On parameters for the group $SO(2n)$

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In memory of Ilya Piatetski-Shapiro.

1. Background

This article is an attempt to refine a property of automorphic representations from the monograph [**A3**]. It concerns the local and global representations of an even, quasisplit, special orthogonal group

$$G = SO(N), \qquad N = 2n, \tag{1.1}$$

over a field F of characteristic 0. The methods of [**A3**] are by comparison with representations of the group $GL(N)$. This eventually leads to results on the representations of the full orthogonal group $O(N)$. But it is the connected subgroup $G = SO(N)$ of $O(N)$ that is the ultimate object of interest.

In [**A3**, §8.4], we were able to characterize certain representations of G from the results obtained earlier for $O(N)$. The representations are those associated to Langlands parameters $\phi \in \Phi_{\mathrm{bdd}}(G)$. In the case of local F, Theorem 8.4.1 of [**A3**] provides an endoscopic classification of the representations of $G(F)$. For global F, however, the associated automorphic representations are governed by a larger class of parameters $\psi \in \Psi(G)$. An understanding of their local components requires supplementary endoscopic character relations for the localizations $\psi_v \in \Psi(G_v)$ of ψ. In this article, we shall establish conditional analogues for ψ_v of the results for ϕ_v in [**A3**, §8.4]. The conditions we impose are local, and include properties established for p-adic F by Moeglin. Such properties seem to be of considerable interest in their own right, as we will try to indicate with a few supplementary remarks in §3. In §4, we will use the conditions to formulate a conjecture on the contribution of a global parameter ψ to the automorphic discrete spectrum of G. We will then sketch a proof of the local results in §5.

Until further notice, the field F will be local. We then have the local Langlands group

$$L_F = \begin{cases} W_F, & \text{if } F \text{ is archimedean,} \\ W_F \times SU(2), & \text{if } F \text{ is } p\text{-adic,} \end{cases}$$

where W_F is the local Weil group of F. Let us recall the basic objects from [**A3**] that we will be working with here.

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We write $\widetilde{\Psi}(G)$ as in [**A3**] for the set of equivalence classes of L-homomorphisms

$$\psi: \ L_F \times SU(2) \longrightarrow {}^L G$$

with bounded image. We are taking the L-group ${}^L G$ here to be the semidirect product

$${}^L G = \widehat{G} \rtimes \Gamma_{E/F} = SO(N, \mathbb{C}) \rtimes \mathrm{Gal}(E/F),$$

where E/F is an extension of degree 2 if G is not split (but of course still quasisplit), and degree 1 if G is split. The equivalence relation that gives the classes in $\widetilde{\Psi}(G)$ is defined by the action of the disconnected group $O(N, \mathbb{C})$ by conjugation on its identity component $\widehat{G} = SO(N, \mathbb{C})$. Notice that the domain of ψ has two copies of the special unitary group $SU(2)$ if F is p-adic, and one copy if F is archimedean. In each case, the condition of bounded image is relevant only to the restriction of ψ to W_F.

The more familiar notation $\Psi(G)$ is reserved for the set of L-homomorphisms of the same sort, but taken up to the finer equivalence relation defined by conjugacy of the connected group $\widehat{G} = SO(N, \mathbb{C})$ on itself. This set obviously comes with a surjective mapping

$$\Psi(G) \ \longrightarrow \ \widetilde{\Psi}(G).$$

We write $\Psi(\psi)$ for the fibre in $\Psi(G)$ of any element $\psi \in \widetilde{\Psi}(G)$. It is an orbit in $\Psi(G)$ under the group

$$\widetilde{O}(G) = O(N, \mathbb{C})/SO(N, \mathbb{C}) = \mathbb{Z}/2\mathbb{Z},$$

of order

$$m(\psi) = |\Psi(\psi)| \in \{1, 2\}.$$

Following [**A3**], we write

$$\widetilde{\Psi}(G) = \widetilde{\Psi}'(G) \coprod \Psi(\widetilde{G}) \tag{1.2}$$

where

$$\widetilde{\Psi}'(G) = \{\psi \in \widetilde{\Psi}(G) : \ m(\psi) = 2\}$$

and

$$\Psi(\widetilde{G}) = \{\psi \in \widetilde{\Psi}(G) : \ m(\psi) = 1\}.$$

The nontrivial element in $\widetilde{O}(G)$ can be identified with the F-automorphism of G that stabilizes an underlying F-stable splitting. The group $\widetilde{O}(G)$ therefore acts on the set $\Pi_{\mathrm{unit}}(G)$ of equivalence classes of irreducible unitary representations of $G(F)$. We can thus form the set $\widetilde{\Pi}_{\mathrm{unit}}(G)$ of $\widetilde{O}(G)$-orbits in $\Pi_{\mathrm{unit}}(G)$. Following the convention above, we write $\Pi(\pi)$ for the fibre in $\Pi_{\mathrm{unit}}(G)$ of any element $\pi \in \widetilde{\Pi}_{\mathrm{unit}}(G)$. Similarly, we write $\Pi = \Pi(\widetilde{\Pi})$ for the preimage in $\Pi_{\mathrm{unit}}(G)$ of any subset $\widetilde{\Pi}$ of $\widetilde{\Pi}_{\mathrm{unit}}(G)$. More generally, suppose that $\widetilde{\Pi}$ is a set *over* $\widetilde{\Pi}_{\mathrm{unit}}(G)$, by which we mean a set equipped with a mapping into $\widetilde{\Pi}_{\mathrm{unit}}(G)$. We again write Π for the correspondence set over $\Pi_{\mathrm{unit}}(G)$. It equals the fibre product of $\widetilde{\Pi}$ and $\Pi_{\mathrm{unit}}(G)$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$.

The main local result of [**A3**], as it applies to the group G here, is the construction of a canonical finite set $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$ for any $\psi \in \widetilde{\Psi}(G)$ [**A3**, Theorem 2.2.1]. It was defined uniquely by the endoscopic transfer of twisted characters from $GL(N)$ to G, and the endoscopic transfer of ordinary characters from G to its endoscopic groups G'. For any given $\psi \in \widetilde{\Psi}(G)$, we can form the finite set

$$\Pi_\psi = (\widetilde{\Pi}_\psi) \times_{\widetilde{\Pi}_{\mathrm{unit}}(G)} \big(\Pi_{\mathrm{unit}}(G)\big), \tag{1.3}$$

over $\Pi_{\mathrm{unit}}(G)$, according to the definition above. The set Π_ψ of course also has a projection onto $\widetilde{\Pi}_\psi$. What we would like here is to construct a section

$$\widetilde{\Pi}_\psi \longrightarrow \Pi_\psi \tag{1.4}$$

that is compatible with endoscopic transfer for G. In fact, we would like a section that is *canonical* up to the action of $\widetilde{O}(G)$ on Π_ψ, which is to say a canonical $\widetilde{O}(G)$-orbit of sections.

If ψ belongs to the subset $\Psi(\widetilde{G})$ of $\widetilde{\Psi}(G)$, we can ask whether the elements in $\widetilde{\Pi}_\psi$ are trivial as $\widetilde{O}(G)$-orbits. This property is not known, but it could perhaps be established if we impose a further condition on ψ. It implies that $\widetilde{\Pi}_\psi$ is equal to Π_ψ, and that the section (1.4) exists and is trivial. For this reason, we shall generally restrict our study to parameters in the complement $\widetilde{\Psi}'(G)$ of $\Psi(\widetilde{G})$ in $\widetilde{\Psi}(G)$.

If ψ belongs to $\widetilde{\Psi}'(G)$, Theorem 2.1 asserts the existence of a compatible section. After stating it in §2, we will describe the minor, local component of its proof, but leave the main, global argument for the final §5. However, the constructions of the theorem are not canonical. In §3, we introduce the notion of a *coherent* parameter in $\widetilde{\Psi}(G)$, motivated by the work [**M**] of Moeglin. In the case $\psi \in \Psi(\widetilde{G})$, this is the condition that might lead to the property above on the elements in $\widetilde{\Pi}_\psi$ (Conjecture 3.1). If ψ is a coherent parameter in $\widetilde{\Psi}'(G)$, Proposition 3.2 asserts that there is at most one compatible section up to the action of $\widetilde{O}(G)$. Theorem 2.1 and Proposition 3.2 together thus imply that for any coherent $\psi \in \widetilde{\Psi}'(G)$, there is a unique $\widetilde{O}(G)$-orbit of sections (1.4) that is compatible with endoscopy. We note that while this $\widetilde{O}(G)$-orbit will be of order 2, it will not come with a canonical $\widetilde{O}(G)$-isomorphism onto the original $\widetilde{O}(G)$-orbit $\Psi(\psi)$ of order 2. The main point, however, is that we do obtain a canonical section (1.4), if we are prepared to impose a condition on ψ that is not known to hold in general. Such a section is of obvious interest for the representation theory of G. It will also be important for the classification of representations of certain inner twists of our quasisplit group G. (See [**A3**, §9.4].)

In §4, the field F is global. There we will state Conjecture 4.1, a global counterpart of Theorem 2.1. It gives a refined decomposition of a part of the automorphic discrete spectrum of G. More precisely, it provides a decomposition of the invariant subspace

$$L^2_{\mathrm{disc},\psi}\big(G(F)\backslash G(\mathbb{A})\big)$$

of the discrete spectrum attached to a global parameter $\psi \in \widetilde{\Psi}'_2(G)$. The parameter is assumed to be locally coherent at a predetermined set of places V, while the decomposition is as a module over the global Hecke algebra $\mathcal{H}_V(G)$ of functions on $G(\mathbb{A})$ that are locally symmetric at places outside of V. We shall say a few words about how one might begin a proof of the conjecture, following that of its generic counterpart [**A3**, Theorem 8.4.2]. These will be used in §5 to introduce the global proof of Theorem 2.1. Our proof of Theorem 2.1 is a little different from that of its generic counterpart [**A3**, Theorem 8.4.1]. It is actually a little simpler, and could perhaps be used as a guide to the earlier proof. However, there are enough common threads between the two proofs that we can allow the discussion of §5 to be quite brief.

I observe in summary that the rest of the article is composed of two noticeably distinct parts. Sections 4 and 5 are somewhat technical. As we have noted, they contain discussion of proofs based on arguments that appear in greater detail in §8.4 of [**A3**]. Sections 2 and 3 are more elementary. Section 3 contains our central themes, and is somewhat speculative.

I do not know whether the notion of coherence in §3 is universal, or even correct. It does seem to bear some relevance to more general groups G. After introducing the parameters $\psi \in \Psi(G)$ in [**A1**], I conjectured the existence of canonical packets Π_ψ under the mistaken assumption that the mapping from Π_ψ to the group of characters on the associated finite group $\mathcal{S}_\psi$ would be injective. For general real groups, the packets were later constructed in a canonical way in [**ABV**] in terms of the geometry on the dual group $\widehat{G}$. For orthogonal and symplectic groups over any local field F, the packets were constructed [**A3**] in terms of twisted characters for general linear groups. At the end of §3, we shall describe how a variant of the notion of coherence might be used to characterize packets Π_ψ for general G and F.

2. Statement of the local theorem

We should recall the local results [**A3**] we are trying to refine. We continue to take F to be a local field, and G to be an even, quasisplit, special orthogonal group (1.1) over F. Since the group $\widetilde{O}(G)$ of order 2 acts on $G(F)$, it acts on the Hecke algebra $\mathcal{H}(G)$ of functions on $G(F)$. We write $\widetilde{\mathcal{H}}(G)$ for the subalgebra of $\widetilde{O}(G)$-symmetric functions in $\mathcal{H}(G)$. If $\pi \in \widetilde{\Pi}_{\mathrm{unit}}(G)$ is an $\widetilde{O}(G)$-orbit of representations π_* in $\Pi_{\mathrm{unit}}(G)$, of order

$$m(\pi) = |\Phi(\pi)| = 1, 2,$$

the distributional character

$$f_G(\pi) = f_G(\pi_*) = \mathrm{tr}\big(\pi_*(f)\big), \qquad f \in \widetilde{\mathcal{H}}(G),\ \pi_* \in \Pi(\pi),$$

on $\widetilde{\mathcal{H}}(G)$ is independent of the representative π_* in $\Pi_{\mathrm{unit}}(G)$.

The endoscopic construction of representations in [**A3**] is founded on the character identity

$$f'(\psi') = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi x, \pi\rangle f_G(\pi), \qquad \psi \in \widetilde{\Psi}(G),\ f \in \widetilde{\mathcal{H}}(G), \tag{2.1}$$

of Theorem 2.2.1. On the right hand side, x is the image in the 2-group

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G})^\Gamma$$

of a given semisimple element s in the centralizer

$$S_\psi = \mathrm{Cent}\big(\mathrm{im}(\psi), \widehat{G}\big)$$

of $\psi\big(L_F \times SU(2)\big)$ in $\widehat{G}$, while s_ψ is the image in $\mathcal{S}_\psi$ of the value of ψ at the central element

$$1 \times \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $L_F \times SU(2)$. The coefficient in the sum is the value at $(s_\psi x, \pi)$ of a pairing

$$\langle \cdot, \cdot \rangle :\ \mathcal{S}_\psi \times \widetilde{\Pi}_\psi \ \longrightarrow\ \{\pm 1\}, \tag{2.2}$$

which is a linear character in the first variable. The left hand side of (2.1) depends on a pair

$$(G', \psi'), \qquad \psi' \in \widetilde{\Psi}(G'),$$

which is determined in the natural way from the given pair (ψ, s). The function $f' = f^{G'}$ is the Langlands-Shelstad transfer of f to a function on the endoscopic group $G'(F)$. Finally, the linear form $f'(\psi')$ is the value of f' at the stable distribution on $G'(F)$ attached to the parameter ψ' by the prescription of Theorem 2.2.1(a) of [**A3**].

The left hand side of (2.1) is thus determined by the given objects ψ, s and f. It is to be regarded as a definition of the objects on the right. In other words, it determines the packet $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$ attached to ψ, and the mapping of $\widetilde{\Pi}_\psi$ into the group of characters on the 2-group $\mathcal{S}_\psi$.

Our description of (2.1) is perhaps too dense for a first time reader. The discussion of §4 of the survey [**A4**] might be of some help, but in any case, the details are not so important at this stage. Our main observation here is just that the test function f in (2.1) is restricted to the symmetric Hecke subalgebra $\widetilde{\mathcal{H}}(G)$ of $\mathcal{H}(G)$. The reason is that the stable distribution on $G(F)$ attached to ψ in [**A3**, Theorem 2.1.1(a)] (whose analogue ψ' for G' appears on the left hand side of (2.1)) is defined only up to the action of $\widetilde{O}(G)$. This essential constraint reflects the fact that the distribution is obtained by transfer of a twisted character on the group $GL(N, F)$. It is why we have to take ψ to be an element in the set $\widetilde{\Psi}(G)$ of $\widetilde{O}(G)$-orbits in $\Psi(G)$, rather than just a parameter in $\Psi(G)$.

The problem is to refine (2.1) to an identity for functions $f \in \mathcal{H}(G)$, and ideally, parameters $\psi \in \Psi(G)$. This would give a packet of representations in $\Pi_{\mathrm{unit}}(G)$, in place of $\widetilde{O}(G)$-orbits of representations from the set $\widetilde{\Pi}_{\mathrm{unit}}(G)$. The problem was essentially solved for the subset of generic parameters ϕ in $\widetilde{\Psi}(G)$ in Theorem 8.4.1 of [**A3**].

Recall [**A3**] that $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ denotes the subset of parameters in $\widetilde{\Psi}(G)$ that are trivial on the supplementary factor $SU(2)$. An element in $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ is therefore a Langlands parameter

$$\phi : \; L_F \; \longrightarrow \; {}^L G$$

with bounded image, taken up to conjugacy in $\widehat{G}$ by the group $O(N, \mathbb{C})$. We write $\widetilde{\Phi}'_{\mathrm{bdd}}(G)$, $\Phi_{\mathrm{bdd}}(\widetilde{G})$, $\Phi_{\mathrm{bdd}}(G)$, etc., for the subsets of parameters in the associated sets $\widetilde{\Psi}'(G)$, $\Psi(\widetilde{G})$, $\Psi(G)$, etc., that are trivial on the factor $SU(2)$. We can also write $\widetilde{\Phi}(G)$ for the set of all Langlands parameters (that is, without the boundedness condition), taken again up to conjugacy by $O(N, \mathbb{C})$. Recall that any $\psi \in \widetilde{\Psi}(G)$ restricts to a parameter

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad w \in L_F, \tag{2.3}$$

in $\widetilde{\Phi}(G)$, where $|w|$ is the pullback to L_F of the absolute value on W_F, and where the domain of ψ is understood to have been extended analytically from $L_F \times SU(2)$ to $L_F \times SL(2, \mathbb{C})$. The mapping $\psi \to \phi_\psi$ is then an injection from $\widetilde{\Psi}(G)$ to $\widetilde{\Phi}(G)$. It gives the second embedding in the chain

$$\widetilde{\Phi}_{\mathrm{bdd}}(G) \subset \widetilde{\Psi}(G) \subset \widetilde{\Phi}(G). \tag{2.4}$$

We have qualified the reference to [**A3**, Theorem 8.4.1] above with the adjective "essentially" because it still contains an internal $(\mathbb{Z}/2\mathbb{Z})$-symmetry. (See the remarks in the middle of §8.3 and the end of §8.4 of [**A3**].) The preimage $\Psi(\psi)$ in $\Psi(G)$ of any parameter $\psi \in \widetilde{\Psi}'(G)$ is a set of order 2, which we continue to call somewhat superfluously an $\widetilde{O}(G)$-torsor. The $(\mathbb{Z}/2\mathbb{Z})$-symmetry can be resolved only if we replace $\Psi(\psi)$ by another $\widetilde{O}(G)$-torsor $T(\psi)$, which is constructed from representations rather than parameters. We shall first review the definition of $T(\psi)$ in case $\phi = \psi$ is generic, and then introduce its extension for arbitrary ψ.

Suppose that ϕ belongs to $\widetilde{\Phi}_{\mathrm{sim}}(G)$, the intersection of $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ with the subset

$$\widetilde{\Psi}_{\mathrm{sim}}(G) = \big\{\psi \in \widetilde{\Psi}(G) :\ \overline{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma = \{1\}\big\}$$

of simple parameters in $\widetilde{\Psi}(G)$. Since its degree is even, ϕ automatically belongs to the subset $\widetilde{\Phi}'_{\mathrm{bdd}}(G)$ of $\widetilde{\Phi}_{\mathrm{bdd}}(G)$. According to the local Langlands classification established in [**A3**, §6.7], ϕ corresponds to an $\widetilde{O}(G)$-orbit $\pi_\phi \in \widetilde{\Pi}(G)$ of irreducible representations in $\Pi_{\mathrm{temp}}(G)$. By Corollary 6.7.3 of [**A3**], the order $m(\pi_\phi)$ of this orbit equals the order $m(\phi) = 2$ of the orbit of ϕ in $\Phi_{\mathrm{bdd}}(G)$. We define $T(\phi)$ in this case to be the $\widetilde{O}(G)$-torsor π_ϕ. Consider next a general element

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r, \qquad \phi_i \in \widetilde{\Phi}_{\mathrm{sim}}(G_i),\ G_i \in \mathcal{E}_{\mathrm{sim}}(N_i),\ N_i = 2n_i,$$

in $\widetilde{\Phi}'_2(G)$. In this case, we take

$$T(\phi) = \{t = t_1 \times \cdots \times t_r :\ t_i \in T(\phi_i)\}/\sim$$

to be a set of equivalence classes in the product over i of the sets $T(\phi_i)$. The equivalence relation is defined by writing $t' \sim t$ if the subset of indices i such that $t'_i \neq t_i$ is even. Finally, if the parameter ϕ lies in the complement of $\widetilde{\Phi}'_2(G)$ in $\widetilde{\Phi}'_{\mathrm{bdd}}(G)$, it has a natural decomposition that was denoted by

$$\phi = \phi^* \oplus \phi_-, \qquad \phi^* \in \widetilde{\Phi}(G^*),\ \phi_- \in \widetilde{\Phi}_2(G_-),$$

in [**A3**, §8.3]. In this last case, the $\widetilde{O}(G)$-torsor is defined

$$T(\phi) = \{t = t^* \times t_- \in T(\phi^*) \times T(\phi_-)\}/\sim$$

as in [**A3**, (8.3.10)], in terms of the components ϕ^* and ϕ_- of ϕ.

We noted the correspondence $(G', \phi') \to (\phi, s)$ (with a parameter $\psi \in \widetilde{\Psi}(G)$ in place of $\phi \in \widetilde{\Phi}(G)$) in our description of (2.1) above. It anchors the spectral theory of endoscopy, and follows immediately from general definitions. To be in step with the general theory, ϕ should really be an element in $\Phi(G)$ rather than $\widetilde{\Phi}(G)$. Since $T(\phi)$ is meant to serve as a substitute for the preimage of ϕ in $\Phi(G)$, it is not surprising that the definitions above lead to an immediate extension

$$(G', \phi', t') \longrightarrow (\phi, s, t), \qquad t \in T(\phi),\ t' \in T(\phi'),$$

of the correspondence. In this form, it is a surjective mapping, on whose fibres the finite abelian group $\mathrm{Out}_G(G')$ acts transitively. (See the remarks at the beginning of §8.4 of [**A3**].)

This extended correspondence gives some perspective on the definition of the torsor $T(\phi)$. Suppose that $G' = G'_1 \times G'_2$ is a proper, elliptic endoscopic group for G. Then there is a surjective mapping

$$\widetilde{O}(G') = \widetilde{O}(G'_1) \times \widetilde{O}(G'_2) = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \widetilde{O}(G) = (\mathbb{Z}/2\mathbb{Z})$$

of automorphism groups. Suppose also that ϕ is an element in $\widetilde{\Phi}'(G)$ that factors through G'. In other words, G' and ϕ are the first components of triplets (G',ϕ',t') and (ϕ,s,t) that correspond as above. We then have parallel $\widetilde{O}(G')$-torsors

$$\Phi(G',\phi) \longrightarrow \Phi(\phi)$$

and

$$T(G',\phi) \longrightarrow T(\phi),$$

where $\Phi(\phi)$ and $T(\phi)$ are the parallel $\widetilde{O}(G)$-torsors we are working with, and $\Phi(G',\phi)$ and $T(G',\phi)$ are their preimages in the sets

$$\Phi(G') = \coprod_{\phi'\in\widetilde{\Phi}(G')} \Phi(\phi')$$

and

$$T(G') = \coprod_{\phi'\in\widetilde{\Phi}(G')} T(\phi').$$

This motivates[1] the equivalence relation in the definition of $\Phi(\phi)$ above.

It is an easy matter to extend these definitions to general parameters ψ. Recall that we have an injective mapping $\psi\to\phi_\psi$. Since it is also injective as a mapping from $\Psi(G)$ to $\Phi(G)$, it takes the subset $\widetilde{\Psi}'(G)$ of $\widetilde{\Psi}(G)$ into the subset $\widetilde{\Phi}'(G)$ of $\widetilde{\Phi}(G)$. Now the definition of the $\widetilde{O}(G)$-torsor

$$T(\phi), \qquad \phi\in\widetilde{\Phi}'_{\mathrm{bdd}}(G),$$

we have just recalled extends directly to parameters in the larger set $\widetilde{\Phi}'(G)$. We can therefore define

$$T(\psi)=T(\phi_\psi), \qquad \psi\in\widetilde{\Psi}'(G). \tag{2.5}$$

This attaches an $\widetilde{O}(G)$-torsor to any $\psi\in\widetilde{\Psi}'(G)$. It is also easy to see that the bijective correspondence

$$(G',\psi') \longrightarrow (\psi,s), \qquad \psi\in\widetilde{\Psi}(G),\ s\in S_\psi,$$

extends to a surjective mapping

$$(G',\psi',t') \longrightarrow (\psi,s,t), \qquad t\in T(\psi),\ t'\in T(\psi'). \tag{2.6}$$

THEOREM 2.1. *Suppose that F is local, and that ψ lies in $\widetilde{\Psi}'(G)$. Then there is an $\widetilde{O}(G)$-equivariant bijection*

$$t \longrightarrow \psi_t, \qquad t\in T(\psi),$$

from $T(\psi)$ onto a pair of stable linear forms

$$f \longrightarrow f^G(\psi_t), \qquad f\in\mathcal{H}(G),\ t\in T(\psi),$$

on $\mathcal{H}(G)$, and a mapping

$$(\pi,t) \longrightarrow \pi_t, \qquad \pi\in\widetilde{\Pi}_\psi,\ t\in T(\psi),$$

from $\widetilde{\Pi}_\psi\times T(\psi)$ to Π_ψ, such that

$$f^G(\psi)=f^G(\psi_t), \qquad f\in\widetilde{\mathcal{H}}(G),\ t\in T(\psi), \tag{2.7}$$

[1] I thank the referee for implicitly suggesting that I include a remark of this nature.

and

$$f_G(\pi) = f_G(\pi_t), \qquad f \in \widetilde{\mathcal{H}}(G),\ t \in T(\psi), \tag{2.8}$$

and such for any $s \in S_{\psi,\mathrm{ss}}$ *with image* x *in* $\mathcal{S}_\psi$*, the identity*

$$f'(\psi'_{t'}) = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi x, \pi \rangle f_G(\pi_t), \qquad f \in \mathcal{H}(G),\ t \in T(\psi), \tag{2.9}$$

is valid for an endoscopic preimage (G', ψ', t') *of* (ψ, s, t).

Remarks. 1. This is essentially a transcription of the statement of Theorem 8.4.1 of [**A3**] from ϕ to ψ. Notice however that the word *bijection* in the fifth line of the earlier statement has now been weakened simply to *mapping*. The reason is that we need to allow here for the possibility of elements $\pi \in \widetilde{\Pi}_\psi$ whose preimage in Π_ψ contains only one element. We shall return briefly to this point in §3.

2. Assume that the theorem has been proved. It then extends by analytic continuation to the standard representations attached to parameters ψ in the larger set

$$(\widetilde{\Psi}^+)'(G) = \{\psi \in \widetilde{\Psi}^+(G) :\ m_\psi = 2\},$$

defined without the boundedness condition on ψ. In particular, the theorem is valid for the packet $\widetilde{\Pi}_\psi$ attached to any parameter in the subset

$$(\widetilde{\Psi}^+_{\mathrm{unit}})'(G) = (\widetilde{\Psi}^+)'(G) \cap \widetilde{\Psi}^+_{\mathrm{unit}}(G)$$

of $(\widetilde{\Psi}^+)'(G)$. We recall that the right hand two sets in the chain

$$\widetilde{\Psi}(G) \subset \widetilde{\Psi}^+_{\mathrm{unit}}(G) \subset \widetilde{\Psi}^+(G)$$

were defined in [**A3**, §1.5]. They are needed for the localizations of global parameters, to account for our lack of a proof for the generalized Ramanujan conjecture for $GL(N)$.

The proof of Theorem 2.1 is similar to that of its generic analogue [**A3**, Theorem 8.4.1]. It is largely global. In common with many of the local results from [**A3**], the proof rests on the stabilization of the trace formula, for a group $\dot{G}$ over a global field $\dot{F}$ such that $(\dot{G}_u, \dot{F}_u) = (G, F)$ for some valuation u of $\dot{F}$. The general idea is to deduce local results at the completion $\dot{G}_u = G$ from an a priori knowledge that they hold at some other completions $\dot{G}_v$. What distinguishes the argument here (and in [**A3**, Theorem 8.4.1]) from those in the earlier parts of [**A3**] is that we require no further reference to the twisted trace formula for $GL(N)$. This leaves us free to work with a test function in the global Hecke algebra $\mathcal{H}(\dot{G})$ that need not be everywhere locally symmetric, and thereby exploit the full trace formula. We shall sketch the main global argument of the proof in §5.

In the last part of this section, we shall describe how to establish Theorem 2.1 for some special parameters $\psi \in \widetilde{\Psi}(G)$. This represents the local part of the general proof of the theorem, and as such, will be considerably simpler than the global part. In fact, it is more or less implicit in the results of [**A3**, §7.1, §8.4]. We will apply the local information so obtained to the global arguments of §5, specifically to a certain completion $(\dot{G}_{u_1}, \dot{\psi}_{u_1})$ of a global pair $(\dot{G}, \dot{\psi})$.

Assume for the rest of this section that the local field F is nonarchimedean. We then have an involution on $\widetilde{\Psi}(G)$, which sends any $\psi \in \widetilde{\Psi}(G)$ to the dual parameter

$$\widehat{\psi}(w, u_1, u_2) = \psi(w, u_2, u_1), \qquad w \in W_F,\ u_1, u_2 \in SU(2), \tag{2.10}$$

in $\widetilde{\Psi}(G)$. In particular, if ϕ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$, $\psi = \widehat{\phi}$ is a parameter in $\widetilde{\Psi}(G)$ whose restriction to L_F is trivial on the subgroup $SU(2)$ of L_F. It is then clear that ψ belongs to the subset $\widetilde{\Psi}'(G)$ of $\widetilde{\Psi}(G)$ if and only if ϕ lies in the subset $\widetilde{\Phi}'_{\mathrm{bdd}}(G)$ of $\widetilde{\Phi}_{\mathrm{bdd}}(G)$.

LEMMA 2.2. *The assertions of Theorem 2.1 are valid for any parameter in $\widetilde{\Psi}'(G)$ of the form*

$$\psi = \widehat{\phi}, \qquad \phi \in \Phi'_{\mathrm{bdd}}(G).$$

PROOF. The lemma is a consequence of three theorems, the special case [**A3**, Theorem 8.4.1] of Theorem 2.1 for the generic parameter ϕ, the compatibility of duality with endoscopic transfer ([**Hi**], [**A5**]) and the endoscopic identity [**A3**, (2.2.6)] for parameters $\psi \in \widetilde{\Psi}(G)$. I will be content just to add a couple of brief comments to this.

The point is that there is another duality operator $D = D_G$, which acts on the Grothendieck group of the category of $G(F)$-modules of finite length. Aubert [**Au**] has shown that it satisfies an identity

$$D[\pi] = \beta(\pi)\,[\widehat{\pi}], \qquad \pi \in \Pi(G),$$

where $\pi \to \widehat{\pi}$ is an involution on $\Pi(G)$, and $\beta(\pi)$ is a certain sign. This is the operator that commutes with the endoscopic transfer of characters, again up to a sign, and also with twisted endoscopic transfer from $GL(N)$. One then obtains a relation between D and the involution (2.10) on parameters, from the corresponding relation for $GL(N)$ (known originally as the Zelevinsky conjecture). To establish the lemma, one applies D_G to each side of the analogue [**A3**, (8.4.3)] for ϕ of the identity (2.9). This gives the formula (2.9) itself, up to a multiplicative sign on each of the irreducible characters that parametrize the summands on the right hand side of (2.9). One can then use the original, unrefined formula [**A3**, (2.6)] to resolve these signs. (We refer the reader to [**A3**, §7.1] for further discussion of some of these points.) □

3. The problem of uniqueness

Theorem 2.1 looks very similar to the earlier result [**A3**, Theorem 8.4.1] for generic parameters, but it has a serious drawback. Without further information, we cannot say that the mappings $t \to \psi_t$ and $(\pi, t) \to \pi_t$ of the theorem are unique. The problem becomes untenable when we try to consider the global implications of the theorem in §4. For example, some local uniqueness assertion will be essential for the necessary refinement of the stable multiplicity formula [**A3**, Theorem 4.1.2].

If $\phi = \psi$ belongs to the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$, the mappings of the theorem are determined by the given conditions. This is an easily verified fact, which we left to the reader in [**A3**]. (See Remark 2 following the statement of Theorem 8.4.1 of [**A3**].) Under what conditions on a general parameter ψ are the mappings unique?

The group G remains an even, quasisplit, special orthogonal group (1.1) over the local field F. Suppose first that ψ lies in the subset

$$\widetilde{\Psi}'_2(G) = \{\psi \in \widetilde{\Psi}(G) :\ |S_\psi| < \infty,\ m(\psi) = 2\}$$

of square integrable parameters in $\widetilde{\Psi}'(G)$. Then

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r, \qquad \psi_i \in \widetilde{\Psi}_{\mathrm{sim}}(G_i),\ G_i \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N_i),$$

where $N_i = 2n_i$ and G_i is a quasisplit special orthogonal group $SO(N_i)$. Let us say that ψ is *coherent* if it satisfies the following two conditions.

(i) If $\psi_- \in \widetilde{\Psi}'_2(G_-)$ is any subparameter of ψ, where $G_- = SO(N_-) = SO(2n_-)$ is an even special orthogonal subgroup of G, the elements in the corresponding packet $\widetilde{\Pi}_{\psi_-}$ occur with multiplicity 1. In other words, $\widetilde{\Pi}_{\psi_-}$ is a subset of $\widetilde{\Pi}_{\mathrm{unit}}(G_-)$, as opposed to a set *over* $\widetilde{\Pi}_{\mathrm{unit}}(G_-)$ with nontrivial fibres.

(ii) If $\psi_i \in \widetilde{\Psi}_{\mathrm{sim}}(G_i)$ is a simple constituent of ψ, there is exactly one $\widetilde{O}(G_i)$-orbit of sections

$$\pi_i \longrightarrow \pi_{i,*}, \qquad \pi_i \in \widetilde{\Pi}_{\psi_i},$$

from $\widetilde{\Pi}_{\psi_i}$ to Π_{ψ_i} such that the distribution

$$f \longrightarrow \sum_{\pi_i \in \widetilde{\Pi}_{\psi_i}} f_{i,G}(\pi_{i,*}), \qquad f_i \in \mathcal{H}(G_i),$$

on $G_i(F)$ is stable. In other words, the mappings of Theorem 2.1 for the pair (G_i, ψ_i) are unique.

If ψ is a general element in the set $\widetilde{\Psi}'(G)$, there is a Levi subgroup M of G and a square integrable parameter $\psi_M \in \widetilde{\Psi}_2(M, \psi)$ for M that maps to ψ. The group M is a product of a general linear factors with a special orthogonal subgroup

$$G_- = SO(N_-) = SO(2n_-).$$

In this case, we shall say that ψ is *coherent* if the subparameter $\psi_- \in \widetilde{\Psi}'_2(G_-)$ of ψ_M is coherent.

We shall see that the mappings of Theorem 2.1 are unique if ψ is coherent. Before doing so, however, we shall first discuss a simpler property, having to do only with the symmetric Hecke algebra $\widetilde{\mathcal{H}}(G)$. The definition of coherent extends to parameters ψ in the complement $\Psi(\widetilde{G})$ of $\widetilde{\Psi}'(G)$ in $\widetilde{\Psi}(G)$, where we understand the condition (ii) above to be vacuous if the rank N_i of ψ_i is odd. The following conjecture is an analogue for coherent parameters $\psi \in \widetilde{\Psi}_2(G)$ of Corollaries 6.6.6 and 6.7.3 of [**A3**] for generic parameters ϕ.

Conjecture 3.1. *If $\psi \in \widetilde{\Psi}_2(G)$ is coherent and π belongs to $\widetilde{\Pi}_\psi$, then*

$$m(\pi) = m(\psi).$$

I have stated the conjecture simply as a point of discussion, rather than for any pressing need. While it seems plausible, it would not be amenable to the methods used for generic parameters ϕ in [**A3**, §8.4]. These rely on orthogonality relations, which to this point have not been of use for the nongeneric parameters ψ. On the other hand, one could perhaps deduce Conjecture 3.1 from twisted endoscopy for the outer automorphism of the group G. The multiplicity 1 condition (i) of coherence would no doubt have to be part of the argument. □

We now consider the uniqueness of the mappings of Theorem 2.1. The theorem was formulated for parameters in the subset $\widetilde{\Psi}'(G)$ of $\widetilde{\Psi}(G)$, following [**A3**, §8.4], since this is the main case. However, it remains valid as stated if ψ belongs to the complementary set $\Psi(\widetilde{G})$. This is the case that the orbit $\Psi(\psi)$ has order 1. A formal application to ψ of the earlier definition (2.5) yields a set $T(\psi)$ that is also of order 1, together with a trivial endoscopic mapping (2.6). The proof we sketch in §5 of Theorem 2.1 is then easily adapted to parameters $\psi \in \Psi(\widetilde{G})$. We may as well therefore state the uniqueness property for the subset $\widetilde{\Psi}_{\mathrm{coh}}(G)$ of coherent parameters in the full set $\widetilde{\Psi}(G)$.

PROPOSITION 3.2. *Suppose that $\psi \in \widetilde{\Psi}_{\mathrm{coh}}(G)$ is a coherent parameter. Then the mappings $t \to \psi_t$ and $\pi \to \pi_t$ of Theorem 2.1 (and their analogues for the complementary set $\Psi(\widetilde{G})$) are unique.*

PROOF. The main point will be to characterize the stable distributions $t \to \psi_t$. We assume inductively that we have done this if ψ is replaced by any proper subparameter $\psi_- \in \widetilde{\Psi}'_2(G_-)$. This takes care of the case that ψ lies in the complement of $\widetilde{\Psi}'_2(G)$ in $\widetilde{\Psi}(G)$. For ψ is then the image of a parameter in a set $\widetilde{\Psi}_2(M, \psi)$, as above, for a proper Levi subgroup M of G. In other words, ψ factors through a proper Levi subgroup ${}^LM$ of ${}^LG$, and ψ_t is the image of the stable linear form attached by our induction hypothesis to M. We can therefore assume that ψ belongs to $\widetilde{\Psi}'_2(G)$.

If ψ is simple, the assertion of the proposition (which of course includes the existence of ψ_t) is just the condition (ii) from the definition of coherence above. We can therefore assume that ψ is not simple. This means that there is an element $s \in S_\psi$ that is not central in ${}^LG$. Given $t \in T(\psi)$, let (G', ψ', t') be the preimage of the triplet (ψ, s, t), according to the understanding of §2. Then $\psi' = \psi'_1 \times \psi'_2$ is a parameter for the proper, elliptic, endoscopic group $G' = G'_1 \times G'_2$ for G. By our induction hypothesis, the stable linear form

$$\psi'_{t'} = \psi'_{1,t'_1} \times \psi'_{2,t'_2}$$

on $\mathcal{H}(G')$ is uniquely determined. This characterizes the left hand side of (2.9).

In all cases, we can then deduce the required uniqueness assertions from (2.9). By condition (i) above for coherence, the set $\widetilde{\Pi}_\psi$ that indexes the summands on the right hand side of (2.9) is a subset of $\widetilde{\Pi}_{\mathrm{unit}}(G)$. The restrictions $f_G(\pi)$ to $\widetilde{\mathcal{H}}(G)$ of the associated linear forms $f_G(\pi_t)$ are therefore linearly independent. The linear forms themselves are given by a section

$$\widetilde{\Pi}_\psi \ \xrightarrow{\sim}\ \Pi_{\psi_t} = \{\pi_t :\ \pi \in \widetilde{\Pi}_\psi\} \subset \Pi_\psi,$$

so they are defined on the larger space $\mathcal{H}(G)$, where they obviously remain linearly independent. Since their coefficients $\langle s_\psi x, \pi\rangle$ are nonzero, the left hand side of (2.9) determines the packet Π_{ψ_t}, and also the corresponding characters $\langle \cdot, \pi\rangle = \langle \cdot, \pi_t\rangle$ on $\widehat{\mathcal{S}}_\psi$ (again by the condition (i) above). These in turn characterize the remaining stable distribution

$$f^G(\psi_t) = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi, \pi\rangle\, f_G(\pi_t), \qquad f \in \mathcal{H}(G),$$

as required. □

Proposition 3.2 places Theorem 2.1 on the same plane as its predecessor [**A3**, Theorem 8.4.1] for generic parameters ϕ. However, it requires that ψ be coherent. This is surely a strong condition, but as far as I can see, it is the best we can do. I would guess that any $\psi \in \widetilde{\Psi}(G)$ is coherent, but I don't really have much evidence. Let us review the examples.

Suppose first that the local field F is arbitrary. If $\phi = \psi$ belongs to the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of generic parameters in $\widetilde{\Psi}(G)$, it is coherent. This follows from the classification established in Chapter 6 of [**A3**], especially the assertion of [**A3**, Theorem 1.5.1(b)]. If the (general linear) rank $N = 2n$ of G equals 2 or 4, it is also easy to see that any $\psi \in \widetilde{\Psi}(G)$ is coherent, but these are pretty trivial examples.

Suppose next that F is p-adic. Any parameter of the form

$$\psi = \widehat{\phi}, \qquad \phi \in \widetilde{\Phi}_{\mathrm{bdd}}(G),$$

is coherent. This follows from the fact that ϕ is coherent, as one can see from the brief discussion of this case in the proof of Lemma 2.2. Consider a general parameter $\psi \in \widetilde{\Psi}(G)$. Moeglin [**M**] has established the condition (i) for coherence, namely that the elements in the packet $\widetilde{\Pi}_\psi$ have multiplicity 1. This is a major result, which applies to parameters for any quasisplit orthogonal or symplectic group. It was the motivation for our definition of coherence. The condition (ii) of the definition is still just a guess. It could be implicit in the work of Moeglin, and is perhaps not too difficult to verify one way or the other. An affirmative answer would tell us that any p-adic parameter for G is coherent.

Suppose finally that F is archimedean. We recall that Adams, Barbasch and Vogan have attached endoscopic packets to parameters for any reductive group over F [**ABV**]. This follows the special cases of general parameters for $F = \mathbb{C}$ [**BV**] and cohomological parameters for $F = \mathbb{R}$ [**AJ**]. The constructions do not include twisted endoscopy, which if it were known even for $GL(N)$, would confirm that these archimedean packets are the same as the ones defined for orthogonal and symplectic groups in [**A3**]. The problem is presumably accessible, at least in the cases [**BV**] and [**AJ**]. In any event, a check of the constructions in [**BV**] and [**AJ**] reveals that for our even orthogonal group G, all parameters $\psi \in \Psi(G)$ for $F = \mathbb{C}$ and all cohomological parameters $\psi \in \widetilde{\Psi}(G)$ for $F = \mathbb{R}$, are coherent (or at least will be once it has been verified that the constructions satisfy twisted endoscopy for $GL(N)$). However, the conditions (i) and (ii) for the general parameter $\psi \in \widetilde{\Psi}(G)$ for $\mathbb{R}$ in [**ABV**] appear to be deeper.

This concludes our discussion of coherent parameters $\psi \in \widetilde{\Psi}(G)$ for the even orthogonal group (1.1). There is something more we could say. The notion of coherence seems to bear some relevance to parameters for more general groups. We shall finish the section with some philosophical remarks in this direction.

In what follows, we could take G to be an arbitrary quasisplit group over the local field F. To keep the discussion a little more concrete, we assume until further notice that G is one of the groups treated in [**A3**], namely a quasisplit, special orthogonal or symplectic group over F. We put aside the question of how an outer automorphism might act, and consider just the usual sets of parameters $\Phi_{\mathrm{bdd}}(G)$, $\Phi(G)$ and $\Psi(G)$, taken up to the equivalence relation defined by $\widehat{G}$-conjugacy. They come with a chain of embeddings

$$\Phi_{\mathrm{bdd}}(G) \subset \Psi(G) \subset \Phi(G), \tag{3.1}$$

as in (2.4). The main reason for introducing the supplementary set $\Psi(G)$ is global, as we have noted. It provides the local framework for describing the multiplicities of automorphic representations attached to a given global family $c = \{c_v\}$ of Hecke eigenvalues. There is also ample local reason to consider local parameters $\psi \in \Psi(G)$, if only because their corresponding packets Π_ψ should give interesting new unitary representations of $G(F)$. Let us, however, try to motivate these objects for a different local reason, related purely to the theory of ordinary (untwisted) endoscopy.

The set $\Phi_{\mathrm{bdd}}(G)$ of bounded Langlands parameters serves two simultaneous ends. It leads to a classification of the set $\Pi_{\mathrm{temp}}(G)$ of irreducible tempered representations of $G(F)$, and at the same time, a collection of endoscopic reciprocity laws among the characters of these representations. Indeed, the two roles coalesce in the analogue

$$f'(\phi') = \sum_{\pi\in\Pi_\phi} \langle x, \pi\rangle f_G(\pi), \qquad f \in \mathcal{H}(G),\ \phi \in \Phi_{\mathrm{bdd}}(G),\ x \in \mathcal{S}_\phi, \tag{3.2}$$

of (2.1). The set $\Phi(G)$ of general Langlands parameters can also play both roles. In this case, however, the roles have to be separated. The L-packets Π_ϕ and the associated pairings $\langle\cdot,\cdot\rangle$ on $\mathcal{S}_\phi \times \Pi_\phi$ provide a classification of the set $\Pi(G)$ of all irreducible representations of $G(F)$, but they do not satisfy (3.2). On the other hand, we obtain endoscopic relations

$$f'(\phi') = \sum_{\rho\in P_\phi} \langle x, \rho\rangle f_G(\rho) \tag{3.3}$$

simply by replacing Π_ϕ by the corresponding packet of standard representations

$$P_\phi = \{\rho \in P(G) :\ \pi_\rho \in \Pi_\phi\},$$

equipped with the induced pairing

$$\langle x, \rho\rangle = \langle x, \pi_\rho\rangle, \qquad x \in \mathcal{S}_\phi,$$

where π_ρ is the Langlands quotient of ρ. The problem now is that the representations in P_ϕ are no longer irreducible. But we can adjust this if we replace P_ϕ with the packet

$$\Pi_\phi^+ = \{[\pi^+] \subset [\rho] :\ \rho \in P_\phi\}$$

of irreducible constituents of the standard representations $\rho \in P_\phi$, repeated according to multiplicity. Then Π_ϕ^+ is a set over $\Pi(G)$, with the induced pairing

$$\langle x, \pi^+\rangle = \langle x, \rho\rangle, \qquad x \in \mathcal{S}_\phi,\ [\pi^+] \subset [\rho],\ \rho \in P_\phi,$$

such that

$$f'(\phi') = \sum_{\pi^+\in\Pi_\phi^+} \langle x, \pi^+\rangle f_G(\pi^+), \qquad f \in \mathcal{H}(G),\ \phi \in \Phi(G),\ x \in \mathcal{S}_\phi. \tag{3.4}$$

Using the Langlands classification established[2] for representations $\Pi(G)$ in [**A3**], we have attached a packet Π_ϕ^+ over $\Pi(G)$ to any $\phi \in \Phi(G)$, which satisfies the endoscopic relation (3.4). This looks familiar. It has the same general structure as the packet Π_ψ of a parameter $\psi \in \Psi(G)$. However, the packets Π_ϕ^+ will be considerably more complicated. The multiplicities of fibres in Π_ϕ^+ over $\Pi(G)$ are determined by the generalized Kazhdan-Lusztig algorithm. They are very complex, largely because they can be arbitrarily large. It is within this context that we can consider the parameters ψ.

For any parameter in a subset of $\Phi(G)$, namely the injective image

$$\{\phi_\psi :\ \psi \in \Psi(G)\}$$

[2] If $G = SO(2n)$ as earlier, we assume we have "broken" the $(\mathbb{Z}/2\mathbb{Z})$-symmetries arbitrarily, by fixing a bijection between the $\widetilde{O}(G)$-torsors $\Phi(\phi)$ and $T(\phi)$ attached to any $\phi \in \widetilde{\Phi}_{\mathrm{sim}}(G)$. (See the end of §8.4 of [**A3**].)

of $\Psi(G)$, we have another packet[3] Π_ψ that is simpler than $\Pi^+_{\phi_\psi}$. Like $\Pi^+_{\phi_\psi}$, it contains the representations from the original packet Π_{ϕ_ψ} [**A3**, Proposition 7.4.1]. But it contains fewer supplementary representations π, and it fibres over the subset $\Pi_{\mathrm{unit}}(G)$ of $\Pi(G)$. It also satisfies richer endoscopic relations

$$f'(\psi') = \sum_{\pi \in \Pi_\psi} \langle s_\psi x, \pi\rangle f_G(\pi), \qquad f \in \mathcal{H}(G),\ \psi \in \Psi(G),\ x \in \mathcal{S}_\psi, \tag{3.5}$$

where $\langle \cdot, \pi\rangle$ is a character on an extension $\mathcal{S}_\psi$ of $\mathcal{S}_{\phi_\psi}$, which is the image of the original character on $\mathcal{S}_{\phi_\psi}$ in case π lies in the subset Π_{ϕ_ψ} of Π_ψ. The theorem of Moeglin, which we hope is also true in the archimedean case, asserts that for p-adic F, Π_ψ is a subset of $\Pi_{\mathrm{unit}}(G)$. In other words, the multiplicity of any fibre in Π_ψ over $\Pi_{\mathrm{unit}}(G)$ equals 1. This seems to be a fundamental defining property for the packet Π_ψ. It makes the endoscopic identity (3.5) look much closer to its classical antecedent (3.3) for tempered parameters $\phi \in \Phi_{\mathrm{bdd}}(G)$ than its more complicated but formally similar analogue (3.4) for general parameters $\phi \in \Phi(G)$. The packets Π_ψ thus achieve a delicate balance between relatively simple endoscopic character relations (3.5) and a role in the (as yet unfinished) classification of $\Pi_{\mathrm{unit}}(G)$.

For the rest of the section, we allow G to range over all connected, quasisplit groups over the fixed local field F. Could some variant of coherence be used to characterize the general packets Π_ψ?

The parameter sets $\Psi_{\mathrm{bdd}}(G)$, $\Psi(G)$ and $\Phi(G)$ are defined, and satisfy (3.1). For any $\psi \in \Psi(G)$, we have the centralizer S_ψ. Its finite quotient $\mathcal{S}_\psi$ maps onto the corresponding group $\mathcal{S}_{\phi_\psi}$ attached to the image of ψ in $\Psi(G)$. We also have the set

$$\Psi_{\mathrm{sim}}(G) = \{\psi \in \Psi(G) :\ \overline{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma = 1\}$$

of simple parameters. We will not assume the full Langlands classification for $\Pi_{\mathrm{temp}}(G)$. We suppose only that for each G, and each ϕ in the subset

$$\Phi_{\mathrm{sim}}(G) = \Psi_{\mathrm{sim}}(G) \cap \Phi_{\mathrm{bdd}}(G) = \Psi_{\mathrm{sim}}(G) \cap \Phi(G)$$

of $\Psi_{\mathrm{sim}}(G)$, we have been given a representation $\pi_\phi \in \Pi_{\mathrm{temp}}(G)$ such that the distribution

$$f \longrightarrow f_G(\pi_\phi), \qquad f \in \mathcal{H}(G),$$

is stable, with the further understanding that these representations collectively satisfy natural relations with respect to central twists. If ψ lies in the larger family $\Psi_{\mathrm{sim}}(G)$, it is then not hard to attach a natural Langlands quotient $\pi_{\phi_\psi} \in \Pi(G)$ to the parameter $\phi_\psi \in \Phi(G)$. We set $\pi_\psi = \pi_{\phi_\psi}$.

Assumption 3.3. *For the given assignment*

$$(G, \phi) \longrightarrow \pi_\phi, \qquad \phi \in \Phi_{\mathrm{sim}}(G),$$

of representations in $\Pi_{\mathrm{temp}}(G)$*, there is a unique assignment*

$$(G, \psi) \longrightarrow n_\psi, \qquad \psi \in \Psi_{\mathrm{sim}}(G)$$

of functions

$$n_\psi :\ \Pi(G) \longrightarrow \{0, 1\}$$

of finite support that satisfies the following conditions.

(i) $n_\psi(\pi_\psi) = 1$.

[3] We are assuming Theorem 2.1 here in case $G = SO(2n)$.

(ii) *The distribution*

$$f \longrightarrow f^G(\psi) = \sum_{\pi\in\Pi(G)} n_\psi(\pi) f_G(\pi), \qquad f \in \mathcal{H}(G),$$

on $G(F)$ is stable.

(iii) *If n^*_ψ is another function that satisfies* (i) *and* (ii)*, then*

$$n_\psi(\pi) \leq n^*_\psi(\pi), \qquad \pi \in \Pi(G).$$

The assignment $\phi \to \pi_\phi$ is supposed to represent a part of the Langlands correspondence, namely the cuspidal L-packets

$$\Pi_\phi = \{\pi_\phi\}, \qquad \phi \in \Phi_{\mathrm{sim}}(G), \tag{3.6}$$

of order 1. To my knowledge, there are no conjectural conditions in general that would characterize it uniquely. In any case, with this interpretation we can regard Assumption 3.3 as a conjecture. It would characterize the A-packets

$$\Pi_\psi = \{\pi \in \Pi(G) : \ n_\psi(\pi) = 1\}, \qquad \psi \in \Psi_{\mathrm{sim}}(G), \tag{3.7}$$

attached to parameters in the subset $\Psi_{\mathrm{sim}}(G)$ of $\Psi(G)$. The function n_ψ is of course uniquely determined by the condition (iii). I have not looked at any of the known examples in this light, so I have no evidence for such a conjecture beyond general aesthetic considerations. Should it prove false, we would still want to look for a natural refinement that would characterize the packets (3.7) in terms of the fundamental L-packets (3.6). We observe that Assumption 3.3 is closely related to the second condition (ii) of the earlier definition of coherence.

It is possible that Assumption 3.3 (or some natural variant/extension) would give a conjectural way to characterize general A-packets Π_ψ (and as a special case, general L-packets Π_ϕ). However, we will not try to characterize the signs

$$\langle s_\psi, \pi\rangle, \qquad \pi \in \Pi_\psi, \tag{3.8}$$

in the general analogue of (3.5). (See [**ABV**] and [**MW**].) We shall instead confine ourselves to the subset

$$\Psi_{\mathrm{even}}(G) = \{\psi \in \Psi(G) : \ s_\psi = 1\}$$

of parameters in $\Psi(G)$ in which s_ψ (as a central element in $\mathcal{S}_\psi$) is trivial.

Proposition 3.4. *Under Assumption 3.3, there is at most one assignment*

$$(G,\psi) \longrightarrow (\Pi_\psi, \langle\cdot,\cdot\rangle), \qquad \psi \in \Psi_{\mathrm{even}}(G),$$

where Π_ψ is a finite subset of $\Pi(G)$, and $\langle\cdot,\cdot\rangle$ is a mapping

$$\pi \longrightarrow \langle\cdot,\pi\rangle, \qquad \pi \in \Pi_\psi,$$

from Π_ψ to the set $\widehat{\mathcal{S}}_\psi$ of irreducible characters

$$x \longrightarrow \langle x,\pi\rangle, \qquad x \in \mathcal{S}_\psi,$$

on $\mathcal{S}_\psi$, that satisfies the following conditions.

(i) *If ψ lies in the subset $\Psi_{\mathrm{sim}}(G)$ of $\Psi_{\mathrm{even}}(G)$, Π_ψ equals the packet* (3.7)*.*

(ii) *The distribution*

$$f \longrightarrow f^G(\psi) = \sum_{\pi\in\Pi_\psi} f_G(\pi), \qquad f \in \mathcal{H}(G),$$

on $G(F)$ is stable.

(iii) *Suppose that s is a semisimple element in S_ψ with image x in $\mathcal{S}_\psi$, and that (G', ψ') is the preimage*[4] *of the pair (ψ, s). Then*

$$f'(\psi') = \sum_{\pi \in \Pi_\psi} \langle x, \pi \rangle f_G(\pi), \qquad f \in \mathcal{H}(G).$$

Proof. We can extend the $\widehat{\mathcal{S}}_\psi$-valued function

$$\xi_\psi : \ \pi \ \longrightarrow \ \xi_\psi(\pi) = \langle \cdot, \pi \rangle, \qquad \pi \in \Pi_\psi,$$

on Π_ψ to an $\widehat{\mathcal{S}}_\psi \cup \{0\}$-valued function that vanishes on the complement of Π_ψ in $\Pi(G)$. Then

$$\Pi_\psi = \{\pi \in \Pi(G) : \ \xi_\psi(\pi) \neq 0\}.$$

Suppose ξ^*_ψ is another family of such functions, parametrized again by the pairs (G, ψ). We fix (G, ψ), and assume inductively that $\xi^*_{\psi'} = \xi_{\psi'}$ for every proper endoscopic pair (G', ψ') for (G, ψ). If there are no proper pairs, $\overline{S}_\psi$ is trivial, and ψ lies in the subset $\Psi_{\mathrm{sim}}(G)$ of $\Psi_{\mathrm{even}}(G)$. In this case, condition (i) above and Assumption 3.3 tell us that $\xi^*_\psi = \xi_\psi$. We may therefore assume that ψ lies in the complement of $\Psi_{\mathrm{sim}}(G)$.

Consider the condition (iii) above, where s is a nontrivial point in $\overline{S}_\psi$. It follows from our induction hypothesis that the left hand side $f'(\psi')$ of the identity in (iii) is the same for the two families ξ^* and ξ. The difference

$$\sum_{\pi \in \Pi(G)} \big(\xi^*_\psi(\pi) - \xi_\psi(\pi)\big) f_G(\pi), \qquad f \in \mathcal{H}(G),$$

of the two right hand sides, regarded as a linear combination of irreducible characters on $\mathcal{S}_\psi$, therefore vanishes for any $s \neq 1$. Given any representation $\pi \in \Pi(G)$, we can choose a function $f \in \mathcal{H}(G)$ such that

$$f_G(\pi') = \begin{cases} 1, & \text{if } \pi' = \pi, \\ 0, & \text{otherwise,} \end{cases}$$

for any π' in the union of the two packets Π_ψ and Π^*_ψ. It follows that the two functions

$$\xi^*_\psi(\pi), \xi_\psi(\pi) : \ \mathcal{S}_\psi \ \longrightarrow \ \mathbb{C}$$

in $\widehat{\mathcal{S}}_\psi \cup \{0\}$ are equal at the image in $\mathcal{S}_\psi$ of any $s \neq 1$. Since any element in $\mathcal{S}_\psi$ can be so represented if the identity component of $\overline{S}^0_\psi$ of $\overline{S}_\psi$ is nontrivial, we can assume that ψ lies in the subset

$$\Psi_2(G) = \big\{\psi \in \Psi(G) : \ |\overline{S}_\psi| < \infty\big\}$$

of $\Psi(G)$. In this case, the difference $\xi^*_\psi(\pi) - \xi_\psi(\pi)$ still vanishes on the complement of 1 in $\mathcal{S}_\psi$. Comparing it with the character of the regular representation on $\mathcal{S}_\psi$, we deduce that this difference must in fact vanish identically on $\mathcal{S}_\psi$.

We have established that $\xi^*_\psi = \xi_\psi$ in all cases. This completes our induction hypothesis, and gives the uniqueness assertion of the proposition. $\square$

[4] We are assuming for simplicity here that the L-group ${}^L G'$ has an L-embedding into ${}^L G$ (which has been fixed). Otherwise, (G', ψ') would have to be replaced by a pair $(\widetilde{G}', \widetilde{\psi}')$, in which $\widetilde{G}'$ is a suitable extension of G'.

REMARKS. 1. The proof of Proposition 3.4 depends on the condition that Π_ψ be a subset of $\Pi(G)$ rather than just a set over $\Pi(G)$. Given this essential requirement, the proposition can also be regarded as an implicit conjecture, that would characterize the A-packets Π_ψ of parameters ψ in the subset $\Psi_{\text{even}}(G)$ of $\Psi(G)$. As such, it is again based on the generalization of a property of coherence, the first condition (i) of the earlier definition. We do have some evidence, namely Moeglin's theorem [**M**] of multiplicity 1 for p-adic orthogonal and symplectic groups.

2. Notice that the subset $\Phi_{\text{bdd}}(G)$ of $\Phi(G)$ is contained in $\Psi_{\text{even}}(G)$. Proposition 3.4 therefore includes a construction of the L-packets Π_ϕ attached to general parameters $\phi \in \Phi_{\text{bdd}}(G)$, given of course the basic objects of Assumption 3.3.

3. Assume that the assignment $(G, \phi) \to \pi_\phi$ of Assumption 3.3 has been provided, and that it leads to the local Langlands classification for G, according to Remark 2. We should then be able to read off the functions n_ψ from the expansion of $f_G(\pi_\psi)$ into standard characters. We would simply subtract the minimal number of irreducible characters (with possible multiplicities) from the expansion so as to transform it into a *stable* combination of standard characters. The implicit condition of Assumption 3.3 is that the multiplicities should all be 1.

4. What about the signs (3.8)? Their presence for a general parameter $\psi \in \Psi(G)$ means that the proof of Proposition 3.4 will fall short of a conjectural characterization of the packet Π_ψ. However, one could consider a generalization of the family of functions n_ψ of Assumption 3.3. Suppose for example that $\mathcal{S}_\psi$ has order 2, and that s_ψ represents the nontrivial element. If (G', ψ') is the preimage of the pair (ψ, s_ψ), the formula (3.5) becomes

$$f'(\psi') = \sum_{\pi \in \Pi_\psi} \langle s_\psi s_\psi, \pi \rangle f_G(\pi) = \sum_{\pi \in \Pi_\psi} f_G(\pi).$$

This expansion, which would be given to us inductively as in the proof, yields the packet Π_ψ but not the pairing $\langle \cdot, \cdot \rangle$. However, we would need the pairing to define the associated stable distribution

$$f^G(\psi) = \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle f_G(\pi),$$

and in particular, to complete the induction argument. To rectify the problem, we could postulate the existence of a unique function

$$n_\psi : \ \Pi(G) \ \longrightarrow \ \{1, 0, -1\},$$

which is supported on the subset Π_ψ of $\Pi(G)$, and satisfies the three conditions of Assumption 3.3 (with ψ now the given parameter in $\Pi(G)$). This is just a guess, to be modified as required. But if it is correct in this case, it could no doubt be formulated for any parameter $\psi \in \Psi(G)$. It could then be used as in the proof of the proposition to give a conjectural characterization of the general packet Π_ψ.

4. The global theorem

Our ultimate interest is in automorphic representations. In this section, we will state a conjecture for automorphic discrete spectra that is both a global counterpart of Theorem 2.1, and an extension of the earlier refinement [**A3**, Theorem 8.4.2] for generic global parameters. In my original submission, I actually stated it as a theorem, incautiously claiming that it could be established by a variant of the proof of Theorem 8.4.2 of [**A3**]. I thank the referee for a well founded note of restraint.

We return therefore to the earlier setting, in which G is an even, quasisplit, special orthogonal group over F. In this section, the field F will be global. Suppose that $\psi \in \widetilde{\Psi}(G)$ is an associated global parameter [**A3**, §1.4]. We can then write

$$I_{\mathrm{disc},\psi}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}'_{\mathrm{disc},\psi}(f'), \qquad f \in \mathcal{H}(G), \tag{4.1}$$

in the notation (3.3.15) that runs throughout [**A3**]. We recall, for example, that the subscript ψ is defined [**A3**, (3.3.12) and (3.3.13)] as the restriction of a given distribution to the subspace of its domain attached to ψ. The main point here is that like its predecessor [**A3**, 8.4.4] for a generic parameter ϕ, the formula (4.1) does not require that the test function f lie in the subspace $\widetilde{\mathcal{H}}(G)$ of locally symmetric functions in the Hecke algebra $\mathcal{H}(G)$ on $G(\mathbb{A})$.

The global conjecture will be founded on the local objects of Theorem 2.1, which will in turn have to be compatible at localizations of global data. This requires the uniqueness property of Proposition 3.2, and hence an assumption of coherence at some given set $V \subset \mathrm{val}(F)$ of valuations of F. We will apply the stabilization (4.1) to the subalgebra

$$\mathcal{H}_V(G) = \mathcal{H}(G_V) \otimes \widetilde{\mathcal{H}}(G^V) = \mathcal{H}\big(G(\mathbb{A}_V)\big) \otimes \widetilde{\mathcal{H}}\big(G(\mathbb{A}^V)\big)$$

of functions in $\mathcal{H}(G)$ that are locally symmetric outside of V. This is of course an intermediate space

$$\widetilde{\mathcal{H}}(G) \subset \mathcal{H}_V(G) \subset \mathcal{H}(G)$$

between the two algebras of functions on $G(\mathbb{A})$ we worked with in [**A3**, §8.4]. We will also work with the set

$$\widetilde{\Psi}_{\mathrm{coh},V}(G) = \{\psi \in \widetilde{\Psi}(G) : \ \psi_v \in \widetilde{\Psi}_{\mathrm{coh}}(G_v), \ v \in V\}$$

of global parameters in $\widetilde{\Psi}(G)$ whose localizations at places in V are coherent. In other words, at any place v of F, either the test function f_v will be symmetric or the parameter will be coherent.

At a minor loss of generality, we shall formulate the conjecture to be compatible with both Theorem 2.1 and [**A3**, Theorem 8.4.2]. That is, we shall restrict it to square integrable global parameters in the subset

$$\widetilde{\Psi}'_2(G) = \{\psi \in \widetilde{\Psi}_2(G) : \ m(\psi) = 2\}$$

of $\widetilde{\Psi}_2(G)$. To any ψ in this set, we can attach a global torsor $T(\psi)$ under the (global) automorphism group $\widetilde{O}(G)$ of order 2. The construction is similar to that of the generic global definition from [**A3**, §8.3]. We shall describe it very briefly.

The main point is again the case that ψ lies in the subset $\widetilde{\Psi}_{\mathrm{sim}}(G)$ of simple global parameters. Given ψ, one can define ϕ_ψ as an element in a set $\widetilde{\Phi}(G)$ of generic global parameters. This gives the canonical element

$$\pi = \pi_\psi = \bigotimes_v \pi_{\psi,v}, \qquad \pi_{\psi,v} \in \widetilde{\Pi}_{\psi,v}, \tag{4.2}$$

in the global packet $\widetilde{\Pi}_\psi$, where π_ψ is the element in the subset $\widetilde{\Pi}_{\phi_\psi,v}$ of $\widetilde{\Pi}_{\psi,v}$ [**A3**, Proposition 7.4.1] corresponding to the trivial character on the group $\mathcal{S}_{\phi_\psi,v}$. Since the parameter ϕ_ψ is the image in $\widetilde{\Phi}(G)$ of a global parameter

$$\{\phi_{M,\lambda} : \ \phi_M \in \widetilde{\Phi}_2(M), \ \lambda \in \mathfrak{a}^*_M\},$$

for a Levi subgroup M of G, it does not generally lie in $\widetilde{\Phi}_2(G)$. Nevertheless, it still has the property that

$$m(\phi_\psi) = m(\psi) = 2,$$

as in the local case from §2. One can use it to construct a canonical $\widetilde{O}(G)$-orbit $T(\psi)$ of irreducible representations $\{\pi_*\}$ of $G(\mathbb{A})$ that map to the element $\pi = \pi_\psi$ in $\widetilde{\Pi}_\psi$. This is obtained from the M-analogue of the $\widetilde{O}(G)$-orbit $T(\phi)$ attached to any global parameter $\phi \in \widetilde{\Phi}_2(G)$, which was defined prior to the statement of Theorem 8.4.2 of [**A3**], and which governs the assertion (8.4.6) of the theorem. Having constructed the $\widetilde{O}(G)$-orbit $T(\psi)$ for any $\psi \in \widetilde{\Psi}_{\mathrm{sim}}(G)$, we define it for general elements $\psi \in \widetilde{\Psi}_2'(G)$ as in the global generic case in [**A3**, §8.3], or for that matter, the local generic case from §2 here.

Consider a localization ψ_v of some global parameter $\psi \in \widetilde{\Psi}_2'(G)$. Then ψ_v lies in the subset $\widetilde{\Psi}^+_{\mathrm{unit}}(G_v)$ of $\widetilde{\Psi}^+(G_v)$. (See Remark 2 following the statement of Theorem 2.1. It is at this point that we have to account for the possible failure of the generalized Ramanujan conjecture.) Suppose that ψ_v lies in the subset $(\widetilde{\Psi}^+_{\mathrm{unit}})'(G_v)$ of $\widetilde{\Psi}^+_{\mathrm{unit}}(G_v)$. Assume further that the assertions of Theorem 2.1 are valid for ψ_v (and its subparameters), and that ψ_v is coherent. We can then define an isomorphism

$$t \longrightarrow t_v$$

between the torsors $T(\psi)$ and $T(\psi_v)$, following the definition for generic parameters near the beginning of §8.4 of [**A3**]. In particular, suppose that $\psi \in \widetilde{\Psi}_{\mathrm{sim}}(G)$ is simple. According to the definition above, an element $t \in T(\psi)$ is represented by an automorphic representation $\pi_* = \pi_t$ attached to the $\widetilde{O}(G)$-orbit π of (4.2) above. We define t_v to be the unique element in $T(\psi_v)$ such that

$$\pi_{v,t_v} = \pi_{t,v},$$

where the representation on the left is defined by the second assertion of Theorem 2.1. We extend this construction to more general $\psi \in \widetilde{\Psi}_2'(G)$ directly from the definitions.

Suppose now that the local Theorem 2.1 holds for the completion F_v of F at every v in the given set $V \subset \mathrm{val}(F)$ of valuations. Let ψ be a global parameter in $\widetilde{\Psi}_2'(G)$, which also lies in the subset $\widetilde{\Psi}_{\mathrm{coh},V}(G)$ of $\widetilde{\Psi}(G)$. The mappings $t \to t_v$ then allow us to globalize the two constructions of the local theorem. The first is the global, $\widetilde{O}(G)$-equivariant mapping

$$t \longrightarrow \psi_t = \psi_{t,V}, \qquad t \in T(\psi),$$

from $T(\psi)$ to the space of stable linear forms on the global space $\mathcal{H}_V(G)$, defined by

$$\psi_{t,V} = \left(\bigotimes_{v \in V} \phi_{v,t_v} \right) \otimes \left(\bigotimes_{w \notin V} \phi_w \right).$$

The second is the $\widetilde{O}(G)$-equivariant mapping

$$(\pi, t) \longrightarrow \pi_t = \pi_{t,V}, \qquad \pi \in \widetilde{\Pi}_\psi,\ t \in T(\psi),$$

from $\widetilde{\Pi}_\psi \times T(\psi)$ to the set

$$\Pi_{\mathrm{unit}}(G_V) \otimes \widetilde{\Pi}_{\mathrm{unit}}(G^V),$$

defined by

$$\pi_{t,V} = \Big(\bigotimes_{v\in V} \pi_{v,t_v}\Big) \otimes \Big(\bigotimes_{w\notin V} \pi_w\Big).$$

The last mapping here gives a linear form

$$f \longrightarrow f_G(\pi_{t,V}), \qquad f \in \mathcal{H}_V(G),$$

on $\mathcal{H}_V(G)$.

CONJECTURE 4.1. *Assume that F is global, and that ψ lies in the subset*

$$\widetilde{\Psi}'_{2,\mathrm{coh},V}(G) = \widetilde{\Psi}'_2(G) \cap \widetilde{\Psi}_{\mathrm{coh},V}(G)$$

of $\widetilde{\Psi}'_2(G)$, for some set $V \subset \mathrm{val}(F)$ of valuations of F. Then

$$S^G_{\mathrm{disc},\psi}(f) = |\mathcal{S}_\psi|^{-1} \sum_{t\in T(\psi)} \varepsilon^G(\psi) f^G(\psi_{t,V}), \qquad f \in \mathcal{H}_V(G), \tag{4.3}$$

and

$$\mathrm{tr}\big(R^G_{\mathrm{disc},\psi}(f)\big) = \sum_{\pi\in\widetilde{\Pi}_\psi(\varepsilon_\psi)} \sum_{t\in T(\psi)} f_G(\pi_{t,V}), \qquad f \in \mathcal{H}_V(G). \tag{4.4}$$

In (4.3), $\varepsilon^G(\psi)$ is the value at s_ψ of the sign character $\varepsilon_\psi = \varepsilon^G_\psi$ on $\mathcal{S}_\psi$ defined following the statement of Theorem 1.5.2 of [**A3**]. In (4.3), $R^G_{\mathrm{disc},\psi}$ is the representation of $G(\mathbb{A})$ on the ψ-component of the automorphic discrete spectrum of G [**A3**, (3.4.5)], while $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ is as in [**A3**, Theorem 1.5.2], the subset of representations π in the global packet $\widetilde{\Pi}_\psi$ such that the character $\langle\cdot,\pi\rangle$ on $\mathcal{S}_\psi$ equals ε_ψ. Finally, we are writing $\phi_{t,V}$ and $\pi_{t,V}$ in (4.3) and (4.4) with the implicit understanding that Theorem 2.1 is valid for the valuations $v \in V$. Since the localizations ϕ_v at $v \in V$ are assumed to be coherent, these mappings are then canonical.

I do not know how to prove this conjecture. However, it might still be instructive to review the beginnings of the argument from the proof of Theorem 8.4.2 of [**A3**]. This allows us at least to point out where the earlier proof fails in this setting.

The starting point is the stabilization (4.1) of the discrete part of the trace formula for G. We suppose that Conjecture 4.1 is valid if N is replaced by any even, positive integer $N_- < N$. We can then apply the analogue of (4.2) for the proper elliptic endoscopic data $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ that index the terms with $G' \neq G$ in (4.1). Making the appropriate substitution, and recalling how we treated the terms in [**A3**, (7.4.7)], for example, we see that the right hand side of (4.1) equals

$$S^G_{\mathrm{disc},\psi}(f) + |\mathcal{S}_\psi|^{-1} \sum_{\substack{x\in\mathcal{S}_\psi\\ x\neq 1}} \sum_{t\in T(\psi)} \varepsilon'(\psi')\, f'(\psi'_{t',V}). \tag{4.5}$$

Following a standard convention from [**A3**], we write ${}^0S^G_{\mathrm{disc},\psi}(f)$ for the difference between the two sides of the putative formula (4.3) for G. The right hand side of (4.1) then equals

$${}^0S^G_{\mathrm{disc},\psi}(f) + |\mathcal{S}_\psi|^{-1} \sum_{x\in\mathcal{S}_\psi} \sum_{t\in T(\psi)} \varepsilon'(\psi')\, f(\psi'_{t',V}).$$

Since ψ lies in the subset $\widetilde{\Psi}_2(G)$ of $\widetilde{\Psi}(G)$, the left hand side of (4.1) reduces to the trace of $R^G_{\mathrm{disc},\psi}(f)$. This follows from the usual arguments, in [**A3**, §7.4] for example. We write ${}^0r^G_{\mathrm{disc},\psi}(f)$ for the difference between the two sides of the desired formula (4.4). If we combine the local endoscopic character expansion (2.9) with a simplified variant of the elementary remarks at the end of [**A3**, §4.7], we see without difficulty that the sum

$$|\mathcal{S}_\psi|^{-1} \sum_{x\in\mathcal{S}_\psi} \sum_{t\in T(\psi)} \varepsilon'(\psi')\, f'(\psi'_{t',V})$$

above equals the right hand side

$$\sum_{\pi\in\widetilde{\Pi}_\psi(\varepsilon_\psi)} \sum_{t\in T(\psi)} f_G(\pi_{t,V})$$

of (4.4). We conclude that

$$ {}^0r^G_{\mathrm{disc},\psi}(f) = {}^0S^G_{\mathrm{disc},\psi}(f), \qquad f\in\mathcal{H}_U(G). \tag{4.6}$$

The analogue of the identity (4.6) for the generic parameters ϕ of [**A3**, Theorem 8.4.4] was the first step (8.4.21) in the proof. Notice that the required assertions (4.3) and (4.4) are equivalent to the vanishing of the right and left hand sides respectively of the identity. It would therefore suffice to establish either one of them. This would resolve the implicit induction hypothesis above, and complete the proof of the proposition.

In the proof of [**A3**, Theorem 8.4.2], we deduced the vanishing of the two linear forms together by playing one off against the other in (8.4.21). An essential ingredient was Corollary 8.4.5. This is a local result that serves to characterize the stable distributions in the span of the characters in a generic packet Π_{ϕ_v}. Combined with global arguments, it leads to the required stable multiplicity formula [**A3**, (8.4.5)], or in other words, the vanishing of the generic analogue of the right hand side of (4.6).

Theorem 8.4.2 of [**A3**] applies to generic global parameters $\psi=\phi$ such that the mapping

$$\mathcal{S}_\psi \;\longrightarrow\; \mathcal{S}_{\psi_{\mathbb{A}}} = \prod_v \mathcal{S}_{\psi_v}$$

is injective. (I am indebted to the referee for an implicit suggestion that the original proof in [**A3**] requires a restriction of this sort.) We assume that the generic global parameter $\psi\in\widetilde{\Psi}(G)$ here satisfies the same condition. However, we would require more. This is because we do not have an analogue of Corollary 8.4.5 of [**A3**]. That is, we cannot rule out the existence of nonstandard stable distributions in the span of a local packet Π_{ψ_v}. Perhaps one could establish the conjecture in the special case that ψ has local constraints, with localizations of the sort treated in Lemma 2.2 for example. For general ψ, I have no ideas. The problem does seem to raise interesting questions concerning stable distributions. These could conceivably be related to questions from §3, such as the condition (ii) for coherence, the conditions of Assumption 3.3, or the questions from Remarks 3 and 4 following Proposition 3.4.

5. Proof of the local theorem

We complete the paper in this section by outlining a proof of Theorem 2.1. The global argument we give is based on that of Theorem 8.4.1 of [**A3**]. It is in fact slightly simpler, since we will be able to apply Lemma 2.2 at a suitable p-adic completion, instead of the local results of Shelstad at the set of all archimedean places. We can offer little motivation here for the various steps, referring instead to the relevant discussion from [**A3**].

We return to the local setting from the earlier parts of the paper. In particular, we take (F, G, ψ) to be as in Theorem 2.1. Then G is an even, quasisplit, special orthogonal group (1.1) over a local field F, and ψ is a parameter in the set $\widetilde{\Psi}'(G)$. The essential case is still that of a parameter

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r, \qquad \psi_i \in \widetilde{\Psi}_{\mathrm{sim}}(G_i),\ G_i = SO(N_i),\ N_i = 2n_i, \tag{5.1}$$

in the subset $\widetilde{\Psi}'_2(G)$ of $\widetilde{\Psi}'(G)$. With this assumption on ψ, we will attach a global triplet $(\dot{F}, \dot{G}, \dot{\psi})$ to the given local triplet (F, G, ψ), as in Proposition 7.2.1 of [**A3**]. Then $\dot{G}$ is an even, quasisplit, special orthogonal group over the global field $\dot{F}$, and $\dot{\psi}$ is a parameter in $\widetilde{\Psi}'_2(\dot{G})$ of the corresponding form

$$\dot{\psi} = \dot{\psi}_1 \oplus \cdots \oplus \dot{\psi}_r, \qquad \dot{\psi}_i \in \widetilde{\Psi}_{\mathrm{sim}}(\dot{G}_i),\ \dot{G}_i = SO(N_i),\ N_i = 2n_i. \tag{5.2}$$

The global triplet has the property that

$$(F, G, \psi) = (\dot{F}_u, \dot{G}_u, \dot{\psi}_u),$$

for some fixed place u of $\dot{F}$.

We can actually work with conditions that are simpler that the special requirements of [**A3**, Proposition 7.2.1]. This is because we have already established Theorem 2.1 for a large family of p-adic parameters in Lemma 2.2. It suffices here to let V be a set consisting of one element, a fixed p-adic valuation $v = u_1$ with large residual characteristic p, rather than the large finite set of nonarchimedean places in [**A3**, §7.2]. To recall the context, we write

$$\psi_i = \mu_i \otimes \nu_i, \qquad N_i = m_i n_i,\ 1 \leq i \leq r,$$

for irreducible representations μ_i and ν_i of L_F and $SU(2)$ respectively. We then apply Corollary 6.2.4 of [**A3**], supplemented by Remark 3 following its proof, as at the beginning of [**A3**, §7.2]. We thereby construct primary global pairs

$$(\dot{H}_i, \dot{\mu}_i), \qquad \dot{H}_i \in \dot{\widetilde{\mathcal{E}}}_{\mathrm{sim}}(m_i),\ \dot{\mu}_i \in \widetilde{\Phi}_{\mathrm{sim}}(\dot{H}_i),$$

over $\dot{F}$ from the given local pairs

$$(H_i, \mu_i), \qquad H_i \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(m_i),\ \mu_i \in \widetilde{\Phi}_{\mathrm{sim}}(H_i),$$

over F. But instead of specifying $\dot{\mu}_{i,v}$ at a large finite set V of nonarchimedean places, as a direct sum of distinct irreducible representations of the subgroup W_F of L_F (of dimension 1 or 2), we simply take $\mu_{i,v}$ to be an irreducible representation of W_F (of dimension m_i) at the one place $v = u_1$ chosen here. Armed with the global pairs $(\dot{H}_i, \dot{\mu}_i)$, we set

$$\dot{\psi}_i = \dot{\mu}_i \otimes \dot{\nu}_i, \qquad 1 \leq i \leq r,$$

where $\dot{\nu}_i$ is the irreducible representation of $SL(2, \mathbb{C})$ of dimension n_i. This leads directly to the required global triplet $(\dot{F}, \dot{G}, \dot{\psi})$. Its corresponding localization

$$(F_1, G_1, \psi_1) = (\dot{F}_{u_1}, \dot{G}_{u_1}, \dot{\psi}_{u_1})$$

then has the property

$$\psi_1 = \widehat{\phi}_1$$

of Lemma 2.2, for a local parameter $\phi_1 \in \widetilde{\Phi}'_2(G_1)$.

We set

$$U = \{u, u_1\} = \{u\} \cup V, \qquad V = \{u_1\}.$$

The completions of ψ and ψ_1 of ψ at the two places in U will have similar endoscopic properties, to the extent that

$$S_\psi \cong S_{\dot\psi} \cong S_{\psi_1}. \tag{5.3}$$

We shall consider the global stabilization (4.1) (with $\dot G$ in place of G), for functions $\dot f$ in the space $\mathcal{H}_U(\dot G)$.

We would like to substitute the analogue

$$S^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f) = |\mathcal{S}_{\dot\psi}|^{-1} \sum_{\dot t \in T(\dot\psi)} \varepsilon^{\dot G}(\dot\psi) \dot f^{\dot G}(\dot\psi_{\dot t}), \qquad f \in \mathcal{H}_U(\dot G), \tag{5.4}$$

of the identity (4.3) of Conjecture 4.1 for the summands on the right hand side of (4.1). However, there are two difficulties. One is that the proof of Conjecture 4.1 presupposes the validity of the local theorem we are trying to prove here. The other is that the localizations of the global parameter $\dot\psi$ at the exceptional set U were assumed to be coherent. The problem is of course at the primary valuation u, since the localization at the other valuation $u_1 \in U$ satisfies the conditions of Lemma 2.2. The parameters $\phi = \dot\phi_u$ and $\dot\phi$ do have completely parallel structures (5.1) and (5.2). This property allows us to relate the sets $T(\phi)$ and $T(\dot\phi)$, which we regard as torsors over the group $\widetilde{O}(G) \cong \widetilde{O}(\dot G)$. Following the definitions, and the generic case treated at the beginning of the proof of Theorem 8.4.1 in §8.4 of [**A3**], we define a canonical $\widetilde{O}(G)$-isomorphism from $T(\psi)$ to $T(\dot\psi)$. We can also include the parameter $\phi_1 = \dot\phi_{u_1}$, since it has exactly the same structure. We obtain canonical $\widetilde{O}(G)$-isomorphisms

$$t \ \xrightarrow{\sim}\ \dot t \ \xrightarrow{\sim}\ t_1 = \dot t_{u_1}, \qquad t \in T(\phi),$$

for the torsors $T(\phi)$, $T(\dot\phi)$ and $T(\phi_1)$. On the other hand, we cannot at this point define the stable linear forms

$$\dot f \ \longrightarrow\ \dot f^{\dot G}(\dot\psi_{\dot t}) = \dot f^{\dot G}(\dot\psi_{\dot t,U}), \qquad \dot f \in \mathcal{H}_U(\dot G),\ \dot t \in T(\dot\psi),$$

on the right hand side of (5.4). We will do so eventually, but in the meantime, we will have to treat (5.4) as something to be established independently of Conjecture 4.1, under the more specialized conditions here.

To this end, we assume inductively that (5.4) is valid if N is replaced by any even positive integer $N_- < N$. We make the same induction assumption locally for Theorem 2.1, as we must in order that the linear forms in (5.4) (with $N_- < N$) be defined. We then apply the formula to the proper summands on the right hand side of (4.1) (with $\dot\psi$ and $\dot f$ in place of ψ and f). As we have already seen, in the expression (4.5) in the last section, the right hand side of (4.1) becomes

$$S^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f) + |\mathcal{S}_{\dot\psi}|^{-1} \sum_{\substack{x \in \mathcal{S}_\psi \\ x \neq 1}} \sum_{t \in T(\psi)} \varepsilon'(\dot\psi') \dot f'(\dot\psi'_{\dot t'}), \tag{5.5}$$

where $(\dot G', \dot\psi', \dot t')$ is a preimage of $(\dot\psi, \dot x, \dot t)$, and $\dot f$ is any function in $\mathcal{H}_U(\dot G)$.

The left hand side of (4.1) reduces to the trace of the relevant operator $R^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f)$ on the discrete spectrum. We write it for now simply in the form

$$\sum_{\dot\pi\in\widetilde\Pi_{\mathrm{unit}}(\dot G)}\ \sum_{\dot\pi_*\in\Pi(\dot\pi)} n_{\dot\psi}(\dot\pi_*)\dot f_{\dot G}(\dot\pi), \tag{5.6}$$

as in [**A3**, (8.4.8)]. Our analysis of this sum will be simpler than that of its earlier counterpart. This is in part because the local packet $\widetilde\Pi_{\psi_1}$ whose behaviour we understand is simpler than the multiple archimedean packet used before. We are also working with a function $\dot f$ that is locally symmetric at the places $v\neq u,u_1$, so the analysis at these places is easy. Of course, it is also true that the local packet $\widetilde\Pi_\psi$ we are trying to understand is potentially more complicated than its tempered analogue from [**A3**, §8.4].

We have to establish the assertions of Theorem 2.1 from the equality of (5.5) and (5.6). We take our test function to be a product

$$\dot f=\dot f_U\dot f^U=f_1\cdot f\cdot \dot f^U,\qquad f_1\in\mathcal H(G_1),\ f\in\mathcal H(G),\ \dot f^U\in\widetilde{\mathcal H}(\dot G^U).$$

We then choose the locally symmetric function $\dot f^U$ to isolate the element

$$\dot\pi^U=\bigotimes_{w\notin U}\dot\pi_w,\qquad \dot\pi_w\in\widetilde\Pi_{\phi_{\dot\psi,w}}.$$

in the packet of $\phi^U_{\dot\psi}$ such that the character $\langle\cdot,\dot\pi_w\rangle$ equals 1 for each $w\notin U$. Here we are using Proposition 7.4.1 of [**A3**] to treat this packet as a subset of the larger packet $\widetilde\Pi^U_{\dot\psi}$. The pullback of the function $\dot f^U_{\dot G}$ from $\widetilde\Pi_{\mathrm{unit}}(\dot G^U)$ to $\widetilde\Pi^U_{\dot\psi}$ then equals 1 at $\dot\pi^U$, and vanishes on the complement of $\dot\pi^U$. The expression (5.5) becomes

$$S^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f)+|\mathcal S_\psi|^{-1}\sum_{\substack{x\in\mathcal S_\psi\\ x\neq 1}}\ \sum_{t\in T(\psi)}\varepsilon'(\dot\psi')f_1'(\psi'_{1,t_1'})f'(\psi'_{t'}), \tag{5.7}$$

as in the reduction of (8.4.7) to (8.4.9) in §8.4 of [**A3**].

The linear form in $f_1\in\mathcal H(G_1)$ satisfies the analogue for G_1 of the expansion (2.9). We write it as

$$f_1'(\psi'_{1,t_1'})=\sum_{\xi_1\in\widehat{\mathcal S}_{\psi_1}}\xi_1(s_{\psi_1}x_1)\,f_{1,G_1}\big(\pi_{1,t_1}(\xi_1)\big),$$

where $\pi_1(\xi_1)$ is the element in $\widetilde\Pi_{\psi_1}$ such that $\langle\cdot,\pi_1(\xi_1)\rangle$ equals the character ξ_1, and $\pi_{1,t_1}(\xi_1)=\pi_1(\xi_1)_{t_1}$ is the representation in Π_{ψ_1} attached to the pair $\big(\pi_1(\xi_1),t_1\big)$. To deal with the other linear form $f'(\psi'_{t'})$, we will use notation from §7.1 and §7.4 of [**A3**]. Specifically, if f belongs to the symmetric subalgebra $\widetilde{\mathcal H}(G)$ of $\mathcal H(G)$, we write

$$f'(\psi')=\sum_{\sigma\in\widetilde\Sigma_\psi}\langle s_\psi x,\sigma\rangle f_G(\sigma)=\sum_{\xi\in\widehat{\mathcal S}_\psi}\xi(s_\psi x)f_G\big(\sigma(\xi)\big),$$

where $\sigma(\xi)$ represents the linear form on $\widetilde{\mathcal H}(G)$ attached to the character $\xi\in\widehat{\mathcal S}_\psi$. The local results of [**A3**, §7.4] establish that

$$f_G\big(\sigma(\xi)\big)=\sum_{\pi\in\widetilde\Pi_\psi(\xi)}f_G(\pi),\qquad f\in\widetilde{\mathcal H}(G),$$

where $\widetilde\Pi_\psi(\xi)$ is the subset of elements $\pi\in\widetilde\Pi_\psi$ such that $\langle\cdot,\pi\rangle$ equals π. This amounts to the assertion (2.2.6) of Theorem 2.2.1 of [**A3**]. Our task will be to establish the generalization of this local expansion to functions in the larger space $\mathcal{H}(G)$.

The expression (5.7) equals (5.6). To deal with the latter, we must use the localization $G_1 = \dot{G}_{u_1}$ of $\dot{G}$. For any character $\xi \in \widehat{\mathcal{S}}_\psi$, we write

$$\xi_1 = \varepsilon_{\dot{\psi}} \xi^{-1},$$

where we have identified the isomorphic centralizers (5.3). We then have the $\widetilde{O}(G_1)$-orbit $\pi_{1,\xi} = \pi_1(\xi_1)$ in $\widetilde{\Pi}_{\psi_1}$, and the associated representation

$$\pi_{1,\xi,t} = \pi_{1,t_1}(\xi_1)$$

in Π_{ψ_1} attached to any $t \in T(\psi)$. Now if the factors f and f_1 of the function $\dot{f}$ chosen above are both symmetric, $\dot{f}$ belongs to the subspace $\widetilde{\mathcal{H}}(G)$ of $\mathcal{H}_U(G)$. Under this condition, (5.6) is just the contribution of $\dot{\psi}$ to the global multiplicity formula [**A3**, (1.5.5)]. We can therefore write it as

$$\begin{aligned}
\operatorname{tr}\big(R_{\mathrm{disc},\dot{\psi}}(\dot{f})\big) &= m(\dot{\psi}) \sum_{\dot{\pi} \in \widetilde{\Pi}_{\dot{\psi}}(\varepsilon_{\dot{\psi}})} \dot{f}_{\dot{G}}(\dot{\pi}) \\
&= 2 \sum_{\xi \in \widehat{\mathcal{S}}_\psi} f_{1,G_1}(\pi_{1,\xi}) f_G\big(\sigma(\xi)\big) \\
&= \sum_{\xi \in \widehat{\mathcal{S}}_\psi} \sum_{t \in T(\psi)} f_{1,G_1}(\pi_{1,\xi,t}) f_G\big(\sigma(\xi)\big),
\end{aligned}$$

given our choice of the factor $\dot{f}^U$, the fact that the character $\xi_1\xi$ on $\mathcal{S}_{\dot{\psi}} = \mathcal{S}_\psi$ equals $\varepsilon_{\dot{\psi}}$, the identity

$$m(\dot{\psi}) = 2 = |T(\psi)|,$$

and the fact that the analogue for G_1 of (2.8) is valid. We shall compare this last double sum with the general expression (5.6), in which f and f_1 are not required to be symmetric. We first observe that if the coefficient $n_{\dot{\psi}}(\dot{\pi}_*)$ of a given $\dot{\pi}_*$ in (5.6) is nonzero, $\dot{f}^U_{\dot{G}}(\dot{\pi}^U)$ equals 1, and $\pi_1 = \dot{\pi}_{*,u_1}$ is equal to one of the distinct representations

$$\pi_{1,\xi,t}, \qquad \xi \in \widehat{\mathcal{S}}_\psi,\ t \in T(\psi).$$

We then see that we can write (5.6) in the form

$$\sum_{\xi \in \widehat{\mathcal{S}}_\psi} \sum_{t \in T(\psi)} f_{1,G_1}(\pi_{1,\xi,t}) f_G\big(\sigma_t(\xi)\big), \tag{5.8}$$

where $\sigma_t(\xi)$ is a uniquely determined, nonnegative integral linear combination of representations in Π_ψ such that

$$f_G\big(\sigma_t(\xi)\big) = f_G\big(\sigma(\xi)\big), \qquad t \in T(\psi),$$

if $f \in \widetilde{\mathcal{H}}(G)$ is symmetric.

The section

$$\sigma(\xi) \longrightarrow \sigma_t(\xi), \qquad \xi \in \widehat{\mathcal{S}}_\psi,\ t \in T(\psi),$$

we have just introduced gives us the two definitions we need for the theorem. For we can write

$$f_G\big(\sigma_t(\xi)\big) = \sum_{\pi \in \widetilde{\Pi}_\psi(\xi)} f_G(\pi_t), \qquad f \in \mathcal{H}(G),$$

where π_t lies in the preimage of π in $\Pi_\psi(\xi)$. This provides the required $\widetilde{O}(G)$-equivariant mapping $(\pi, t) \to \pi_t$ from $\widetilde{\Pi}_\psi$ to Π_ψ. We define the other mapping by setting

$$f^G(\psi_t) = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi, \pi \rangle f_G(\pi_t), \qquad t \in T(\psi),\ f \in \mathcal{H}(G). \tag{5.9}$$

The required identities (2.7) and (2.8) then follow from these definitions. However, we must still show that the right hand side of (5.9) is stable in f. We have also to establish (2.9).

We note in passing that while $\sigma_t(\xi)$ is characterized by (5.8), the associated section $\pi \to \pi_t$ is not uniquely determined at the orbits π with multiplicities greater than 1 in $\widetilde{\Pi}_\psi(\xi)$. This illustrates a minor discrepancy in the terminology used to formulate the original Theorem 1.5.1 of [**A3**]. Strictly speaking, the local packet $\widetilde{\Pi}_\psi$ is neither a "multiset in" $\widetilde{\Pi}_{\mathrm{unit}}(G)$ nor a "set over" $\widetilde{\Pi}_{\mathrm{unit}}(G)$, but something intermediate between the two. For there is nothing in its defining property [**A3**, (2.2.6)] that distinguishes among elements in $\widetilde{\Pi}_\psi$ with higher multiplicities over $\widehat{\mathcal{S}}_\psi$. If we wanted to be completely precise, we would say that $\widetilde{\Pi}_\psi$ is a "multiset in $\widetilde{\Pi}_{\mathrm{unit}}(G) \times \widehat{\mathcal{S}}_\psi$". This slightly arcane point does not arise if the sets $\widetilde{\Pi}_\psi(\xi)$ are multiplicity free, as expected. We shall continue to ignore it[5], as we have up until now.

We still have the two assertions of Theorem 2.1, the stability of $f^G(\psi_t)$ and the validity of (2.9), to establish for our parameter $\psi \in \widetilde{\Psi}'_2(G)$. We follow the argument from [**A3**, §8.4], modified by the presence here of the point $s_\psi \in \mathcal{S}_\psi$ and the character $\varepsilon_{\dot{\psi}} \in \widehat{\mathcal{S}}_\psi$. We will deduce what is needed from the identity of (5.8) with (5.7).

The analogue for G_1 of (2.9), established in Lemma 2.2, tells us that for any $x \in \mathcal{S}_\psi$ and $t \in T(\psi)$, the linear forms

$$f_{1,G_1}(\psi_{1,t_1}, s_\psi x) = \sum_{\xi \in \widehat{\mathcal{S}}_\psi} \langle s_\psi x, \pi_{1,\xi,t} \rangle\, f_{1,G_1}(\pi_{1,\xi,t})$$

and

$$f'_{1,G_1}(\psi_{1,t_1}, x) = f'_1(\psi'_{1,t'_1})$$

in $f_1 \in \mathcal{H}(G_1)$ are equal. As x and t vary, the linear forms in either of these two families are linearly independent. We can therefore choose f_1 so that $f_{1,G_1}(\psi_{1,t_1}, s_\psi x)$ vanishes if $x \neq 1$, but so that its two values

$$f_{1,G_1}(\psi_{1,t_1}, s_\psi) = f'_{1,G_1}(\psi_{1,t_1}, 1) = f_1^{G_1}(\psi_{1,t_1}), \qquad t \in T(\psi),$$

at $x = 1$ are arbitrary. We observe from the definitions $\pi_{1,\xi,t} = \pi_{1,t_1}(\xi_1)$ and $\xi_1 = \varepsilon_{\dot{\psi}} \xi^{-1} = \varepsilon_{\dot{\psi}}^{-1} \xi^{-1}$ that

$$f_{1,G_1}(\psi_{1,t_1}, s_\psi x) = \sum_{\xi_1} \xi_1(s_\psi x) f_{1,G_1}(\pi_{1,\xi,t}),$$

for any x and t, since

$$\langle s_\psi x, \pi_{1,\xi,t} \rangle = \langle s_\psi x, \pi_{1,t_1}(\xi_1) \rangle = \xi_1(s_\psi x).$$

[5] This minor point of uniqueness is separate from the broader question of §3, treated specifically in Proposition 3.2. The latter is a desired universal property governed only by the assertions of Theorem 2.1, which is independent of the chosen global triplet $(\dot{F}, \dot{G}, \dot{F})$, for example.

It follows by inversion on $\mathcal{S}_\psi$ and the properties of f_1 that

$$\begin{aligned} f_{1,G_1}(\pi_{1,\xi,t}) &= |\mathcal{S}_\psi|^{-1} \sum_x f_{1,G_1}(\psi_{1,t_1}, s_\psi x)\xi_1(s_\psi x)^{-1} \\ &= |\mathcal{S}_\psi|^{-1} f_{1,G_1}(\psi_{1,t_1}, s_\psi)\xi_1(s_\psi)^{-1} \\ &= |\mathcal{S}_\psi|^{-1} \varepsilon_{\dot\psi}(s_{\dot\psi}) f_1^{G_1}(\psi_{1,t_1})\xi(s_\psi). \end{aligned}$$

We will substitute the chosen function f_1 into (5.7) and (5.8), and then examine the resulting identity of linear forms in $f \in \mathcal{H}(G)$.

For the given function f_1, the expression (5.7) reduces to $S^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f)$, since the factors

$$f_1'(\psi'_{1,t_1'}) = f_{1,G_1}(\psi_{1,t_1}, s_\psi x), \qquad x \neq 1,$$

in the proper summands all vanish. The expression (5.8) becomes

$$\begin{aligned} &\sum_{\xi \in \widehat{\mathcal{S}}_\psi} \sum_{t \in T(\psi)} f_{1,G_1}(\pi_{1,\xi,t}) f_G\big(\sigma_t(\xi)\big) \\ &= \sum_{\xi,t} |\mathcal{S}_\psi|^{-1} \varepsilon_{\dot\psi}(s_{\dot\psi}) \xi(s_\psi) f_1^G(\psi_{1,t_1}) f_G\big(\sigma_t(\xi)\big) \\ &= |\mathcal{S}_\psi|^{-1} \varepsilon_{\dot\psi}(s_{\dot\psi}) \sum_{t \in T(\psi)} f_1^{G_1}(\psi_{1,t_1}) \sum_{\xi \in \widehat{\mathcal{S}}_\psi} \sum_{\pi \in \widetilde{\Pi}_\psi(\xi)} \xi(s_\psi)\, f_G(\pi_t) \\ &= |\mathcal{S}_\psi|^{-1} \xi_{\dot\psi}(s_{\dot\psi}) \sum_t f_1^{G_1}(\psi_{1,t_1}) \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi, \pi\rangle f_G(\pi_t) \\ &= |\mathcal{S}_\psi|^{-1} \varepsilon_{\dot\psi}(s_{\dot\psi}) \sum_t f_1^{G_1}(\psi_{1,t_1}) f^G(\psi_t), \end{aligned}$$

by (5.9). We have shown that

$$S^{\dot G}_{\mathrm{disc},\dot\psi}(\dot f) = |\mathcal{S}_\psi|^{-1} \varepsilon_{\dot\psi}(s_{\dot\psi}) \sum_{t \in T(\psi)} f_1^{G_1}(\psi_{1,t_1}) f^G(\psi_t), \tag{5.10}$$

for the variable function $f \in \mathcal{H}(G)$. Since we can choose f_1 so that $f_1^{G_1}(\psi_{1,t_1})$ is arbitrary, we can arrange that the coefficients of $f^G(\psi_t)$ vary independently of t. We conclude that for any $t \in T(\psi)$, $f^G(\psi_t)$ is a stable linear form in $f \in \mathcal{H}(G)$, as required.

The left hand side of (5.10) of course includes the fixed components $\dot f^U$ and $f_1 = \dot f_{u_1}$ of $\dot f$ we have chosen. The summand on the right hand side can be written

$$f_1^{G_1}(\psi_{1,t_1}) f^G(\psi_t) = f_1^{G_1}(\psi_{1,t_1}) f^G(\psi_t) (\dot f^U)^{\dot G}(\psi_{\dot t}^U) = \dot f^{\dot G}(\dot\psi_{\dot t}),$$

by the properties we have imposed on $\dot f^U$. The formula (5.10) therefore amounts to a special case of the putative identity (5.4). We can treat it as a resolution of our global induction hypothesis, for functions $\dot f \in \mathcal{H}_U(\dot G)$ of the sort we have been using. As a matter of fact, (5.10) is still valid without the condition on f_1 under which it was derived. This follows as in the discussion of [**A3**, (8.4.17)], namely from the stability of each side, and the fact that the subspace of functions $f_1 \in \mathcal{H}(G_1)$ that satisfy the condition maps *onto* the stable space

$$\mathcal{S}(G_1) = \big\{ f_1^{G_1} :\ f_1 \in \mathcal{H}(G_1) \big\}.$$

It is then not hard to see that (5.4) is valid for *any* function $\dot f \in \mathcal{H}_U(\dot G)$.

It remains to establish (2.9). We have agreed that (5.10) holds for any functions $f_1 \in \mathcal{H}(G_1)$ and $f \in \mathcal{H}(G)$ (and the function $\dot{f}^U \in \widetilde{\mathcal{H}}(\dot{G}^U)$ fixed earlier). Substituting it for the leading term in (5.7), we obtain an expression

$$|\mathcal{S}_\psi|^{-1} \sum_{x \in \mathcal{S}_\psi} \sum_{t \in T(\psi)} \varepsilon'(\dot{\psi}')\, f_1'(\psi_{1,t_1'})\, f'(\psi'_{t'}),$$

which is therefore equal to the sum

$$\sum_{\xi \in \widehat{\mathcal{S}}_\psi} \sum_{t \in T(\psi)} f_{1,G_1}(\pi_{1,\xi,t})\, f_G\big(\sigma_t(\xi)\big)$$

we labeled (5.8) above. In the first expression, we can write

$$\begin{aligned} \varepsilon'(\dot{\psi}') f_1'(\psi'_{1,t_1'}) &= \varepsilon'(\dot{\psi}') \sum_{\xi_1 \in \widehat{\mathcal{S}}_{\psi_1}} \xi_1(s_\psi x) f_{1,G_1}\big(\pi_{1,t_1}(\xi_1)\big) \\ &= \varepsilon'(\dot{\psi}') \sum_{\xi \in \widehat{\mathcal{S}}_\psi} \varepsilon_{\dot{\psi}}(s_{\dot{\psi}} x)^{-1} \xi(s_\psi x)^{-1} f_{1,G_1}(\pi_{1,t,\xi}) \\ &= \sum_{\xi \in \widehat{\mathcal{S}}_\psi} \xi(s_\psi x)^{-1} f_{1,G_1}(\pi_{1,\xi,t}), \end{aligned}$$

by the analogue of (2.9) for (G_1, ψ_1), and the sign Lemma 4.4.1 of [**A3**]. As ξ and t vary, the linear forms $f_{1,G_1}(\pi_{1,t,\xi})$ in $f_1 \in \mathcal{H}(G_1)$ are linearly independent. We fix $t \in T(\psi)$, and also a point $x \in \mathcal{S}_\psi$, and then choose f_1 so that

$$f_{1,G_1}(\pi_{1,\xi,u}) = \begin{cases} \xi(s_\psi x), & \text{if } u = t, \\ 0, & \text{if } u \neq t, \end{cases}$$

for any $\xi \in \widehat{\mathcal{S}}_\psi$ and $u \in T(\psi)$. The two sums in our last expression for (5.7) are over variable points in $\mathcal{S}_\psi$ and $T(\psi)$. The given substitution introduces a third sum over $\widehat{\mathcal{S}}_\psi$. By inversion on the group $\mathcal{S}_\psi$, and with the understanding that (G', ψ', t') maps to the triplet (ψ, x, t) we have fixed, we see that this expression reduces to the left hand side $f'(\psi'_{t'})$ of (2.9). On the other hand, our last expression for (5.8) becomes

$$\sum_{\xi \in \widehat{\mathcal{S}}_\psi} \xi(s_\psi x) f_G\big(\sigma_t(\xi)\big) = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi x, \pi \rangle\, f_G(\pi_t),$$

the right hand side of (2.9). The formula (2.9) is therefore valid for (G, ψ).

We have completed the proof of Theorem 2.1 for local parameters in the subset $\widetilde{\Psi}'_2(G)$ of $\widetilde{\Psi}'(G)$. It remains to deal with parameters ψ in the complement of $\widetilde{\Psi}'_2(G)$ of $\widetilde{\Psi}'(G)$. In this case, ψ is the image of a square integrable parameter $\psi_M \in \widetilde{\Psi}_2(M, \psi)$ for a proper Levi subgroup M of G. It is then possible to define the mappings $t \to \psi_t$ and $(\pi, t) \to \pi_t$ directly from their analogues for M, with the requirement that they be compatible with induction. This leads in turn to analogues $f'_G(\psi_t, s_\psi s)$ and $f_G(\psi_t, u)$ of the two sides of the local intertwining relation [**A3**, (2.4.7)]. The general identity (2.9) of Theorem 2.1 can then be established from the following variant of the local intertwining relation.

Proposition 5.1. *For the given group G of the form* (1.1) *over the local field F, assume that ψ lies in the complement of $\widetilde{\Psi}'_2(G)$ in $\widetilde{\Psi}'(G)$. Then*

$$f'_G(\psi_t, s_\psi s) = f_G(\psi_t, u), \qquad f \in \mathcal{H}(G),\ t \in T(\psi), \tag{5.11}$$

for u and s as in Theorem 2.4.1 of [**A3**].

This is the general analogue of Proposition 8.4.4 for generic parameters ϕ. We leave the details to the reader.

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