Hence

$$2\sin \pi s\,\Pi(s-1)\zeta(s) = i\int\limits_{\infty}^{\infty} \frac{(-x)^{s-1}\,dx}{e^x - 1},$$

where the integral has the meaning just specified.

This equation now gives the value of the function $\zeta(s)$ for all complex numbers s and shows that this function is one-valued and finite for all finite values of s with the exception of 1, and also that it is zero if s is equal to a negative even integer.

If the real part of s is negative, then, instead of being taken in a positive sense around the specified domain, this integral can also be taken in a negative sense around that domain containing all the remaining complex quantities, since the integral taken though values of infinitely large modulus is then infinitely small. However, in the interior of this domain, the integrand has discontinuities only where x becomes equal to a whole multiple of $\pm 2\pi i$, and the integral is thus equal to the sum of the integrals taken in a negative sense around these values. But the integral around the value $n2\pi i$ one obtains from this

$$2\sin \pi s\,\Pi(s-1)\zeta(s) = (2\pi)^s \textstyle\sum n^{s-1}((-i)^{s-1} + i^{s-1}),$$

thus a relation between $\zeta(s)$ and $\zeta(1-s)$, which, through the use of known properties of the function Π, may be expressed as follows:

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-\frac{s}{2}}\zeta(s)$$

remains unchanged when s is replaced by $1-s$.

This property of the function induced me to introduce, in place of $\Pi(s-1)$, the integral $\Pi\left(\frac{s}{2} - 1\right)$ into the general term of the series $\sum \frac{1}{n^s}$, whereby one obtains a very convenient expression for the function $\zeta(s)$. In fact

$$\frac{1}{n^s}\Pi\left(\frac{s}{2} - 1\right)\pi^{-\frac{s}{2}} = \int\limits_0^{\infty} e^{-nn\pi x} x^{\frac{s}{2}-1}\,dx,$$

thus, if one sets

$$\sum_1^{\infty} e^{-nn\pi x} = \psi(x)$$

then

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-\frac{s}{2}}\zeta(s) = \int\limits_0^{\infty} \psi(x) x^{\frac{s}{2}-1}\,dx,$$

or since

$$2\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right), \text{ (Jacobi, Fund. S. 184)}$$

$$\begin{aligned}
\Pi\left(\frac{s}{2} - 1\right) \pi^{-\frac{s}{2}} \zeta(s) &= \int\limits_1^\infty \psi(x) x^{\frac{s}{2}-1}\, dx + \int\limits_1^\infty \psi\left(\frac{1}{x}\right) x^{\frac{s-3}{2}}\, dx \\
&\quad + \tfrac{1}{2} \int\limits_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1} \right) dx \\
&= \frac{1}{s(s-1)} + \int\limits_1^\infty \psi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}} \right) dx.
\end{aligned}$$

I now set $s = \frac{1}{2} + ti$ and

$$\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = \xi(t),$$