (\Theta_A \setminus \{W\})$: since Θ_A is finite the planes $\mathbb{P}(W)$ and $\mathbb{P}(W')$ intersect in a single point, see Corollary 2.3.

It follows that for generic $[v] \in \mathcal{B}(W,A)$ there exists

$$\alpha \in \left((A \cap F_v \cap S_W) \setminus \bigwedge^3 W\right). \tag{4.3}$$

Given such α there is a unique $[v] \in \mathbb{P}(W)$ such that (4.3) holds. In fact suppose the contrary: then α is a decomposable element whose support is a $W' \in (\Theta_A \setminus \{W\})$ intersecting W in a 2-dimensional subspace, that contradicts the hypothesis that Θ_A is finite (see above). Since $\dim \mathcal{B}(W,A) \geq 1$ it follows that

$$\dim(A \cap S_W) \geq 3. \tag{4.4}$$

Thus $\mathbb{P}(A)$ intersects the projective tangent space to $\mathrm{Gr}(2,\mathbb{P}^5)$ (embedded by Plücker) at $\mathbb{P}(W)$ in a linear space of dimension at least 2. Now let $\Omega \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)$ be a generic 10-dimensional projective space containing $\mathbb{P}(A)$. Notice that

$$\dim \Omega + \dim \mathrm{Gr}(2,\mathbb{P}^5) = 19 = \dim \mathbb{P}(\bigwedge^3 \mathbb{C}^6).$$

The intersection $\Omega \cap \mathrm{Gr}(2,\mathbb{P}^5)$ is finite because by hypothesis $\mathbf{\Theta}_A = \mathbb{P}(A) \cap \mathrm{Gr}(2,\mathbb{P}^5)$ is finite. By (4.4) we get that Ω intersects the projective tangent space to $\mathrm{Gr}(2,\mathbb{P}^5)$ at $\mathbb{P}(W)$ in a linear space of dimension at least 2: thus

$$\mathrm{mult}_{\mathbb{P}(W)}\, \Omega \cdot \mathrm{Gr}(2,\mathbb{P}^5) \geq 3. \tag{4.5}$$

Since the cardinality of $\mathbf{\Theta}_A$ is at least 15 we get that $\Omega \cdot \mathrm{Gr}(2,\mathbb{P}^5) \geq 45$, that is a contradiction because $\deg \mathrm{Gr}(2,\mathbb{P}^5) = 42$, see p. 247 of [**5**]. $\square$

Now let T be a finite complete family of pairwise incident planes in $\mathbb{P}^5$. By Claim 3.2 there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that $\mathbf{\Theta}_A = T$. Suppose that T has cardinality at least 15: by Lemma 4.3 there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced sextic curves. Let $W' \in (\Theta_A \setminus \{W\})$: by Corollary 2.3 the intersection $\mathbb{P}(W) \cap \mathbb{P}(W')$ is a point. By Proposition 4.2 the curve $C_{W,A}$ is singular at $\mathbb{P}(W) \cap \mathbb{P}(W')$. Thus we have a map

$$\begin{array}{ccc} \Theta_A \setminus \{W\} & \xrightarrow{\varphi} & \mathrm{sing}\, C_{W,A} \\ W' & \mapsto & \mathbb{P}(W') \cap \mathbb{P}(W) \end{array} \tag{4.6}$$

There are at most 15 singular points of $C_{W,A}$ (the maximum 15 is achieved by sextics which are the union of 6 generic lines): it follows that if φ is injective then $\mathbf{\Theta}_A = T$ has at most 16 elements. Since φ is not necessarily injective we will need to answer the following question: what is the relation between the cardinality of $\varphi^{-1}(p)$ and the singularity of $C_{W,A}$ at p? First we will recall how to compute the initial terms in the Taylor expansion of a local equation of $C_{W,A}$ at a given point $[v_0] \in \mathbb{P}(W)$ - here $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $W \in \Theta_A$ are arbitrary. Let $[w] \in \mathbb{P}(W)$; we let

$$G_w := F_w / \bigwedge^3 W. \tag{4.7}$$

Let $W_0 \subset W$ be a subspace complementary to $[v_0]$. We have an isomorphism

$$\begin{array}{ccc} W_0 & \xrightarrow{\sim} & \mathbb{P}(W) \setminus \mathbb{P}(W_0) \\ w & \mapsto & [v_0 + w] \end{array} \tag{4.8}$$

onto a neighborhood of $[v_0]$; thus $0 \in W_0$ is identified with $[v_0]$. We have

$$C_{W,A} \cap W_0 = V(g_0 + g_1 + \cdots + g_6), \qquad g_i \in \mathrm{S}^i\, W_0^\vee. \tag{4.9}$$

Given $w \in W$ we define the Plücker quadratic form $\psi_w^{v_0}$ on G_{v_0} as follows. Let $\overline{\alpha} \in G_{v_0}$ be the equivalence class of $\alpha \in F_{v_0}$. Thus $\alpha = v_0 \wedge \beta$ where $\beta \in \bigwedge^2 V$ is defined modulo $(\bigwedge^2 W + [v_0] \wedge V)$: we let

$$\psi_w^{v_0}(\overline{\alpha}) := \mathrm{vol}(v_0 \wedge w \wedge \beta \wedge \beta). \tag{4.10}$$

Proposition 4.4 (Prop. 3.1.2 of [**11**]). *Keep notation and hypotheses as above. Let $\overline{K} := A \cap F_{v_0} / \bigwedge^3 W$ (notice that $\overline{K} \subset G_{v_0}$) and $\overline{k} := \dim \overline{K} = \dim(A \cap F_{v_0}) - 1$. Then the following hold:*

(1) *$g_i = 0$ for $i < \overline{k}$.*
(2) *There exists $\mu \in \mathbb{C}^*$ such that*

$$g_{\overline{k}}(w) = \mu \det(\psi_w^{v_0}|_{\overline{K}}), \quad w \in W_0. \tag{4.11}$$

Next we will give a geometric interpretation of the right-hand side of (4.11). Choose a subspace $V_0 \subset \mathbb{C}^6$ complementary to $[v_0]$ and such that $V_0 \cap W = W_0$. Thus have isomorphisms

$$\begin{array}{ccc} \bigwedge^2 V_0 & \xrightarrow{\sim} & F_{v_0} \\ \beta & \mapsto & v_0 \wedge \beta \end{array} \tag{4.12}$$

and

$$\begin{array}{ccc} \bigwedge^2 V_0 / \bigwedge^2 W_0 & \xrightarrow{\sim} & G_{v_0} \\ \overline{\beta} & \mapsto & \overline{v_0 \wedge \beta}. \end{array} \tag{4.13}$$

Let $\psi_w^{v_0}$ be as in (4.10): we will view it as a quadratic form on $\bigwedge^2 V_0 / \bigwedge^2 W_0$ via (4.13). Let $V(\psi_w^{v_0}) \subset \mathbb{P}(\bigwedge^2 V_0 / \bigwedge^2 W_0)$ be the zero-locus of $\psi_w^{v_0}$. Let

$$\widetilde{\rho} \colon \mathbb{P}(\bigwedge^2 V_0) \dashrightarrow \mathbb{P}(\bigwedge^2 V_0 / \bigwedge^2 W_0) \tag{4.14}$$

be projection with center $\bigwedge^2 W_0$. Let

$$\mathbb{G}r(2, V_0)_{W_0} := \widetilde{\rho}(\mathbb{G}r(2, V_0)). \tag{4.15}$$

(The right-hand side is to be interpreted as the closure of $\widetilde{\rho}(\mathbb{G}r(2, V_0) \setminus \{\bigwedge^2 W_0\})$.) Let ρ be the restriction of $\widetilde{\rho}$ to $\mathbb{G}r(2, V_0)$. The rational map

$$\rho \colon \mathbb{G}r(2, V_0) \dashrightarrow \mathbb{G}r(2, V_0)_{W_0} \tag{4.16}$$

is birational because $\mathbb{G}r(2, V_0)$ is cut out by quadrics. The following is an easy exercise, see Claim 3.5 of [**11**].

Claim 4.5. *Keep notation as above. Then*

$$\bigcap_{w \in W_0} V(\psi_w^{v_0}) = \mathbb{G}r(2, V_0)_{W_0} \tag{4.17}$$

and the scheme-theoretic intersection on the left is reduced.

Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. Let $p \in \mathbb{P}(W)$. We let

$$n_p := \#\{W' \in (\Theta_A \setminus \{W\}) \mid p \in \mathbb{P}(W')\}. \tag{4.18}$$

Notice that if $n_p > 0$ then $p \in C_{W,A}$.

Proposition 4.6. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that Θ_A is finite. Assume that $W \in \Theta_A$. Let $p \in \mathbb{P}(W)$.*

(1) *$n_p \leq 4$.*

(2) *Assume in addition that $C_{W,A}$ is a curve. Then the following hold:*
  (2a) *If $n_p = 2$ then either $C_{W,A}$ has a cusp*[3] *at p or else* $\operatorname{mult}_p C_{W,A} \geq 3$.
  (2b) *If $n_p = 3$ or $n_p = 4$ then* $\operatorname{mult}_p C_{W,A} \geq 3$

PROOF. Throughout the proof we will let $p = [v_0]$. Let $K := A \cap F_{v_0}$: we will view K as a subspace of $\bigwedge^2 V_0$ via Isomorphism (4.12).

(1): Suppose that $n_p > 4$. We claim that $\dim K \geq 4$. In fact suppose that $\dim K \leq 3$ i.e. $\dim \mathbb{P}(K) \leq 2$. Since $n_p \geq 5$ the intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ contains at least 6 points: that is absurd because $\mathrm{Gr}(2, V_0)$ is cut out by quadrics and the intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite (recall that Θ_A is finite by hypothesis). This proves that $\dim K \geq 4$. Since $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite we get that $\dim \mathbb{P}(K) \leq 3$ and hence $\dim \mathbb{P}(K) = 3$. Since the degree of $\mathrm{Gr}(2, V_0)$ is 5 and $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ contains at least 6 points we get that $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is infinite: that is a contradiction.

(2a): If $\dim K \geq 4$ then $\operatorname{mult}_p C_{W,A} \geq 3$ by Item (1) of Proposition 4.4. Suppose that $\dim K < 4$ i.e. $\dim \mathbb{P}(K) \leq 2$. By hypothesis $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite and contains 3 points. Since $\mathrm{Gr}(2, V_0)$ is cut out by quadrics it follows that $\dim \mathbb{P}(K) = 2$. Let $g_0, \ldots, g_6$ be as in (4.9). Then $0 = g_0 = g_1$ because $\dim K = 3$ (see Item (1) of Proposition 4.4) and g_2 is given by (4.11). Let $\widetilde{\rho}$ be the projection of (4.14). The closure of $\widetilde{\rho}(\mathbb{P}(K) \setminus \bigwedge^2 W_0)$ is a line intersecting $\mathrm{Gr}(2, V_0)_{W_0}$ in two distinct points, namely the images under projection of the two points belonging to $(\mathbb{P}(K) \setminus \bigwedge^2 W_0) \cap \mathrm{Gr}(2, V_0)$. By (4.9) and Claim 4.5 we get that $g_2 = l^2$ where $0 \neq l \in W_0^\vee$: thus $C_{W,A}$ has a cusp at p.

(2b): We will prove that $\dim K \geq 4$ - then $\operatorname{mult}_p C_{W,A} \geq 3$ will follow from Item (1) of Proposition 4.4. Assume that $\dim K < 4$. Suppose that $n_p = 3$. Then $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ has cardinality 4. Since $\mathrm{Gr}(2, V_0)$ is cut out by quadrics we get that $\dim \mathbb{P}(K) = 2$ and no three among the points of $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ are collinear. Now project $\mathbb{P}(K)$ from $\bigwedge^2 W_0$ - see (4.14): we get that $\widetilde{\rho}(\mathbb{P}(K) \setminus \bigwedge^2 W_0)$ is a line intersecting $\mathrm{Gr}(2, V_0)_{W_0}$ in three distinct points, that contradicts Claim 4.5. We have proved that if $n_p = 3$ then $\operatorname{mult}_p C_{W,A} \geq 3$. Lastly suppose that $n_p = 4$. Then $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ has cardinality 5 and $\dim \mathbb{P}(K) \leq 2$: that is absurd because $\mathrm{Gr}(2, V_0)$ is cut out by quadrics. □

Now let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and assume that Θ_A is finite of cardinality at least 15. By Lemma 4.3 there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced curve. We let

$$L_j := \{p \in \mathbb{P}(W) \mid n_p = j\}, \qquad \ell_j := \# L_j. \tag{4.19}$$

By Proposition 4.6 we have that $\ell_j = 0$ for $j > 4$ and hence

$$\# \Theta_A = 1 + \ell_1 + 2\ell_2 + 3\ell_3 + 4\ell_4. \tag{4.20}$$

LEMMA 4.7. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and assume that Θ_A is finite of cardinality at least 15. Let $W \in \Theta_A$ be such that $C_{W,A}$ is a reduced curve and keep notation as above. Let s be the number of irreducible components of $C_{W,A}$. Then*

$$\ell_1 + \ell_2 + 3\ell_3 + 3\ell_4 \leq 9 + s. \tag{4.21}$$

PROOF. Let $C := C_{W,A}$ and $\mu \colon Z \to \mathbb{P}^2$ be a series of blow-ups that desingularize C i.e. such that the strict transform $\widetilde{C} \subset Z$ is smooth. Then

$$-2s \leq 2(h^0(K_{\widetilde{C}}) - h^1(K_{\widetilde{C}})) = 2\chi(K_{\widetilde{C}}) = \widetilde{C} \cdot \widetilde{C} + \widetilde{C} \cdot K_Z. \tag{4.22}$$

[3]By cusp we mean a plane curve singularity with tangent cone which is quadratic of rank 1.

On the other hand let $p\in\mathbb{P}(W)$: if $n_p\geq 1$ then C is singular at p and if $n_p\geq 3$ then the multiplicity of C at p is at least 3, see Proposition 4.6. It follows that

$$\widetilde{C}\cdot\widetilde{C}+\widetilde{C}\cdot K_Z\leq (C\cdot C+C\cdot K_{\mathbb{P}^2})-2(\ell_1+\ell_2)-6(\ell_3+\ell_4)=18-2(\ell_1+\ell_2)-6(\ell_3+\ell_4). \tag{4.23}$$

The proposition follows from (4.22) and (4.23). □

The result below completes the proof of Theorem 1.1.

PROPOSITION 4.8. *Let $A\in\mathbb{LG}(\bigwedge^3\mathbb{C}^6)$ and assume that Θ_A is finite. Then*

$$\#\Theta_A\leq 20 \tag{4.24}$$

PROOF. We may assume that $\#\Theta_A>16$. By Lemma 4.3 there exists $W\in\Theta_A$ such that $C_{W,A}$ is a reduced curve. Let L_j and ℓ_j be as in (4.19) and s be the number of irreducible components of $C_{W,A}$. We recall that $C_{W,A}$ is singular at each point of L_1, it has either a cusp or a point of multiplicity at least 3 at each point of L_2 and it has multiplicity at least 3 at each point of $L_3\cup L_4$, see Proposition 4.6. By (4.20) we have that

$$16\leq\ell_1+2\ell_2+3\ell_3+4\ell_4 \tag{4.25}$$

The proof consists of a case-by-case analysis. Suppose first that $s=1$. Assume that $(\ell_3+\ell_4)=0$. Applying Plücker's formulae to $C_{W,A}$ (notice that $\deg C^{\vee}_{W,A}\geq 3$) we get that $(2\ell_1+3\ell_2)\leq 27$. It follows that $\#\Theta_A\leq 19$ (recall (4.20)) - the "worst" case being $C_{W,A}$ the dual of a smooth cubic i.e. a sextic with 9 cusps. Assume that $(\ell_3+\ell_4)=1$. Then $(\ell_1+\ell_2)\leq 7$ by Lemma 4.7: it follows that $\#\Theta_A\leq 19$. If $(\ell_3+\ell_4)=2$ then $(\ell_1+\ell_2)\leq 4$ by Lemma 4.7: it follows that $\#\Theta_A=17$. Next suppose that $s=2$. A similar analysis shows that necessarily[4] $C_{W,A}=D+L$ where D is an irreducible quintic with 4 cusps (the points of L_2) and 2 nodes (the points of L_4), L is the line through the nodes of D: thus $\#\Theta_A=17$. Lastly suppose that $s\geq 3$. Then $C_{W,A}=D_1+D_2+D_3$ where D_1, D_2 and D_3 are reduced conics (eventually reducible) belonging to the same pencil with reduced base locus (which is equal to $L_3\cup L_4$). We have $\#\Theta_A\leq(17+\delta)$ where δ is the number of singular conics among $\{D_1,D_2,D_3\}$. □

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Clay Mathematics Proceedings
Volume **18**, 2013

Derived equivalence and non-vanishing loci

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To Joe Harris, with great admiration.

1. The conjecture and its variants

The purpose of this note is to propose and motivate a conjecture on the behavior of cohomological support loci for topologically trivial line bundles under derived equivalence, to verify it in the case of surfaces, and to explain further developments. The reason for such a conjecture is the desire to understand the relationship between the cohomology groups of (twists of) the canonical line bundles of derived equivalent varieties. This in turn is motivated by the following well-known problem, stemming from a prediction of Kontsevich in the case of Calabi-Yau manifolds (and which would also follow from the main conjecture in Orlov [**Or2**]).

Problem 1.1. *Let X and Y be smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Is it true that $h^{p,q}(X) = h^{p,q}(Y)$ for all p and q?*

Here, given a smooth projective complex variety X, we denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$. For surfaces the answer is yes, for instance because of the derived invariance of Hochschild homology [**Or1**], [**Ca**]. This is also true for threefolds, again using the invariance of Hochschild homology, together with the behavior of the Picard variety under derived equivalence [**PS**]. In general, even the invariance of $h^{0,q}$ with $1 < q < \dim X$ is not known at the moment, and this leads to the search for possible methods for circumventing the difficult direct study of the cohomology groups $H^i(X, \omega_X)$.

More precisely, in [**PS**] it is shown that if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are isogenous. This opens the door towards studying the behavior or more refined objects associated to irregular varieties (i.e. those with $q(X) = h^0(X, \Omega^1_X) > 0$) under derived equivalence. Among the most important such objects are the *cohomological support loci* of the canonical bundle: given a smooth projective X, for $i = 0, \ldots, \dim X$ one defines

$$V^i(\omega_X) := \{\alpha \mid H^i(X, \omega_X \otimes \alpha) \neq 0\} \subseteq \mathrm{Pic}^0(X).$$

By semicontinuity, these are closed algebraic subsets of $\mathrm{Pic}^0(X)$. It has become clear in recent years that these loci are the foremost tool in studying the special

2010 *Mathematics Subject Classification.* 14F05, 14K30.

Key words and phrases. Derived categories, Picard variety, cohomological support loci.

The author was partially supported by the NSF grant DMS-1101323.

birational geometry of irregular varieties, with applications ranging from results about singularities of theta divisors [**EL**] to the proof of Ueno's conjecture [**ChH**]. They are governed by the following fundamental results of generic vanishing theory ([**GL1**], [**GL2**], [**Ar**], [**Si**]):

• If $a : X \to \mathrm{Alb}(X)$ is the Albanese map of X, then

$$\text{codim } V^i(\omega_X) \geq i - \dim X + \dim a(X) \quad \text{for all i,} \tag{1.1}$$

and there exists an i for which this is an equality.

• The irreducible components of each $V^i(\omega_X)$ are torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

• Each positive dimensional component of some $V^i(\omega_X)$ corresponds to a fibration $f : X \to Y$ onto a normal variety with $0 < \dim Y \leq \dim X - i$ and with generically finite Albanese map.

The main point of this note is the following conjecture, saying that cohomological support loci should be preserved by derived equivalence. In the next sections I will explain that the conjecture holds for surfaces, and that is almost known to hold for threefolds.

Conjecture 1.2. *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then*

$$V^i(\omega_X) \simeq V^i(\omega_Y) \quad \text{for all } i \geq 0.$$

Note that I am proposing isomorphism, even though the ambient spaces $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ may only be isogenous. There are roughly speaking three main reasons for this: (1) the conjecture is known to hold for surfaces and for most threefolds, as explained in §2 and §3; (2) it holds for V^0 in arbitrary dimension, as explained at the beginning of §3; (3) more heuristically, according to [**PS**] the failure of isomorphism at the level of Pic^0 is induced by the presence of abelian varieties in the picture, and for these all cohomological support loci consist only of the origin.

Furthermore, denote by $V^i(\omega_X)_0$ the union of the irreducible components of $V^i(\omega_X)$ passing through the origin. Generic vanishing theory tells us that in many applications one only needs to control well $V^i(\omega_X)_0$. In fact, for all applications I currently have in mind, the following variant of Conjecture 1.2 suffices.

Variant 1.3. *Under the same hypothesis,*

$$V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0 \quad \text{for all } i \geq 0.$$

The key fact implied by this variant is that, excepting perhaps surjective maps to abelian varieties, roughly speaking two derived equivalent varieties must have the same types of fibrations onto lower dimensional irregular varieties (see Corollary 3.4). This would hopefully allow for further geometric tools in the classification of irregular derived partners. Even weaker versions of Conjecture 1.2 and Variant 1.3 are of interest, as they are all that is needed in other applications.

Variant 1.4. *Under the same hypothesis,*

$$\dim V^i(\omega_X) = \dim V^i(\omega_Y) \quad \text{for all } i \geq 0.$$

Variant 1.5. *Under the same hypothesis,*

$$\dim V^i(\omega_X)_0 = \dim V^i(\omega_Y)_0 \quad \text{for all } i \geq 0.$$

For instance, Variant 1.5 implies the derived invariance of the Albanese dimension (see Corollary 3.2). Other numerical applications, and progress due to Lombardi [**Lo**] in the case of V^0 and V^1, and on the full conjecture for threefolds, are described in §3. In §2 I present a proof of Conjecture 1.2 in the case of surfaces.

2. A proof of Conjecture 1.2 for surfaces

Due to the classification of Fourier-Mukai equivalences between surfaces, in this case the conjecture reduces to a calculation of all possible cohomological support loci via a case by case analysis, combined with some elliptic surface theory, generic vanishing theory, well-known results of Kollár on higher direct images of dualizing sheaves, and of Beauville on the positive dimensional components of $V^1(\omega_X)$. This of course does not have much chance to generalize to higher dimensions. In the next section I will point to more refined techniques developed by Lombardi [**Lo**], which recover the case of surfaces, but do address higher dimensions as well.

THEOREM 2.1. *Conjecture* 1.2 *holds when X and Y are smooth projective surfaces.*

PROOF. The first thing to note is that, due to the work of Bridgeland-Maciocia [**BM**] and Kawamata [**Ka**], Fourier-Mukai equivalences of surfaces are completely classified. According to [**Ka**] Theorem 1.6, the only non-minimal surfaces that can have derived partners are rational elliptic, and therefore regular. Hence we can restrict to minimal surfaces. Among these on the other hand, according to [**BM**] Theorem 1.1, only abelian, $K3$ and elliptic surfaces can have distinct derived partners.

Now $K3$ surfaces are again regular, hence for these the problem is trivial. On the other hand, on any abelian variety A (of arbitrary dimension) one has

$$V^i(\omega_A) = \{0\} \ \text{ for all } i,$$

and since the only derived partners of abelian varieties are again abelian varieties (see [**HN**] Proposition 3.1; cf. also [**PS**], end of §3), the problem is again trivial. Therefore our question is truly a question about elliptic surfaces which are not rational. Moreover, according to [**BM**], bielliptic surfaces do not have non-trivial derived partners. We are left with certain elliptic fibrations over $\mathbf{P}^1$, and with elliptic fibrations over smooth projective curves of genus at least 2. I will try in each case to present the most elementary proof I am aware of.

Let first $f : X \to \mathbf{P}^1$ be an elliptic surface over $\mathbf{P}^1$. Since our problem is non-trivial only for irregular surfaces, requiring $q(X) \neq 0$ we must then have that $q(X) = 1$, which implies that f is isotrivial, and in fact that X is a $\mathbf{P}^1$-bundle

$$\pi : X \longrightarrow E$$

over an elliptic curve E. We can now compute the cohomological support loci $V^i(\omega_X)$ explicitly. Note first that $V^2(\omega_X) = \{0\}$ for any smooth projective surface, by Serre duality. In the case at hand, note also that π is the Albanese map of X. Therefore, identifying line bundles in $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(E)$, for every $\alpha \in \mathrm{Pic}^0(X)$ we have

$$H^0(X, \omega_X \otimes \alpha) \simeq H^0(E, \pi_*\omega_X \otimes \alpha),$$

which implies that $V^0(\omega_X) = V^0(E, \pi_*\omega_X)$.[1] But given that $\pi_*\omega_X$ must be torsion-free, and the fibers of π are rational curves, we have $\pi_*\omega_X = 0$, and so $V^0(\omega_X) = \emptyset$. We are left with computing $V^1(\omega_X)$. For this, recall that by [**Ko2**] Theorem 3.1, in $\mathbf{D}(E)$ we have the decomposition

$$\mathbf{R}\pi_*\omega_X \simeq \pi_*\omega_X \oplus R^1\pi_*\omega_X[-1] \simeq R^1\pi_*\omega_X[-1],$$

where the second isomorphism follows from what we said above. Therefore, for any $\alpha \in \mathrm{Pic}^0(X)$, we have

$$H^1(X, \omega_X \otimes \alpha) \simeq H^0(E, R^1\pi_*\omega_X \otimes \alpha).$$

Finally, [**Ko1**] Proposition 7.6 implies that, as the top non-vanishing higher direct image,

$$R^1\pi_*\omega_X \simeq \omega_E \simeq \mathcal{O}_E,$$

which immediately gives that $V^1(\omega_X) = \{0\}$. In conclusion, we have obtained that for the type of surface under discussion we have

$$V^0(\omega_X) = \emptyset, \quad V^1(\omega_X) = \{0\}, \quad V^2(\omega_X) = \{0\}.$$

Finally, if Y is another smooth projective surface such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then due to [**BM**] Proposition 4.4 we have that Y is another elliptic surface over $\mathbf{P}^1$ with the same properties as X, which leads therefore to the same cohomological support loci.

Assume now that $f : X \to C$ is an elliptic surface over a smooth projective curve C of genus $g \geq 2$ (so that $\kappa(X) = 1$). By the same [**BM**] Proposition 4.4, if Y is another smooth projective surface such that $\mathbf{D}(Y) \simeq \mathbf{D}(X)$, then Y has an elliptic fibration structure $h : Y \to C$ over the same curve (and with isomorphic fibers over a Zariski open set in C; in fact it is a relative Picard scheme associated to f). There are two cases, namely when f is isotrivial, and when it is not. It is well known (see e.g. [**Be1**] Exercise IX.1 and [**Fr**] Ch.7) that f is isotrivial if and only if $q(X) = g + 1$ (in which case the only singular fibers are multiple fibers with smooth reduction), and it is not isotrivial if and only if $q(X) = g$. Since we know that $q(X) = q(Y)$, we conclude that h must be of the same type as f. We will again compute all $V^i(\omega_X)$ in the two cases.

Let's assume first that f is not isotrivial. As mentioned above, in this case $q(X) = g$, and in fact $f^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(X)$ is an isomorphism. To compute $V^1(\omega_X)$, we use again [**Ko2**] Theorem 3.1, saying that in $\mathbf{D}(C)$ there is a direct sum decomposition

$$\mathbf{R}f_*\omega_X \simeq f_*\omega_X \oplus R^1f_*\omega_X[-1],$$

and therefore for each $\alpha \in \mathrm{Pic}^0(X)$ one has

$$H^1(X, \omega_X \otimes \alpha) \simeq H^1(C, f_*\omega_X \otimes \alpha) \oplus H^0(C, R^1f_*\omega_X \otimes \alpha).$$

Once again using [**Ko1**] Proposition 7.6, we also have $R^1f_*\omega_X \simeq \omega_C$. This, combined with the decomposition above, gives the inclusion

$$f^* : \mathrm{Pic}^0(C) = V^0(C, \omega_C) \hookrightarrow V^1(\omega_X),$$

finally implying $V^1(\omega_X) = \mathrm{Pic}^0(X)$. Finally, note that by the Castelnuovo inequality [**Be1**] Theorem X.4, we have $\chi(\omega_X) \geq 0$. Now the Euler characteristic is a

[1] In general, for any coherent sheaf $\mathcal{F}$ on a smooth projective variety Z, and any integer i, we denote $V^i(Z, \mathcal{F}) := \{\alpha \in \mathrm{Pic}^0(Z) \mid h^i(Z, \mathcal{F} \otimes \alpha) \neq 0\}$.

deformation invariant, hence $\chi(\omega_X \otimes \alpha) \geq 0$ for all $\alpha \in \mathrm{Pic}^0(X)$. For $\alpha \neq \mathcal{O}_X$, this gives

$$h^0(X, \omega_X \otimes \alpha) \geq h^1(X, \omega_X \otimes \alpha),$$

so that $V^1(\omega_X) \subset V^0(\omega_X)$. By the above, we obtain $V^0(\omega_X) = \mathrm{Pic}^0(X)$ as well. We rephrase the final result as saying that

$$V^0(\omega_X) = V^1(\omega_X) \simeq \mathrm{Pic}^0(C), \quad V^2(\omega_X) = \{0\}.$$

The preceding paragraph says that the exact same calculation must hold for a Fourier-Mukai partner Y.

Let's now assume that f is isotrivial. First note that for such an X we have $q(X) = g + 1$, and in fact the Albanese variety of X is an extension of abelian varieties

$$1 \to F \to \mathrm{Alb}(X) \to J(C) \to 1$$

with F an elliptic curve isogenous to the general fiber of f, though this will not play an explicit role in the calculation. Moreover, we have $\chi(\omega_X) = 0$ (see [**Fr**] Ch.7, Lemma 14 and Corollary 17).

We now use a result of Beauville [**Be2**] [**Be3**], characterizing the positive dimensional irreducible components of $V^1(\omega_X)$. Concretely, by [**Be3**] Corollaire 2.3, any such positive dimensional component would have to come either from a fiber space $h : X \to B$ over a curve of genus at least 2, or from a fiber space $p : X \to F$ over an elliptic curve, with at least one multiple fiber. Regarding the first type, the union of all such components is shown in *loc. cit.* to be equal to

$$\mathrm{Pic}^0(X, h) := \mathrm{Ker}\big(\mathrm{Pic}^0(X) \xrightarrow{i^*} \mathrm{Pic}^0(F)\big),$$

where i^* is the restriction map to any smooth fiber F of h. But since f is an elliptic fibration, it is clear that there is exactly one such fiber space, namely f itself (otherwise the elliptic fibers of any other fibration would have to dominate C, which is impossible). Therefore the union of the components coming from fibrations over curves of genus at least 2 is $\mathrm{Pic}^0(X, f)$. On the other hand, for elliptic surfaces of the type we are currently considering, fibrations $p : X \to F$ over elliptic curves as described above do not exist. (Any such would have to come from a group action on a product between an elliptic curve F' and another of genus at least 2, with the action on the elliptic component having no fixed points, therefore leading to an étale cover $F' \to F$; in the language of [**Be3**], we are saying that $\Gamma^0(p) = \{0\}$.)

Using once more the deformation invariance of the Euler characteristic, we have $\chi(\omega_X \otimes \alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0(X)$, which gives

$$h^1(X, \omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha), \quad \text{for all } \alpha \neq \mathcal{O}_X. \tag{2.1}$$

This implies that $\mathrm{Pic}^0(X, f)$ is also the union of all positive dimensional components of $V^0(\omega_X)$.

We are left with considering nontrivial isolated points in $V^1(\omega_X)$ (or equivalently in $V^0(\omega_X)$ by (2.1)). These can be shown not to exist by means of a different argument: by a variant of the higher dimensional Castelnuovo-de Franchis inequality, see [**LP**] Remark 4.13, an isolated point $\alpha \neq 0$ in $V^1(\omega_X)$ forces the inequality

$$\chi(\omega_X) \geq q(X) - 1 = g \geq 2,$$

which contradicts the fact that $\chi(\omega_X) = 0$.[2]

Putting everything together, we obtain

$$V^0(\omega_X) = V^1(\omega_X) = \mathrm{Pic}^0(X, f), \quad V^2(\omega_X) = \{0\}.$$

Recall that a Fourier-Mukai partner of X must be an elliptic fibration of the same type over C. Now one of the main results of [**Ph**], Theorem 5.2.7, says that for derived equivalent elliptic fibrations $f : X \to C$ and $h : Y \to C$ which are isotrivial with only multiple fibers, one has

$$\mathrm{Pic}^0(X, f) \simeq \mathrm{Pic}^0(Y, h),$$ [3]

which allows us to conclude that $V^i(\omega_X)$ and $V^i(\omega_Y)$ are isomorphic. □

3. Further evidence and applications

Progress. Progress towards the conjectures in §1 has been made by Lombardi [**Lo**]. The crucial point is to come up with an explicit mapping realizing the potential isomorphisms in Conjecture 1.2. This is done by means of the *Rouquier isomorphism*; namely, given a Fourier-Mukai equivalence $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ induced by an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, Rouquier [**Ro**] Théorème 4.18 shows that there is an induced isomorphism of algebraic groups

$$F : \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \longrightarrow \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$

given by a concrete formula involving $\mathcal{E}$ (usually mixing the two factors), [**PS**] Lemma 3.1. A key result in [**Lo**] is that if $\alpha \in V^0(\omega_X)$ and

$$F(\mathrm{id}_X, \alpha) = (\varphi, \beta),$$

then in fact $\varphi = \mathrm{id}_Y$, $\beta \in V^0(\omega_Y)$, and moreover

$$H^0(X, \omega_X \otimes \alpha) \simeq H^0(Y, \omega_Y \otimes \beta). \tag{3.1}$$

One of the main tools used there is the derived invariance of a generalization of Hochschild homology taking into account the Rouquier isomorphism. This implies the invariance of V^0, while further work using a variant of the Hochschild-Kostant-Rosenberg isomorphism gives the following, again the isomorphisms being induced by the Rouquier mapping.

Theorem 3.1 (Lombardi [**Lo**]). *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then:*
(i) $V^0(\omega_X) \simeq V^0(\omega_Y)$.
(ii) $V^1(\omega_X) \cap V^0(\omega_X) \simeq V^1(\omega_Y) \cap V^0(\omega_Y)$.
(iii) $V^1(\omega_X)_0 \simeq V^1(\omega_Y)_0$.

This result recovers Theorem 2.1 in a more formal way. In the case when $\dim X = \dim Y = 3$, with extra work one shows that this has the following consequences, verifying or getting close to verifying the various conjectures:

- Variant 1.3 holds.
- For any i, $V^i(\omega_X)$ is positive dimensional if and only if $V^i(\omega_Y)$ is positive dimensional, and of the same dimension. Therefore Variant 1.4 holds, except for

[2] As L. Lombardi points out, a variant of the derivative complex argument in [**LP**] leading to this inequality can also be used, as an alternative to Beauville's argument, in order to show that positive dimensional components not passing through the origin do not exist in the case of surfaces of maximal Albanese dimension with $\chi(\omega_X) = 0$.

[3] This also follows from [**Lo**], via Theorem 3.1 below.

the possible case where for some $i > 0$, $V^i(\omega_X)$ is finite, while $V^i(\omega_Y) = \emptyset$. This last case can possibly happen only when $q(X) = 1$.

- Conjecture 1.2 is true when:
  (1) X is of maximal Albanese dimension (i.e. the Albanese map of X is generically finite onto its image).
  (2) $V^0(\omega_X) = \mathrm{Pic}^0(X)$ – for instance, by [**PP**] Theorem E, this condition holds whenever the Albanese image $a(X)$ is not fibered in subtori of $\mathrm{Alb}(X)$, and $V^0(\omega_X) \neq \emptyset$.
  (3) $\mathrm{Aut}^0(X)$ is affine – this holds for varieties which are not isotrivially fibered over a positive dimensional abelian variety (see [**Br**] p.2 and §3), for instance again when the Albanese image is not fibered in subtori of $\mathrm{Alb}(X)$ according to a theorem of Nishi (cf. [**Ma**] Theorem 2).

These conditions together impose very strong restrictions on the threefolds for which the conjecture is not yet known. Note finally that in [**Lo**] there are further extensions involving cohomological support loci for $\omega_X^{\otimes m}$ with $m \geq 2$, and for Ω_X^p with $p < \dim X$.

Some first applications. Let X be a smooth projective complex variety of dimension d, and let $a : X \to A = \mathrm{Alb}(X)$ be the Albanese map of X. A first consequence of the weakest version of the conjectures would be the derived invariance of the Albanese dimension $\dim a(X)$.

COROLLARY 3.2 (**assuming Variant 1.5**). *If X and Y are smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then*

$$\dim a(X) = \dim a(Y).$$

This follows from the fact that, according to [**LP**] Remark 2.4, the Albanese dimension can be computed from the dimension of the cohomological support loci around the origin, according to the formula

$$\dim a(X) = \min_{i=0,\ldots,d} \{d - i + \mathrm{codim}\ V^i(\omega_X)_0\}.$$

Note that Lombardi [**Lo**] is in fact able to prove Corollary 3.2 when $\kappa(X) \geq 0$ by relying on different tools from birational geometry. The only progress when $\kappa(X) = -\infty$, namely a solution for surfaces and threefolds that can also be found in *loc. cit.*, involves the approach described here.

Another numerical application involves the holomorphic Euler characteristic. While the individual Hodge numbers are not yet known to be preserved by derived equivalence, the Euler characteristic can be attacked in some cases by using generic vanishing theory and the derived invariance of $V^0(\omega_X)$ established in Theorem 3.1.

COROLLARY 3.3 (*of Theorem* 3.1, [**Lo**]). *If X and Y are smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, and X is of maximal Albanese dimension, then $\chi(\omega_X) = \chi(\omega_Y)$.*

This follows from the fact that, according to (1.1), for generic $\alpha \in \mathrm{Pic}^0(X)$ one has

$$\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha),$$

combined with (3.1). The argument is extended in [**Lo**] to other cases as well. Going back to Hodge numbers, this implies for instance that if X and Y are derived equivalent 4-folds of maximal Albanese dimension, then

$$h^{0,2}(X) = h^{0,2}(Y),^4$$

since in the case of 4-folds all the other $h^{0,q}$ Hodge numbers are known to be preserved.

Perhaps the main point in this picture is the fact that the positive dimensional components of the cohomological support loci $V^i(\omega_X)$ reflect the nontrivial fibrations of X over irregular varieties. Therefore, roughly speaking, the key geometric significance of Conjecture 1.2 is that derived equivalent varieties should have the same type of fibrations over lower dimensional irregular varieties, thus allowing for more geometric tools in the study of Fourier-Mukai partners. One version of this principle can be stated as follows:

Corollary 3.4 (**assuming Variant 1.3**). *Let X and Y be smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Fix an integer $m > 0$, and assume that X admits a morphism $f : X \to Z$ with connected fibers, onto a normal irregular variety of dimension m whose Albanese map is not surjective. Then Y admits a morphism $h : Y \to W$ with connected fibers, onto a positive dimensional normal irregular variety of dimension $\leq m$. Moreover, if $m = 1$, then W can also be taken to be a curve of genus at least 2.*

This is due to the fact that, by the degeneration of the Leray spectral sequence for $\mathbf{R}f_*\omega_X$ due to Kollár [**Ko1**], one has

$$f^*V^0(\omega_Z) \subset V^{n-m}(\omega_X),$$

where $n = \dim X = \dim Y$. Now in [**EL**] Proposition 2.2 it is shown that if 0 is an isolated point in $V^0(\omega_Z)$, then the Albanese map of Z must be surjective. Thus the hypothesis implies that we obtain a positive dimensional component in $V^{n-m}(\omega_X)_0$, hence by Variant 1.3 also in $V^{n-m}(\omega_Y)_0$. Going in reverse, recall now from §1 that, according to one of the main results of [**GL2**], a positive dimensional component of $V^{n-m}(\omega_Y)$ produces a fiber space $h : Y \to W$, with W a positive dimensional normal irregular variety (with generically finite Albanese map) and $\dim W \leq m$. The slightly stronger statement in the case of fibrations over curves follows from the precise description of the positive dimensional components of $V^{n-1}(\omega_X)$ given in [**Be3**].

I suspect that one should be able to remove the non-surjective Albanese map hypothesis (in other words allow maps onto abelian varieties), but this must go beyond the methods described here.

Acknowledgements. I thank L. Lombardi and C. Schnell for numerous useful conversations on the topic presented here.

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Clay Mathematics Proceedings
Volume **18**, 2013

Degenerations of rationally connected varieties and PAC fields

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This paper is dedicated to Joe Harris, with congratulations on his 60th birthday, for too many kindnesses to list.

Abstract. A degeneration of a separably rationally connected variety over a field k contains a geometrically irreducible subscheme if k contains the algebraic closure of its prime subfield. If k is a perfect PAC field, the degeneration has a k-point. This generalizes (and gives a purely algebraic proof of) a known result of field arithmetic, [**4**, Theorem 21.3.6(a)]: a degeneration of a Fano complete intersection over k has a k-point if k is a perfect PAC field containing the algebraic closure of its prime subfield.

1. Statement of results

Recently, a number of fields long known to be C_1 were proved to satisfy an *a priori* stronger property related to rationally connected varieties.

(i) Every rationally connected variety defined over the function field of a curve over a characteristic 0 algebraically closed field has a rational point, [**6**].
(ii) Every separably rationally connected variety defined over the function field of a curve over an algebraically closed field of arbitrary characteristic has a closed point, [**2**].
(iii) Every smooth, rationally chain connected variety over a finite field has a rational point, [**3**].

Moreover, in each of these cases degenerations of these varieties also have rational points, at least under some mild hypotheses on the degeneration.

This article considers the same problem for perfect PAC fields containing an algebraically closed field. A field K is *pseudo-algebraically closed*, or *PAC*, if every integral, finite type K-scheme has a K-rational point. Such fields are known to be C_1, [**4**, Theorem 21.3.6(a)]. The main theorem is the following.

Theorem 1.1. *Let k be a perfect PAC field containing the algebraic closure of its prime subfield. Let X_k be the closed fiber of a proper, flat algebraic space over a DVR with residue field k. If the geometric generic fiber is separably rationally connected (in the sense of* [**2**]*), then X_k has a k-point.*

2010 *Mathematics Subject Classification.* Primary 14G05, 14G27.

Key words and phrases. Rational point, PAC field, rationally connected variety.

The author was supported in part by NSF Grants DMS-0553921, DMS-0758521, and DMS-0846972.

In particular, this gives a new proof of a theorem of Fried-Jarden. [**4**, Theorem 21.3.6(a)], i.e., every perfect PAC field containing an algebraically closed field is C_1.

Corollary 1.2. [**4**, Theorem 21.3.6(a)] *Every perfect PAC field containing an algebraically closed field is C_1.*

The original proof uses the Chebotarev density theorem and the Riemann hypothesis for curves. Our proof uses only results of algebraic geometry, particularly the results (i) and (ii) above. Apart from (i) and (ii), we use only foundational results of algebraic geometry, e.g., results from EGA and Bertini theorems.

Theorem 1.1 should be compared to the following theorems of Kollár and de Jong respectively.

Theorem 1.3. [**10**] *Let k be a characteristic 0 PAC field. Let X_k be the closed fiber of a projective, flat scheme over a DVR with residue field k. If the geometric generic fiber is a Fano manifold, then X_k has a k-point. In particular, k is C_1.*

Theorem 1.4 (de Jong). *Let k be a characteristic 0 field containing $\overline{\mathbb{Q}}$ and having a point in every rationally connected k-scheme. Let X_k be the closed fiber of a proper, flat algebraic space over a DVR with residue field k. If the geometric generic fiber is rationally connected, then X_k has a k-point. In particular, k is C_1.*

1.1. The basic argument. As mentioned above, the proof uses very little besides (i) and (ii) above. Here we give the basic argument, postponing some technical lemmas to a later section. First of all, Theorem 1.1, which applies only to perfect, PAC fields, follows from a result that applies to all fields.

Theorem 1.5. *Let R be a DVR with residue field k_R, and let X_R be a proper, flat R-algebraic space. whose geometric generic fiber is separably rationally connected (in the sense of [**2**]). Denote by L_R the compositum of k_R with the algebraic closure of the prime subfield. If the geometric generic fiber of X_R over R is separably rationally connected (in the sense of [**2**]), then there exists a closed subspace Y_{L_R} of $X_R \otimes_R L_R$ which is geometrically irreducible over L_R, i.e., $Y_{L_R} \otimes_{L_R} \overline{k}_R$ is irreducible.*

Proof of Theorem 1.1 assuming Theorem 1.5. Because $k = k_R$ contains the algebraic closure of its prime subfield, k_R equals L_R. By Theorem 1.5, X_k contains a closed subspace Y_k which is geometrically irreducible over k. Because k is a perfect PAC field, every geometrically irreducible k-scheme has a k-point. In particular, Y has a k-point. Therefore X_k has a k-point. □

While we are at it, here is the proof of Corollary 1.2.

Proof of Corollary 1.2 assuming Theorem 1.1. Every field k is the residue field (i.e., closed point) of a DVR R whose fraction field (i.e., generic point) has characteristic 0. Every complete intersection in $\mathbb{P}^n_k$ is the closed fiber of a complete intersection in $\mathbb{P}^n_R$ whose generic fiber is smooth. If the complete intersection satisfies the C_1 inequality, the generic fiber is a Fano manifold. By [**11**] and [**1**], a Fano manifold in characteristic 0 is rationally connected. Since rationally connected varieties in characteristic 0 are separably rationally connected, it is even separably rationally connected. Therefore Theorem 1.1 implies the complete intersection in $\mathbb{P}^n_k$ has a k-point if k is a perfect PAC field containing the algebraic closure of its prime subfield. In other words, every perfect PAC field containing an algebraically closed field is C_1, cf. [**4**, Theorem 21.3.6(a)]. □

The main step in the proof of Theorem 1.5 is the rationally connected fibration theorem of [**6**] and [**2**].

THEOREM 1.6. [**6**],[**2**] *Let κ be an algebraically closed field, let C_κ be a smooth, connected curve over κ, and let*

$$\pi : X_C \to C_\kappa$$

be a proper algebraic space over C_κ whose geometric generic fiber is separably rationally connected (in the sense of [**2**]*). Then there exists a section of π.*

Because every finite type algebraic space over a ring R is the base change of a finite type algebraic space defined over a finitely generated ring, the general case of Theorem 1.5 reduces to the case where R is "essentially of finite type". The next definition makes this precise.

DEFINITION 1.7. A *finite type datum* is a datum

$$((P, Q) \to (S, \mathfrak{s}), X_P)$$

of

(i) a Dedekind domain S of finite type over $\mathbb{Z}$ or over $\mathbb{Q}$ and a maximal ideal $\mathfrak{s}$ of S,
(ii) a flat, quasi-projective S-scheme P whose geometric generic fiber is integral and normal,
(iii) an integral, normal Weil divisor Q of P contained in $P_\mathfrak{s}$,
(iv) and a P-algebraic space X_P.

The datum is *proper* if X_P is proper over P and it is *strict* if Q is geometrically irreducible over $\kappa(\mathfrak{s})$, i.e., if $Q \otimes_{\kappa(\mathfrak{s})} \overline{\kappa(\mathfrak{s})}$ is irreducible.

LEMMA 1.8. *Theorem* 1.5 *for arbitrary pairs (R, X_R) follows from Theorem* 1.5 *for those pairs $(\mathcal{O}, X_\mathcal{O})$ arising from strict, proper, finite type data by setting $\mathcal{O}$ to be the stalk of P at the generic point of Q.*

If Q is finite over S, then the residue field $k_\mathcal{O}$ of $\mathcal{O}$ is a finite extension of the prime subfield, i.e., $L_\mathcal{O} = \overline{k_\mathcal{O}}$. In this case Theorem 1.5 is trivial. Therefore assume that Q has positive dimension $d + 1$.

The proper morphism of S-spaces

$$X_P \to P$$

gives rise to two proper morphisms of algebraic spaces over algebraically closed fields by forming the geometric generic fiber and the geometric closed fiber over $(S, \mathfrak{s})$.

(i) First, let $P_{\overline{\eta_S}}$ be the base change of P to $\overline{K(S)}$, i.e., $P_{\overline{\eta_S}} := P \otimes_S \overline{K(S)}$. By hypothesis this is integral and normal. Form the corresponding base change of X_P, i.e.,

$$X_{P,\overline{\eta_S}} = X_P \times_P P_{\overline{\eta_S}} \xrightarrow{\mathrm{pr}_{P_{\overline{\eta_S}}}} P_{\overline{\eta_S}}.$$

(ii) Second, let $Q_{\overline{\mathfrak{s}}}$ be the base change of Q to $\overline{\kappa(\mathfrak{s})}$, i.e., $Q_{\overline{\mathfrak{s}}} = Q \otimes_{\kappa(\mathfrak{s})} \overline{\kappa(\mathfrak{s})}$. Because $\kappa(\mathfrak{s})$ is perfect, $Q_{\overline{\mathfrak{s}}}$ is still normal. Form the corresponding base change of X_P, i.e.,

$$X_{Q,\overline{\mathfrak{s}}} = X_P \times_P Q_{\overline{\mathfrak{s}}} \xrightarrow{\mathrm{pr}_{Q_{\overline{\mathfrak{s}}}}} Q_{\overline{\mathfrak{s}}}.$$

Each of these is an example of a proper morphism of algebraic spaces over an algebraically closed field κ,

$$X_D \to D,$$

where

(i) X_D is an algebraic space, $X_D = X_{P,\overline{\eta_S}}$, resp. $X_D = X_{Q,\overline{\mathfrak{s}}}$,
(ii) D, embedded in some projective space $\mathbb{P}^N_\kappa$, is an integral, normal, quasi-projective variety over κ, $D = P_{\overline{\eta_S}}$, resp. $D = Q_{\overline{\mathfrak{s}}}$, and
(iii) κ is an algebraically closed field, $\kappa = \overline{K(S)}$, resp. $\kappa = \overline{\kappa(\mathfrak{s})}$.

For each triple $(X_D/D/\kappa)$, we associate a family of curves in D and then give a criterion for the existence of a proper subspace Y of X_D as in Theorem 1.5. Denote by $\mathrm{Grass}(d,N)$ the Grassmannian scheme parametrizing codimension-d linear subspaces of $\mathbb{P}^N_\kappa$, and denote by Λ_d the universal codimension-d linear subspace of $\mathbb{P}^N_k \times_k \mathrm{Grass}(d,N)$.

Definition 1.9. The *base of the universal family*, V_D, is the maximal open subscheme of $\mathrm{Grass}(d,N)$ over which the following projection morphism is flat,

$$\mathrm{pr}_{\mathrm{Grass}(d,N)} \circ \mathrm{pr}_{\Lambda_d} : D \times_{\mathbb{P}^N_k} \Lambda_d \to \Lambda_d \to \mathrm{Grass}(d,N).$$

The *universal family of codimension-d linear sections of* D, C_D, is the fiber product

$$C_D := D \times_{\mathbb{P}^N_k} \Lambda_d \times_{\mathrm{Grass}(d,N)} V_D.$$

Denote the two projection morphisms by

$$\mathrm{pr}_D : C_D \to D, \quad \mathrm{pr}_{V_D} : C_D \to V_D.$$

The *generic codimension-d linear section* $C_{\overline{\eta}}$ is the geometric generic fiber of the projection morphism pr_{V_D}.

Proposition 1.10. *Let κ be an algebraically closed field. Let $D \subset \mathbb{P}^N_\kappa$ be a normal, irreducible, quasi-projective κ-scheme of positive dimension $d+1$. Let X_D be a proper algebraic space over D.*

If there exists a section of the base change

$$X_D \times_D C_{\overline{\eta}} \to C_{\overline{\eta}},$$

then there exists an irreducible, closed subspace Y of X_D such that $Y \times_D \mathrm{Spec}\ \overline{\kappa(D)}$ is irreducible.

The proof uses a lemma which is an immediate consequence of the limit theorems of [**8**, §8].

Lemma 1.11. *If there exists a section of the base change*

$$X_D \times_D C_{\overline{\eta}} \to C_{\overline{\eta}},$$

then there exists an integral scheme E, a finite type, dominant, affine morphism

$$g : E \to V_D$$

and a section of the base change

$$X_D \times_D C_D \times_{V_D} E \to C_D \times_{V_D} E.$$

Taking $\kappa = \overline{\kappa(\mathfrak{s})}$, $D = Q_{\overline{\mathfrak{s}}}$ and $X_D = X_{Q,\overline{\mathfrak{s}}}$, the *conclusion* of Proposition 1.10 is the existence of a subspace Y of X_D whose base change

$$Y_{L_{\mathcal{O}}} := Y \times_D \mathrm{Spec}\ \kappa(D)$$

inside $X_D \times_D \mathrm{Spec}\ \kappa(D) = X_{\mathcal{O}} \otimes_{\mathcal{O}} L_{\mathcal{O}}$ is a subspace as in Theorem 1.5. Therefore, to complete the proof of Theorem 1.5 it suffices to verify the *hypothesis* of Proposition 1.10.

On the other hand, for $\kappa = \overline{K(S)}$, $D = P_{\overline{\eta_S}}$ and $X_D = X_{P,\overline{\eta_S}}$, the geometric generic fiber of X_D over D is separably rationally connected by hypothesis. Therefore Theorem 1.6 implies the restriction of X_D over $C_{\overline{\eta}}$ has a section, i.e., the *hypothesis* of Proposition 1.10 are satisfied. Thus Theorem 1.5 finally follows from a specialization lemma showing the hypothesis for $(X_{P,\overline{\eta_S}}/P_{\overline{\eta_S}}/\overline{K(S)})$ implies the hypothesis for $(X_{Q,\overline{\mathfrak{s}}}/Q_{\overline{\mathfrak{s}}}/\overline{\kappa(\mathfrak{s})})$. The lemma is well-known: one version is [**5**, Lemma 2.5].

LEMMA 1.12. *For a proper, strict, finite type datum such that P/S has relative dimension $d+1$, if the restriction of $X_{P,\overline{\eta_S}}$ over the generic codimension-d linear section of $P_{\overline{\eta_S}}$ has a section, then the restriction of $X_{Q,\overline{\mathfrak{s}}}$ over the generic codimension-d linear section of $Q_{\overline{\mathfrak{s}}}$ also has a section.*

Mostly Proposition 1.10 is a consequence of the limit theorems in [**8**, §8]. However, we need to use one irreducibility result about linear sections of varieties which is somewhat more precise than the usual Bertini theorem. Since we could find no reference, we include a proof. This result might be of independent interests to those who study Bertini theorems. We end the article with a question as to generalizations of this proposition.

PROPOSITION 1.13. *Let κ be an algebraically closed field. Denote by $\Lambda_d \subset \mathbb{P}^N_\kappa \times_\kappa Grass(d,N)$ the universal codimension-d linear subspace. Let E be an irreducible, finite type κ-scheme and let $g : E \to Grass(d,N)$ be a dominant morphism. After replacing E by a dense open subset, for every irreducible closed subset D of $\mathbb{P}^N_\kappa$ of dimension $\geq d+1$ the geometric generic fiber of*

$$pr_D : D \times_{\mathbb{P}^N_\kappa} \Lambda_d \times_{Grass(d,N)} E \to D$$

is irreducible.

Acknowledgments. I thank A. J. de Jong for explaining his proof of Theorem 1.4 some years ago. I thank János Kollár for encouraging me to write-up this article and pointing out [**4**, Theorem 21.3.6(a)]. I thank the referees, of this version and an earlier version, for their helpful comments.

2. Proof of Lemma 1.8

Step 1. Reduction of X_R to a finite type subring A. Because X_R is a finite type algebraic space, there exists a finitely generated subring A of R and a proper algebraic space X_A over A such that $X_A \otimes_A R \cong X_R$. If k_R contains $\mathbb{Q}$, replace A by $A \otimes \mathbb{Q}$. Also increase A if necessary so that $\mathfrak{m}_R \cap A$ is principally generated by an element π which is a uniformizer of $\mathfrak{m}_R$. Finally, by the Noether normalization theorem, the integral closure of A in its fraction field is still finitely generated. Increase A if necessary so that A is normal.

Step 2. Compatibility with base change. Denote by $\mathfrak{m}$ the prime ideal $\mathfrak{m}_R \cap A$ of A. By construction $\mathfrak{m}$ is principal and A is normal. Thus the local ring $A_{\mathfrak{m}}$

is a DVR. By construction $A_{\mathfrak{m}}$ is contained in R and the inclusion homomorphism is a local homomorphism. Denote by $X_{A_{\mathfrak{m}}}$ the base change $X_A \otimes_A A_{\mathfrak{m}}$. Associated to every closed subspace $Y_{A_{\mathfrak{m}}}$ of $X_{A_{\mathfrak{m}}} \otimes_{A_{\mathfrak{m}}} L_{A_{\mathfrak{m}}}$, the image of

$$Y_{A_{\mathfrak{m}}} \otimes_{L_{A_{\mathfrak{m}}}} L_R \subset X_{A_{\mathfrak{m}}} \otimes_{A_{\mathfrak{m}}} L_R \xrightarrow{\cong} X_R \otimes_R L_R$$

is a closed subspace Y_R of $X_R \otimes_R L_R$. Moreover,

$$Y_R \otimes_{L_R} \overline{k_R} \cong (Y_{A_{\mathfrak{m}}} \otimes_{L_{A_{\mathfrak{m}}}} \overline{k_{A_{\mathfrak{m}}}}) \otimes_{\overline{k_{A_{\mathfrak{m}}}}} \overline{k_R}.$$

By [**8**, Théorème 4.4.4], if $Y_{A_{\mathfrak{m}}} \otimes_{L_{A_{\mathfrak{m}}}} \overline{k_{A_{\mathfrak{m}}}}$ is irreducible, so is $Y_R \otimes_{L_R} \overline{k_R}$. Thus, to prove Theorem 1.5 for R and X_R, it suffices to prove Theorem 1.5 for $A_{\mathfrak{m}}$ and $X_{A_{\mathfrak{m}}}$.

Step 3. Reduction to the strict case. There is a slight bifurcation at this point. If A contains $\mathbb{Q}$, respectively a finite field $\mathbb{F}_p$, then denote by S^{pre} the integral closure in A of $\mathbb{Q}[\pi]$, resp. $\mathbb{F}_p[\pi]$. If A contains $\mathbb{Z}$ and $pA \subset \mathfrak{m}$, denote by S^{pre} the integral closure of $\mathbb{Z}$ in A. In each case, by the Noether normalization theorem, S^{pre} is a finitely generated $\mathbb{Z}$-algebra. And, of course, $\mathfrak{m} \cap S^{\text{pre}}$ contains a primary element, i.e., an element ρ such that ρA is an $\mathfrak{m}$-primary ideal. (In the first two cases S^{pre} even contains π, and in the third case $\rho = p$ is a primary element.)

Because S^{pre} is a Dedekind domain and $\mathfrak{m} \cap S^{\text{pre}}$ is a nonzero prime ideal, it is a maximal ideal. Denote the residue field by

$$k_{S^{\text{pre}}} := S^{\text{pre}}/\mathfrak{m} \cap S^{\text{pre}}.$$

It may happen that $A/\mathfrak{m}$ is not geometrically irreducible over $k_{S^{\text{pre}}}$, i.e., $k_{S^{\text{pre}}}$ is not algebraically closed in the fraction field of $A/\mathfrak{m}$. Denote by k_S the algebraic closure of $k_{S^{\text{pre}}}$ in the fraction field of $A/\mathfrak{m}$. Because $k_{S^{\text{pre}}}$ is either a number field or a finite field, the field extension $k_S/k_{S^{\text{pre}}}$ is finite and separable. By the primitive element theorem, there exists $\overline{f} \in k_{S^{\text{pre}}}[T]$ such that $k_S \cong k_{S^{\text{pre}}}[T]/\langle f \rangle$. Let $f \in S^{\text{pre}}[T]$ be an element mapping to $\overline{f}$. After inverting the discriminant of f in S^{pre}, the ring

$$S := S^{\text{pre}}[T]/\langle f \rangle$$

is finite and étale over S^{pre}. Denote by $\mathfrak{m}_S$ the maximal ideal in S generated by $\mathfrak{m} \cap S^{\text{pre}}$. Denote by A_S the normalization of $S \otimes_{S^{\text{pre}}} A$. Then $A_S/\mathfrak{m}_S A_S$ is isomorphic to $A/\mathfrak{m}$.

Denote by X_{A_S} the base change of X_A to A_S. The geometric generic fiber of X_{A_S} over A_S equals the geometric generic fiber of X_A over A. And the closed fiber of X_{A_S} over A_S equals the closed fiber of X_A over A. Therefore, to prove the theorem for A, it suffices to prove the theorem for A_S. And k_S is algebraically closed in the fraction field of $A_S/\mathfrak{m}_S A_S$, i.e., $A_S/\mathfrak{m}_S A_S$ is geometrically irreducible over k_S.

Step 4. Existence of a finite type datum. It is a straightforward consequence of Nagata compactification, [**12**], etc., that there exists a strict, proper, finite type datum over S such that $\mathcal{O}$ equals the localization of A_S at $\mathfrak{m}_S A_S$ and $X_{\mathcal{O}}$ equals the base change of X_{A_S} over the localization.

3. A symmetry for Bertini theorems. Proof of Proposition 1.13

At the end of this section we prove Proposition 1.13. In this section, κ denotes an algebraically closed field. The classical Bertini theorem concerns the general hyperplane section of a quasi-projective κ-variety. More generally, one could consider a family of cycles C in an ambient variety A parametrized by a base B, i.e., a finite type morphism

$$\iota = (\iota_A, \iota_B) : C \to A \times_k B.$$

The classical case is $A = \mathbb{P}^N_\kappa$, $B = (\mathbb{P}^N_\kappa)^\vee$, the dual projective space, and C is the universal hyperplane in $A \times_\kappa B$.

In this setting, a Bertini theorem asserts that some property of a morphism $f : D \to A$ implies a similar property for the geometric generic fiber of the associated morphism

$$\iota_B \circ \mathrm{pr}_C : D \times_A C \to C \to B.$$

The proposition below proves that when (A, B, C) satisfies a Bertini theorem, the "opposite triple" (B, A, C^{opp}) satisfies a similar Bertini theorem.

Proposition 3.1. *Assume that for every pair D, E of irreducible, finite type κ-schemes and for every pair*

$$f : D \to A, \quad g : E \to B$$

of morphisms of κ-schemes with

$$dim(f(D)) \geq d, \quad dim(g(E)) \geq e,$$

after replacing D by a dense open subset, the geometric generic fiber of

$$pr_E : D \times_A C \times_B E \to E$$

is irreducible. Then after replacing E by a dense open subset also the geometric generic fiber of

$$pr_D : D \times_A C \times_B E \to D$$

is irreducible.

The proof uses a criterion from [**8**, §4] for irreducibility of the geometric generic fiber of a morphism.

Lemma 3.2. (i) *The geometric generic fiber of a dominant morphism of irreducible schemes*

$$t : W \to Z$$

is integral if there exists a rational section s of t mapping the generic point of Z to a normal point of W.

(ii) *The geometric generic fiber of a dominant morphism of irreducible schemes*

$$u : Z \to X$$

is irreducible if and only if $Z \times_X Z$ has a unique irreducible component W dominating Z.

Proof. **(i).** Denote by η_Z the generic point of Z and denote by η_s the image $s(\eta_Z)$ in W. Then $\mathcal{O}_{W,\eta_s}$ is a normal local domain containing its residue field $\kappa(\eta_Z)$. In particular, $\kappa(\eta_Z)$ is separably algebraic closed (even algebraically closed) in the fraction field $K(\mathcal{O}_{W,\eta_s}) = K(W)$. Therefore, by [**8**, Corollaire 4.5.10], $K(W)$ is geometrically irreducible over κ.

(ii). It suffices to prove this after replacing Z and X by their generic points. Moreover, because purely inseparable field extensions are universal homeomorphisms, [**8**, Proposition 4.5.21], it suffices to consider the case when $Z \to X$ is the morphism of schemes associated to a separable field extension.

If the geometric generic fiber of Z over X is irreducible, then by [**8**, Théorème 4.4.4], the geometric generic fiber of $Z \times_X Z$ over Z is irreducible. Hence $Z \times_X Z$ is irreducible, being the image of the geometric generic fiber. Conversely, assume $Z \times_X Z$ is irreducible. By [**8**, Proposition 6.14.1], $Z \times_X Z$ is also normal. The diagonal morphism $\Delta_{Z/X}$ gives a Z-point of $Z \times_X Z$. Therefore, by (i), the geometric generic fiber of $Z \times_X Z$ over Z is irreducible. By [**8**, Théorème 4.4.4] again, the geometric generic fiber of Z over X is irreducible. □

Proof of Proposition 3.1. Denote the fiber product

$$E \times_A C \times_B D$$

by $C_{E,D}$ and denote the projections onto E and D by

$$p_{(D,E);D} : C_{D,E} = D \times_A C \times_B E \to D,$$

and

$$p_{(D,E);D} : C_{D,E} = D \times_A C \times_B E \to E.$$

The goal is to prove the geometric generic fiber of $p_{(D,E);E}$ is irreducible after replacing D by a suitable open, dense subset. In fact, it suffices to replace D, respectively E, by the maximal open subsets over which $p_{(D,E);D}$ is flat, resp. $p_{(D,E);E}$ is flat. This is dense by [**8**, Théorème 6.9.1]. Thus assume $p_{(D,E);D}$ and $p_{(D,E);E}$ are flat. There is one observation: because flatness is preserved by base change and because compositions of flat morphisms are flat, the following two associated morphisms are also flat:

$$\mathrm{pr}_1 : C_{D,E} \times_D C_{D,E} \to C_{D,E}, \text{ and}$$

$$p_{(D,E);E} \circ \mathrm{pr}_2 : C_{D,E} \times_D C_{D,E} \to C_{D,E} \to E.$$

By Lemma 3.2(ii), to prove the geometric generic fiber of $p_{(D,E);D}$ is irreducible, it suffices to prove that $C_{D,E}$ and $C_{D,E} \times_D C_{D,E}$ are irreducible. Because E is irreducible and $p_{(D,E);E}$ is flat, every component of $C_{D,E}$ intersects the generic fiber of $p_{(D,E);E}$. Because the geometric generic fiber is irreducible, also the generic fiber is irreducible. Therefore $C_{D,E}$ is irreducible.

But then, denoting $C_{D,E}$ by D' and denoting by f' the composition,

$$f' = f \circ p_{(D,E);D} : C_{D,E} \to D \to A,$$

the pairs (D', E) and (f', g) satisfy the same conditions as (D, E) and (f, g). Indeed, D' is irreducible by the last paragraph. Because $p_{(D,E);D}$ is flat $f'(D')$ is dense in $f(D)$, thus has dimension $\geq d$. As observed above, flatness of $p_{(D,E);D}$ and $p_{(D,E);E}$ implies flatness of the associated morphisms pr_1 and $p_{(D,E);E} \circ \mathrm{pr}_2$. By a straightforward diagram chase,

$$C_{D',E} = C_{D,E} \times_D C_{D,E}, \quad p_{(D',E);D'} = \mathrm{pr}_1, \text{ and } p_{(D',E);E} = p_{(D,E);E} \circ \mathrm{pr}_2.$$

Thus the argument above proves $C_{D,E} \times_D C_{D,E}$ is irreducible. Therefore Lemma 3.2(ii) implies irreducibility of the geometric generic fiber of $p_{(D,E);D}$. □

Proof of Proposition 1.13. The classical Bertini theorem implies the triple $(\mathbb{P}^N_\kappa, \mathrm{Grass}(d, N), \Lambda_d)$ satisfies the conditions of Proposition 3.1 with

$$\dim(f(D)) \geq d + 1, \quad \dim(g(E)) = \dim(\mathrm{Grass}(d, N)), \text{ i.e., } g \text{ is dominant},$$

cf. [**9**, Théorème I.6.10]. Thus, for every irreducible closed subset D of $\mathbb{P}^N_\kappa$ of dimension $\geq d+1$, there exists an open subset V of E such that the geometric generic fiber of

$$\mathrm{pr}_D : D \times_{\mathbb{P}^N_\kappa} \Lambda_d \times_{\mathrm{Grass}(d,N)} V \to D$$

is irreducible.

For every open subset V of E, the subset

$$R_V = \{x \in \mathbb{P}^N_\kappa | \mathrm{Spec}\ \overline{\kappa}(x) \times_{\mathbb{P}^N_\kappa} \Lambda_d \times_{\mathrm{Grass}(d,N)} V \text{ reducible } \}$$

is constructible by [**9**, Théorème I.4.10]. Because $\mathbb{P}^N_\kappa$ is Noetherian, the collection of closed subsets

$$\{\overline{R}_V | V \text{ dense, open in } E\}$$

has a minimal element $\overline{R}_V$. This V works for all D simultaneously. $\square$

Remark 3.3. **(i)** One example of a morphism $g : E \to \mathrm{Grass}(d,N)$ where the conclusion holds only after replacing E by an open dense subset is the blowing up of $\mathrm{Grass}(d,N)$ at the ideal sheaf of a closed point (assuming $d > 1$ and $N > 2d$).

(ii) There are also examples where the conclusion fails if one allows morphisms f with $\dim(f(D)) = d$.

4. Limit theorems. Proof of Propositon 1.10

Proof of Lemma 1.11. Denote by X_{C_D} the base change $X_D \times_D C_D$. This is a finite type algebraic space over V_D. Denote by $X_{C_{\overline{\eta}}}$ the base change $X_D \times_D C_{\overline{\eta}}$, which also equals $X_{C_D} \times_{V_D} \mathrm{Spec}\ \overline{K(V_D)}$. Associate to each section

$$\sigma : C_{\overline{\eta}} \to X_{C_{\overline{\eta}}}$$

the image Γ_σ of σ,

$$\Gamma_\sigma := \sigma(C_{\overline{\eta}}) \subset X_{C_{\overline{\eta}}}.$$

This is a closed subspace.

Denote by S_0 the algebraic space X_{C_D}. Denote the coherent $\mathcal{O}_{V_D}$-subalgebras of the constant $\mathcal{O}_{V_D}$-algebra $\overline{K(V_D)}$ by R_λ. For every λ, denote by A_λ the pullback of R_λ to C_D. And denote

$$S_\lambda = \mathrm{Spec}\ A_\lambda.$$

Because $\overline{K(V_D)}$ is the filtering direct limit of the algebras R_λ, the inverse limit of the schemes S_λ is

$$\varprojlim S_\lambda \cong X_{C_D} \times_{V_D} \mathrm{Spec}\ \overline{K(V_D)} = X_{\overline{\eta}}.$$

By the limit theorems of EGA IV, particularly [**8**, Proposition 8.6.3], there exists λ and a closed subspace Γ_λ of S_λ such that

$$\Gamma_\sigma = \Gamma_\lambda \times_{S_\lambda} X_{\overline{\eta}}.$$

To be precise, EGA IV deals only with schemes, not algebraic spaces. However it is straightforward to generalize the proposition to algebraic spaces.

Denote by

$$g : E \to V_D \subset \mathrm{Grass}(d,N)$$

the relative spectrum of R_λ. Then Γ_λ is equivalent to a closed subspace

$$\Gamma_E \subset X_{C_D} \times_{V_D} E.$$

Consider the projection

$$\pi_E : \Gamma_E \to C_D \times_{V_D} E.$$

After further base change from E to Spec $\overline{K(V_D)}$, this morphism becomes an isomorphism. Therefore, by [**8**, Théorème 8.8.2(i)], after replacing R_λ by a larger $\mathcal{O}_{V_D}$-algebra, we may assume that the projection is an isomorphism. Denote by

$$\sigma_E : C_D \times_{V_D} E \to X_{C_D} \times_{V_D} E$$

the unique morphism such that

$$\sigma_E \circ \pi_E : \Gamma_E \to X_{C_D} \times_{V_D} E$$

is the inclusion morphism. Then σ_E is a section of the projection

$$X_{C_D} \times_{V_D} E \to C_D \times_{V_D} E.$$

Because R_λ is an $\mathcal{O}_{V_D}$-subalgebra of $\overline{K(V_D)}$, in particular it is integral and contains $\mathcal{O}_{V_D}$. Thus E is integral and g is a finite type, affine, dominant morphism. $\square$

Proof of Proposition 1.10. Let E, g and σ_E be as in Lemma 1.11. Observe that E and g satisfy all the hypotheses of Proposition 1.13. Therefore, possibly after replacing E by a dense open subset, the geometric generic fiber of

$$\mathrm{pr}_D : C_D \times_{V_D} E = D \times_{\mathbb{P}^N_\kappa} \Lambda_d \times_{\mathrm{Grass}(d,N)} E \to D$$

is irreducible. In other words, denoting by Z the unique irreducible component of $D \times_{\mathbb{P}^N_\kappa} \Lambda_d \times_{\mathrm{Grass}(d,N)} E$ dominating D, the geometric generic fiber of

$$\mathrm{pr}_D|_Z : Z \to D$$

is irreducible.

The restriction of σ_E to Z is an E-morphism

$$\sigma_Z : Z \subset C_D \times_{V_D} E \to X_{C_D} \times_{V_D} E$$

which is a section of $\mathrm{pr}_{C_E}|_Z$. Denote by s_Z the composition

$$s_Z : Z \xrightarrow{\sigma_Z} X_{C_D} \times_{V_D} E \xrightarrow{\mathrm{pr}_{X_{C_D}}} X_{C_D} \xrightarrow{\mathrm{pr}_{X_D}} X_D.$$

By a straightforward diagram chase, this is a morphism of D-schemes. Denote by Y the closure of the image of s_Z,

$$Y := \overline{s_Z(Z)}.$$

The geometric generic fiber of Z over D dominates the geometric generic fiber of Y over D. Therefore, since the geometric generic fiber of Z over D is irreducible, also the geometric generic fiber of Y over D is irreducible. $\square$

5. Proof of Lemma 1.12

As mentioned, this is essentially the same argument as in [**5**, Lemma 2.5]. Embed P in $\mathbb{P}^N_S$. Denote by $\mathrm{Grass}(d,N)$ the Grassmannian scheme over S parametrizing codimension-d linear subspaces of fibers of $\mathbb{P}^N_S$ over S. Denote by Λ_d the universal codimension-d linear subspace of $\mathbb{P}^N_S \times_S \mathrm{Grass}(d,N)$. Just as above, denote by V_P the maximal open subscheme of $\mathrm{Grass}(d,N)$ over which

$$P \times_{\mathbb{P}^N_S} \Lambda_d \to \Lambda_d \to \mathrm{Grass}(d,N)$$

is flat. Because $\mathrm{Grass}(d,N)$ is regular, and because $P \times_{\mathbb{P}^N_S} \Lambda_d$ is integral, V_P contains all the codimension one points of $\mathrm{Grass}(d,N)$, cf. [**7**, p. 57, §6.3]. In particular, V_P is dense in the closed fiber of $\mathrm{Grass}(d,N)$ over S.

By hypothesis there exists a section of the restriction of X_P to the generic codimension-d linear section of $P_{\overline{\eta_S}}$. By an argument almost identical to the proof of Lemma 1.11, there exists an integral scheme E, a finite type, dominant, affine morphism

$$g : E \to V_P,$$

and a section

$$\sigma_E : C_P \times_{V_P} E \to X_P \times_P C_P \times_{V_P} E$$

of the projection morphism

$$\pi_E : X_P \times_P C_P \times_{V_P} E \to C_P \times_{V_P} E.$$

Because g is affine and finite type, there exists an integral, normal scheme $\overline{E}$, a proper, surjective morphism

$$\overline{g} : \overline{E} \to V_P$$

and an open immersion $\iota : E \hookrightarrow \overline{E}$ such that $g = \overline{g} \circ \iota$. Denote by W the maximal open subscheme

$$W \subset C_P \times_{V_P} \overline{E}$$

over which σ_E extends to a morphism

$$\sigma_W : W \to X_P \times_P C_P \times_{V_P} \overline{E}.$$

Because $X_P \times_P C_P \times_{V_P} \overline{E}$ is proper, the valuative criterion of propeness implies that W contains every normal, codimension 1 point of $C_P \times_{V_P} \overline{E}$.

There exists a normal, codimension 1 point ζ of $\overline{E}$ mapping to the generic point of the closed fiber of $\mathrm{Grass}(d, N)$ over S. Denote by $C_{Q,\overline{\eta}}$ the generic codimension-d linear section of Q. There is a generic point ζ' of $C_P \times_{V_P} \zeta$ whose image in C_P equals the generic point of $C_{Q,\overline{\eta}}$. Because $C_{Q,\overline{\eta}}$ is smooth over $\overline{K(V_Q)}$, ζ' is a normal, codimension 1 point of $C_P \times_{V_P} \overline{E}$. Therefore the image of W in C_P contains the image of the generic point of the generic codimension-d linear section $C_{Q,\overline{\eta}}$ of Q.

By the last paragraph, the image of

$$W \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)} \to C_P \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)}$$

contains the generic point of $C_{Q,\overline{\eta}}$. Therefore there exists a $\overline{K(V_Q)}$-point of $E \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)}$,

$$e : \mathrm{Spec}\, \overline{K(V_Q)} \to E \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)}$$

whose associated morphism

$$\widetilde{e} : C_P \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)} \to C_P \times_{V_P} E \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)}$$

pulls back W to an open subset $\widetilde{e}^{-1}(W)$ of $C_P \times_{V_P} \mathrm{Spec}\, \overline{K(V_Q)}$ containing the generic point of $C_{Q,\overline{\eta}}$.

Denote by W' the open subset

$$W' := \widetilde{e}^{-1}(W) \cap C_{Q,\overline{\mathfrak{s}}}.$$

This is a dense open subset of $C_{Q,\overline{\mathfrak{s}}}$. The composition,

$$\mathrm{pr}_{X_P} \circ \sigma_W \circ \widetilde{e} : W' \to W \to X_P \times_P C_P \times_{V_P} \overline{E} \to X_P$$

is a P-morphism giving a section of

$$X_Q \times_Q C_{Q,\overline{\eta}} \to C_{Q,\overline{\eta}}$$

over W'. But $C_{Q,\overline{\eta}}$ is a normal curve, and $X_Q \times_Q C_{Q,\overline{\eta}}$ is proper over $C_{Q,\overline{\eta}}$. Therefore, by the valuative criterion of properness, this extends to a section of the restriction of $X_{Q,\overline{\mathfrak{s}}}$ over the generic codimension-d linear section $C_{Q,\overline{\eta}}$ of $Q_{\overline{\mathfrak{s}}}$.

6. Final question

QUESTION 6.1. Let $f : D \to \mathbb{P}^N_\kappa$ and $g : E \to \mathrm{Grass}(d, N)$ be morphisms from irreducible, finite type κ-schemes as in Proposition 3.1. Assume that

$$\dim(f(D)) \geq d + a, \quad \dim(g(E)) \geq \dim(\mathrm{Grass}(d, N)) - b.$$

Also assume (shrinking D and E if necessary) that the projections

$$\mathrm{pr}_D : D \times_{\mathbb{P}^N} \Lambda_d \times_{\mathrm{Grass}(d,N)} E \to D,$$

$$\mathrm{pr}_E : D \times_{\mathbb{P}^N} \Lambda_d \times_{\mathrm{Grass}(d,N)} E \to E,$$

are both flat. Assuming $a > b$, does it follow that the geometric generic fibers of both pr_D and pr_E are irreducible?

By Proposition 3.1, if for one of pr_D or pr_E the geometric generic fiber is always irreducible, then the same holds for the other morphism as well. Moreover, Proposition 1.13 is precisely the case when $a = 1$ and $b = 0$. However, as far as we know, the question is open except in this case.

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Clay Mathematics Proceedings
Volume **18**, 2013

Remarks on curve classes on rationally connected varieties

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This note is dedicated to Joe Harris, whose influence on the subject of curves on rationally connected algebraic varieties (among other topics!) is invaluable.

Abstract. We study for rationally connected varieties X the group of degree 2 integral homology classes on X modulo those which are algebraic. We show that the Tate conjecture for divisor classes on surfaces defined over finite fields implies that this group is trivial for any rationally connected variety.

1. Introduction

Let X be a smooth complex projective variety. Define

$$Z^{2i}(X) = \frac{\mathrm{Hdg}^{2i}(X,\mathbb{Z})}{H^{2i}(X,\mathbb{Z})_{alg}}, \tag{1.1}$$

where $\mathrm{Hdg}^{2i}(X,\mathbb{Z})$ is the space of integral Hodge classes on X and $H^{2i}(X,\mathbb{Z})_{alg}$ is the subgroup of $H^{2i}(X,\mathbb{Z})$ generated by classes of codimension i closed algebraic subsets of X.

These groups measure the defect of the Hodge conjecture for integral Hodge classes, hence they are trivial for $i = 0$, 1 and $n = \dim X$, but in general they can be nonzero by [**1**]. Furthermore they are torsion if the Hodge conjecture for *rational* Hodge classes on X of degree $2i$ holds. In addition to the previously mentioned case, this happens when $i = n-1$, $n = \dim X$, due to the Lefschetz theorem on $(1,1)$-classes and the hard Lefschetz isomorphism (cf. [**23**]). We will call classes in $\mathrm{Hdg}^{2n-2}(X,\mathbb{Z})$ "curve classes", as they are also degree 2 homology classes.

Note that the Kollár counterexamples (cf. [**14**]) to the integral Hodge conjecture already exist for curve classes (that is degree 4 cohomology classes in this case) on projective threefolds, unlike the Atiyah-Hirzebruch examples which work for degree 4 integral Hodge classes in higher dimension.

It is remarked in [**21**], [**23**] that the two groups

$$Z^4(X),\ Z^{2n-2}(X),\ n := \dim X$$

2010 *Mathematics Subject Classification.* Primary 14C25, 14F99, 14K99.

Key words and phrases. Integral Hodge class, integral Tate class, rationally connected varieties.

The author thanks the organizers of the beautiful conference "A celebration of algebraic geometry" for inviting me there. The author also thanks Jean-Louis Colliot-Thélène, Olivier Debarre and János Kollár for useful discussions.

are birational invariants. (For threefolds, this is the same group, but not in higher dimension.) The nontriviality of these birational invariants for rationally connected varieties is asked in [**23**]. Still more interesting is the nontriviality of these invariants for unirational varieties, having in mind the Lüroth problem (cf. [**3**], [**2**], [**4**]).

Concerning the group $Z^4(X)$, Colliot-Thélène and the author proved in [**8**], building on the work of Colliot-Thélène and Ojanguren [**5**], that it can be nonzero for unirational varieties starting from dimension 6. What happens in dimensions 5 and 4 is unknown (the four dimensional case being particularly challenging in our mind), but in dimension 3, there is the following result proved in [**22**]:

Theorem 1.1. *(Voisin 2006) Let X be a smooth projective threefold which is either uniruled or Calabi-Yau (meaning that K_X is trivial and $H^1(X,\mathcal{O}_X)=0$). Then the group $Z^4(X)$ is equal to 0.*

This result, and in particular the Calabi-Yau case, implies that the group $Z^6(X)$ is also 0 for a Fano fourfold X which admits a smooth anticanonical divisor. Indeed, a smooth anticanonical divisor $j : Y \hookrightarrow X$ is a Calabi-Yau threefold, so that we have $Z^4(Y)=0$ by Theorem 1.1 above. As $H^2(Y,\mathcal{O}_Y)=0$, every class in $H^4(Y,\mathbb{Z})$ is a Hodge class, and it follows that $H^4(Y,\mathbb{Z})=H^4(Y,\mathbb{Z})_{alg}$. As the Gysin map $j_* : H^4(Y,\mathbb{Z}) \to H^6(X,\mathbb{Z})$ is surjective by the Lefschetz theorem on hyperplane sections, it follows that $H^6(X,\mathbb{Z})=H^6(X,\mathbb{Z})_{alg}$, and thus $Z^6(X)=0$.

In the paper [**11**], it was proved more generally that if X is any Fano fourfold, the group $Z^6(X)$ is trivial. Similarly, if X is a Fano fivefold of index 2, the group $Z^8(X)$ is trivial.

These results have been generalized to higher dimensional Fano manifolds of index $n-3$ and dimension ≥ 8 by Enrica Floris [**9**] who proves the following result:

Theorem 1.2. *Let X be a Fano manifold over $\mathbb{C}$ of dimension $n \geq 8$ and index $n-3$. Then the group $Z^{2n-2}(X)$ is equal to 0: Equivalently, any integral cohomology class of degree $2n-2$ on X is algebraic.*

The purpose of this note is to provide evidence for the vanishing of the group $Z^{2n-2}(X)$, for any rationally connected variety over $\mathbb{C}$. Note that in this case, since $H^2(X,\mathcal{O}_X)=0$, the Hodge structure on $H^2(X,\mathbb{Q})$ is trivial, and so is the Hodge structure on $H^{2n-2}(X,\mathbb{Q})$, so that $Z^{2n-2}(X)=H^{2n-2}(X,\mathbb{Z})/H^{2n-2}(X,\mathbb{Z})_{alg}$. We will first prove the following two results.

Proposition 1.3. *The group $Z^{2n-2}(X)$ is locally a deformation invariant for rationally connected manifolds X.*

Let us explain the meaning of the statement. Consider a smooth projective morphism $\pi : \mathcal{X} \to B$ between connected quasi-projective complex varieties, with n dimensional fibers. Recall from [**15**] that if one fiber $X_b := \pi^{-1}(b)$ is rationally connected, so is every fiber. Let us endow everything with the usual topology. Then the sheaf $R^{2n-2}\pi_*\mathbb{Z}$ is locally constant on B. On any Euclidean open set $U \subset B$ where this local system is trivial, the group $Z^{2n-2}(X_b)$, $b \in U$, is the finite quotient of the *constant* group $H^{2n-2}(X_b,\mathbb{Z})$ by its subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$. To say that $Z^{2n-2}(X_b)$ is locally constant means that on open sets U as above, the subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$ of the constant group $H^{2n-2}(X_b,\mathbb{Z})$ does not depend on b.

It follows from the above result that the vanishing of the group $Z^{2n-2}(X)$ for X a rationally connected manifold reduces to the similar statement for X defined over a number field.

Let us now define an l-adic analogue $Z^{2n-2}(X)_l$ of the group $Z^{2n-2}(X)$ (cf. [**6**], [**7**]). Let X be a smooth projective variety defined over a field K which in the sequel will be either a finite field or a number field. Let $\overline{K}$ be an algebraic closure of K. Any cycle $Z \in CH^s(X_{\overline{K}})$ is defined over a finite extension of K. Let l be a prime integer different from $p = \mathrm{char}\, K$ if K is finite. It follows that the cycle class

$$cl(Z) \in H^{2s}_{et}(X_{\overline{K}}, \mathbb{Q}_l(s))$$

is invariant under an open subgroup of $\mathrm{Gal}(\overline{K}/K)$.

Classes satisfying this property are called Tate classes. The Tate conjecture for finite fields asserts the following:

Conjecture 1.4. *(cf.* [**18**] *for a recent account) Let X be smooth and projective over a finite field K. The cycle class map gives for any s a surjection*

$$cl : CH^s(X_{\overline{K}}) \otimes \mathbb{Q}_l \to H^{2s}(X_{\overline{K}}, \mathbb{Q}_l(s))_{Tate}.$$

Note that the cycle class map defined on $CH^s(X_{\overline{K}})$ in fact takes values in $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))$, and more precisely in the subgroup $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))_{Tate}$ of classes invariant under an open subgroup of $\mathrm{Gal}(\overline{K}/K)$. We thus get for each i a morphism

$$cl^i : CH^i(X_{\overline{K}}) \otimes \mathbb{Z}_l \to H^{2i}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}.$$

We can thus introduce the following variant of the groups $Z^{2i}(X)$:

$$Z^{2i}_{et}(X)_l := H^{2i}_{et}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}/\mathrm{Im}\, cl^i.$$

An argument similar to the one used for the proof of Proposition 1.3 will lead to the following result:

Proposition 1.5. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over* $\mathrm{Spec}\,\mathcal{O}_K$. *Fix a prime integer l. Then except for finitely many $p \in \mathrm{Spec}\,\mathcal{O}_K$, the group $Z^{2n-2}_{et}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et}(X_p)_l$.*

In the course of the paper, we will also consider variants $Z^{2n-2}_{rat}(X)$, resp. $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}(X)$, resp. $Z^{2n-2}_{et}(X)_l$, obtained by taking the quotient of the group of integral Hodge classes (resp. integral l-adic Tate classes) by the subgroup generated by classes of *rational* curves. This variant is suggested by Kollár's paper (cf. [**16**, Question 3, (1)]). By the same arguments, these groups are also deformation and specialization invariants for rationally connected varieties.

Our last result is conditional but it strongly suggests the vanishing of the group $Z^{2n-2}(X)$ for X a smooth rationally connected variety over $\mathbb{C}$. Indeed, we will prove using the main result of [**19**] and the two propositions above the following consequence of Theorem 1.5:

Theorem 1.6. *Assume Tate's conjecture* 1.4 *holds for degree* 2 *Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

2. Deformation and specialization invariance

Proof of Proposition 1.3. We first observe that, due to the fact that relative Hilbert schemes parameterizing curves in the fibers of B are a countable union of varieties which are projective over B, given a simply connected open set $U \subset B$ (in the classical topology of B), and a class $\alpha \in \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ such that α_t is algebraic for $t \in V$, where V is a smaller nonempty open set $V \subset U$, then α_t is algebraic for any $t \in U$.

To prove the deformation invariance, we just need using the above observation to prove the following:

LEMMA 2.1. *Let $t \in U \subset B$, and let $C \subset X_t$ be a curve and let $[C] \in H^{2n-2}(X_t, \mathbb{Z}) \cong \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ be its cohomology class. Then the class $[C]_s$ is algebraic for s in a neighborhood of t in U.*

PROOF. By results of [**15**], there are rational curves $R_i \subset X_t$ with ample normal bundle which meet C transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number D of such curves with generically chosen tangent directions at the attachment points. We then know by [**10**, §2.1] that the curve $C' = C \cup_{i \leq D} R_i$ is smoothable in X_t to a smooth unobstructed curve $C'' \subset X_t$, that is $H^1(C'', N_{C''/X_t}) = 0$. This curve C'' then deforms with X_t (cf. [**12**], [**13**, II.1]) in the sense that the morphism from the deformation of the pair (C'', X_t) to B is smooth, and in particular open. So there is a neighborhood of V of t in U such that for $s \in V$, there is a curve $C''_s \subset X_s$ which is a deformation of $C'' \subset X_t$. The class $[C''_s] = [C'']_s$ is thus algebraic on X_s. On the other hand, we have

$$[C''] = [C'] = [C] + \sum_i [R_i].$$

As the R_i's are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_i]_s$ also are algebraic on X_s for s in a neighborhood of t in U. Thus $[C]_s = [C'']_s - \sum_i [R_i]_s$ is algebraic on X_s for s in a neighborhood of t in U. The lemma, hence also the proposition, is proved.

□

REMARK 2.2. There is an interesting variant of the group $Z^{2n-2}(X)$, which is suggested by Kollár (cf. [**16**]) given by the following groups:

$$Z^{2n-2}_{rat}(X) := H^{2n-2}(X, \mathbb{Z})/\langle [C], C \text{ rational curve in } X\rangle.$$

Here, by a rational curve, we mean an irreducible curve whose normalization is rational. These groups are torsion for X rationally connected, as proved by Kollár ([**13**, Theorem 3.13 p 206]). It is quite easy to prove that they are birationally invariant.

The proof of Proposition 1.3 gives as well the following result (already noticed by Kollár [**16**]) :

VARIANT 2.3. *If $\mathcal{X} \to B$ is a smooth projective morphism with rationally connected fibers, the groups $Z^{2n-2}_{rat}(\mathcal{X}_t)$ are local deformation invariants.*

Let us give one application of Proposition 1.3 (or rather its proof) and/or its variant 2.3. Let X be a smooth projective variety of dimension $n + r$, with $n \geq 3$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let $C_1, \ldots, C_k$ be smooth curves in X whose cohomology classes generate the group $H^{2n+2r-2}(X, \mathbb{Z})$. For $\sigma \in H^0(X, \mathcal{E})$, we denote by X_σ the zero locus of σ. When $\mathcal{E}$ is generated by sections, X_σ is smooth of dimension n for general σ.

THEOREM 2.4. *1) Assume that the sheaves $\mathcal{E} \otimes \mathcal{I}_{C_i}$ are generated by global sections for $i = 1, \ldots, k$. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.*

2) Under the same assumptions as in 1), assume the curves $C_i \subset X$ are rational. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}_{rat}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.

PROOF. 1) Let $j_\sigma : X_\sigma \to X$ be the inclusion map. Since $n \geq 3$ and $\mathcal{E}$ is ample, by Sommese's theorem [**20**], the Gysin map $j_{\sigma *} : H^{2n-2}(X_\sigma, \mathbb{Z}) \to H^{2n+2r-2}(X, \mathbb{Z})$ is an isomorphism. It follows that the group $H^{2n-2}(X_\sigma, \mathbb{Z})$ is a constant group. In order to show that $Z^{2n-2}(X_\sigma)$ is trivial, it suffices to show that the classes $(j_{\sigma *})^{-1}([C_i])$ are algebraic on X_σ since they generate $H^{2n-2}(X_\sigma, \mathbb{Z})$. Since the X_σ's are rationally connected, Theorem 1.3 tells us that it suffices to show that for each i, there exists a $\sigma(i)$ such that $X_{\sigma(i)}$ is smooth n-dimensional and that the class $(j_{\sigma(i)*})^{-1}([C_i])$ is algebraic on $X_{\sigma(i)}$.

It clearly suffices to exhibit one smooth $X_{\sigma(i)}$ containing C_i, which follows from the following lemma:

LEMMA 2.5. *Let X be a variety of dimension $n + r$ with $n \geq 2$, $C \subset X$ be a smooth curve, $\mathcal{E}$ be a rank r vector bundle on X such that $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global section. Then for a generic $\sigma \in H^0(X, \mathcal{E} \otimes \mathcal{I}_C)$, the zero set X_σ is smooth of dimension n.*

PROOF. The fact that X_σ is smooth of dimension n away from C is standard and follows from the fact that the incidence set $(\sigma, x) \in \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)) \times (X \setminus C)$, $\sigma(x) = 0\}$ is smooth of dimension $n + N$, where $N := \dim \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$. It thus suffices to check the smoothness along C for generic σ.

This is checked by observing that since $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global sections, its restriction $\mathcal{E} \otimes N^*_{C/X}$ is also generated by global sections. This implies that for each point $c \in C$, the condition that X_σ is singular at c defines a codimension n closed algebraic subset P_c of $P := \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$, determined by the condition that $d\sigma_c : N_{C/X,c} \to \mathcal{E}_c$ is not surjective. Since $\dim C = 1$, the union of the P_c's cannot be equal to P if $n \geq 2$. $\square$

This concludes the proof of 1) and the proof of 2) works exactly in the same way. $\square$

REMARK 2.6. (Added in proof.) After this paper was accepted, it has been proved by Runpu Zong [**24**] that every curve on a rationally connected variety over $\mathbb{C}$ is algebraically equivalent, hence in particular cohomologous, to a (non-effective) integral sum of rational curves. This shows that the groups $Z^{2n-2}(X)$ and $Z^{2n-2}(X)_{rat}$ are in fact isomorphic for rationally connected n-folds X over $\mathbb{C}$.

Let us finish this section with the proof of Proposition 1.5.

PROOF OF PROPOSITION 1.5. Let $p \in \operatorname{Spec} \mathcal{O}_K$, with residue field $k(p)$. Assume $\mathcal{X}_p$ is smooth. For l prime to $\operatorname{char} k(p)$, the (adequately constructed) specialization map

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)) \to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1)) \tag{2.1}$$

is then an isomorphism (cf. [**17**, Chapter VI, §4]).

Observe also that since $X_{\overline{K}}$ is rationally connected, the rational étale cohomology group $H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Q}_l(n-1))$ is generated over $\mathbb{Q}_l$ by curve classes. Hence the same is true for $H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Q}_l(n-1))$. Thus the whole cohomology groups

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)),\ H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))$$

consist of Tate classes, and (2.1) gives an isomorphism

$$H^{2n-2}_{et}(X_{\overline{K}},\mathbb{Z}_l(n-1))_{Tate}\to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}},\mathbb{Z}_l(n-1))_{Tate}. \tag{2.2}$$

In order to prove Proposition 1.5, it thus suffices to prove the following:

Lemma 2.7. *1) For almost every $p\in\operatorname{Spec}\mathcal{O}_K$, the fiber $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected.*

2) If $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected, for any curve $C_{\overline{p}}\subset\mathcal{X}_{\overline{p}}$, the inverse image $[C_{\overline{p}}]_{\overline{K}}\in H^{2n-2}_{et}(X_{\overline{K}},\mathbb{Z}_l(n-1))$ of the class $[C_{\overline{p}}]\in H^{2n-2}_{et}(\mathcal{X}_{\overline{p}},\mathbb{Z}_l(n-1))$ via the isomorphism (2.2) is the class of a 1-cycle on $X_{\overline{K}}$.

Proof. 1) When the fiber $\mathcal{X}_p$ is smooth, the separable rational connectedness of $\mathcal{X}_{\overline{p}}$ is equivalent to the existence of a smooth rational curve $C_{\overline{p}}\cong\mathbb{P}^1_{\overline{k(p)}}$ together with a morphism $\phi:C_{\overline{p}}\to\mathcal{X}_{\overline{p}}$ such that the vector bundle $\phi^*T_{\mathcal{X}_{\overline{p}}}$ on $\mathbb{P}^1_{\overline{k(p)}}$ is a direct sum $\oplus_i\mathcal{O}_{\mathbb{P}^1_{\overline{k(p)}}}(a_i)$ where all a_i are positive. Equivalently

$$H^1(\mathbb{P}^1_{\overline{k(p)}},\phi^*T_{\mathcal{X}_{\overline{p}}}(-2))=0. \tag{2.3}$$

The smooth projective variety $X_{\overline{K}}$ being rationally connected in characteristic 0, it is separably rationally connected, hence there exist a finite extension K' of K, a curve C and a morphism $\phi:C\to X$ defined over K', such that $C\cong\mathbb{P}^1_{K'}$ and $H^1(\mathbb{P}^1_{K'},\phi^*T_{X_{K'}}(-2))=0$.

We choose a model

$$\Phi:\mathcal{C}\cong\mathbb{P}^1_{\mathcal{O}_{K'}}\to\mathcal{X}'$$

of C and ϕ defined over a Zariski open set of $\operatorname{Spec}\mathcal{O}_{K'}$. By upper-semi-continuity of cohomology, the vanishing (2.3) remains true after restriction to almost every closed point $p\in\operatorname{Spec}\mathcal{O}_{K'}$, which proves 1).

2) The proof is identical to the proof of Proposition 1.3: we just have to show that the curve $C_{\overline{p}}\subset\mathcal{X}_{\overline{p}}$ is algebraically equivalent in $\mathcal{X}_{\overline{p}}$ to a difference $C''_{\overline{p}}-\sum_i R_{i,\overline{p}}$, where each curve $C''_{\overline{p}}$, resp. $R_{i,\overline{p}}$ (they are in fact defined over a finite extension $k(p)'$ of $k(p)$), lifts to a curve C'', resp. R_i in $X_{K'}$ for some finite extension K' of K.

Assuming the curves $C''_{\overline{p}}$, $R_{i,\overline{p}}$ are smooth, the existence of such a lifting is granted by the condition $H^1(C''_{\overline{p}},N_{C''_{\overline{p}}/\mathcal{X}_{\overline{p}}})=0$, resp. $H^1(R_{i,\overline{p}},N_{R_{i,\overline{p}}/\mathcal{X}_{\overline{p}}})=0$.

Starting from $C\subset\mathcal{X}_{\overline{p}}$ where $\mathcal{X}_{\overline{p}}$ is separably rationally connected over $\overline{p}$, we obtain such curves $C''_{\overline{p}}$, $R_{i,\overline{p}}$ as in the previous proof, applying [**10**, §2.1]. ∎

The proof of Proposition 1.5 is finished. ∎

Again, this proof leads as well to the proof of the specialization invariance of the l-adic analogues $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}_{rat}(X)$ introduced in Remark 2.2.

Variant 2.8. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over $\operatorname{Spec}\mathcal{O}_K$. Fix a prime integer l. Then for any $p\in\operatorname{Spec}\mathcal{O}_K$ such that $\mathcal{X}_{\overline{p}}$ is smooth separably connected, the group $Z^{2n-2}_{et,rat}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et,rat}(X_p)_l$.*

3. Consequence of a result of Chad Schoen

In [**19**], Chad Schoen proves the following theorem:

Theorem 3.1. *Let X be a smooth projective variety of dimension n defined over a finite field k of characteristic p. Assume that the Tate conjecture holds for degree 2 Tate classes on smooth projective surfaces defined over a finite extension of k. Then the étale cycle class map:*

$$cl : CH^{n-1}(X_{\overline{k}}) \otimes \mathbb{Z}_l \to H^{2n-2}(X_{\overline{k}}, \mathbb{Z}_l(n-1))_{Tate}$$

is surjective, that is $Z^{2n-2}_{et}(X)_l = 0$.

In other words, the Tate conjecture 1.4 for degree 2 *rational* Tate classes implies that the groups $Z^{2n-2}_{et}(X)_l$ should be trivial for all smooth projective varieties defined over finite fields. This is of course very different from the situation over $\mathbb{C}$ where the groups $Z^{2n-2}(X)$ are known to be possibly nonzero.

Remark 3.2. There is a similarity between the proof of Theorem 3.1 and the proof of Theorem 1.1. Schoen proves that given an integral Tate class α on X (defined over a finite field), there exist a smooth complete intersection surface $S \subset X$ and an integral Tate class β on S such that $j_{S*}\beta = \alpha$, where j_S is the inclusion of S in X. The result then follows from the fact that if the Tate conjecture holds for degree 2 rational Tate classes on S, it holds for degree 2 integral Tate classes on S.

I prove that for X a uniruled or Calabi-Yau, and for $\beta \in Hdg^4(X, \mathbb{Z})$ there exist surfaces $S_i \overset{j_{S_i}}{\hookrightarrow} X$ (in an adequately chosen linear system on X) and integral Hodge classes $\beta_i \in Hdg^2(S_i, \mathbb{Z})$ such that $\alpha = \sum_i j_{S_i*}\beta$. The result then follows from the Lefschetz theorem on $(1,1)$-classes applied to the β_i.

We refer to [**7**] for some comments on and other applications of Schoen's theorem, and conclude this note with the proof of the following theorem (cf. Theorem 1.6 of the introduction).

Theorem 3.3. *Assume Tate's conjecture* 1.4 *holds for degree* 2 *Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

Proof. We first recall that for a smooth rationally connected variety X, the group $Z^{2n-2}(X)$ is equal to the quotient $H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{alg}$, due to the fact that the Hodge structure on $H^{2n-2}(X, \mathbb{Q})$ is trivial. In fact, we have more precisely

$$H^{2n-2}(X, \mathbb{Q}) = H^{2n-2}(X, \mathbb{Q})_{alg}$$

by hard Lefschetz theorem and the fact that

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{alg}$$

by the Lefschetz theorem on $(1,1)$-classes.

Next, in order to prove that $Z^{2n-2}(X)$ is trivial, it suffices to prove that for each l, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\operatorname{Im} cl) \otimes \mathbb{Z}_l$ is trivial.

We apply Proposition 1.3 which tells as well that over $\mathbb{C}$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is locally deformation invariant for families of smooth rationally connected varieties. Note that our smooth projective rationally connected variety X is the fiber X_t of a smooth projective morphism $\phi : \mathcal{X} \to B$ defined over a number field, where

$\mathcal{X}$ and B are quasiprojective, geometrically connected and defined over a number field. By local deformation invariance, the vanishing of $Z^{2n-2}(X)\otimes\mathbb{Z}_l$ is equivalent to the vanishing of $Z^{2n-2}(X_{t'})\otimes\mathbb{Z}_l$ for any point $t'\in B(\mathbb{C})$. Taking for t' a point of B defined over a number field, $X_{t'}$ is defined over a number field. Hence it suffices to prove the vanishing of $Z^{2n-2}(X)\otimes\mathbb{Z}_l$ for X rationally connected defined over a number field L.

We have

$$Z^{2n-2}(X)\otimes\mathbb{Z}_l = H^{2n-2}(X,\mathbb{Z}_l)/(\operatorname{Im} cl)\otimes\mathbb{Z}_l,$$

and by the Artin comparison theorem (cf. [**17**, Chapter III,§3]), this is equal to

$$\frac{H^{2n-2}_{et}(X,\mathbb{Z}_l(n-1))}{(\operatorname{Im} cl)\otimes\mathbb{Z}_l} = Z^{2n-2}_{et}(X)_l$$

since $H^{2n-2}_{et}(X,\mathbb{Z}_l(n-1))$ consists of Tate classes. Hence it suffices to prove that for X rationally connected defined over a number field and for any l, the group $Z^{2n-2}_{et}(X)_l$ is trivial.

We now apply Proposition 1.5 to X and its reduction X_p for almost every closed point $p\in\operatorname{Spec}\mathcal{O}_L$. It follows that the vanishing of $Z^{2n-2}_{et}(X)_l$ is implied by the vanishing of $Z^{2n-2}_{et}(X_p)_l$. According to Schoen's theorem 3.1, the last vanishing is implied by the Tate conjecture for degree 2 Tate classes on smooth projective surfaces. □

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ISBN-13: 978-0-8218-8983-1

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