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An Introduction to the Trace Formula

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Foreword

These notes are an attempt to provide an entry into a subject that has not been very accessible. The problems of exposition are twofold. It is important to present motivation and background for the kind of problems that the trace formula is designed to solve. However, it is also important to provide the means for acquiring some of the basic techniques of the subject. I have tried to steer a middle course between these two sometimes divergent objectives. The reader should refer to earlier articles [**Lab2**], [**Lan14**], and the monographs [**Sho**], [**Ge**], for different treatments of some of the topics in these notes.

I had originally intended to write fifteen sections, corresponding roughly to fifteen lectures on the trace formula given at the Summer School. These sections comprise what has become Part I of the notes. They include much introductory material, and culminate in what we have called the coarse (or unrefined) trace formula. The coarse trace formula applies to a general connected, reductive algebraic group. However, its terms are too crude to be of much use as they stand.

Part II contains fifteen more sections. It has two purposes. One is to transform the trace formula of Part I into a refined formula, capable of yielding interesting information about automorphic representations. The other is to discuss some of the applications of the refined formula. The sections of Part II are considerably longer and more advanced. I hope that a familiarity with the concepts of Part I will allow a reader to deal with the more difficult topics in Part II. In fact, the later sections still include some introductory material. For example, §16, §22, and §27 contain heuristic discussions of three general problems, each of which requires a further refinement of the trace formula. Section 26 contains a general introduction to Langlands' principle of functoriality, to which many of the applications of the trace formula are directed.

We begin with a discussion of some constructions that are part of the foundations of the subject. In §1 we review the Selberg trace formula for compact quotient. In §2 we introduce the ring $\mathbb{A} = \mathbb{A}_F$ of adeles. We also try to illustrate why adelic algebraic groups $G(\mathbb{A})$, and their quotients $G(F)\backslash G(\mathbb{A})$, are more concrete objects than they might appear at first sight. Section 3 is devoted to examples related to §1 and §2. It includes a brief description of the Jacquet-Langlands correspondence between quaternion algebras and $GL(2)$. This correspondence is a striking example of the kind of application of which the trace formula is capable. It also illustrates the need for a trace formula for noncompact quotient.

In §4, we begin the study of noncompact quotient. We work with a general algebraic group G, since this was a prerequisite for the Summer School. However, we have tried to proceed gently, giving illustrations of a number of basic notions. For example, §5 contains a discussion of roots and weights, and the related objects needed for the study of noncompact quotient. To lend Part I an added appearance of simplicity, we work over the ground field $\mathbb{Q}$, instead of a general number field F.

The rest of Part I is devoted to the general theme of truncation. The problem is to modify divergent integrals so that they converge. At the risk of oversimplifying matters, we have tried to center the techniques of Part I around one basic result, Theorem 6.1. Corollary 10.1 and Theorem 11.1, for example, are direct corollaries of Theorem 6.1, as well as essential steps in the overall construction. Other results in Part I also depend in an essential way on either the statement of Theorem 6.1 or a key aspect of its proof. Theorem 6.1 itself asserts that a truncation of the function

$$K(x,x) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma x), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),$$

is integrable. It is the integral of this function over $G(\mathbb{Q})\backslash G(\mathbb{A})$ that yields a trace formula in the case of compact quotient. The integral of its truncation in the general case is what leads eventually to the coarse trace formula at the end of Part I.

After stating Theorem 6.1 in §6, we summarize the steps required to convert the truncated integral into some semblance of a trace formula. We sketch the proof of Theorem 6.1 in §8. The arguments here, as well as in the rest of Part I, are both geometric and combinatorial. We present them at varying levels of generality. However, with the notable exception of the review of Eisenstein series in §7, we have tried in all cases to give some feeling for what is the essential idea. For example, we often illustrate geometric points with simple diagrams, usually for the special case $G = SL(3)$. The geometry for $SL(3)$ is simple enough to visualize, but often complicated enough to capture the essential point in a general argument. I am indebted to Bill Casselman, and his flair for computer graphics, for the diagrams. The combinatorial arguments are used in conjunction with the geometric arguments to eliminate divergent terms from truncated functions. They rely ultimately on that simplest of cancellation laws, the binomial identity

$$\sum_{F \subset S} (-1)^{|F|} = \begin{cases} 0, & \text{if } S \neq \emptyset, \\ 1, & \text{if } S = \emptyset, \end{cases}$$

which holds for any finite set S (Identity 6.2).

The parallel sections §11 and §15 from the later stages of Part I anticipate the general discussion of §16–21 in Part II. They provide refined formulas for "generic" terms in the coarse trace formula. These formulas are explicit expressions, whose local dependence on the given test function f is relatively transparent. The first problem of refinement is to establish similar formulas for all of the terms. Because the remaining terms are indexed by conjugacy classes and representations that are singular, this problem is more difficult than any encountered in Part I. The solution requires new analytic techniques, both local and global. It also requires extensions of the combinatorial techniques of Part I, which are formulated in §17 as properties of (G, M)-families. We refer the reader to §16–21 for descriptions of the various results, as well as fairly substantial portions of their proofs.

The solution of the first problem yields a refined trace formula. We summarize this new formula in §22, in order to examine why it is still not satisfactory. The problem here is that its terms are not invariant under conjugation of f by elements in $G(\mathbb{A})$. They are in consequence not determined by the values taken by f at irreducible characters. We describe the solution of this second problem in §23. It yields an invariant trace formula, which we derive by modifying the terms in the refined, noninvariant trace formula so that they become invariant in f.

In §24–26 we pause to give three applications of the invariant trace formula. They are, respectively, a finite closed formula for the traces of Hecke operators on certain spaces, a term by term comparison of invariant trace formulas for general linear groups and central simple algebras, and cyclic base change of prime order for $GL(n)$. It is our discussion of base change that provides the opportunity to review Langlands' principle of functoriality.

The comparisons of invariant trace formulas in §25 and §26 are directed at special cases of functoriality. To study more general cases of functoriality, one requires a third refinement of the trace formula.

The remaining problem is that the terms of the invariant trace formula are not stable as linear forms in f. Stability is a subtler notion than invariance, and is part of Langlands' conjectural theory of endoscopy. We review it in §27. In §28 and §29 we describe the last of our three refinements. This gives rise to a stable trace formula, each of whose terms is stable in f. Taken together, the results of §29 can be regarded as a stabilization process, by which the invariant trace formula is decomposed into a stable trace formula, and an error term composed of stable trace formulas on smaller groups. The results are conditional upon the fundamental lemma. The proofs, conditional as they may be, are still too difficult to permit more than passing comment in §29.

The general theory of endoscopy includes a significant number of cases of functoriality. However, its avowed purpose is somewhat different. The principal aim of the theory is to analyze the internal structure of representations of a given group. Our last application is a broad illustration of what can be expected. In §30 we describe a classification of representations of quasisplit classical groups, both local and global, into packets. These results depend on the stable trace formula, and the fundamental lemma in particular. They also presuppose an extension of the stabilization of §29 to twisted groups.

As a means for investigating the general principle of functoriality, the theory of endoscopy has very definite limitations. We have devoted a word after §30 to some recent ideas of Langlands. The ideas are speculative, but they seem also to represent the best hope for attacking the general problem. They entail using the trace formula in ways that are completely new.

These notes are really somewhat of an experiment. The style varies from section to section, ranging between the technical and the discursive. The more difficult topics typically come in later sections. However, the progression is not always linear, or even monotonic. For example, the material in §13–§15, §19–§21, §23, and §25 is no doubt harder than much of the broader discussion in §16, §22, §26, and §27. The last few sections of Part II tend to be more discursive, but they are also highly compressed. This is the price we have had to pay for trying to get close to the frontiers. The reader should feel free to bypass the more demanding passages, at least initially, in order to develop an overall sense of the subject.

It would not have been possible to go very far by insisting on complete proofs. On the other hand, a survey of the results might have left a reader no closer to acquiring any of the basic techniques. The compromise has been to include something representative of as many arguments as possible. It might be a sketch of the general proof, a suggestive proof of some special case, or a geometric illustration by a diagram. For obvious reasons, the usual heading "PROOF" does not appear in the notes. However, each stated result is eventually followed by a small box

□, when the discussion that passes for a proof has come to an end. This ought to make the structure of each section more transparent. My hope is that a determined reader will be able to learn the subject by reinforcing the partial arguments here, when necessary, with the complete proofs in the given references.

Part I. The Unrefined Trace Formula

1. The Selberg trace formula for compact quotient

Suppose that H is a locally compact, unimodular topological group, and that Γ is a discrete subgroup of H. The space $\Gamma\backslash H$ of right cosets has a right H-invariant Borel measure. Let R be the unitary representation of H by right translation on the corresponding Hilbert space $L^2(\Gamma\backslash H)$. Thus,

$$\big(R(y)\phi\big)(x) = \phi(xy), \qquad \phi \in L^2(\Gamma\backslash H),\ x, y \in H.$$

It is a fundamental problem to decompose R explicitly into irreducible unitary representations. This should be regarded as a theoretical guidepost rather than a concrete goal, since one does not expect an explicit solution in general. In fact, even to state the problem precisely requires the theory of direct integrals.

The problem has an obvious meaning when the decomposition of R is discrete. Suppose for example that H is the additive group $\mathbb{R}$, and that Γ is the subgroup of integers. The irreducible unitary representations of $\mathbb{R}$ are the one dimensional characters $x \to \mathrm{e}^{\lambda x}$, where λ ranges over the imaginary axis $i\mathbb{R}$. The representation R decomposes as direct sum over such characters, as λ ranges over the subset $2\pi i\mathbb{Z}$ of $i\mathbb{R}$. More precisely, let $\widehat{R}$ be the unitary representation of $\mathbb{R}$ on $L^2(\mathbb{Z})$ defined by

$$\big(\widehat{R}(y)c\big)(n) = \mathrm{e}^{2\pi i n y} c(n), \qquad c \in L^2(\mathbb{Z}).$$

The correspondence that maps $\phi \in L^2(\mathbb{Z}\backslash\mathbb{R})$ to its set of Fourier coefficients

$$\widehat{\phi}(n) = \int_{\mathbb{Z}\backslash\mathbb{R}} \phi(x)\mathrm{e}^{-2\pi i n x}\mathrm{d}x, \qquad n \in \mathbb{Z},$$

is then a unitary isomorphism from $L^2(\mathbb{Z}\backslash\mathbb{R})$ onto $L^2(\mathbb{Z})$, which intertwines the representations R and $\widehat{R}$. This is of course the Plancherel theorem for Fourier series.

The other basic example to keep in mind occurs where $H = \mathbb{R}$ and $\Gamma = \{1\}$. In this case the decomposition of R is continuous, and is given by the Plancherel theorem for Fourier transforms. The general intuition that can inform us is as follows. For arbitrary H and Γ, there will be some parts of R that decompose discretely, and therefore behave qualitatively like the theory of Fourier series, and others that decompose continuously, and behave qualitatively like the theory of Fourier transforms.

In the general case, we can study R by integrating it against a test function $f \in C_c(H)$. That is, we form the operator

$$R(f) = \int_H f(y)R(y)\mathrm{d}y$$

on $L^2(\Gamma\backslash H)$. We obtain

$$\begin{aligned}\big(R(f)\phi\big)(x) &= \int_H \big(f(y)R(y)\phi\big)(x)\mathrm{d}y \\ &= \int_H f(y)\phi(xy)\mathrm{d}y \\ &= \int_H f(x^{-1}y)\phi(y)dy \\ &= \int_{\Gamma\backslash H} \Big(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\Big)\phi(y)\mathrm{d}y,\end{aligned}$$

for any $\phi \in L^2(\Gamma\backslash H)$ and $x \in H$. It follows that $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y), \qquad x,y \in \Gamma\backslash H. \tag{1.1}$$

The sum over γ is finite for any x and y, since it may be taken over the intersection of the discrete group Γ with the compact subset

$$x\,\mathrm{supp}(f)y^{-1}$$

of H.

For the rest of the section, we consider the special case that $\Gamma\backslash H$ is *compact*. The operator $R(f)$ then acquires two properties that allow us to investigate it further. The first is that R decomposes discretely into irreducible representations π, with finite multiplicities $m(\pi, R)$. This is not hard to deduce from the spectral theorem for compact operators. Since the kernel $K(x,y)$ is a continuous function on the compact space $(\Gamma\backslash H)\times(\Gamma\backslash H)$, and is hence square integrable, the corresponding operator $R(f)$ is of Hilbert-Schmidt class. One applies the spectral theorem to the compact self adjoint operators attached to functions of the form

$$f(x) = (g * g^*)(x) = \int_H g(y)\overline{g(x^{-1}y)}\mathrm{d}y, \qquad g \in C_c(H).$$

The second property is that for many functions, the operator $R(f)$ is actually of trace class, with

$$\mathrm{tr}\, R(f) = \int_{\Gamma\backslash H} K(x,x)\mathrm{d}x. \tag{1.2}$$

If H is a Lie group, for example, one can require that f be smooth as well as compactly supported. Then $R(f)$ becomes an integral operator with smooth kernel on the compact manifold $\Gamma\backslash H$. It is well known that (1.2) holds for such operators.

Suppose that f is such that (1.2) holds. Let $\{\Gamma\}$ be a set of representatives of conjugacy classes in Γ. For any $\gamma \in \Gamma$ and any subset Ω of H, we write Ω_γ for the centralizer of γ in Ω. We can then write

$$\begin{aligned}
\operatorname{tr}\bigl(R(f)\bigr) &= \int_{\Gamma\backslash H} K(x,x)\mathrm{d}x \\
&= \int_{\Gamma\backslash H} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\mathrm{d}x \\
&= \int_{\Gamma\backslash H} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\mathrm{d}x \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash H} f(x^{-1}\gamma x)\mathrm{d}x \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{H_\gamma\backslash H} \int_{\Gamma_\gamma\backslash H_\gamma} f(x^{-1}u^{-1}\gamma u x)\mathrm{d}u\,\mathrm{d}x \\
&= \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash H_\gamma) \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)\mathrm{d}x.
\end{aligned}$$

These manipulations follow from Fubini's theorem, and the fact that for any sequence $H_1 \subset H_2 \subset H$ of unimodular groups, a right invariant measure on $H_1\backslash H$ can be written as the product of right invariant measures on $H_2\backslash H$ and $H_1\backslash H_2$ respectively. We have obtained what may be regarded as a geometric expansion of $\operatorname{tr}\bigl(R(f)\bigr)$ in terms of conjugacy classes γ in Γ. By restricting $R(f)$ to the irreducible subspaces of $L^2(\Gamma\backslash H)$, we obtain a spectral expansion of $R(f)$ in terms of irreducible unitary representations π of H.

The two expansions $\operatorname{tr}\bigl(R(f)\bigr)$ provide an identity of linear forms

$$\sum_\gamma a^H_\Gamma(\gamma) f_H(\gamma) = \sum_\pi a^H_\Gamma(\pi) f_H(\pi), \tag{1.3}$$

where γ is summed over (a set of representatives of) conjugacy classes in Γ, and π is summed over (equivalence classes of) irreducible unitary representatives of H. The linear forms on the geometric side are invariant orbital integrals

$$f_H(\gamma) = \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)\mathrm{d}x, \tag{1.4}$$

with coefficients

$$a^H_\Gamma(\gamma) = \operatorname{vol}(\Gamma_\gamma\backslash H_\gamma),$$

while the linear forms on the spectral side are irreducible characters

$$f_H(\pi) = \operatorname{tr}\bigl(\pi(f)\bigr) = \operatorname{tr}\Bigl(\int_H f(y)\pi(y)\mathrm{d}y\Bigr), \tag{1.5}$$

with coefficients

$$a^H_\Gamma(\pi) = m(\pi, R).$$

This is the Selberg trace formula for compact quotient.

We note that if $H = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, the trace formula (1.3) reduces to the Poisson summation formula. For another example, we could take H to be a finite group and $f(x)$ to be the character $\operatorname{tr}\pi(x)$ of an irreducible representation π of H. In this case, (1.3) reduces to a special case of the Frobenius reciprocity theorem, which applies to the trivial one dimensional representation of the subgroup Γ of H. (A minor extension of (1.3) specializes to the general form of Frobenius reciprocity.)

Some of Selberg's most striking applications of (1.3) were to the group $H = SL(2,\mathbb{R})$ of real, (2×2)-matrices of determinant one. Suppose that X is a compact Riemann surface of genus greater than 1. The universal covering surface of X is then the upper half plane, which we identify as usual with the space of cosets $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. (Recall that the compact orthogonal group $K = SO(2,\mathbb{R})$ is the stabilizer of $\sqrt{-1}$ under the transitive action of $SL(2,\mathbb{R})$ on the upper half plane by linear fractional transformations.) The Riemann surface becomes a space of double cosets

$$X = \Gamma\backslash H/K,$$

where Γ is the fundamental group of X, embedding in $SL(2,\mathbb{R})$ as a discrete subgroup with compact quotient. By choosing left and right K-invariant functions $f \in C_c^\infty(H)$, Selberg was able to apply (1.3) to both the geometry and analysis of X.

For example, closed geodesics on X are easily seen to be bijective with conjugacy classes in Γ. Given a large positive integer N, Selberg chose f so that the left hand side of (1.3) approximated the number $g(N)$ of closed geodesics of length less than N. An analysis of the corresponding right hand side gave him an asymptotic formula for $g(N)$, with a sharp error term. Another example concerns the Laplace-Beltrami operator Δ attached to X. In this case, Selberg chose f so that the right hand side of (1.3) approximated the number $h(N)$ of eigenvalues of Δ less than N. An analysis of the corresponding left hand side then provided a sharp asymptotic estimate for $h(N)$.

The best known discrete subgroup of $H = SL(2,\mathbb{R})$ is the group $\Gamma = SL(2,\mathbb{Z})$ of unimodular integral matrices. In this case, the quotient $\Gamma\backslash H$ is not compact. The example of $\Gamma = SL(2,\mathbb{Z})$ is of special significance because it comes with the supplementary operators introduced by Hecke. Hecke operators include a family of commuting operators $\{T_p\}$ on $L^2(\Gamma\backslash H)$, parametrized by prime numbers p, which commute also with the action of the group $H = SL(2,\mathbb{R})$. The families $\{c_p\}$ of simultaneous eigenvalues of Hecke operators on $L^2(\Gamma\backslash H)$ are known to be of fundamental arithmetic significance. Selberg was able to extend his trace formula (1.3) to this example, and indeed to many other quotients of rank 1. He also included traces of Hecke operators in his formulation. In particular, he obtained a finite closed formula for the trace of T_p on any space of classical modular forms.

Selberg worked directly with Riemann surfaces and more general locally symmetric spaces, so the role of group theory in his papers is less explicit. We can refer the reader to the basic articles [**Sel1**] and [**Sel2**]. However, many of Selberg's results remain unpublished. The later articles [**DL**] and [**JL**, §16] used the language of group theory to formulate and extend Selberg's results for the upper half plane.

In the next section, we shall see how to incorporate the theory of Hecke operators into the general framework of (1.1). The connection is through adele groups, where Hecke operators arise in a most natural way. Our ultimate goal is to describe a general trace formula that applies to any adele group. The modern role of such a trace formula has changed somewhat from the original focus of Selberg. Rather than studying geometric and spectral data attached to a given group in isolation, one tries to compare such data for different groups. In particular, one would like to establish reciprocity laws among the fundamental arithmetic data associated to Hecke operators on different groups.

2. Algebraic groups and adeles

Suppose that G is a connected reductive algebraic group over a number field F. For example, we could take G to be the multiplicative group $GL(n)$ of invertible $(n \times n)$-matrices, and F to be the rational field $\mathbb{Q}$. Our interest is in the general setting of the last section, with Γ equal to $G(F)$. It is easy to imagine that this group could have arithmetic significance. However, it might not be at all clear how to embed Γ discretely into a locally compact group H. To do so, we have to introduce the adele ring of F.

Suppose for simplicity that F equals the rational field $\mathbb{Q}$. We have the usual absolute value $v_\infty(\cdot) = |\cdot|_\infty$ on $\mathbb{Q}$, and its corresponding completion $\mathbb{Q}_{v_\infty} = \mathbb{Q}_\infty = \mathbb{R}$. For each prime number p, there is also a p-adic absolute value $v_p(\cdot) = |\cdot|_p$ on $\mathbb{Q}$, defined by

$$|t|_p = p^{-r}, \qquad t = p^r ab^{-1},$$

for integers r, a and b with $(a,p) = (b,p) = 1$. One constructs its completion $\mathbb{Q}_{v_p} = \mathbb{Q}_p$ by a process identical to that of $\mathbb{R}$. As a matter of fact, $|\cdot|_p$ satisfies an enhanced form of the triangle inequality

$$|t_1 + t_2|_p \leq \max\{|t_1|_p, |t_2|_p\}, \qquad t_1, t_2 \in \mathbb{Q}.$$

This has the effect of giving the compact "unit ball"

$$\mathbb{Z}_p = \{t_p \in \mathbb{Q}_p : \ |t_p|_p \leq 1\}$$

in $\mathbb{Q}_p$ the structure of a *subring* of $\mathbb{Q}_p$. The completions $\mathbb{Q}_v$ are all locally compact fields. However, there are infinitely many of them, so their direct product is not locally compact. One forms instead the restricted direct product

$$\begin{aligned} \mathbb{A} = \prod_v^{\text{rest}} \mathbb{Q}_v &= \mathbb{R} \times \prod_p^{\text{rest}} \mathbb{Q}_p = \mathbb{R} \times \mathbb{A}_{\text{fin}} \\ &= \{t = (t_v) : \ t_p = t_{v_p} \in \mathbb{Z}_p \text{ for almost all } p\}. \end{aligned}$$

Endowed with the natural direct limit topology, $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ becomes a locally compact ring, called the adele ring of $\mathbb{Q}$. The diagonal image of $\mathbb{Q}$ in $\mathbb{A}$ is easily seen to be discrete. It follows that $H = G(\mathbb{A})$ is a locally compact group, in which $\Gamma = G(\mathbb{Q})$ embeds as a discrete subgroup. (See [**Tam2**].)

A similar construction applies to a general number field F, and gives rise to a locally compact ring $\mathbb{A}_F$. The diagonal embedding

$$\Gamma = G(F) \subset G(\mathbb{A}_F) = H$$

exhibits $G(F)$ as a discrete subgroup of the locally compact group $G(\mathbb{A}_F)$. However, we may as well continue to assume that $F = \mathbb{Q}$. This represents no loss of generality, since one can pass from F to $\mathbb{Q}$ by restriction of scalars. To be precise, if G_1 is the algebraic group over $\mathbb{Q}$ obtained by restriction of scalars from F to $\mathbb{Q}$, then $\Gamma = G(F) = G_1(\mathbb{Q})$, and $H = G(\mathbb{A}_F) = G_1(\mathbb{A})$.

We can define an *automorphic representation* π of $G(\mathbb{A})$ informally to be an irreducible representation of $G(\mathbb{A})$ that "occurs in" the decomposition of R. This definition is not precise for the reason mentioned in §1, namely that there could be a part of R that decomposes continuously. The formal definition [**Lan6**] is in fact quite broad. It includes not only irreducible unitary representations of $G(\mathbb{A})$ in the continuous spectrum, but also analytic continuations of such representations.

The introduction of adele groups appears to have imposed a new and perhaps unwelcome level of abstraction onto the subject. The appearance is illusory. Suppose for example that G is a simple group over $\mathbb{Q}$. There are two possibilities: either $G(\mathbb{R})$ is noncompact (as in the case $G = SL(2)$), or it is not. If $G(\mathbb{R})$ is noncompact, the adelic theory for G may be reduced to the study of of arithmetic quotients of $G(\mathbb{R})$. As in the case $G = SL(2)$ discussed at the end of §1, this is closely related to the theory of Laplace-Beltrami operators on locally symmetric Riemannian spaces attached to $G(\mathbb{R})$. If $G(\mathbb{R})$ is compact, the adelic theory reduces to the study of arithmetic quotients of a p-adic group $G(\mathbb{Q}_p)$. This in turn is closely related to the spectral theory of combinatorial Laplace operators on locally symmetric hypergraphs attached to the Bruhat-Tits building of $G(\mathbb{Q}_p)$.

These remarks are consequences of the theorem of strong approximation. Suppose that S is a finite set of valuations of $\mathbb{Q}$ that contains the archimedean valuation v_∞. For any G, the product

$$G(\mathbb{Q}_S) = \prod_{v \in S} G(\mathbb{Q}_v)$$

is a locally compact group. Let K^S be an open compact subgroup of $G(\mathbb{A}^S)$, where

$$\mathbb{A}^S = \big\{ t \in \mathbb{A} : \ t_v = 0, \ v \in S \big\}$$

is the ring theoretic complement of $\mathbb{Q}_S$ in $\mathbb{A}$. Then $G(F_S)K^S$ is an open subgroup of $G(\mathbb{A})$.

Theorem 2.1. (a) *(Strong approximation) Suppose that G is simply connected, in the sense that the topological space $G(\mathbb{C})$ is simply connected, and that $G'(\mathbb{Q}_S)$ is noncompact for every simple factor G' of G over $\mathbb{Q}$. Then*

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{Q}_S) K^S.$$

(b) *Assume only that $G'(\mathbb{Q}_S)$ is noncompact for every simple quotient G' of G over $\mathbb{Q}$. Then the set of double cosets*

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{Q}_S) K^S$$

is finite.

For a proof of (a) in the special case $G = SL(2)$ and $S = \{v_\infty\}$, see [**Shim**, Lemma 6.15]. The reader might then refer to [**Kne**] for a sketch of the general argument, and to [**P**] for a comprehensive treatment. Part (b) is essentially a corollary of (a). □

According to (b), we can write $G(\mathbb{A})$ as a disjoint union

$$G(\mathbb{A}) = \coprod_{i=1}^{n} G(\mathbb{Q}) \cdot x^i \cdot G(\mathbb{Q}_S) K^S,$$

for elements $x^1 = 1, x^2, \dots, x^n$ in $G(\mathbb{A}^S)$. We can therefore write

$$\begin{aligned} G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^S &= \coprod_{i=1}^{n} \big(G(\mathbb{Q}) \backslash G(\mathbb{Q}) \cdot x^i \cdot G(\mathbb{Q}_S) K^S / K^S \big) \\ &\cong \coprod_{i=1}^{n} \big(\Gamma_S^i \backslash G(\mathbb{Q}_S) \big), \end{aligned}$$

for discrete subgroups

$$\Gamma_S^i = G(\mathbb{Q}_S) \cap \big(G(\mathbb{Q}) \cdot x^i K^S (x^i)^{-1}\big)$$

of $G(\mathbb{Q}_S)$. We obtain a $G(\mathbb{Q}_S)$-isomorphism of Hilbert spaces

$$L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})/K^S\big) \cong \bigoplus_{i=1}^{n} L^2\big(\Gamma_S^i \backslash G(\mathbb{Q}_S)\big). \tag{2.1}$$

The action of $G(\mathbb{Q}_S)$ on the two spaces on each side of (2.1) is of course by right translation. It corresponds to the action by right convolution on either space by functions in the algebra $C_c\big(G(\mathbb{Q}_S)\big)$. There is a supplementary convolution algebra, the Hecke algebra $\mathcal{H}\big(G(\mathbb{A}^S), K^S\big)$ of compactly supported functions on $G(\mathbb{A}^S)$ that are left and right invariant under translation by K^S. This algebra acts by right convolution on the left hand side of (2.1), in a way that clearly commutes with the action of $G(\mathbb{Q}_S)$. The corresponding action of $\mathcal{H}\big(G(\mathbb{A}^S), K^S\big)$ on the right hand side of (2.1) includes general analogues of the operators defined by Hecke on classical modular forms.

This becomes more concrete if $S = \{v_\infty\}$. Then $\mathbb{A}^S$ equals the subring $\mathbb{A}_{\text{fin}} = \{t \in \mathbb{A} : \ t_\infty = 0\}$ of "finite adeles" in $\mathbb{A}$. If G satisfies the associated noncompactness criterion of Theorem 2.1(b), and K_0 is an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$, we have a $G(\mathbb{R})$-isomorphism of Hilbert spaces

$$L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})/K_0\big) \cong \bigoplus_{i=1}^{n} L^2\big(\Gamma^i \backslash G(\mathbb{R})\big),$$

for discrete subgroups $\Gamma^1, \dots, \Gamma^n$ of $G(\mathbb{R})$. The Hecke algebra $\mathcal{H}\big(G(\mathbb{A}_{\text{fin}}), K_0\big)$ acts by convolution on the left hand side, and hence also on the right hand side.

Hecke operators are really at the heart of the theory. Their properties can be formulated in representation theoretic terms. Any automorphic representation π of $G(\mathbb{A})$ can be decomposed as a restricted tensor product

$$\pi = \bigotimes_v \pi_v, \tag{2.2}$$

where π_v is an irreducible representation of the group $G(\mathbb{Q}_v)$. Moreover, for every valuation $v = v_p$ outside some finite set S, the representation $\pi_p = \pi_{v_p}$ is *unramified*, in the sense that its restriction to a suitable maximal compact subgroup K_p of $G(\mathbb{Q}_p)$ contains the trivial representation. (See [**F**]. It is known that the trivial representation of K_p occurs in π_p with multiplicity at most one.) This gives rise to a maximal compact subgroup $K^S = \prod_{p \notin S} K_p$, a Hecke algebra

$$\mathcal{H}^S = \bigotimes_{p \notin S} \mathcal{H}_p = \bigotimes_{p \notin S} \mathcal{H}\big(G(\mathbb{Q}_p), K_p\big)$$

that is actually abelian, and an algebra homomorphism

$$c(\pi^S) = \bigotimes_{p \notin S} c(\pi_p) : \ \mathcal{H}^S = \bigotimes_{p \notin S} \mathcal{H}_p \ \longrightarrow \ \mathbb{C}. \tag{2.3}$$

Indeed, if $v^S = \bigotimes_{p\notin S} v_p$ belongs to the one-dimensional space of K^S-fixed vectors for the representation $\pi^S = \bigotimes_{p\notin S} \pi_p$, and $h^S = \bigotimes_{p\notin S} h_p$ belongs to $\mathcal{H}^S$, the vector

$$\pi^S(h^S)v^S = \bigotimes_{p\notin S} \big(\pi_p(h_p)v_p\big)$$

equals

$$c(\pi^S, h^S)v^S = \bigotimes_{p\notin S} \big(c(\pi_p, h_p)v_p\big).$$

This formula defines the homomorphism (2.3) in terms of the unramified representation π^S. Conversely, for any homomorphism $\mathcal{H}^S \to \mathbb{C}$, it is easy to see that there is a unique unramified representation π^S of $G(\mathbb{A}^S)$ for which the formula holds.

The decomposition (2.2) actually holds for general irreducible representations π of $G(\mathbb{A})$. In this case, the components can be arbitrary. However, the condition that π be automorphic is highly rigid. It imposes deep relationships among the different unramified components π_p, or equivalently, the different homomorphisms $c(\pi_p):\ \mathcal{H}_p \to \mathbb{C}$. These relationships are expected to be of fundamental arithmetic significance. They are summarized by Langlands's principle of functoriality [**Lan3**], and his conjecture that relates automorphic representations to motives [**Lan7**]. (For an elementary introduction to these conjectures, see [**A28**]. We shall review the principle of functoriality and its relationship with unramified representations in §26.) The general trace formula provides a means for analyzing some of the relationships.

The group $G(\mathbb{A})$ can be written as a direct product of the real group $G(\mathbb{R})$ with the totally disconnected group $G(\mathbb{A}_{\mathrm{fin}})$. We define

$$C_c^\infty\big(G(\mathbb{A})\big) = C_c^\infty\big(G(\mathbb{R})\big) \otimes C_c^\infty\big(G(\mathbb{A}_{\mathrm{fin}})\big),$$

where $C_c^\infty\big(G(\mathbb{R})\big)$ is the usual space of smooth, compactly supported functions on the Lie group $G(\mathbb{R})$, and $C_c^\infty\big(G(\mathbb{A}_{\mathrm{fin}})\big)$ is the space of locally constant, compactly supported, complex valued functions on the totally disconnected group $G(\mathbb{A}_{\mathrm{fin}})$. The vector space $C_c^\infty\big(G(\mathbb{A})\big)$ is an algebra under convolution, which is of course contained in the algebra $C_c\big(G(\mathbb{A})\big)$ of continuous, compactly supported functions on $G(\mathbb{A})$.

Suppose that f belongs to $C_c^\infty\big(G(\mathbb{A})\big)$. We can choose a finite set of valuations S satisfying the condition of Theorem 2.1(b), an open compact subgroup K^S of $G(\mathbb{A}^S)$, and an open compact subgroup $K_{0,S}$ of the product

$$G(\mathbb{Q}_S^\infty) = \prod_{v\in S-\{v_\infty\}} G(\mathbb{Q}_v)$$

such that f is bi-invariant under the open compact subgroup $K_0 = K_{0,S}K^S$ of $G(\mathbb{A}_{\mathrm{fin}})$. In particular, the operator $R(f)$ vanishes on the orthogonal complement of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})/K^S\big)$ in $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. We leave the reader the exercise of using (1.1) and (2.1) to identify $R(f)$ with an integral operator with smooth kernel on a finite disjoint union of quotients of $G(\mathbb{R})$.

Suppose, in particular, that $G(\mathbb{Q})\backslash G(\mathbb{A})$ happens to be compact. Then $R(f)$ may be identified with an integral operator with smooth kernel on a compact manifold. It follows that $R(f)$ is an operator of trace class, whose trace is given by

(1.2). The Selberg trace formula (1.3) is therefore valid for f, with $\Gamma = G(\mathbb{Q})$ and $H = G(\mathbb{A})$. (See [**Tam1**].)

3. Simple examples

We have tried to introduce adele groups as gently as possible, using the relations between Hecke operators and automorphic representations as motivation. Nevertheless, for a reader unfamiliar with such matters, it might take some time to feel comfortable with the general theory. To supplement the discussion of §2, and to acquire some sense of what one might hope to obtain in general, we shall look at a few concrete examples.

Consider first the simplest example of all, the case that G equals the multiplicative group $GL(1)$. Then $G(\mathbb{Q}) = \mathbb{Q}^*$, while

$$G(\mathbb{A}) = \mathbb{A}^* = \{x \in \mathbb{A} : \ |x| \neq 0, \ |x_p|_p = 1 \text{ for almost all } p\}$$

is the multiplicative group of *ideles* for $\mathbb{Q}$. If N is a positive integer with prime factorization $N = \prod_p p^{\mathrm{e}_p(N)}$, we write

$$K_N = \{k \in G(\mathbb{A}_{\mathrm{fin}}) = \mathbb{A}^*_{\mathrm{fin}} : \ |k_p - 1|_p \leq p^{-\mathrm{e}_p(N)} \text{ for all } p\}.$$

A simple exercise for a reader unfamiliar with adeles is to check directly that K_N is an open compact subgroup of $\mathbb{A}^*_{\mathrm{fin}}$, that any open compact subgroup K_0 contains K_N for some N, and that the abelian group

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_N = \mathbb{Q}^*\backslash\mathbb{A}^*/\mathbb{R}^*K_N$$

is finite. The quotient $G(\mathbb{Q})\backslash G(\mathbb{A}) = \mathbb{Q}^*\backslash\mathbb{A}^*$ is not compact. This is because the mapping

$$x \longrightarrow |x| = \prod_v |x_v|_v, \qquad x \in \mathbb{A}^*,$$

is a continuous surjective homomorphism from $\mathbb{A}^*$ to the multiplicative group $(\mathbb{R}^*)^0$ of positive real numbers, whose kernel

$$\mathbb{A}^1 = \{x \in \mathbb{A} : \ |x| = 1\}$$

contains $\mathbb{Q}^*$. The quotient $\mathbb{Q}^*\backslash\mathbb{A}^1$ is compact. Moreover, we can write the group $\mathbb{A}^*$ as a canonical direct product of $\mathbb{A}^1$ with the group $(\mathbb{R}^*)^0$. The failure of $\mathbb{Q}^*\backslash\mathbb{A}^*$ to be compact is therefore entirely governed by the multiplicative group $(\mathbb{R}^*)^0$ of positive real numbers.

An irreducible unitary representation of the abelian group $GL(1,\mathbb{A}) = \mathbb{A}^*$ is a homomorphism

$$\pi : \ \mathbb{A}^* \longrightarrow U(1) = \{z \in \mathbb{C}^* : \ |z| = 1\}.$$

There is a free action

$$s : \ \pi \longrightarrow \pi_s(x) = \pi(x)|x|^s, \qquad s \in i\mathbb{R},$$

of the additive group $i\mathbb{R}$ on the set of such π. The orbits of $i\mathbb{R}$ are bijective under the restriction mapping from $\mathbb{A}^*$ to $\mathbb{A}^1$ with the set of irreducible unitary representations of $\mathbb{A}^1$. A similar statement applies to the larger set of irreducible (not necessarily unitary) representations of $\mathbb{A}^*$, except that one has to replace $i\mathbb{R}$ with the additive group $\mathbb{C}$.

Returning to the case of a general group over $\mathbb{Q}$, we write A_G for the largest central subgroup of G over $\mathbb{Q}$ that is a $\mathbb{Q}$-split torus. In other words, A_G is $\mathbb{Q}$-isomorphic to a direct product $GL(1)^k$ of several copies of $GL(1)$. The connected component $A_G(\mathbb{R})^0$ of 1 in $A_G(\mathbb{R})$ is isomorphic to the multiplicative group $\big((\mathbb{R}^*)^0\big)^k$, which in turn is isomorphic to the additive group $\mathbb{R}^k$. We write $X(G)_{\mathbb{Q}}$ for the additive group of homomorphisms $\chi:\ g \to g^\chi$ from G to $GL(1)$ that are defined over $\mathbb{Q}$. Then $X(G)_{\mathbb{Q}}$ is a free abelian group of rank k. We also form the real vector space

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}\big(X(G)_{\mathbb{Q}}, \mathbb{R}\big)$$

of dimension k. There is then a surjective homomorphism

$$H_G:\ G(\mathbb{A}) \longrightarrow \mathfrak{a}_G,$$

defined by

$$\big\langle H_G(x), \chi\big\rangle = \big|\log(x^\chi)\big|, \qquad x \in G(\mathbb{A}),\ \chi \in X(G)_{\mathbb{Q}}.$$

The group $G(\mathbb{A})$ is a direct product of the normal subgroup

$$G(\mathbb{A})^1 = \big\{x \in G(\mathbb{A}):\ H_G(x) = 0\big\}$$

with $A_G(\mathbb{R})^0$.

We also have the dual vector space $\mathfrak{a}_G^* = X(G)_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{R}$, and its complexification $\mathfrak{a}_{G,\mathbb{C}}^* = X(G)_{\mathbb{Q}} \otimes \mathbb{C}$. If π is an irreducible unitary representation of $G(\mathbb{A})$ and λ belongs to $i\mathfrak{a}_G^*$, the product

$$\pi_\lambda(x) = \pi(x)e^{\lambda(H_G(x))}, \qquad x \in G(\mathbb{A}),$$

is another irreducible unitary representation of $G(\mathbb{A})$. The set of associated $i\mathfrak{a}_G^*$-orbits is in bijective correspondence under the restriction mapping from $G(\mathbb{A})$ to $G(\mathbb{A})^1$ with the set of irreducible unitary representations of $G(\mathbb{A})^1$. A similar assertion applies the larger set of irreducible (not necessary unitary) representations, except that one has to replace $i\mathfrak{a}_G^*$ with the complex vector space $\mathfrak{a}_{G,\mathbb{C}}^*$.

In the case $G = GL(n)$, for example, we have

$$A_{GL(n)} = \left\{\begin{pmatrix} z & & 0 \\ & \ddots & \\ 0 & & z \end{pmatrix}:\ z \in GL(1)\right\} \cong GL(1).$$

The abelian group $X\big(GL(n)\big)_{\mathbb{Q}}$ is isomorphic to $\mathbb{Z}$, with canonical generator given by the determinant mapping from $GL(n)$ to $GL(1)$. The adelic group $GL(n,\mathbb{A})$ is a direct product of the two groups

$$GL(n,\mathbb{A})^1 = \big\{x \in GL(n,\mathbb{A}):\ |\det(x)| = 1\big\}$$

and

$$A_{GL(n)}(\mathbb{R})^0 = \left\{\begin{pmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{pmatrix}:\ r \in (\mathbb{R}^*)^0\right\}.$$

In general, $G(\mathbb{Q})$ is contained in the subgroup $G(\mathbb{A})^1$ of $G(\mathbb{A})$. The group $A_G(\mathbb{R})^0$ is therefore an immediate obstruction to $G(\mathbb{Q})\backslash G(\mathbb{A})$ being compact, as indeed it was in the simplest example of $G = GL(1)$. The real question is then whether the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact. When the answer is affirmative, the discussion above tells us that the trace formula (1.3) can be applied. It holds for $\Gamma = G(\mathbb{Q})$ and $H = G(\mathbb{A})^1$, with f being the restriction to $G(\mathbb{A})^1$ of a function in $C_c^\infty\big(G(\mathbb{A})\big)$.

The simplest nonabelian example that gives compact quotient is the multiplicative group

$$G = \{x \in A : \ x \neq 0\}$$

of a quaternion algebra over $\mathbb{Q}$. By definition, A is a four dimensional division algebra over $\mathbb{Q}$, with center $\mathbb{Q}$. It can be written in the form

$$A = \{x = x_0 + x_1 i + x_2 j + x_3 k : \ x_\alpha \in \mathbb{Q}\},$$

where the basis elements 1, i, j and k satisfy

$$ij = -ji = k, \quad i^2 = a, \quad j^2 = b,$$

for nonzero elements $a, b \in \mathbb{Q}^*$. Conversely, for any pair $a, b \in \mathbb{Q}^*$, the $\mathbb{Q}$-algebra defined in this way is either a quaternion algebra or is isomorphic to the matrix algebra $M_2(\mathbb{Q})$. For example, if $a = b = -1$, A is a quaternion algebra, since $A \otimes_{\mathbb{Q}} \mathbb{R}$ is the classical Hamiltonian quaternion algebra over $\mathbb{R}$. On the other hand, if $a = b = 1$, the mapping

$$x \longrightarrow x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is an isomorphism from A onto $M_2(\mathbb{Q})$. For any A, one defines an automorphism

$$x \longrightarrow \bar{x} = x_0 - x_1 i - x_2 j - x_3 k$$

of A, and a multiplicative mapping

$$x \longrightarrow N(x) = x\bar{x} = x_0 - a x_1^2 - b x_2^2 + ab x_3^2$$

from A to $\mathbb{Q}$. If $N(x) \neq 0$, x^{-1} equals $N(x)^{-1}\bar{x}$. It follows that $x \in A$ is a unit if and only if $N(x) \neq 0$.

The description of a quaternion algebra A in terms of rational numbers $a, b \in \mathbb{Q}^*$ has the obvious attraction of being explicit. However, it is ultimately unsatisfactory. Among other things, different pairs a and b can yield the same algebra A. There is a more canonical characterization in terms of the completions $A_v = A \otimes_{\mathbb{Q}} \mathbb{Q}_v$ at valuations v of $\mathbb{Q}$. If $v = v_\infty$, we know that A_v is isomorphic to either the matrix ring $M_2(\mathbb{R})$ or the Hamiltonian quaternion algebra over $\mathbb{R}$. A similar property holds for any other v. Namely, there is exactly one isomorphism class of quaternion algebras over $\mathbb{Q}_v$, so there are again two possibilities for A_v. Let V be the set of valuations v such that A_v is a quaternion algebra. It is then known that V is a finite set of *even* order. Conversely, for any nonempty set V of even order, there is a unique isomorphism class of quaternion algebras A over $\mathbb{Q}$ such that A_v is a quaternion algebra for each $v \in V$ and a matrix algebra $M_2(\mathbb{Q}_v)$ for each v outside V.

We digress for a moment to note that this characterization of quaternion algebras is part of a larger classification of reductive algebraic groups. The general classification over a number field F, and its completions F_v, is a beautiful union of class field theory with the structure theory of reductive groups. One begins with a group G_s^* over F that is split, in the sense that it has a maximal torus that splits over F. By a basic theorem of Chevalley, the groups G_s^* are in bijective correspondence with reductive groups over an algebraic closure $\overline{F}$ of F, the classification of which reduces largely to that of complex semisimple Lie algebras. The general group G over F is obtained from G_s^* by twisting the action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ by automorphisms of G_s^*. It is a two stage process. One first constructs an "outer twist" G^* of G_s^* that is quasisplit, in the sense that it has a Borel subgroup that is defined over F. This is the easier step. It reduces to a knowledge of the group of outer automorphisms of G_s^*, something that is easy to describe in terms of the general structure of reductive groups. One then constructs an "inner twist" $G \xrightarrow{\psi} G^*$, where ψ is an isomorphism such that for each $\sigma \in \mathrm{Gal}(\overline{F}/F)$, the composition

$$\alpha(\sigma) = \psi \circ \sigma(\psi)^{-1}$$

belongs to the group $\mathrm{Int}(G^*)$ of inner automorphisms of G^*. The role of class field theory is to classify the functions $\sigma \to \alpha(\sigma)$. More precisely, class field theory allows us to characterize the equivalence classes of such functions defined by the Galois cohomology set

$$H^1\big(F, \mathrm{Int}(G^*)\big) = H^1\big(\mathrm{Gal}(\overline{F}/F), \mathrm{Int}(G)^*(\overline{F})\big).$$

It provides a classification of the finite sets of local inner twists $H^1\big(F_v, \mathrm{Int}(G_v^*)\big)$, and a characterization of the image of the map

$$H^1\big(F, \mathrm{Int}(G^*)\big) \hookrightarrow \prod_v H^1\big(F, \mathrm{Int}(G_v^*)\big)$$

in terms of an explicit generalization of the parity condition for quaternion algebras. The map is injective, by the Hasse principle for the adjoint group $\mathrm{Int}(G^*)$. Its image therefore classifies the isomorphism classes of inner twists G of G^* over F.

In the special case above, the classification of quaternion algebras A is equivalent to that of the algebraic groups A^*. In this case, $G^* = G_s^* = GL(2)$. In general, the theory is not especially well known, and goes beyond what we are assuming for this course. However, as a structural foundation for the Langlands program, it is well worth learning. A concise reference for a part of the theory is [**Ko5**, §1-2].

Let G be the multiplicative group of a quaternion algebra A over $\mathbb{Q}$, as above. The restriction of the norm mapping N to G is a generator of the group $X(G)_{\mathbb{Q}}$. In particular,

$$G(\mathbb{A})^1 = \big\{x \in G(\mathbb{A}) : \ |N(x)| = 1\big\}.$$

It is then not hard to see that the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact. (The reason is that G has no proper parabolic subgroup over $\mathbb{Q}$, a point we shall discuss in the next section.) The Selberg trace formula (1.3) therefore holds for $\Gamma = G(\mathbb{Q})$, $H = G(\mathbb{A})^1$, and f the restriction to $G(\mathbb{A})^1$ of a function in $C_c^\infty\big(G(\mathbb{A})\big)$. If $\Gamma(G)$ denotes the set of conjugacy classes in $G(\mathbb{Q})$, and $\Pi(G)$ is the set of equivalence classes of automorphic representations of G (or more properly, restrictions to $G(\mathbb{A})^1$ of automorphic representations of $G(\mathbb{A})$), we have

$$\sum_{\gamma \in \Gamma(G)} a^G(\gamma) f_G(\gamma) = \sum_{\pi \in \Pi(G)} a^G(\pi) f_G(\pi), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big), \tag{3.1}$$

for the volume $a^G(\gamma) = a_\Gamma^H(\gamma)$, the multiplicity $a^G(\pi) = a_\Gamma^H(\pi)$, the orbital integral $f_G(\gamma) = f_H(\gamma)$, and the character $f_G(\pi) = f_H(\pi)$. Jacquet and Langlands gave a striking application of this formula in §16 of their monograph [**JL**].

Any function in $C_c^\infty\big(G(\mathbb{A})\big)$ is a finite linear combination of products

$$f = \prod_v f_v, \qquad f_v \in C_c^\infty\big(G(\mathbb{Q}_v)\big).$$

Assume that f is of this form. Then $f_G(\gamma)$ is a product of local orbital integrals $f_{v,G}(\gamma_v)$, where γ_v is the image of γ in the set $\Gamma(G_v)$ of conjugacy classes in $G(\mathbb{Q}_v)$, and $f_G(\pi)$ is a product of local characters $f_{v,G}(\pi_v)$, where π_v is the component of π in the set $\Pi(G_v)$ of equivalence classes of irreducible representations of $G(\mathbb{Q}_v)$. Let V be the even set of valuations v such that G is not isomorphic to the group $G^* = GL(2)$ over $\mathbb{Q}_v$. If v does not belong to V, the $\mathbb{Q}_v$-isomorphism from G to G^* is determined up to inner automorphisms. There is consequently a canonical bijection $\gamma_v \to \gamma_v^*$ from $\Gamma(G_v)$ to $\Gamma(G_v^*)$, and a canonical bijection $\pi_v \to \pi_v^*$ from $\Pi(G_v)$ to $\Pi(G_v^*)$. One can therefore define a function $f_v^* \in C_c^\infty(G_v^*)$ for every $v \notin V$ such that

$$f_{v,G^*}^*(\gamma_v^*) = f_{v,G}(\gamma_v)$$

and

$$f_{v,G^*}^*(\pi_v^*) = f_{v,G}(\pi_v),$$

for every $\gamma_v \in \Gamma(G_v)$ and $\pi_v \in \Pi(G_v)$. This suggested to Jacquet and Langlands the possibility of comparing (3.1) with the trace formula Selberg had obtained for the group $G^* = GL(2)$ with noncompact quotient.

If v belongs to V, $G(\mathbb{Q}_v)$ is the multiplicative group of a quaternion algebra over $\mathbb{Q}_v$. In this case, there is a canonical bijection $\gamma_v \to \gamma_v^*$ from $\Gamma(G_v)$ onto the set $\Gamma_{\mathrm{ell}}(G_v^*)$ of semisimple conjugacy classes in $G^*(\mathbb{Q}_v)$ that are either central, or do not have eigenvalues in $\mathbb{Q}_v$. Moreover, there is a global bijection $\gamma \to \gamma^*$ from $\Gamma(G)$ onto the set of semisimple conjugacy classes $\gamma^* \in \Gamma(G^*)$ such that for every $v \in V$, γ_v^* belongs to $\Gamma_{\mathrm{ell}}(G_v^*)$. For each $v \in V$, Jacquet and Langlands assigned a function $f_v^* \in C_c^\infty\big(G^*(\mathbb{Q}_v)\big)$ to f_v such that

$$f_{v,G^*}^*(\gamma_v^*) = \begin{cases} f_{v,G}(\gamma_v), & \text{if } \gamma_v^* \in \Gamma_{\mathrm{ell}}(G_v^*), \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

for every (strongly) regular class $\gamma_v^* \in \Gamma_{\mathrm{reg}}(G_v^*)$. (An element is *strongly regular* if its centralizer is a maximal torus. The strongly regular orbital integrals of f_v^* are known to determine the value taken by f_v^* at any invariant distribution on $G^*(\mathbb{Q}_v)$.) This allowed them to attach a function

$$f^* = \prod_v f_v^*$$

in $C_c^\infty\big(G^*(\mathbb{A})\big)$ to the original function f. They then observed that

$$f_{G^*}^*(\gamma^*) = \begin{cases} f_G(\gamma), & \text{if } \gamma^* \text{ is the image of } \gamma \in \Gamma(G), \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

for any class $\gamma^* \in \Gamma(G^*)$.

It happens that Selberg's formula for the group $G^* = GL(2)$ contains a number of supplementary terms, in addition to analogues of the terms in (3.1). However, Jacquet and Langlands observed that the local vanishing conditions (3.2) force all of the supplementary terms to vanish. They then used (3.3) to deduce that the remaining terms on the geometric side equaled the corresponding terms on the geometric side of (3.1). This left only a spectral identity

$$\sum_{\pi \in \Pi(G)} m(\pi, R)\mathrm{tr}\big(\pi(f)\big) = \sum_{\pi^* \in \Pi(G^*)} m(\pi^*, R_{\mathrm{disc}}^*)\mathrm{tr}\big(\pi^*(f^*)\big), \tag{3.4}$$

where R_{disc}^* is the subrepresentation of the regular representation of $G^*(\mathbb{A})^1$ on $L^2\big(G^*(\mathbb{Q})\backslash G^*(\mathbb{A})^1\big)$ that decomposes discretely. By setting $f = f_S f^S$, for a fixed finite set S of valuations containing $V \cup \{v_\infty\}$, and a fixed function $f_S \in C_c^\infty\big(G(\mathbb{Q}_S)\big)$, one can treat (3.4) as an identity of linear forms in a variable function f^S belonging to the Hecke algebra $\mathcal{H}(G^S, K^S)$. Jacquet and Langlands used it to establish an injective global correspondence $\pi \to \pi^*$ of automorphic representations, with $\pi_v^* = \pi_v$ for each $v \notin V$. They also obtained an injective local correspondence $\pi_v \to \pi_v^*$ of irreducible representations for each $v \in V$, which is compatible with the global correspondence, and also the local correspondence $f_v \to f_v^*$ of functions. Finally, they gave a simple description of the images of both the local and global correspondences of representations.

The Jacquet-Langlands correspondence is remarkable for both the power of its assertions and the simplicity of its proof. It tells us that the arithmetic information carried by unramified components π_p of automorphic representations π of $G(\mathbb{A})$, whatever form it might take, is included in the information carried by automorphic representations π^* of $G^*(\mathbb{A})$. In the case $v_\infty \notin V$, it also implies a correspondence between spectra of Laplacians on certain compact Riemann surfaces, and discrete spectra of Laplacians on noncompact surfaces. The Jacquet-Langlands correspondence is a simple prototype of the higher reciprocity laws one might hope to deduce from the trace formula. In particular, it is a clear illustration of the importance of having a trace formula for noncompact quotient.

4. Noncompact quotient and parabolic subgroups

If $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is not compact, the two properties that allowed us to derive the trace formula (1.3) fail. The regular representation R does not decompose discretely, and the operators $R(f)$ are not of trace class. The two properties are closely related, and are responsible for the fact that the integral (1.2) generally diverges. To see what goes wrong, consider the case that $G = GL(2)$, and take f to be the restriction to $H = G(\mathbb{A})^1$ of a nonnegative function in $C_c^\infty\big(G(\mathbb{A})\big)$. If the integral (1.2) were to converge, the double integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x$$

would be finite. Using Fubini's theorem to justify again the manipulations of §1, we would then be able to write the double integral as

$$\sum_{\gamma \in \{G(\mathbb{Q})\}} \mathrm{vol}\big(G(\mathbb{Q})_\gamma \backslash G(\mathbb{A})^1_\gamma\big) \int_{G(\mathbb{A})^1_\gamma \backslash G(\mathbb{A})^1} f(x^{-1}\gamma x)\mathrm{d}x.$$

As it happens, however, the summand corresponding to γ is often infinite.

Sometimes the volume of $G(\mathbb{Q})_\gamma\backslash G(\mathbb{A})^1_\gamma$ is infinite. Suppose that $\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$, for a pair of distinct elements γ_1 and γ_2 in $\mathbb{Q}^*$. Then

$$G_\gamma = \left\{ \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} : \ y_1, y_2 \in GL(1) \right\} \cong GL(1) \times GL(1),$$

so that

$$G(\mathbb{A})^1_\gamma \cong \big\{(y_1, y_2) \in (\mathbb{A}^*)^2 : \ |y_1||y_2| = 1\big\},$$

and

$$G(\mathbb{Q})_\gamma\backslash G(\mathbb{A})^1_\gamma \cong (\mathbb{Q}^*\backslash\mathbb{A}^1) \times (\mathbb{Q}^*\backslash\mathbb{A}^*) \cong (\mathbb{Q}^*\backslash\mathbb{A}^1)^2 \times (\mathbb{R}^*)^0.$$

An invariant measure on the left hand quotient therefore corresponds to a Haar measure on the abelian group on the right. Since this group is noncompact, the quotient has infinite volume.

Sometimes the integral over $G(\mathbb{A})^1_\gamma \backslash G(\mathbb{A})^1$ diverges. Suppose that $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$G(\mathbb{A})_\gamma = \left\{ \begin{pmatrix} z & y \\ 0 & z \end{pmatrix} : \ y \in \mathbb{A}, \ z \in \mathbb{A}^* \right\}$$

The computation of the integral

$$\int_{G(\mathbb{A})^1_\gamma \backslash G(\mathbb{A})^1} f(x^{-1}\gamma x) \mathrm{d}x = \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1}\gamma x) \mathrm{d}x$$

is a good exercise in understanding relations among the Haar measures d^*a, du and dx on $\mathbb{A}^*$, $\mathbb{A}$, and $G(\mathbb{A})$, respectively. One finds that the integral equals

$$\int_{G_\gamma(\mathbb{A}) \backslash P_0(\mathbb{A})} \int_{P_0(\mathbb{A}) \backslash G(\mathbb{A})} f(k^{-1}p^{-1}\gamma pk) d_\ell p \mathrm{d}k,$$

where $P_0(\mathbb{A})$ is the subgroup of upper triangular matrices

$$\left\{ p = \begin{pmatrix} a^* & u \\ 0 & b^* \end{pmatrix} : \ a^*, b^* \in \mathbb{A}^*, \ u \in \mathbb{A} \right\},$$

with left Haar measure

$$d_\ell p = |a^*|^{-1} da^* db^* du,$$

and dk is a Borel measure on the compact space $P_0(\mathbb{A}) \backslash G(\mathbb{A})$. The integral then reduces to an expression

$$c(f) \prod_p (1 - p^{-1})^{-1} = c(f) \left(\sum_{n=1}^{\infty} \frac{1}{n} \right),$$

where

$$c(f) = c_0 \int_{P_0(\mathbb{A}) \backslash G(\mathbb{A})} \int_{\mathbb{A}} f \left(k^{-1} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k \right) \mathrm{d}u \mathrm{d}k,$$

for a positive constant c_0. In particular, the integral is generally infinite.

Observe that the nonconvergent terms in the case $G = GL(2)$ both come from conjugacy classes in $GL(2, \mathbb{Q})$ that intersect the parabolic subgroup P_0 of upper triangular matrices. This suggests that rational parabolic subgroups are responsible for the difficulties encountered in dealing with noncompact quotient. Our suspicion is reinforced by the following characterization, discovered independently by Borel and Harish-Chandra [**BH**] and Mostow and Tamagawa [**MT**]. For a general group G over $\mathbb{Q}$, the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$ is noncompact if and only if G has a proper parabolic subgroup P defined over $\mathbb{Q}$.

We review some basic properties of parabolic subgroups, many of which are discussed in the chapter [**Mur**] in this volume. We are assuming now that G is a general connected reductive group over $\mathbb{Q}$. A *parabolic subgroup* of G is an algebraic subgroup P such that $P(\mathbb{C}) \backslash G(\mathbb{C})$ is compact. We consider only parabolic subgroups P that are defined over $\mathbb{Q}$. Any such P has a Levi decomposition $P = MN_P$, which is a semidirect product of a reductive subgroup M of G over $\mathbb{Q}$ with a normal unipotent subgroup N_P of G over $\mathbb{Q}$. The *unipotent radical* N_P is uniquely determined by P, while the *Levi component* M is uniquely determined up to conjugation by $P(\mathbb{Q})$.

Let P_0 be a fixed minimal parabolic subgroup of G over $\mathbb{Q}$, with a fixed Levi decomposition $P_0 = M_0 N_0$. Any subgroup P of G that contains P_0 is a parabolic subgroup that is defined over $\mathbb{Q}$. It is called a *standard parabolic subgroup* (relative to P_0). The set of standard parabolic subgroups of G is finite, and is a set of representatives of the set of all $G(\mathbb{Q})$-conjugacy classes of parabolic subgroups over $\mathbb{Q}$. A standard parabolic subgroup P has a *canonical* Levi decomposition $P = M_P N_P$, where M_P is the unique Levi component of P that contains M_0. Given P, we can form the central subgroup $A_P = A_{M_P}$ of M_P, the real vector space $\mathfrak{a}_P = \mathfrak{a}_{M_P}$, and the surjective homomorphism $H_P = H_{M_P}$ from $M_P(\mathbb{A})$ onto $\mathfrak{a}_P$. In case $P = P_0$, we often write $A_0 = A_{P_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ and $H_0 = H_{P_0}$.

In the example $G = GL(n)$, one takes P_0 to be the Borel subgroup of upper triangular matrices. The unipotent radical N_0 of P_0 is the subgroup of unipotent upper triangular matrices. For the Levi component M_0, one takes the subgroup of diagonal matrices. There is then a bijection

$$P \longleftrightarrow (n_1, \dots, n_p)$$

between standard parabolic subgroups P of $G = GL(n)$ and partitions $(n_1, \dots, n_p)$ of n. The group P is the subgroup of block upper triangular matrices associated to $(n_1, \dots, n_p)$. The unipotent radical of P is the corresponding subgroup

$$N_P = \left\{ \begin{pmatrix} \underline{I_{n_1}|} & & * \\ & \ddots & \\ 0 & & |\overline{I_{n_p}} \end{pmatrix} \right\}$$

of block unipotent matrices, the canonical Levi component is the subgroup

$$M_P = \left\{ m = \begin{pmatrix} \underline{m_1|} & & 0 \\ & \ddots & \\ 0 & & |\overline{m_p} \end{pmatrix} :\ m_i \in GL(n_i) \right\}$$

of block diagonal matrices, while

$$A_P = \left\{ a = \begin{pmatrix} \underline{a_1 I_{n_1}|} & & 0 \\ & \ddots & \\ 0 & & |\overline{a_p I_{n_p}} \end{pmatrix} :\ a_i \in GL(1) \right\}.$$

Naturally, I_k stands here for the identity matrix of rank k. The free abelian group $X(M_P)_{\mathbb{Q}}$ attached to M_P has a canonical basis of rational characters

$$\chi_i :\ m \longrightarrow \det(m_i), \qquad m \in M_P,\ 1 \leq i \leq p.$$

We are free to use the basis $\frac{1}{n_1}\chi_1, \dots, \frac{1}{n_p}\chi_p$ of the vector space $\mathfrak{a}_P^*$, and the corresponding dual basis of $\mathfrak{a}_P$, to identify both $\mathfrak{a}_P^*$ and $\mathfrak{a}_P$ with $\mathbb{R}^p$. With this interpretation, the mapping H_P takes the form

$$H_P(m) = \left(\frac{1}{n_1} \log |\det m_1|, \dots, \frac{1}{n_p} \log |\det m_p| \right), \qquad m \in M_P(\mathbb{A}).$$

It follows that

$$H_P(a) = \big(\log |a_1|, \dots, \log |a_p| \big), \qquad a \in A_P(\mathbb{A}).$$

For general G, we have a variant of the regular representation R for any standard parabolic subgroup P. It is the regular representation R_P of $G(\mathbb{A})$ on $L^2\big(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})\big)$, defined by

$$\big(R_P(y)\phi\big)(x) = \phi(xy), \qquad \phi \in L^2\big(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})\big),\ x, y \in G(\mathbb{A}).$$

Using the language of induced representations, we can write

$$R_P = \mathrm{Ind}_{N_P(\mathbb{A})M_P(\mathbb{Q})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})M_P(\mathbb{Q})}) \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})} \otimes R_{M_P}),$$

where $\mathrm{Ind}_K^H(\cdot)$ denotes a representation of H induced from a subgroup K, and 1_K denotes the trivial one dimensional representation of K. We can of course integrate R_P against any function $f \in C_c^\infty\big(G(\mathbb{A})\big)$. This gives an operator $R_P(f)$ on the Hilbert space $L^2\big(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})\big)$. Arguing as in the special case $R = R_G$ of §1, we find that $R_P(f)$ is an integral operator with kernel

$$K_P(x, y) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(x^{-1}\gamma n y)\mathrm{d}n, \qquad x, y \in N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}). \tag{4.1}$$

We have seen that the diagonal value $K(x, x) = K_G(x, x)$ of the original kernel need not be integrable over $x \in G(\mathbb{Q})\backslash G(\mathbb{A})^1$. We have also suggested that parabolic subgroups are somehow responsible for this failure. It makes sense to try to modify $K(x, x)$ by adding correction terms indexed by proper parabolic subgroups P. The correction terms ought to be supported on some small neighbourhood of infinity, so that they do not affect the values taken by $K(x, x)$ on some large compact subset of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. The diagonal value $K_P(x, x)$ of the kernel of $R_P(f)$ provides a natural function for any P. However, $K_P(x, x)$ is invariant under left translation of x by the group $N_P(\mathbb{A})M_P(\mathbb{Q})$, rather than $G(\mathbb{Q})$. One could try to rectify this defect by summing $K_P(\delta x, \delta x)$ over elements δ in $P(\mathbb{Q})\backslash G(\mathbb{Q})$. However, this sum does not generally converge. Even if it did, the resulting function on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ would not be supported on a small neighbourhood of infinity. The way around this difficulty will be to multiply $K_P(x, x)$ by a certain characteristic function on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$ that is supported on a small neighbourhood of infinity, and which depends on a choice of maximal compact subgroup K of $G(\mathbb{A})$.

In case $G = GL(n)$, the product

$$K = O(n, \mathbb{R}) \times \prod_p GL(n, \mathbb{Z}_p)$$

is a maximal compact subgroup of $G(\mathbb{A})$. According to the Gramm-Schmidt orthogonalization lemma of linear algebra, we can write

$$GL(n, \mathbb{R}) = P_0(\mathbb{R})O(n, \mathbb{R}).$$

A variant of this process, applied to the height function

$$\|v\|_p = \max\{|v_i|_p :\ 1 \leq i \leq n\}, \qquad v \in \mathbb{Q}_p^n,$$

on $\mathbb{Q}_p^n$ instead of the standard inner product on $\mathbb{R}^n$, gives a decomposition

$$GL(n, \mathbb{Q}_p) = P_0(\mathbb{Q}_p)GL(n, \mathbb{Z}_p),$$

for any p. It follows that $GL(n, \mathbb{A})$ equals $P_0(\mathbb{A})K$.

These properties carry over to our general group G. We choose a suitable maximal compact subgroup

$$K = \prod_v K_v, \qquad K_v \subset G(\mathbb{Q}_v),$$

of $G(\mathbb{A})$, with $G(\mathbb{A}) = P_0(\mathbb{A})K$ [**Ti**, (3.3.2), (3.9], [**A5**, p. 9]. We fix K, and consider a standard parabolic subgroup P of G. Since P contains P_0, we obtain a decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K = N_P(\mathbb{A})M_P(\mathbb{A})K = N_P(\mathbb{A})M_P(\mathbb{A})^1 A_P(\mathbb{R})^0 K.$$

We then define a continuous mapping

$$H_P : \ G(\mathbb{A}) \longrightarrow \mathfrak{a}_P$$

by setting

$$H_P(nmk) = H_{M_P}(m), \qquad n \in N_P(\mathbb{A}),\ m \in M_P(\mathbb{A}),\ k \in K.$$

We shall multiply the kernel $K_P(x,x)$ by the preimage under H_P of the characteristic function of a certain cone in $\mathfrak{a}_P$.

5. Roots and weights

We have fixed a minimal parabolic subgroup P_0 of G, and a maximal compact subgroup K of $G(\mathbb{A})$. We want to use these objects to modify the kernel function $K(x,x)$ so that it becomes integrable. To prepare for the construction, as well as for future geometric arguments, we review some properties of roots and weights.

The restriction homomorphism $X(G)_{\mathbb{Q}} \to X(A_G)_{\mathbb{Q}}$ is injective, and has finite cokernel. If $G = GL(n)$, for example, the homomorphism corresponds to the injection $z \to nz$ of $\mathbb{Z}$ into itself. We therefore obtain a canonical linear isomorphism

$$\mathfrak{a}_P^* = X(M_P)_{\mathbb{Q}} \otimes \mathbb{R} \xrightarrow{\ \sim\ } X(A_P)_{\mathbb{Q}} \otimes \mathbb{R}. \tag{5.1}$$

Now suppose that P_1 and P_2 are two standard parabolic subgroups, with $P_1 \subset P_2$. There are then $\mathbb{Q}$-rational embeddings

$$A_{P_2} \subset A_{P_1} \subset M_{P_1} \subset M_{P_2}.$$

The restriction homomorphism $X(M_{P_2})_{\mathbb{Q}} \to X(M_{P_1})_{\mathbb{Q}}$ is injective. It provides a linear injection $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$ and a dual linear surjection $\mathfrak{a}_{P_1} \mapsto \mathfrak{a}_{P_2}$. We write $\mathfrak{a}_{P_1}^{P_2} \subset \mathfrak{a}_{P_1}$ for the kernel of the latter mapping. The restriction homomorphism $X(A_{P_1})_{\mathbb{Q}} \to X(A_{P_2})_{\mathbb{Q}}$ is surjective, and extends to a surjective mapping from $X(A_{P_1})_{\mathbb{Q}} \otimes \mathbb{R}$ to $X(A_{P_2})_{\mathbb{Q}} \otimes \mathbb{R}$. It thus provides a linear surjection $\mathfrak{a}_{P_1}^* \mapsto \mathfrak{a}_{P_2}^*$, and a dual linear injection $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$. Taken together, the four linear mappings yield split exact sequences

$$0 \longrightarrow \mathfrak{a}_{P_2}^* \rightleftarrows \mathfrak{a}_{P_1}^* \longrightarrow \mathfrak{a}_{P_1}^*/\mathfrak{a}_{P_2}^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{a}_{P_1}^{P_2} \longrightarrow \mathfrak{a}_{P_1} \rightleftarrows \mathfrak{a}_{P_2} \longrightarrow 0$$

of real vector spaces. We may therefore write

$$\mathfrak{a}_{P_1} = \mathfrak{a}_{P_2} \oplus \mathfrak{a}_{P_1}^{P_2}$$

and

$$\mathfrak{a}_{P_1}^* = \mathfrak{a}_{P_2}^* \oplus (\mathfrak{a}_{P_1}^{P_2})^*.$$

For any P, we write Φ_P for the set of roots of (P, A_P). We also write $\mathfrak{n}_P$ for the Lie algebra of N_P. Then Φ_P is a finite subset of nonzero elements in $X(A_P)_{\mathbb{Q}}$ that parametrizes the decomposition

$$\mathfrak{n}_P = \bigoplus_{\alpha \in \Phi_P} \mathfrak{n}_\alpha$$

of $\mathfrak{n}_P$ into eigenspaces under the adjoint action

$$\mathrm{Ad}:\ A_P \longrightarrow GL(\mathfrak{n}_P)$$

of A_P. By definition,

$$\mathfrak{n}_\alpha = \big\{X_\alpha \in \mathfrak{n}_P :\ \mathrm{Ad}(a)X_\alpha = a^\alpha X_\alpha,\ a \in A_P\big\},$$

for any $\alpha \in \Phi_P$. We identify Φ_P with a subset of $\mathfrak{a}_P^*$ under the canonical mappings

$$\Phi_P \subset X(A_P)_{\mathbb{Q}} \subset X(A_P)_{\mathbb{Q}} \otimes \mathbb{R} \simeq \mathfrak{a}_P^*.$$

If H belongs to the subspace $\mathfrak{a}_G$ of $\mathfrak{a}_P$, $\alpha(H) = 0$ for each $\alpha \in \Phi_P$, so Φ_P is contained in the subspace $(\mathfrak{a}_P^G)^*$ of $\mathfrak{a}_P^*$. As is customary, we define a vector

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P} (\dim \mathfrak{n}_\alpha)\alpha$$

in $(\mathfrak{a}_P^G)^*$. We leave the reader to check that left and right Haar measures on the group $P(\mathbb{A})$ are related by

$$d_\ell p = \mathrm{e}^{2\rho(H_P(p))} d_r p, \qquad p \in P(\mathbb{A}).$$

In particular, the group $P(\mathbb{A})$ is not unimodular, if $P \neq G$.

We write $\Phi_0 = \Phi_{P_0}$. The pair

$$(V, R) = \big((\mathfrak{a}_{P_0}^G)^*, \Phi_0 \cup (-\Phi_0)\big)$$

is a root system [**Ser2**], for which Φ_0 is a system of positive roots. We write $W_0 = W_0^G$ for the Weyl group of (V, R). It is the finite group generated by reflections about elements in Φ_0, and acts on the vector spaces $V = (\mathfrak{a}_{P_0}^G)^*$, $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$, and $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$. We also write $\Delta_0 \subset \Phi_0$ for the set of simple roots attached to Φ_0. Then Δ_0 is a basis of the real vector space $(\mathfrak{a}_0^G)^* = (\mathfrak{a}_{P_0}^G)^*$. Any element $\beta \in \Phi_0$ can be written uniquely

$$\beta = \sum_{\alpha \in \Delta_0} n_\alpha \alpha,$$

for nonnegative integers n_α. The corresponding set

$$\Delta_0^\vee = \{\alpha^\vee :\ \alpha \in \Delta_0\}$$

of simple coroots is a basis of the vector space $\mathfrak{a}_0^G = \mathfrak{a}_{P_0}^G$. We write

$$\widehat{\Delta}_0 = \{\varpi_\alpha :\ \alpha \in \Delta_0\}$$

for the set of simple weights, and

$$\widehat{\Delta}_0^\vee = \{\varpi_\alpha^\vee :\ \alpha \in \Delta_0\}$$

for the set of simple co-weights. In other words, $\widehat{\Delta}_0$ is the basis of $(\mathfrak{a}_0^G)^*$ dual to $\Delta_0^\vee$, and $\widehat{\Delta}_0^\vee$ is the basis of $\mathfrak{a}_0^G$ dual to Δ_0.

Standard parabolic subgroups are parametrized by subsets of Δ_0. More precisely, there is an order reversing bijection $P \leftrightarrow \Delta_0^P$ between standard parabolic subgroups P of G and subsets Δ_0^P of Δ_0, such that

$$\mathfrak{a}_P = \{H \in \mathfrak{a}_0 :\ \alpha(H) = 0,\ \alpha \in \Delta_0^P\}.$$

For any P, Δ_0^P is a basis of the space $\mathfrak{a}_{P_0}^P = \mathfrak{a}_0^P$. Let Δ_P be the set of linear forms on $\mathfrak{a}_P$ obtained by restriction of elements in the complement $\Delta_0 - \Delta_0^P$ of Δ_0^P in Δ_0. Then Δ_P is bijective with $\Delta_0 - \Delta_0^P$, and any root in Φ_P can be written uniquely as a nonnegative integral linear combination of elements in Δ_P. The set Δ_P is a basis of $(\mathfrak{a}_P^G)^*$. We obtain a second basis of $(\mathfrak{a}_P^G)^*$ by taking the subset

$$\widehat{\Delta}_P = \{\varpi_\alpha :\ \alpha \in \Delta_0 - \Delta_0^P\}$$

of $\widehat{\Delta}_0$. We shall write

$$\Delta_P^\vee = \{\alpha^\vee :\ \alpha \in \Delta_P\}$$

for the basis of $\mathfrak{a}_P^G$ dual to $\widehat{\Delta}_P$, and

$$\widehat{\Delta}_P^\vee = \{\varpi_\alpha^\vee :\ \alpha \in \Delta_P\}$$

for the basis of $\mathfrak{a}_P^G$ dual to Δ_P. We should point out that this notation is not standard if $P \neq P_0$. For in this case, a general element $\alpha \in \Delta_P$ is not part of a root system (as defined in [**Ser2**]), so that $\alpha^\vee$ is not a coroot. Rather, if α is the restriction to $\mathfrak{a}_P$ of the simple root $\beta \in \Delta_0 - \Delta_0^P$, $\alpha^\vee$ is the projection onto $\mathfrak{a}_P$ of the coroot $\beta^\vee$.

We have constructed two bases Δ_P and $\widehat{\Delta}_P$ of $(\mathfrak{a}_P^G)^*$, and corresponding dual bases $\widehat{\Delta}_P^\vee$ and $\Delta_P^\vee$ of $\mathfrak{a}_P^G$, for any P. More generally, suppose that $P_1 \subset P_2$ are two standard parabolic subgroups. Then we can form two bases $\Delta_{P_1}^{P_2}$ and $\widehat{\Delta}_{P_1}^{P_2}$ of $(\mathfrak{a}_{P_1}^{P_2})^*$, and corresponding dual bases $(\widehat{\Delta}_{P_1}^{P_2})^\vee$ and $(\Delta_{P_1}^{P_2})^\vee$ of $\mathfrak{a}_{P_1}^{P_2}$. The construction proceeds in the obvious way from the bases we have already defined. For example, $\Delta_{P_1}^{P_2}$ is the set of linear forms on the subspace $\mathfrak{a}_{P_1}^{P_2}$ of $\mathfrak{a}_{P_1}$ obtained by restricting elements in $\Delta_0^{P_2} - \Delta_0^{P_1}$, while $\widehat{\Delta}_{P_1}^{P_2}$ is the set of linear forms on $\mathfrak{a}_{P_1}^{P_2}$ obtained by restricting elements in $\widehat{\Delta}_{P_1} - \widehat{\Delta}_{P_2}$. We note that $P_1 \cap M_{P_2}$ is a standard parabolic subgroup of the reductive group M_{P_2}, relative to the fixed minimal parabolic subgroup $P_0 \cap M_{P_2}$. It follows from the definitions that

$$\mathfrak{a}_{P_1 \cap M_{P_2}} = \mathfrak{a}_{P_1}, \quad \mathfrak{a}_{P_1 \cap M_{P_2}}^{M_{P_2}} = \mathfrak{a}_{P_1}^{P_2}, \quad \Delta_{P_1 \cap M_{P_2}} = \Delta_{P_1}^{P_2},$$

and

$$\widehat{\Delta}_{P_1 \cap M_{P_2}} = \widehat{\Delta}_{P_1}^{P_2}.$$

Consider again the example of $G = GL(n)$. Its Lie algebra is the space M_n of $(n \times n)$-matrices, with the Lie bracket

$$[X, Y] = XY - YX,$$

and the adjoint action

$$\mathrm{Ad}(g):\ X \longrightarrow gXg^{-1}, \qquad g \in G,\ X \in M_n,$$

of G. The group

$$A_0 = \left\{ a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} :\ a_i \in GL(1) \right\}$$

acts by conjugation on the Lie algebra

$$\mathfrak{n}_0 = \mathfrak{n}_{P_0} = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix} \right\}$$

of N_{P_0}, and

$$\Phi_0 = \{\beta_{ij} : \ a \longrightarrow a_i a_j^{-1}, \ i < j\}.$$

As linear functionals on the vector space

$$\mathfrak{a}_0 = \left\{ u : \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix} : \ u_i \in \mathbb{R} \right\},$$

the roots Φ_0 take the form

$$\beta_{ij}(u) = u_i - u_j, \qquad i < j.$$

The decomposition of a general root in terms of the subset

$$\Delta_0 = \{\beta_i = \beta_{i,i+1} : \ 1 \le i \le n-1\},$$

of simple roots is given by

$$\beta_{ij} = \beta_i + \cdots + \beta_{j-1}, \qquad i < j.$$

The set of coroots equals

$$\Phi_0^\vee = \big\{\beta_{ij}^\vee = e_i - e_j = (\overbrace{\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots, 0, -1}^{j}, 0, \ldots, 0) : \ i < j\big\},$$

where we have identified $\mathfrak{a}_0$ with the vector space $\mathbb{R}^n$, equipped with the standard basis $\mathrm{e}_1, \ldots, \mathrm{e}_n$. The simple coroots form the basis

$$\Delta_0^\vee = \{\beta_i^\vee = \mathrm{e}_i - \mathrm{e}_{i+1} : \ 1 \le i \le n-1\}$$

of the subspace

$$\mathfrak{a}_0^G = \{u \in \mathbb{R}^n : \ \sum u_i = 0\}.$$

The simple weights give the dual basis

$$\widehat{\Delta}_0 = \{\varpi_i : \ 1 \le i \le n-1\},$$

where

$$\varpi_i(u) = \frac{n-i}{n}(u_1 + \cdots + u_i) - \Big(\frac{i}{n}\Big)(u_{i+1} + \cdots + u_n).$$

The Weyl group W_0 of the root system for $GL(n)$ is the symmetric group S_n, acting by permutation of the coordinates of vectors in the space $\mathfrak{a}_0 \cong \mathbb{R}^n$. The dot product on $\mathbb{R}^n$ give a W-invariant inner product $\langle \cdot, \cdot \rangle$ on both $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. It is obvious that

$$\langle \beta_i, \beta_j \rangle \le 0, \qquad i \ne j.$$

We leave to the reader the exercise of showing that

$$\langle \varpi_i, \varpi_j \rangle \ge 0, \qquad 1 \le i, \ j \le n-1.$$

Suppose that $P \subset GL(n)$ corresponds to the partition $(n_1, \ldots, n_p)$ of n. The general embedding $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_0$ we have defined corresponds to the embedding

$$t \longrightarrow \big(\underbrace{t_1, \ldots, t_1}_{n_1}, \underbrace{t_2, \ldots, t_2}_{n_2}, \ldots, \underbrace{t_p, \ldots, t_p}_{n_p}\big), \qquad t \in \mathbb{R}^p,$$

of $\mathbb{R}^p$ into $\mathbb{R}^n$. It follows that

$$\Delta_0^P = \{\beta_i :\ i \neq n_1 + \cdots + n_k,\ 1 \leq k \leq p-1\}.$$

Since Δ_P is the set of restrictions to $\mathfrak{a}_P \subset \mathfrak{a}_0$ of elements in the set

$$\Delta_0 - \Delta_0^P = \{\beta_{n_1}, \beta_{n_1+n_2}, \ldots\},$$

we see that

$$\Delta_P = \{\alpha_i :\ t \rightarrow t_i - t_{i+1},\ 1 \leq i \leq p-1,\ t \in \mathbb{R}^p\}.$$

The example of $G = GL(n)$ provides algebraic intuition. It is useful for readers less familiar with general algebraic groups. However, the truncation of the kernel also requires geometric intuition. For this, the example of $G = SL(3)$ is often sufficient.

The root system for $SL(3)$ is the same as for $GL(3)$. In other words, we can identify $\mathfrak{a}_0$ with the two dimensional subspace

$$\{u \in \mathbb{R}^3 :\ \sum u^i = 0\}$$

of $\mathbb{R}^3$, in which case

$$\Delta_0 = \{\beta_1, \beta_2\} \subset \Phi_0 = \{\beta_1, \beta_2, \beta_1 + \beta_2\},$$

in the notation above. We can also identify $\mathfrak{a}_0$ isometrically with the two dimension Euclidean plane. The singular (one-dimensional) hyperplanes, the coroots $\Phi_0^\vee$, and the simple coweights $(\widehat{\Delta}^0)^\vee$ are then illustrated in the familiar Figures 5.1 and 5.2.

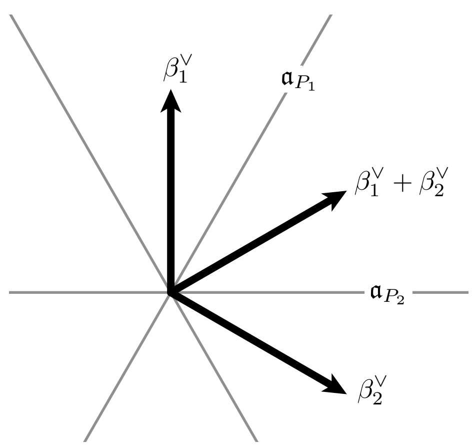

FIGURE 5.1. The two simple coroots $\beta_1^\vee$ and $\beta_2^\vee$ are orthogonal to the respective subspaces $\mathfrak{a}_{P_2}$ and $\mathfrak{a}_{P_1}$ of $\mathfrak{a}_0$. Their inner product is negative, and they span an obtuse angled cone.

There are four standard parabolic subgroups P_0, P_1, P_2, and G, with P_1 and P_2 being the maximal parabolic subgroups such that $\Delta_0^{P_1} = \{\beta_2\}$ and $\Delta_0^{P_2} = \{\beta_1\}$.

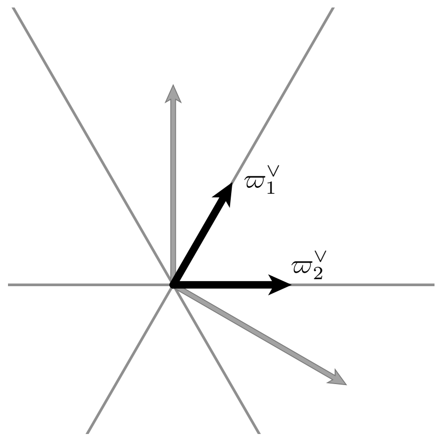

FIGURE 5.2. The two simple coweights $\varpi_1^\vee$ and $\varpi_2^\vee$ lie in the respective subspaces $\mathfrak{a}_{P_1}$ and $\mathfrak{a}_{P_2}$. Their inner product is positive, and they span an acute angled cone.

6. Statement and discussion of a theorem

Returning to the general case, we can now describe how to modify the function $K(x,x)$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$. For a given standard parabolic subgroup P, we write τ_P for the characteristic function of the subset

$$\mathfrak{a}_P^+ = \{t \in \mathfrak{a}_P :\ \alpha(t) > 0,\ \alpha \in \Delta_P\}$$

of $\mathfrak{a}_P$. In the case $G = SL(3)$, this subset is the open cone generated by $\varpi_1^\vee$ and $\varpi_2^\vee$ in Figure 5.2 above. We also write $\widehat{\tau}_P$ for the characteristic function of the subset

$$\{t \in \mathfrak{a}_P :\ \varpi(t) > 0,\ \varpi \in \widehat{\Delta}_P\}$$

of $\mathfrak{a}_P$. In case $G = SL(3)$, this subset is the open cone generated by $\beta_1^\vee$ and $\beta_2^\vee$ in Figure 5.1.

The truncation of $K(x,x)$ depends on a parameter T in the cone $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$ that is suitably regular, in the sense that $\beta(T)$ is large for each root $\beta \in \Delta_0$. For any given T, we define

(6.1)
$$k^T(x) = k^T(x,f) = \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x, \delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big).$$

This is the modified kernel, on which the general trace formula is based. A few remarks might help to put it into perspective.

One has to show that for any x, the sum over δ in (6.1) may be taken over a finite set. In the case $G = SL(2)$, the reader can verify the property as an exercise in reduction theory for modular forms. In general, it is a straightforward consequence [**A3**, Lemma 5.1] of the Bruhat decomposition for G and the construction by Borel and Harish-Chandra of an approximate fundamental domain for $G(\mathbb{Q})\backslash G(\mathbb{A})$. (We shall recall both of these results later.) Thus, $k^T(x)$ is given by a double sum over (P, δ) in a finite set. It is a well defined function of $x \in G(\mathbb{Q})\backslash G(\mathbb{A})$.

Observe that the term in (6.1) corresponding to $P = G$ is just $K(x,x)$. In case $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact, there are no proper parabolic subgroups P (over

$\mathbb{Q})$. Therefore $k^T(x)$ equals $K(x,x)$ in this case, and the truncation operation is trivial. In general, the terms with $P \neq G$ represent functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ that are supported on some neighbourhood of infinity. Otherwise said, $k^T(x)$ equals $K(x,x)$ for x in some large compact subset of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ that depends on T.

Recall that $G(\mathbb{A})$ is a direct product of $G(\mathbb{A})^1$ with $A_G(\mathbb{R})^0$. Observe also that $k^T(x)$ is invariant under translation of x by $A_G(\mathbb{R})^0$. It therefore suffices to study $k^T(x)$ as a function of x in $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

THEOREM 6.1. *The integral*

$$J^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x,f)\mathrm{d}x \tag{6.2}$$

converges absolutely.

Theorem 6.1 does not in itself provide a trace formula. It is really just a first step. We are giving it a central place in our discussion for two reasons. The statement of the theorem serves as a reference point for outlining the general strategy. In addition, the techniques required to prove it will be an essential part of many other arguments.

Let us pause for a moment to outline the general steps that will take us to the end of Part I. We shall describe informally what needs to be done in order to convert Theorem 6.1 into some semblance of a trace formula.

Step 1. *Find spectral expansions for the functions $K(x,y)$ and $k^T(x)$ that are parallel to the geometric expansions (1.1) and (6.1).*

This step is based on Langlands's theory of Eisenstein series. We shall describe it in the next section.

Step 2. *Prove Theorem 6.1.*

We shall sketch the argument in §8.

Step 3. *Show that the function*

$$T \longrightarrow J^T(f),$$

defined a priori for points $T \in \mathfrak{a}_0^+$ that are highly regular, extends to a polynomial in $T \in \mathfrak{a}_0$.

This step allows us to define $J^T(f)$ for any $T \in \mathfrak{a}_0$. It turns out that there is a canonical point $T_0 \in \mathfrak{a}_0$, depending on the choice of K, such that the distribution $J(f) = J^{T_0}(f)$ is independent of the choice of P_0 (though still dependent of the choice of K). For example, if $G = GL(n)$ and K is the standard maximal compact subgroup of $GL(n,\mathbb{A})$, $T_0 = 0$. We shall discuss these matters in §9, making full use of Theorem 6.1.

Step 4. *Convert the expansion (6.1) of $k^T(x)$ in terms of rational conjugacy classes into a geometric expansion of $J(f) = J^{T_0}(f)$.*

We shall give a provisional solution to this problem in §10, as a direct corollary of the proof of Theorem 6.1.

Step 5. *Convert the expansion of $k^T(x)$ in §7 in terms of automorphic representations into a spectral expansion of $J(f) = J^{T_0}(f)$.*

This problem turns out to be somewhat harder than the last one. We shall give a provisional solution in §14, as an application of a truncation operator on functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

We shall call the provisional solutions we obtain for the problems of Steps 4 and 5 the coarse geometric expansion and the coarse spectral expansion, following [**CLL**]. The identity of these two expansions can be regarded as a first attempt at a general trace formula. However, because the terms in the two expansions are still of an essentially global nature, the identity is of little use as it stands. The general problem of refining the two expansions into more tractible local terms will be left until Part II. In order to give some idea of what to expect, we shall deal with the easiest terms near the end of Part I.

In §11, we will rewrite the geometric terms attached to certain semisimple conjugacy classes in $G(\mathbb{Q})$. The distributions so obtained are interesting new linear forms in f, known as *weighted orbital integrals*. In §15, we will rewrite the spectral terms attached to certain induced cuspidal automorphic representations of $G(\mathbb{A})$. The resulting distributions are again new linear forms in f, known as *weighted characters*. This will set the stage for Part II, where one of the main tasks will be to write the entire geometric expansion in terms of weighted orbital integrals, and the entire spectral expansion in terms of weighted characters.

There is a common thread to Part I. It is the proof of Theorem 6.1. For example, the proofs of Corollary 10.1, Theorem 11.1, Proposition 12.2 and parts (ii) and (iii) of Theorem 14.1 either follow directly from, or are strongly motivated by, the proof of Theorem 6.1. Moreover, the actual assertion of Theorem 6.1 is the essential ingredient in the proofs of Theorems 9.1 and 9.4, as well as their geometric analogues in §10 and their spectral analogues in §14. We have tried to emphasize this pattern in order to give the reader some overview of the techniques.

The proof of Theorem 6.1 itself has both geometric and analytic components. However, its essence is largely combinatorial. This is due to the cancellation in (6.1) implicit in the alternating sum over P. At the heart of the proof is the simplest of all cancellation laws, the identity obtained from the binomial expansion of $\big(1+(-1)\big)^n$.

IDENTITY 6.2. *Suppose that S is a finite set. Then*

$$\sum_{F\subset S}(-1)^{|S|-|F|} = \begin{cases} 1 & \text{if } S=\emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

□

7. Eisenstein series

Eisenstein series are responsible for the greatest discrepancy between what we need and what we can prove here. Either of the two main references [**Lan5**] or [**MW2**] presents an enormous challenge to anyone starting to learn the subject. Langlands's survey article [**Lan1**] is a possible entry point. For the trace formula, one can usually make do with a statement of the main theorems on Eisenstein series. We give a summary, following [**A2**, §2].

The role of Eisenstein series is to provide a spectral expansion for the kernel $K(x,y)$. In general, the regular representation R of $G(\mathbb{A})$ on $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ does not decompose discretely. Eisenstein series describe the continuous part of the spectrum.

We write $R_{G,\mathrm{disc}}$ for the restriction of the regular representation of $G(\mathbb{A})^1$ to the subspace $L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ that decomposes discretely.

Since $G(\mathbb{A})$ is a direct product of $G(\mathbb{A})^1$ with $A_G(\mathbb{R})^0$, we can identify $R_{G,\mathrm{disc}}$ with the representation of $G(\mathbb{A})$ on the subspace $L^2_{\mathrm{disc}}\big(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A})\big)$ of $L^2\big(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A})\big)$ that decomposes discretely. For any point $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, the tensor product

$$R_{G,\mathrm{disc},\lambda}(x) = R_{G,\mathrm{disc}}(x)\mathrm{e}^{\lambda(H_G(x))}, \qquad x \in G(\mathbb{A}),$$

is then a representation of $G(\mathbb{A})$, which is unitary if λ lies in $i\mathfrak{a}^*_G$.

We have assumed from the beginning that the invariant measures in use satisfy any obvious compatibility conditions. For example, if P is a standard parabolic subgroup, it is easy to check that the Haar measures on the relevant subgroups of $G(\mathbb{A})$ can be chosen so that

$$\begin{aligned}
&\int_{G(\mathbb{A})} f(x)\mathrm{d}x \\
&= \int_K \int_{P(\mathbb{A})} f(pk)\mathrm{d}_\ell p\mathrm{d}k \\
&= \int_K \int_{M_P(\mathbb{A})} \int_{N_P(\mathbb{A})} f(mnk)\mathrm{d}n\mathrm{d}m\mathrm{d}k \\
&= \int_K \int_{M_P(\mathbb{A})^1} \int_{A_P(\mathbb{R})^0} \int_{N_P(\mathbb{A})} f(mank)\mathrm{d}n\mathrm{d}a\mathrm{d}m\mathrm{d}k,
\end{aligned}$$

for any $f \in C^\infty_c\big(G(\mathbb{A})\big)$. We are assuming implicitly that the Haar measures on K and $N_P(\mathbb{A})$ are normalized so that the spaces K and $N_P(\mathbb{Q})\backslash N_P(\mathbb{A})$ each have volume 1. The Haar measure $\mathrm{d}x$ on $G(\mathbb{A})$ is then determined by Haar measures $\mathrm{d}m$ and $\mathrm{d}a$ on the groups $M_P(\mathbb{A})^1$ and $A_P(\mathbb{R})^0$. We write $\mathrm{d}H$ for the Haar measure on $\mathfrak{a}_P$ that corresponds to $\mathrm{d}a$ under the exponential map. We then write $\mathrm{d}\lambda$ for the Haar measure on $i\mathfrak{a}^*_P$ that is dual to $\mathrm{d}H$, in the sense that

$$\int_{i\mathfrak{a}^*_P} \int_{\mathfrak{a}_P} h(H)\mathrm{e}^{-\lambda(H)}\mathrm{d}H\mathrm{d}\lambda = h(0),$$

for any function $h \in C^\infty_c(\mathfrak{a}_P)$.

Suppose that P is a standard parabolic subgroup of G, and that λ lies in $\mathfrak{a}^*_{P,\mathbb{C}}$. We write

$$y \longrightarrow \mathcal{I}_P(\lambda, y), \qquad y \in G(\mathbb{A}),$$

for the induced representation

$$\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}(I_{N_P(\mathbb{A})} \otimes R_{M_P,\mathrm{disc},\lambda})$$

of $G(\mathbb{A})$ obtained from λ and the discrete spectrum of the reductive group M_P. This representation acts on the Hilbert space $\mathcal{H}_P$ of measurable functions

$$\phi : \; N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that the function

$$\phi_x : \; m \longrightarrow \phi(mx), \qquad m \in M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1,$$

belongs to $L^2_{\mathrm{disc}}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$ for any $x \in G(\mathbb{A})$, and such that

$$\|\phi\|^2 = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2\mathrm{d}m\mathrm{d}k < \infty.$$

For any $y \in G(\mathbb{A})$, $\mathcal{I}_P(\lambda, y)$ maps a function $\phi \in \mathcal{H}_P$ to the function

$$\big(\mathcal{I}_P(\lambda, y)\phi\big)(x) = \phi(xy)\mathrm{e}^{(\lambda+\rho_P)(H_P(xy))}\mathrm{e}^{-(\lambda+\rho_P)(H_P(x))}.$$

We have put the twist by λ into the operator $\mathcal{I}_P(\lambda, y)$ rather than the underlying Hilbert space $\mathcal{H}_P$, in order that $\mathcal{H}_P$ be independent of λ. Recall that the function $\mathrm{e}^{\rho_P(H_P(\cdot))}$ is the square root of the modular function of the group $P(\mathbb{A})$. It is included in the definition in order that the representation $\mathcal{I}_P(\lambda)$ be unitary whenever the inducing representation is unitary, which is to say, whenever λ belongs to the subset $i\mathfrak{a}_P^*$ of $\mathfrak{a}_{P,\mathbb{C}}^*$.

Suppose that

$$R_{M_P,\mathrm{disc}} \cong \bigoplus_\pi \pi \cong \bigoplus_\pi \Big(\bigotimes_v \pi_v\Big)$$

is the decomposition of $R_{M_P,\mathrm{disc}}$ into irreducible representations $\pi = \bigotimes_v \pi_v$ of $M_P(\mathbb{A})/A_P(\mathbb{R})^0$. The induced representation $\mathcal{I}_P(\lambda)$ then has a corresponding decomposition

$$\mathcal{I}_P(\lambda) \cong \bigoplus_\pi \mathcal{I}_P(\pi_\lambda) \cong \bigoplus_\pi \Big(\bigotimes_v \mathcal{I}_P(\pi_{v,\lambda})\Big)$$

in terms of induced representations $\mathcal{I}_P(\pi_{v,\lambda})$ of the local groups $G(\mathbb{Q}_v)$. This follows from the definition of induced representation, and the fact that

$$\mathrm{e}^{\lambda(H_{M_P}(m))} = \prod_v \mathrm{e}^{\lambda(H_{M_P}(m_v))},$$

for any point $m = \prod_v m_v$ in $M_P(\mathbb{A})$. If $\lambda \in i\mathfrak{a}_P^*$ is in general position, all of the induced representations $\mathcal{I}_P(\pi_{v,\lambda})$ are irreducible. Thus, if we understand the decomposition of the discrete spectrum of M_P into irreducible representations of the local groups $M_P(\mathbb{Q}_v)$, we understand the decomposition of the generic induced representations $\mathcal{I}_P(\lambda)$ into irreducible representations of the local groups $G(\mathbb{Q}_v)$.

The aim of the theory of Eisenstein series is to construct intertwining operators between the induced representations $\mathcal{I}_P(\lambda)$ and the continuous part of the regular representation R of $G(\mathbb{A})$. The problem includes being able to construct intertwining operators among the representations $\mathcal{I}_P(\lambda)$, as P and λ vary. The symmetries among pairs (P, λ) are given by the Weyl sets $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ of Langlands. For a given pair P and P' of standard parabolic subgroups, $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is defined as the set of distinct linear isomorphisms from $\mathfrak{a}_P \subset \mathfrak{a}_0$ onto $\mathfrak{a}_{P'} \subset \mathfrak{a}_0$ obtained by restriction of elements in the Weyl group W_0. Suppose, for example that $G = GL(n)$. If P and P' correspond to the partitions $(n_1, \dots, n_p)$ and $(n_1', \dots, n_{p'}')$ of n, the set $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is empty unless $p = p'$, in which case

$$W(\mathfrak{a}_P, \mathfrak{a}_{P'}) \cong \{s \in S_p : \ n_i' = n_{s(i)}, \ 1 \le i \le p\}.$$

In general, we say that P and P' are *associated* if the set $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is nonempty. We would expect a pair of induced representations $\mathcal{I}_P(\lambda)$ and $\mathcal{I}_{P'}(\lambda')$ to be equivalent if P and P' belong to the same associated class, and $\lambda' = s\lambda$ for some element $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$.

The formal definitions apply to any elements $x \in G(\mathbb{A})$, $\phi \in \mathcal{H}_P$, and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. The associated Eisenstein series is

$$E(x, \phi, \lambda) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x)\mathrm{e}^{(\lambda+\rho_P)(H_P(\delta x))}. \tag{7.1}$$

If s belongs to $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, the operator

$$M(s,\lambda):\ \mathcal{H}_P \longrightarrow \mathcal{H}_{P'}$$

that intertwines $\mathcal{I}_P(\lambda)$ with $\mathcal{I}_{P'}(s\lambda)$ is defined by

$$\big(M(s,\lambda)\phi\big)(x) = \int \phi(w_s^{-1}nx)\mathrm{e}^{(\lambda+\rho_P)(H_P(w_s^{-1}nx))}\mathrm{e}^{(-s\lambda+\rho_{P'})(H_{P'}(x))}\mathrm{d}n, \tag{7.2}$$

where the integral is taken over the quotient

$$N_{P'}(\mathbb{A}) \cap w_s N_P(\mathbb{A}) w_s^{-1} \backslash N_{P'}(\mathbb{A}),$$

and w_s is any representative of s in $G(\mathbb{Q})$. A reader so inclined could motivate both definitions in terms of finite group theory. Each definition is a formal analogue of a general construction by Mackey [**Ma**] for the space of intertwining operators between two induced representations $\mathrm{Ind}_{H_1}^H(\rho_1)$ and $\mathrm{Ind}_{H_2}^H(\rho_2)$ of a finite group H.

It follows formally from the definitions that

$$E\big(x, \mathcal{I}_P(\lambda,y)\phi,\lambda\big) = E(xy,\phi,\lambda)$$

and

$$M(s,\lambda)\mathcal{I}_P(\lambda,y) = \mathcal{I}_{P'}(s\lambda,y)M(s,\lambda).$$

These are the desired intertwining properties. However, (7.1) and (7.2) are defined by sums and integrals over noncompact spaces. They do not generally converge. It is this fact that makes the theory of Eisenstein series so difficult.

Let $\mathcal{H}_P^0$ be the subspace of vectors $\phi \in \mathcal{H}_P$ that are K-finite, in the sense that the subset

$$\{\mathcal{I}_P(\lambda,k)\phi :\ k \in K\}$$

of $\mathcal{H}_P$ spans a finite dimensional space, and that lie in a finite sum of irreducible subspaces of $\mathcal{H}_P$ under the action $\mathcal{I}_P(\lambda)$ of $G(\mathbb{A})$. The two conditions do not depend on the choice of λ. Taken together, they are equivalent to the requirement that the function

$$\phi(x_\infty x_{\mathrm{fin}}), \qquad x_\infty \in G(\mathbb{R}),\ x_{\mathrm{fin}} \in G(\mathbb{A}_{\mathrm{fin}}),$$

be locally constant in x_{fin}, and smooth, $K_{\mathbb{R}}$-finite and $\mathcal{Z}_\infty$-finite in x_∞, where $\mathcal{Z}_\infty$ denotes the algebra of bi-invariant differential operators on $G(\mathbb{R})$. The space $\mathcal{H}_P^0$ is dense in $\mathcal{H}_P$.

For any P, we can form the chamber

$$(\mathfrak{a}_P^*)^+ = \big\{\Lambda \in \mathfrak{a}_P^* :\ \Lambda(\alpha^\vee) > 0,\ \alpha \in \Delta_P\big\}$$

in $\mathfrak{a}_P^*$.

LEMMA 7.1 (Langlands). *Suppose that $\phi \in \mathcal{H}_P^0$ and that λ lies in the open subset*

$$\big\{\lambda \in \mathfrak{a}_{P,\mathbb{C}}^* :\ \mathrm{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+\big\}$$

of $\mathfrak{a}_{P,\mathbb{C}}^*$. Then the sum (7.1) and integral (7.2) that define $E(x,\phi,\lambda)$ and $\big(M(s,\lambda)\phi\big)(x)$ both converge absolutely to analytic functions of λ. □

For spectral theory, one is interested in points λ such that $\mathcal{I}_P(\lambda)$ is unitary, which is to say that λ belongs to the real subspace $i\mathfrak{a}_P^*$ of $\mathfrak{a}_{P,\mathbb{C}}^*$. This is outside the domain of absolute convergence for (7.1) and (7.2). The problem is to show that the functions $E(x,\phi,\lambda)$ and $M(s,\lambda)\phi$ have analytic continuation to this space. The following theorem summarizes Langlands' main results on Eisenstein series.

THEOREM 7.2 (Langlands). (a) *Suppose that* $\phi \in \mathcal{H}_P^0$. *Then* $E(x, \phi, \lambda)$ *and* $M(s, \lambda)\phi$ *can be analytically continued to meromorphic functions of* $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ *that satisfy the functional equations*

$$E\big(x, M(s, \lambda)\phi, s\lambda\big) = E(x, \phi, \lambda) \tag{7.3}$$

and

$$M(ts, \lambda) = M(t, s\lambda)M(s, \lambda), \qquad t \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{P''}). \tag{7.4}$$

If $\lambda \in i\mathfrak{a}^*_P$, *both* $E(x, \phi, \lambda)$ *and* $M(s, \lambda)$ *are analytic, and* $M(s, \lambda)$ *extends to a unitary operator from* $\mathcal{H}_P$ *to* $\mathcal{H}_{P'}$.

(b) *Given an associated class* $\mathcal{P} = \{P\}$, *define* $\widehat{L}_{\mathcal{P}}$ *to be the Hilbert space of families of measurable functions*

$$F = \{F_P : \ i\mathfrak{a}^*_P \longrightarrow \mathcal{H}_P, \ P \in \mathcal{P}\}$$

that satisfy the symmetry condition

$$F_{P'}(s\lambda) = M(s, \lambda)F_P(\lambda), \qquad s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'}),$$

and the finiteness condition

$$\|F\|^2 = \sum_{P \in \mathcal{P}} n_P^{-1} \int_{i\mathfrak{a}^*_P} \|F_P(\lambda)\|^2 d\lambda < \infty,$$

where

$$n_P = \sum_{P' \in \mathcal{P}} |W(\mathfrak{a}_P, \mathfrak{a}_{P'})|$$

for any $P \in \mathcal{P}$. *Then the mapping that sends* F *to the function*

$$\sum_{P \in \mathcal{P}} n_P^{-1} \int_{i\mathfrak{a}^*_P} E\big(x, F_P(\lambda), \lambda\big) \mathrm{d}\lambda, \qquad x \in G(\mathbb{A}),$$

defined whenever $F_P(\lambda)$ *is a smooth, compactly supported function of* λ *with values in a finite dimensional subspace of* $\mathcal{H}_P^0$, *extends to a unitary mapping from* $\widehat{L}_{\mathcal{P}}$ *onto a closed* $G(\mathbb{A})$*-invariant subspace* $L^2_{\mathcal{P}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ *of* $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. *Moreover, the original space* $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ *has an orthogonal direct sum decomposition*

$$L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big) = \bigoplus_{\mathcal{P}} L^2_{\mathcal{P}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big). \tag{7.5}$$

□

Theorem 7.2(b) gives a qualitative description of the decomposition of R. It provides a finite decomposition

$$R = \bigoplus_{\mathcal{P}} R_{\mathcal{P}},$$

where $R_{\mathcal{P}}$ is the restriction of R to the invariant subspace $L^2_{\mathcal{P}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. It also provides a unitary intertwining operator from $R_{\mathcal{P}}$ onto the representation $\widehat{R}_{\mathcal{P}}$ of $G(\mathbb{A})$ on $\widehat{L}_{\mathcal{P}}$ defined by

$$\big(\widehat{R}_{\mathcal{P}}(y)F\big)_P(\lambda) = \mathcal{I}_P(\lambda, y)F_P(\lambda), \qquad F \in \widehat{L}^2_{\mathcal{P}}, \ P \in \mathcal{P}.$$

The theorem is thus compatible with the general intuition we retain from the theory of Fourier series and Fourier transforms.

Let $\mathcal{B}_P$ be an orthonormal basis of the Hilbert space $\mathcal{H}_P$. We assume that every $\phi \in \mathcal{B}_P$ lies in the dense subspace $\mathcal{H}_P^0$. It is a direct consequence of Theorem 7.2 that the kernel

$$K(x,y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),$$

of $R(f)$ also has a formal expansion

$$\sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{\phi \in \mathcal{B}_P} E\big(x, \mathcal{I}_P(\lambda, f)\phi, \lambda\big)\overline{E(y,\phi,\lambda)}\mathrm{d}\lambda \tag{7.6}$$

in terms of Eisenstein series. A reader to whom this assertion is not clear might consider the analogous assertion for the case $H = \mathbb{R}$ and $\Gamma = \{1\}$. If f belongs to $C_c^\infty(\mathbb{R})$, the spectral expansion

$$K(x,y) = f(-x+y) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \pi_\lambda(f)\mathrm{e}^{\lambda x}\overline{e^{\lambda y}}\mathrm{d}\lambda, \qquad f \in C_c^\infty(\mathbb{R}),$$

of the kernel of $R(f)$, in which

$$\pi_\lambda(f) = \int_{\mathbb{R}} f(u)\mathrm{e}^{\lambda u}\mathrm{d}u,$$

is just the inverse Fourier transform of f.

In the case of Eisenstein series, one has to show that the spectral expansion of $K(x,y)$ converges in order to make the formal argument rigorous. In general, it is not feasible to estimate $E(x,\phi,\lambda)$ as a function of $\lambda \in i\mathfrak{a}_P^*$. What saves the day is the following simple idea of Selberg, which exploits only the underlying functional analysis.

One first shows that f may be written as a finite linear combination of convolutions $h_1 * h_2$ of functions $h_i \in C_c^r\big(G(\mathbb{A})\big)$, whose archimedean components are differentiable of arbitrarily high order r. An application of the Holder inequality to the formal expansion (7.6) establishes that it is enough to prove the convergence in the special case that $f = h_i * h_i^*$, where $h_i^*(x) = \overline{h_i(x^{-1})}$, and $x = y$. The integrand in (7.6) is then easily seen to be nonnegative. In fact, the double integral over λ and ϕ can be expressed as an increasing limit of nonnegative functions, each of which is the kernel of the restriction of $R(f)$ to an invariant subspace. Since this limit is bounded by the nonnegative function

$$K_i(x,x) = \sum_{\gamma \in G(\mathbb{Q})} (h_i * h_i^*)(x^{-1}\gamma x),$$

the integral converges. (See [**A3**, p. 928–934].)

There is also a spectral expansion for the kernel

$$K_Q(x,y) = \int_{N_Q(\mathbb{A})} \sum_{\gamma \in M_Q(\mathbb{Q})} f(x^{-1}\gamma n y)\mathrm{d}n$$

of $R_Q(f)$, for any standard parabolic subgroup Q. One has only to replace the multiplicity $n_P = n_P^G$ and the Eisenstein series $E(x,\phi,\lambda) = E_P^G(x,\phi,\lambda)$ in (7.6) by their relative analogues $n_P^Q = n_{M_Q \cap P}$ and

$$E_P^Q(x,\phi,\lambda) = \sum_{\delta \in P(\mathbb{Q})\backslash Q(\mathbb{Q})} \phi(\delta x)\mathrm{e}^{(\lambda+\rho_P)(H_P(\delta x))},$$

for each $P \subset Q$. Since $P\backslash Q = M_Q \cap P\backslash M_Q$, the analytic continuation of $E_P^Q(x,\phi,\lambda)$ follows from Theorem 7.2(a), with $(M_Q, M_Q \cap P)$ in place of (G,P). The spectral expansion of $K_Q(x,y)$ is

$$\sum_{P\subset Q}(n_P^Q)^{-1}\int_{i\mathfrak{a}_P^*}\sum_{\phi\in\mathcal{B}_P}E_P^Q\big(x,\mathcal{I}_P(\lambda,f)\phi,\lambda\big)\overline{E_P^Q(y,\phi,\lambda)}\mathrm{d}\lambda.$$

If we substitute this formula into (6.1), we obtain a spectral expansion for the truncated kernel $k^T(x)$. The two expansions of $k^T(x)$ ultimately give rise to two formulas for the integral $J^T(f)$. They are thus the source of the trace formula.

8. On the proof of the theorem

Theorem 6.1 represents a significant step in the direction of a trace formula. It is time now to discuss its proof. We shall outline the main argument, proving as much as possible. There are some lemmas whose full justification will be left to the references. However, in these cases we shall try to give the basic geometric idea behind the proof.

Suppose that T_1 belongs to the real vector space $\mathfrak{a}_0$, and that ω is a compact subset of $N_{P_0}(\mathbb{A})M_{P_0}(\mathbb{A})^1$. The subset

$$\begin{aligned}\mathcal{S}^G(T_1) &= \mathcal{S}^G(T_1,\omega)\\ &= \big\{x = pak:\ p\in\omega,\ a\in A_0(\mathbb{R})^0,\ k\in K,\ \beta\big(H_{P_0}(a)-T_1\big)>0,\ \beta\in\Delta_0\big\}\end{aligned}$$

of $G(\mathbb{A})$ is called the *Siegel set* attached to T_1 and ω. The inequality in the definition amounts to the assertion that

$$\tau_{P_0}\big(H_{P_0}(x)-T_1\big) = \tau_{P_0}\big(H_{P_0}(a)-T_1\big) = 1.$$

For example, if $G = SL(3)$, the condition is that the point $H_{P_0}(x)$ in the two dimensional vector space $\mathfrak{a}_0$ lies in the open cone in Figure 8.1.

THEOREM 8.1 (Borel, Harish-Chandra). *One can choose T_1 and ω so that*

$$G(\mathbb{A}) = G(\mathbb{Q})\mathcal{S}^G(T_1,\omega).$$

This is one of the main results in the foundational paper [**BH**] of Borel and Harish-Chandra. It was formulated in the adelic terms stated here in [**Bor1**]. The best reference might be the monograph [**Bor2**]. □

From now on, T_1 and ω are to be fixed as in Theorem 8.1. Suppose that $T \in \mathfrak{a}_0$ is a truncation parameter, in the earlier sense that $\beta(T)$ is large for each $\beta \in \Delta_0$. We then form the truncated Siegel set

$$\mathcal{S}^G(T_1,T) = \mathcal{S}^G(T_1,T,\omega) = \big\{x\in\mathcal{S}^G(T_1,\omega):\ \varpi\big(H_{P_0}(x)-T\big)\le 0,\ \varpi\in\widehat{\Delta}_0\big\}.$$

For example, if $G = SL(3)$, $\mathcal{S}^G(T_1,T)$ is the set of elements $x \in \mathcal{S}^G(T_1)$ such that $H_{P_0}(x)$ lies in the relatively compact subset of $\mathfrak{a}_0$ illustrated in Figure 8.2.

We write $F^G(x,T)$ for the characteristic function in x of the projection of $\mathcal{S}^G(T_1,T)$ onto $G(\mathbb{Q})\backslash G(\mathbb{A})$. Since $G(\mathbb{A})^1\cap\mathcal{S}^G(T_1,T)$ is compact, $F^G(\cdot,T)$ has compact support on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$, and is invariant under translation by $A_G(\mathbb{R})^0$.

More generally, suppose that P is a standard parabolic subgroup. We define the sets $\mathcal{S}^P(T_1) = \mathcal{S}^P(T_1,\omega)$ and $\mathcal{S}^P(T_1,T) = \mathcal{S}^P(T_1,T,\omega)$ and the characteristic function $F^P(x,T)$ exactly as above, but with Δ_{P_0}, $\widehat{\Delta}_{P_0}$ and $G(\mathbb{Q})\backslash G(\mathbb{A})$ replaced by

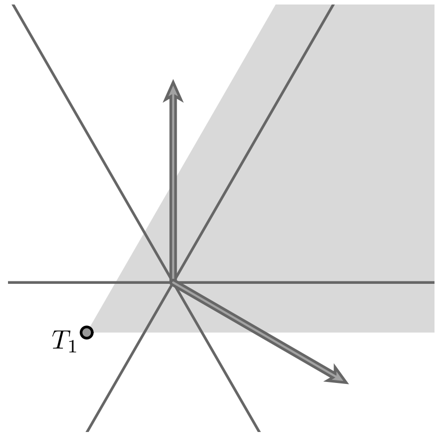

FIGURE 8.1. The shaded region is the projection onto $\mathfrak{a}_0$ of a Siegel set for $G = SL(3)$. It is the translate of the open cone $\mathfrak{a}_{P_0}^+$ by a point $T_1 \in \mathfrak{a}_0$. If T_1 is sufficiently regular in the negative cone $(-\mathfrak{a}_{P_0}^+)$, the Siegel set is an approximate fundamental domain.

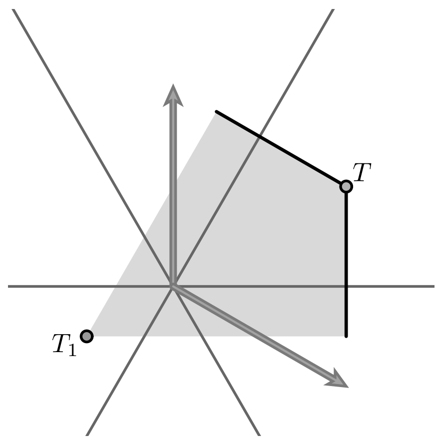

FIGURE 8.2. The shaded region represents a truncation of the Siegel set at a point $T \in \mathfrak{a}_{P_0}^+$. The image of the truncated Siegel set in $SL(3,\mathbb{Q})\backslash SL(3,\mathbb{A})$ is compact.

$\Delta_{P_0}^P$, $\widehat{\Delta}_{P_0}^P$ and $P(\mathbb{Q})\backslash G(\mathbb{A})$ respectively. In particular, $F^P(x,T)$ is the characteristic function of a subset of $P(\mathbb{Q})\backslash G(\mathbb{A})$. More precisely, if

$$x = nmak, \qquad n \in N_P(\mathbb{A}),\ m \in M_P(\mathbb{A})^1,\ a \in A_P(\mathbb{R})^0,\ k \in K,$$

then

$$F^P(x,T) = F^P(m,T) = F^{M_P}(m,T).$$

LEMMA 8.2. *For any $x \in G(\mathbb{A})$, we have*

$$\sum_P \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} F^P(\delta x, T)\tau_P\big(H_P(\delta x)\big) = 1.$$

In case $G = SL(2)$, the lemma follows directly from classical reduction theory, as we shall see in Figure 8.3 below. The general proof is established from properties of finite dimensional $\mathbb{Q}$-rational representations of G. (See [**A3**, Lemma 6.4], a result that is implicit in Langlands monograph, for example in [**Lan5**, Lemma 2.12].)

Lemma 8.2 can be restated geometrically in terms of the subsets

$$G_P(T) = \big\{x \in P(\mathbb{Q})\backslash G(\mathbb{A}) : \ F^P(x,T) = 1, \ \tau_P\big(H_P(x) - T\big) = 1\big\}$$

of $P(\mathbb{Q})\backslash G(\mathbb{A})$. The lemma asserts that for any P, the projection of $P(\mathbb{Q})\backslash G(\mathbb{A})$ onto $G(\mathbb{Q})\backslash G(\mathbb{A})$ maps $G_P(T)$ *injectively* onto a subset $\overline{G}_P(T)$ of $G(\mathbb{Q})\backslash G(\mathbb{A})$, and that $G(\mathbb{Q})\backslash G(\mathbb{A})$ is a disjoint union over P of the sets $\overline{G}_P(T)$. Otherwise said, $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ has a partition parametrized by the set of standard parabolic subgroups, which separates the problem of noncompactness from the topological complexity of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. The subset corresponding to $P = G$ is compact but topologically complex, while the subset corresponding to $P = P_0$ is topologically simple but highly noncompact. The subset corresponding to a group $P \notin \{P_0, G\}$ is mixed, being a product of a compact set of intermediate complexity with a simple set of intermediate degree of noncompactness. The partition of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is, incidentally, closely related to the compactification of this space defined by Borel and Serre.

Consider the case that $G = SL(2)$. If K is the standard maximal compact subgroup of $SL(2,\mathbb{A})$, Theorem 2.1(a) tells us that

$$SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})/K \cong SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2) \cong SL(2,\mathbb{Z})\backslash\mathcal{H},$$

where $\mathcal{H} \cong SL(2,\mathbb{R})/SO(2)$ is the upper half plane. Since they are right K-invariant, the two sets $\overline{G}_P(T)$ in this case may be identified with subsets of $SL(2,\mathbb{Z})\backslash\mathcal{H}$, which we illustrate in Figure 8.3. The darker region in the figure represents the standard fundamental domain for $SL(2,\mathbb{Z})$ in $\mathcal{H}$. Its intersection with the lower bounded rectangle equals $\overline{G}_G(T)$, while its intersection with the upper unbounded rectangle equals $\overline{G}_{P_0}(T)$. The larger unbounded rectangle represents a Siegel set, and its associated truncation. These facts, together with Lemma 8.2, follow in this case from a basic fact from classical reduction theory. Namely, if $\gamma \in SL(2,\mathbb{Z})$ and $z \in \mathcal{H}$ are such that the y-coordinates of both z and γz are greater than e^T, then γ is upper triangular.

For another example, consider the case that $G = SL(3)$. In this case there are four sets, corresponding to the four standard parabolic subgroups P_0, P_1, P_2 and G. In Figure 8.4, we illustrate the partition of $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ by describing the corresponding partition of the image in $\mathfrak{a}_0$ of the Siegel set $\mathcal{S}(T_1)$. □

Lemma 8.2 is a critical first step in the proof of Theorem 6.1. We shall actually apply it in a slightly different form. Suppose that $P_1 \subset P$. Then

$$P_1\backslash P = (P_1 \cap M_P)N_P\backslash M_P N_P \cong P_1 \cap M_P\backslash M_P.$$

We write $\tau_{P_1}^P = \tau_{P_1\cap M_P}$ and $\widehat{\tau}_{P_1}^P = \widehat{\tau}_{P_1\cap M_P}$. We shall regard these two functions as characteristic functions on $\mathfrak{a}_0$ that depend only on the projection of $\mathfrak{a}_0$ onto $\mathfrak{a}_{P_1}^{P_2}$, relative to the decomposition

$$\mathfrak{a}_0 = \mathfrak{a}_0^{P_1} \oplus \mathfrak{a}_{P_1}^{P_2} \oplus \mathfrak{a}_{P_2}.$$

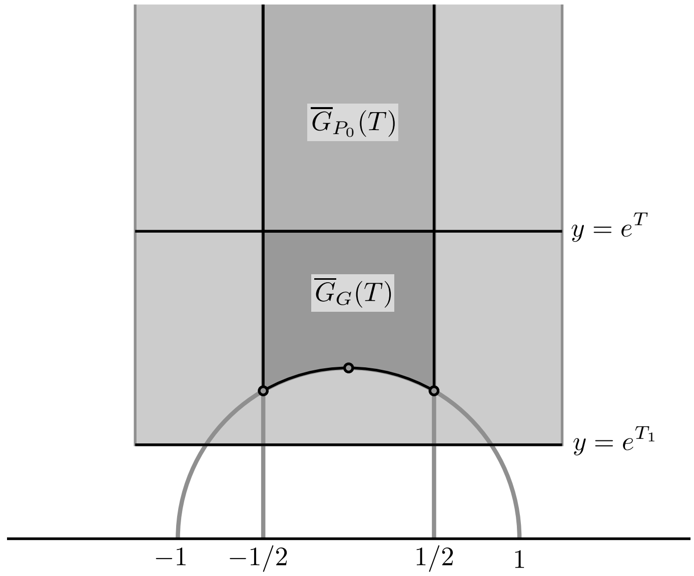

FIGURE 8.3. An illustration for $\mathcal{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$ of a standard fundamental domain and its truncation at a large positive number T, together with the more tractible Siegel set and its associated truncation.

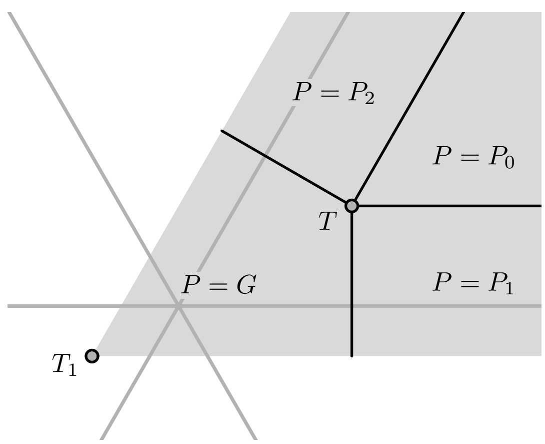

FIGURE 8.4. A partition of the region in Figure 8.1 into four sets, parametrized by the four standard parabolic subgroups P of $SL(3)$. The set corresponding to $P = P_0$ is the truncated region in Figure 8.2.

If P is fixed, we obtain the identity

$$\sum_{\{P_1 : P_1 \subset P\}} \sum_{\delta_1 \in P_1(\mathbb{Q})\backslash P(\mathbb{Q})} F^{P_1}(\delta_1 x, T)\tau_{P_1}^{P}\big(H_{P_1}(\delta_1 x) - T\big) = 1 \tag{8.1}$$

by applying Lemma 8.2 to M_P instead of G, noting at the same time that

$$F^{P_1}(y,T) = F^{M_{P_1}}(m,T)$$

and

$$H_{P_1}(y) = H_{M_{P_1}}(m),$$

for any point

$$y = nmk, \qquad n \in N_P(\mathbb{A}),\ m \in M_P(\mathbb{A}),\ k \in K.$$

We can now begin the proof of Theorem 6.1. We write

$$\begin{aligned} k^T(x) &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x, \delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big) \\ &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_\delta \Big(\sum_{P_1 \subset P} \sum_{\delta_1 \in P_1(\mathbb{Q})\backslash P(\mathbb{Q})} F^{P_1}(\delta_1\delta x, T)\tau_{P_1}^P\big(H_{P_1}(\delta_1\delta x) - T\big)\Big) \\ &\qquad \cdot \widehat{\tau}_P\big(H_P(\delta x) - T\big)K_P(\delta x, \delta x), \end{aligned}$$

by substituting (8.1) into the definition of $k^T(x)$. We then write

$$K_P(\delta x, \delta x) = K_P(\delta_1\delta x, \delta_1\delta x)$$

and

$$\widehat{\tau}_P\big(H_P(\delta x) - T\big) = \widehat{\tau}_P\big(H_P(\delta_1\delta x) - T\big),$$

since both functions are left $P(\mathbb{Q})$-invariant. Combining the double sum over δ and δ_1 into a single sum over $\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$, we write $k^T(x)$ as the sum over pairs $P_1 \subset P$ of the product of $(-1)^{\dim(A_P/A_G)}$ with

$$\sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^{P_1}(\delta x, T)\tau_{P_1}^P\big(H_{P_1}(\delta x) - T\big)\widehat{\tau}_P\big(H_P(\delta x) - T\big)K_P(\delta x, \delta x).$$

The next step is to consider the product

$$\tau_{P_1}^P\big(H_{P_1}(\delta x) - T\big)\widehat{\tau}_P\big(H_P(\delta x) - T\big) = \tau_{P_1}^P(H_1)\widehat{\tau}_P(H_1),$$

for the vector

$$H_1 = H_{P_1}(\delta x) - T_{P_1}$$

in $\mathfrak{a}_{P_1}$. (We have written T_{P_1} for the projection of T onto $\mathfrak{a}_{P_1}$.) We claim that

$$\tau_{P_1}^P(H_1)\widehat{\tau}_P(H_1) = \sum_{\{P_2, Q : P \subset P_2 \subset Q\}} (-1)^{\dim(A_{P_2}/A_Q)}\tau_{P_1}^Q(H_1)\widehat{\tau}_Q(H_1),$$

for fixed groups $P_1 \subset P$. Indeed, for a given pair of parabolic subgroups $P \subset Q$, the set of P_2 with $P \subset P_2 \subset Q$ is bijective with the collection of subsets $\Delta_P^{P_2}$ of Δ_P^Q. Since

$$(-1)^{\dim(A_{P_2}/A_Q)} = (-1)^{|\Delta_P^Q| - |\Delta_P^{P_2}|},$$

the claim follows from Identity 6.2. We can therefore write

$$\tau_{P_1}^P(H_1)\widehat{\tau}_P(H_1) = \sum_{\{P_2 : P_2 \supset P\}} \sigma_{P_1}^{P_2}(H_1), \tag{8.2}$$

where

$$\sigma_{P_1}^{P_2}(H_1) = \sum_{\{Q : Q \supset P_2\}} (-1)^{\dim(A_{P_2}/A_Q)}\tau_{P_1}^Q(H_1)\widehat{\tau}_Q(H_1).$$

Lemma 8.3. *Suppose that* $P_1 \subset P_2$, *and that*

$$H_1 = H_1^2 + H_2, \qquad H_1^2 \in \mathfrak{a}_{P_1}^{P_2},\ H_2 \in \mathfrak{a}_{P_2}^{G},$$

is a point in the space $\mathfrak{a}_{P_1}^G = \mathfrak{a}_{P_1}^{P_2} \oplus \mathfrak{a}_{P_2}^G$. *The function* $\sigma_{P_1}^{P_2}(H_1)$ *then has the following properties.*

(a) $\sigma_{P_1}^{P_2}(H_1)$ *equals* 0 *or* 1.

(b) *If* $\sigma_{P_1}^{P_2}(H_1) = 1$, *then* $\tau_{P_1}^{P_2}(H_1^2) = 1$, *and* $\|H_2\| \leq c\|H_1^2\|$, *for a positive constant* c *that depends only on* P_1 *and* P_2.

The proof of Lemma 8.3 is a straightforward analysis of roots and weights. It is based on the intuition gained from the example of $G = SL(3)$, $P_1 = P_0$, and P_2 a (standard) maximal parabolic subgroup. For the general case, we refer the reader to Lemma 6.1 of [**A3**], which gives an explicit description of the function $\sigma_{P_1}^{P_2}$ from which the conditions (a) and (b) are easily inferred. In the case of the example, Q is summed over the set $\{P_2, G\}$, and we obtain a difference

$$\sigma_{P_1}^{P_2}(H_1) = \sigma_{P_0}^{P_2}(H_1) = \tau_{P_0}^{P_2}(H_1)\widehat{\tau}_{P_2}(H_1) - \tau_{P_0}(H_1)$$

of two characteristic functions. The first characteristic function is supported on the open cone generated by the vectors $\beta_1^\vee$ and $\varpi_2^\vee$ in Figure 8.5. The second characteristic function is supported on the open cone generated by $\varpi_1^\vee$ and $\varpi_2^\vee$. The difference $\sigma_{P_1}^{P_2}(H_1)$ is therefore the characteristic function of the half open cone generated by $\beta_1^\vee$ and $\varpi_1^\vee$, the region shaded in Figure 8.5. It is obvious that this function satisfies the conditions (i) and (ii).

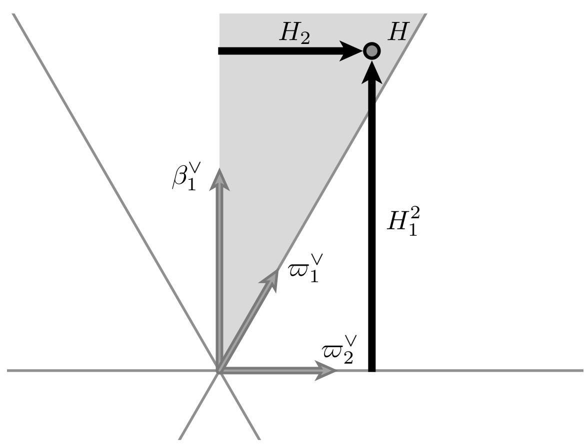

Figure 8.5. The shaded region is the complement in the upper right hand quadrant of the acute angled cone spanned by $\varpi_1^\vee$ and $\varpi_2^\vee$. It represents the support of the characteristic function $\sigma_{P_1}^{P_2}(H_1)$ attached to $G = SL(3)$, $P_1 = P_0$ minimal, and P_2 maximal. This function has compact support in the horizontal component H_2 of H_1, and semi-infinite support in the vertical component H_1^2.

□

We have established that $k^T(x)$ equals

$$\sum_{P_1 \subset P} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^{P_1}(\delta x, T) \cdot \cdot \Big(\sum_{\{P_2 : P_2 \supset P\}} \sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x) - T\big)\Big) K_P(\delta x, \delta x).$$

Therefore $k^T(x) = k^T(x, f)$ has an expansion

$$\sum_{P_1 \subset P_2} \sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^{P_1}(\delta x, T)\sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x) - T\big) k_{P_1,P_2}(\delta x), \tag{8.3}$$

where $k_{P_1,P_2}(x) = k_{P_1,P_2}(x, f)$ is the value at $y = x$ of the alternating sum

$$\begin{aligned} K_{P_1,P_2}(x,y) &= \sum_{\{P : P_1 \subset P \subset P_2\}} (-1)^{\dim(A_P/A_G)} K_P(x,y) \\ &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\gamma \in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f(x^{-1}\gamma n y) \mathrm{d}n. \end{aligned} \tag{8.4}$$

The function

$$\chi^T(x) = \chi^T_{P_1,P_2}(x) = F^{P_1}(x, T)\sigma_{P_1}^{P_2}\big(H_{P_1}(x) - T\big)$$

takes values 0 or 1. We can therefore write

$$|k^T(x)| \le \sum_{P_1 \subset P_2} \sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \chi^T(\delta x) |k_{P_1,P_2}(\delta x)|.$$

It follows that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |k^T(x)| \mathrm{d}x \le \sum_{P_1 \subset P_2} \int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \chi^T(x) |k_{P_1,P_2}(x)| \mathrm{d}x. \tag{8.5}$$

Suppose that the variable of integration $x \in P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$ on the right hand side of this inequality is decomposed as

$$x = p_1 a_1 k, \tag{8.6}$$

and

$$H_{P_1}(a_1) = H_1^2 + H_2, \qquad H_1^2 \in \mathfrak{a}_{P_1}^{P_2},\ H_2 \in \mathfrak{a}_{P_2}^G, \tag{8.7}$$

where $p_1 \in P_1(\mathbb{Q})\backslash M_{P_1}(\mathbb{A})^1 N_{P_1}(\mathbb{A})$, $a_1 \in A_{P_1}(\mathbb{R})^0 \cap G(\mathbb{A})^1$, and $k \in K$. The integrand is then compactly supported in p_1, k and H_2. We need only study its behaviour in H_1^2, for points H_1^2 with $\tau_{P_1}^{P_2}(H_1^2 - T) > 0$. This is the heart of the proof. It is where we exploit the cancellation implicit in the alternating sum over P.

We claim that the sum over $\gamma \in M_P(\mathbb{Q})$ in the formula for

$$k_{P_1,P_2}(x) = K_{P_1,P_2}(x, x)$$

can be restricted to the subset $P_1(\mathbb{Q}) \cap M_P(\mathbb{Q})$ of $M_P(\mathbb{Q})$. More precisely, given standard parabolic subgroups $P_1 \subset P \subset P_2$, a point $T \in \mathfrak{a}_0^+$ with $\beta(T)$ large (relative to the support of f) for each $\beta \in \Delta_0$, and a point $x \in P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$ with $\chi^T(x) \neq 0$, we claim that

$$\int_{N_P(\mathbb{A})} f(x^{-1}\gamma n x) \mathrm{d}n = 0,$$

for any element γ in the complement of $P_1(\mathbb{Q})$ in $M_P(\mathbb{Q})$.

Consider the example that $G = SL(2)$, $P_1 = P_0$, and $P = P_2 = G$. Then $N_P = N_G = \{1\}$. Suppose that γ belongs to the set

$$M_P(\mathbb{Q}) - P_1(\mathbb{Q}) = G(\mathbb{Q}) - P_0(\mathbb{Q}).$$

Then γ is of the form $\begin{pmatrix} * & * \\ c & * \end{pmatrix}$, for some element $c \in \mathbb{Q}^*$. Suppose that x is such that $\chi^T(x) \neq 0$. Then

$$x = p_1 a_1 k, \qquad\qquad p_1 = \begin{pmatrix} u_1 & * \\ 0 & u_1^{-1} \end{pmatrix},\ a_1 = \begin{pmatrix} \mathrm{e}^r & 0 \\ 0 & \mathrm{e}^{-r} \end{pmatrix},\ k \in K,$$

for an element $u_1 \in \mathbb{A}^*$ with $|u_1| = 1$ and a real number r that is large. We see that

$$\begin{aligned}
\int_{N_P(\mathbb{A})} & f(x^{-1}\gamma n x)\mathrm{d}n = f(x^{-1}\gamma x) \\
&= f\left(k^{-1}\begin{pmatrix} \mathrm{e}^r & 0 \\ 0 & \mathrm{e}^{-r} \end{pmatrix}^{-1}\begin{pmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{pmatrix}^{-1}\begin{pmatrix} * & * \\ c & * \end{pmatrix}\begin{pmatrix} u_1 & * \\ 0 & u_1^{-1} \end{pmatrix}\begin{pmatrix} \mathrm{e}^r & 0 \\ 0 & \mathrm{e}^{-r} \end{pmatrix} k\right) \\
&= f\left(k^{-1}\begin{pmatrix} * & * \\ u_1^2\mathrm{e}^{2r}c & * \end{pmatrix} k\right).
\end{aligned}$$

Since f is compactly supported, and $|u_1^2\mathrm{e}^{2r}c| = \mathrm{e}^{2r}$ is large, the last expression vanishes. The claim therefore holds in the special case under consideration.

The claim in general is established on p. 944 of [**A3**]. Taking it now for granted, we can then replace the sum over $M_P(\mathbb{Q})$ in the expression for $k_{P_1,P_2}(x)$ by a sum over $P_1(\mathbb{Q}) \cap M_P(\mathbb{Q})$. But $P_1(\mathbb{Q}) \cap M_P(\mathbb{Q})$ equals $M_{P_1}(\mathbb{Q})N_{P_1}^P(\mathbb{Q})$, where $N_{P_1}^P = N_{P_1} \cap M_P$ is the unipotent radical of the parabolic subgroup $P_1 \cap M_P$ of M_P. We may therefore write $k_{P_1,P_2}(x)$ as

$$\sum_{\{P : P_1 \subset P \subset P_2\}} (-1)^{\dim(A_P/A_G)} \sum_{\mu \in M_{P_1}(\mathbb{Q})} \sum_{\nu \in N_{P_1}^P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f(x^{-1}\mu\nu n x)\mathrm{d}n.$$

Now the restriction of the exponential map

$$\exp :\ \mathfrak{n}_{P_1} = \mathfrak{n}_{P_1}^P \oplus \mathfrak{n}_P \ \longrightarrow\ N_{P_1} = N_{P_1}^P N_P$$

is an isomorphism of algebraic varieties over $\mathbb{Q}$, which maps the Haar measure $\mathrm{d}x_1$ on $\mathfrak{n}_{P_1}(\mathbb{A})$ to the Haar measure $\mathrm{d}n_1$ on $N_{P_1}(\mathbb{A})$. This allows us to write $k_{P_1,P_2}(x)$ as

$$\sum_{\mu \in M_{P_1}(\mathbb{Q})} \Big(\sum_{P : P_1 \subset P \subset P_2\}} (-1)^{\dim(A_P/A_G)} \sum_{\zeta \in \mathfrak{n}_{P_1}^P(\mathbb{Q})} \int_{\mathfrak{n}_P(\mathbb{A})} f\big(x^{-1}\mu\exp(\zeta + X)x\big)\mathrm{d}X\Big).$$

There is one more operation to be performed on our expression for $k_{P_1,P_2}(x)$. We shall apply the Poisson summation formula for the locally compact abelian group $\mathfrak{n}_{P_1}^P(\mathbb{A})$ to the sum over the discrete cocompact subgroup $\mathfrak{n}_{P_1}^P(\mathbb{Q})$. We identify $\mathfrak{n}_{P_1}^P$ with $\dim(\mathfrak{n}_{P_1}^P)$-copies of the additive group by choosing a rational basis of root vectors. We can then identify $\mathfrak{n}_{P_1}^P(\mathbb{A})$ with its dual group by means of the standard bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{A}^{\dim(\mathfrak{n}_{P_1}^P)}$ and a nontrivial additive character ψ on $\mathbb{A}/\mathbb{Q}$. We obtain an expression

$$\sum_{\mu}\sum_{P}(-1)^{\dim(A_P/A_G)}\sum_{\xi\in\mathfrak{n}_{P_1}^{P}(\mathbb{Q})}\int_{\mathfrak{n}_{P_1}(\mathbb{A})} f\big(x^{-1}\mu\exp(X_1)x\big)\psi\big(\langle\xi,X_1\rangle\big)\mathrm{d}X_1$$

for $k_{P_1,P_2}(x)$. But $\mathfrak{n}_{P_1}^{P}(\mathbb{Q})$ is contained in $\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})$, for any P with $P_1\subset P\subset P_2$. As P varies, certain summands will occur more than once, with differing signs. This allows us at last to effect the cancellation given by the alternating sum over P. Set

$$\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})' = \big\{\xi\in\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q}):\ \xi\notin\mathfrak{n}_{P_1}^{P}(\mathbb{Q}),\ \text{for any } P\subsetneq P_2\big\}.$$

It then follows from Identity 6.2 that $k_{P_1,P_2}(x)$ equals

$$(-1)^{\dim(A_{P_2}/A_G)}\sum_{\mu\in M_{P_1}(\mathbb{Q})}\sum_{\xi\in\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})'}\left(\int_{\mathfrak{n}_{P_1}(\mathbb{A})} f(x^{-1}\mu\exp X_1x)\psi\big(\langle\xi,X_1\rangle\big)\mathrm{d}X_1\right). \tag{8.8}$$

We have now obtained an expression for $k_{P_1,P_2}(x,x)$ that will be rapidly decreasing in the coordinate H_1^2 of x, relative to the decompositions (8.6) and (8.7). The main reason is that the integral

$$h_{x,\mu}(Y_1)=\int_{\mathfrak{n}_{P_1}(\mathbb{A})} f(x^{-1}\mu\exp X_1x)\psi\big(\langle Y_1,X_1\rangle\big)\mathrm{d}X_1$$

is a Schwartz-Bruhat function of $Y_1\in\mathfrak{n}_{P_1}(\mathbb{A})$. This function varies smoothly with $x\in G(\mathbb{A})$, and is finitely supported in $\mu\in M_{P_1}(\mathbb{Q})$, independently of x in any compact set.

We substitute the formula (8.8) for $k_{P_1,P_2}(x)$ into the right hand side of (8.5), and then decompose the integral over x according to the (8.6). We deduce that the the integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}|k^T(x)|\mathrm{d}x$$

is bounded by a constant multiple of

$$\sum_{P_1\subset P_2}\sum_{\mu\in M_{P_1}(\mathbb{Q})}\sum_{\xi\in\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})'}\sup_{y}\int\big|h_{y,\mu}\big(\mathrm{Ad}(a_1)\xi\big)\big|\mathrm{d}a_1, \tag{8.9}$$

where the integral is taken over the set of elements a_1 in $A_{P_1}(\mathbb{R})^0\cap G(\mathbb{A})^1$ with $\sigma_{P_1}^{P_2}\big(H_{P_1}(a_1)-T\big)=1$, and the supremum is taken over the compact subset of elements

$$y=a_1^{-1}p_1a_1k,\qquad p_1\in P_1(\mathbb{Q})\backslash M_{P_1}(\mathbb{A})^1N_{P_1}(\mathbb{A}),\ a\in A_{P_1}(\mathbb{R})^0\cap G(\mathbb{A})^1,\ k\in K,$$

in $G(\mathbb{A})^1$ with $F^{P_1}(p_1,T)=\sigma_{P_1}^{P_2}\big(H_{P_1}(a_1)-T\big)=1$. We have used two changes of variables of integration here, with complementary Radon-Nikodym derivatives, which together have allowed us to write

$$\mathrm{d}X_1\mathrm{d}x=\mathrm{d}(a_1^{-1}X_1a_1)\mathrm{d}p_1\mathrm{d}a_1\mathrm{d}k,\qquad\qquad x=p_1a_1k.$$

The mapping $\mathrm{Ad}(a_1)$ in (8.9) acts by dilation on ξ. We leave the reader to show that this property implies that (8.9) is finite, and hence that the integral of $|k^T(x)|$ converges. (See [**A3**, Theorem 7.1].) This completes our discussion of the proof of Theorem 6.1. □

We have seen that Lemma 8.3 is an essential step in the proof of Theorem 6.1. There is a particularly simple case of this lemma that is important for other combinatorial arguments. It is the identity

$$\sum_{\{P:P_1\subset P\}}(-1)^{\dim(A_{P_1}/A_P)}\tau_{P_1}^P(H_1)\widehat{\tau}_P(H_1)=\begin{cases}0, & \text{if } P_1\neq G,\\ 1, & \text{if } P_1=G,\end{cases}\tag{8.10}$$

obtained by setting $P_2=P_1$. The identity holds for any standard parabolic subgroup P_1 and any point $H_1\in\mathfrak{a}_{P_1}$. Indeed, the left hand side of (8.10) equals $\sigma_{P_1}^{P_1}(H_1)$, so the identity follows from condition (ii) of Lemma 8.3.

There is also a parallel identity

$$\sum_{\{P:P_1\subset P\}}(-1)^{\dim(A_{P_1}/A_P)}\widehat{\tau}_{P_1}^P(H_1)\tau_P(H_1)=\begin{cases}0, & \text{if } P_1\neq G,\\ 1, & \text{if } P_1=G,\end{cases}\tag{8.11}$$

related by inversion to (8.10). To see this, it is enough to consider the case that P_1 is proper in G. One can then derive (8.11) from (8.10) by evaluating the expression

$$\sum_{\{P,Q:P_1\subset P\subset Q\}}(-1)^{\dim(A_P/A_Q)}\widehat{\tau}_{P_1}^P(H_1)\tau_P^Q(H_1)\widehat{\tau}_Q(H_1)$$

as two different iterated sums. For if one takes Q to index the inner sum, and assumes inductively that (8.11) holds whenever G is replaced by a proper Levi subgroup, one finds that the expression equals the sum of $\widehat{\tau}_{P_1}(H_1)$ with the left hand side of (8.11). On the other hand, by taking the inner sum to be over P, one sees from (8.10) that the expression reduces simply to $\widehat{\tau}_{P_1}(H_1)$. It follows that the left hand side of (8.11) vanishes, as required. In the case that $G=SL(3)$ and $P_1=P_0$ is minimal, the reader can view the left hand side of (8.11) (or of (8.10)) as an algebraic sum of four convex cones, formed in the obvious way from Figure 5.1. In general, (8.11) is only one of several identities that can be deduced from (8.10). We shall describe these identities, known collectively as Langlands' combinatorial lemma, in §17.

9. Qualitative behaviour of $J^T(f)$

Theorem 6.1 allows us to define the linear form

$$J^T(f)=J^{G,T}(f)=\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}k^T(x,f)\mathrm{d}x,\qquad f\in C_c^\infty\big(G(\mathbb{A})\big),$$

on $C_c^\infty\big(G(\mathbb{A})\big)$. We are still a long way from converting the geometric and spectral expansions of $k^T(x,f)$ to an explicit trace formula. We put this question aside for the moment, in order to investigate two qualitative properties of $J^T(f)$.

The first property concerns the behaviour of $J^T(f)$ as a function of T.

Theorem 9.1. *For any $f\in C_c^\infty\big(G(\mathbb{A})\big)$, the function*

$$T\longrightarrow J^T(f),$$

defined for $T\in\mathfrak{a}_0^+$ sufficiently regular, is a polynomial in T whose degree is bounded by the dimension of $\mathfrak{a}_0^G$.

We shall sketch the proof of Theorem 9.1. Let T_1 be a fixed point in $\mathfrak{a}_0$ with $\beta(T_1)$ large for every $\beta \in \Delta_0$, and let $T \in \mathfrak{a}_0$ be a variable point with $\beta(T - T_1) > 0$ for each β. It would be enough to show that the function

$$T \longrightarrow J^T(f) - J^{T_1}(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \big(k^T(x) - k^{T_1}(x)\big)\mathrm{d}x$$

is a polynomial in T. If we substitute the definition (6.1) for the two functions in the integrand, we see that the only terms in the resulting expression that depend on T and T_1 are differences of characteristic functions

$$\widehat{\tau}_P\big(H_P(\delta x) - T\big) - \widehat{\tau}_P\big(H_P(\delta x) - T_1\big).$$

We need to compare the supports of these two functions. We shall do so by expanding the first function in terms of analogues of the second function for smaller groups.

Suppose that H and X range over points in $\mathfrak{a}_0^G$. We define functions

$$\Gamma'_P(H, X), \qquad P \supset P_0,$$

inductively on $\dim(A_P/A_G)$ by setting

$$\widehat{\tau}_P(H - X) = \sum_{\{Q:Q\supset P\}} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}_P^Q(H)\Gamma'_Q(H, X), \tag{9.1}$$

for any P. Since the summand with $Q = P$ equals the product of $(-1)^{\dim(A_P/A_G)}$ with $\Gamma'_P(H, X)$, (9.1) does indeed give an inductive definition of $\Gamma'_P(H, X)$ in terms of functions $\Gamma'_Q(H, X)$ with $\dim(A_Q/A_G)$ less than $\dim(A_P/A_G)$. It follows inductively from the definition that $\Gamma'_P(H, X)$ depends only on the projections H_P and T_P of H and T onto $\mathfrak{a}_P^G$.

LEMMA 9.2. (a) *For any X and P, the function*

$$H \longrightarrow \Gamma'_P(H, X), \qquad H \in \mathfrak{a}_P^G,$$

is compactly supported.

(b) *The function*

$$X \longrightarrow \int_{\mathfrak{a}_P^G} \Gamma'_P(H, X)\mathrm{d}H, \qquad X \in \mathfrak{a}_P^G,$$

is a homogeneous polynomial of degree equal to $\dim(\mathfrak{a}_P^G)$.

Once again, we shall be content to motivate the lemma geometrically in some special cases. For the general case, we refer the reader to [**A5**, Lemmas 2.1 and 2.2].

The simplest case is when $\mathfrak{a}_P^G$ is one-dimensional. Suppose for example that $G = SL(3)$ and $P = P_1$ is a maximal parabolic subgroup. Then Q is summed over the set $\{P_1, G\}$. Taking X to be a fixed point in positive chamber in $\mathfrak{a}_P^G$, we see that $H \to \Gamma'_P(H, X)$ is the difference of characteristic functions of two open half lines, and is hence the characteristic function of the bounded half open interval in Figure 9.1.

Suppose that $G = SL(3)$ and $P = P_0$. Then Q is summed over the set $\{P_0, P_1, P_2, G\}$, where P_1 and P_2 are the maximal parabolic subgroups represented in Figure 5.1. If X is a fixed point in the positive chamber $\mathfrak{a}_0^+$ in $\mathfrak{a}_P^G = \mathfrak{a}_0$, we can describe the summands in (9.1) corresponding to P_1 and P_2 with the help of

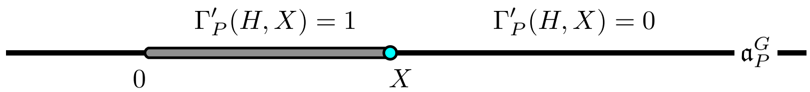

FIGURE 9.1. The half open, bounded interval represents the support of a characteristic function $\Gamma'_P(H,X)$ of H, for a maximal parabolic subgroup $P \subset G$. It is the complement of one open half line in another.

Figure 9.1. We see that the function $H \to \Gamma'_P(H,X)$ is a signed sum of characteristic functions of four regions, two obtuse cones and two semi-infinite rectangles. Keeping track of the signed contribution of each region in Figure 9.2, we see that $\Gamma'_P(H,X)$ is the characteristic function of the bounded shaded region in the figure. It is clear that the area of this figure is a homogeneous polynomial of degree 2 in the coordinates of X.

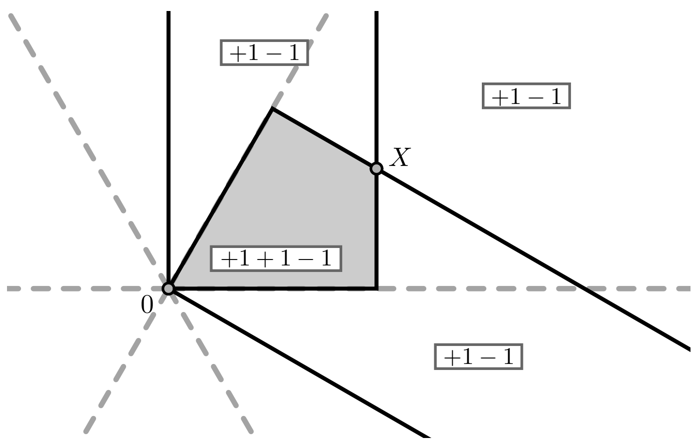

FIGURE 9.2. The bounded shaded region represents the support of the characteristic function $\Gamma'_P(H,X)$ of H, for the minimal parabolic subgroup $P = P_0$ of $SL(3)$. It is an algebraic sum of four unbounded regions, the two obtuse angled cones with vertices 0 and X, and the two semi-infinite rectangles defined by 0 and the projections of X onto the two spaces $\mathfrak{a}_{P_1}$ and $\mathfrak{a}_{P_2}$.

□

Let us use Lemma 9.2 to prove Theorem 9.1. We set $H = H_P(\delta x) - T_1$ and $X = T - T_1$. Then $H - X$ equals $H_P(\delta x) - T$, and the expansion (9.1) is

$$\widehat{\tau}_P\big(H_P(\delta x) - T\big) = \sum_{Q \supset P} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}_P^Q\big(H_P(\delta x) - T_1\big)\Gamma'_Q\big(H_P(\delta x) - T_1, T - T_1\big).$$

Substituting the right hand side of this formula into the definition of $J^T(f)$, we obtain

$$\begin{aligned} J^T(f) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{Q\supset P} (-1)^{\dim(A_Q/A_G)} C(\delta x)\mathrm{d}x \\ &= \sum_Q \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{P\subset Q} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in Q(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{\eta\in P(\mathbb{Q})\backslash Q(\mathbb{Q})} C(\eta\delta x)\mathrm{d}x \\ &= \sum_Q \int_{Q(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{P\subset Q} (-1)^{\dim(A_P/A_Q)} \sum_{\eta\in P(\mathbb{Q})\cap M_Q(\mathbb{Q})\backslash M_Q(\mathbb{Q})} C(\eta x)\mathrm{d}x, \end{aligned}$$

where

$$C(y) = K_P(y,y)\widehat{\tau}_P^Q\big(H_P(y)-T_1\big)\Gamma'_Q\big(H_Q(y)-T_1, T-T_1\big).$$

We are going to make a change of variables in the integral over x in $Q(\mathbb{Q})\backslash G(\mathbb{A})^1$. Since the expression we ultimately obtain will be absolutely convergent, this change of variables, as well as the ones above, will be justified by Fubini's theorem.

We write $x = n_Q m_Q a_Q k$, for variables n_Q, m_Q, a_Q and k in $N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})$, $M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A})^1$, $A_Q(\mathbb{R})^0 \cap G(\mathbb{A})^1$, and K respectively. The invariant measures are then related by

$$\mathrm{d}x = \delta_Q(a_Q)\mathrm{d}n_Q\mathrm{d}m_Q\mathrm{d}a_Q\mathrm{d}k.$$

The three factors in the product $C(\eta x)$ become

$$\begin{aligned} \Gamma'_Q\big(H_Q(\eta x)-T_1, T-T_1\big) &= \Gamma'_Q\big(H_Q(x)-T_1, T-T_1\big) = \Gamma'_Q\big(H_Q(a_Q)-T_1, T-T_1\big), \\ \widehat{\tau}_P^Q\big(H_P(\eta x)-T_1\big) &= \widehat{\tau}_P^Q\big(H_P(\eta m_Q)-T_1\big), \end{aligned}$$

and

$$K_P(\eta x, \eta x) = \int_{N_P(\mathbb{A})} \sum_{\gamma\in M_P(\mathbb{Q})} f(k^{-1}m_Q^{-1}a_Q^{-1}n_Q^{-1}\eta^{-1}\cdot\gamma n\cdot\eta n_Q a_Q m_Q k)\mathrm{d}n.$$

In this last integrand, the element η normalizes the variables n_Q and a_Q without changing the measures. The same is true of the element γ. We can therefore absorb both variables in the integral over n. Since

$$\delta_Q(a_Q)\mathrm{d}n = \mathrm{d}(a_Q^{-1}n_Q^{-1}nn_Qa_Q),$$

the product of $\delta_Q(a_Q)$ with $K_P(\eta x, \eta x)$ equals

$$\int_{N_P(\mathbb{A})} \sum_{\gamma\in M_P(\mathbb{Q})} f(k^{-1}m_Q^{-1}\eta^{-1}\cdot\gamma n\cdot\eta m_Q k)\mathrm{d}n. \tag{9.2}$$

The original variable n_Q has now disappeared from all three factors, so we may as well write

$$\mathrm{d}n = \mathrm{d}(n^Q n_Q) = \mathrm{d}n^Q\mathrm{d}n_Q, \qquad n^Q \in N_P^Q(\mathbb{A}),\ n_Q\in N_Q(\mathbb{A}),$$

for the decomposition of the measure in $N_P(\mathbb{A})$. The last expression (9.2) is the only factor that depends on the original variable k. Its integral over k equals

$$\begin{aligned}
&\int_K \int_{N_Q(\mathbb{A})} \int_{N_P^Q(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(k^{-1} m_Q^{-1} \eta^{-1} \cdot \gamma n^Q n_Q \cdot \eta m_Q k) \mathrm{d}n^Q \mathrm{d}n_Q \mathrm{d}k \\
&= \int_{N_P^Q(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} \int_K \int_{N_Q(\mathbb{A})} f(k^{-1} m_Q^{-1} \eta^{-1} \cdot \gamma n^Q \cdot \eta m_Q n_Q k) \mathrm{d}n_Q \mathrm{d}k \mathrm{d}n^Q \\
&= \int_{N_P^Q(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f_Q(m_Q^{-1} \eta^{-1} \cdot \gamma n^Q \cdot \eta m_Q) \mathrm{d}n^Q \\
&= K_{P \cap M_Q}(\eta m_Q, \eta m_Q),
\end{aligned}$$

where

$$f_Q(m) = \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(\mathbb{A})} f(k^{-1} m n_Q k) \mathrm{d}n_Q \mathrm{d}k, \qquad m \in M_Q(\mathbb{A}),$$

and $K_{P \cap M_Q}(\cdot, \cdot)$ is the induced kernel (4.1), but with G, P, and f replaced by M_Q, $P \cap M_Q$, and f_Q respectively. We have used the facts that

$$\mathrm{d}n_Q = \mathrm{d}\big((\eta m_Q)^{-1} n_Q (\eta m_Q)\big),$$

for η and m_Q as above, and that

$$\delta_Q(m) = \mathrm{e}^{2\rho_Q(H_Q(m))} = 1,$$

when $m = \gamma$ lies in $M_Q(\mathbb{Q})$. The correspondence $f \to f_Q$ is a continuous linear mapping from $C_c^\infty\big(G(\mathbb{A})\big)$ to $C_c^\infty\big(M_Q(\mathbb{A})\big)$. It was introduced originally by Harish-Chandra to study questions of descent.

We now collect the various terms in the formula for $J^T(f)$. We see that $J^T(f)$ equals the sum over Q and the integral over m_Q in $M_Q(\mathbb{Q}) \backslash M_Q(\mathbb{A})^1$ of the product of

$$\sum_{P \subset Q} (-1)^{\dim(A_P/A_Q)} \sum_{\eta \in P(\mathbb{Q}) \cap M_Q(\mathbb{Q}) \backslash M_Q(\mathbb{Q})} K_{P \cap M_Q}(\eta m_Q, \eta m_Q) \widehat{\tau}_P^Q \big(H_P(\eta m_Q) - T_1\big)$$

with the factor

$$\begin{aligned}
p_Q(T_1, T) &= \int_{A_Q(\mathbb{R})^0 \cap G(\mathbb{R})^1} \Gamma'_Q\big(H_Q(a_Q) - T_1, T - T_1\big) \mathrm{d}a \\
&= \int_{\mathfrak{a}_Q^G} \Gamma'_Q(H - T_1, T - T_1) \mathrm{d}H.
\end{aligned}$$

By Lemma 9.2, the last factor is a polynomial in T of degree equal to $\dim(\mathfrak{a}_Q^G)$. To analyze the first factor, we note that

$$\dim(A_P/A_Q) = \dim(A_{P \cap M_Q}/A_{M_Q})$$

and

$$\widehat{\tau}_P^Q\big(H_P(\eta m_Q) - T_1\big) = \widehat{\tau}_{P \cap M_Q}^{M_Q}\big(H_{P \cap M_Q}(\eta m_Q) - T_1\big),$$

and that the mapping $P \to P \cap M_Q$ is a bijection from the set of standard parabolic subgroups P of G with $P \subset Q$ onto the set of standard parabolic subgroups of M_Q. The first factor therefore equals the analogue $k^{T_1}(m_Q, f_Q)$ for T_1, m_Q and f_Q of the truncated kernel $k^T(x,f)$. Its integral over m_Q equals $J^{M_Q,T_1}(f_Q)$. We conclude that

$$J^T(f) = \sum_{Q\supset P_0} J^{M_Q,T_1}(f_Q)p_Q(T_1,T). \tag{9.3}$$

Therefore $J^T(f)$ is a polynomial in T whose degree is bounded by the dimension of $\mathfrak{a}_0^G$. This completes the proof of Theorem 9.1. □

Having established Theorem 9.1, we are now free to define $J^T(f)$ at any point T in $\mathfrak{a}_0$. We could always set $T=0$. However, it turns out that there is a better choice in general. The question is related to the choice of minimal parabolic subgroup P_0.

We write $\mathcal{P}(M_0)$ for the set of (minimal) parabolic subgroups of G with Levi component M_0. The mapping

$$s \longrightarrow sP_0 = w_sP_0w_s^{-1}, \qquad s\in W_0,$$

is then a bijection from W_0 to $\mathcal{P}(M_0)$. We recall that w_s is a representative of s in $G(\mathbb{Q})$. If $G=GL(n)$, we can take w_s to be a permutation matrix, an element in $G(\mathbb{Q})$ that also happens to lie in the standard maximal compact subgroup K of $G(\mathbb{A})$. In general, however, s might require a separate representative $\widetilde{w}_s$ in K. The quotient $w_s^{-1}\widetilde{w}_s$ does belong to $M_0(\mathbb{A})$, so the point

$$H_{P_0}(w_s^{-1}) = H_{M_0}(w_s^{-1}\widetilde{w}_s)$$

in $\mathfrak{a}_0$ is independent of the choice of P_0. By arguing inductively on the length of $s\in W_0$, one shows that there is a unique point $T_0\in\mathfrak{a}_0^G$ such that

$$H_{P_0}(w_s^{-1}) = T_0 - s^{-1}T_0, \tag{9.4}$$

for every $s\in W_0$. (See [**A5**, Lemma 1.1].) In the case that G equals $GL(n)$ and K is the standard maximal compact subgroup of $GL(n,\mathbb{A})$, $T_0=0$.

Proposition 9.3. *The linear form*

$$J(f) = J^G(f), \qquad f\in C_c^\infty\big(G(\mathbb{A})\big),$$

defined as the value of the polynomial

$$J^T(f) = J^{G,T}(f)$$

at $T=T_0$, is independent of the choice of $P_0\in\mathcal{P}(M_0)$.

The proof of Proposition 9.3 is a straightforward exercise. If $T\in\mathfrak{a}_0$ is highly regular relative to P_0, sT is highly regular relative to the group $P_0'=sP_0$ in $\mathcal{P}(M_0)$. The mapping

$$P \ \longrightarrow\ P' = sP = w_sPw_s^{-1}, \qquad P\supset P_0,$$

is a bijection between the relevant families $\{P\supset P_0\}$ and $\{P'\supset P_0'\}$ of standard parabolic subgroups. For any P, the mapping $\delta\to\delta' = w_s\delta$ is a bijection from $P(\mathbb{Q})\backslash G(\mathbb{Q})$ onto $P'(\mathbb{Q})\backslash G(\mathbb{Q})$. It follows from the definitions that

$$\begin{aligned}\widehat{\tau}_P\big(H_P(\delta x)-T\big) &= \widehat{\tau}_{P'}\big(sH_P(w_s^{-1}\delta' x) - sT\big)\\ &= \widehat{\tau}_{P'}\big(sH_P(\widetilde{w}_s^{-1}\delta' x) + sH_{P_0}(w_s^{-1}) - sT\big)\\ &= \widehat{\tau}_{P'}\big(H_{P'}(\delta' x) - (sT - sT_0 + T_0)\big).\end{aligned}$$

Comparing the definition (6.1) of the truncated kernel with its analogue for $P_0' = sP_0$, we see that

$$J_{P_0}^T(f) = J_{sP_0}^{sT-sT_0+T_0}(f),$$

where the subscripts indicate the minimal parabolic subgroups with respect to which the linear forms have been defined. Each side of this identity extends to a polynomial function of $T \in \mathfrak{a}_0$. Setting $T = T_0$, we see that the linear form

$$J(f) = J_{P_0}^{T_0}(f) = J_{sP_0}^{T_0}(f)$$

is indeed independent of the choice of P_0. (See [**A5**, p. 18–19].) $\square$

The second qualitative property of $J^T(f)$ concerns its behaviour under conjugation by $G(\mathbb{A})$. A *distribution* on $G(\mathbb{A})$ is a linear form I on $C_c^\infty\big(G(\mathbb{A})\big)$ that is continuous with respect to the natural topology. The distribution is said to be *invariant* if

$$I(f^y) = I(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),\ y \in G(\mathbb{A}),$$

where

$$f^y(x) = f(yxy^{-1}).$$

The proof of Theorem 6.1 implies that $f \to J^T(f)$ is a distribution if $T \in \mathfrak{a}_{P_0}^+$ is sufficiently regular. Since $J^T(f)$ is a polynomial in T, its coefficients are also distributions. In particular, $f \to J(f)$ is a distribution on $G(\mathbb{A})$, which is independent of the choice of $P_0 \in \mathcal{P}(M_0)$. We would like to compute its obstruction to being invariant.

Consider a point $y \in G(\mathbb{A})$, a function $f \in C_c^\infty\big(G(\mathbb{A})\big)$, and a highly regular point $T \in \mathfrak{a}_0^+$. We are interested in the difference $J^T(f^y) - J^T(f)$.

To calculate $J^T(f^y)$, we have to replace the factor

$$K_P(\delta x, \delta x) = \sum_{\gamma \in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f(x^{-1}\delta^{-1}\gamma n \delta x)\mathrm{d}n$$

in the truncated kernel (6.1) by the expression

$$\sum_{\gamma \in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f^y(x^{-1}\delta^{-1}\gamma n \gamma x)\mathrm{d}n = K_P(\delta x y^{-1}, \delta x y^{-1}).$$

The last expression is invariant under translation of y by the central subgroup $A_G(\mathbb{R})^0$. We may as well therefore assume that y belongs to the subgroup $G(\mathbb{A})^1$ of $G(\mathbb{A})$. With this condition, we can make a change of variables $x \to xy$ in the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ that defines $J^T(f^y)$. We see that $J^T(f^y)$ equals

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Big(\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x, \delta x) \widehat{\tau}_P\big(H_P(\delta x y) - T\big)\Big)\mathrm{d}x.$$

If $\delta x = nmak$, for elements n, m, a, and k in $N_P(\mathbb{A})$, $M_P(\mathbb{A})^1$, $A_P(\mathbb{R})^0 \cap G(\mathbb{A})^1$, and K respectively, set $k_P(\delta x) = k$. We can then write

$$\begin{aligned} \widehat{\tau}_P\big(H_P(\delta x y) - T\big) &= \widehat{\tau}_P\big(H_P(a) + H_P(ky) - T\big) \\ &= \widehat{\tau}_P\big(H_P(\delta x) - T + H_P(k_P(\delta x)y)\big). \end{aligned}$$

The last expression has an expansion

$$\sum_{Q \supset P} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}_P^Q\big(H_P(\delta x) - T\big)\Gamma'_Q\big(H_P(\delta x) - T, -H_P(k_P(\delta x)y)\big)$$

given by (9.1), which we can substitute into the formula above for $J^T(f^y)$.

The discussion now is identical to that of the proof of Theorem 9.1. Set

$$u'_Q(k,y) = \int_{\mathfrak{a}_Q^G} \Gamma'_Q\big(H, -H_Q(ky)\big)\mathrm{d}H, \qquad k \in K,$$

and

$$f_{Q,y}(m) = \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(\mathbb{A})} f(k^{-1}mnk)u'_Q(k,y)\mathrm{d}n\mathrm{d}k, \qquad m \in M_Q(\mathbb{A}).$$

The transformation $f \to f_{Q,y}$ is a continuous linear mapping from $C_c^\infty\big(G(\mathbb{A})\big)$ to $C_c^\infty\big(M_Q(\mathbb{A})\big)$, which varies smoothly with $y \in G(\mathbb{A})$, and depends only on the image of y in $G(\mathbb{A})^1$. The proof of Theorem 9.1 then leads directly to the following analogue

$$J^T(f^y) = \sum_{Q \supset P_0} J^{M_Q,T}(f_{Q,y}) \tag{9.5}$$

of (9.3). Since we have taken $K_Q = K \cap M_Q(\mathbb{A})$ as maximal compact subgroup of $M_Q(\mathbb{A})$, $H_{P_0}(w_s^{-1})$ equals $H_{P_0 \cap M_Q}(w_s^{-1})$ for any s in the subgroup W_0^M of $W_0 = W_0^G$. The canonical point $T_0 \in \mathfrak{a}_0^G$, defined for G by (9.4), therefore projects onto the canonical point in $\mathfrak{a}_0^Q$ attached to M_Q. Setting $T = T_0$ in (9.5), we obtain the following result.

THEOREM 9.4. *The distribution J satisfies the formula*

$$J(f^y) = \sum_{Q \supset P_0} J^{M_Q}(f_{Q,y})$$

for conjugation of $f \in C_c^\infty\big(G(\mathbb{A})\big)$ by $y \in G(\mathbb{A})$. □

10. The coarse geometric expansion

We have constructed a distribution J on $G(\mathbb{A})$ from the truncated kernel $k^T(x) = k^T(x,f)$. The next step is to transform the geometric expansion for $k^T(x)$ into a geometric expansion for $J(f)$. The problem is more subtle than it might first appear. This is because the truncation $k^T(x)$ of $K(x,x)$ is not completely compatible with the decomposition of $K(x,x)$ according to conjugacy classes. The difficulty comes from those conjugacy classes in $G(\mathbb{Q})$ that are particular to the case of noncompact quotient, namely the classes that are not semisimple.

In this section we shall deal with the easy part of the problem. We shall give a geometric expansion of $J(f)$ into terms parametrized by semisimple conjugacy classes in $G(\mathbb{Q})$. The proof requires only minor variations of the discussion of the last two sections.

Recall that any element γ in $G(\mathbb{Q})$ has a *Jordan decomposition* $\gamma = \mu\nu$. It is the unique decomposition of γ into a product of a semisimple element $\mu = \gamma_s$ in $G(\mathbb{Q})$, with a unipotent element $\nu = \gamma_u$ in $G(\mathbb{Q})$ that commutes with γ_s. We define two elements γ and γ' in $G(\mathbb{Q})$ to be $\mathcal{O}$-equivalent if their semisimple parts γ_s and γ'_s are $G(\mathbb{Q})$-conjugate. We then write $\mathcal{O} = \mathcal{O}^G$ for the set of such equivalence classes. A class $\mathfrak{o} \in \mathcal{O}$ is thus a union of conjugacy classes in $G(\mathbb{Q})$.

The set $\mathcal{O}$ is in obvious bijection with the semisimple conjugacy classes in $G(\mathbb{Q})$. We shall say that a semisimple conjugacy class in $G(\mathbb{Q})$ is *anisotropic* if it does not intersect $P(\mathbb{Q})$, for any $P \subsetneq G$. Then $\gamma \in G(\mathbb{Q})$ represents an anisotropic class if and only if A_G is the maximal $\mathbb{Q}$-split torus in the connected centralizer H of γ in

G. (Such classes were called elliptic in [**A3**, §2]. However, the term elliptic is better reserved for semisimple elements γ in $G(\mathbb{Q})$ such as 1, for which A_G is the maximal split torus in the *center* of H.) We can define an *anisotropic rational datum* to be an equivalence class of pairs (P, α), where $P \subset G$ is a standard parabolic subgroup, and α is an anisotropic conjugacy class in $M_P(\mathbb{Q})$. The equivalence relation is just conjugacy, which for standard parabolic subgroups is given by the Weyl sets $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ of §7. In other words, (P', α') is equivalent to (P, α) if $\alpha = w_s^{-1}\alpha' w_s$ for some element $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$. The mapping that sends $\{(P, \alpha)\}$ to the conjugacy class of α in $G(\mathbb{Q})$ is a bijection onto the set of semisimple conjugacy classes in $G(\mathbb{Q})$. We therefore have a canonical bijection from the set of anisotropic rational data and our set $\mathcal{O}$. Anisotropic rational data will not be needed for the constructions of this section. We mention them in order to be able to recognize the formal relations between these constructions and their spectral analogues in §12.

In case $G = GL(n)$, the classes $\mathcal{O}$ are related to basic notions from linear algebra. The Jordan decomposition is given by Jordan normal form. Two elements γ and γ' in $GL(n, \mathbb{Q})$ are $\mathcal{O}$-equivalent if and only if they have the same set of complex eigenvalues (with multiplicity). This is the same as saying that γ and γ' have the same characteristic polynomial. The set $\mathcal{O}$ of equivalence classes in $GL(n, \mathbb{Q})$ is thus bijective with the set of rational monic polynomials of degree n with nonzero constant term. If $\mathfrak{o} \in \mathcal{O}$ is an equivalence class, the intersection $\mathfrak{o} \cap P(\mathbb{Q})$ is empty for all $P \neq G$ if and only if the characteristic polynomial of $\mathfrak{o}$ is irreducible over $\mathbb{Q}$. This is the condition that $\mathfrak{o}$ consist of a single anisotropic conjugacy class in $G(\mathbb{Q})$. A general equivalence class $\mathfrak{o} \in \mathcal{O}$ consists of only one conjugacy class if and only if the elements in $\mathfrak{o}$ are all semisimple, which in turn is equivalent to saying that the characteristic polynomial of $\mathfrak{o}$ has distinct irreducible factors over $\mathbb{Q}$. We leave the reader to verify these properties from linear algebra.

If G is arbitrary, we have a decomposition

$$K(x, x) = \sum_{\mathfrak{o} \in \mathcal{O}} K_{\mathfrak{o}}(x, x), \tag{10.1}$$

where

$$K_{\mathfrak{o}}(x, x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x).$$

More generally, we can write

$$K_P(x, x) = \sum_{\gamma \in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f(x^{-1}\gamma n x) \mathrm{d}n = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x, x)$$

for any P, where

$$K_{P,\mathfrak{o}}(x, x) = \sum_{\gamma \in M_P(\mathbb{Q}) \cap \mathfrak{o}} \int_{N_P(\mathbb{A})} f(x^{-1}\gamma n x) \mathrm{d}n.$$

We therefore have a decomposition

$$k^T(x) = \sum_{\mathfrak{o} \in \mathcal{O}} k_{\mathfrak{o}}^T(x) \tag{10.2}$$

of the truncated kernel, where

$$\begin{aligned}k^T_{\mathfrak{o}}(x) &= k^T_{\mathfrak{o}}(x,f)\\ &= \sum_P (-1)^{\dim(A_P/A_G)} \int_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\mathfrak{o}}(\delta x,\delta x)\widehat{\tau}_P\big(H_P(\delta x)-T\big).\end{aligned}$$

The following extension of Theorem 6.1 can be regarded as a corollary of its proof.

COROLLARY 10.1. *The double integral*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T_{\mathfrak{o}}(x,f)\mathrm{d}x \tag{10.3}$$

converges absolutely.

The proof of Corollary 10.1 is in fact identical to the proof of Theorem 6.1 sketched in §8, but for one point. The discrepancy arises when we apply the Poisson summation formula to the lattice $\mathfrak{n}^P_{P_1}(\mathbb{Q})$, for standard parabolic subgroups $P_1 \subset P$. To do so, we require a sum over the lattice, or what amounts to the same thing, a sum over elements $\nu \in N^P_{P_1}(\mathbb{Q})$. In the proof of Theorem 6.1, we recall that such a sum arose from the property

$$P_1(\mathbb{Q}) \cap M_P(\mathbb{Q}) = M_{P_1}(\mathbb{Q})N^P_{P_1}(\mathbb{Q}).$$

That it also occurs in treating a class $\mathfrak{o}\in\mathcal{O}$ is a consequence of the parallel property

$$P_1(\mathbb{Q})\cap M_P(\mathbb{Q})\cap\mathfrak{o} = \big(M_{P_1}(\mathbb{Q})\cap\mathfrak{o}\big)N^P_{P_1}(\mathbb{Q}). \tag{10.4}$$

This is in turn a consequence of the first assertion of the next lemma.

LEMMA 10.2. *Suppose that $P\supset P_0$, $\gamma\in M(\mathbb{Q})$, and $\phi\in C_c\big(N_P(\mathbb{A})\big)$. Then*

$$\sum_{\delta\in N_P(\mathbb{Q})_{\gamma_s}\backslash N_P(\mathbb{Q})} \sum_{\eta\in N_P(\mathbb{Q})_{\gamma_s}} \phi(\gamma^{-1}\delta^{-1}\gamma\eta\delta) = \sum_{\nu\in N_P(\mathbb{Q})}\phi(\nu)$$

and

$$\int_{N_P(\mathbb{A})_{\gamma_s}\backslash N_P(\mathbb{A})}\int_{N_P(\mathbb{A})_{\gamma_s}} \phi(\gamma^{-1}n_1^{-1}\gamma n_2 n_1)\mathrm{d}n_2\mathrm{d}n_1 = \int_{N_P(\mathbb{A})}\phi(n)\mathrm{d}n,$$

where $N_P(\cdot)_{\gamma_s}$ denotes the centralizer of γ_s in $N_P(\cdot)$.

The proof of Lemma 10.2 is a typical change of variable argument for unipotent groups. The first assertion represents a decomposition of a sum over $N_P(\mathbb{Q})$, while the second is the corresponding decomposition of an adelic integral over $N_P(\mathbb{A})$. (See [**A3**, Lemmas 2.1 and 2.2].) □

The first assertion of the lemma implies that $P(\mathbb{Q})\cap\mathfrak{o}$ equals $\big(M_P(\mathbb{Q})\cap\mathfrak{o}\big)N_P(\mathbb{Q})$. If we apply it to the pair $(M_P, P_1\cap M_P)$ in place of (G,P), we obtain the required relation (10.4). We then obtain Corollary 10.1 by following step by step the proof of Theorem 6.1. (Theorem 7.1 of [**A3**] was actually stated and proved directly for the functions $k^T_{\mathfrak{o}}(x)$ rather than their sum $k^T(x)$.) □

Once we have Corollary 10.1, we can apply Fubini's theorem to double integral (10.3). We obtain an absolutely convergent expansion

$$J^T(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J^T_{\mathfrak{o}}(f),$$

whose terms are given by absolutely convergent integrals

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)\mathrm{d}x, \qquad \mathfrak{o} \in \mathcal{O}. \tag{10.5}$$

The behaviour of $J_{\mathfrak{o}}^T(f)$ as a function of T is similar to that of $J^T(f)$. We have only to apply the proof of Theorem 9.1 to the absolutely convergent integral (10.5). This tells us that for any $f \in C_c^\infty\big(G(\mathbb{A})\big)$ and $\mathfrak{o} \in \mathcal{O}$, the function

$$T \longrightarrow J_{\mathfrak{o}}^T(f),$$

defined for $T \in \mathfrak{a}_0^+$ sufficiently regular in a sense that is independent of $\mathfrak{o}$, is a polynomial in T of degree bounded by the dimension of $\mathfrak{a}_0^G$. We can therefore define $J_{\mathfrak{o}}^T(f)$ for all values of $T \in \mathfrak{a}_0$ by its polynomial extension. We then set

$$J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^{T_0}(f), \qquad \mathfrak{o} \in \mathcal{O},$$

for the point $T_0 \in \mathfrak{a}_0^G$ given by (9.4). The proof of Proposition 9.3 tells us that $J_{\mathfrak{o}}(f)$ is independent of the choice of minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$.

The distributions $J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^G(f)$ can sometimes be invariant, though they are not generally so. To see this, we apply the proof of Theorem 9.4 to the absolutely convergent integral (10.5). For any $Q \supset P_0$ and $h \in C_c^\infty\big(M_Q(\mathbb{A})\big)$, set

$$J_{\mathfrak{o}}^{M_Q}(h) = \sum_{\mathfrak{o}_Q} J_{\mathfrak{o}_Q}^{M_Q}(h), \qquad \mathfrak{o} \in \mathcal{O},$$

where $\mathfrak{o}_Q$ ranges over the finite preimage of $\mathfrak{o}$ in $\mathcal{O}^{M_Q}$ under the obvious mapping of $\mathcal{O}^{M_Q}$ into $\mathcal{O} = \mathcal{O}^G$. We then obtain the variance property

$$J_{\mathfrak{o}}(f^y) = \sum_{Q \supset P_0} J_{\mathfrak{o}}^{M_Q}(f_{Q,y}), \qquad \mathfrak{o} \in \mathcal{O},\ y \in G(\mathbb{A}), \tag{10.6}$$

in the notation of Theorem 9.4. Observe that $\mathfrak{o}$ need not lie in the image the map $\mathcal{O}^{M_Q} \to \mathcal{O}$ attached to any proper parabolic subgroup $Q \subsetneq G$. This is so precisely when $\mathfrak{o}$ is *anisotropic*, in the sense that it consists of a single anisotropic (semisimple) conjugacy class. It is in this case that the distribution $J_{\mathfrak{o}}(f)$ is invariant.

The expansion of $J^T(f)$ in terms of distributions $J_{\mathfrak{o}}^T(f)$ extends by polynomial interpolation to all values of T. Setting $T = T_0$, we obtain an identity

$$J(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big), \tag{10.7}$$

of distributions. This is what we will call the coarse geometric expansion. The distributions $J_{\mathfrak{o}}(f)$ for which $\mathfrak{o}$ is anisotropic are to be regarded as general analogues of the geometric terms in the trace formula for compact quotient.

11. Weighted orbital integrals

The summands $J_{\mathfrak{o}}(f)$ in the coarse geometric expansion of $J(f)$ were defined in global terms. We need ultimately to describe them more explicitly. For example, we would like to have a formula for $J_{\mathfrak{o}}(f)$ in which the dependence on the local components f_v of f is more transparent. In this section, we shall solve the problem for "generic" classes $\mathfrak{o} \in \mathcal{O}$. For such classes, we shall express $J_{\mathfrak{o}}(f)$ as a weighted orbital integral of f.

We fix a class $\mathfrak{o} \in \mathcal{O}$, which for the moment we take to be arbitrary. Recall that

$$K_{P,\mathfrak{o}}(x,y) = \sum_{\gamma \in M_P(\mathbb{Q}) \cap \mathfrak{o}} \int_{N_P(\mathbb{A})} f(x^{-1}\gamma n y) \mathrm{d}n,$$

for any $P \supset P_0$. Lemma 10.2 provides a decomposition of the integral over $N_P(\mathbb{A})$ onto a double integral. We define a modified function

$$\widetilde{K}_{P,\mathfrak{o}}(x,y) = \sum_{\gamma \in M_P(\mathbb{Q}) \cap \mathfrak{o}} \sum_{\eta \in N_P(\mathbb{Q})_{\gamma_s} \backslash N_P(\mathbb{Q})} \int_{N_P(\mathbb{A})_{\gamma_s}} f(x^{-1}\eta^{-1}\gamma n \eta y) \mathrm{d}n \tag{11.1}$$

by replacing the outer adelic integral of the lemma with a corresponding sum of rational points. We then define a modified kernel $\widetilde{k}^T_{\mathfrak{o}}(x) = \widetilde{k}^T_{\mathfrak{o}}(x,f)$ by replacing the function $K_{P,\mathfrak{o}}(\delta x, \delta x)$ in the formula for $k^T_{\mathfrak{o}}(x)$ with the modified function $\widetilde{K}_{P,\mathfrak{o}}(\delta x, \delta x)$. That is,

$$\widetilde{k}^T_{\mathfrak{o}}(x,f) = \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \widetilde{K}_{P,\mathfrak{o}}(\delta x, \delta x) \widehat{\tau}_P\big(H_P(\delta x) - T\big).$$

THEOREM 11.1. *If $T \in \mathfrak{a}^+_{P_0}$ is highly regular, the integral*

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} \widetilde{k}^T_{\mathfrak{o}}(x,f) \mathrm{d}x \tag{11.2}$$

converges absolutely, and equals $J^T_{\mathfrak{o}}(f)$.

The proof of Theorem 11.1 is again similar to that of Theorem 6.1, or rather its modification for the class $\mathfrak{o}$ discussed in §10. Copying the formal manipulations from the first half of §8, we write

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} \widetilde{k}^T_{\mathfrak{o}}(x) \mathrm{d}x = \sum_{P_1 \subset P_2} \int_{P_1(\mathbb{Q}) \backslash G(\mathbb{A})^1} \chi^T(x) \widetilde{k}_{P_1,P_2,\mathfrak{o}}(x) \mathrm{d}x, \tag{11.3}$$

where $\chi^T(x)$ is as in (8.5), and

$$\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x) = \sum_{\{P : P_1 \subset P \subset P_2\}} (-1)^{\dim(A_P/A_G)} \widetilde{K}_{P,\mathfrak{o}}(x,x).$$

To justify these manipulations, we have to show that for any $P_1 \subset P_2$, the integral

$$\int_{P_1(\mathbb{Q}) \backslash G(\mathbb{A})^1} \chi^T(x) |\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x)| \mathrm{d}x \tag{11.4}$$

is finite. This would also establish the absolute convergence assertion of the theorem.

We estimate the integral (11.4) as in the second half of §8. We shall be content simply to mention the main steps. The first is to show that if T is sufficiently regular and $\chi^T(x) \neq 0$, the summands in the formula for $\widetilde{K}_{P,\mathfrak{o}}(x,x)$ vanish for elements γ in the complement of $P_1(\mathbb{Q}) \cap M_P(\mathbb{Q}) \cap \mathfrak{o}$ in $M_P(\mathbb{Q}) \cap \mathfrak{o}$. The next step is to write $P_1(\mathbb{Q}) \cap M_P(\mathbb{Q}) \cap \mathfrak{o}$ as a product $\big(M_{P_1}(\mathbb{Q}) \cap \mathfrak{o}\big) N^P_{P_1}(\mathbb{Q})$, by appealing to Lemma 10.2. We then have to apply Lemma 10.2 again, with $(M_P, P_1 \cap M_P)$ in place of (G,P), to the resulting sum over (μ,ν) in the product of $M_{P_1}(\mathbb{Q}) \cap \mathfrak{o}$ with $N^P_{P_1}(\mathbb{Q})$. This yields a threefold sum, one of which is taken over the set

$$N^P_{P_1}(\mathbb{Q})_{\mu_s} = \exp\big(\mathfrak{n}^P_{P_1}(\mathbb{Q})_{\mu_s}\big),$$

where $\mathfrak{n}_{P_1}^P(\mathbb{Q})_{\mu_s}$ denotes the centralizer of μ_s in the Lie algebra $\mathfrak{n}_{P_1}^P(\mathbb{Q})$. The last step is to apply the Poisson summation formula to the lattice $\mathfrak{n}_{P_1}^P(\mathbb{Q})_{\mu_s}$ in $\mathfrak{n}_{P_1}^P(\mathbb{A})_{\mu_s}$. The resulting cancellation from the alternating sum over P then yields a formula for $\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x)$ analogous to the formula (8.8) for $k_{P_1,P_2}(x)$. Namely, $\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x)$ equals the product of $(-1)^{\dim(A_{P_2}/A_G)}$ with the sum over $\mu \in M_{P_1}(\mathbb{Q}) \cap \mathfrak{o}$ of

$$\sum_{\eta \in N_{P_1}(\mathbb{Q})_{\mu_s} \backslash N_{P_1}(\mathbb{Q})} \sum_{\xi \in \mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})'_{\mu_s}} \Big(\int_{\mathfrak{n}_{P_1}(\mathbb{A})_{\mu_s}} f\big(x^{-1}\eta^{-1}\mu \exp(X_1)\eta x\big)\psi\big(\langle \xi, X_1\rangle\big)\mathrm{d}X_1\Big),$$

where $\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})'_{\mu_s}$ is the intersection of $\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})_{\mu_s}$ with the set $\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})'$ in (8.8). The convergence of the integral (11.4) is then proved as at the end of §8. (See [**A3**, p. 948–949].)

Once we have shown that the integrals (11.4) are finite, we know that the identity (11.3) is valid. The remaining step is to compare it with the corresponding identity

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x)\mathrm{d}x = \sum_{P_1 \subset P_2} \int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \chi^T(x) k_{P_1,P_2,\mathfrak{o}}(x)\mathrm{d}x,$$

which we obtain by modifying the proof of Theorem 6.1 as in the last section.

Suppose that $P_1 \subset P_2$ are fixed. We can then write

$$\begin{aligned}
&\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \chi^T(x)\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x)\mathrm{d}x \\
&= \int_{M_{P_1}(\mathbb{Q})N_{P_1}(\mathbb{A})\backslash G(\mathbb{A})^1} \chi^T(x)\Big(\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} \widetilde{k}_{P_1,P_2,\mathfrak{o}}(n_1 x)\mathrm{d}n_1\Big)\mathrm{d}x,
\end{aligned}$$

since $\chi^T(x)$ is left $N_{P_1}(\mathbb{A})$-invariant. The integral of $\widetilde{k}_{P_1,P_2,\mathfrak{o}}(n_1 x)$ over n_1 is equal to the sum over pairs

$$(P, \mu), \qquad\qquad P_1 \subset P \subset P_2,\ \mu \in M_{P_1}(\mathbb{Q}) \cap \mathfrak{o},$$

of the product of the sign $(-1)^{\dim(A_P/A_G)}$ with the expression

$$\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} \sum_{\eta \in N_P(\mathbb{Q})_{\mu_s}\backslash N_P(\mathbb{Q})} \Big(\int_{N_P(\mathbb{A})_{\mu_s}} f(x^{-1}n_1^{-1}\eta^{-1}\mu n \eta n_1 x)\mathrm{d}n\Big)\mathrm{d}n_1.$$

If we replace the variable n_1 by νn_1, and then integrate over ν in $N_P(\mathbb{Q})\backslash N_P(\mathbb{A})$, we can change the sum over η to an integral over ν in $N_P(\mathbb{Q})_{\mu_s}\backslash N_P(\mathbb{A})$. Since the resulting integrand is invariant under left translation of ν by elements in the larger group $N_P(\mathbb{A})_{\mu_s}$, we can in fact integrate ν over $N_P(\mathbb{A})_{\mu_s}\backslash N_P(\mathbb{A})$. We can thus change the sum of η in the expression to an adelic integral over ν. Applying Lemma 10.2 to the resulting double integral over ν and n, we see that the expression equals

$$\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} \int_{N_P(\mathbb{A})} f(x^{-1}n_1^{-1}\mu n n_1 x)\mathrm{d}n\mathrm{d}n_1.$$

The signed sum over (P, μ) of this last expression equals

$$\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} k_{P_1,P_2,\mathfrak{o}}(n_1 x)\mathrm{d}n_1.$$

We conclude that

$$\begin{aligned}
&\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \chi^T(x)\widetilde{k}_{P_1,P_2,\mathfrak{o}}(x)\mathrm{d}x \\
&= \int_{M_{P_1}(\mathbb{Q})N_{P_1}(\mathbb{A})\backslash G(\mathbb{A})^1} \chi^T(x)\Big(\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} k_{P_1,P_2,\mathfrak{o}}(n_1x)\mathrm{d}n_1\Big)\mathrm{d}x \\
&= \int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \chi^T(x)k_{P_1,P_2,\mathfrak{o}}(x)\mathrm{d}x.
\end{aligned}$$

We have shown that the summands corresponding to $P_1 \subset P_2$ in the two identities are equal. It follows that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \widetilde{k}^T_{\mathfrak{o}}(x)\mathrm{d}x = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T_{\mathfrak{o}}(x)\mathrm{d}x = J^T_{\mathfrak{o}}(f).$$

This is the second assertion of Theorem 11.1. □

The formula (11.2) for $J^T_{\mathfrak{o}}(f)$ is better suited to computation. As an example, we consider the special case that the class $\mathfrak{o} \in \mathcal{O}$ consists entirely of semisimple elements. Then $\mathfrak{o}$ is a semisimple conjugacy class in $G(\mathbb{Q})$, and for any element $\gamma \in \mathfrak{o}$, the centralizer $G(\mathbb{Q})_\gamma$ of $\gamma = \gamma_s$ contains no nontrivial unipotent elements. In particular, the group $N_P(\mathbb{Q})_{\gamma_s} = N_P(\mathbb{Q})_\gamma$ attached to any P is trivial. It follows that

$$\widetilde{K}_{P,\mathfrak{o}}(x,x) = \sum_{\gamma\in M_P(\mathbb{Q})\cap\mathfrak{o}} \sum_{\eta\in N_P(\mathbb{Q})} f(x^{-1}\eta^{-1}\gamma\eta x).$$

To proceed, we need to characterize the intersection $M_P(\mathbb{Q}) \cap \mathfrak{o}$.

In §7, we introduced the Weyl set $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P'_1})$ attached to any pair of standard parabolic subgroups P_1 and P'_1. Suppose that P_1 is fixed. If P is any other standard parabolic subgroup, we define $W(P_1;P)$ to be the set of elements s in the union over $P'_1 \subset P$ of the sets $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P'_1})$ such that $s^{-1}\alpha > 0$ for every root α in the subset $\Delta^P_{P'_1}$ of $\Delta_{P'_1}$. In other words, $s^{-1}\alpha$ belongs to the set Φ_{P_1} for every such α. Suppose for example that $G = GL(n)$, and that P_1 corresponds to the partition $(\nu_1, \dots, \nu_{p_1})$ of n. We noted in §7 that each of the sets $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P'_1})$ is identified with a subset of the symmetric group S_{p_1}. The union over P'_1 of these sets is identified with the full group S_{p_1}. If P corresponds to the partition $(n_1, \dots, n_p)$ of n, $W(P_1;P)$ becomes the set of elements $s \in S_{p_1}$ such that $(\nu_{s(1)}, \dots, \nu_{s(p_1)})$ is finer than $(n_1, \dots, n_p)$, and such that $s^{-1}(i) < s^{-1}(i+1)$, for any i that is not of the form $n_1 + \cdots + n_k$ for some k.

The problem is simpler if we impose a second condition on $\mathfrak{o}$. Suppose that (P_1, α_1) represents the anisotropic rational datum attached to $\mathfrak{o}$ in the last section, and that γ_1 belongs to the anisotropic conjugacy class α_1 in $M_{P_1}(\mathbb{Q})$. Then γ_1 represents the semisimple conjugacy class in $\mathfrak{o}$. We know that the group H, obtained by taking the connected component of 1 in the centralizer of γ_1 in G, is contained in M_{P_1}. For H would otherwise have a proper parabolic subgroup over $\mathbb{Q}$, and $H(\mathbb{Q})$ would contain a nontrivial unipotent element, contradicting the condition that $\mathfrak{o}$ consist entirely of semisimple elements. The group $H(\mathbb{Q})$ is of finite index in $G(\mathbb{Q})_\gamma$. We shall say that $\mathfrak{o}$ is *unramified* if $G(\mathbb{Q})_\gamma$ is also contained in M_{P_1}. This is equivalent to asking that the stabilizer of the conjugacy class α_1 in $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1})$ be equal to $\{1\}$. In the case $G = GL(n)$, the condition is automatically satisfied, since any centralizer is connected.

Assume that $\mathfrak{o}$ is unramified, and that (P_1, α_1) and $\gamma_1 \in \alpha_1$ are fixed as above. The condition that $\mathfrak{o}$ be unramified implies that if (P_1', α_1') is any other representative of the anisotropic rational datum of $\mathfrak{o}$, there is a unique element in $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})$ that maps α_1 to α_1'. Suppose that P is any standard parabolic subgroup and that γ is an element in $M_P \cap \mathfrak{o}$. It follows easily from this discussion that γ can be expressed uniquely in the form

$$\gamma = \mu^{-1} w_s \gamma_1 w_s^{-1} \mu, \qquad s \in W(P_1; P),\ \mu \in M_P(\mathbb{Q})_{w_s\gamma_1 w_s^{-1}} \backslash M_P(\mathbb{Q}),$$

where as usual,

$$M_P(\mathbb{Q})_{w_s\gamma_1 w_s^{-1}} = M_{P, w_s\gamma_1 w_s^{-1}}(\mathbb{Q})$$

is the centralizer of $w_s\gamma_1 w_s^{-1}$ in $M_P(\mathbb{Q})$. (See [**A3**, p. 950].)

Having characterized the intersection $M_P(\mathbb{Q}) \cap \mathfrak{o}$, we can write

$$\begin{aligned}
&\widetilde{K}_{P,\mathfrak{o}}(x,x)\\
&= \sum_{s \in W(P_1;P)} \sum_{\mu} \sum_{\eta \in N_P(\mathbb{Q})} f(x^{-1}\eta^{-1}\mu^{-1} w_s\gamma_1 w_s^{-1}\mu\eta x)\\
&= \sum_{s}\sum_{\pi} f(x^{-1}\pi^{-1} w_s\gamma_1 w_s^{-1}\pi x),
\end{aligned}$$

where μ and π are summed over the right cosets of $M_P(\mathbb{Q})_{w_s\gamma w_s^{-1}}$ in $M_P(\mathbb{Q})$ and $P(\mathbb{Q})$ respectively. Therefore $\widetilde{k}^T_{\mathfrak{o}}(x)$ equals the expression

$$\begin{aligned}
&\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \widetilde{K}_{P,\mathfrak{o}}(\delta x, \delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big)\\
&= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{s \in W(P_1;P)} \sum_{\delta} f(x^{-1}\delta^{-1} w_s\gamma_1 w_s^{-1}\delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big),
\end{aligned}$$

where δ is summed over the right cosets of $M_P(\mathbb{Q})_{w_s\gamma_1 w_s^{-1}}$ in $G(\mathbb{Q})$. Set $\delta_1 = w_s^{-1}\delta$. Since

$$w_s^{-1}\big(M_P(\mathbb{Q})_{w_s\gamma_1 w_s^{-1}}\big)w_s = G(\mathbb{Q})_{\gamma_1} = M_{P_1}(\mathbb{Q})_{\gamma_1},$$

we obtain

$$\begin{aligned}
&\widetilde{k}^T_{\mathfrak{o}}(x)\\
&= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{s \in W(P_1;P)} \sum_{\delta_1} f(x^{-1}\delta_1^{-1}\gamma_1\delta_1 x)\widehat{\tau}_P\big(H_P(w_s\delta_1 x) - T\big)\\
&= \sum_{\delta_1} f(x^{-1}\delta_1^{-1}\gamma_1\delta_1 x)\psi^T(\delta_1 x),
\end{aligned}$$

where δ_1 is summed over right cosets of $M_{P_1}(\mathbb{Q})_{\gamma_1}$ in $G(\mathbb{Q})$, and

$$\begin{aligned}
\psi^T(y) = \psi^T_{P_1}(y) &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{s \in W(P_1;P)} \widehat{\tau}_P\big(H_P(w_s y) - T\big)\\
&= \sum_{P_1'} \sum_{s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})} \sum_{\{P : s \in W(P_1;P)\}} (-1)^{\dim(A_P/A_G)}\widehat{\tau}_P\big(H_{P_1'}(w_s y) - T\big).
\end{aligned}$$

Therefore

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \widetilde{k}_{\mathfrak{o}}^T(x,f)\mathrm{d}x$$
$$= \int_{M_{P_1}(\mathbb{Q})_{\gamma_1}\backslash G(\mathbb{A})^1} f(x^{-1}\gamma x)\psi^T(x)\mathrm{d}x.$$

The convergence of the second integral follows from the convergence of the first integral (Theorem 11.1), and the fact (implied by Lemma 11.2 below) that the function χ_T is nonnegative.

We can write

$$M_{P_1}(\mathbb{Q})_{\gamma_1}\backslash G(\mathbb{A})^1 \cong \big(M_{P_1}(\mathbb{Q})_{\gamma_1}\backslash M_{P_1}(\mathbb{A})^1_{\gamma_1}\big) \times \big(M_{P_1}(\mathbb{A})^1_{\gamma_1}\backslash G(\mathbb{A})^1\big),$$

where $M_{P_1}(\mathbb{A})^1_{\gamma_1}$ is the centralizer of γ_1 in the group $M_{P_1}(\mathbb{A})^1$. Since the centralizer of γ_1 in $M_{P_1}(\mathbb{A})$ equals its centralizer $G(\mathbb{A})_{\gamma_1}$ in $G(\mathbb{A})$, we can also write

$$M_{P_1}(\mathbb{A})^1_{\gamma_1}\backslash G(\mathbb{A})^1 \cong \big(A_{P_1}(\mathbb{R})^0 \cap G(\mathbb{R})^1\big) \times \big(G(\mathbb{A})_{\gamma_1}\backslash G(\mathbb{A})\big).$$

In the formula for $J_{\mathfrak{o}}^T(f)$ we have just obtained, we are therefore free to decompose the variable of integration as

$$x = may, \quad m \in M_{P_1}(\mathbb{Q})_{\gamma_1}\backslash M_{P_1}(\mathbb{A})^1_{\gamma_1},\ a \in A_{P_1}(\mathbb{R})^0 \cap G(\mathbb{R})^1,\ y \in G(\mathbb{A})_{\gamma_1}\backslash G(\mathbb{A}).$$

Then $f(x^{-1}\gamma_1 x) = f(y^{-1}\gamma_1 y)$ and $\psi^T(x) = \psi^T(ay)$. Therefore

$$J_{\mathfrak{o}}^T(f) = \mathrm{vol}\big(M_{P_1}(\mathbb{Q})_{\gamma_1}\backslash M_{P_1}(\mathbb{A})^1_{\gamma_1}\big)\int_{G(\mathbb{A})_{\gamma_1}\backslash G(\mathbb{A})} f(y^{-1}\gamma_1 y)v_{P_1}^T(y)\mathrm{d}y, \tag{11.5}$$

where

$$v_{P_1}^T(y) = \int_{A_{P_1}(\mathbb{R})^0 \cap G(\mathbb{R})^1} \psi^T(ay)\mathrm{d}a = \int_{\mathfrak{a}_{P_1}^G} \psi^T(\exp H \cdot y)\mathrm{d}H.$$

It remains to evaluate the function $v_{P_1}^T(y)$.

For any parabolic subgroup $Q \supset P_0$ and any point $\Lambda \in \mathfrak{a}_Q^*$, define $\varepsilon_Q(\Lambda)$ to be the sign $+1$ or -1 according to whether the number of roots $\alpha \in \Delta_Q$ with $\Lambda(\alpha^\vee) \leq 0$ is even or odd. Let

$$H \longrightarrow \phi_Q(\Lambda, H), \qquad H \in \mathfrak{a}_Q,$$

be the characteristic function of the set of H such that for any $\alpha \in \Delta_Q$, $\varpi_\alpha(H) > 0$ if $\Lambda(\alpha^\vee) \leq 0$, and $\varpi_\alpha(H) \leq 0$ if $\Lambda(\alpha^\vee) > 0$. These functions were introduced by Langlands [**Lan1**], and are useful for studying certain convex polytopes. We apply them to the discussion above by taking $Q = P_1'$ and $\Lambda = s\Lambda_1$, for an element $s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})$ and a point Λ_1 in the chamber

$$(\mathfrak{a}_{P_1}^*)^+ = \big\{\Lambda_1 \in \mathfrak{a}_{P_1}^* :\ \Lambda_1(\alpha^\vee) > 0,\ \alpha \in \Delta_{P_1}\big\}.$$

Suppose that s belongs to any one of the sets $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})$. We claim that for any point $H' \in \mathfrak{a}_{P_1'}$, the expression

$$\sum_{\{P : s \in W(P_1;P)\}} (-1)^{\dim(A_P/A_G)}\widehat{\tau}_P(H') \tag{11.6}$$

that occurs in the definition of $\psi^T(y)$ equals

$$\varepsilon_{P_1'}(s\Lambda_1)\phi_{P_1'}(s\Lambda_1, H'), \qquad \Lambda_1 \in (\mathfrak{a}_{P_1}^*)^+. \tag{11.7}$$

To see this, define a parabolic subgroup $P^s \supset P_1'$ by setting

$$\Delta_{P_1'}^{P^s} = \{\alpha \in \Delta_{P_1'} : \ s^{-1}\alpha > 0\}.$$

The element s then lies in $W(P_1; P)$ if and only if $P_1' \subset P \subset P^s$. The expression (11.6) therefore equals

$$\sum_{\{P: P_1' \subset P \subset P^s\}} (-1)^{\dim(A_P/A_G)} \widehat{\tau}_P(H').$$

If we write the projection of H' onto $\mathfrak{a}_{P_1'}^G$ in the form

$$\sum_{\alpha} c_\alpha \alpha^\vee, \qquad \alpha \in \Delta_{P_1'}, \ c_\alpha \in \mathbb{R},$$

we can apply (6.3) to the alternating sum over P. We see that the expression equals the sign $\varepsilon_{P_1'}(s\Lambda_1)$ if H' lies in the support of the function $\phi_{P_1'}(s\Lambda_1, H')$, and vanishes otherwise. The claim is therefore valid.

The function $\psi^T(\exp H \cdot y)$ equals

$$\sum_{P_1'} \sum_{s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})} \sum_{\{P: s \in W(P_1; P)\}} (-1)^{\dim(A_P/A_G)} \widehat{\tau}_P\big(sH + H_{P_1'}(w_s y) - T_{P_1'}\big),$$

where $T_{P_1'}$ is the projection of T onto $\mathfrak{a}_{P_1'}$. This in turn equals

$$(11.8) \qquad \sum_{P_1'} \sum_{s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'})} \varepsilon_{P_1'}(s\Lambda_1) \phi_{P_1'}\big(s\Lambda_1, sH + H_{P_1'}(w_s y) - T_{P_1'}\big),$$

by what we have just established. Now as a function $H \in \mathfrak{a}_{P_1}^G$, (11.8) would appear to be complicated. It is not! One shows in fact that (11.8) equals the characteristic function of the projection onto $\mathfrak{a}_{P_1}^G$ of the convex hull of

$$\big\{Y_s = s^{-1}\big(T_{P_1'} - H_{P_1'}(w_s y)\big) : \ s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_1'}), \ P_1' \supset P_0\big\}.$$

The proof of this fact [**A1**, Lemma 3.2] uses elementary properties of convex hulls and a combinatorial lemma of Langlands [**A1**, §2]. We shall discuss it in greater generality later, in §17. In the meantime, we shall illustrate the property geometrically in the special case that $G = SL(3)$.

Assume for the moment then that $G = SL(3)$ and $P_1 = P_0$. In this case, the signed sum of characteristic functions

$$\phi_{P_1'}\big(s\Lambda_1, sH + H_{P_1'}(w_s y) - T\big) = \phi_{P_1'}\big(s\Lambda_1, s(H - Y_s)\big), \qquad H \in \mathfrak{a}_{P_1} = \mathfrak{a}_{P_1}^G,$$

is over elements s parametrized by the symmetric group S_3. We have of course the simple roots $\Delta_{P_1} = \{\alpha_1, \alpha_2\}$, and the basis $\{\alpha_1^\vee, \alpha_2^\vee\}$ of $\mathfrak{a}_{P_1}$ dual to $\widehat{\Delta}_{P_1}$. Writing

$$s(H - Y_s) = t_1\alpha_1^\vee + t_2\alpha_2^\vee, \qquad t_i \in \mathbb{R},$$

we see that $\phi_{P_1'}\big(s\Lambda_1, s(H - Y_s)\big)$ is the characteristic function of the affine cone

$$\big\{H = Y_s + t_1 s^{-1}(\alpha_1^\vee) + t_2 s^{-1}(\alpha_2^\vee) : \ t_i > 0 \text{ if } s^{-1}(\alpha_i) < 0; \ t_i \leq 0 \text{ if } s^{-1}(\alpha_i) > 0\big\}.$$

In Figure 11.1, we plot the six vertices $\{Y_s\}$, the associated six cones, and the signs

$$\varepsilon_{P_1'}(s\Lambda_1) = (-1)^{|\{i: s^{-1}(\alpha_i) < 0\}|}, \qquad s \in S_3,$$

by which the corresponding characteristic functions have to be multiplied. We then observe that the signs cancel in every region of the plane except the convex hull of the set of vertices.

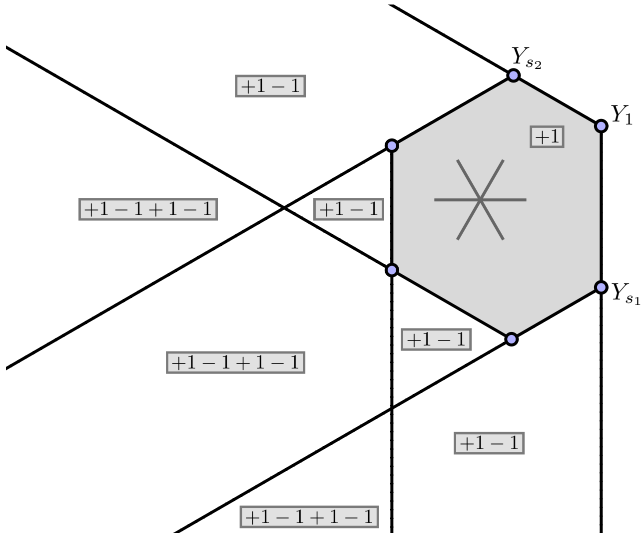

FIGURE 11.1. The shaded region is the convex hull of six points $\{Y_s\}$ in the two dimensional vector space $\mathfrak{a}_0$ attached to $SL(3)$. It is a signed sum of six cones, with vertices at each of the six points.

Returning to the general case, we take for granted the assertion that (11.8) is equal to the characteristic function of the convex hull. Then $v^T_{P_1}(y)$ equals the volume of the given convex hull. In particular, the manipulations used to derive the formula (11.5) for $J^T_{\mathfrak{o}}(f)$ are justified. Observe that

$$\begin{aligned} Y_s &= s^{-1}\big(T_{P_1'} - H_{P_1'}(w_s y)\big) \\ &= s^{-1}\big(T_{P_1'} - H_{P_1'}(\widetilde{w}_s y) - H_{P_1'}(w_s \widetilde{w}_s^{-1})\big) \\ &= s^{-1}\big(T_{P_1'} - H_{P_1'}(\widetilde{w}_s y) - (T_0)_{P_1'} + s(T_0)_{P_1}\big). \end{aligned}$$

When $T = T_0$, the point Y_s equals

$$-s^{-1} H_{P_1'}(\widetilde{w}_s y) + (T_0)_{P_1}.$$

The point $(T_0)_{P_1}$ is independent of s, and consequently represents a fixed translate of the convex hull. Since it has no effect on the volume, it may be removed from consideration.

We have established the following result, which we state with P and γ in place of P_1 and γ_1.

THEOREM 11.2. *Suppose that $\mathfrak{o} \in \mathcal{O}$ is an unramified class, with anisotropic rational datum represented by a pair (P, α). Then*

$$J_{\mathfrak{o}}(f) = \mathrm{vol}\big(M_P(\mathbb{Q})_\gamma \backslash M_P(\mathbb{A})^1_\gamma\big) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1}\gamma x) v_P(x) \mathrm{d}x, \tag{11.9}$$

where γ is any element in the $M_P(\mathbb{Q})$-conjugacy class α, and $v_P(x)$ is the volume of the projection onto $\mathfrak{a}^G_P$ of the convex hull of

$$\big\{ - s^{-1} H_{P'}(\widetilde{w}_s x) : \ s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'}), \ P' \supset P_0 \big\}.$$

□

12. Cuspidal automorphic data

We shall temporarily put aside the finer analysis of the geometric expansion in order to develop the spectral side. We are looking for spectral analogues of the geometric results we have already obtained. In this section, we introduce a set $\mathfrak{X}$ that will serve as the analogue of the set $\mathcal{O}$ of §10. Its existence is a basic consequence of Langlands' theory of Eisenstein series.

A function ϕ in $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ is said to be *cuspidal* if

$$\int_{N_P(\mathbb{A})} \phi(nx)\mathrm{d}n = 0, \tag{12.1}$$

for every $P \neq G$ and almost every $x \in G(\mathbb{A})^1$. This condition is a general analogue of the vanishing of the constant term of a classical modular form, which characterizes space of cusp forms. The subspace $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ of cuspidal functions in $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ is closed and invariant under right translation by $G(\mathbb{A})^1$. The following property of this subspace is one of the foundations of the subject.

THEOREM 12.1 (Gelfand, Piatetski-Shapiro). *The space $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ decomposes under the action of $G(\mathbb{A})^1$ into a discrete sum of irreducible representations with finite multiplicities. In particular, $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ is a subspace of $L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$.*

The proof is similar to that of the discreteness of the decomposition of R, in the case of compact quotient. For if $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact, there are no proper parabolic subgroups, by the criterion of Borel and Harish-Chandra, and $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ equals $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$. In general, one combines the vanishing condition (12.1) with the approximate fundamental domain of Theorem 8.1 to show that for any $f \in C_c^\infty\big(G(\mathbb{A})^1\big)$, the restriction $R_{\mathrm{cusp}}(f)$ of $R(f)$ to $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ is of Hilbert-Schmidt class. In particular, if $f(x) = \overline{f(x^{-1})}$, $R_{\mathrm{cusp}}(f)$ is a compact self-adjoint operator. One then uses the spectral theorem to show that $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ decomposes discretely. See [**Lan5**] and [**Har4**]. □

The theorem provides a $G(\mathbb{A})^1$-invariant orthogonal decomposition

$$L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big) = \bigoplus_\sigma L^2_{\mathrm{cusp},\sigma}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big),$$

where σ ranges over irreducible unitary representations of $G(\mathbb{A})^1$, and $L^2_{\mathrm{cusp},\sigma}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ is $G(\mathbb{A})^1$-isomorphic to a finite number of copies of σ. We define a *cuspidal automorphic datum* to be an equivalence class of pairs (P, σ), where $P \subset G$ is a standard parabolic subgroup of G, and σ is an irreducible representation of $M_P(\mathbb{A})^1$ such that the space $L^2_{\mathrm{cusp},\sigma}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$ is nonzero. The equivalence relation is defined by the conjugacy, which for standard parabolic groups is given by the Weyl sets $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$. In other words, (P', σ') is equivalent to (P, σ) if there is an element $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ such that the representation

$$s^{-1}\sigma' : \; m \longrightarrow \sigma'(w_s m w_s^{-1}), \qquad m \in M_P(\mathbb{A})^1,$$

of $M_P(\mathbb{A})^1$ is equivalent to σ. We write $\mathfrak{X} = \mathfrak{X}^G$ for the set of cuspidal automorphic data $\chi = \{(P, \sigma)\}$.

Cuspidal functions do not appear explicitly in Theorem 7.2, but they are an essential ingredient of Langlands's proof. For example, they give rise to a decomposition

$$L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big) = \bigoplus_{\mathcal{P}} L^2_{\mathcal{P}\text{-cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big), \tag{12.2}$$

which is based on cuspidal automorphic data, and is more elementary than the spectral decomposition (7.5). Let us describe it.

For any P, we have defined the right $G(\mathbb{A})$-invariant Hilbert space $\mathcal{H}_P$ of functions on $G(\mathbb{A})$, and the dense subspace $\mathcal{H}_P^0$. Let $\mathcal{H}_{P,\mathrm{cusp}}$ be the subspace of vectors $\phi \in \mathcal{H}_P$ such that for almost all $x \in G(\mathbb{A})$, the function $\phi_x(m) = \phi(mx)$ on $M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1$ lies in the space $L^2_{\mathrm{cusp}}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$. Then

$$\mathcal{H}_{P,\mathrm{cusp}} = \bigoplus_{\sigma} \mathcal{H}_{\mathcal{P},\mathrm{cusp},\sigma},$$

where for any irreducible unitary representation σ of $M_P(\mathbb{A})^1$, $\mathcal{H}_{P,\mathrm{cusp},\sigma}$ is the subspace of vectors $\phi \in \mathcal{H}_{\mathcal{P},\mathrm{cusp}}$ such that each of the functions ϕ_x lies in the space $L^2_{\mathrm{cusp},\sigma}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$. We write $\mathcal{H}^0_{P,\mathrm{cusp}}$ and $\mathcal{H}^0_{\mathcal{P},\mathrm{cusp},\sigma}$ for the respective intersections of $\mathcal{H}_{P,\mathrm{cusp}}$ and $\mathcal{H}_{P,\mathrm{cusp},\sigma}$ with $\mathcal{H}^0_P$.

Suppose that $\Psi(\lambda)$ is an entire function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ of Paley-Wiener type, with values in a finite dimensional subspace of functions $x \to \Psi(\lambda, x)$ in $\mathcal{H}^0_{P,\mathrm{cusp},\sigma}$. Then $\Psi(\lambda, x)$ is the Fourier transform in λ of a smooth, compactly supported function on $\mathfrak{a}_P$. This means that for any point $\Lambda \in \mathfrak{a}^*_P$, the function

$$\psi(x) = \int_{\Lambda + i\mathfrak{a}^*_P} \mathrm{e}^{(\lambda+\rho_P)(H_P(x))}\Psi(\lambda, x)\mathrm{d}\lambda$$

of $x \in N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$ is compactly supported in $H_P(x)$.

Lemma 12.2 (Langlands). *The function*

$$(E\psi)(x) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \psi(\delta x), \qquad x \in G(\mathbb{Q})\backslash G(\mathbb{A}),$$

lies in $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$.

Lemma 12.3 (Langlands). *Suppose that* $\Psi'(\lambda', x)$ *is a second such function, attached to a pair* (P', σ'). *Then the inner product formula*

$$(E\psi, E\psi') = \int_{\Lambda + i\mathfrak{a}^*_P} \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})} \big(M(s,\lambda)\Psi(\lambda), \Psi'(-s\overline{\lambda})\big)\mathrm{d}\lambda \tag{12.3}$$

holds if Λ *is any point in* $\mathfrak{a}^*_P$ *such that* $(\Lambda - \rho_P)(\alpha^\vee) > 0$ *for every* $\alpha \in \Delta_P$.

If χ is the class in $\mathfrak{X}$ represented by a pair (P, σ), let $L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ be the closed, $G(\mathbb{A})$-invariant subspace of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ generated by the functions $E\psi$ attached to (P, σ).

Lemma 12.4 (Langlands). *There is an orthogonal decomposition*

$$L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big) = \bigoplus_{\chi \in \mathfrak{X}} L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big). \tag{12.4}$$

Lemmas 12.2–12.4 are discussed in the early part of Langlands's survey article [**Lan1**]. They are the foundations for the rest of the theory, and for Theorem 7.2 in particular. We refer the reader to [**Lan1**] for brief remarks on the proofs, which are relatively elementary. □

The inner product formula (12.3) is especially important. It is used in the proof of both the analytic continuation (a) and the spectral decomposition (b) in Theorem 7.2. Observe that the domain of integration in (12.3) is contained in the region of absolute convergence of the cuspidal operator valued function $M(s,\lambda)$ in the integrand. Once he had proved the meromorphic continuation of this function, Langlands was able to use (12.3) to establish the remaining analytic continuation assertions of Theorem 7.2(a), and the spectral decomposition of the space $L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. His method was based on a change contour of integration from $\Lambda + i\mathfrak{a}_P^*$ to $i\mathfrak{a}_P^*$, and an elaborate analysis of the resulting residues. It was a tour de force, the details of which comprise the notoriously difficult Chapter 7 of [**Lan5**].

Any class $\chi = \{(\mathcal{P},\sigma)\}$ in $\mathfrak{X}$ determines an associated class $\mathcal{P}_\chi = \{P\}$ of standard parabolic subgroups. We then obtain a decomposition (12.2) from (12.4) by setting

$$L^2_{\mathcal{P}\text{-cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big) = \bigoplus_{\{\chi\in\mathfrak{X}:\mathcal{P}_\chi=\mathcal{P}\}} L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big).$$

However, it is the finer decomposition (12.4) that is more often used. We shall actually apply the obvious variant of (12.4) that holds for $G(\mathbb{A})^1$ in place of $G(\mathbb{A})$, or rather its restriction

$$L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big) = \bigoplus_{\chi\in\mathfrak{X}} L^2_{\mathrm{disc},\chi}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big) \tag{12.5}$$

to the discrete spectrum, in which

$$L^2_{\mathrm{disc},\chi}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big) = L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)\cap L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big).$$

If P is a standard parabolic subgroup, the correspondence

$$(P_1\cap M_P,\sigma_1) \longrightarrow (P_1,\sigma_1), \qquad P_1\subset P,\ \{(P_1\cap M_P,\sigma_1)\}\in\mathfrak{X}^{M_P},$$

yields a mapping $\chi_P\to\chi$ from $\mathfrak{X}^{M_P}$ to the set $\mathfrak{X}=\mathfrak{X}^G$. We can then write

$$L^2_{\mathrm{disc}}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big) = \bigoplus_{\chi\in\mathfrak{X}} L^2_{\mathrm{disc},\chi}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big),$$

where $L^2_{\mathrm{disc},\chi}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$ is the sum of those subspaces of $L^2_{\mathrm{disc}}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$ attached to classes $\chi_P\in\mathfrak{X}^{M_P}$ in the fibre of χ. Let $\mathcal{H}_{P,\chi}$ be the subspace of functions ϕ in the Hilbert space $\mathcal{H}_P$ such that for almost all x, the function $\phi_x(m)=\phi(mx)$ lies in $L^2_{\mathrm{disc},\chi}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$. There is then an orthogonal direct sum

$$\mathcal{H}_P = \bigoplus_\chi \mathcal{H}_{\mathcal{P},\chi}.$$

There is also an algebraic direct sum

$$\mathcal{H}^0_P = \bigoplus_\chi \mathcal{H}^0_{P,\chi}, \tag{12.6}$$

where $\mathcal{H}^0_{P,\chi}$ is the intersection of $\mathcal{H}_{P,\chi}$ with $\mathcal{H}^0_P$. For any λ and f, we shall write $\mathcal{I}_{P,\chi}(\lambda,f)$ for the restriction of the operator $\mathcal{I}_P(\lambda,f)$ to the invariant subspace $\mathcal{H}_{P,\chi}$ of $\mathcal{H}_P$.

At the end of §7, we described the spectral expansions for both the kernel $K(x,y)$ and the truncated function $k^T(x)$ in terms of Eisenstein series. They were defined by means of an orthonormal basis $\mathcal{B}_P$ of $\mathcal{H}_P$. We can assume that $\mathcal{B}_P$ is compatible with the algebraic direct sum (12.6). In other words,

$$\mathcal{B}_P = \coprod_{\chi\in\mathfrak{X}} \mathcal{B}_{\mathcal{P},\chi},$$

where $\mathcal{B}_{\mathcal{P},\chi}$ is the intersection of $\mathcal{B}_P$ with $\mathcal{H}^0_{\mathcal{P},\chi}$. For any $\chi \in \mathfrak{X}$ we set

$$K_\chi(x,y) = \sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{\phi\in\mathcal{B}_{P,\chi}} E\big(x, \mathcal{I}_{P,\chi}(\lambda,f)\phi,\lambda\big)\overline{E(y,\phi,\lambda)}\mathrm{d}\lambda, \tag{12.7}$$

where n_P is the integer defined in Theorem 7.2(b). It is a consequence of Langlands' construction of the spectral decomposition (7.5) from the more elementary decomposition (12.4) that $K_\chi(x,y)$ is the kernel of the restriction of $R(f)$ to the invariant subspace $L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. It follows, either from this or from the definition (12.7), that

$$K(x,y) = \sum_{\chi\in\mathfrak{X}} K_\chi(x,y). \tag{12.8}$$

This is the spectral analogue of the geometric decomposition (10.1).

More generally, suppose that we fix P, and use $P_1 \subset P$ in place of P to index the orthonormal bases. Then we have

$$K_P(x,y) = \sum_{\chi\in\mathfrak{X}} K_{P,\chi}(x,y),$$

where $K_{P,\chi}(x,y)$ is equal to

$$\sum_{P_1\subset P} (n^P_{P_1})^{-1} \int_{i\mathfrak{a}_P^*} \sum_{\phi\in\mathcal{B}_{P_1,\chi}} E^P_{P_1}\big(x,\mathcal{I}_{P_1,\chi}(\lambda,f)\phi,\lambda\big)\overline{E^P_{P_1}(y,\phi,\lambda)}\mathrm{d}\lambda.$$

We obtain a decomposition

$$k^T(x) = \sum_{\chi\in\mathfrak{X}} k^T_\chi(x), \tag{12.9}$$

where

$$\begin{aligned} k^T_\chi(x) &= k^T_\chi(x,f) \\ &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\chi}(\delta x,\delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big). \end{aligned}$$

This is the spectral analogue of the geometric decomposition (10.2) of the truncated kernel.

We have given spectral versions of the constructions at the beginning of §10. However, the spectral analogue of the coarse geometric expansion (10.7) is more difficult. The problem is to obtain an analogue of Corollary 10.1. We know from Theorem 6.1 that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Big|\sum_\chi k^T_\chi(x)\Big|\mathrm{d}x = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |k^T(x)|\mathrm{d}x < \infty.$$

To obtain a corresponding expansion for $J^T(f)$, we would need the stronger assertion that the double integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_\chi |k_\chi^T(x)|\mathrm{d}x$$

is finite. Unlike the geometric case of Corollary 10.1, this is not an immediate consequence of the proof of Theorem 6.1. It requires some new methods.

13. A truncation operator

The process that assigns the modified function $k^T(x) = k^T(x, f)$ to the original kernel $K(x, x)$ can be regarded as a construction that is based on the adjoint action of G on itself. It is compatible with the geometry of classes $\mathfrak{o} \in \mathcal{O}$. The process is less compatible with the spectral properties of classes $\chi \in \mathfrak{X}$. However, we still have to deal with the spectral expansion (12.9) of $k^T(x)$. We do so by introducing an operator that systematically truncates functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

The operator depends on the same parameter T used to define $k^T(x)$. It acts on the space $\mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ of locally bounded, measurable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. For any suitably regular point $T \in \mathfrak{a}_0^+$ and any function $\phi \in \mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$, we define $\Lambda^T\phi$ to be the function in $\mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ whose value at x equals

$$(13.1)\qquad \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi(n\delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big)\mathrm{d}n.$$

The inner sum may be taken over a finite set (that depends on x), while the integrand is a bounded function of n. Notice the formal similarity of the definition with that of $k^T(x)$ in §6. Notice also that if ϕ belongs to $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$, then $\Lambda^T\phi = \phi$.

There are three basic properties of the operator Λ^T to be discussed in this section. The first is that Λ^T is an orthogonal projection.

Proposition 13.1. (a) *For any P_1, any $\phi_1 \in \mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$, and any $x_1 \in G(\mathbb{A})^1$, the integral*

$$\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} (\Lambda^T\phi_1)(n_1x_1)\mathrm{d}n_1$$

vanishes unless $\varpi\big(H_{P_1}(x_1) - T\big) \leq 0$ for every $\varpi \in \widehat{\Delta}_{P_1}$.

(b) $\Lambda^T \circ \Lambda^T = \Lambda^T$.

(c) *The operator Λ^T is self-adjoint, in the sense that it satisfies the inner product formula*

$$(\Lambda^T\phi_1, \phi_2) = (\phi_1, \Lambda^T\phi_2),$$

for functions $\phi_1 \in \mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ and $\phi_2 \in C_c\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$.

The first assertion of the proposition is Lemma 1.1 of [**A4**]. (The symbol $<$ in the statement of this lemma should in fact be $\leq$.) In the case $G = SL(2)$, it follows directly from classical reduction theory, as illustrated in the earlier Figure 8.3. In general, one has to apply the Bruhat decomposition to elements in the sum over $P(\mathbb{Q})\backslash G(\mathbb{Q})$ that occurs in the definition of $\Lambda^T\phi$. We recall that the Bruhat decomposition is a double coset decomposition

$$G(\mathbb{Q}) = \coprod_{s\in W_0} \big(B_0(\mathbb{Q})w_s N_0(\mathbb{Q})\big)$$

of $G(\mathbb{Q})$, which in turn leads easily to a characterization

$$P(\mathbb{Q})\backslash G(\mathbb{Q}) \cong \coprod_{s\in W_0^M\backslash W_0} \big(w_s^{-1}N_0(\mathbb{Q})w_s \cap N_0(\mathbb{Q})\backslash N_0(\mathbb{Q})\big)$$

of $P(\mathbb{Q})\backslash G(\mathbb{Q})$. Various manipulations, which we will not reproduce here, reduce the assertion of (i) to Identity 6.2.

The assertion (ii) follows from (i). Indeed, $\big(\Lambda^T(\Lambda^T\phi)\big)(x)$ equals the sum over $P_1 \supset P_0$ and $\delta_1 \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$ of

$$\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} (\Lambda^T\phi)(n_1\delta_1 x)\widehat{\tau}_{P_1}\big(H_{P_1}(\delta_1 x) - T\big)\mathrm{d}n_1.$$

The term corresponding to $P_1 = G$ equals $(\Lambda^T\phi)(x)$, while if $P_1 \neq G$, the term vanishes by (i) and the definition of $\widehat{\tau}_{P_1}$.

To establish (iii), we observe that

$$\begin{aligned}
&(\Lambda^T\phi_1,\phi_2)\\
&= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \cdot\\
&\qquad \cdot \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi_1(n\delta x)\widehat{\tau}_P\big(H_P(\delta x)-T\big)\overline{\phi_2(x)}\mathrm{d}n\mathrm{d}x\\
&= \sum_P (-1)^{\dim(A_P/A_G)} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \int_{P(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi_1(nx)\overline{\phi_2(x)}\widehat{\tau}_P\big(H_P(x)-T\big)\mathrm{d}x\mathrm{d}n\\
&= \sum_P (-1)^{\dim(A_P/A_G)} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \int_{P(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi_1(x)\overline{\phi_2(nx)}\widehat{\tau}_P\big(H_P(x)-T\big)\mathrm{d}x\mathrm{d}n\\
&= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \cdot\\
&\qquad \cdot \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi_1(x)\overline{\phi_2(n\delta x)}\widehat{\tau}_P\big(H_P(\delta x)-T\big)\mathrm{d}n\mathrm{d}x\\
&= (\phi_1,\Lambda^T\phi_2).
\end{aligned}$$

□

It is not hard to show from (ii) and (iii) that Λ^T extends to an orthogonal projection from the space $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ to itself. It is also easy to see that Λ^T preserves each of the spaces $L^2_{\mathcal{P}\text{-cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ in the cuspidal decomposition (12.2). On the other hand, Λ^T is decidedly not compatible with the spectral decomposition (7.5). It is an operator built upon the cuspidal properties of §12, rather than the more sensitive spectral properties of Theorem 7.2.

The second property of the operator Λ^T is that it transforms uniformly tempered functions to rapidly decreasing functions. To describe this property quantitatively, we need to choose a height function $\|\cdot\|$ on $G(\mathbb{A})$.

Suppose first that G is a general linear group $GL(m)$, and that $x = (x_{ij})$ is a matrix in $GL(m, \mathbb{A})$. We define

$$\|x_v\|_v = \max_{i,j} |x_{ij,v}|_v$$

if v is a p-adic valuation, and

$$\|x_v\|_v = \Big(\sum_{i,j} |x_{ij,v}|_v^2 \Big)^{\frac{1}{2}}$$

if v is the archimedean valuation. Then $\|x_v\|_v = 1$ for almost all v. The height function

$$\|x\| = \prod_v \|x_v\|_v$$

is therefore defined by a finite product. For arbitrary G, we fix a $\mathbb{Q}$-rational injection $r\colon G \to GL(m)$, and define

$$\|x\| = \|r(x)\|.$$

By choosing r appropriately, we can assume that the set of points $x \in G(\mathbb{A})$ with $\|x\| \leq t$ is compact, for any $t > 0$. The chosen height function $\|\cdot\|$ on $G(\mathbb{A})$ then satisfies

$$\|xy\| \leq \|x\|\|y\|, \qquad x, y \in G(\mathbb{A}), \tag{13.2}$$

$$\|x^{-1}\| \leq C_0 \|x\|^{N_0}, \qquad x \in G(\mathbb{A}), \tag{13.3}$$

and

$$\big|\{x \in G(\mathbb{Q}) :\ \|x\| \leq t\}\big| \leq C_0 t^{N_0}, \qquad t \geq 0, \tag{13.4}$$

for positive contants C_0 and N_0. (See [**Bor2**].)

We shall say that a function ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is *rapidly decreasing* if for any positive integer N and any Siegel set $\mathcal{S} = \mathcal{S}^G(T_1)$ for $G(\mathbb{A})$, there is a positive constant C such that

$$|\phi(x)| \leq C\|x\|^{-N}$$

for every x in $\mathcal{S}^1 = \mathcal{S} \cap G(\mathbb{A})^1$. The notion of uniformly tempered applies to the space of smooth functions

$$C^\infty\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big) = \varinjlim_{K_0} C^\infty\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big).$$

By definition, $C^\infty\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$ is the space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ that are right invariant under the open compact subgroup K_0 of $G(\mathbb{A}_{\mathrm{fin}})$, and are infinitely differentiable as functions on the subgroup $G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$ of $G(\mathbb{A})^1$. We can of course also define the larger space $C^r\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ of functions of differentiability class C^r in the same way. If X is a left invariant differential operator on $G(\mathbb{R})^1$ of degree $k \leq r$, and ϕ lies in $C^r\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$, $X\phi$ is a function in $C^{r-k}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$. Let us say that a function $\phi \in C^\infty\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ is *uniformly tempered* if there is an $N_0 \geq 0$ with the property that for every left invariant differentiable operator X on $G(\mathbb{R})^1$, there is a constant c_X such that

$$|(X\phi)(x)| \leq c_X \|x\|^{N_0},$$

for every $x \in G(\mathbb{A})^1$.

PROPOSITION 13.2. (a) *If* $\phi \in C^\infty\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ *is uniformly tempered, the function* $\Lambda^T\phi$ *is rapidly decreasing.*

(b) *Given a Siegel set* $\mathcal{S}$, *positive integers* N *and* N_0, *and an open compact subgroup* K_0 *of* $G(\mathbb{A}_{\mathrm{fin}})$, *we can choose a finite set* $\{X_i\}$ *of left invariant differential operators on* $G(\mathbb{R})^1$ *and a positive integer* r *with the property that if* $(\Omega, \mathrm{d}\omega)$ *is a measure space, and* $\phi(\omega)$: $x \to \phi(\omega, x)$ *is any measurable function from* Ω *to* $C^r\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$, *the supremum*

$$\sup_{x\in\mathcal{S}^1}\Big(\|x\|^N\int_\Omega|\Lambda^T\phi(\omega,x)|\mathrm{d}\omega\Big) \tag{13.5}$$

is bounded by

$$\sup_{y\in G(\mathbb{A})^1}\Big(\|y\|^{-N_0}\sum_i\int_\Omega|X_i\phi(\omega,y)|\mathrm{d}\omega\Big). \tag{13.6}$$

It is enough to prove (ii), since it is a refined version of (i). This assertion is Lemma 1.4 of [**A4**], the proof of which is reminiscent of that of Theorem 6.1. The initial stages of the two proofs are in fact identical. We multiply the summand corresponding to P in

$$\Lambda^T\phi(\omega,x)$$
$$= \sum_P(-1)^{\dim(A_P/A_G)}\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})}\phi(\omega,n\delta x)\mathrm{d}n\cdot\widehat{\tau}_P\big(H_P(\delta x)-T\big)$$

by the left hand side of (8.1). We then apply the definition (8.2) to the product of functions $\tau_{P_1}^P$ and $\widehat{\tau}_P$ that occurs in the resulting expansion. The function $\Lambda^T\phi(\sigma,x)$ becomes the sum over pairs $P_1 \subset P_2$ and elements $\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$ of the product

$$F^{P_1}(\delta x,T)\sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x)-T\big)\phi_{P_1,P_2}(\omega,\delta x),$$

where

$$\phi_{P_1,P_2}(\omega,y) = \sum_{\{P:P_1\subset P\subset P_2\}}(-1)^{\dim(A_P/A_G)}\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})}\phi(\omega,ny)\mathrm{d}n. \tag{13.7}$$

Suppose that $y = \delta x$ is such that the first two factors in the last product are both nonzero. Replacing δ by a left $P_1(\mathbb{Q})$-translate, if necessary, we can assume that

$$y = \delta x = n_* n^* mak,$$

for $k \in K$, elements n_*, n^* and m in fixed compact subsets of $N_{P_2}(\mathbb{A})$, $N_{P_1}^{P_2}(\mathbb{A})$ and $M_{P_1}(\mathbb{A})^1$ respectively, and a point $a \in A_{P_1}(\mathbb{R})^0$ with $\sigma_{P_1}^{P_2}\big(H_{P_1}(a)-T\big) \neq 0$. Therefore

$$y = \delta x = n_* a\cdot a^{-1}n^* amk = n_* ab,$$

where b belongs to a fixed compact subset of $G(\mathbb{A})^1$ that depends only on G. The next step is to extract an estimate of rapid decrease for the function

$$\phi_{P_1,P_2}(\omega,y) = \phi_{P_1,P_2}(\omega,\delta x) = \phi_{P_1,P_2}(\omega,ab)$$

from the alternating sum over P in (13.7).

At this point the argument diverges slightly from that of Theorem 6.1. The quantitative nature of the assertion (ii) represents only a superficial difference, since similar estimates are implicit in the discussion of §8. However, the integrals in (13.7) are over quotients $N_P(\mathbb{Q})\backslash N_P(\mathbb{A})$ rather than groups $N_P(\mathbb{A})$, a reflection of the left $G(\mathbb{Q})$-invariance of the underlying function $y \to \phi(\sigma, y)$. This alters the way we realize the cancellation in the alternating sum over P. It entails having to apply the Fourier inversion formula to a product of groups $\mathbb{Q}\backslash\mathbb{A}$, in place of the Poisson summation formula for a product of groups $\mathbb{A}$. The problem is that the quotient $\mathfrak{n}_{P_1}^{P_2}(\mathbb{Q})\backslash\mathfrak{n}_{P_1}^{P_2}(\mathbb{A})$ does not correspond with $N_{P_1}^{P_2}(\mathbb{Q})\backslash N_{P_1}^{P_2}(\mathbb{A})$ under the exponential mapping. However, the problem may be resolved by a straightforward combinatorial argument that appears in [**Har4**, Lemma 11]. One constructs a finite set of pairs

$$(N_I^-, N_I), \qquad\qquad N_{P_2} \subset N_I^- \subset N_I \subset N_{P_1},$$

of $\mathbb{Q}$-rational groups, where N_I^- is normal in N_I with abelian quotient N^I. Each index I parametrizes a subset

$$\left\{\beta_{I,\alpha} \in \Phi_{P_1}^{P_2} :\ \alpha \in \Delta_{P_1}^{P_2}\right\}$$

of roots of the parabolic subgroup $M_{P_2} \cap P_1$ of M_{P_2} such that $\beta_{I,\alpha}$ contains α in its decomposition into simple roots. If $X_{I,\alpha} \in \mathfrak{n}_{P_1}(\mathbb{Q})$ stands for a root vector relative to $\beta_{I,\alpha}$, the space

$$\mathfrak{n}^I(\mathbb{Q}) = \bigoplus_{\alpha\in\Delta_{P_1}^{P_2}} \mathbb{Q}X_{I,\alpha}$$

becomes a linear complement for the Lie algebra of $N_I^-(\mathbb{Q})$ in that of $N_I(\mathbb{Q})$. The combinatorial argument yields an expansion of $\phi_{P_1,P_2}(\omega, ab)$ as linear combination over I of functions

$$\sum_{\xi\in\mathfrak{n}^I(\mathbb{Q})'} \int_{\mathfrak{n}^I(\mathbb{Q})\backslash\mathfrak{n}^I(\mathbb{A})} \int_{N_I^-(\mathbb{Q})\backslash N_I^-(\mathbb{A})} \phi\big(\omega, u\exp(X)ab\big)\psi\big(\langle X,\xi\rangle\big)\mathrm{d}u\mathrm{d}X, \tag{13.8}$$

where

$$\mathfrak{n}^I(\mathbb{Q})' = \Big\{\xi = \sum_{\alpha\in\Delta_{P_1}^{P_2}} r_\alpha X_{I,\alpha} :\ r_\alpha \in \mathbb{Q}^*\Big\}.$$

(See [**A4**, p. 94].)

One can estimate (13.8) as in the proof of Theorem 6.1. In fact, it is not hard to show that for any positive integer n, the product of $\mathrm{e}^{n\|H_{P_1}(a)\|}$ with the integral of the absolute value of (13.8) over ω has a bound of the form (13.6). But

$$\mathrm{e}^{n\|H_{P_0}(a)\|} \geq c_1\|a\|^{n\varepsilon} \geq c_2\|n^*ab\|^{n\varepsilon} = c_2\|\delta x\|^{n\varepsilon},$$

for positive constants c_1, c_2 and ε. Moreover, it is known that there is a positive constant c such that

$$\|\delta x\| \geq c\|x\|,$$

for any x in the Siegel set $\mathcal{S}$, and any $\delta \in G(\mathbb{Q})$. It follows that the supremum

$$\sup_{x\in\mathcal{S}} \sup_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \Big(\|x\|^{n\varepsilon}\int_\Omega \big|\phi_{P_1,P_2}(\omega,\delta x)\big|\mathrm{d}\omega\Big)$$

has a bound of the form (13.6). Since this supremum is independent of δ, we have only to estimate the sum

$$\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^{P_1}(\delta x, T)\sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x) - T\big).$$

It follows from the definition (8.3) and the fact that both $F^{P_1}(\cdot, T)$ and $\sigma_{P_1}^{P_2}(\cdot)$ are characteristic functions that the summand corresponding to δ is bounded by

$\widehat{\tau}_{P_1}\big(H_{P_1}(\delta x)-T\big)$. In §6 we invoked Lemma 5.1 of [**A3**] in order to say that the sum over δ in (6.1) could be taken over a finite set. The lemma actually asserts that

$$\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \widehat{\tau}_{P_1}\big(H_{P_1}(\delta x)-T\big)\leq c_T\|x\|^{N_1},$$

for positive constants c_T and N_1. We obtain an estimate (13.6) for (13.5) by choosing $n\geq\varepsilon^{-1}(N+N_1)$. □

The proof of Proposition 13.2 we have just sketched is that of [**A4**, Lemma 1.4]. The details in [**A4**] are a little hard to follow, thanks to less than perfect exposition and some typographical errors. Perhaps the discussion above will make them easier to read.

The most immediate application of Proposition 13.2 is to an Eisenstein series $x\to E(x,\phi,\lambda)$. Among the many properties established by Langlands in the course of proving Theorem 7.2 was the fact that Eisenstein series are uniformly slowly increasing. More precisely, there is a positive integer N_0 such that for any vector $\phi\in\mathcal{H}_P^0$ and any left invariant differential operator X on $G(\mathbb{R})^1$, there is an inequality

$$|XE(x,\phi,\lambda)|\leq c_{X,\phi}(\lambda)\|x\|^{N_0},\qquad x\in G(\mathbb{A}),$$

in which $c_{X,\phi}(\lambda)$ is a locally bounded function on the set of $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$ at which $E(x,\phi,\lambda)$ is analytic. It follows from Proposition 13.2 that for any N and any Siegel set $\mathcal{S}$, there is a locally bounded function $c_{N,\phi}(\lambda)$ on the set of λ at which $E(x,\phi,\lambda)$ is analytic such that

$$|\Lambda^TE(x,\phi,\lambda)|\leq c_{N,\phi}(\lambda)\|x\|^{-N}, \tag{13.9}$$

for every $x\in\mathcal{S}^1$. In particular, the truncated Eisenstein series $\Lambda^TE(x,\phi,\lambda)$ is square integrable on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. As we shall see, the spectral expansion of the trace formula depends on being able to evaluate the inner product of two truncated Eisenstein series.

The third property of the truncation operator is one of cancellation. It concerns the partial truncation operator Λ^{T,P_1} attached to a standard parabolic subgroup $P_1\supset P_0$. If ϕ is any function in $\mathcal{B}_{\mathrm{loc}}\big(P_1(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$, we define $\Lambda^{T,P_1}\phi$ to be the function in $\mathcal{B}_{\mathrm{loc}}\big(M_{P_1}(\mathbb{Q})N_{P_1}(\mathbb{A})\backslash G(\mathbb{A})^1\big)$ whose value at x equals

$$\sum_{\{Q:P_0\subset Q\subset P_1\}}(-1)^{\dim(A_Q/A_{P_1})}\sum_{\delta\in Q(\mathbb{Q})\backslash P_1(\mathbb{Q})}\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})}\phi(n\delta x)\widehat{\tau}_Q^{P_1}\big(H_Q(\delta x)-T\big).$$

PROPOSITION 13.3. *If ϕ belongs to $\mathcal{B}_{\mathrm{loc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$, then*

$$\sum_{P_1\supset P_0}\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})}\Lambda^{T,P_1}\phi(\delta x)\tau_{P_1}\big(H_{P_1}(\delta x)-T\big)=\phi(x).$$

More generally, if ϕ belongs to $\mathcal{B}_{\mathrm{loc}}\big(P(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$ for some $P\supset P_0$, the sum

$$\sum_{\{P_1:P_0\subset P_1\subset P\}}\sum_{\delta\in P_1(\mathbb{Q})\backslash P(\mathbb{Q})}\Lambda^{T,P_1}\phi(\delta x)\tau_{P_1}^P\big(H_{P_1}(\delta x)-T\big) \tag{13.10}$$

equals

$$\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})}\phi(nx)\mathrm{d}n.$$

If we substitute the definition of $\Lambda^{T,P_1}\phi$ into (13.10), we obtain a double sum over Q and P_1. Combining the double sum over $Q(\mathbb{Q})\backslash P_1(\mathbb{Q})$ and $P_1(\mathbb{Q})\backslash P(\mathbb{Q})$ into a single sum over $Q(\mathbb{Q})\backslash P(\mathbb{Q})$, we write (13.10) as the sum over parabolic subgroups Q, with $P_0 \subset Q \subset P$, and elements $\delta \in Q(\mathbb{Q})\backslash P(\mathbb{Q})$ of the product of

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \phi(n\delta x)$$

with

$$\sum_{\{P_1 : Q \subset P_1 \subset P\}} (-1)^{\dim(A_Q/A_{P_1})} \widehat{\tau}_Q^{P_1}\big(H_Q(\delta x) - T\big)\tau_{P_1}^P\big(H_{P_1}(\delta x) - T\big).$$

Since $\tau_{P_1}^P\big(H_{P_1}(\delta x) - T\big) = \tau_{P_1}^P\big(H_Q(\delta x) - T\big)$, we can apply (8.11) to the alternating sum over P_1. This proves that the alternating sum vanishes unless $Q = P$, in which case it is trivially equal to 1. The formula of the lemma follows. (See [**A4**, Lemma 1.5].) □

14. The coarse spectral expansion

The truncation operator Λ^T acts on functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. If h is a function of two variables and Λ is a linear operator on any space of functions in $G(\mathbb{A})$, we write $\Lambda_1 h$ and $\Lambda_2 h$ for the transforms of h obtained by letting Λ act separately on the first and second variables respectively. We want to consider the case that $\Lambda = \Lambda^T$, and $h(x, y)$ equals the χ-component $K_\chi(x, y)$ of the kernel $K(x, y)$ of $R(f)$. We recall that the parameter T in both the operator Λ^T and the modified kernel $k^T(x)$ is a suitably regular point in $\mathfrak{a}_0^+$.

Theorem 14.1. (a) *The double integral*

$$\sum_{\chi \in \mathfrak{X}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_2^T K_\chi(x, x)\mathrm{d}x \tag{14.1}$$

converges absolutely.

(b) *If T is suitably regular, in a sense that depends only on the support of f, the double integral*

$$\sum_{\chi \in \mathfrak{X}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x)\mathrm{d}x \tag{14.2}$$

also converges absolutely.

(c) *If T is as in (ii), we have*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x)\mathrm{d}x = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_2^T K_\chi(x, x)\mathrm{d}x,$$

for any $\chi \in \mathfrak{X}$.

The assertions of Theorem 14.1 are among the main results of [**A4**]. Their proof is given in §2 of that paper. We shall try to give some idea of the argument.

The assertion (i) requires a quantitative estimate for the spectral expansion of the kernel

$$K(x, y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y).$$

The sum here can obviously be taken over elements γ in the support of the function $u \to f(x^{-1}uy)$. Since the support equals $x \cdot \mathrm{supp} f \cdot y^{-1}$, we can apply the properties (13.2)–(13.4) of the height function $\| \cdot \|$. We see that

$$|K(x,y)| \leq c(f)\|x\|^{N_1}\|y\|^{N_1},$$

for a positive number N_1 that depends only on G. For any $\chi \in \mathfrak{X}$, $K_\chi(x,y)$ is the kernel of the restriction of $R(f)$ to the invariant subspace $L^2_\chi\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ of $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$. It follows from the discussion at the end of §7 that the sum

$$\sum_{\chi \in \mathfrak{X}} |K_\chi(x,y)|$$

of absolute values is bounded by a finite sum of products

$$\Big(\sum_\chi K_{\chi,1}(x,x)\Big)^{\frac{1}{2}}\Big(\sum_\chi K_{\chi,2}(y,y)\Big)^{\frac{1}{2}} = \big(K_1(x,x)\big)^{\frac{1}{2}}\big(K_2(y,y)\big)^{\frac{1}{2}}.$$

of kernels $K_i(\cdot,\cdot)$ attached positive definite functions

$$f_i = h_i * h_i^*, \qquad h_i \in C_c^r\big(G(\mathbb{A})\big).$$

It follows that

$$\sum_\chi |K_\chi(x,y)| = c(f)\|x\|^{N_1}\|y\|^{N_1}, \qquad x,y \in G(\mathbb{A}),$$

for some constant $c(f)$ depending on f.

A similar estimate holds for derivatives of the kernel. Suppose that X and Y are left invariant differential operators on $G(\mathbb{R})$ of degrees d_1 and d_2. Suppose that f belongs to $C_c^r\big(G(\mathbb{A})\big)$, for some large positive integer r. The corresponding kernel then satisfies

$$X_1Y_2K_\chi(x,y) = K_\chi^{X,Y}(x,y), \qquad \chi \in \mathfrak{X},$$

where $K^{X,Y}(x,y)$ is the kernel attached to a function $f_{X,Y}$ in $C_c^{r-d_1-d_2}\big(G(\mathbb{A})\big)$. It follows that

$$\sum_{\chi\in\mathfrak{X}} |X_1Y_2K_\chi(x,y)| \leq c(f_{X,Y})\|x\|^{N_1}\|y\|^{N_1},$$

for all $x,y \in G(\mathbb{A})$.

We combine the last estimate with Proposition 13.2(b). Choose the objects $\mathcal{S}$, N, N_0 and K_0 of Proposition 13.2(b) so that $G(\mathbb{A}) = G(\mathbb{Q})\mathcal{S}$, N is large, $N_0 = N_1$, and f is biinvariant under K_0. We can then find a finite set $\{Y_i\}$ of left invariant differential operators on $G(\mathbb{R})$ such that

$$\begin{aligned}
&\sup_{y\in\mathcal{S}}\Big(\|y\|^N\sum_\chi|\Lambda_2^T K_\chi(x,y)|\Big)\\
\le&\sup_{y\in G(\mathbb{A})}\Big(\sum_\chi\|y\|^{-N_0}\Big|\sum_i(Y_i)_2K_\chi(x,y)\Big|\Big)\\
\le&\sup_{y\in G(\mathbb{A})}\Big(\sum_i\|y\|^{-N_0}\sum_\chi|(Y_i)_2K_\chi(x,y)|\Big)\\
\le&\sup_{y\in G(\mathbb{A})}\Big(\sum_i\|y\|^{-N_0}c(f_{1,Y_i})\|x\|^{N_1}\|y\|^{N_1}\Big)\\
\le&\Big(\sum_i c(f_{1,Y_i})\Big)\|x\|^{N_1},
\end{aligned}$$

for any $x\in G(\mathbb{A})$. Setting $x=y$, we see that there is a constant $c_1=c_1(f)$ such that

$$\sum_\chi|\Lambda_2^T K_\chi(x,x)|\le c_1\|x\|^{N_1-N},$$

for any $x\in\mathcal{S}$. Since any bounded function is integrable over $\mathcal{S}^1=\mathcal{S}\cap G(\mathbb{A})^1$, we conclude that the sum over χ of the functions $|\Lambda_2^T K_\chi(x,x)|$ is integrable over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. This is the assertion (a).

The proof of (b) and (c) begins with an expansion of the function $k_\chi^T(x)=k_\chi^T(x,f)$. We are not thinking of the χ-form of the expansion (8.3) of $k^T(x)$, but rather a parallel expansion in terms of partial truncation operators. We shall derive it as in §8, using Proposition 13.3 in place of Lemma 8.2.

The kernel $K_{P,\chi}(x,y)$ defined in §12 is invariant under left translation of either variable by $N_P(\mathbb{A})$. In particular, we can write

$$K_{P,\chi}(x,y)=\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})}K_{P,\chi}(x,ny)\mathrm{d}n.$$

It follows from the definition in §12 that $k_\chi^T(x)$ equals

$$\sum_P(-1)^{(A_P/A_G)}\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\widehat{\tau}_P\big(H_P(\delta x)-T\big)\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})}K_{P,\chi}(\delta x,n\delta x)\mathrm{d}n.$$

The integral over n can then be expanded according to Proposition 13.3. The resulting sum over $P_1(\mathbb{Q})\backslash P(\mathbb{Q})$ combines with that over $P(\mathbb{Q})\backslash G(\mathbb{Q})$ to give an expression

$$\sum_{P_1\subset P}(-1)^{\dim(A_P/A_G)}\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})}\widehat{\tau}_P\big(H_P(\delta x)-T\big)\tau_{P_1}^P\big(H_{P_1}(\delta x)-T\big)\Lambda_2^{T,P_1}K_{P,\chi}(\delta x,\delta x)$$

for $k_\chi^T(x)$. Applying the expansion (8.2), we write

$$\begin{aligned}
&\widehat{\tau}_P\big(H_P(\delta x)-T\big)\tau_{P_1}^P\big(H_{P_1}(\delta x)-T\big)\\
=&\widehat{\tau}_P\big(H_{P_1}(\delta x)-T\big)\tau_{P_1}^P\big(H_{P_1}(\delta x)-T\big)\\
=&\sum_{\{P_2:P_2\supset P\}}\sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x)-T\big).
\end{aligned}$$

It follows that $k_\chi^T(x)$ has an expansion

$$\sum_{P_1\subset P_2}\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})}\sigma_{P_1}^{P_2}\big(H_{P_1}(\delta x)-T\big)\Lambda_2^{T,P_1}K_{P_1,P_2,\chi}(\delta x,\delta x), \tag{14.3}$$

where

$$K_{P_1,P_2,\chi}(x,y)=\sum_{\{P:P_1\subset P\subset P_2\}}(-1)^{\dim(A_P/A_G)}K_{P,\chi}(x,y).$$

Observe that (14.3) is the same as the expansion (8.3) (or rather its χ-analogue), except that the partial "cut-off" function $F^{P_1}(\cdot,T)$ has been replaced by the partial truncation operator Λ_2^{T,P_1}.

We recall from Lemma 8.3 that $\sigma_{P_1}^{P_2}$ vanishes if $P_1=P_2\neq G$, so the corresponding summand in (14.3) equals 0. If $P_1=P_2=G$, $\sigma_{P_1}^{P_2}$ equals 1, and the corresponding summand in (14.3) equals $\Lambda_2^T K_\chi(x,x)$. It follows that the difference

$$k_\chi^T(x)-\Lambda_2^T K_\chi(x,x)$$

equals the modified expression (14.3) obtained by taking the first sum over $P_1\subsetneq P_2$. Consider the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ of the absolute value of this difference. The absolute value is of course bounded by the corresponding double sum of absolute values, in which we can combine the integral with the sum over $P_1(\mathbb{Q})\backslash G(\mathbb{Q})$. It follows that the double integral

$$\sum_{\chi\in\mathfrak{X}}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}|k_\chi^T(x)-\Lambda_2^T K_\chi(x,x)|\mathrm{d}x$$

is bounded by

$$\sum_{\chi\in\mathfrak{X}}\sum_{P_1\subsetneq P_2}\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1}\sigma_{P_1}^{P_2}\big(H_{P_1}(x)-T\big)|\Lambda_2^{T,P_1}K_{P_1,P_2,\chi}(x,x)|\mathrm{d}x. \tag{14.4}$$

The assertion (ii) would follow from (i) if it could be shown that (14.4) is finite. In fact, one shows that for T highly regular, the integrand in (14.4) actually vanishes. This obviously suffices to establish both (ii) and (iii).

Consider the integrand in (14.4) attached to a fixed pair $P_1\subsetneq P_2$. In order to treat the factor $\Lambda_2^{T,P_1}K_{P_1,P_2,\chi}$, one studies the function

$$\begin{aligned}&\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})}K_{P_1,P_2}(x,n_1y)\mathrm{d}n_1\\ &=\sum_{\{P:P_1\subset P\subset P_2\}}(-1)^{\dim(A_P/A_G)}\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})}K_P(x,n_1y)\mathrm{d}n_1\\ &=\sum_P(-1)^{\dim(A_P/A_G)}\int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})}\int_{N_P(\mathbb{A})}\sum_{\gamma\in M_P(\mathbb{Q})}f(x^{-1}\gamma nn_1y)\mathrm{d}n\mathrm{d}n_1.\end{aligned}$$

In the last summand corresponding to P, we change the triple integral to a double integral over the product

$$M_P(\mathbb{Q})N_P(\mathbb{A})/N_{P_1}(\mathbb{Q})\times N_{P_1}(\mathbb{A}).$$

This in turn can be written as a triple integral over the product

$$\big(M_P(\mathbb{Q})/M_P(\mathbb{Q})\cap N_{P_1}(\mathbb{Q})\big)\times\big(N_P(\mathbb{A})/N_P(\mathbb{Q})\big)\times N_{P_1}(\mathbb{A}).$$

The integral over $N_P(\mathbb{A})/N_P(\mathbb{Q})$ can then be absorbed in the integral over $N_{P_1}(\mathbb{A})$. Since

$$M_P(\mathbb{Q})/M_P(\mathbb{Q})\cap N_{P_1}(\mathbb{Q}) \cong P(\mathbb{Q})/P_1(\mathbb{Q}) \times M_{P_1}(\mathbb{Q}),$$

the sum over P takes the form

$$\begin{aligned}&\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\gamma\in P_1(\mathbb{Q})\backslash P(\mathbb{Q})} \int_{N_{P_1}(\mathbb{A})} \sum_{\gamma_1\in M_{P_1}(\mathbb{Q})} f(x^{-1}\gamma^{-1}\gamma_1 n_1 y)\mathrm{d}n_1\\ =&\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\gamma\in P_1(\mathbb{Q})\backslash P(\mathbb{Q})} K_{P_1}(\gamma x, y).\end{aligned}$$

Let $F(P_1,P_2)$ be the set of elements in $P_1(\mathbb{Q})\backslash P_2(\mathbb{Q})$ which do not lie in $P_1(\mathbb{Q})\backslash P(\mathbb{Q})$ for any P with $P_1 \subset P \subsetneq P_2$. The alternating sum over P and γ then reduces to a sum over $\gamma \in F(P_1,P_2)$, by Identity 6.2. We have established that

$$(14.5)\quad \int_{N_{P_1}(\mathbb{Q})\backslash N_{P_1}(\mathbb{A})} K_{P_1,P_2}(x,n_1y)\mathrm{d}n_1 = (-1)^{\dim(A_{P_2}/A_G)} \sum_{\gamma\in F(P_1,P_2)} K_{P_1}(\gamma x, y).$$

There remain two steps to showing that the integrand in (14.4) vanishes. The first is to show that for any x and y, $\Lambda_2^{T,P_1} K_{P_1,P_2,\chi}(x,y)$ depends linearly on the function of $m \in M_{P_1}(\mathbb{Q})\backslash M_{P_1}(\mathbb{A})^1$ obtained from the left hand side of (14.5) by replacing y by my. This is related to the decompositions of §12, and is easily established from the estimates we have discussed. The other is to show that if T is highly regular relative to $\mathrm{supp}(f)$, and $\sigma_{P_1}^{P_2}\big(H_{P_1}(x)-T\big) \neq 0$, then $K_{P_1}(\gamma x, mx) = 0$ for all m and any $\gamma \in F(P_1,P_2)$. This is a consequence of the Bruhat decomposition for $G(\mathbb{Q})$. In the interests of simplicity (rather than efficiency), we shall illustrate the ideas in the concrete example of $G = GL(2)$, referring the reader to [**A4**, §2] for the general case.

Assume that $G = GL(2)$, $P_1 = P_0$ and $P_2 = G$. The partial truncation operator Λ^{T,P_1} is then given simply by an integral over $N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})$. Therefore

$$\Lambda_2^{T,P_1} K_{P_1,P_2,\chi}(x,y) = \int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} \big(K_\chi(x,ny) - K_{P_0,\chi}(x,ny)\big)\mathrm{d}n.$$

If $\chi = (G,\pi)$, the integral of $K_\chi(x,ny)$ over n vanishes, since π is a cuspidal automorphic representation of $G(\mathbb{A})$, while $K_{P_0,\chi}(x,ny)$ vanishes by definition. The integrand in (14.4) thus vanishes in this case for any T.

For $G = GL(2)$, we have reduced the problem to the remaining case that χ is represented by a pair (P_0,σ_0). Since M_{P_0} is the group of diagonal matrices in $GL(2)$, we can identify σ_0 with a pair of characters on the group $\mathbb{Q}^*\backslash\mathbb{A}^1$. It follows directly from the definitions that

$$\begin{aligned}\int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} K_{P_0,\chi}(x,ny)\mathrm{d}n &= K_{P_0,\chi}(x,y)\\ &= \int_{M_{P_0}(\mathbb{Q})\backslash M_{P_0}(\mathbb{A})^1} K_{P_0}(x,my)\sigma_0(m)\mathrm{d}m.\end{aligned}$$

The spectral decomposition of the kernel $K(x, y)$ also leads to a formula

$$\int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} K_\chi(x, ny)\mathrm{d}n$$
$$= \int_{M_{P_0}(\mathbb{Q})\backslash M_{P_0}(\mathbb{A})^1} \int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} K(x, nmy)\sigma_0(m)\mathrm{d}n\mathrm{d}m.$$

Indeed, the required contribution from the terms in $K(x, y)$ corresponding to the Hilbert space $\mathcal{H}_G$ can be inferred from the fact that the representation of $G(\mathbb{A})$ on $\mathcal{H}_G$ is a sum of cuspidal automorphic representations and one-dimensional automorphic representations. To obtain the contribution from the terms in $K(x, y)$ corresponding to $\mathcal{H}_{P_0}$, we use the fact that for any $\phi \in \mathcal{H}^0_{P_0}$, the function

$$y \longrightarrow \int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} E(ny, \phi, \lambda)\mathrm{d}n$$

also belongs to $\mathcal{H}^0_{P_0}$. Combining the two formulas, we see that

$$\Lambda_2^{T,P_1} K_{P_1,P_2,\chi}(x, y)$$
$$= \int_{M_{P_0}(\mathbb{Q})\backslash M_{P_0}(\mathbb{A})^1} \int_{N_{P_0}(\mathbb{Q})\backslash N_{P_0}(\mathbb{A})} K_{P_1,P_2}(x, nmy)\sigma_0(m)\mathrm{d}n\mathrm{d}m$$
$$= \int_{M_{P_0}(\mathbb{Q})\backslash M_{P_0}(\mathbb{A})^1} \sum_{\gamma \in F(P_1,P_2)} K_{P_1}(\gamma x, my)\sigma_0(m)\mathrm{d}m,$$

for any x and y. This completes the first step in the case of $G = GL(2)$.

For the second step, we note that

$$F(P_1, P_2) = F(P_0, G) = P_0(\mathbb{Q})\backslash\big(G(\mathbb{Q}) - P_0(\mathbb{A})\big)$$
$$= \left\{ M_{P_0}(\mathbb{Q}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_{P_0}(\mathbb{Q}) \right\},$$

by the Bruhat decomposition for $GL(2)$. Setting $y = x$, we write

$$\sum_{\gamma \in F(P_1,P_2)} K_{P_1}(\gamma x, mx)$$
$$= \sum_{\gamma \in F(P_1,P_2)} \int_{N_{P_0}(\mathbb{A})} f(x^{-1}\gamma^{-1}nmx)\mathrm{d}n$$
$$= \int_{N_{P_0}(\mathbb{A})} \sum_{\nu \in N_{P_0}(\mathbb{Q})} \sum_{\mu \in M_{P_0}(\mathbb{Q})} f\left(x^{-1}\nu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mu n m x\right)\mathrm{d}n,$$

for any $m \in M_P(\mathbb{A})^1$. We need to show that if T is highly regular relative to $\mathrm{supp}(f)$, the product of any summand with $\sigma_{P_1}^{P_2}\big(H_{P_1}(x) - T\big)$ vanishes for each $x \in G(\mathbb{A})^1$. Assume the contrary, and write

$$x = n_1 \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} m_1 k_1, \qquad n_1 \in N_{P_0}(\mathbb{A}),\ r \in (\mathbb{R}^*)^0,\ m_1 \in M_{P_0}(\mathbb{A})^1,\ k_1 \in K.$$

On the one hand, the number

$$\sigma_{P_1}^{P_2}\big(H_{P_1}(x) - T\big) = \sigma_{P_0}^{G}\big(H_{P_0}(x) - T\big) = \tau_{P_0}\left(H_{P_0} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} - T\right)$$

is positive, so that r is large relative to $\mathrm{supp}(f)$. On the other hand, it follows from the discussion above that the point

$$x^{-1}\nu\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mu n m x$$

belongs to $\mathrm{supp}(f)$, for some $\nu \in N_{P_0}(\mathbb{Q})$, $\mu \in M_{P_0}(\mathbb{Q})$, $n \in N_{P_0}(\mathbb{A})$, and $m \in M_{P_0}(\mathbb{A})^1$. Substituting for x, we see that there is a point $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $GL(2,\mathbb{A})^1$, with $|c| = r^2$, which lies in the fixed compact set $K \cdot \mathrm{supp}\, f \cdot K$. This is a contradiction. The argument in the case of $G = GL(2)$ is thus complete. $\square$

We have finished our remarks on the proof of Theorem 14.1. We can now treat the double integral (14.2) as we did its geometric analogue (10.3) in §10. By Fubini's theorem, we obtain an absolutely convergent expression

$$J^T(f) = \sum_{\chi\in\mathfrak{X}} J^T_\chi(f)$$

whose terms are given by absolutely convergent integrals

$$J^T_\chi(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T_\chi(x,f)\mathrm{d}x, \qquad \chi \in \mathfrak{X}. \tag{14.6}$$

Following the discussion of §10, we analyze $J^T_\chi(f)$ as a function of T by means of the proof of Theorem 9.1. Defined initially for $T \in \mathfrak{a}_0^+$ sufficiently regular, we see that $J^T_\chi(f)$ extends to any $T \in \mathfrak{a}_0$ as a polynomial function whose degree is bounded by the dimension of $\mathfrak{a}_0^G$. We then set

$$J_\chi(f) = J^{T_0}_\chi(f), \qquad \chi \in \mathfrak{X},$$

for the point $T_0 \in \mathfrak{a}_0^G$ given by (9.4). By the proof of Proposition 9.3, each distribution $J_\chi(f)$ is independent of the choice of minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$.

The new distributions $J_\chi(f) = J^G_\chi(f)$ are again generally not invariant. Applying the proof of Theorem 9.4 to the absolutely convergent integral (14.6), we obtain the variance property

$$J_\chi(f^y) = \sum_{Q\supset P_0} J^{M_Q}_\chi(f_{Q,y}), \qquad \chi \in \mathfrak{X},\ y \in G(\mathbb{A}). \tag{14.7}$$

As before, $J^{M_Q}_\chi(f_{Q,y})$ is defined as a finite sum of distributions $J^{M_Q}_{\chi_Q}(f_{Q,y})$, in which χ_Q ranges over the preimage of χ in $\mathfrak{X}^{M_Q}$ under the mapping of $\mathfrak{X}^{M_Q}$ to $\mathfrak{X}$. Once again, χ need not lie in the image of the map $\mathfrak{X}^{M_Q} \to \mathfrak{X}$ attached to any proper parabolic subgroup $Q \subsetneq G$. This is the case precisely when χ is *cuspidal*, in the sense that it is defined by a pair (G,σ). When χ is cuspidal, the distribution $J_\chi(f)$ is in fact invariant.

The expansion of $J^T(f)$ in terms of distributions $J^T_\chi(f)$ extends by polynomial interpolation to all values of T. Setting $T = T_0$, we obtain an identity

$$J(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f), \qquad f \in C^\infty_c\big(G(\mathbb{A})\big). \tag{14.8}$$

This is what we will call the coarse spectral expansion. The distributions $J_\chi(f)$ for which χ is cuspidal are to be regarded as general analogues of the spectral terms in the trace formula for compact quotient.

15. Weighted characters

This section is parallel to §11. It is aimed at the problem of describing the summands $J_\chi(f)$ in the coarse spectral expansion more explicitly. At this point, we can give a partial solution. We shall express $J_\chi(f)$ as a weighted character for "generic" classes $\chi \in \mathfrak{X}$.

For any $\chi \in \mathfrak{X}$, $J_\chi^T(f)$ is defined by the formula (14.6). However, Theorem 14.1(iii) and the definition (12.7) provide another expression

$$\begin{aligned} J_\chi^T(f) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_2^T K_\chi(x,x)\mathrm{d}x \\ &= \sum_P n_P^{-1} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Big(\int_{i\mathfrak{a}_P^*} \sum_{\phi \in \mathcal{B}_{P,\chi}} E\big(x, \mathcal{I}_P(\lambda, f)\phi, \lambda\big) \overline{\Lambda^T E(x,\phi,\lambda)} \mathrm{d}\lambda \Big) \mathrm{d}x \end{aligned}$$

for $J_\chi^T(f)$. This second formula is better suited to computation.

Suppose that $\lambda \in i\mathfrak{a}_P^*$. The function $E(x,\phi',\lambda)$ is slowly increasing for any $\phi' \in \mathcal{H}_{P,\chi}^0$, while the function $\Lambda^T E(x,\phi,\lambda)$ is rapidly decreasing by (13.9). The integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} E(x,\phi',\lambda)\overline{\Lambda^T E(x,\phi,\lambda)}\mathrm{d}x$$

therefore converges, and consequently defines a Hermitian bilinear form on $\mathcal{H}_{P,\chi}^0$. By the intertwining property of Eisenstein series, this bilinear form behaves in the natural way under the the actions of K and $\mathcal{Z}_\infty$ on $\mathcal{H}_{P,\chi}^0$. It may therefore be written as

$$\big(M_{P,\chi}^T(\lambda)\phi', \phi\big),$$

for a linear operator $M_{P,\chi}^T(\lambda)$ on $\mathcal{H}_{P,\chi}^0$. Since Λ^T is a self-adjoint projection, by Proposition 13.1, we see that

$$\big(M_{P,\chi}^T(\lambda)\phi', \phi\big) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi',\lambda)\overline{\Lambda^T E(x,\phi,\lambda)}\mathrm{d}x, \tag{15.1}$$

for any vectors ϕ' and ϕ in $\mathcal{H}_{P,\chi}^0$. It follows that the operator $M_{P,\chi}^T(\lambda)$ is self-adjoint and positive definite.

The following result can be regarded as a spectral analogue of Theorem 11.1.

THEOREM 15.1. *If $T \in \mathfrak{a}_{P_0}^+$ is suitably regular, in a sense that depends only on the support of f, the double integral*

$$\sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \mathrm{tr}\big(M_{P,\chi}^T(\lambda)\mathcal{I}_{P,\chi}(\lambda, f)\big)\mathrm{d}\lambda \tag{15.2}$$

converges absolutely, and equals $J_\chi^T(f)$.

This is Theorem 3.2 of [**A4**]. It includes the implicit assertion that the operator in the integrand is of trace class, as well as that of the absolute convergence of the integral. The precise assertion is Theorem 3.1 of [**A4**], which states that the expression

$$\sum_\chi \sum_P \int_{i\mathfrak{a}_P^*} \|M_{P,\chi}^T(\lambda)\mathcal{I}_{P,\chi}(\lambda, f)\|_1 \mathrm{d}\lambda$$

is finite. As usual, $\|\cdot\|_1$ denotes the trace class norm, taken here for operators on the Hilbert space $\mathcal{H}_{P,\chi}$.

Apart from the last convergence assertion, Theorem 15.1 is a formal consequence of the expression above for $J^T_\chi(f)$. It follows from the definition of $M^T_{P,\chi}(\lambda)$, once we know that the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ in the expression can be taken inside the integral over λ and the sum over ϕ. The convergence assertion is a modest extension of Theorem 14.1(i). Its proof combines the same two techniques, namely the estimates for $K(x,y)$ obtained from Selberg's positivity argument, and the estimates for Λ^T given by Proposition 13.2. We refer the reader to §3 of [**A4**]. $\square$

Suppose that P is fixed. Since the inner product (15.1) depends only on the image of T in the intersection $(\mathfrak{a}^G_{P_0})^+$ of $\mathfrak{a}^+_{P_0}$ with $\mathfrak{a}^G_{P_0}$, we shall assume for the rest of this section that T actually lies in $(\mathfrak{a}^G_{P_0})^+$. It turns out that the inner product can be computed explicitly for cuspidal Eisenstein series. The underlying reason for this is that the constant term

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E(nx,\phi,\lambda)\mathrm{d}n, \qquad \phi\in\mathcal{H}^0_P,\ \lambda\in\mathfrak{a}^*_{P,\mathbb{C}},$$

defined for any standard $Q\supset P_0$, has a relatively simple formula if ϕ is cuspidal.

Suppose that ϕ belongs to $\mathcal{H}^0_{P,\mathrm{cusp}}$ and that λ lies in $\mathfrak{a}^*_{P,\mathbb{C}}$. If Q is associated to P, we have the basic formula

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E(nx,\phi,\lambda)\mathrm{d}n = \sum_{s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)} \big(M(s,\lambda)\phi\big)(x)\mathrm{e}^{(s\lambda+\rho_Q)(H_Q(x))}$$

This is established in the domain of absolute convergence of Eisenstein series from the integral formula for $M(s,\lambda)\phi$ and the Bruhat decomposition for $G(\mathbb{Q})$ [**Lan1**, Lemma 3]. More generally, suppose that Q is arbitrary. Then

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E(nx,\phi,\lambda)\mathrm{d}n = \sum_{s\in W(P;Q)} E^Q\big(x,M(s,\lambda)\phi,s\lambda\big), \tag{15.3}$$

where we have written $E^Q(\cdot,\cdot,\cdot)=E^Q_{P_1}(\cdot,\cdot,\cdot)$, for the group P_1 such that s belongs to $W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$. This is established inductively from the first formula by showing that for any $Q'\subsetneq Q$, the Q'-constant terms of each sides are equal. The formula (15.3) allows us to express the truncated Eisenstein series $\Lambda^T E(x,\phi,\lambda)$, for λ in its domain of absolute convergence, in terms of the signs ε_Q and characteristic functions ϕ_Q defined in §11.

Lemma 15.2. *Suppose that $\phi\in\mathcal{H}^0_{P,\mathrm{cusp}}$ and $\lambda\in\Lambda+i\mathfrak{a}^*_P$, where Λ is any point in the affine chamber $\rho_P+(\mathfrak{a}^*_P)^+$. Then*

$$\Lambda^T E(x,\phi,\lambda) = \sum_{Q\supset P_0}\sum_{\delta\in Q(\mathbb{Q})\backslash G(\mathbb{Q})} \psi_Q(\delta x), \tag{15.4}$$

where for any $y\in G(\mathbb{A})$, $\psi_Q(y)$ is the sum over $s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)$ of the expression

$$\varepsilon_Q(s\Lambda)\phi_Q\big(s\Lambda,H_Q(\delta x)-T_Q\big)\mathrm{e}^{(s\lambda+\rho_Q)(H_Q(y))}\big(M(s,\lambda)\phi\big)(y). \tag{15.5}$$

This is Lemma 4.1 of [**A4**]. To prove it, we note that for any Q, s, and δ, the expression

$$\varepsilon_Q(s\Lambda)\phi_Q\big(s\Lambda,H_Q(\delta x)-T_Q\big)$$

equals

$$\sum_{\{R\supset Q:s\in W(P;R)\}} (-1)^{\dim(A_R/A_G)}\widehat{\tau}_R\big(H_R(\delta x)-T_R\big),$$

by the identity of (11.7) and (11.6) established in §11. We substitute this into the formula (15.5) for $\psi_Q(\delta x)$. We then take the sum over δ in (15.4) inside the resulting sums over s and R. This allows us to decompose it into a double sum over $\xi \in Q(\mathbb{Q})\backslash R(\mathbb{Q})$ and $\delta \in R(\mathbb{Q})\backslash G(\mathbb{Q})$. The sum

$$\sum_{\xi \in Q(\mathbb{Q})\backslash R(\mathbb{Q})} \mathrm{e}^{(s\lambda+\rho_Q)(H_Q(\xi\delta x)}\big(M(s,\lambda)\phi\big)(\xi\delta x)$$

converges absolutely to $E^R\big(\delta x, M(s,\lambda)\phi, s\lambda\big)$. It follows that the right hand side of (15.4) equals

$$\sum_R (-1)^{\dim(A_R/A_G)} \sum_\delta \Big\{ \sum_s E^R\big(\delta x, M(s,\lambda)\phi, s\lambda\big)\Big\}\widehat{\tau}_R\big(H_R(\delta x) - T_R\big),$$

with δ and s summed over $R(\mathbb{Q})\backslash G(\mathbb{Q})$ and $W(P;R)$ respectively. Moreover, the last expression in the brackets equals

$$\int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A})} E(n\delta x, \phi, \lambda)\mathrm{d}n,$$

by (15.3). It then follows from the definition (13.1) that the right hand side of (15.4) equals the truncated Eisenstein series on the right hand side of (15.4). (The elementary convergence arguments needed to justify these manipulations are given on p. 114 of [**A4**].) □

For any Q, we treat the sum ψ_Q in the last lemma as a function on $N_Q(\mathbb{A})M_Q(\mathbb{Q})\backslash G(\mathbb{A})^1$. It then follows from the definition of the characteristic functions $\phi_Q(s\Lambda, \cdot)$ and our choice of Λ that $\psi_Q(x)$ is rapidly decreasing in $H_Q(x)$. This is slightly weaker than the condition of compact support imposed on the function ψ in §12. However, we shall still express the right hand side of (15.4) as the sum over Q of functions $(E\psi_Q)(x)$, following the notation of Lemma 12.2. In fact, the inner product formula (12.3) is easily seen to hold under the slightly weaker conditions here. We shall sketch how to use it to compute the inner product of truncated Eisenstein series.

One has first to compute the Fourier transform

$$\Psi_Q(\mu, x) = \int_{A_Q(\mathbb{R})^0 \cap G(\mathbb{A})^1} \mathrm{e}^{-(\mu+\rho_Q)(H_Q(ax))}\psi_Q(ax)\mathrm{d}a,$$

for any $\mu \in i\mathfrak{a}_Q^*$. This entails computing the integral

$$\int_{A_Q(\mathbb{R})^0 \cap G(\mathbb{A})^1} \mathrm{e}^{(s\lambda-\mu)(H_Q(ax))}\varepsilon_Q(s\Lambda)\phi_Q\big(s\Lambda, H_Q(ax) - T_Q\big)\mathrm{d}a,$$

which can be written as

$$\int_{\mathfrak{a}_Q^G} \mathrm{e}^{(s\lambda-\mu)(H)}\varepsilon_Q(s\Lambda)\phi_Q(s\Lambda, H - T_Q)\mathrm{d}H,$$

after the obvious change of variables. A second change of variables

$$H = \sum_{\alpha \in \Delta_Q} t_\alpha \alpha^\vee, \qquad t_\alpha \in \mathbb{R},$$

simplifies the integral further. It becomes a product of integrals of rapidly decreasing exponential functions over half lines, each of which contributes a linear form in $s\lambda - \mu$ to the denominator. We have of course to multiply the resulting expression by the relevant Jacobian determinant, which equals the volume of $\mathfrak{a}_Q^G$ modulo the lattice $\mathbb{Z}(\Delta_Q^\vee)$ generated by $\Delta_Q^\vee$. The result is

$$\Psi_Q(\mu, x) = \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)} \mathrm{e}^{(s\lambda-\mu)(T)} \big(M(s,\lambda)\phi\big)(x) \theta_Q(s\lambda - \mu)^{-1}, \tag{15.6}$$

where

$$\theta_Q(s\lambda - \mu) = \mathrm{vol}\big(\mathfrak{a}_Q^G / \mathbb{Z}(\Delta_Q^\vee)\big)^{-1} \prod_{\alpha \in \Delta_Q} (s\lambda - \mu)(\alpha^\vee). \tag{15.7}$$

It is worth emphasizing that $\Psi_Q(\mu, x)$ is a rather simple function of μ, namely a linear combination of products of exponentials with quotients of polynomials. We have taken the real part Λ of λ to be any point in $\rho_P + (\mathfrak{a}_P^*)^+$. Assume from now on that it is also highly regular, in the sense that $\Lambda(\alpha^\vee)$ is large for every $\alpha \in \Delta_P$. Then $\Psi_Q(\mu, x)$ is an analytic function of μ in the tube in $\mathfrak{a}_{Q,\mathbb{C}}^*$ over a ball B_Q around 0 in $\mathfrak{a}_Q^*$ of large radius. Moreover, for any $\Lambda_Q \in B_Q$,

$$\Psi_Q(\mu) : \ x \longrightarrow \Psi_Q(\mu, x), \qquad\qquad \mu \in \Lambda_Q + i(\mathfrak{a}_Q^G)^*,$$

is a square integrable function of μ with values in a finite dimensional subspace of $\mathcal{H}_{Q,\mathrm{cusp}}^0$.

Consider another set of data P', $\phi' \in \mathcal{H}_{P',\mathrm{cusp}}^0$ and $\lambda' \in \Lambda' + i\mathfrak{a}_{P'}^*$, where P' is associated to P and Λ' is a highly regular point in $\rho_{P'} + (\mathfrak{a}_{P'}^*)^+$. These give rise to a corresponding pair of functions $\psi_{Q'}(x)$ and $\Psi_{Q'}(\mu', x)$, for each standard Q' associated to P'. Following the notation of Lemma 12.2, we write the inner product

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x, \phi, \lambda) \overline{\Lambda^T E(x, \phi', \lambda')} \mathrm{d}x \tag{15.8}$$

as

$$\sum_{Q,Q'} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} (E\psi_Q)(x) \overline{(E\psi_{Q'})(x)} \mathrm{d}x.$$

We are taking for granted the extension of Lemma 12.3 to the rapidly decreasing functions ψ_Q and $\psi_{Q'}$. It yields the further expression

$$\sum_{Q,Q'} \int_{\Lambda_Q + i(\mathfrak{a}_Q^G)^*} \sum_{t \in W(\mathfrak{a}_Q, \mathfrak{a}_{Q'})} \big(M(t,\mu)\Psi_Q(\mu), \Psi_{Q'}(-t\overline{\mu})\big) \mathrm{d}\mu$$

for the inner product, where Λ_Q is any point in the intersection of $\rho_Q + (\mathfrak{a}_Q^G)^*$ with the ball B_Q. It follows from (15.6) (and its analogue for P') that the inner product (15.8) equals the sum over Q and $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$, and the integral over $\mu \in \Lambda_Q + i(\mathfrak{a}_Q^G)^*$, of the product of

$$\theta_Q(s\lambda - \mu)^{-1} \mathrm{e}^{(s\lambda - \mu)(T)} \tag{15.9}$$

with

$$\sum_{Q'} \sum_{t} \sum_{s'} \theta_{Q'}(s'\bar{\lambda}' + t\mu)^{-1} \mathrm{e}^{(s'\bar{\lambda}' + t\mu)(T)} \big(M(t,\mu)M(s,\lambda)\phi, M(s',\lambda')\phi'\big). \tag{15.10}$$

The inner sums in (15.10) are over elements $t \in W(\mathfrak{a}_Q, \mathfrak{a}_{Q'})$ and $s' \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{Q'})$.

There are three more steps. The first is to show that (15.10) is an analytic function of μ if the real part of μ is any point in $\rho_Q + (\mathfrak{a}_Q^*)^+$. The operator valued functions $M(t, \mu)$ are certainly analytic, since the integral formula (7.2) converges uniformly in the given domain. The remaining functions $\theta_Q(s'\bar{\lambda}' + t\mu)^{-1}$ of μ have singularities along hyperplanes

$$\{\mu : (s'\bar{\lambda}' + t\mu)(\alpha^\vee) = 0\}, \qquad \alpha \in \Delta_{Q'},$$

for fixed Q', t, s' and λ'. However, each such hyperplane occurs twice in the sum (15.10), corresponding to a pair of multi-indices (Q', t, s') and $(Q'_\alpha, s_\alpha t, s_\alpha s')$ that differ by a simple reflection about α. (By definition, Q'_α is the standard parabolic subgroup such that s_α belongs to $W(\mathfrak{a}_{Q'}, \mathfrak{a}_{Q'_\alpha})$.) It is a consequence of the functional equations (7.4) that

$$\big(M(s_\alpha t, \mu)M(s, \lambda)\phi, M(s_\alpha s', \lambda')\phi'\big) = \big(M(t, \mu)M(s, \lambda)\phi, M(s', \lambda')\phi'\big),$$

whenever $(s'\bar{\lambda}' + t\mu)(\alpha^\vee) = 0$. It then follows that the singularities cancel from the sum (15.10), and therefore that (15.10) is analytic in the given domain. (This argument is a basic part of the theory of (G, M)-families, to be discussed in §17.)

The second step is to show that if $s \neq 1$, the integral over μ of the product of (15.9) and (15.10) vanishes. For any such s, there is a root $\alpha \in \Delta_{Q'}$ such that $(s\Lambda_Q)(\alpha^\vee) < 0$. As a function of μ, (15.9) is analytic on any of the affine spaces

$$(\Lambda_Q + r\varpi_\alpha) + i(\mathfrak{a}_Q^G)^*, \qquad 0 \le r < \infty.$$

We have just seen that the same property holds for the function (15.10). We can therefore deform the contour of integration from $\Lambda_Q + i(\mathfrak{a}_Q^G)^*$ to the affine space attached to any r. The function $M(t, \mu)$ is bounded independently of r on this affine space, as is the product

$$e^{-\mu(T)}e^{(t\mu)(T)}.$$

This leaves only the product

$$\theta_Q(s\lambda - \mu)^{-1}\theta_{Q'}(s'\bar{\lambda}' + t\mu)^{-1},$$

which is the inverse of a polynomial in μ of degree twice the dimension of the affine space. The integral attached to r therefore approaches 0 as r approaches infinity. The original integral therefore vanishes.

The final step is to set $s = 1$ in (15.9) and (15.10), and then integrate the product of the resulting two expressions over μ in $\Lambda_Q + i(\mathfrak{a}_Q^G)^*$. The group Q actually equals P when s equals 1. However, the point Λ_Q in $(\mathfrak{a}_Q^G)^* = (\mathfrak{a}_P^G)^*$ does not equal the real part Λ of λ. Indeed, the conditions we have imposed imply that $(\Lambda - \Lambda_Q)(\alpha^\vee) > 0$ for each $\alpha \in \Delta_Q$. We change the contour of integration from $\Lambda_Q + i(\mathfrak{a}_Q^G)^*$ to the affine space

$$\Lambda_Q + r\rho_P + i(\mathfrak{a}_Q^G)^*,$$

for a large positive number r. As in the second step, the integral approaches 0 as r approaches infinity. In this case, however, the function

$$\theta_Q(s\lambda - \mu) = \theta_P(\lambda - \mu)$$

contributes a multidimensional residue at $\mu = \lambda$. Using a change of variables

$$\mu = \sum_{\alpha \in \Delta_P} z_\alpha \varpi_\alpha, \qquad z_\alpha \in \mathbb{C},$$

one sees without difficulty that the residue equals the value of (15.10) at $s = 1$ and $\mu = \lambda$. This value is therefore equal to the original inner product (15.8). Since the original indices of summation Q and s have disappeared, we may as well re-introduce them in place of the indices Q' and t in (15.9). We then have the following inner product formula.

PROPOSITION 15.3 (Langlands). *Suppose that* $\phi \in \mathcal{H}^0_{P,\mathrm{cusp}}$ *and* $\phi' \in \mathcal{H}^0_{P',\mathrm{cusp}}$, *for standard parabolic subgroups* P *and* P'. *The inner product*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi,\lambda)\overline{\Lambda^T E(x,\phi',\lambda')}\mathrm{d}x$$

is then equal to the sum

$$\sum_Q \sum_s \sum_{s'} \theta_Q(s\lambda + s'\bar{\lambda}')^{-1}\mathrm{e}^{(s\lambda+s'\bar{\lambda}')(T)}\big(M(s,\lambda)\phi, M(s',\lambda')\phi'\big), \tag{15.11}$$

taken over $Q \supset P_0$, $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ *and* $s' \in W(\mathfrak{a}_{P'}, \mathfrak{a}_Q)$, *as meromorphic functions of* $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ *and* $\lambda' \in \mathfrak{a}^*_{P',\mathbb{C}}$.

The discussion above has been rather dense. However, it does yield the required formula if the real parts of λ and λ' are suitably regular points in $(\mathfrak{a}^*_P)^+$ and $(\mathfrak{a}^*_{P'})^+$ respectively. Since both sides are meromorphic in λ and $\bar{\lambda}'$, the formula holds in general. □

The argument we have given was taken from §4 of [**A4**]. The formula stated by Langlands [**Lan1**, §9] actually differs slightly from (15.11). It contains an extra signed sum over the ordered partitions $\mathfrak{p}$ of the set Δ_Q. The reader might find it an interesting combinatorial exercise to prove directly that this formula reduces to (15.11).

We shall say that a class $\chi \in \mathfrak{X}$ is *unramified* if for every pair (P,π) in χ, the stabilizer of π in $W(\mathfrak{a}_P, \mathfrak{a}_P)$ is $\{1\}$. This is obviously completely parallel to the corresponding geometric definition in §11. Assume that χ is unramified, and that (P,π) is a fixed pair in χ. We shall use Proposition 15.3 to evaluate the distribution $J_\chi(f)$.

Suppose that ϕ and ϕ' are two vectors in the subspace $\mathcal{H}^0_{P,\mathrm{cusp},\pi}$ of $\mathcal{H}_P$. This represents the special case of Proposition 15.3 with $P' = P$. The factor

$$\big(M(s,\lambda)\phi, M(s',\lambda')\phi'\big)$$

in (15.11) vanishes if $s \neq s'$, since $M(s,\lambda)\phi$ and $M(s',\lambda')\phi'$ lie in the orthogonal subspaces $\mathcal{H}_{Q,\mathrm{cusp},s\pi}$ and $\mathcal{H}_{Q,\mathrm{cusp},s'\pi}$ of $\mathcal{H}_Q$. We use the resulting simplification to compute the inner product (15.1). We have of course to interchange the roles of (ϕ,λ) and (ϕ',λ'), and then let λ' approach a fixed point $\lambda \in i\mathfrak{a}^*_P$. Writing $\lambda' = \lambda + \zeta$, for a small point $\zeta \in i\mathfrak{a}^*_P$ in general position, we obtain

$$\begin{aligned}\big(M^T_{P,\chi}(\lambda)\phi',\phi\big) &= \lim_{\zeta\to 0}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi',\lambda+\zeta)\overline{\Lambda^T E(x,\phi,\lambda)}\mathrm{d}x \\ &= \lim_{\zeta\to 0}\sum_Q \sum_{s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)} \theta_Q(s\zeta)^{-1}\mathrm{e}^{(s\zeta)(T)}\big(M(s,\lambda+\zeta)\phi', M(s,\lambda)\phi\big).\end{aligned}$$

In particular, the last limit exists, and takes values in a finite dimensional space of functions of the highly regular point $T \in (\mathfrak{a}^G_0)^+$. (This is also easy to show directly.) We can therefore extend both the limit and the operator $M^T_{P,\chi}(\lambda)$ to all values of $T \in \mathfrak{a}^G_{P_0}$ so that the identity remains valid. Now, let $M(\widetilde{w}_s,\lambda)$ be the operator on $\mathcal{H}_P$ defined by analytic continuation from the analogue of (7.2) in which w_s has been replaced by the representative $\widetilde{w}_s$ of s in K. Since $M(s,\lambda)$ is unitary, we see easily from the definition (9.4) that

$$\begin{aligned}\big(M(s,\lambda+\zeta)\phi', M(s,\lambda)\phi\big) &= \big(M(s,\lambda)^{-1}M(s,\lambda+\zeta)\phi',\phi\big)\\ &= \mathrm{e}^{-(s\zeta)(T_0)}\big(M(\widetilde{w}_s,\lambda)^{-1}M(\widetilde{w}_s,\lambda+\zeta)\phi',\phi\big).\end{aligned}$$

It follows that

$$\big(M^{T_0}_{P,\chi}(\lambda)\phi',\phi\big) = \lim_{\zeta\to 0}\sum_Q\sum_s \theta_Q(s\zeta)^{-1}\big(M(\widetilde{w}_s,\lambda)^{-1}M(\widetilde{w}_s,\lambda+\zeta)\phi',\phi\big).$$

This formula does not depend on the choice of π. To compute the value

$$\mathrm{tr}\big(M^{T_0}_{P,\chi}(\lambda)\mathcal{I}_{P,\chi}(\lambda,f)\big) \tag{15.12}$$

at $T=T_0$ of the integrand in (15.2), we need only replace ϕ' by $\mathcal{I}_{P,\chi}(\lambda,f)\phi$, and then sum ϕ over a suitable orthonormal basis of $\mathcal{H}_{P,\chi}$.

Recall that $\mathcal{I}_P(\pi_\lambda)$ denotes the representation of $G(\mathbb{A})$ obtained by parabolic induction from the representation

$$\pi_\lambda(m) = \pi(m)\mathrm{e}^{\lambda(H_P(m))}, \qquad m\in M(\mathbb{A}),$$

of $M_P(\mathbb{A})$. We can also write $M(\widetilde{w}_s,\pi_\lambda)$ for the intertwining operator from $\mathcal{I}_P(\pi_\lambda)$ to $\mathcal{I}_Q(s\pi_\lambda)$ associated to an element $s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)$. Finally, let $m_{\mathrm{cusp}}(\pi)$ denote the multiplicity of π in the representation $R_{M_P,\mathrm{cusp}}$. Since

$$\mathcal{H}_{P,\chi} = \bigoplus_{s\in W(\mathfrak{a}_P,\mathfrak{a}_P)}\mathcal{H}_{P,\mathrm{cusp},s\pi},$$

the representation $\mathcal{I}_{P,\chi}(\lambda)$ is then isomorphic to a direct sum of

$$|W(\mathfrak{a}_P,\mathfrak{a}_P)|m_{\mathrm{cusp}}(\pi) \tag{15.13}$$

copies of the representation $\mathcal{I}_P(\pi_\lambda)$. The trace (15.12) is therefore equal to the product of (15.13) with

$$\mathrm{tr}\big(\mathcal{M}_P(\pi_\lambda)\mathcal{I}_P(\pi_\lambda,f)\big),$$

where $\mathcal{M}_P(\pi_\lambda)$ is the operator on underlying Hilbert space of $\mathcal{I}_P(\pi_\lambda)$ defined explicitly in terms of intertwining operators by

$$\mathcal{M}_P(\pi_\lambda) = \lim_{\zeta\to 0}\Big(\sum_Q\sum_{s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)}\theta_Q(s\lambda)^{-1}M(\widetilde{w}_s,\pi_\lambda)^{-1}M(\widetilde{w}_s,\pi_{\lambda+\zeta})\Big). \tag{15.14}$$

Since P has been fixed, we shall let P_1 index the sum over standard parabolic subgroups in the formula (15.2) for $J^T_\chi(f)$. If P_1 does not belong to $\mathcal{P}_\chi$, it turns out that $\mathcal{H}_{P_1,\chi}=\{0\}$. This is a consequence of Langlands's construction [**Lan5**, §7] of the full discrete spectrum in terms of residues of cuspidal Eisenstein series. For the construction includes a description of the inner product on the residual discrete spectrum in terms of residues of cuspidal self-intertwining operators. Since χ is unramified, there are no such operators, and the residual discrete spectrum associated to χ is automatically zero. This leaves only groups P_1 in the set $\mathcal{P}_\chi$. For any such P_1, the value at $T=T_0$ of the corresponding integral in (15.2) equals the integral over $\lambda\in i\mathfrak{a}^*_M$ of (15.11). Since

$$n_P^{-1}|\mathcal{P}_\chi||W(\mathfrak{a}_P,\mathfrak{a}_P)| = 1,$$

we obtain the following theorem.

THEOREM 15.4. *Suppose that* $\chi = \{(P, \pi)\}$ *is unramified. Then*

$$J_\chi(f) = m_{\mathrm{cusp}}(\pi) \int_{i\mathfrak{a}_P^*} \mathrm{tr}\big(\mathcal{M}_P(\pi_\lambda)\mathcal{I}_P(\pi_\lambda, f)\big)\,\mathrm{d}\lambda. \tag{15.15}$$

□

Part II. Refinements and Applications

16. The first problem of refinement

We have completed the general steps outlined in §6. The coarse geometric expansion of §10 and the coarse spectral expansion of §14 give us an identity

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J_\chi(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big), \tag{16.1}$$

that holds for any reductive group G. We have also seen how to evaluate the distributions $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ explicitly for unramified classes $\mathfrak{o}$ and χ.

From now on, we shall generally work over an arbitrary number field F, whose adele ring $\mathbb{A}_F$ we denote simply by $\mathbb{A}$. We write S_∞ for the set of archimedean valuations of F, and we let q_v denote the order of the residue class field of the nonarchimedean completion F_v attached to any $v \notin S_\infty$. We are now taking G to be a fixed, connected reductive algebraic group over F. We write $S_{\mathrm{ram}} = S_{\mathrm{ram}}(G)$ for the finite set of valuations of F outside of which G is unramified. Thus, for any $v \notin S_{\mathrm{ram}}$, G is quasisplit over F_v, and splits over some finite unramified extension of F_v.

The notation of Part I carries over with F in place of $\mathbb{Q}$. So do the results, since they are valid for the group $G_1 = R_{F/\mathbb{Q}}G$ over $\mathbb{Q}$ obtained from G by restriction of scalars. For example, the real vector space $\mathfrak{a}_{G_1}$ is canonically isomorphic to its analogue $\mathfrak{a}_G$ for G. The kernel $G(\mathbb{A})^1$ of the canonical mapping $H_G\colon G(\mathbb{A}) \to \mathfrak{a}_G$ is isomorphic to $G_1(\mathbb{A})^1$. It is a factor in a direct product decomposition

$$G(\mathbb{A}) = G(\mathbb{A})^1 \times A_\infty^+,$$

whose other factor

$$A_\infty^+ = A_{G_1}(\mathbb{R})^0$$

embeds diagonally in the connected, abelian Lie group

$$\prod_{v\in S_\infty} A_G(F_v)^0.$$

We shall apply the notation and results of Part I without further comment.

The results in Part I that culminate in the identity (16.1) are the content of the papers [**A3**], [**A4**] and [**A5**, §1–3], and a part of [**A1**, §1–3]. We note in passing that there is another possible approach to the problem, which was used more recently in a local context [**A19**]. It exploits the cruder truncation operation of simply multiplying functions by the local analogue of the characteristic function $F^G(\cdot, T)$. Although the methods of [**A19**] have not been applied globally, they could conceivably shorten some of the arguments. On the other hand, such methods are perhaps less natural in the global context. They would lead to functions of T that are asymptotic to the relevant polynomials, rather than being actually equal to them.

The identity (16.1) can be regarded as a first approximation to a general trace formula. Let us write $\mathcal{X}_{\mathrm{cusp}}$ for the set of cuspidal classes in $\mathcal{X}$. A class $\chi \in \mathcal{X}_{\mathrm{cusp}}$ is thus of the form (G, π), where π is a cuspidal automorphic representation of $G(\mathbb{A})^1$. For any such χ, the explicit formula of §15 specializes to

$$J_\chi(f) = a^G(\pi) f_G(\pi),$$

where

$$f_G(\pi) = \mathrm{tr}\big(\pi(f)\big) = \mathrm{tr}\Big(\int_{G(\mathbb{A})^1} f(x)\pi(x)\mathrm{d}x\Big)$$

and

$$a^G(\pi) = m_{\mathrm{cusp}}(\pi).$$

Recall that $m_{\mathrm{cusp}}(\pi)$ is the multiplicity of π in the representation R_{cusp} of $G(\mathbb{A})^1$ on $L^2_{\mathrm{cusp}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$. In particular,

$$\mathrm{tr}\big(R_{\mathrm{cusp}}(f)\big) = \sum_{\chi\in\mathcal{X}_{\mathrm{cusp}}} J_\chi(f).$$

The identity (16.1) can thus be written as a trace formula

$$\mathrm{tr}\big(R_{\mathrm{cusp}}(f)\big) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) - \sum_{\chi\in\mathcal{X}-\mathcal{X}_{\mathrm{cusp}}} J_\chi(f). \tag{16.1$'$}$$

The problem is that the explicit formulas we have obtained so far do not apply to all of the terms on the right.

It is also easy to see that (16.1) generalizes the Selberg trace formula (1.3) for compact quotient. Let us write $\mathcal{O}_{\mathrm{anis}}$ for the set of anisotropic classes in $\mathcal{O}$. A class $\mathfrak{o}\in\mathcal{O}_{\mathrm{anis}}$ is thus of form $\{\gamma\}$, where γ represents an anisotropic conjugacy class in $G(\mathbb{Q})$. (Recall that an anisotropic class is one that does not intersect $P(\mathbb{Q})$ for any proper $P \subsetneq G$.) For any such $\mathfrak{o}$, the explicit formula of §11 specializes to

$$J_{\mathfrak{o}}(f) = a^G(\gamma) f_G(\gamma),$$

where

$$f_G(\gamma) = \int_{G(\mathbb{A})_\gamma\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\mathrm{d}x$$

and

$$a^G(\gamma) = \mathrm{vol}\big(G(F)_\gamma\backslash G(\mathbb{A})^1_\gamma\big).$$

The identity (16.1) can therefore be written

(16.1)$''$

$$\sum_{\gamma\in\Gamma_{\mathrm{anis}}(G)} a^G(\gamma)f_G(\gamma) + \sum_{\mathfrak{o}\in\mathcal{O}-\mathcal{O}_{\mathrm{anis}}} J_{\mathfrak{o}}(f) = \sum_{\pi\in\Pi_{\mathrm{cusp}}(G)} a^G(\pi)f_G(\pi) + \sum_{\chi\in\mathcal{X}-\mathcal{X}_{\mathrm{cusp}}} J_\chi(f),$$

where $\Gamma_{\mathrm{anis}}(G)$ is the set of conjugacy classes in $G(F)$ that do not intersect any proper group $P(F)$, and $\Pi_{\mathrm{cusp}}(G)$ is the set of equivalence classes of cuspidal automorphic representations of $G(\mathbb{A})^1$. Recall that $G(F)\backslash G(\mathbb{A})^1$ is compact if and only if G has no proper rational parabolic subgroup P. In this case $\mathcal{O} = \mathcal{O}_{\mathrm{anis}}$ and $\mathcal{X} = \mathcal{X}_{\mathrm{cusp}}$, and (16.1)$''$ reduces to the trace formula for compact quotient discussed in §1.

For general G, the equivalent formulas (16.1), (16.1)$'$, and (16.1)$''$ are of limited use as they stand. Without explicit expressions for all of the distributions $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$, one cannot get much information about the discrete (or cuspidal) spectrum. In the language of [**CLL**], we need to refine the coarse geometric and spectral expansions we have constructed.

What exactly are we looking for? The unramified cases solved in §11 and §15 will serve as guidelines.

The weighted orbital integral on the right hand side of the formula (11.9) is defined explicitly in terms of f. It is easier to handle than the original global construction of the distribution $J_{\mathfrak{o}}(f)$ on the left hand side of the formula. We would like to have a similar formula in general. The problem is that the right hand side of (11.9) does not make sense for more general classes $\mathfrak{o} \in \mathcal{O}$. It is in fact not so simple to define weighted orbital integrals for arbitrary elements in M. We shall do so in §18. Then in §19, we shall describe a general formula for $J_{\mathfrak{o}}(f)$ as a linear combination of weighted orbital integrals.

The weighted character on the right hand side of (15.15) is also defined explicitly in terms of f. It is again easier to handle than the global construction of the distribution $J_\chi(f)$ on the left hand side. Weighted characters are actually rather easy to define in general. However, this advantage is accompanied by a delicate analytic problem that does not occur on the geometric side. It concerns an interchange of two limits that arises when one tries to evaluate $J_\chi(f)$ for general classes $\chi \in \mathcal{X}$. We shall describe the solution of the analytic problem in §20. In §21 we shall give a general formula for $J_\chi(f)$ as a linear combination of weighted characters.

We adjust our focus slightly in Part II, which is to say, for the rest of the paper. We have already agreed to work over a general number field F instead of $\mathbb{Q}$. We shall make three further changes, all minor, in the conventions of Part I.

The first is a small change of notation. If H is a connected algebraic group over a given field k, and γ belongs to $H(k)$, we shall denote the centralizer of γ in H by $H_{\gamma,+}$ instead of H_γ. We reserve the symbol H_γ for the Zariski connected component of 1 in $H_{\gamma,+}$. Then H_γ is a connected algebraic group over k, which is reductive if H is reductive and γ is semisimple. This convention leads to a slightly different way of writing the formula (11.9) for unramified classes $\mathfrak{o} \in \mathcal{O}$. In particular, suppose that $\mathfrak{o}$ is anisotropic. Then

$$J_{\mathfrak{o}}(f) = a^G(\gamma) f_G(\gamma),$$

where we now write

$$a^G(\gamma) = \mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1\big)$$

and

$$f_G(\gamma) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\mathrm{d}x.$$

This would seem to be in conflict with the notation of $(16.1)''$, since the group $G_\gamma(\mathbb{A})^1$ here is of finite index in the group denoted $G(\mathbb{A})^1_\gamma$ above. There is in fact no discrepancy, for the reason that the two factors $a^G(\gamma)$ and $f_G(\gamma)$ depend in either case on an implicit and unrestricted choice of Haar measure on the given isotropy group.

The second change is to make the discussion more canonical by allowing the minimal parabolic subgroup P_0 to vary. We have, after all, shown that the distributions $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ are independent of P_0. Some new notation is required, which we may as well formulate for an arbitrary field k that contains F. We can of course regard G as a reductive algebraic group over k. Parabolic subgroups certainly make sense in this context, as do other algebraic objects we have discussed.

By a Levi subgroup of G over k, we mean an k-rational Levi component of some k-rational parabolic subgroup of G. Any such group M is reductive, and comes with a maximal k-split central torus A_M, and a corresponding real vector space $\mathfrak{a}_M$. (A Levi subgroup M of G over F is also a Levi subgroup over k, but A_M and $\mathfrak{a}_M$ depend on the choice of base field. Failure to remember this can lead to embarrassing errors!) Given M, we write $\mathcal{L}(M) = \mathcal{L}^G(M)$ for the set of Levi subgroups of G over k that contain M, and $\mathcal{F}(M) = \mathcal{F}^G(M)$ for the set of parabolic subgroups of G over k that contain M. Any element $Q \in \mathcal{F}(M)$ has a unique Levi component M_Q in $\mathcal{L}(M)$, and hence a canonical Levi decomposition $Q = M_Q N_Q$. We write $\mathcal{P}(M)$ for the subset of groups $Q \in \mathcal{F}(M)$ such that $M_Q = M$. For any $P \in \mathcal{P}(M)$, the roots of (P, A_M) determine an open chamber $\mathfrak{a}_P^+$ in the vector space $\mathfrak{a}_M$. Similarly, the corresponding coroots determine a chamber $(\mathfrak{a}_M^*)_P^+$ in the dual space $\mathfrak{a}_M^*$.

The sets $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\mathcal{F}(M)$ are all finite. They can be described in terms of the geometry on the space $\mathfrak{a}_M$. To see this, we use the singular hyperplanes in $\mathfrak{a}_M$ defined by the roots of (G, A_M). For example, the correspondence $P \to \mathfrak{a}_P^+$ is a bijection from $\mathcal{P}(M)$ onto the set of connected components in the complement in $\mathfrak{a}_M$ of the set of singular hyperplanes. We shall say that two groups $P, P' \in \mathcal{P}(M)$ are adjacent if their chambers share a common wall. The mapping $L \to \mathfrak{a}_L$ is a bijection from $\mathcal{L}(M)$ onto the set of subspaces of $\mathfrak{a}_M$ obtained by intersecting singular hyperplanes. The third set $\mathcal{F}(M)$ is clearly the disjoint union over $L \in \mathcal{L}(M)$ of the sets $\mathcal{P}(L)$. The mapping $Q \to \mathfrak{a}_Q^+$ is therefore a bijection from $\mathcal{F}(M)$ onto the set of "facets" in $\mathfrak{a}_M$, obtained from chambers of subspaces $\mathfrak{a}_L$. Since any element in $\mathfrak{a}_M$ belongs to a unique facet, there is a surjective mapping from $\mathfrak{a}_M$ to $\mathcal{F}(M)$.

Suppose for example that G is the split group $SL(3)$, that k is any field, and that $M = M_0$ is the standard minimal Levi subgroup. The singular hyperplanes in the two dimensional space $\mathfrak{a}_M$ are illustrated in Figure 16.1. The set $\mathcal{P}(M)$ is bijective with the six open chambers in the diagram. The set $\mathcal{L}(M)$ has five elements, consisting of the two-dimensional space $\mathfrak{a}_M$, the three one-dimensional lines, and the zero-dimensional origin. The set $\mathcal{F}(M)$ has thirteen elements, consisting of six open chambers, six half lines, and the origin. The intuition gained from Figure 16.1, simple though it is, is often useful in understanding operations we perform in general.

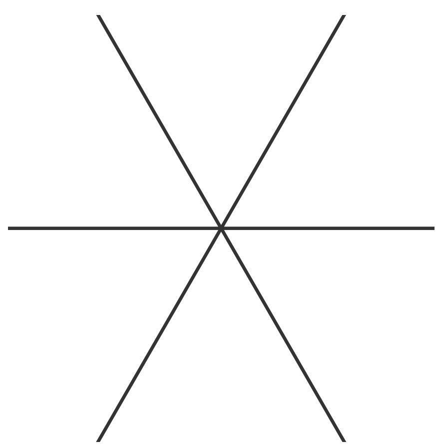

FIGURE 16.1. The three singular hyperplanes in the two dimensional space $\mathfrak{a}_M = \mathfrak{a}_0$ attached to $G = SL(3)$.

Suppose now that $k = F$. Even though we do not fix the minimal parabolic subgroup as in Part I, we shall work with a fixed minimal Levi subgroup M_0 of G

over $\mathbb{Q}$. We denote the associated sets $\mathcal{L}(M_0)$ and $\mathcal{F}(M_0)$ by $\mathcal{L} = \mathcal{L}^G$ and $\mathcal{F} = \mathcal{F}^G$, respectively.

The variance formulas (10.6) and (14.7) can be written without reference to P_0. The reason is that for a given $P_0 \in \mathcal{P}(M_0)$, any group $R \in \mathcal{F}$ is the image under some element in the restricted Weyl group $W_0 = W_0^G$ of a unique group $Q \in \mathcal{F}$ with $Q \supset P_0$. It is an easy consequence of the definitions that $J_{\mathfrak{o}}^{M_R}(f_{R,y})$ equals $J_{\mathfrak{o}}^{M_Q}(f_{Q,y})$ for any $\mathfrak{o}$, and that $J_{\chi}^{M_R}(f_{R,y})$ equals $J_{\chi}^{M_Q}(f_{Q,y})$ for any χ. The order of the preimage of Q in $\mathcal{F}$ is equal to the quotient $|W_0^{M_Q}||W_0^G|^{-1}$. Letting Q now stand for an arbitrary group in $\mathcal{F}$, we can write the earlier formulas as

$$J_{\mathfrak{o}}(f^y) = \sum_{Q \in \mathcal{F}} |W_0^{M_Q}||W_0^G|^{-1} J_{\mathfrak{o}}^{M_Q}(f_{Q,y}), \qquad \mathfrak{o} \in \mathcal{O}, \tag{16.2}$$

and

$$J_{\chi}(f^y) = \sum_{Q \in \mathcal{F}} |W_0^{M_Q}||W_0^G|^{-1} J_{\chi}^{M_Q}(f_{Q,y}), \qquad \chi \in \mathcal{X}. \tag{16.3}$$

The third point is a slight change of emphasis. The distributions $J_{\mathfrak{o}}(f)$ and $J_{\chi}(f)$ in (16.1) depend only on the restriction of f to $G(\mathbb{A})^1$. We have in fact identified f implicitly with its restriction to $G(\mathbb{A})^1$, in writing $R_{\mathrm{cusp}}(f)$ above for example. Let us now formalize the convention by setting $C_c^{\infty}\big(G(\mathbb{A})^1\big)$ equal to the space of functions on $G(\mathbb{A})^1$ obtained by restriction of functions in $C_c^{\infty}\big(G(\mathbb{A})\big)$. We can then take the test function f to be an element in $C_c^{\infty}\big(G(\mathbb{A})^1\big)$ rather than $C_c^{\infty}\big(G(\mathbb{A})\big)$, thereby regarding (16.1) as an identity of distributions on $G(\mathbb{A})^1$. This adjustment is obviously quite trivial. However, as we shall see in §22, it raises an interesting philosophical question that is at the heart of some key operations on the trace formula.

17. (G, M)-families

The terms in the refined trace formula will have some interesting combinatorial properties. To analyze them, one introduces the notion of a (G, M)-family of functions. We shall see that among other things, (G, M)-families provide a partial unification of the study of weighted orbital integrals and weighted characters.

We are now working in the setting of the last section. Then G is defined over the fixed number field F, and hence over any given extension k of F. Let M be a Levi subgroup of G over k. Suppose that for each $P \in \mathcal{P}(M)$,

$$c_P(\lambda), \qquad \lambda \in i\mathfrak{a}_M^*,$$

is a smooth function on the real vector space $i\mathfrak{a}_M^*$. The collection

$$\{c_P(\lambda) : \ P \in \mathcal{P}(M)\}$$

is called a (G, M)-*family* if $c_P(\lambda) = c_{P'}(\lambda)$, for any pair of adjacent groups $P, P' \in \mathcal{P}(M)$, and any point λ in the hyperplane spanned by the common wall of the chambers $i(\mathfrak{a}_M^*)_P^+$ and $i(\mathfrak{a}_M^*)_{P'}^+$. We shall describe a basic operation that assigns a supplementary smooth function $c_M(\lambda)$ on $i\mathfrak{a}_M^*$ to any (G, M)-family $\{c_P(\lambda)\}$.

The algebraic definitions of §4 and §5 of course hold with the field k in place of $\mathbb{Q}$. In particular, for any $P \in \mathcal{P}(M)$ we have the simple roots Δ_P of (P, A_M), and the associated sets $\Delta_P^{\vee}$, $\widehat{\Delta}_P$ and $(\widehat{\Delta}_P)^{\vee}$. We are assuming we have fixed a suitable

Haar measure on the subspace $\mathfrak{a}_M^G = \mathfrak{a}_P^G$ of $\mathfrak{a}_M$. We then define a homogeneous polynomial

$$\theta_P(\lambda) = \mathrm{vol}\big(\mathfrak{a}_M^G/\mathbb{Z}(\Delta_P^\vee)\big)^{-1} \cdot \prod_{\alpha\in\Delta_P} \lambda(\alpha^\vee), \qquad \lambda \in i\mathfrak{a}_M^*,$$

on $i\mathfrak{a}_M^*$, where $\mathbb{Z}(\Delta_P^\vee)$ is the lattice spanned by the basis $\Delta_P^\vee$ of $\mathfrak{a}_M^G$.

LEMMA 17.1. *For any (G, M)-family $\{c_P(\lambda)\}$, the sum*

$$c_M(\lambda) = \sum_{P\in\mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1} \tag{17.1}$$

extends to a smooth function of $\lambda \in i\mathfrak{a}_M^*$.

The only possible singularities of $c_M(\lambda)$ are simple poles along hyperplanes in $i\mathfrak{a}_M^*$ of the form $\lambda(\alpha^\vee) = 0$. These in turn come from adjacent pairs P and P' for which α and $\alpha' = (-\alpha)$ are respective simple roots. Using the fact that $c_P(\lambda) = c_{P'}(\lambda)$ for any λ on the hyperplane, one sees directly that the simple poles cancel, and therefore that $c_M(\lambda)$ does extend to a smooth function. (See [**A5**, Lemma 6.2].) □

We often write $c_M = c_M(0)$ for the value of $c_M(\lambda)$ at $\lambda = 0$. It is in this form that the (G, M)-families from harmonic analysis usually appear.

We shall first describe a basic example that provides useful geometric intuition. Suppose that

$$\mathcal{Y} = \{Y_P : \ P \in \mathcal{P}(M)\}$$

is a family of points in $\mathfrak{a}_M$ parametrized by $\mathcal{P}(M)$. We say that $\mathcal{Y}$ is a *positive, (G, M)-orthogonal set* if for every pair P and P' of adjacent groups in $\mathcal{P}(M)$, whose chambers share the wall determined by the uniquely determined simple root $\alpha \in \Delta_P$,

$$Y_P - Y_{P'} = r_\alpha\alpha^\vee,$$

for a nonnegative number r_α. Assume that this condition holds. The collection

$$c_P(\lambda, \mathcal{Y}) = \mathrm{e}^{\lambda(Y_P)}, \qquad \lambda \in i\mathfrak{a}_M^*, \ P \in \mathcal{P}(M), \tag{17.2}$$

is then a (G, M)-family of functions, which extend analytically to all points λ in the complex space $\mathfrak{a}_{M,\mathbb{C}}^*$. As with any (G, M)-family, the associated smooth function $c_M(\lambda, \mathcal{Y})$ depends on the choice of Haar measure on $\mathfrak{a}_M^G$. In this case, the function has a simple interpretation.

Observe first that

$$Y_P = Y_P^G + Y_G, \qquad Y_P^G \in \mathfrak{a}_P^G, \ Y_G \in \mathfrak{a}_G,$$

where Y_G is independent of the choice of $P \in \mathcal{P}(M)$. Subtracting the fixed point $Y_G \in \mathfrak{a}_G$ from each Y_P, we can assume that $Y_P \in \mathfrak{a}_M^G$. Now in §11, we attached a sign $\varepsilon_P(\Lambda)$ and a characteristic function $\phi_P(\Lambda, \cdot)$ on $\mathfrak{a}_M$ to each $P \in \mathcal{P}(M)$ and $\Lambda \in \mathfrak{a}_M$. Suppose that Λ is in general position, and that λ is any point in $\mathfrak{a}_{M,\mathbb{C}}^*$ whose real part equals Λ. The function

$$\varepsilon_P(\Lambda)\phi_P(\Lambda, H - Y_P)\mathrm{e}^{\lambda(H)}, \qquad H \in \mathfrak{a}_M^G,$$

is then rapidly decreasing. By writing

$$H = \sum_{\alpha\in\Delta_P} t_\alpha\alpha^\vee, \qquad t_\alpha \in \mathbb{R},$$

we deduce easily that the integral of this function over H equals

$$e^{\lambda(Y_P)}\theta_P(\lambda)^{-1} = c_P(\lambda, \mathcal{Y})\theta_P(\lambda)^{-1}.$$

It then follows that

$$\sum_{P\in\mathcal{P}(M)} e^{\lambda(Y_P)}\theta_P(\lambda)^{-1} = \int_{\mathfrak{a}_M^G} \psi_M(H,\mathcal{Y})e^{\lambda(H)}dH, \tag{17.3}$$

where

$$\psi_M(H,\mathcal{Y}) = \sum_{P\in\mathcal{P}(M)} \varepsilon_P(\Lambda)\phi_P(\Lambda, H - Y_P).$$

LEMMA 17.2. *The function*

$$H \longrightarrow \psi_M(H,\mathcal{Y}), \qquad H \in \mathfrak{a}_M^G,$$

is the characteristic function of the convex hull in $\mathfrak{a}_M^G$ *of* $\mathcal{Y}$.

The main step in the proof of Lemma 17.2 is the combinatorial lemma of Langlands mentioned at the end of §8. This result asserts that

$$\sum_{Q\supset P} \varepsilon_P^Q(\Lambda)\phi_P^Q(\Lambda, H)\tau_Q(H) = \begin{cases} 1, & \text{if } \Lambda(\alpha^\vee) > 0,\ \alpha\in\Delta_P, \\ 0, & \text{otherwise,}\end{cases} \tag{17.4}$$

for any $P \in \mathcal{P}(M)$ and $H \in \mathfrak{a}_M$, where ε_P^Q and ϕ_P^Q denote objects attached to the parabolic subgroup $P \cap M_Q$ of M_Q. Langlands's geometric proof of (17.4) was reproduced in [**A1**, §2]. There is a different combinatorial proof [**A3**, Corollary 6.3], which combines an induction argument with (8.10) and Identity 6.2. Given the formula (17.4), one then observes that $\psi_M(H,\mathcal{Y})$ is independent of the point Λ. This follows inductively from the expression obtained by summing the left hand side of (17.4) over $P \in \mathcal{P}(M)$ [**A1**, Lemma 3.1]. Finally, by varying Λ, one shows that

$$\psi_M(H,\mathcal{Y}) = \begin{cases} 1, & \text{if } \varpi(H - Y_P) \le 0,\ \varpi = \widehat{\Delta}_P,\ P\in\mathcal{P}(M), \\ 0, & \text{otherwise.}\end{cases}$$

The inequalities on the right characterize the convex hull of $\mathcal{Y}$, according to the Krein-Millman theorem. (See [**A1**, Lemma 3.2].) □

The convex hull of $\mathcal{Y}$ is of course compact. It follows that the integral on the right hand side of (17.3) converges absolutely, uniformly for $\lambda \in i\mathfrak{a}_{M,\mathbb{C}}^*$. We can therefore identify the smooth function $c_M(\lambda,\mathcal{Y})$ with the Fourier transform of the characteristic function of the convex hull of $\mathcal{Y}$. Its value $c_M(\mathcal{Y})$ at $\lambda = 0$ is simply the volume of the convex hull. We have actually been assuming that the point $Y_G \in \mathfrak{a}_G$ attached to $\mathcal{Y}$ equals zero. However, if Y_G is nonzero, the convex hull of $\mathcal{Y}$ represents a compactly supported distribution in the affine subspace $Y_G + \mathfrak{a}_M^G$ of $\mathfrak{a}_M$. The last two assertions therefore remain valid for any $\mathcal{Y}$.

Consider the case that $G = SL(3)$ and M equals the standard minimal Levi subgroup. The convex hull of a typical set $\mathcal{Y}$ is illustrated in Figure 17.1, a diagram on which one could superimpose six convex cones, as in the earlier special case of Figure 11.1. The six points Y_P are the six vertices in the diagram. We have chosen them here to lie in the associated chambers $\mathfrak{a}_P^+$. Notice that with this condition, the intersection of the convex hull with the closure of a chamber $\mathfrak{a}_P^+$ equals a set of the kind illustrated in Figure 9.2. This suggests that the characteristic function $\psi_M(H,\mathcal{Y})$ is closely related to the functions $\Gamma'_P(\cdot, Y_P)$ defined in §9.

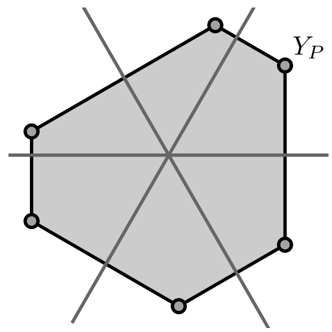

FIGURE 17.1. The convex hull of six points $\{Y_P\}$ in the two dimensional space $\mathfrak{a}_0$ attached to $SL(3)$. Observe that its intersection with any of the six chambers $\mathfrak{a}_P^+$ in the diagram is a region like that in Figure 9.2.

Suppose that X is any point in $\mathfrak{a}_M^G$. According to Lemma 9.2, the function $H \to \Gamma'_P(H,X)$ on $\mathfrak{a}_M^G$ is compactly supported for any $P \in \mathcal{P}(M)$. The integral

$$\int_{\mathfrak{a}_M^G} \Gamma'_P(H,X) e^{\lambda(H)} dH \tag{17.5}$$

therefore converges uniformly to an analytic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. To compute it, we first note that for any $P \in \mathcal{P}(M)$,

$$\begin{aligned}&\sum_{Q \supset P} (-1)^{\dim(A_Q/A_G)} \tau_P^Q(H) \widehat{\tau}_Q(H-X)\\ =&\sum_{Q \supset P} (-1)^{\dim(A_Q/A_G)} \tau_P^Q(H) \sum_{Q' \supset Q} (-1)^{\dim(A_{Q'}/A_G)} \widehat{\tau}_Q^{Q'}(H) \Gamma'_{Q'}(H,X)\\ =&\sum_{Q' \supset P} \Big(\sum_{\{Q: P \subset Q \subset Q'\}} (-1)^{\dim(A_Q/A_{Q'})} \tau_P^Q(H) \widehat{\tau}_Q^{Q'}(H) \Big) \Gamma'_Q(H,X)\\ =&\Gamma'_P(H,X),\end{aligned}$$

by the inductive definition (9.1) and the formula (8.10). Suppose that the real part of λ lies in the negative chamber $-(\mathfrak{a}_M^*)_P^+$. Then the integral

$$\int_{\mathfrak{a}_M^G} \tau_P^Q(H) \widehat{\tau}_Q(H-X) e^{\lambda(H)} dH$$

converges. Changing variables by writing

$$H = \sum_{\varpi \in \widehat{\Delta}_P^Q} t_\varpi \varpi^\vee + \sum_{\alpha \in \Delta_Q} t_\alpha \alpha^\vee, \qquad t_\varpi, t_\alpha \in \mathbb{R},$$

one sees without difficulty that the integral equals

$$(-1)^{\dim(A_P/A_G)} e^{\lambda_Q(X)} \widehat{\theta}_P^Q(\lambda)^{-1} \theta_Q(\lambda_Q)^{-1},$$

where λ_Q is the projection of λ onto $\mathfrak{a}^*_{Q,\mathbb{C}}$, and

$$\widehat{\theta}_P^Q(\lambda) = \widehat{\theta}_{P \cap M_Q}(\lambda) = \mathrm{vol}\big(\mathfrak{a}_P^Q / \mathbb{Z}\big((\widehat{\Delta}_P^Q)^\vee\big)\big)^{-1} \prod_{\varpi \in \widehat{\Delta}_P^Q} \lambda(\varpi^\vee).$$

(See [**A5**, p. 15].) It follows that the original integral (17.5) equals

$$\sum_{Q \supset P} (-1)^{\dim(A_P/A_Q)} e^{\lambda_Q(X)} \widehat{\theta}_P^Q(\lambda)^{-1} \theta_Q(\lambda_Q)^{-1}. \tag{17.6}$$

In particular, the function (17.6) extends to an analytic function of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.

Suppose now that for a given $P \in \mathcal{P}(M)$, $c_P(\lambda)$ is an arbitrary smooth function of $\lambda \in i\mathfrak{a}_M^*$. Motivated by the computation above, we set

$$c'_P(\lambda) = \sum_{Q \supset P} (-1)^{\dim(A_P/A_Q)} c_Q(\lambda_Q) \widehat{\theta}_P^Q(\lambda)^{-1} \theta_Q(\lambda_Q)^{-1}, \tag{17.7}$$

where c_Q is the restriction of c_P to $i\mathfrak{a}_Q^*$, and λ_Q is again the projection of λ onto $i\mathfrak{a}_Q^*$. Then c'_P is defined on the complement of a finite set of hyperplanes in $i\mathfrak{a}_M^*$.

Lemma 17.3. *$c'_Q(\lambda)$ extends to a smooth function of $\lambda \in i\mathfrak{a}_M^*$.*

The lemma is not surprising, given what we have established in the special case that $c_P(\lambda) = e^{\lambda(X)}$. One can either adapt the discussion above to the more general case, as in [**A3**, Lemma 6.1], or approximate $c_P(\lambda)$ by functions of the form $e^{\lambda(X)}$, and apply the results above directly. □

Assume now that $\{c_P(\lambda) : P \in \mathcal{P}(M)\}$ is a general (G,M)-family. There are two restriction operations that give rise to two new families. Suppose that $Q \in \mathcal{F}(M)$. If R belongs to $\mathcal{P}^{M_Q}(M)$, we set

$$c_R^Q(\lambda) = c_{Q(R)}(\lambda),$$

where $Q(R)$ is the unique group in $\mathcal{P}(M)$ that is contained in Q, and whose intersection with M_Q equals R. Then $\{c_R^Q(\lambda) : R \in \mathcal{P}^{M_Q}(M)\}$ is an (M_Q,M)-family. The other restriction operation applies to a given group $L \in \mathcal{L}(M)$. If λ lies in the subspace $i\mathfrak{a}_L^*$ of $i\mathfrak{a}_M^*$, and Q is any group in $\mathcal{P}(L)$, we set

$$c_Q(\lambda) = c_P(\lambda),$$

for any group $P \in \mathcal{P}(M)$ with $P \subset Q$. Since we started with a (G,M)-family, this function is independent of the choice of P, and the resulting collection $\{c_Q(\lambda) : Q \in \mathcal{P}(L)\}$ is a (G,L)-family. Observe that the definition (17.7) can be applied to any Q. It yields a smooth function $c'_Q(\lambda)$ on $i\mathfrak{a}_L^*$ that depends only on $c_Q(\lambda)$. Again, we often write $d'_Q = d'_Q(0)$ for the value of $d'_Q(\lambda)$ at $\lambda = 0$.

Let $\{d_P(\lambda) : P \in \mathcal{P}(M)\}$ be a second (G,M)-family. Then the pointwise product

$$(cd)_P(\lambda) = c_P(\lambda) d_P(\lambda), \qquad P \in \mathcal{P}(M),$$

is also a (G,M)-family.

Lemma 17.4. *The product (G,M)-family satisfies the splitting formula*

$$(cd)_M(\lambda) = \sum_{Q \in \mathcal{P}(M)} c_M^Q(\lambda) d'_Q(\lambda_Q).$$

In particular the values at $\lambda = 0$ of the functions in the formula satisfy

$$(cd)_M = \sum_{Q \in \mathcal{F}(M)} c_M^Q d'_Q. \tag{17.8}$$

The lemma is an easy consequence of a formula

$$c_P(\lambda)\theta_P(\lambda)^{-1} = \sum_{Q\supset P} c'_Q(\lambda_Q)\theta_P^Q(\lambda)^{-1}, \qquad P \in \mathcal{P}(M), \tag{17.9}$$

where $\theta_P^Q = \theta_{P\cap M_Q}$, which we obtain by inverting the definition (17.7). To derive (17.9), we write

$$\begin{aligned}
&\sum_{Q\supset P} c'_Q(\lambda_Q)\theta_P^Q(\lambda)^{-1}\\
&= \sum_{Q\supset P}\sum_{Q'\supset Q}(-1)^{\dim(A_Q/A_{Q'})}c_{Q'}(\lambda_{Q'})\widehat{\theta}_Q^{Q'}(\lambda_Q)^{-1}\theta_{Q'}(\lambda_{Q'})^{-1}\theta_P^Q(\lambda)^{-1}\\
&= \sum_{Q'\supset P} c_{Q'}(\lambda_{Q'})\theta_{Q'}(\lambda_{Q'})^{-1}\Big(\sum_{\{Q:P\subset Q\subset Q'\}}(-1)^{\dim(A_Q/A_{Q'})}\theta_P^Q(\lambda)^{-1}\widehat{\theta}_Q^{Q'}(\lambda_Q)^{-1}\Big).
\end{aligned}$$

The expression in the brackets may be written as a Fourier transform

$$\int_{\mathfrak{a}_P^{Q'}}\Big(\sum_{\{Q:P\subset Q\subset Q'\}}(-1)^{\dim(A_P/A_Q)}\widehat{\tau}_P^Q(H)\tau_Q^{Q'}(H)\Big)e^{\lambda(H)}dH,$$

provided that the real part of λ lies in $-(\mathfrak{a}_M^*)_P^+$. The identity (8.11) tells us that the expression equals 0 or 1, according to whether Q' properly contains P or not. The formula (17.9) follows. Once we have (17.9), we see that

$$\begin{aligned}
(cd)_M(\lambda) &= \sum_{P\in\mathcal{P}(M)} c_P(\lambda)d_P(\lambda)\theta_P(\lambda)^{-1}\\
&= \sum_P c_P(\lambda)\sum_{Q\supset P} d'_Q(\lambda_Q)\theta_P^Q(\lambda)^{-1}\\
&= \sum_{Q\in\mathcal{F}(M)}\Big(\sum_{\{P\in\mathcal{P}(M):P\subset Q\}} c_P(\lambda)\theta_P^Q(\lambda)^{-1}\Big)d'_Q(\lambda_Q)\\
&= \sum_{Q\in\mathcal{F}(M)} c_M^Q(\lambda)d'_Q(\lambda_Q),
\end{aligned}$$

as required. (See [**A3**, Lemma 6.3].) □

Suppose for example that $c_P(\lambda) = 1$ for each P and λ. This is the family attached to the trivial positive (G, M)-orthogonal set $\mathcal{Y} = 0$. Then $c_M^Q(\lambda)$ equals 0 unless Q lies in the subset $\mathcal{P}(M)$ of $\mathcal{F}(M)$, in which case it equals 1. It follows that

$$d_M(\lambda) = \sum_{P\in\mathcal{P}(M)} d'_P(\lambda). \tag{17.10}$$

In the case that $d_P(\lambda)$ is of the special form (17.2), this formula matches the intuition we obtained from Figure 17.1 and Figure 9.2. For general $\{d_P(\lambda)\}$, and for $\{c_P(\lambda)\}$ subject only to a supplementary condition that the numbers

$$c_M^L = c_M^Q, \qquad L \in \mathcal{L}(M),\ Q \in \mathcal{P}(L), \tag{17.11}$$

be independent of the choice of Q, (17.10) can be applied to the splitting formula (17.8). We obtain a simpler splitting formula

$$(cd)_M = \sum_{L\in\mathcal{L}(M)} c_M^L d_L \tag{17.12}$$

Suppose that $\{c_P(\lambda)\}$ and $\{d_P(\lambda)\}$ correspond to positive (G, M)-orthogonal sets $\mathcal{Y} = \{Y_P\}$ and $\mathcal{Z} = \{Z_P\}$. Then the product family $\{(cd)_P(\lambda)\}$ corresponds to the sum $\mathcal{Y} + \mathcal{Z} = \{Y_P + Z_P\}$. In this case, (17.8) is similar to a classical formula for mixed volumes. In the case that $G = SL(3)$ and M is minimal, it is illustrated in Figure 17.2.

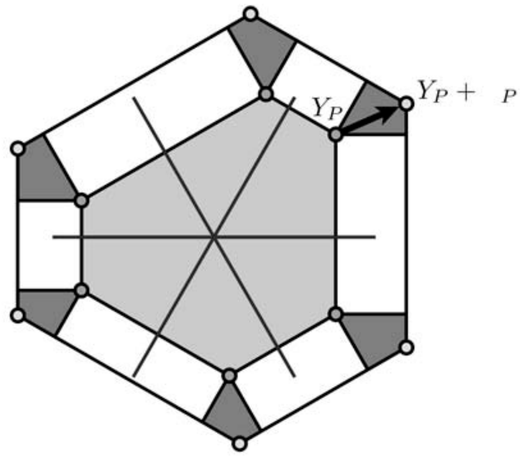

FIGURE 17.2. The entire region is the convex hull of six points $\{Y_P + Z_P\}$ in the two dimensional space $\mathfrak{a}_0$ attached to $SL(3)$. The inner shaded region is the convex hull of the six points $\{Y_P\}$. For any P, the area of the darker shaded region with vertex Y_P equals the area of a region in Figure 9.2. The areas of the six rectangular regions represent mixed volumes between the sets $\{Y_P\}$ and $\{Z_P\}$.

In addition to the splitting formula (17.8), there is a descent formula that relates the two restriction operations we have defined. It applies in fact to a generalization of the second operation.

Suppose that M contains a Levi subgroup M_1 of G over some extension k_1 of k. Then $\mathfrak{a}_M$ is contained in the vector space $\mathfrak{a}_{M_1}$ attached to M_1. Suppose that $\{c_{P_1}(\lambda_1) : P_1 \in \mathcal{P}(M_1)\}$ is a (G_1, M_1)-family. If P belongs to $\mathcal{P}(M)$ and λ lies in the subspace $i\mathfrak{a}_M^*$ of $i\mathfrak{a}_{M_1}^*$, we set

$$c_P(\lambda) = c_{P_1}(\lambda)$$

for any $P_1 \in \mathcal{P}(M_1)$ with $P_1 \subset P$. This function is independent of the choice of P_1, and the resulting collection $\{c_P(\lambda) : P \in \mathcal{P}(M)\}$ is a (G, M)-family. We would like to express the supplementary function $c_M(\lambda)$ in terms of corresponding functions $c_{M_1}^{Q_1}(\lambda_1)$ attached to groups $Q_1 \in \mathcal{F}(M_1)$. A necessary step is of course to fix Haar measures on each of the spaces $\mathfrak{a}_{M_1}^{L_1}$, as $L_1 = L_{Q_1}$ ranges over $\mathcal{L}(M_1)$. For example, we could fix a suitable Euclidean inner product on the space $\mathfrak{a}_{M_1}$, and then take the Haar measure on $\mathfrak{a}_{M_1}^{L_1}$ attached to the restricted inner product. For each L_1, we introduce a nonnegative number $d_{M_1}^G(M, L_1)$ to make the relevant measures compatible. We define $d_{M_1}^G(M, L_1)$ to be 0 unless the natural map

$$\mathfrak{a}_{M_1}^M \oplus \mathfrak{a}_{M_1}^{L_1} \longrightarrow \mathfrak{a}_{M_1}^G$$

is an isomorphism, in which case $d^G_{M_1}(M, L_1)$ is the factor by which the product Haar measure on $\mathfrak{a}^L_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1}$ must be multiplied in order to be equal to the Haar measure on $\mathfrak{a}^G_{M_1}$. (The measure on $\mathfrak{a}^M_{M_1}$ is the quotient of the chosen measures on $\mathfrak{a}^G_{M_1}$ and $\mathfrak{a}^G_M$.)

There is one other choice to be made. Given M and M_1, we select a small vector ξ in general position in $\mathfrak{a}^M_{M_1}$. If L_1 is any group $\mathcal{L}(M_1)$ with $d^G_{M_1}(M, L_1) \neq 0$, the affine space $\xi + \mathfrak{a}^G_M$ intersects $\mathfrak{a}^G_{L_1}$ at one point. This point is nonsingular, and so belongs to a chamber $\mathfrak{a}^+_{Q_1}$, for a unique group $Q_1 \in \mathcal{P}(L_1)$. The point ξ thus determines a section

$$L_1 \longrightarrow Q_1, \qquad\qquad L_1 \in \mathcal{L}(M_1),\ d^G_{M_1}(M, L_1) \neq 0,$$

from L_1 to its fibre $\mathcal{P}(L_1)$.

Lemma 17.5. *For $F_1 \supset F$, $M_1 \subset M$, and $\{c_{P_1}(\lambda_1)\}$ as above, we have*

$$c_M(\lambda) = \sum_{L_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, L_1) c^{Q_1}_{M_1}(\lambda), \qquad\qquad \lambda \in i\mathfrak{a}^*_M.$$

In particular, the values at $\lambda = 0$ of these functions satisfy

$$c_M = \sum_{L_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, L_1) c^{Q_1}_{M_1}. \tag{17.13}$$

Lemma 17.5 is proved under slightly more general conditions in [**A13**, Proposition 7.1]. We shall be content to illustrate it geometrically in a very special case. Suppose that $k = k_1$, $G = SL(3)$, M is a maximal Levi subgroup, M_1 is a minimal Levi subgroup, and $\{c_{P_1}(\lambda_1)\}$ is of the special form (17.2). The points $\{Y_{P_1}\}$ are the six vertices of the polytope in Figure 17.3. They are of course bijective with the set of minimal parabolic subgroups $P_1 \in \mathcal{P}(M_1)$. The six edges in the polytope are bijective with the six maximal parabolic subgroups $Q_1 \in \mathcal{F}(M_1)$. The two vertical edges are perpendicular to $\mathfrak{a}_M$, so the corresponding coefficients $d^G_{M_1}(M, L_1)$ vanish. The remaining four edges occur in pairs, corresponding to two pairs of groups $Q_1 \in \mathcal{P}(L_1)$ attached to the two maximal Levi subgroups $L_1 \neq M$. However, the upward pointing vector $\xi \in \mathfrak{a}^M_{M_1}$ singles out the upper two edges. The projections of these two edges onto the line $\mathfrak{a}_M$ are disjoint (apart from the interior vertex), with union equal to the line segment obtained by intersecting $\mathfrak{a}_M$ with the polytope. The length of this line segment is the sum of the lengths of the two upper edges, scaled in each case by the associated coefficient $d^G_{M_1}(M, L_1)$.

If this simple example is not persuasive, the reader could perform some slightly more complicated geometric experiments. Suppose that $\dim(\mathfrak{a}_{M_1}) = 3$ and $\{c_{P_1}(\lambda_1)\}$ is of the special form (17.2), but that k, G, k_1, and M_1 are otherwise arbitrary. It is interesting to convince oneself geometrically of the validity of the lemma in the two cases $\dim \mathfrak{a}_M = 1$ and $\dim \mathfrak{a}_M = 2$. The motivation for the general proof is based on these examples. □

We sometimes use a variant of Lemma 17.5, which is included in the general formulation of [**A13**, Propositon 7.1]. It concerns the case that $F = F_1$, but where M is embedded diagonally in the Levi subgroup $\mathcal{M} = M \times M$ of $\mathcal{G} = G \times G$. Then $\mathfrak{a}_M$ is embedded diagonally in the space $\mathfrak{a}_{\mathcal{M}} = \mathfrak{a}_M \oplus \mathfrak{a}_M$. Elements in $\mathcal{L}(\mathcal{M})$ consist of pairs $\mathcal{L} = (L_1, L_2)$, for Levi subgroups $L_1, L_2 \in \mathcal{L}(M)$ of G. (We have written

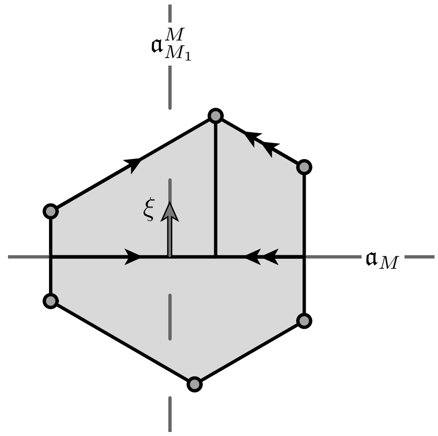

FIGURE 17.3. An illustration of the proof of Lemma 17.5, with $G = SL(3)$, M maximal, and $M_1 = M_0$ minimal. The two upper edges of the polytope project onto the two interior intervals on the horizontal axis. In each case, the projection contracts the length by the appropriate determinant $d_{M_1}^G(M, L_1)$.

$\mathcal{M}$, $\mathcal{G}$, and $\mathcal{L}$ in place of M_1, G_1, and L_1, since we are now using L_1 to denote the first component of $\mathcal{L}$.) The corresponding coefficient in (17.13) satisfies

$$d_{\mathcal{M}}^{\mathcal{G}}(M, \mathcal{L}) = 2^{\frac{1}{2}\dim(\mathfrak{a}_M^G)} d_M^G(L_1, L_2),$$

while if P belongs to $\mathcal{P}(M)$, the pair $\mathcal{P} = (P, P)$ in $\mathcal{P}(\mathcal{M})$ satisfies

$$\theta_{\mathcal{P}}(\lambda) = 2^{\frac{1}{2}\dim(\mathfrak{a}_M^G)} \theta_P(\lambda), \qquad \lambda \in i\mathfrak{a}_M^*.$$

We choose a small point ξ in general position in the space

$$\mathfrak{a}_{\mathcal{M}}^M = \{(H, -H) : \ H \in \mathfrak{a}_M\},$$

and let

$$(L_1, L_2) \longrightarrow (Q_1, Q_2), \qquad L_1, L_2 \in \mathcal{L}(M),\ d_M^G(L_1, L_2) \neq 0,$$

be the corresponding section from (L_1, L_2) to its fibre $\mathcal{P}(L_1) \times \mathcal{P}(L_2)$. If ξ is written in the form $\frac{1}{2}\xi_1 - \frac{1}{2}\xi_2$, Q_i is in fact the group in $\mathcal{P}(L_i)$ such that ξ_i belongs to $\mathfrak{a}_{Q_i}^+$.

LEMMA 17.6. *The product (G, M)-family of Lemma 17.4 satisfies the alternate splitting formula*

$$(cd)_M(\lambda) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) c_M^{Q_1}(\lambda) c_M^{Q_2}(\lambda).$$

In particular, the values at $\lambda = 0$ of the functions in the formula satisfy

$$(cd)_M = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) c_M^{Q_1} d_M^{Q_1}. \qquad (17.14)$$

(See [**A13**, Corollary 7.4].) □

18. Local behaviour of weighted orbital integrals

We now consider the refinement of the coarse geometric expansion (10.7). In this section, we shall construct the general weighted orbital integrals that are to be the local ingredients. In the next section, we shall describe how to expand $J(f)$ as a linear combination of weighted orbital integrals, with certain global coefficients.

Recall that invariant orbital integrals (1.4) arose naturally at the beginning of the article. Weighted orbital integrals are noninvariant analogues of these distributions. We define them by scaling the invariant measure dx with a function $v_M(x)$ obtained from a certain (G, M)-family.

The simplest case concerns the setting at the end of §16, in which k is a completion F_v of F. Then M is a Levi subgroup of G over F_v. We also have to fix a suitable maximal compact subgroup K_v of $G(k) = G(F_v)$. If x_v is an element in $G(F_v)$, and P belongs to $\mathcal{P}(M)$, we form the point $H_P(x_v)$ in $\mathfrak{a}_M$ as in §4. It is a consequence of the definitions that

$$\{Y_P = -H_P(x_v) : \; P \in \mathcal{P}(M)\}.$$

is a positive (G, M)-orthogonal set. The functions

$$v_P(\lambda, x_v) = e^{-\lambda(H_P(x_v))}, \qquad \lambda \in i\mathfrak{a}_M^*, \; P \in \mathcal{P}(M),$$

then form a (G, M)-family. The associated smooth function

$$v_M(\lambda, x_v) = \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x_v)\theta_P(\lambda)^{-1}$$

is the Fourier transform of the characteristic function of the convex hull in $\mathfrak{a}_M^G$ of the projection onto $\mathfrak{a}_M^G$ of the points $\{-H_P(x_v) : \; P \in \mathcal{P}(M)\}$. The number

$$v_M(x_v) = v_M(0, x_v) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x_v)\theta_P(\lambda)^{-1}$$

equals the volume of this convex hull.

For the trace formula, we need to consider the global case that $k = F$. Until further notice, the maximal compact subgroup $K = \prod K_v$ of $G(\mathbb{A})$ will remain fixed. Suppose that M is a Levi subgroup in the finite set $\mathcal{L} = \mathcal{L}(M_0)$, and that x belongs to $G(\mathbb{A})$. The collection

$$v_P(\lambda, x) = e^{-\lambda(H_P(x))}, \qquad \lambda \in i\mathfrak{a}_M^*, \; P \in \mathcal{P}(M), \tag{18.1}$$

is then a (G, M)-family of functions. The limit

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x)\theta_P(\lambda)^{-1} \tag{18.2}$$

exists and equals the volume of the convex hull in $\mathfrak{a}_M^G$ of the projection of the points $\{-H_P(x) : \; P \in \mathcal{P}(M)\}$. To see how this function is related to the discussion of §11, choose a parabolic subgroup $P \in \mathcal{P}(M)$, and a minimal parabolic subgroup P_0 of G over $\mathbb{Q}$ that is contained in P. The correspondence

$$(P', s) \longrightarrow Q = w_s^{-1} P' w_s, \qquad P' \supset P_0, \; s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'}),$$

is then a bijection from the disjoint union over P' of the sets $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ onto the set $\mathcal{P}(M)$, with the property that

$$s^{-1} H_{P'}(\widetilde{w}_s x) = H_Q(x).$$

It follows that $v_M(x)$ equals the weight function $v_P(x)$ of Theorem 11.2.

The local and global cases are of course related. For any $x \in G(\mathbb{A})$, we can write

$$H_P(x) = \sum_v H_P(x_v), \qquad P \in \mathcal{P}(M),$$

where x_v is the component of x in $G(F_v)$. For almost every valuation v, x_v lies in K_v, and $H_P(x_v) = 0$. We obtain a finite sum

$$H_P(x) = \sum_{v \in S} H_P(x_v),$$

where S is a finite set of valuations that contains the set S_∞ of archimedean valuations. We may therefore fix S, and take x to be a point in the product

$$G(F_S) = \prod_{v \in S} G(F_v).$$

The (G, M)-family $\{v_P(\lambda, x)\}$ decomposes into a pointwise product

$$v_P(\lambda, x) = \prod_{v \in S} v_P(\lambda, x_v), \qquad \lambda \in i\mathfrak{a}_M^*, \ P \in \mathcal{P}(M),$$

of (G, M)-families $\{v_P(\lambda, x_v)\}$. We can therefore use the splitting formula (17.14) and the descent formula (17.13) (with $k = F$ and $k_1 = F_v$) to express the volume $v_M(x)$ in terms of volumes associated to the points $x_v \in G(F_v)$.

We fix the Levi subgroup M of G over F. We also fix an arbitrary finite set S of valuations, and write $K_S = \prod_{v \in S} K_v$ for the maximal compact subgroup of $G(F_S)$. Suppose that $\gamma = \prod \gamma_v$ is an element in $M(F_S)$. Our goal is to construct a weighted orbital integral of a function $f \in C_c^\infty\big(G(F_S)\big)$ over the space of F_S-valued points in the conjugacy class of G induced from γ. More precisely, let γ^G be the union of those conjugacy classes in $G(F_S)$ that for any $P \in \mathcal{P}(M)$ intersect $\gamma N_P(F_S)$ in a nonempty open set. We shall define the weighted orbital integral attached to M and γ by means of a canonical, noninvariant Borel measure on γ^G.

For any v, the connected centralizer G_{γ_v} is an algebraic group over F_v. We regard the product $G_\gamma = \prod_{v \in S} G_{\gamma_v}$ as a scheme over F_S, which is to say simply that

$$G_\gamma(F_S) = \prod_{v \in S} G_{\gamma_v}(F_v).$$

It is known [**R**] that this group is unimodular, and hence that there is a right invariant measure dx on the quotient $G_\gamma(F_S)\backslash G(F_S)$. The correspondence $x \to x^{-1}\gamma x$ is a surjective mapping from $G_\gamma(F_S)\backslash G(F_S)$ onto the conjugacy class of γ in $G(F_S)$, with finite fibres (corresponding to the connected components in the full centralizer $G_{\gamma,+}(F_S)$). Now if γ is not semisimple, the preimage in $G_\gamma(F_S)\backslash G(F_S)$ of a compact subset of the conjugacy class of γ (in the topology induced from $G(F_S)$) need not be compact. Nevertheless, a theorem of Deligne and Rao [**R**] asserts that the measure dx defines a $G(F_S)$-invariant Borel measure on the conjugacy class of γ. We obtain a continuous $G(F_S)$-invariant linear form

$$f \longrightarrow \int_{G_\gamma(F_S)\backslash G(F_S)} f(x^{-1}\gamma x)\mathrm{d}x, \qquad f \in C_c^\infty(F_S),$$

on $C_c^\infty\big(G(F_S)\big)$.

Suppose first that G_γ is contained in M. In other words, $G_\gamma = M_\gamma$. This condition holds for example if γ is the image in $M(F_S)$ of an element in $M(F)$ that represents an unramified class $\mathfrak{o} \in \mathcal{O}$, as in Theorem 11.2. With this condition, we define the weighted orbital integral

$$J_M(\gamma, f) = J_M^G(\gamma, f)$$

of $f \in C_c^\infty\big(G(F_S)\big)$ at γ by

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F_S)\backslash G(F_S)} f(x^{-1}\gamma x) v_M(x) \mathrm{d}x. \tag{18.3}$$

The normalizing factor

$$D(\gamma) = D^G(\gamma) = \prod_{v \in S} D^G(\gamma_v)$$

is the generalized Weyl discriminant

$$\prod_{v \in S} \det\big(1 - \mathrm{Ad}(\sigma_v)\big)_{\mathfrak{g}/\mathfrak{g}_{\sigma_v}},$$

where σ_v is the semisimple part of γ_v, and $\mathfrak{g}_{\sigma_v}$ is the Lie algebra of G_{σ_v}. Its presence in the definition simplifies some formulas. Since G_γ is contained in M, and $v_M(mx)$ equals $v_M(x)$ for any $m \in M(F_S)$, the integral is well defined.

Lemma 18.1. *Suppose that y is any point in $G(F_S)$. Then*

$$J_M(\gamma, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}), \tag{18.4}$$

where

$$f_{Q,y}(m) = \delta_Q(m)^{\frac{1}{2}} \int_{K_S} \int_{N_Q(F_S)} f(k^{-1}mnk) u'_Q(k,y) \mathrm{d}n \mathrm{d}k, \tag{18.5}$$

for $m \in M_Q(F_S)$, and

$$u'_Q(k,y) = \int_{\mathfrak{a}_Q^G} \Gamma'_Q\big(H, -H_Q(ky)\big) \mathrm{d}H. \tag{18.6}$$

This formula is Lemma 8.2 of [**A5**]. It probably does not come as a surprise, since the global distributions $J_\mathfrak{o}(f)$ satisfy a similar formula (16.2), and Theorem 11.2 tells us that for many $\mathfrak{o}$, $J_\mathfrak{o}(f)$ is a weighted orbital integral.

To prove the lemma, we first write

$$\begin{aligned} J_M(\gamma, f^y) &= |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F_S)\backslash G(F_S)} f(yx^{-1}\gamma xy^{-1}) v_M(x) \mathrm{d}x \\ &= |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F_S)\backslash G(F_S)} f(x^{-1}\gamma x) v_M(xy) \mathrm{d}x. \end{aligned}$$

We then observe that

$$\begin{aligned} v_P(\lambda, xy) = \mathrm{e}^{-\lambda(H_P(xy))} &= \mathrm{e}^{-\lambda(H_P(x))} \mathrm{e}^{-\lambda(H_P(k_P(x)y))} \\ &= v_P(\lambda, x) u_P(\lambda, x, y), \end{aligned}$$

where

$$u_P(\lambda, x, y) = \mathrm{e}^{-\lambda(H_P(k_P(x)y))},$$

and $k_P(x)$ is the point K_S such that $xk_P(x)^{-1}$ belongs to $P(F_S)$. It is a consequence of Lemma 17.4 that

$$v_M(xy) = \sum_{Q\in\mathcal{F}(M)} v_M^Q(x)u'_Q(x,y).$$

If k belongs to K_S, it follows from the definition (18.6) of $u'_Q(k,y)$, and the equality of (17.5) with (17.6) established in §17, that $u'_Q(k,y)$ is indeed of the form (17.7). Making two standard changes of variables in the integral over x in $G_\gamma(F_S)\backslash G(F_S)$, we write

$$\begin{aligned}
&|D(\gamma)|^{\frac{1}{2}}\int f(x^{-1}\gamma x)v_M(xy)\mathrm{d}x\\
&=\sum_{Q\in\mathcal{F}(M)}|D(\gamma)|^{\frac{1}{2}}\int f(x^{-1}\gamma x)v_M^Q(x)u'_Q(x,y)\mathrm{d}x\\
&=\sum_{Q}|D(\gamma)|^{\frac{1}{2}}\int\int\int f(k^{-1}n^{-1}m^{-1}\gamma mnk)v_M^Q(m)u'_Q(k,y)\mathrm{d}m\mathrm{d}n\mathrm{d}k\\
&=\sum_{Q}|D^M(\gamma)|^{\frac{1}{2}}\delta_Q(\gamma)^{\frac{1}{2}}\int\int\int f(k^{-1}m^{-1}\gamma mnk)v_M^Q(m)u'_Q(k,y)\mathrm{d}n\mathrm{d}k\mathrm{d}m\\
&=\sum_{Q}|D^M(\gamma)|^{\frac{1}{2}}\int f_{Q,y}(m^{-1}\gamma m)v_M^Q(m)\mathrm{d}m,
\end{aligned}$$

for integrals over m, n, and k in $M_{Q,\gamma}(F_S)\backslash M_Q(F)$, $N_Q(F_S)$, and K_S respectively. This equals the right hand side of (18.4), as required. □

The distribution (18.3) is to be regarded as a local object, despite the fact that M is a Levi subgroup of G over F. It can be reduced to the more elementary distributions

$$J_{M_v}(\gamma_v, f_v), \qquad \gamma_v\in M_v(F_v),\ f_v\in C_c^\infty\big(G(F_v)\big),$$

defined for Levi subgroups M_v of G over F_v by the obvious analogues of (18.3).

Suppose for example that S is a disjoint union of two sets of valuations S_1 and S_2. Suppose that

$$f = f_1f_2, \qquad f_i\in C_c^\infty\big(G(F_{S_i})\big)$$

and that

$$\gamma = \gamma_1\gamma_2, \qquad \gamma_i\in M(F_{S_i}).$$

We continue to assume that $G_\gamma = M_\gamma$, so that $G_{\gamma_i} = M_{\gamma_i}$ for $i = 1, 2$. We apply the general splitting formula (17.14) to the (G, M)-family

$$v_P(\lambda, x_1, x_2) = v_P(\lambda, x_1)v_P(\lambda, x_2), \qquad P\in\mathcal{P}(M),\ x_i\in G(F_{S_i}).$$

We then deduce from (18.3) that

$$J_M(\gamma, f) = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2)J_M^{L_1}(\gamma_1, f_{Q_1})J_M^{L_2}(\gamma_2, f_{Q_2}), \tag{18.7}$$

where $(L_1, L_2)\to(Q_1, Q_2)$ is the section in (17.14), and

$$f_{i,Q_i}(m_i) = \delta_{Q_i}(m_i)^{\frac{1}{2}}\int_{K_{S_i}}\int_{N_{Q_i}(F_{S_i})} f_i(k_i^{-1}m_in_ik_i),\mathrm{d}n_i\mathrm{d}k_i,$$

for $m_i \in M_{Q_i}(F_{S_i})$. If we apply this result inductively, we can reduce the compound distributions (18.3) to the simple case that S contains one element.

Suppose that S does consist of one element v. Assume that M_v is a Levi subgroup of G over F_v, and that γ_v is an element in $M_v(F_v)$ with $G_{\gamma_v} = M_{v,\gamma_v}$. Then $M_{v,\gamma_v} = M_{\gamma_v}$ and $M_{\gamma_v} = G_{\gamma_v}$. The first of these conditions implies that the induced class γ_v^M equals the conjugacy class of γ_v in $M(F_v)$. The second implies that the distribution

$$J_M(\gamma_v^M, f_v) = J_M(\gamma_v, f_v)$$

is defined by (18.3), for any $f_v \in C_c^\infty\big(G(F_v)\big)$. We apply the general descent formula (17.13) to the (G, M)-family

$$v_P(\lambda, x_v), \qquad\qquad P \in \mathcal{P}(M),\ x_v \in G(F_v).$$

We then deduce from (18.3) that

$$J_M(\gamma_v^M, f_v) = \sum_{L_v \in \mathcal{L}(M_v)} d_{M_v}^G(M, L_v) J_{M_v}^{L_v}(\gamma_v, f_{v,Q_v}), \tag{18.8}$$

where $L_v \to Q_v$ is the section in (17.13). The two formulas (18.7) and (18.8) together provide the required reduction of (18.3).

Suppose now that $\gamma \in M(F_S)$ is arbitrary. In the most extreme case, for example, γ could be the identity element in $M(F_S)$. The problem of defining a weighted orbital integral is now much harder. We cannot form the integral (18.3), since $v_M(x)$ is no longer a well defined function on $G_\gamma(F_S)\backslash G(F_S)$. Nor can we change the domain of integration to $M_\gamma(F_S)\backslash G(F_S)$, since the integral might then not converge.

What we do instead is to replace γ by a point $a\gamma$, for a small variable point $a \in A_M(F_S)$ in general position. Then $G_{a\gamma} = M_{a\gamma}$, so we can define $J_M(a\gamma, f)$ by the integral (18.3). The idea is to construct a distribution $J_M(\gamma, f)$ from the values taken by $J_M(a\gamma, f)$ around $a = 1$. This is somewhat subtle. To get an idea of what happens, let us consider the special case of $GL(2)$.

Assume that $F = \mathbb{Q}$, $G = GL(2)$, $M = M_0$ is minimal, S is the archimedean valuation v_∞, and $\gamma = 1$. Then $a\gamma = a = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, for distinct positive real numbers t_1 and t_2. Since

$$G_{a\gamma}(\mathbb{R})\backslash G(\mathbb{R}) = M(\mathbb{R})\backslash P(\mathbb{R})K_\mathbb{R} \cong N_P(\mathbb{R})K_\mathbb{R},$$

where P is the standard Borel subgroup of upper triangular matrices, the integral (18.3) can be written as

$$J_M(a, f) = |D(a)|^{\frac{1}{2}} \int_{K_\mathbb{R}} \int_{N_P(\mathbb{R})} f(k^{-1}n^{-1}ank) v_M(n) \mathrm{d}n \mathrm{d}k. \tag{18.9}$$

It is easy to compute the function $v_M(n)$. We first write

$$\begin{aligned} v_M(n) &= \lim_{\lambda \to 0} \big(\mathrm{e}^{-\lambda(H_P(n))}\theta_P(\lambda)^{-1} + \mathrm{e}^{-\lambda(H_{\overline{P}}(n))}\theta_{\overline{P}}(\lambda)^{-1}\big) \\ &= \lim_{\lambda \to 0} (1 - \mathrm{e}^{-\lambda(H_{\bar{P}}(n))})\theta_P(\lambda)^{-1} \\ &= \lim_{\lambda \to 0} \lambda\big(H_{\overline{P}}(n)\big)\lambda(\alpha^\vee)^{-1} \mathrm{vol}\big(\mathfrak{a}_M^G/\mathbb{Z}(\alpha^\vee)\big) \\ &= \mathrm{e}_1^*\big(H_{\overline{P}}(n)\big), \end{aligned}$$

where $\overline{P}$ is the Borel subgroup of lower triangular matrices, α is the simple root of (P, A_P), e_1^* is the linear form on $\mathfrak{a}_M \cong \mathbb{R}^2$ defined by projecting $\mathbb{R}^2$ onto the first component, and the measure on $\mathfrak{a}_M^G = \{(H, -H) : H \in \mathbb{R}\}$ is defined by Lebesgue measure on $\mathbb{R}$. We then note that n lies in a set $N_{\overline{P}}(\mathbb{R}) \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} K_{\mathbb{R}}$, for a positive real number u, and hence that

$$e_1^*\big(H_{\overline{P}}(n)\big) = \log|u| = \log\|(1,0)n\|.$$

It follows that if $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then

$$v_M(n) = \log\|(1,x)\| = \frac{1}{2}\log(1+x^2). \tag{18.10}$$

We make the standard change of variables

$$n \longrightarrow \nu = a^{-1}n^{-1}an = \begin{pmatrix} 1 & x(1-t_1^{-1}t_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{18.11}$$

in the last integral over $N_P(\mathbb{R})$. This entails multiplying the factor $|D(a)|^{\frac{1}{2}}$ by the Jacobian determinant

$$|D(a)|^{-\frac{1}{2}}e^{\rho_P(a)} = |D(a)|^{-\frac{1}{2}}(t_1t_2^{-1})^{\frac{1}{2}}$$

of the transformation. We conclude that $J_M(a,f)$ equals

$$(t_1t_2^{-1})^{\frac{1}{2}}\int_{K_{\mathbb{R}}}\int_{\mathbb{R}} f\Big(k^{-1}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}k\Big)\Big(\frac{1}{2}\log\big(1+\xi^2(1-t_1^{-1}t_2)^{-2}\big)\Big)d\xi dk.$$

The logarithmic factor in the last expression for $J_M(a,f)$ blows up at $a=1$. However, we can modify it by adding a logarithmic factor

$$r_M^G(a) = \log|\alpha(a)-\alpha(a)^{-1}| = \log|t_1t_2^{-1} - t_1^{-1}t_2|$$

that is independent of ξ. This yields a locally integrable function

$$\xi \longrightarrow \frac{1}{2}\log\Big((t_1t_2^{-1}+1)^2\big((1-t_1^{-1}t_2)^2+\xi^2\big)\Big), \qquad \xi \in \mathbb{R},$$

whose integral over any compact subset of $\mathbb{R}$ is bounded near $a=1$. Observe that

$$\begin{aligned}
&(t_1t_2^{-1})^{\frac{1}{2}}\int_{K_{\mathbb{R}}}\int_{\mathbb{R}} f\Big(k^{-1}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}k\Big)d\xi dk \\
&= |D(a)|^{\frac{1}{2}}\int_{K_{\mathbb{R}}}\int_{N_P(\mathbb{R})} f(k^{-1}n^{-1}ank)dndk \\
&= J_G(a,f).
\end{aligned}$$

It follows from the dominated convergence theorem that the limit

$$\lim_{a\to 1}\big(J_M(a,f) + r_M^G(a)J_G(a,f)\big)$$

exists, and equals the integral

$$J_M(1,f) = \int_{K_{\mathbb{R}}}\int_{\mathbb{R}} f\Big(k^{-1}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}k\Big)\log(2|\xi|)d\xi dk.$$

This is how we define the weighted orbital integral in the case $G = GL(2)$. As a distribution on $GL(2,\mathbb{R})$, it is given by a noninvariant Borel measure on the conjugacy class 1^G of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For arbitrary F, G, M, S, and γ, the techniques are more elaborate. However, the basic method is similar. One begins with the general analogue of the formula (18.9), valid for a fixed group $P \in \mathcal{P}(M)$. One then computes the function $v_M(n)$ as above, using a variable irreducible right G-module over F in place of the standard two-dimensional $GL(2)$-module, and a highest weight vector in place of $(1,0)$. If

$$\nu \longrightarrow n = n(\nu, \gamma a)$$

is the inverse of the bijection $n \to (\gamma a)^{-1} n^{-1} (\gamma a) n$ of $N_P(\mathbb{R})$, the problem becomes that of understanding the behaviour of the function

$$v_M\big(n(\nu, \gamma a)\big)$$

near $a = 1$. This leads to general analogues of the factor $r_M^G(a)$ defined above for $GL(2)$.

THEOREM 18.2. *For any F, G, M, S, and $\gamma \in M(F_S)$, there are canonical functions*

$$r_M^L(\gamma, a), \qquad\qquad L \in \mathcal{L}(M),$$

defined for small points $a \in A_M(F_S)$ in general position, such that the limit

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f) \tag{18.12}$$

exists and equals the integral of f with respect to a Borel measure on the set γ^G.

This is Theorem 5.2 of [**A12**], one of the principal results of [**A12**]. There are two basic steps in its proof. The first is construct the functions $r_M^L(\gamma, a)$. The second is to establish the existence and properties of the limit.

The function $r_M^L(\gamma, a)$ is understood to depend only on L, M, γ, and a (and not G), so we need only construct it when $L = G$. In this case, the function is defined as the limit

$$r_M^G(\gamma, a) = \lim_{\lambda \to 0} \Big(\sum_{P \in \mathcal{P}(M)} r_P(\lambda, \gamma, a) \theta_P(\lambda)^{-1} \Big)$$

associated to a certain (G, M)-family

$$r_P(\lambda, \gamma, a) = \prod_{v \in S} \prod_{\beta_v} r_{\beta_v}\Big(\tfrac{1}{2}\lambda, u_v, a_v\Big), \qquad\qquad \lambda \in i\mathfrak{a}_M^*.$$

The factors in this last product are defined in terms of the Jordan decomposition $\gamma_v = \sigma_v u_v$ of the v-component of γ. Let P_{σ_v} be the parabolic subgroup $P \cap G_{\sigma_v}$ of G_{σ_v}. The indices β_v then range over the reduced roots of $(P_{\sigma_v}, A_{M_{\sigma_v}})$. Any such β_v determines a Levi subgroup G_{σ_v, β_v} of G_{σ_v}, and a maximal parabolic subgroup $P_{\sigma_v, \beta_v} = P_{\sigma_v} \cap G_{\sigma_v, \beta_v}$ of G_{σ_v, β_v} with Levi component M_{σ_v}. We will not describe the factors in the product further, except to say that they are of the form

$$r_{\beta_v}(\Lambda, u_v, a_v) = |a_v^{\beta_v} - a_v^{-\beta_v}|^{\rho(\beta_v, u_v)\Lambda(\beta_v^\vee)}, \qquad\qquad \Lambda = \tfrac{1}{2}\lambda,$$

for positive constants $\rho(\beta_v, u_v)$, and that they are defined by subjecting G_{σ_v, β_v}, M_{σ_v}, and u_v to an analysis similar to that of the special case $GL(2)$, M_0, and 1 (with $v = v_\infty$) above.

The existence of the limit (18.12) is more subtle. The functions $r_{\beta_v}(\Lambda, u_v, a_v)$ are defined so as to make the associated limits for G_{σ_v, β_v}, M_{σ_v}, and u_v exist.

However, these limits are simpler. They concern a variable a_v that is essentially one-dimensional (since M_{σ_v} is a maximal Levi subgroup in G_{σ_v,β_v}), while the variable a in (18.12) is multidimensional (since M is an arbitrary Levi subgroup of G). The existence of the general limit depends on algebraic geometry, specifically a surprising application by Langlands of Zariski's main theorem [**A12**, §4], and some elementary analysis [**A12**, Lemma 6.1]. The fact that the resulting distribution $f \to J_M(\gamma, f)$ is a measure is a consequence of the proof of the existence of the limit. □

Once we have defined the general distributions $J_M(\gamma, f)$, we can extend the properties established in the special case that $G_\gamma = M_\gamma$. First of all, we note that $J_M(\gamma, f)$ depends only on the conjugacy class of γ in $M(F_S)$. It is also easy to see from the definition (18.12) that

$$J_{M_1}(\gamma_1, f) = J_M(\gamma, f),$$

where $\gamma_1 = w_s \gamma w_s^{-1} = \widetilde{w}_s \gamma \widetilde{w}_s^{-1}$ and $M_1 = w_s M w_s^{-1}$, for elements $\gamma \in M(F_S)$ and $s \in W_0$.

Suppose that y lies in $G(F_S)$, and that $\gamma \in M(F_S)$ is arbitrary. It then follows from (18.12) and Lemma 18.1 that

$$\begin{aligned} J_M(\gamma, f^y) &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f^y) \\ &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} \sum_{Q \in \mathcal{F}(L)} r_M^L(\gamma, a) J_L^{M_Q}(a\gamma, f_{Q,y}) \\ &= \lim_{a \to 1} \sum_{Q \in \mathcal{F}(M)} \Big(\sum_{L \in \mathcal{L}^{M_Q}(M)} r_M^L(\gamma, a) J_L^{M_Q}(a\gamma, f_{Q,y}) \Big) \\ &= \sum_{Q \in \mathcal{F}(L)} J_M^{M_Q}(\gamma, f_{Q,y}). \end{aligned}$$

The formula (18.4) therefore holds in general.

The splitting formula (18.7) and the descent formula (18.8) also hold in general. In particular, the general distributions $J_M(\gamma, f)$ can be reduced to the more elementary local distributions $J_{M_v}(\gamma_v, f_v)$. The proof entails application to the general definition (18.12) of the special cases of these formulas already established. One has to also apply Lemmas 17.5 and 17.6 to the coefficients $r_M^L(\gamma, a)$ in (18.12). The argument is not difficult, but is more complicated than the general proof of (18.4) above. We refer the reader to the proofs of Theorem 8.1 and Proposition 9.1 of [**A13**].

19. The fine geometric expansion

We now turn to the global side of the problem. It would be enough to express the distribution $J_{\mathfrak{o}}(f)$ in explicit terms, for any $\mathfrak{o} \in \mathcal{O}$. We solved the problem for unramified classes $\mathfrak{o}$ in §11 by writing $J_{\mathfrak{o}}(f)$ as a weighted orbital integral. We would like to have a similar formula that applies to an arbitrary class $\mathfrak{o}$.

The general weighted orbital integrals defined in the last section are linear forms on the space $C_c^\infty\big(G(F_S)\big)$, where S is any finite set of valuations. Assume that S is a large finite set that contains the archimedean valuations S_∞, and write $C_c^\infty\big(G(F_S)^1\big)$ for the space of functions on $G(F_S)^1 = G(F_S) \cap G(\mathbb{A})^1$ obtained by restriction of functions in $C_c^\infty\big(G(F_S)\big)$. If γ belongs to the intersection of $M(F_S)$

with $G(F_S)^1$, we can obviously define the corresponding weighted orbital integral as a linear form on $C_c^\infty\big(G(F_S)^1\big)$. Let

$$\chi^S = \prod_{v \notin S} \chi_v$$

be the characteristic function of the maximal compact subgroup

$$K^S = \prod_{v \notin S} K_v$$

of $G(\mathbb{A}^S)$. The mapping $f \to f\chi^S$ is then an injection of $C_c^\infty\big(G(F_S)^1\big)$ into $C_c^\infty\big(G(\mathbb{A})^1\big)$. We shall identify $C_c^\infty\big(G(F_S)^1\big)$ with its image in $C_c^\infty\big(G(\mathbb{A})^1\big)$. We can thus form the distribution $J_{\mathfrak{o}}(f)$ for any $f \in C_c^\infty\big(G(F_S)^1\big)$. Our goal is to write it explicitly in terms of weighted orbital integrals of f.

Suppose first that $\mathfrak{o}$ consists entirely of unipotent elements. Then $\mathfrak{o} = \mathfrak{o}_{\mathrm{unip}} = \mathcal{U}_G(F)$, where $\mathcal{U}_G$ is the closed variety of unipotent elements in G. It is this class in $\mathcal{O}$ that is furthest from being unramified, and which is consequently the most difficult to handle. In general, there are infinitely many $G(F)$-conjugacy classes in $\mathcal{U}_G(F)$. However, we say that two elements $\gamma_1, \gamma_2 \in \mathcal{U}_G(F)$ are (G,S)-*equivalent* if they are $G(F_S)$-conjugate. The associated set $\big(\mathcal{U}_G(F)\big)_{G,S}$ of equivalence classes is then finite. The next theorem gives an expansion of the distribution

$$J_{\mathrm{unip}}(f) = J^G_{\mathrm{unip}}(f) = J^G_{\mathfrak{o}_{\mathrm{unip}}}(f)$$

whose terms are indexed by the finite sets $\big(\mathcal{U}_M(F)\big)_{M,S}$.

THEOREM 19.1. *For any S as above, there are uniquely determined coefficients*

$$a^M(S,u), \qquad M \in \mathcal{L},\ u \in \big(\mathcal{U}_M(F)\big)_{M,S},$$

with

$$a^M(S,1) = \mathrm{vol}\big(M(F)\backslash M(\mathbb{A})^1\big), \tag{19.1}$$

such that

$$J_{\mathrm{unip}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{u \in (\mathcal{U}_M(F))_{M,S}} a^M(S,u) J_M(u,f), \tag{19.2}$$

for any $f \in C_c^\infty\big(G(F_S)^1\big)$.

This is the main result, Theorem 8.1, of the paper [**A10**]. The full proof is too long for the space we have here. However, the basic idea is easy to describe.

Assume inductively that the theorem is valid if G is replaced by any proper Levi subgroup. It is understood that the coefficients $a^M(S,u)$ depend only on M (and not G). The induction hypothesis therefore implies that the coefficients have been defined whenever M is proper in G. We can therefore set

$$T_{\mathrm{unip}}(f) = J_{\mathrm{unip}}(f) - \sum_{\substack{M \in \mathcal{L} \\ M \neq G}} |W_0^M||W_0^G|^{-1} \sum_{u \in (\mathcal{U}_M(F))_{M,S}} a^M(S,u) J_M(u,f),$$

for any $f \in C_c^\infty\big(G(F_S)^1\big)$. Suppose that $y \in G(F_S)$. By (16.2) and (18.4), we can write the difference

$$T_{\mathrm{unip}}(f^y) - T_{\mathrm{unip}}(f)$$

as the difference between the global expression

$$J_{\mathrm{unip}}(f^y) - J_{\mathrm{unip}}(f) = \sum_{\substack{Q\in\mathcal{F}\\ Q\neq G}} |W_0^{M_Q}||W_0^G|^{-1} J_{\mathrm{unip}}^{M_Q}(f_{Q,y})$$

and the local expression

$$\begin{aligned}&\sum_{M\neq G} |W_0^M||W_0^G|^{-1} \sum_{u\in(\mathcal{U}_M(\mathbb{Q}))_{M,S}} a^M(S,u)\big(J_M(u,f^y) - J_M(u,f)\big)\\ &= \sum_{M\neq G}\sum_{\substack{Q\in\mathcal{F}(M)\\ Q\neq G}} |W_0^{M_Q}||W_0^G|^{-1}\sum_u |W_0^M||W_0^{M_Q}|^{-1} a^M(S,u) J_M^{M_Q}(u,f_{Q,y}).\end{aligned}$$

The difference between $T_{\mathrm{unip}}(f^y)$ and $T_{\mathrm{unip}}(f)$ is therefore equal to the sum over $Q\in\mathcal{F}$ with $Q\neq G$ of the product of $|W_0^{M_Q}||W_0^G|^{-1}$ with the expression

$$J_{\mathrm{unip}}^{M_Q}(f_{Q,y}) - \sum_{M\in\mathcal{L}^{M_Q}} |W_0^M||W_0^{M_Q}|^{-1} \sum_{u\in(\mathcal{U}_M(F))_{M,S}} a^M(S,u) J_M^{M_Q}(u,f_{Q,y}).$$

The last expression vanishes by our induction assumption. It follows that $T_{\mathrm{unip}}(f^y)$ equals $T_{\mathrm{unip}}(f)$, and therefore that the distribution T_{unip} on $G(F_S)^1$ is invariant.

Recall that $J_{\mathrm{unip}}(f)$ is the value at $T=T_0$ of the polynomial

$$J_{\mathrm{unip}}^T(f) = \int_{G(F)\backslash G(\mathbb{A})^1} k_{\mathrm{unip}}^T(x,f)\mathrm{d}x,$$

where

$$k_{\mathrm{unip}}^T(x,f) = \sum_{P\supset P_0} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\mathrm{unip}}(\delta x,\delta x)\widehat{\tau}_P\big(H_P(\delta x) - T\big)$$

and

$$K_{P,\mathrm{unip}}(\delta x,\delta x) = \sum_{u\in\mathcal{U}_M(F)} \int_{N_P(\mathbb{A})} f(x^{-1}\delta^{-1}un\delta x)\mathrm{d}n.$$

It follows that $J_{\mathrm{unip}}(f)$ vanishes for any function $f\in C_c^\infty\big(G(\mathbb{A})^1\big)$ that vanishes on the unipotent set in $G(F_S)^1$. For any such function, the distributions $J_M(u,f)$ all vanish as well, according to Theorem 18.2. We conclude that the invariant distribution T_{unip} annihilates any function in $C_c^\infty\big(G(F_S)^1\big)$ that vanishes on the unipotent set. It follows from this that

$$T_{\mathrm{unip}}(f) = \sum_u a^G(S,u) J_G(u,f),$$

for coefficients $a^G(S,u)$ parametrized by unipotent classes u in $G(F_S)$.

It remains to show that $a^G(S,u)$ vanishes unless u is the image of a unipotent class in $G(F)$, and to evaluate $a^G(S,u)$ explicitly as a Tamagawa number in the case that $u=1$. This is the hard part. The two assertions are plausible enough. The integrand $k_{\mathrm{unip}}^T(x,f)$ above is supported on the space of $G(\mathbb{A})$-conjugacy classes that come from F-rational unipotent classes. Moreover, the contribution to $k_{\mathrm{unip}}^T(x,f)$ from the class 1 equals $f(1)$, which is obviously independent of x and T. The integral over $G(F)\backslash G(\mathbb{A})^1$ of this contribution converges, and equals the product

$$\mathrm{vol}\big(G(F)\backslash G(\mathbb{A})^1\big) f(1) = \mathrm{vol}\big(G(F)\backslash G(\mathbb{A})^1\big) J_G(1,f).$$

However, $J_{\mathrm{unip}}(f)$ is defined in terms of the polynomial $J_{\mathrm{unip}}^T(f)$, which depends on a fixed minimal parabolic subgroup $P_0\in\mathcal{P}(M_0)$, and is equal to an integral whose convergence we can control only for suitably regular points $T \in \mathfrak{a}_{P_0}^+$. Among other difficulties, the dependence of $J_{\mathrm{unip}}^T(f)$ on the local components f_v is not at all transparent. It is therefore not trivial to deduce the remaining two assertions from the intuition we have.

There are two steps. The first is to approximate $J_{\mathrm{unip}}^T(f)$ by the integral of the function

$$K_{\mathrm{unip}}(x,x) = \sum_{u \in \mathcal{U}_G(F)} f(x^{-1}ux)$$

over a compact set. The assertion is that

$$\Big| J_{\mathrm{unip}}^T(f) - \int_{G(F)\backslash G(\mathbb{A})^1} F^G(x,T) K_{\mathrm{unip}}(x,x)\mathrm{d}x \Big| \leq \mathrm{e}^{-\frac{1}{2}d_{P_0}(T)}, \tag{19.3}$$

where $F^G(\cdot,T)$ is the compactly supported function on $G(F)\backslash G(\mathbb{A})^1$ defined in §8, and

$$d_{P_0}(T) = \inf_{\alpha \in \Delta_{P_0}} \alpha(T).$$

This inequality is Theorem 3.1 of [**A10**]. Its proof includes an assertion that $F^G(\cdot,T)$ equals the image of the constant function 1 on $G(F)\backslash G(\mathbb{A})^1$ under the truncation operator Λ^T [**A10**, Lemma 2.1]. The estimate (19.3), incidentally, is reminiscent of our remarks on the local trace formula at the beginning of §16.

The second step is to solve a kind of lattice point problem. Let U be a unipotent conjugacy class in $G(F)$. If v is a valuation in S and $\varepsilon > 0$, one can define a function $f_{U,v}^{\varepsilon} \in C_c^{\infty}\big(G(\mathbb{A})^1\big)$ that, roughly speaking, truncates the function $f(x)$ whenever the distance from x_v to the $G(F_v)$-conjugacy class of U is greater than ε. (See the beginning of §4 of [**A10**]. The function $f_{U,v}^{\varepsilon}$ equals f at any point in $G(\mathbb{A})^1$ that is conjugate to any point in $\overline{U}(F)$, where $\overline{U}$ is the Zariski closure of U.) One then establishes an inequality

$$\int_{G(F)\backslash G(\mathbb{A})^1} F^G(x,T) \sum_{\gamma \in G(F) - \overline{U}(F)} |f_{U,v}^{\varepsilon}(x^{-1}\gamma x)|\mathrm{d}x \leq \varepsilon^r \|f\|(1+\|T\|)^{d_0}, \tag{19.4}$$

where $\|\cdot\|$ is a continuous seminorm on $C_c^{\infty}\big(G(\mathbb{A})^1\big)$, and $d_0 = \dim(\mathfrak{a}_0)$. This inequality is the main technical result, Lemma 4.1, of the paper [**A10**]. Its proof in §5-6 of [**A10**] relies on that traditional technique for lattice point problems, the Poisson summation formula.

The inequalities (19.3) and (19.4) are easily combined. By letting ε approach 0, one deduces the remaining two assertions of Theorem 19.1 from the definition of $J_{\mathrm{unip}}(f) = J_{\mathrm{unip}}^{T_0}(f)$ in terms of $J_{\mathrm{unip}}^T(f)$. (See [**A10**, §4].) □

Remark. The explicit formula (19.1) for $a^M(S,1)$ is independent of the set S. For nontrivial elements $u \in \mathcal{U}_M(F)$, the coefficients $a^M(S,u)$ do depend on S. One sees this in the case $G = GL(2)$ from the term (v) on p. 516 of [**JL**]. As a matter of fact, it is only in the case $G = GL(2)$ that the general coefficients $a^M(S,u)$ have been evaluated. It would be very interesting to understand them better in other examples, although this does not seem to be necessary for presently conceived applications of the trace formula.

The case $\mathfrak{o} = \mathfrak{o}_{\mathrm{unip}}$ we have just discussed is the the most difficult. It is the furthest from the unramified case solved explicitly in §11. For a general class $\mathfrak{o}$, one fashions a descent argument from the techniques of §11. This reduces the problem of computing $J_{\mathfrak{o}}(f)$ to the unipotent case of Theorem 19.1.

We need a couple of definitions before we can state the general result. We say that a semisimple element $\sigma \in G(F)$ is *F-elliptic* if A_{G_σ} equals A_G. In the case $G = GL(n)$, for example, a diagonal element σ in $G(F)$ is F-elliptic if and only if it is a scalar.

Suppose that γ is an element in $G(F)$ with semisimple Jordan component σ, and that S is a large finite set of valuations of F that contains S_∞. We shall say that a second element γ' in $G(F)$ is (G, S)-equivalent to γ if there is a $\delta \in G(F)$ with the following two properties.

(i) σ is also the semisimple Jordan component of $\delta^{-1}\gamma'\delta$.
(ii) The unipotent elements $\sigma^{-1}\gamma$ and $\sigma^{-1}\delta^{-1}\gamma'\delta$ in $G_\sigma(F)$ are (G_σ, S)-equivalent, in the sense of the earlier definition.

There could be several classes $u \in \big(\mathcal{U}_{G_\sigma}(F)\big)_{G_\sigma,S}$ such that σu is (G, S)-equivalent to γ. The set of such u, which we write simply as $\{u : \ \sigma u \sim \gamma\}$, has a transitive action under the finite group

$$\iota^G(\sigma) = G_{\sigma,+}(F)/G_\sigma(F).$$

We define

$$a^G(S, \gamma) = \varepsilon^G(\sigma)|\iota^G(\sigma)|^{-1} \sum_{\{u:\sigma u\sim\gamma\}} a^{G_\sigma}(S, u), \tag{19.5}$$

where

$$\varepsilon^G(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is } F\text{-elliptic in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Then $a^G(S, \gamma)$ depends only on the (G, S)-equivalence class of γ. If γ is semisimple, we can use (19.1) to express $a^G(S, \gamma)$. In this case, we see that

$$a^G(S, \gamma) = \varepsilon^G(\gamma)|\iota^G(\gamma)|^{-1}\mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1\big), \tag{19.6}$$

and in particular, that $a^G(S, \gamma)$ is independent of S.

THEOREM 19.2. *Suppose that $\mathfrak{o}$ is any class in $\mathcal{O}$. Then there is a finite set $S_{\mathfrak{o}}$ of valuations of F that contains S_∞ such that for any finite set $S \supset S_{\mathfrak{o}}$ and any function $f \in C_c^\infty\big(G(F_S)^1\big)$,*

$$J_{\mathfrak{o}}(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F)\cap\mathfrak{o})_{M,S}} a^M(S, \gamma)J_M(\gamma, f), \tag{19.7}$$

where $\big(M(F) \cap \mathfrak{o}\big)_{M,S}$ is the finite set of (M, S)-equivalence classes in $M(F) \cap \mathfrak{o}$, and $J_M(\gamma, f)$ is the general weighted orbital integral of f defined in §18.

This is the main result, Theorem 8.1, of the paper [**A11**]. The strategy is to establish formulas of descent that reduce each side of the putative formula (19.7) to the unipotent case (19.2). We are speaking of what might be called "semisimple descent" here. It pertains to the Jordan decomposition, and is therefore different from the property of "parabolic descent" in the formula (18.8). We shall attempt to give a brief idea of the proof.

The reduction is actually a generalization of the unramified case treated in §11. In particular, it begins with the formula

$$J_{\mathfrak{o}}^T(f) = \int_{G(F)\backslash G(\mathbb{A})^1} \widetilde{k}_{\mathfrak{o}}^T(x,f)\mathrm{d}x$$

of Theorem 11.1. We recall that

$$\widetilde{k}_{\mathfrak{o}}^T(x,f) = \sum_{P\supset P_0} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(F)\backslash G(F)} \widetilde{K}_{P,\mathfrak{o}}(\delta x,\delta x)\widehat{\tau}_P\big(H_P(\delta x)-T\big),$$

where $P_0 \in \mathcal{P}(M_0)$ is a fixed minimal parabolic subgroup. The definition (11.1) expresses $\widetilde{K}_{P,\mathfrak{o}}(\delta x,\delta x)$ in terms of f, and the Jordan decomposition of elements $\gamma \in M_P \cap \mathfrak{o}$. The formula contains integrals over unipotent adelic groups $N_R(\mathbb{A}) = N_P(\mathbb{A})_{\gamma_s}$, where R is the parabolic subgroup $P \cap G_{\gamma_s}$ of G_{γ_s}. It is therefore quite plausible that $J_{\mathfrak{o}}^T(f)$ can be reduced to unipotent distributions $J_{\mathrm{unip}}^{H,T_H}(\Phi)$ attached to reductive subgroups H of G, and functions $\Phi \in C_c^\infty\big(H(\mathbb{A})^1\big)$ obtained from f and T by descent. However, the combinatorics of the reduction are somewhat complicated.

One begins as in §11 by fixing a pair (P_1,α_1) that represents the anisotropic rational datum of $\mathfrak{o}$. Then P_1 is a parabolic subgroup, which is standard relative to the fixed minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$ used to construct $J_{\mathfrak{o}}(f)$. One also fixes an element $\sigma = \gamma_1$ in the anisotropic (semisimple) conjugacy class α_1 in $M_{P_1}(F)$. Then $P_{1\sigma} = P_1 \cap G_\sigma$ is a minimal parabolic subgroup of G_σ, with Levi component $M_{1\sigma} = M_{P_1} \cap G_\sigma$. The groups H above are Levi subgroups M_σ of G_σ in the finite set $\mathcal{L}^\sigma = \mathcal{L}^{G_\sigma}(M_{1\sigma})$. The corresponding functions $\Phi = \Phi_y$ of descent in $C_c^\infty\big(M_\sigma(\mathbb{A})^1\big)$ depend on T, and among other things, a set of representatives y of $G_\sigma(\mathbb{A})\backslash G(\mathbb{A})$ in $G(\mathbb{A})$. (See [**A11**, p. 199].)

We take $S_{\mathfrak{o}}$ to be any finite set of valuations of F that contains S_∞, and such that any $v \notin S_{\mathfrak{o}}$ satisfies the following four conditions.

(i) $|D^G(\sigma)|_v = 1$.
(ii) The intersection $K_{\sigma,v} = K_v \cap G_\sigma(F_v)$ is an admissible maximal compact subgroup of $G_\sigma(F_v)$.
(iii) $\sigma K_v \sigma^{-1} = K_v$.
(iv) If $y_v \in G(F_v)$ is such that $y_v^{-1}\sigma\mathcal{U}_{G_\sigma}(F_v)y_v$ meets σK_v, then y_v belongs to $G_\sigma(F_v)K_v$.

(See [**A11**, p. 203].) We choose $S \supset S_{\mathfrak{o}}$ and $f \in C_c^\infty\big(G(F)^1\big)$, as in the statement of the theorem. It then turns out that for any group $M_\sigma \in \mathcal{L}^\sigma$, the corresponding functions of descent Φ_y all lie in the subspace $C_c^\infty\big(M_\sigma(F_S)^1\big)$ of $C_c^\infty\big(M_\sigma(\mathbb{A})^1\big)$.

Recall that $J_{\mathfrak{o}}(f)$ is the value at $T = T_0$ of the polynomial $J^T(f)$. The unipotent distribution $J_{\mathrm{unip}}^{M_\sigma}(\Phi_y)$ is the value of a polynomial $J_{\mathrm{unip}}^{M_\sigma,T_\sigma}(\Phi_y)$ of T_σ in a subspace $\mathfrak{a}_{1\sigma}$ of $\mathfrak{a}_0$ at a fixed point $T_{0\sigma}$. In the descent formula, the groups M_σ are of the form M_R, where R ranges over the set $\mathcal{F}^\sigma = \mathcal{F}^{G_\sigma}(M_{1\sigma})$. The formula is

$$J_{\mathfrak{o}}(f) = |\iota^G(\sigma)|^{-1} \int_{G_\sigma(\mathbb{A})\backslash G(\mathbb{A})} \Big(\sum_{R\in\mathcal{F}^\sigma} |W_0^{M_R}||W_0^{G_\sigma}|^{-1} J_{\mathrm{unip}}^{M_R}(\Phi_{R,y,T_1})\Big)\mathrm{d}y, \tag{19.8}$$

where Φ_{R,y,T_1} is obtained from the general descent function Φ_y by specializing T to the point $T_1 = T_0 - T_{0\sigma}$ [**A11**, Lemma 6.2]. Since the general functions Φ_y and their specializations Φ_{R,y,T_1} are somewhat technical, we have not attempted to define them. However, their construction is formally like that of the functions $f_{Q,y}$ in (18.5). In particular, it relies on the splitting formula of Lemma 17.4.

The formula (19.8) of geometric descent has an analogue for weighted orbital integrals. Suppose that M is a Levi subgroup of G that contains $M_1 = M_{P_1}$. Then σ is contained in $M(F)$. Set $\gamma = \sigma u$, where u is a unipotent element in $M_\sigma(F_S)$. The formula is

$$J_M(\gamma, f) = \int_{G_\sigma(F_S)\backslash G(F_S)} \Big(\sum_{R \in \mathcal{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(u, \Phi_{R,y,T_1}) \Big) \mathrm{d}y, \tag{19.9}$$

where f is any function in $C_c^\infty\big(G(F_S)^1\big)$, and $\mathcal{F}^\sigma(M_\sigma) = \mathcal{F}^{G_\sigma}(M_\sigma)$ ([**A11**, Corollary 8.7]).

The formulas (19.8) and (19.9) of geometric descent must seem rather murky, given the limited extent of our discussion. However, the reader will no doubt agree that the existence of such formulas is plausible. Taking them for granted, one can well imagine that an application of Theorem 19.1 to the distributions in these formulas would lead to an expansion of $J_\mathfrak{o}(f)$. The required formula (19.7) for $J_\mathfrak{o}(f)$ does indeed follow from Theorem 19.1, used in conjunction with the definition (19.5) of the coefficients $a^M(S, \gamma)$. □

If Δ is a compact neighbourhood of 1 in $G(\mathbb{A})^1$, we write $C_\Delta^\infty\big(G(\mathbb{A})^1\big)$ for the subspace of functions in $C_c^\infty\big(G(\mathbb{A})^1\big)$ that are supported on Δ. For example, we could take Δ to be the set

$$\Delta_N = \{x \in G(\mathbb{A}) : \ \log \|x\| \le N\}$$

attached to a positive number N. In this case we write $C_N^\infty\big(G(\mathbb{A})^1\big)$ in place of $C_{\Delta_N}^\infty\big(G(\mathbb{A})^1\big)$. For any Δ, we can certainly find a finite set S of valuations of F containing S_∞, such that Δ is the product of a compact neighbourhood of 1 in $G(F_S)^1$ with K^S. We write S_Δ^0 for the minimal such set. We also write

$$C_\Delta^\infty\big(G(F_S)^1\big) = C_\Delta^\infty\big(G(\mathbb{A})^1\big) \cap C_c^\infty\big(G(F_S)^1\big),$$

for any finite set $S \supset S_\Delta^0$. The fine geometric expansion is given by the following corollary of the last theorem.

Corollary 19.3. *Given a compact neighbourhood Δ of 1 in $G(\mathbb{A})^1$, we can find a finite set $S_\Delta \supset S_\Delta^0$ of valuations of F such that for any finite set $S \supset S_\Delta$, and any $f \in C_\Delta^\infty\big(G(F_S)^1\big)$,*

$$J(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma) J_M(\gamma, f), \tag{19.10}$$

where $\big(M(F)\big)_{M,S}$ is the set of (M,S)-equivalence classes in $M(F)$. The summands on the right hand side of (19.10) vanish for all but finite many γ.

The corollary is Theorem 9.2 of [**A11**]. It follows immediately from Theorem 19.2 above, once we know that there is a finite subset of $\mathcal{O}$ outside of which $J_\mathfrak{o}(f)$ vanishes for any $f \in C_\Delta^\infty\big(G(\mathbb{A})^1\big)$. This property follows immediately from [**A11**, Lemma 9.1], which asserts that there are only finitely many classes $\mathfrak{o} \in \mathcal{O}$ such that the set

$$\{x^{-1}\gamma x : \ x \in G(\mathbb{A}), \ \gamma \in \mathfrak{o}\}$$

meets Δ, and is proved in the appendix of [**A11**]. □

20. Application of a Paley-Wiener theorem

The next two sections will be devoted to the refinement of the coarse spectral expansion (14.8). These sections are longer and more intricate than anything so far. For one reason, there are results from a number of different sources that we need to discuss. Moreover, we have included more details than in some of the earlier arguments. The refined spectral expansion is deeper than its geometric counterpart, dependent as it is on Eisenstein series, and we need to get a feeling for the techniques. In particular, it is important to understand how global intertwining operators intervene in the "discrete part" of the spectral expansion.

The spectral side is complicated by the presence of a delicate analytic problem, with origins in the theory of Eisenstein series. It can be described as that of interchanging two limits. We shall see how to resolve the problem in this section. The computations of the fine spectral expansion will then be treated in the next section.

In order to use the results of Part I, we shall work for the time being with a fixed minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Suppose that $\chi \in \mathcal{X}$ indexes one of the summands in the coarse spectral expansion. According to Theorem 15.1,

$$J_\chi^T(f) = \sum_{P \supset P_0} n_P^{-1} \int_{i\mathfrak{a}_P^*} \operatorname{tr}\big(M_{P,\chi}^T(\lambda)\mathcal{I}_{P,\chi}(\lambda, f)\big)d\lambda,$$

where $T \in \mathfrak{a}_{P_0}^+$ is suitably regular, and $M_{P,\chi}^T(\lambda)$ is the operator on $\mathcal{H}_{P,\chi}$ defined by the inner product (15.1) of truncated Eisenstein series. In the next section, we shall see that the explicit inner product formula for truncated Eisenstein series in Proposition 15.3 holds in general, provided it is interpreted as an asymptotic formula in T. We might therefore hope to compute $J_\chi^T(f)$ as an explicit polynomial in T by letting the distance

$$d_{P_0}(T) = \inf_{\alpha \in \Delta_{P_0}} \alpha(T)$$

approach infinity. However, any such computation seems to require estimates for the derivatives of $M_{P,\chi}^T(\lambda)$ that are uniform in λ. This would amount to estimating derivatives in λ of Eisenstein series outside the domain of absolute convergence, something that is highly problematical. On the other hand, if we could multiply the integrand in the formula for $J_\chi^T(f)$ above by a smooth, compactly supported cut-off function in λ, the computations ought to be manageable. The analytic problem is to show that one can indeed insert such a cut-off function.

In the formula for $J_\chi^T(f)$ we have just quoted from Part I, f belongs to $C_c^\infty\big(G(\mathbb{A})\big)$. We are now taking f to be a function in $C_c^\infty\big(G(\mathbb{A})^1\big)$. For any such f, the integrand in the formula is a well defined function of λ in $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$. The formula remains valid for $f \in C_c^\infty\big(G(\mathbb{A})^1\big)$, so long as we take the integral over $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$.

The class $\chi \in \mathcal{X}$ will be fixed for the rest of this section. We shall first state three preliminary lemmas, all of which are consequences of Theorem 14.1 and its proof. For any $P \supset P_0$, we write

$$\mathcal{H}_{P,\chi} = \bigoplus_\pi \mathcal{H}_{P,\chi,\pi},$$

where π ranges over the set $\Pi_{\mathrm{unit}}\big(M_P(\mathbb{A})^1\big)$ of equivalence classes of irreducible unitary representations of $M_P(\mathbb{A})^1$, and $\mathcal{H}_{P,\chi,\pi}$ is the intersection of $\mathcal{H}_{P,\chi}$ with the subspace $\mathcal{H}_{P,\pi}$ of vectors $\phi \in \mathcal{H}_P$ such that for each $x \in G(\mathbb{A})$, the function $\phi_x(m) = \phi(mx)$ in $L^2_{\mathrm{disc}}\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)$ is a matrix coefficient of π. We write $\mathcal{I}_{P,\chi,\pi}(\lambda, f)$ for the restriction of $\mathcal{I}_{P,\chi}(\lambda, f)$ to $\mathcal{H}_{P,\chi,\pi}$. We then set

$$\Psi^T_\pi(\lambda, f) = n_P^{-1}\mathrm{tr}\big(M^T_{P,\chi}(\lambda)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big),$$

for any $f \in C^\infty_c\big(G(\mathbb{A})^1\big)$ and $\lambda \in i\mathfrak{a}^*_P/i\mathfrak{a}^*_G$.

Lemma 20.1. *There are positive constants C_0 and d_0 such that for any $f \in C^\infty_c\big(G(\mathbb{A})^1\big)$, any $n \geq 0$, and any $T \in \mathfrak{a}_0$ with $d_{P_0}(T) > C_0$,*

$$\sum_{P\supset P_0} \int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G} \sum_\pi |\Psi^T_\pi(\lambda, f)|(1 + \|\lambda\|)^n \mathrm{d}\lambda \leq c_{n,f}(1 + \|T\|)^{d_0},$$

for a constant $c_{n,f}$ that is independent of T.

The lemma is a variant of Proposition 14.1(a). One obtains the factor $(1 + \|\lambda\|)^n$ in the estimate by choosing a suitable differentiable operator Δ on $G(\mathbb{R})$, and applying the arguments of Theorem 14.1(a) to Δf in place of f. (See [**A7**, Proposition 2.1]. One can in fact take $d_0 = \dim\, \mathfrak{a}_0$.) □

Lemma 20.2. *There is a constant C_0 such that for any $N > 0$ and any $f \in C^\infty_N\big(G(\mathbb{A})^1\big)$, the expression*

$$\sum_{P\supset P_0} \sum_\pi \int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G} \Psi^T_\pi(\lambda, f)\mathrm{d}\lambda \tag{20.1}$$

equals $J^T_\chi(f)$, and is hence a polynomial in T of degree bounded by $d_0 = \dim\, \mathfrak{a}_0$, whenever

$$d_{P_0}(T) > C_0(1 + N).$$

The expression equals

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T_1\Lambda^T_2 K_\chi(x, x)\mathrm{d}x = \int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T_2 K_\chi(x, x)\mathrm{d}x.$$

The lemma follows from Theorem 14.1(c), and an analysis of how the proof of this result depends quantitatively on the support of f. (See [**A7**, Proposition 2.2].) □

If $\tau_1, \tau_2 \in \Pi_{\mathrm{unit}}(K_\mathbb{R})$ are irreducible unitary representations of $K_\mathbb{R}$, set

$$f_{\tau_1,\tau_2}(x) = \int_{K_\mathbb{R}} \int_{K_\mathbb{R}} \mathrm{tr}\big(\tau_1(k_1)\big)f(k_1^{-1}xk_2^{-1})\mathrm{tr}\big(\tau_2(k_2)\big)\mathrm{d}k_1\mathrm{d}k_2,$$

for any function $f \in C^\infty_c\big(G(\mathbb{A})^1\big)$. Then

$$f(x) = \sum_{\tau_1,\tau_2} f_{\tau_1,\tau_2}(x).$$

Lemma 20.3. *There is a decomposition*

$$J^T_\chi(f) = \sum_{\tau_1,\tau_2} J^T_\chi(f_{\tau_1,\tau_2}).$$

The lemma follows easily from an inspection of how the estimates of the proof of Theorem 14.1 depend on left and right translation of f by $K_{\mathbb{R}}$. (See [**A7**, Proposition 2.3].) $\square$

The last three lemmas form the backdrop for our discussion of the analytic problem. The third lemma allows us to assume that f belongs to the Hecke algebra

$$\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})^1\big) = \mathcal{H}\big(G(\mathbb{A})^1, K\big)$$

of K-finite functions in $C_c^\infty\big(G(\mathbb{A})^1\big)$. We recall that f is K-finite if the space of functions on $G(\mathbb{A})^1$ spanned by left and right K-translates of f is finite dimensional. The second lemma describes the qualitative behaviour of $J_\chi^T(f)$ as a function of T, quantitatively in terms of the support of f. If we could somehow construct a family of new functions in $\mathcal{H}(G)$ in terms of the operators $\mathcal{I}_{P,\chi,\pi}(f)$, with some control over their supports, we might be able to bring this lemma to bear on our analytic difficulties.

Our rescue comes in the form of a Paley-Wiener theorem, or rather a corollary of the theorem that deals with multipliers. Multipliers are defined in terms of infinitesimal characters. To describe them, we have to fix an appropriate Cartan subalgebra.

For each archimedean valuation $v \in S_\infty$ of F, we fix a real vector space

$$\mathfrak{h}_v = i\mathfrak{b}_v \oplus \mathfrak{a}_0,$$

where $\mathfrak{b}_v$ is a Cartan subalgebra of the compact Lie group $K_v \cap M_0(F_v)$. We then set

$$\mathfrak{h} = \mathfrak{h}_\infty = \bigoplus_{v \in S_\infty} \mathfrak{h}_v.$$

This space can be identified with a split Cartan subalgebra of the Lie group $G_s^*(F_\infty)$, where

$$F_\infty = F_{S_\infty} = \bigoplus_{v \in S_\infty} F_v,$$

and G_s^* is a split F-form of the group G. In particular, the complex Weyl group $W = W_\infty$ of the Lie group $G(F_\infty)$ acts on $\mathfrak{h}$. The space $\mathfrak{h}$ comes with a canonical projection $\mathfrak{h} \mapsto \mathfrak{a}_P$, for any standard parabolic subgroup $P \supset P_0$, whose transpose is an injection $\mathfrak{a}_P^* \subset \mathfrak{h}^*$ of dual spaces. It is convenient to fix a positive definite, W-invariant inner product $(\cdot,\cdot)$ of $\mathfrak{h}$. The corresponding Euclidean norm $\|\cdot\|$ on $\mathfrak{h}$ restricts to a W_0-invariant Euclidean norm on $\mathfrak{a}_0$. We assume that it is dominated by the height function on $G(\mathbb{A})$ fixed earlier, in the sense that

$$\|H\| \leq \log \|\exp H\|, \qquad H \in \mathfrak{a}_0.$$

The infinitesimal character of an irreducible representation $\pi_\infty \in \Pi\big(G(F_\infty)\big)$ is represented by a W-orbit ν_{π_∞} in the complex dual space $\mathfrak{h}_{\mathbb{C}}^*$ of $\mathfrak{h}$. It satisfies

$$\pi_\infty(zf_\infty) = \langle h(z), \nu_{\pi_\infty}\rangle \pi_\infty(f_\infty), \qquad z \in \mathcal{Z}_\infty,\ f_\infty \in C_c^\infty\big(G(F_\infty)\big),$$

where $h\colon \mathcal{Z}_\infty \to S(\mathfrak{h}_{\mathbb{C}})^W$ is the isomorphism of Harish-Chandra, from the algebra $\mathcal{Z}_\infty$ of bi-invariant differential operators on $G(F_\infty)$ onto the algebra of W-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}^*$, that plays a central role in his work on representations of real groups. The algebra $\mathcal{Z}_\infty$ acts on the Hecke algebra $\mathcal{H}\big(G(\mathbb{A})\big)$ of $G(\mathbb{A})$ through the $G(F_\infty)$-component of a given function f. However, the space of functions zf, $z \in \mathcal{Z}_\infty$, is not rich enough for us to exploit Lemma 20.2.

Let $\mathcal{E}(\mathfrak{h})^W$ be the convolution algebra of W-invariant, compactly supported distributions on $\mathfrak{h}$. According to the classical Paley-Wiener theorem, the adjoint Fourier transform $\alpha \to \widehat{\alpha}$ is an isomorphism from $\mathcal{E}(\mathfrak{h})^W$ onto the algebra of entire, W-invariant functions $\widehat{\alpha}(\nu)$ on $\mathfrak{h}^*_{\mathbb{C}}$ of exponential type that are slowly increasing on cylinders

$$\{\nu \in \mathfrak{h}^*_{\mathbb{C}} : \ \|\mathrm{Re}(\nu)\| \leq r\}, \qquad r \geq 0.$$

The subalgebra $C_c^\infty(\mathfrak{h})^W$ is mapped onto the subalgebra of functions $\widehat{\alpha}$ that are rapidly decreasing on cylinders. (By the adjoint Fourier transform $\widehat{\alpha}$ we mean the transpose-inverse of the standard Fourier transform on functions, rather than simply the transpose. In other words,

$$\widehat{\alpha}(\nu) = \int_{\mathfrak{h}} \alpha(H) \mathrm{e}^{\nu(H)} \mathrm{d}H,$$

in case α is a function.)

We write $\mathcal{H}\big(G(F_\infty)\big) = \mathcal{H}\big(G(F_\infty), K_\infty\big)$ for the Hecke algebra of $K_\infty = \prod_{v \in S_\infty} K_v$ finite functions in $C_c^\infty\big(G(F_\infty)\big)$, and $\mathcal{H}_N\big(G(F_\infty)\big)$ for the subspace of functions in $\mathcal{H}\big(G(F_\infty)\big)$ supported on the set

$$\{x_\infty \in G(F_\infty) : \ \log \|x_\infty\| \leq N\}.$$

THEOREM 20.4. *There is a canonical action*

$$\alpha : \ f_\infty \ \longrightarrow \ f_{\infty,\alpha}, \qquad \alpha \in \mathcal{E}(\mathfrak{h})^W, \ f_\infty \in \mathcal{H}\big(G(F_\infty)\big),$$

of $\mathcal{E}(\mathfrak{h})^W$ *on* $\mathcal{H}\big(G(F_\infty)\big)$ *with the property that*

$$\pi_\infty(f_{\infty,\alpha}) = \widehat{\alpha}(\nu_{\pi_\infty}) \pi_\infty(f_\infty),$$

for any $\pi_\infty \in \Pi\big(G(F_\infty)\big)$. *Moreover, if* f_∞ *belongs to* $\mathcal{H}_N\big(G(F_\infty)\big)$ *and* α *is supported on the subset of points* $H \in \mathfrak{h}$ *with* $\|H\| \leq N_\alpha$, *then* $f_{\infty,\alpha}$ *lies in* $\mathcal{H}_{N+N_\alpha}\big(G(F_\infty)\big)$.

(See [**A9**, Theorem 4.2].) □

This is the multiplier theorem we will apply to the expression (20.1). We shall treat (20.1) as a linear functional of f in the Hecke algebra $\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})^1\big)$. If $\mathfrak{h}^1$ is the subspace of points in $\mathfrak{h}$ whose projection onto $\mathfrak{a}_G$ vanishes, we shall take α to be in the subspace $\mathcal{E}(\mathfrak{h}^1)^W$ of distributions in $\mathcal{E}(\mathfrak{h})^W$ supported on $\mathfrak{h}^1$. If f belongs to the Hecke algebra $\mathcal{H}\big(G(\mathbb{A})\big)$ on $G(\mathbb{A})$, we define f_α to be the function in $\mathcal{H}\big(G(\mathbb{A})\big)$ obtained by letting α act on the archimedean component of f. The restriction of f_α to $G(\mathbb{A})^1$ will then depend only on the restriction of f to $G(\mathbb{A})^1$. In other words, $f_\alpha \in \mathcal{H}(G)$ is defined for any $f \in \mathcal{H}(G)$. We shall substitute functions of this form into (20.1).

Suppose that $P \supset P_0$ and $\pi \in \Pi_{\mathrm{unit}}\big(M_P(\mathbb{A})^1\big)$ are as in (20.1). Then π is the restriction to $M_P(\mathbb{A})^1$ of a unitary representation

$$\pi_\infty \otimes \pi_{\mathrm{fin}}, \qquad \pi_\infty \in \Pi_{\mathrm{unit}}\big(M_P(F_\infty)\big), \ \pi_{\mathrm{fin}} \in \Pi_{\mathrm{unit}}\big(M_P(\mathbb{A}_{\mathrm{fin}})\big),$$

of $M_P(\mathbb{A})$. We obtain a linear form $\nu_\pi = \nu_{\pi_\infty}$ on $\mathfrak{h}_{\mathbb{C}}$, which we decompose

$$\nu_\pi = X_\pi + iY_\pi, \qquad X_\pi, Y_\pi \in \mathfrak{h}^*,$$

into real and imaginary parts. These points actually stand for orbits in $\mathfrak{h}^*$ of the complex Weyl group of $M_P(F_\infty)$, but we can take them to be fixed representatives of the corresponding orbits. Then X_π is uniquely determined by π, while the imaginary part Y_π is determined by π only modulo $\mathfrak{a}_P^*$. However, we may as well identify Y_π with the unique representative in $\mathfrak{h}^*$ of the coset in $\mathfrak{h}^*/\mathfrak{a}_P^*$ of smallest norm $\|Y_\pi\|$. This amounts to taking the representation π_∞ of $M_P(F_\infty)$ to be invariant under the subgroup $A^+_{M_P,\infty}$ of $M_P(F_\infty)$, a convention that is already implicit in the notation $\mathcal{H}_{P,\pi}$ above.

If B is any W-invariant function on $i\mathfrak{h}^*$, we define a function

$$B_\pi(\lambda) = B(iY_\pi + \lambda), \qquad \lambda \in i\mathfrak{a}_P^*,$$

on $i\mathfrak{a}_P^*$. We also write

$$B^\varepsilon(\nu) = B(\varepsilon\nu), \qquad \nu \in i\mathfrak{h}^*,$$

for any $\varepsilon > 0$. We shall want B to be rapidly decreasing on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. An obvious candidate would be the Paley-Wiener function $\widehat{\alpha}$ attached to a function $\alpha \in C_c^\infty(\mathfrak{h}^1)^W$. However, the point of this exercise is to allow B to be an arbitrary element in the space $\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ of W-invariant Schwartz functions on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$.

The next theorem provides the way out of our analytic difficulties.

Theorem 20.5. (a) *For any $B \in \mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ and $f \in \mathcal{H}(G)$, there is a unique polynomial $P^T(B,f)$ in T such that the difference*

$$\sum_{P\supset P_0} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_\pi \Psi_\pi^T(\lambda,f) B_\pi(\lambda) \mathrm{d}\lambda - P^T(B,f) \tag{20.2}$$

approaches 0 *as T approaches infinity in any cone*

$$\mathfrak{a}_{P_0}^r = \{T \in \mathfrak{a}_0 : \ d_{P_0}(T) > r\|T\|\}, \qquad r > 0.$$

(b) *If $B(0) = 1$, then*

$$J_\chi^T(f) = \lim_{\varepsilon\to 0} P^T(B^\varepsilon, f).$$

This is the main result, Theorem 6.3, of the paper [**A7**]. We shall sketch the proof.

The idea is to approximate B by Paley-Wiener functions $\widehat{\alpha}$, for $\alpha \in C_c^\infty(\mathfrak{h}^1)^W$. Assume that f belongs to the space

$$\mathcal{H}_N(G) = \mathcal{H}(G) \cap C_N^\infty\big(G(\mathbb{A})^1\big),$$

for some fixed $N > 0$, and that α is a general element in $\mathcal{E}(\mathfrak{h}^1)^W$. Then f_α lies in $\mathcal{H}_{N+N_\alpha}(G)$. For any $P \supset P_0$ and $\lambda \in i\mathfrak{a}_P^*$, $\mathcal{I}_P(\lambda, f_\alpha)$ is an operator on $\mathcal{H}_P$ whose restriction to $\mathcal{H}_{P,\chi,\pi}$ equals

$$\widehat{\alpha}(\nu_\pi + \lambda)\mathcal{I}_{P,\chi,\pi}(\lambda, f).$$

Applying Lemma 20.2 with f_α in place of f, we see that the expression

$$\sum_{P\supset P_0} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_\pi \widehat{\alpha}(\nu_\pi + \lambda)\Psi_\pi^T(\lambda,f)\mathrm{d}\lambda \tag{20.3}$$

equals $J_\chi^T(f_\alpha)$ whenever $d_{P_0}(T) > C_0(1 + N + N_\alpha)$, and is hence a polynomial in T in this range. The sum over π in (20.3) can actually be taken over a finite set that depends only on χ and f. This is implicit in Langlands's proof of Theorem 7.2, specifically his construction of the full discrete spectrum from residues of cuspidal Eisenstein series.

Suppose that α belongs to the subspace $C_c^\infty(\mathfrak{h}^1)^W$ of $\mathcal{E}(\mathfrak{h}^1)^W$. Then $J_\chi^T(f_\alpha)$ equals

$$\sum_{P\supset P_0}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_\pi\int_{\mathfrak{h}^1}\Psi_\pi^T(\lambda,f)\mathrm{e}^{(\nu_\pi+\lambda)(H)}\alpha(H)\mathrm{d}H\mathrm{d}\lambda.$$

By Lemma 20.1, integral

$$\psi_\pi^T(H,f)=\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\Psi_\pi^T(\lambda,f)\mathrm{e}^{\lambda(H)}d\lambda$$

converges to a bounded, smooth function of $H\in\mathfrak{h}^1$. It follows that

$$J_\chi^T(f_\alpha)=\int_{\mathfrak{h}^1}\Big(\sum_{P\supset P_0}\sum_\pi\psi_\pi^T(H,f)\mathrm{e}^{\nu_\pi(H)}\Big)\alpha(H)\mathrm{d}H,$$

whenever $d_{P_0}(T)>C_0(1+N+N_\alpha)$. Since $C_c^\infty(\mathfrak{h}^1)^W$ is dense in $\mathcal{E}(\mathfrak{h}^1)$ (in the weak topology), the assertion actually holds for any $\alpha\in\mathcal{E}(\mathfrak{h}^1)^W$ (with the integral being interpreted as evaluation of the distribution α).

If H is any point in $\mathfrak{h}^1$, let δ_H be the Dirac measure on $\mathfrak{h}^1$ at H. The symmetrization

$$\alpha_H=|W|^{-1}\sum_{s\in W}\delta_{s^{-1}H}$$

belongs to $\mathcal{E}(\mathfrak{h}^1)^W$. The function

$$p^T(H,f)=J_\chi^T(f_{\alpha_H})$$

is therefore a well defined polynomial in T, of degree bounded by d_0. The support of α_H is contained in the ball about the origin of radius $\|H\|$, so we can take $N_{\alpha_H}=\|H\|$. It follows that

$$p^T(H,f)=\sum_{P\supset P_0}\sum_\pi|W|^{-1}\sum_{s\in W}\psi_\pi^T(s^{-1}H,f)\mathrm{e}^{\nu_\pi(s^{-1}H)},\tag{20.4}$$

for all H and T with $d_{P_0}(T)>C_0(1+N+\|H\|)$. The right hand expression may be regarded as a triple sum over a finite set. It follows that $p^T(H,f)$ is a smooth function of $H\in\mathfrak{h}^1$ for all T in the given domain, and hence for all T, by polynomial interpolation. Observe that $\alpha_0=\delta_0$, and therefore that $f_{\alpha_0}=f$. It follows that

$$p^T(0,f)=J_\chi^T(f).$$

To study the right hand side of (20.4), we group the nonzero summands with a given real exponent X_π. More precisely, we define an equivalence relation on the triple indices of summation in (20.4) by setting $(P',\pi',s')\sim(P,\pi,s)$ if $s'X_{\pi'}=sX_\pi$. If Γ is any equivalence class, we set $X_\Gamma=sX_\pi$, for any $(P,\pi,s)\in\Gamma$. We also define

$$\psi_\Gamma^T(H,f)=|W|^{-1}\sum_{(P,\pi,s)\in\Gamma}\mathrm{e}^{iY_\pi(s^{-1}H)}\psi_\pi^T(s^{-1}H,f).$$

Then $\psi_\Gamma^T(H,f)$ is a bounded, smooth function of $H\in\mathfrak{h}^1$ that is defined for all T with $d_{P_0}(T)$ greater than some absolute constant. In fact, Lemma 20.1 implies that for any invariant differential operator D on $\mathfrak{h}^1$, there is a constant $c_{D,f}$ such that

$$|D\psi_\Gamma^T(H,f)|\le c_{D,f}(1+\|T\|)^{d_0},\qquad H\in\mathfrak{h}^1,\ d_{P_0}(T)>C_0,\tag{20.5}$$

for constants C_0 and d_0 independent of f. In particular, we can assume that the constants C_0 in (20.4) and (20.5) are the same. Let $\mathcal{E}=\mathcal{E}_f$ be the finite set of equivalence classes Γ such that the function $\psi^T_\Gamma(H,f)$ is not identically zero. It then follows from (20.4) that

$$\sum_{\Gamma\in\mathcal{E}} \mathrm{e}^{X_\Gamma(H)}\psi^T_\Gamma(H,f) - p^T(H,f) = 0,$$

whenever $d_{P_0}(T) > C_0(1+N+\|H\|)$. The proof of Theorem 20.5 rests on an argument that combines this last identity with the inequality (20.5). We shall describe it in detail for a special case.

Suppose that there is only one class Γ, and that $X_\Gamma = 0$. In other words, if π indexes a nonzero summand in (20.4), X_π vanishes. The identity (20.4) becomes

$$\psi^T_\Gamma(H,f) - p^T(H,f) = 0, \qquad d_{P_0}(T) > C_0(1+N+\|H\|). \tag{20.6}$$

It is easy to deduce in this case that $p^T(H,f)$ is a slowly increasing function of H. In fact, we claim that for every invariant differential operator D on $\mathfrak{h}^1$, there is a constant $c_{D,f}$ such that

$$|Dp^T(H,f)| \leq c_{D,f}(1+\|H\|)^{d_0}(1+\|T\|)^{d_0}, \tag{20.7}$$

for all $H \in \mathfrak{h}^1$ and $T \in \mathfrak{a}_0$. Since $p^T(H,f)$ is a polynomial in T whose degree is bounded by d_0, it would be enough to establish an estimate for each of the coefficients of $p^T(H,f)$ as functions of H. For any H, we choose T so that $d_{P_0}(T)$ is greater than $C_0(1+N+\|H\|)$, but so that $\|T\|$ is less than $C_1(1+\|H\|)$, for some large constant C_1 (depending on C_0 and N). It follows from (20.6) and (20.5) that

$$\begin{aligned}|Dp^T(H,f)| &= |D\psi^T_\Gamma(H,f)| \leq c_{D,f}(1+\|T\|)^{d_0}\\ &\leq c_{D,f}\big(1+C_1(1+\|H\|)\big)^{d_0} \leq c'_{D,f}(1+\|H\|)^{d_0},\end{aligned}$$

for some constant $c'_{D,f}$. Letting T vary within the chosen domain, we obtain a similar estimate for each of the coefficients of $p^T(H,f)$ by interpolation. The claimed inequality (20.7) follows.

We shall now prove Theorem 20.5(a), in the special case under consideration. We can write

$$B(\nu) = \int_{\mathfrak{h}_1} \mathrm{e}^{\nu(H)}\beta(H)\mathrm{d}H,$$

where $\beta \in \mathcal{S}(\mathfrak{h}^1)^W$ is the standard Fourier transform $\widehat{B}$ of the given function $B \in \mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}^*_G)^W$. We then form the integral

$$P^T(B,f) = p^T(\beta,f) = \int_{\mathfrak{h}^1} p^T(H,f)\beta(H)\mathrm{d}H,$$

which converges by (20.7). This is the required polynomial in T. We have to show that it is asymptotic to the expression

$$\sum_{P\supset P_0}\int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G}\sum_\pi \Psi^T_\pi(\lambda,f)B_\pi(\lambda)\mathrm{d}\lambda. \tag{20.8}$$

We write the expression (20.8) as

$$\begin{aligned}
&\sum_P \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_\pi \Psi_\pi^T(\lambda, f) \int_{\mathfrak{h}^1} \mathrm{e}^{(iY_\pi+\lambda)(H)}\beta(H)\mathrm{d}H\mathrm{d}\lambda \\
&= \int_{\mathfrak{h}^1} \sum_P \sum_\pi \psi_\pi^T(H, f)\mathrm{e}^{iY_\pi(H)}\beta(H)\mathrm{d}H \\
&= \int_{\mathfrak{h}^1} \sum_P \sum_\pi |W|^{-1} \sum_{s\in W} \psi_\pi^T(s^{-1}H, f)\mathrm{e}^{iY_\pi(s^{-1}H)}\beta(H)\mathrm{d}H \\
&= \int_{\mathfrak{h}^1} \psi_\Gamma^T(H, f)\beta(H)\mathrm{d}H,
\end{aligned}$$

by the definition of $B_\pi(\lambda)$, the definition of $\psi_\pi^T(H, f)$, the fact that $\beta(H)$ is W-symmetric, and our assumption that $X_\Gamma = 0$. It follows that the difference (20.2) between $p^T(B, f)$ and (20.8) has absolute value bounded by the integral

$$\int_{\mathfrak{h}^1} |\psi_\Gamma^T(H, f) - p^T(H, f)||\beta(H)|\mathrm{d}H.$$

We can assume that T lies in a fixed cone $\mathfrak{a}_{P_0}^r$, and is large. If $d_{P_0}(T)$ is greater than $C_0(1+N+\|H\|)$, the integrand vanishes by (20.6). We may therefore restrict the domain of integration to the subset of points $H \in \mathfrak{h}^1$ with

$$\|H\| \geq C_0^{-1}d_{P_0}(T) - (1 + N) \geq C_0^{-1}r\|T\| - (1 + N) \geq r_1\|T\|,$$

for some fixed positive number r_1. For any such H, we have

$$\begin{aligned}
|\psi_\Gamma^T(H, f) - p^T(H, f)| &\leq |\psi_\Gamma^T(H, f)| + |p^T(H, f)| \\
&\leq c_1(1 + \|H\|)^{2d_0},
\end{aligned}$$

for some $c_1 > 0$, by (20.5) and (20.7). We also have

$$|\beta(H)| \leq c_2(1 + \|H\|)^{-(1+2d_0+2\dim\,\mathfrak{h}^1)}, \qquad H \in \mathfrak{h}^1,$$

for some $c_2 > 0$. The integral is therefore bounded by

$$c_1c_2 \int_{\|H\|\geq r_1\|T\|} (1 + \|H\|)^{2d_0}(1 + \|H\|)^{-(1+2d_0+2\dim\,\mathfrak{h}^1)}\mathrm{d}H,$$

a quantity that is in turn bounded by an expression

$$c_1c_2r_1^{-1}\|T\|^{-1} \int_{\mathfrak{h}^1} (1 + \|H\|)^{-2\dim\,\mathfrak{h}^1}\mathrm{d}H$$

that approaches 0 as T approaches infinity. It follows that the difference (20.2) approaches 0 as T approaches infinity in $\mathfrak{a}_{P_0}^r$. We have established Theorem 20.5(a), in the special case under consideration, by combining (20.5), (20.6), and (20.7).

Next we prove Theorem 20.5(b), in the given special case. Recall that $J_\chi^T(f)$ is the value of $p^T(H, f)$ at $H = 0$. We have to show that this equals the limit of $P^T(B^\varepsilon, f)$ as ε approaches 0, under the assumption that $B(0) = 1$. Now

$$(\widehat{B}^\varepsilon)(H) = (\widehat{B})_\varepsilon(H) = \beta_\varepsilon(H),$$

where

$$\beta_\varepsilon(H) = \varepsilon^{-(\dim\,\mathfrak{h}^1)}\beta(\varepsilon^{-1}H).$$

Therefore

$$P^T(B^\varepsilon, f) = p^T(\beta_\varepsilon, f) = \int_{\mathfrak{h}^1} p^T(H,f)\beta_\varepsilon(H)\mathrm{d}H$$
$$= J^T_\chi(f) + \int_{\mathfrak{h}^1} \big(p^T(H,f) - p^T(0,f)\big)\beta_\varepsilon(H)\mathrm{d}H,$$

since $J^T_\chi(f) = p^T(0,f)$, and $\int \beta_\varepsilon = B^\varepsilon(0) = 1$. But if we combine the mean value theorem with (20.7), we see that

$$|p^T(H,f) - p^T(0,f)| \leq c\|H\|(1+\|H\|)^{d_0}(1+\|T\|)^{d_0},$$

for some fixed $c > 0$, and all H and T. We can assume that $\varepsilon \leq 1$. Then

$$\begin{aligned}
&\int_{\mathfrak{h}^1} |p^T(H,f) - p^T(0,f)||\beta_\varepsilon(H)|\mathrm{d}H \\
&= \varepsilon^{-\dim(\mathfrak{h}^1)} \int_{\mathfrak{h}^1} |p^T(H,f) - p^T(0,f)||\beta(\varepsilon^{-1}H)|\mathrm{d}H \\
&= \int_{\mathfrak{h}^1} |p^T(\varepsilon H,f) - p^T(0,f)||\beta(H)|\mathrm{d}H \\
&\leq c\int_{\mathfrak{h}^1} \varepsilon\|H\|(1+\|\varepsilon H\|)^{d_0}(1+\|T\|)^{d_0}|\beta(H)|\mathrm{d}H \\
&\leq c'\varepsilon(1+\|T\|)^{d_0},
\end{aligned}$$

where

$$c' = c\int_{\mathfrak{h}^1} \|H\|(1+\|H\|)^{d_0}|\beta(H)|\mathrm{d}H.$$

It follows that

$$\lim_{\varepsilon\to 0} \big(p^T(B^\varepsilon, f) - J^T_\chi(f)\big) = 0,$$

as required.

We have established Theorem 20.5 in the special case that there is only one class $\Gamma \in \mathcal{E}$, and that $X_\Gamma = 0$. In general, there are several classes, so there can be nonzero points X_Γ. In place of (20.6), we have the more general identity

$$\sum_\Gamma \mathrm{e}^{X_\Gamma(H)}\psi^T_\Gamma(H,f) - p^T(H,f) = 0, \qquad d_{P_0}(T) > C_0(1+N+\|H\|).$$

In particular, $p^T(H,f)$ can have exponential growth in H, and need not be tempered. It cannot be integrated against a Schwartz function β of H. Now each function $\psi^T_\Gamma(H,f)$ is tempered in H, by (20.7). The question is whether it is asymptotic to a polynomial in T. In other words, does the polynomial $p^T(H,f)$ have a Γ-component $\mathrm{e}^{X_\Gamma(H)}p^T_\Gamma(H,f)$?

To answer the question, we take H_Γ to be the point in $\mathfrak{h}^1$ such that the inner product (H_Γ, H) equals $X_\Gamma(H)$, for each $H \in \mathfrak{h}$. We claim that for fixed H, the function

$$t \longrightarrow \psi^T_\Gamma(tH_\Gamma + H, f), \qquad t \in \mathbb{R},$$

is a finite linear combination of unitary exponential functions. To see this, we first note that the function equals

$$|W|^{-1} \sum_{(P,\pi,s)\in\Gamma} \mathrm{e}^{iY_\pi(s^{-1}(tH_\Gamma+H))}\psi^T_\pi\big(s^{-1}(tH_\Gamma+H), f\big).$$

For any $(P, \pi, s) \in \Gamma$, the linear form $X_\pi = s^{-1}X_\Gamma$ is the real part of the infinitestimal character of a unitary representation π of $M_P(\mathbb{A})^1$. It follows that the corresponding point $s^{-1}H_\Gamma$ in $\mathfrak{h}^1$ lies in the kernel $\mathfrak{h}^P$ of the projection of $\mathfrak{h}$ onto $\mathfrak{a}_P$. On the other hand, the function

$$\psi_\pi^T(H, f) = \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f)\mathrm{e}^{\lambda(H)} d\lambda$$

is invariant under translation by $\mathfrak{h}^P$. Consequently

$$\psi_\pi^T\big(s^{-1}(tH_\Gamma + H)\big) = \psi_\pi^T(s^{-1}H).$$

The claim follows.

It is now pretty clear that we can construct the Γ-component of the polynomial

$$p^T(H, f) = \sum_{\Gamma \in \mathcal{E}} \mathrm{e}^{X_\Gamma(H)} \psi_\Gamma^T(H, f), \qquad d_{P_0}(T) \geq C_0(1 + N + \|H\|),$$

in terms of its direction of real exponential growth. If one examines the question more closely, taking into consideration the derivation of (20.7) above, one obtains the following lemma.

Lemma 20.6. *There are functions*

$$p_\Gamma^T(H, f) \qquad H \in \mathfrak{h}^1, \ \Gamma \in \mathcal{E},$$

which are smooth in H and polynomials in T of degree at most d_0, such that

$$p^T(H, f) = \sum_{\Gamma \in \mathcal{E}} \mathrm{e}^{X_\Gamma(H)} p_\Gamma^T(H, f),$$

and such that if D is any invariant differential operator on $\mathfrak{h}^1$, then

$$(20.6)' \qquad \big|D\big(\psi_\Gamma^T(H, f) - p_\Gamma^T(H, f)\big)\big| \leq c_{D,f}\mathrm{e}^{-\delta d_{P_0}(T)}(1 + \|T\|)^{d_0},$$

for all H and T with $d_{P_0}(T) > C_0(1 + N + \|H\|)$, and

$$(20.7)' \qquad |Dp_\Gamma^T(H, f)| \leq c_{D,f}(1 + \|H\|)^{d_0}(1 + \|T\|)^{d_0},$$

for all H and T, with C_0, δ and $c_{D,f}$ being positive constants.

See [**A7**, Proposition 5.1]. □

Given Lemma 20.6, we set

$$p_\Gamma^T(\beta, f) = \int_{\mathfrak{h}^1} p_\Gamma^T(H, f)\beta(H)\mathrm{d}H,$$

for any function $\beta \in \mathcal{S}(\mathfrak{h}^1)^W$ and any $\Gamma \in \mathcal{E}$. We then argue as above, using the inequalities (20.5), (20.6)′ and (20.7)′ in place of (20.5), (20.6), and (20.7). We deduce that for any Γ and β,

$$(20.9(\mathrm{a})) \qquad \lim_{T \to \infty} \Big(\int_{\mathfrak{h}^1} \psi_\Gamma^T(H, f)\beta(H)\mathrm{d}H - p_\Gamma^T(\beta, f)\Big) = 0, \qquad T \in \mathfrak{a}_{P_0}^r,$$

and that

$$(20.9(\mathrm{b})) \qquad \lim_{\varepsilon \to 0} p_\Gamma^T(\beta_\varepsilon, f) = p_\Gamma^T(0, f),$$

if $\int \beta = 1$, exactly as in the proofs of (a) and (b) in the special case of Theorem 20.5 above. (See [**A7**, Lemmas 6.2 and 6.1].)

To establish Theorem 20.5 in general, we set

$$P^T(B,f) = \sum_{\Gamma\in\mathcal{E}} p^T_\Gamma(\beta,f), \qquad \beta = \widehat{B}.$$

Then, as in the proof of the special case of Theorem 20.5(a) above, we deduce that

$$\sum_P \int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G} \sum_\pi \Psi^T_\pi(\lambda,f)B_\pi(\lambda)\mathrm{d}\lambda = \sum_{\Gamma\in\mathcal{E}} \int_{\mathfrak{h}^1} \psi^T_\Gamma(H,f)\beta(H)\mathrm{d}H.$$

It follows from (20.9(a)) that the difference between the expression on the right hand side of this identity and $P^T(B,f)$ approaches 0 as T approaches infinity in $\mathfrak{a}^r_{P_0}$. The same is therefore true of the difference between the expression on the left hand side of the identity and $P^T(B,f)$. This gives Theorem 20.5(a). For Theorem 20.5(b), we use (20.9(b)) to write

$$\begin{aligned}\lim_{\varepsilon\to 0} P^T(B^\varepsilon,f) &= \lim_{\varepsilon\to 0}\sum_{\Gamma\in\mathcal{E}} p^T_\Gamma(\beta_\varepsilon,f)\\ &= \sum_{\Gamma\in\mathcal{E}} p^T_\Gamma(0,f) = p^T(0,f) = J^T_\chi(f),\end{aligned}$$

if $B(0) = \int\beta = 1$. This completes our discussion of the proof of Theorem 20.5. □

21. The fine spectral expansion

We have taken care of the primary analytic obstruction to computing the distributions $J_\chi(f)$. Its resolution is contained in Theorem 20.5, which applies to objects $\chi\in\mathcal{X}$, $P_0\in\mathcal{P}(M_0)$, $f\in\mathcal{H}(G)$, and $B\in\mathcal{S}(i\mathfrak{h}^*/i\mathfrak{a}^*_G)^W$, with $B(0)=1$. We take B to be compactly supported. The function

$$B_\pi(\lambda) = B(iY_\pi+\lambda), \qquad \lambda\in i\mathfrak{a}^*_P,$$

attached to any $P\supset P_0$ and $\pi\in\Pi_{\mathrm{unit}}\big(M_P(\mathbb{A})^1\big)$ then belongs to $C^\infty_c(i\mathfrak{a}^*_P/i\mathfrak{a}^*_G)$.

Suppose that a^T and b^T are two functions defined on some cone $d_{P_0}(T) > C_0$ in $\mathfrak{a}_0$. We shall write $a^T\sim b^T$ if a^T-b^T approaches 0 as T approaches infinity in any cone $\mathfrak{a}^r_{P_0}$. Theorem 20.5(a) tells us that

$$\begin{aligned}P^T(B,f) &\sim \sum_{P\supset P_0}\int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G}\sum_\pi \Psi^T_\pi(\lambda,f)B_\pi(\lambda)\mathrm{d}\lambda\\ &= \sum_{P\supset P_0} n_P^{-1}\sum_\pi\int_{i\mathfrak{a}^*_P/i\mathfrak{a}^*_G}\mathrm{tr}\big(M^T_{P,\chi}(\lambda)\mathcal{I}_{P,\chi,\pi}(\lambda,f)\big)B_\pi(\lambda)\mathrm{d}\lambda,\end{aligned}$$

where $P^T(B,f)$ is a polynomial in T that depends linearly on B. The fact that each $B_\pi(\lambda)$ has compact support is critical. It removes the analytic problem of reconciling an asymptotic limit in T with an integral in λ over a noncompact space. Our task is to compute $P^T(B,f)$ explicitly, as a bilinear form in the functions $\{B_\pi(\lambda)\}$ and the operators $\{\mathcal{I}_{P,\chi,\pi}(\lambda,f)\}$. We will then obtain an explicit formula for $J^T_\chi(f)$ from the assertion

$$J^T_\chi(f) = \lim_{\varepsilon\to 0} P^T(B^\varepsilon,f)$$

of Theorem 20.5(b).

The operator $M^T_{P,\chi}(\lambda)$ is defined by (15.1) in terms of an inner product of truncated Eisenstein series attached to P. Proposition 15.3 gives the explicit inner product formula of Langlands, which applies to the special case that the Eisenstein series are cuspidal. It turns out that the same formula holds asymptotically in T for arbitrary Eisenstein series.

THEOREM 21.1. *Suppose that $\phi \in \mathcal{H}_P^0$ and $\phi' \in \mathcal{H}_{P'}^0$, for standard parabolic subgroups $P, P' \supset P_0$. Then the difference between the inner product*

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi,\lambda)\overline{\Lambda^T E(x,\phi',\lambda')}\mathrm{d}x$$

and the sum

$$\sum_Q \sum_s \sum_{s'} \theta_Q(s\lambda + s'\overline{\lambda}')^{-1}\mathrm{e}^{(s\lambda+s'\overline{\lambda}')(T)}\big(M(s,\lambda)\phi, M(s',\lambda')\phi'\big) \tag{21.1}$$

over $Q \supset P_0$, $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$, and $s' \in W(\mathfrak{a}_{P'}, \mathfrak{a}_Q)$ is bounded by a product

$$c(\lambda, \lambda', \phi, \phi')\mathrm{e}^{-\varepsilon d_{P_0}(T)},$$

where $\varepsilon > 0$, and $c(\lambda,\lambda',\phi,\phi')$ is a locally bounded function on the set of points $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ and $\lambda' \in \mathfrak{a}_{P',\mathbb{C}}^*$ at which the Eisenstein series are analytic.

This is [**A6**, Theorem 9.1], which is the main result of the paper [**A6**]. The proof begins with the special case already established for cuspidal Eisenstein series in Proposition 15.3. One then uses the results of Langlands in [**Lan5**, §7], which express arbitrary Eisenstein series in terms of residues of cuspidal Eisenstein series. This process is not canonical in general. Nevertheless, one can still show that (21.1) is an asymptotic approximation for the expression obtained from the appropriate residues of the corresponding formula for cuspidal Eisenstein series. □

Let us write $\omega^T(\lambda,\lambda',\phi,\phi')$ for the expression (21.1). If B_χ is any function in $C_c^\infty(i\mathfrak{a}_P^*/i\mathfrak{a}_G^*)$, the theorem tells us that

$$\begin{aligned}&\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \big(M_{P,\chi}^T(\lambda)\mathcal{I}_{P,\chi}(\lambda,f)\phi,\phi\big)B_\chi(\lambda)\mathrm{d}\lambda\\ &\sim \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \omega^T\big(\lambda,\lambda,\mathcal{I}_{P,\chi}(\lambda,f)\phi,\phi\big)B_\chi(\lambda)\mathrm{d}\lambda.\end{aligned}$$

We shall apply this asymptotic formula to the functions $B_\chi = B_\pi$. Since f is K-finite, $\mathcal{I}_{P,\chi}(\lambda,f)\phi$ vanishes for all but finitely many vectors ϕ in the orthonormal basis $\mathcal{B}_{P,\chi}$ of $\mathcal{H}_{P,\chi}$. This is a consequence of Langlands' construction of the discrete spectrum, as we have noted earlier. We assume that $\mathcal{B}_{P,\chi}$ is a disjoint union of orthonormal bases $\mathcal{B}_{P,\chi,\pi}$ of the spaces $\mathcal{H}_{P,\chi,\pi}$. It then follows that

$$\begin{aligned}&P^T(B,f)\\ &\sim \sum_{P\supset P_0} n_P^{-1}\sum_\pi \Big(\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathcal{B}_{P,\chi,\pi}} \omega^T\big(\lambda,\lambda,\mathcal{I}_{P,\chi,\pi}(\lambda,f)\phi,\phi\big)B_\pi(\lambda)\mathrm{d}\lambda\Big).\end{aligned}$$

The problem is to find an explicit polynomial function of T, for any P and π, which is asymptotic in T to the expression in the brackets.

Suppose that P, π, and ϕ are fixed, and that λ lies in $i\mathfrak{a}_P^*$. Changing the indices of summation in the definition (21.1), we write

$$\omega^T\big(\lambda,\lambda,\mathcal{I}_{P,\chi,\pi}(\lambda,f)\phi,\phi\big)$$

as the limit as λ' approaches λ of the expression

$$\begin{aligned}&\omega^T\big(\lambda',\lambda,\mathcal{I}_P(\lambda,f)\phi,\phi\big)\\ =&\sum_{P_1}\sum_{t'}\sum_{t}\theta_{P_1}(t'\lambda'-t\lambda)^{-1}\mathrm{e}^{(t'\lambda'-t\lambda)(T)}\big(M(t',\lambda')\mathcal{I}_P(\lambda,f)\phi,M(t,\lambda)\phi\big)\\ =&\sum_{s}\sum_{P_1}\sum_{t}\theta_{P_1}\big(t(s\lambda'-\lambda)\big)^{-1}\mathrm{e}^{(t(s\lambda'-\lambda))(T)}\big(M(ts,\lambda')\mathcal{I}_P(\lambda,f)\phi,M(t,\lambda)\phi\big),\end{aligned}$$

for sums over $P_1 \supset P_0$ and $t', t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, and for $s = t^{-1}t'$ ranging over the group $W(M_P) = W(\mathfrak{a}_P, \mathfrak{a}_P)$. Since λ is purely imaginary, the adjoint of the operator $M(t,\lambda)$ equals $M(t,\lambda)^{-1}$. The sum

$$\sum_{\phi\in\mathcal{B}_{P,\chi,\pi}}\omega^T\big(\lambda,\lambda,\mathcal{I}_{P,\chi,\pi}(\lambda,f)\phi,\phi\big) \tag{21.2}$$

therefore equals the limit as λ' approaches λ of

$$\sum_{s}\sum_{(P_1,t)}\theta_{P_1}\big(t(s\lambda'-\lambda)\big)^{-1}\mathrm{e}^{(t(s\lambda'-\lambda))(T)}\mathrm{tr}\big(M(t,\lambda)^{-1}M(ts,\lambda')\mathcal{I}_{P,\chi,\pi}(\lambda,f)\big).$$

Set $M = M_P$. The correspondence

$$(P_1,t)\ \longrightarrow\ Q=w_t^{-1}P_1w_t,\qquad\qquad P_1\supset P_0,\ t\in W(\mathfrak{a}_P,\mathfrak{a}_{P_1}),$$

is then a bijection from the set of pairs (P_1, t) in the last sum onto the set $\mathcal{P}(M)$. For any group $Q \in \mathcal{P}(M)$ and any element $s \in W(M)$, there is a unitary intertwining operator

$$M_{Q|P}(s,\lambda):\ \mathcal{H}_P\ \longrightarrow\ \mathcal{H}_Q,\qquad\qquad \lambda\in i\mathfrak{a}_M^*.$$

It is defined by analytic continuation from the analogue of the integral formula (7.2), in which P' is replaced by Q. If (P_1, t) is the preimage of Q, it is easy to see from the definitions that

$$M(ts,\lambda')=tM_{Q|P}(s,\lambda')\mathrm{e}^{(s\lambda'+\rho_Q)(T_0-t^{-1}T_0)},$$

where t: $\mathcal{H}_Q \to \mathcal{H}_{P_1}$ is the operator defined by

$$(t\phi)(x)=\phi(w_t^{-1}x),\qquad\qquad \phi\in\mathcal{H}_P.$$

The point T_0 is used as in §15 to measure the discrepancy between the two representatives w_t and $\widetilde{w}_t$ of the element $t \in W_0$. (See [**A8**, (1.4)].) It follows that

$$M(t,\lambda)^{-1}M(ts,\lambda')=M_{Q|P}(\lambda)^{-1}M_{Q|P}(s,\lambda')\mathrm{e}^{(s\lambda'-\lambda)(T_0-t^{-1}T_0)},$$

where $M_{Q|P}(\lambda) = M_{Q|P}(1, \lambda)$. Next, we define a point $Y_Q(T)$ to be the projection onto $\mathfrak{a}_M$ of the point

$$t^{-1}(T-T_0)+T_0.$$

Then

$$\mathrm{e}^{(t(s\lambda'-\lambda))(T)}\mathrm{e}^{(s\lambda'-\lambda)(T_0-t^{-1}T_0)}=\mathrm{e}^{(s\lambda'-\lambda)(Y_Q(T))}.$$

Finally, it is clear that

$$\theta_{P_1}\big(t(s\lambda'-\lambda)\big)^{-1}=\theta_Q(s\lambda'-\lambda)^{-1}.$$

It follows that (21.2) equals the limit as λ' approaches λ of the sum over $s \in W(M)$ of

$$\sum_{Q\in\mathcal{P}(M)}\mathrm{tr}\big(M_{Q|P}(\lambda)^{-1}M_{Q|P}(s,\lambda')\mathcal{I}_{P,\chi,\pi}(\lambda,f)\big)\mathrm{e}^{(s\lambda'-\lambda)(Y_Q(T))}\theta_Q(s\lambda'-\lambda)^{-1}. \tag{21.3}$$

The expression (21.3) looks rather like the basic function (17.1) we have attached to any (G, M)-family. We shall therefore study it as a function of the variable

$$\Lambda = s\lambda' - \lambda.$$

The expression becomes

$$\sum_{Q \in \mathcal{P}(M)} c_Q(\Lambda) d_Q(\Lambda) \theta_Q(\Lambda)^{-1},$$

where

$$c_Q(\Lambda) = \mathrm{e}^{\Lambda(Y_Q(T))},$$

and

$$d_Q(\Lambda) = \mathrm{tr}\big(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda') \mathcal{I}_{\mathcal{P},\chi,\pi}(\lambda, f)\big).$$

It follows easily from the definition of $Y_Q(T)$ that $\{c_Q(\Lambda)\}$ is a (G, M)-family. The operators $M_{Q|P}(s, \lambda')$ in the second factor satisfy a functional equation

$$M_{Q'|P}(s, \lambda') = M_{Q'|Q}(s\lambda') M_{Q|P}(s, \lambda'), \qquad Q' \in \mathcal{P}(M).$$

It follows easily from this that $\{d_Q(\Lambda)\}$ is also a (G, M)-family. (See [**A8**, p. 1298]. Of course $d_Q(\Lambda)$ depends on the kernel of the mapping $(\lambda', \lambda) \to \Lambda$ as well as on Λ, but at the moment we are only interested in the variable Λ.) The expression (21.3) therefore reduces to something we have studied, namely the function $(cd)_M(\Lambda)$ attached to the (G, M)-family $\{(cd)_Q(\Lambda)\}$. By Lemma 17.1, the function has no singularities in Λ. It follows that the expression (21.3) extends to a smooth function of (λ', λ) in $i\mathfrak{a}_M^* \times i\mathfrak{a}_M^*$.

Remember that we are supposed to take the limit, as λ' approaches λ, of the sum over $s \in W(M)$ of (21.3). We will then want to integrate the product of $B_\pi(\lambda)$ with the resulting function of λ over the space $i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$. From what we have just observed, the integral and limit may be taken inside the sum over s. It turns out that the asymptotic limit in T may also be taken inside the sum over s. In other words, it is possible to find an explicit polynomial in T that is asymptotic to the integral over λ of the product $B_\pi(\lambda)$ with value at $\lambda' = \lambda$ of (21.3). We shall describe how to do this, using the product formula of Lemma 17.4.

Suppose that $s \in W(M)$ is fixed. Let L be the smallest Levi subgroup in $\mathcal{L}(M)$ that contains a representative of s. Then $\mathfrak{a}_L$ equals the kernel of s in $\mathfrak{a}_M$. The element s therefore belongs to the subset

$$W^L(M)_{\mathrm{reg}} = \{t \in W^L(M) : \ \ker(t) = \mathfrak{a}_L\},$$

of regular elements in $W^L(M)$. Given s, we set $\lambda' = \lambda + \zeta$, where ζ is restricted to lie in the subspace $i\mathfrak{a}_L^*$ of $i\mathfrak{a}_M^*$ associated to s. Then $s\zeta = \zeta$, and

$$\Lambda = (s\lambda - \lambda) + \zeta$$

is the decomposition of Λ relative to the direct sum

$$i\mathfrak{a}_M^* = i(\mathfrak{a}_M^L)^* \oplus i\mathfrak{a}_L^*.$$

If λ_L is the projection of λ onto $i\mathfrak{a}_L^*$, the mapping

$$(\lambda, \zeta) \longrightarrow (\Lambda, \lambda_L), \qquad \lambda \in i\mathfrak{a}_M^*, \ \zeta \in i\mathfrak{a}_L^*,$$

is a linear automorphism of the vector space $i\mathfrak{a}_M^* \oplus i\mathfrak{a}_L^*$. In particular, the points λ and $\lambda' = \lambda + \zeta$ are uniquely determined by Λ and λ_L. Let us write

$$c_Q(\Lambda, T) = \mathrm{e}^{\Lambda(Y_Q(T))}$$

and

$$d_Q(\Lambda, \lambda_L) = \mathrm{tr}\big(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, \lambda + \zeta)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big),$$

in order to keep track of our two (G, M)-families on the supplementary variables. They of course remain (G, M)-families in the variable Λ. For $\lambda' = \lambda + \zeta$ as above, the expression (21.3) equals

$$\sum_{Q\in\mathcal{F}(M)} c_Q(\Lambda, T) d_Q(\Lambda, \lambda_L)\theta_Q(\Lambda)^{-1} = \sum_{S\in\mathcal{F}(M)} c_M^S(\Lambda, T) d'_S(\Lambda_S, \lambda_L),$$

by the product formula of Lemma 17.4. To evaluate (21.3) at $\lambda' = \lambda$, we set $\zeta = 0$. This entails simply replacing Λ by $s\lambda - \lambda$. The value of (21.3) at $\lambda' = \lambda$ therefore equals

$$\sum_{S\in\mathcal{F}(M)} c_M^S(s\lambda - \lambda, T) d'_S\big((s\lambda - \lambda)_S, \lambda_L\big).$$

We have therefore to consider the integral

$$\int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \Big(\sum_{S\in\mathcal{F}(M)} c_M^S(s\lambda - \lambda, T) d'_S\big((s\lambda - \lambda)_S, \lambda_L\big)\Big) B_\pi(\lambda)\mathrm{d}\lambda, \tag{21.4}$$

for $M = M_P$, $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$, $L \in \mathcal{L}(M)$ and $s \in W^L(M)_{\mathrm{reg}}$, and for T in a fixed domain $\mathfrak{a}_{P_0}^r$. We need to show that the integral is asymptotic to an explicit polynomial in T. This will allow us to construct $P^T(B)$ simply by summing the product of this polynomial with n_P^{-1} over $P \supset P_0$, π, L, and s.

We first decompose the integral (21.4) into a double integral over $i(\mathfrak{a}_M^L)^*$ and $i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$. If λ belongs to $i\mathfrak{a}_M^*$, $s\lambda - \lambda$ depends only on the projection μ of λ onto $i(\mathfrak{a}_M^L)^*$. Since the mapping

$$F_s: \ \mu \longrightarrow s\mu - \mu$$

is a linear isomorphism of $i(\mathfrak{a}_M^L)^*$, (21.4) equals the product of the inverse

$$|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

of the determinant of this mapping with the sum over $S \in \mathcal{F}(M)$ of

$$\int_{i(\mathfrak{a}_M^L)^*}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} c_M^S(\mu, T) d'_S(\mu_S, \lambda) B_\pi\big(F_s^{-1}(\mu) + \lambda\big)\mathrm{d}\lambda\mathrm{d}\mu. \tag{21.5}$$

Next, we note that the dependence of the integral on T is through the term $c_M^S(\mu, T)$. For fixed S, the set

$$\mathcal{Y}_M^S(T) = \{Y_{S(R)}(T) : \ R \in \mathcal{P}^{M_S}(M)\}$$

is a positive (M_S, M)-orthogonal set of points in $\mathfrak{a}_M$, which all project to a common point $Y_S(T)$ in $\mathfrak{a}_S$. It follows from Lemma 17.2 that

$$c_M^S(\mu, T) = \int_{Y_S(T)+\mathfrak{a}_M^{M_S}} \psi_M^S(H, T)\mathrm{e}^{\mu(H)}\mathrm{d}H,$$

where $\psi_M^S(\cdot, T)$ is the characteristic function of the convex hull in $\mathfrak{a}_M$ of $\mathcal{Y}_M^S(T)$. We can therefore write (21.5) as

$$\int_{Y_S(T)+\mathfrak{a}_M^{M_S}} \psi_M^S(H, T)\phi_S(H)\mathrm{d}H, \tag{21.6}$$

where

$$\phi_S(H) = \int_{i(\mathfrak{a}_M^L)^*} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{e}^{\mu(H)} d'_S(\mu_S,\lambda) B_\pi\big(F_s^{-1}(\mu)+\lambda\big)\mathrm{d}\lambda \mathrm{d}\mu,$$

for any $H \in \mathfrak{a}_M$. Since $d'_S(\cdot,\cdot)$ is smooth, and $B_\pi(\cdot)$ is both smooth and compactly supported, $\phi_S(H)$ is a Schwartz function on $\mathfrak{a}_M/\mathfrak{a}_L$.

There are two cases to consider. Suppose first that S does not belong to the subset $\mathcal{F}(L)$ of $\mathcal{F}(M)$. Then $\mathfrak{a}_S$ is not contained in $\mathfrak{a}_L$, and $Y_S(T)$ projects to a nonzero point $Y_S(T)_M^L$ in $\mathfrak{a}_M^L$. In fact, it follows easily from the fact that T lies in $\mathfrak{a}_{P_0}^r$ that

$$\|Y_S(T)_M^L\| \geq r_1\|T\|,$$

for some $r_1 > 0$. The function $\psi_M^S(\cdot,T)$ is supported on a compact subset of the affine space $Y_S(T) + \mathfrak{a}_M^{M_S}$ whose volume is bounded by a polynomial in T. One combines this with the fact that $\phi_S(H)$ is a Schwartz function on $\mathfrak{a}_M/\mathfrak{a}_L$ to show that (21.6) approaches 0 as T approaches infinity in $\mathfrak{a}_{P_0}^r$. (See [**A8**, p. 1306].)

We can therefore assume that S belongs to $\mathcal{F}(L)$. Then

$$\mathfrak{a}_M^{M_S} = \mathfrak{a}_M^L \oplus \mathfrak{a}_L^{M_S}.$$

Since ϕ_S is $\mathfrak{a}_L$-invariant, we are free to write (21.6) as

$$\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \Big(\int_{\mathfrak{a}_M^L} \phi_S(U)\psi_M^S(U+H,T)\mathrm{d}U\Big)\mathrm{d}H.$$

As it turns out, we can simplify matters further by replacing

$$\psi_M^S(U+H,T)$$

with $\psi_L^S(H,T)$, where $\psi_L^S(H,T)$ is the characteristic function in $\mathfrak{a}_L$ of the set $\mathcal{Y}_L^S(T)$ obtained in the obvious way from $\mathcal{Y}_M^S(T)$. More precisely, the difference between the last expression and the product

$$\text{(21.7)} \qquad \int_{\mathfrak{a}_M^L} \phi_S(U)\mathrm{d}U \cdot \int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \psi_L^S(H,T)\mathrm{d}H$$

approaches 0 as T approaches infinity in $\mathfrak{a}_{P_0}^r$. Suppose for example that $G = SL(3)$, $M = M_0$ is minimal, $M_S = G$, and that L is a standard maximal Levi subgroup M_1. Then $Y_S(T) = 0$, and the difference

$$\psi_L^S(H,T) - \psi_M^S(U+H,T), \qquad U \in \mathfrak{a}_M^L,\ H \in \mathfrak{a}_L,$$

is the characteristic function of the darker shaded region in Figure 21.1. Since $\phi_S(U)$ is rapidly decreasing on the vertical $\mathfrak{a}_M^L$-axis in the figure, the integral over (U,H) of its product with the difference above does indeed approach 0. In the general case, the lemmas in [**A8**, §3] show that the convex hull of $\mathcal{Y}_M^S(T)$ has the same qualitative behaviour as is Figure 21.1. (See [**A8**, p. 1307–1308].)

The problem thus reduces to the computation of the product (21.7), for any element $S \in \mathcal{F}(L)$. The first factor in the product can be written as

$$\int_{\mathfrak{a}_M^L} \phi_S(U)\mathrm{d}U = \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} d'_S(0,\lambda)B_\pi(\lambda)\mathrm{d}\lambda,$$

by the Fourier inversion formula in $\mathfrak{a}_M^L$. The second factor equals

$$\int_{Y_S(T)+\mathfrak{a}_L^{M_S}} \psi_L^S(H,T)\mathrm{d}H = c_L^S(0,T)$$

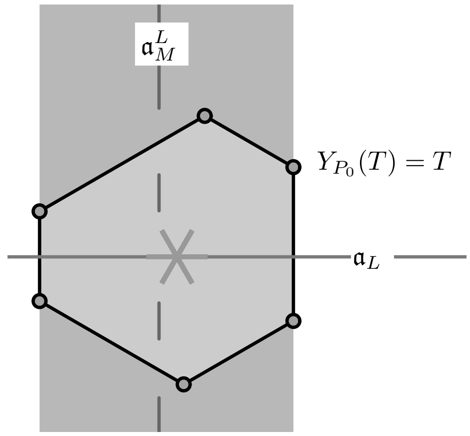

FIGURE 21.1. The vertices represent the six points $Y_P(T)$, as P ranges over $\mathcal{P}(M_0)$. Since T ranges over a set $\mathfrak{a}_{P_0}^r$, the distance from any vertex to the horizontal $\mathfrak{a}_L$-axis is bounded below by a positive multiple of $\|T\|$.

by Lemma 17.2, and is therefore a polynomial in T. In particular, (21.7) is already a polynomial in T. To express its contribution to the asymptotic value of (21.4), we need only sum S over $\mathcal{F}(L)$. We conclude that (21.4) differs from the polynomial

$$\big|\det(s-1)_{\mathfrak{a}_M^L}\big|^{-1}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*}\Big(\sum_{S\in\mathcal{F}(L)} c_L^S(0,T)d_S'(0,\lambda)\Big)B_\pi(\lambda)\mathrm{d}\lambda \tag{21.8}$$

by an expression that approaches 0 as T approaches infinity in $\mathfrak{a}_{P_0}^r$.

The sum

$$\sum_{S\in\mathcal{F}(L)} c_L^S(0,T)d_S'(0,\lambda) \tag{21.9}$$

in (21.8) comes from a product

$$c_{Q_1}(\Lambda,T)d_{Q_1}(\Lambda,\lambda), \qquad Q_1\in\mathcal{P}(L),\ \Lambda\in i\mathfrak{a}_L^*,$$

of (G,L)-families. By Lemma 17.4, it equals the value at $\Lambda=0$ of the sum

$$\sum_{Q_1\in\mathcal{P}(L)} c_{Q_1}(\Lambda,T)d_{Q_1}(\Lambda,\lambda)\theta_{Q_1}(\Lambda)^{-1}.$$

Recall the definition of the (G,M)-family $\{d_Q(\Lambda,\lambda)\}$ of which the (G,L)-family $\{d_{Q_1}(\Lambda,\lambda)\}$ is the restriction. Since λ and Λ lie in the subspace $i\mathfrak{a}_L^*$ of $i\mathfrak{a}_M^*$, λ_L equals λ, and

$$\zeta = \Lambda - (s\lambda - \lambda) = \Lambda.$$

It follows from the definitions and the functional equations of the global intertwining operators that

$$\begin{aligned} d_Q(\Lambda,\lambda) &= \mathrm{tr}\big(M_{Q|P}(\lambda)^{-1}M_{Q|P}(s,\lambda+\Lambda)\mathcal{I}_{P,\chi,\pi}(\lambda,f)\big)\\ &= \mathrm{tr}\big(M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda)M_{P|P}(s,\lambda+\Lambda)\mathcal{I}_{P,\chi,\pi}(\lambda,f)\big), \end{aligned}$$

for any $Q \in \mathcal{P}(M)$. Since the point $\lambda + \Lambda$ lies in the space $i\mathfrak{a}_L^*$ fixed by s, the operator

$$M_P(s, 0) = M_{P|P}(s, \lambda + \Lambda)$$

is independent of λ and Λ. To deal with the other operators, we define

$$\mathcal{M}_Q(\Lambda, \lambda, P) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda),$$

and

$$\begin{aligned} \mathcal{M}_Q^T(\Lambda, \lambda, P) &= \mathrm{e}^{\Lambda(Y_Q(T))} M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda) \\ &= c_Q(\Lambda, T)\mathcal{M}_Q(\Lambda, \lambda, P), \end{aligned}$$

for any $Q \in \mathcal{P}(M)$. As functions of Λ in the larger domain $i\mathfrak{a}_M^*$, these objects form two (G, M)-families as Q varies over $\mathcal{P}(M)$. With Λ restricted to $i\mathfrak{a}_L^*$ as above, the functions

$$\mathcal{M}_{Q_1}^T(\Lambda, \lambda, P) = \mathcal{M}_Q^T(\Lambda, \lambda, P), \qquad Q_1 \in \mathcal{P}(L),\ Q \subset Q_1,$$

form a (G, L)-family as Q_1 varies over $\mathcal{P}(L)$. It follows from the definitions that (21.9) equals

$$\begin{aligned} &\lim_{\Lambda \to 0} \sum_{Q_1 \in \mathcal{P}(L)} c_{Q_1}(\Lambda, T) d_{Q_1}(\Lambda, \lambda) \theta_{Q_1}(\Lambda)^{-1} \\ &= \lim_{\Lambda \to 0} \sum_{Q_1 \in \mathcal{P}(L)} \mathrm{tr}\big(\mathcal{M}_{Q_1}^T(\Lambda, \lambda, P) M_P(s, 0) \mathcal{I}_{P,\chi,\pi}(\lambda, f)\big) \theta_{Q_1}(\Lambda)^{-1} \\ &= \lim_{\Lambda \to 0} \mathrm{tr}\big(\mathcal{M}_L^T(\Lambda, \lambda, P) M_P(s, 0) \mathcal{I}_{P,\chi,\pi}(\lambda, f)\big) \\ &= \mathrm{tr}\big(\mathcal{M}_L^T(\lambda, P) M_P(s, 0) \mathcal{I}_{P,\chi,\pi}(\lambda, f)\big). \end{aligned}$$

We substitute this formula into (21.8). The resulting expression is the required polynomial approximation to (21.4).

The following proposition is Theorem 4.1 of [**A8**]. We have completed a reasonably comprehensive sketch of its proof.

PROPOSITION 21.2. *For any $f \in \mathcal{H}(G)$ and $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$, the polynomial $P^T(B, f)$ equals the sum over $P \supset P_0$, $\pi \in \Pi_{\mathrm{unit}}\big(M_P(\mathbb{A})^1\big)$, $L \in \mathcal{L}(M_P)$, and $s \in W^L(M_P)_{\mathrm{reg}}$ of the product of*

$$n_P^{-1} |\det(s - 1)_{\mathfrak{a}_P^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\mathcal{M}_L^T(\lambda, P) M_P(s, 0) \mathcal{I}_{P,\chi,\pi}(\lambda, f)\big) B_\pi(\lambda) \mathrm{d}\lambda. \qquad \square$$

Recall that

$$J_\chi^T(f) = \lim_{\varepsilon \to 0} P^T(B^\varepsilon, f),$$

where $B^\varepsilon(\nu) = B(\varepsilon\nu)$, and $B(0)$ is assumed to be 1. Therefore

$$J_\chi(f) = J_\chi^{T_0}(f) = \lim_{\varepsilon \to 0} P^{T_0}(B^\varepsilon, f).$$

Now

$$\begin{aligned}\mathcal{M}_L^{T_0}(\lambda, P) &= \lim_{\Lambda \to 0} \sum_{Q_1 \in \mathcal{P}(L)} c_{Q_1}(\Lambda, T_0)\mathcal{M}_{Q_1}(\Lambda, \lambda, P)\theta_{Q_1}(\Lambda)^{-1} \\ &= \lim_{\Lambda \to 0} \sum_{Q_1 \in \mathcal{P}(L)} \mathrm{e}^{\Lambda(Y_{Q_1}(T_0))}\mathcal{M}_{Q_1}(\Lambda, \lambda, P)\theta_{Q_1}(\Lambda)^{-1} \\ &= \lim_{\Lambda \to 0} \mathrm{e}^{\Lambda(T_0)} \sum_{Q_1 \in \mathcal{P}(L)} \mathcal{M}_{Q_1}(\Lambda, \lambda, P)\theta_{Q_1}(\Lambda)^{-1} \\ &= \mathcal{M}_L(\lambda, P),\end{aligned}$$

since $Y_{Q_1}(T_0)$ is just the projection of T_0 onto $\mathfrak{a}_L$. We substitute this into the formula above. The canonical point $T_0 \in \mathfrak{a}_0$ is independent of the minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$ we fixed at the beginning of the section. Moreover, if $M = M_P$, the function

$$\mathrm{tr}\big(\mathcal{M}_L(\lambda, P)M_P(s, 0)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big)$$

is easily seen to be independent of the choice of $P \in \mathcal{P}(M)$. We can therefore rewrite the formula of Proposition 21.2 in terms of Levi subgroups $M \in \mathcal{L}$ rather than standard parabolic subgroups $P \supset P_0$. Making the appropriate adjustments to the coefficients, one obtains the following formula as a corollary of the last one. (See [**A8**, Theorem 5.2].)

Corollary 21.3. *For any $f \in \mathcal{H}(G)$, the linear form $J_\chi(f)$ equals the limit as ε approaches 0 of the expression obtained by taking the sum over $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$, and $s \in W^L(M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\mathcal{M}_L(\lambda, P)M_P(s, 0)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big)B_\pi^\varepsilon(\lambda)\mathrm{d}\lambda. \qquad \square$$

The final step is to get rid of the function B_π^ε and the associated limit in ε. Recall that B had the indispensable role of truncating the support of integrals that would otherwise be unmanageable. The function

$$B_\pi^\varepsilon(\lambda) = B\big(\varepsilon(iY_\pi + \lambda)\big)$$

is compactly supported in $\lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$, but converges pointwise to 1 as ε approaches 0. If we can show that the integral

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\mathcal{M}_L(\lambda, P)M_P(s, 0)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big)\mathrm{d}\lambda \tag{21.10}$$

converges absolutely, we could remove the limit in ε by an appeal to the dominated convergence theorem. One establishes absolute convergence by normalizing the intertwining operators from which the operator $\mathcal{M}_L(\lambda, P)$ is constructed.

Suppose that $\pi_v \in \Pi\big(M(F_v)\big)$ is an irreducible representation of $M(F_v)$, for a Levi subgroup $M \in \mathcal{L}$ and a valuation v of F. We write

$$\pi_{v,\lambda}(m_v) = \pi_v(m_v)\mathrm{e}^{\lambda(H_M(m_v))}, \qquad m_v \in M(F_v),$$

as usual, for the twist of π_v by an element $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. If $P \in \mathcal{P}(M)$, $\mathcal{I}_P(\pi_{v,\lambda})$ denotes the corresponding induced representation of $G(F_v)$, acting on a Hilbert space $\mathcal{H}_P(\pi_v)$ of vector valued functions on K_v. If $Q \in \mathcal{P}(M)$ is another parabolic subgroup, and ϕ belongs to $\mathcal{H}_P(\pi_v)$, the integral

$$\int_{N_Q(F_v)\cap N_P(F_v)\backslash N_Q(F_v)} \phi(n_v x_v) \mathrm{e}^{(\lambda+\rho_P)(H_P(n_v x_v))} \mathrm{e}^{-(\lambda+\rho_Q)(H_Q(x_v))} \mathrm{d}n_v$$

converges if the real part of λ is highly regular in the chamber $(\mathfrak{a}_M^*)_P^+$. It defines an operator

$$J_{Q|P}(\pi_{v,\lambda}):\ \mathcal{H}_P(\pi_v) \longrightarrow \mathcal{H}_Q(\pi_v)$$

that intertwines the local induced representations $\mathcal{I}_P(\pi_{v,\lambda})$ and $\mathcal{I}_Q(\pi_{v,\lambda})$. One knows that $J_{Q|P}(\pi_{v,\lambda})$ can be analytically continued to a meromorphic function of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ with values in the corresponding space of intertwining operators. (See [**Har5**], [**KnS**], and [**Sha1**].) This is a local analogue of Langlands' analytic continuation of the global operators $M_{Q|P}(\lambda)$. Unlike the operators $M_{Q|P}(\lambda)$, however, the local operators $J_{Q|P}(\pi_{v,\lambda})$ are not transitive in Q and P. For example, if $\overline{P}$ is the group in $\mathcal{P}(M)$ opposite to P, Harish-Chandra has proved that

$$J_{P|\bar{P}}(\pi_{v,\lambda}) J_{\bar{P}|P}(\pi_{v,\lambda}) = \mu_M(\pi_{v,\lambda})^{-1},$$

where $\mu_M(\pi_{v,\lambda})$ is a meromorphic scalar valued function that is closely related to the Plancherel density. To make the operators $J_{Q|P}(\pi_{v,\lambda})$ have better properties, one must multiply them by suitable scalar normalizing factors.

THEOREM 21.4. *For any M, v, and $\pi_v \in \Pi\big(M(F_v)\big)$, one can choose meromorphic scalar valued functions*

$$r_{Q|P}(\pi_{v,\lambda}), \qquad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*,\ P, Q \in \mathcal{P}(M),$$

such that the normalized intertwining operators

$$R_{Q|P}(\pi_{v,\lambda}) = r_{Q|P}(\pi_{v,\lambda})^{-1} J_{Q|P}(\pi_{v,\lambda}) \tag{21.11}$$

have the following properties.

(i) $R_{Q'|P}(\pi_{v,\lambda}) = R_{Q'|Q}(\pi_{v,\lambda}) R_{Q|P}(\pi_{v,\lambda}), \qquad Q', Q, P \in \mathcal{P}(M).$

(ii) *The K_v-finite matrix coefficients of $R_{Q|P}(\pi_{v,\lambda})$ are rational functions of the variables $\{\lambda(\alpha^\vee) :\ \alpha \in \Delta_P\}$ if v is archimedean, and the variables $\{q_v^{-\lambda(\alpha^\vee)} :\ \alpha \in \Delta_P\}$ if v is nonarchimedean.*

(iii) *If π_v is unitary, the operator $R_{Q|P}(\pi_{v,\lambda})$ is unitary for $\lambda \in i\mathfrak{a}_M^*$, and hence analytic.*

(iv) *If G is unramified at v, and $\phi \in \mathcal{H}(\pi_v)$ is the characteristic function of K_v, $R_{Q|P}(\pi_{v,\lambda})\phi$ equals ϕ.*

See [**A15**, Theorem 2.1] and [**CLL**, Lecture 15]. The factors $r_{Q|P}(\pi_{v,\lambda})$ are defined as products, over reduced roots β of (Q, A_M) that are not roots of (P, A_M), of meromorphic functions $r_\beta(\pi_\lambda)$ that depend only on $\lambda(\beta^\vee)$. The main step is to establish the property

$$r_{P|\bar{P}}(\pi_{v,\lambda}) r_{\bar{P}|P}(\pi_{v,\lambda}) = \mu_M(\pi_{v,\lambda})^{-1}, \tag{21.12}$$

in the case that M is maximal. □

Remarks. 1. The assertions of the theorem are purely local. They can be formulated for Levi subgroups and parabolic subgroups that are defined over F_v.

2. Suppose that $\bigotimes_v \pi_v$ is an irreducible representation of $M(\mathbb{A})$, whose restriction to $M(\mathbb{A})^1$ we denote by π. The product

$$R_{Q|P}(\pi_\lambda) = \bigotimes_v R_{Q|P}(\pi_{v,\lambda}) \tag{21.13}$$

is then a well defined transformation of the dense subspace $\mathcal{H}^0_P(\pi)$ of K-finite vectors in $\mathcal{H}_P(\pi)$. Indeed, for any $\phi \in \mathcal{H}^0_P(\pi)$, $R_{Q|P}(\pi_\lambda)\phi$ can be expressed as a finite product by (iv). If π is unitary and $\lambda \in i\mathfrak{a}^*_M$, $R_{Q|P}(\pi_\lambda)$ extends to a unitary transformation of the entire Hilbert space $\mathcal{H}_P(\pi)$.

Suppose that $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$ is any representation that occurs in the discrete part $R_{M,\mathrm{disc}}$ of R_M. In other words, the subspace $\mathcal{H}_{P,\pi}$ of $\mathcal{H}_P$ is nonzero. The restriction of the global intertwining operator $M_{Q|P}(\lambda)$ to $\mathcal{H}_{P,\pi}$ can be expressed in terms of the local intertwining operators above. It is isomorphic to $m_{\mathrm{disc}}(\pi)$-copies of the operator

$$J_{Q|P}(\pi_\lambda) = \bigotimes_v J_{Q|P}(\pi_{v,\lambda}),$$

defined for any unitary extension $\bigotimes_v \pi_v$ of π to $M(\mathbb{A})$ by analytic continuation in λ. If $\{r_{Q|P}(\pi_{v,\lambda})\}$ is any family of local normalizing factors that for each v satisfy the conditions of Theorem 21.4, the scalar-valued product

$$r_{Q|P}(\pi_\lambda) = \prod_v r_{Q|P}(\pi_{v,\lambda}) \tag{21.14}$$

is also defined by analytic continuation in λ, and is analytic for $\lambda \in i\mathfrak{a}^*_M$. Let $R_{Q|P}(\lambda)$ be the operator on $\mathcal{H}_P$ whose restriction to any subspace $\mathcal{H}_{P,\pi}$ equals the product of $r_{Q|P}(\pi_\lambda)^{-1}$ with the restriction of $M_{Q|P}(\lambda)$. In other words, the restriction of $R_{Q|P}(\lambda)$ to $\mathcal{H}_{P,\pi}$ is isomorphic to $m_{\mathrm{disc}}(\pi)$-copies of the operator (2.13). We define

$$r_Q(\Lambda, \pi_\lambda, P) = r_{Q|P}(\pi_\lambda)^{-1} r_{Q|P}(\pi_{\lambda+\Lambda}), \qquad \mathcal{H}_{P,\pi} \neq \{0\}, \tag{21.15}$$

and

$$\mathcal{R}_Q(\Lambda, \lambda, P) = R_{Q|P}(\lambda)^{-1} R_{Q|P}(\lambda + \Lambda),$$

for points Λ and λ in $i\mathfrak{a}^*_M$. Then $\{r_Q(\Lambda, \pi_\lambda)\}$ and $\{\mathcal{R}_Q(\Lambda, \lambda, P)\}$ are new (G, M)-families of Λ. They give rise to functions $r_L(\pi_\lambda, P)$ and $\mathcal{R}_L(\lambda, P)$ of λ for any $L \in \mathcal{L}(M)$. We write $r_L(\pi_\lambda) = r_L(\pi_\lambda, P)$, since this function is easily seen to be independent of the choice of P.

Lemma 21.5. (a) *There is a positive integer n such that*

$$\int_{i\mathfrak{a}^*_M/i\mathfrak{a}^*_G} |r_L(\pi_\lambda)|(1 + \|\lambda\|)^{-n} d\lambda < \infty.$$

(b) *The integral (21.10) converges absolutely.*

The integrand in (21.10) depends only on the restriction $\mathcal{M}_L(\lambda, P)_{\chi,\pi}$ of the operator $\mathcal{M}_L(\lambda, P)$ to $\mathcal{H}_{P,\chi,\pi}$. But $\mathcal{M}_L(\lambda, P)_{\chi,\pi}$ can be defined in terms of the product of the two new (G, M)-families above. Moreover, we are free to apply the simpler version (17.12) of the usual splitting formula. This is because for any $S \in \mathcal{F}(L)$ and $Q \in \mathcal{P}(S)$, the number

$$r^S_L(\pi_\lambda) = r^Q_L(\pi_\lambda)$$

is independent of the choice of Q [**A8**, Corollary 7.4]. Therefore

$$\mathcal{M}_L(\lambda, P)_{\chi,\pi} = \sum_{S\in\mathcal{F}(L)} r_L^S(\pi_\lambda)\mathcal{R}_S(\lambda, P)_{\chi,\pi},$$

where $\mathcal{R}_S(\lambda, P)_{\chi,\pi}$ denotes the restriction of $\mathcal{R}_S(\lambda, P)$ to $\mathcal{H}_{P,\chi,\pi}$. The integral (21.10) can therefore be decomposed as a sum

$$\sum_{S\in\mathcal{F}(L)} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} r_L^S(\pi_\lambda)\mathrm{tr}\big(\mathcal{R}_S(\lambda, P)M_P(s,0)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big)\mathrm{d}\lambda. \tag{21.16}$$

Since f lies in the Hecke algebra $\mathcal{H}(G)$, the operator $\mathcal{I}_{P,\chi,\pi}(\lambda, f)$ is supported on a finite dimensional subspace of $\mathcal{H}_{P,\chi,\pi}$. Moreover, it is an easy consequence of the conditions (ii)–(iv) of Theorem 21.4 that any matrix coefficient of the operator $\mathcal{R}_S(\lambda, P)$ is a rational function in finitely many complex variables $\{\lambda(\alpha^\vee), q_v^{-\lambda(\alpha^\vee)}\}$, which is analytic for $\lambda \in i\mathfrak{a}_M^*$. Since $\mathcal{I}_{P,\chi,\pi}(\lambda, f)$ is rapidly decreasing in λ, part (b) of the lemma follows inductively from (a). (See [**A8**, §8].)

It is enough to establish part (a) in the case that M is a maximal Levi subgroup. This is because for general M and L, $r_M^L(\pi_\lambda)$ can be written as a finite linear combination of products

$$r_M^{M_1}(\pi_\lambda)\dots r_M^{M_p}(\pi_\lambda),$$

for Levi subgroups $M_1,\dots,M_p$ in $\mathcal{L}(M)$, with $\dim(\mathfrak{a}_M/\mathfrak{a}_{M_i}) = 1$, such that the mapping

$$\mathfrak{a}_M/\mathfrak{a}_G \longrightarrow \bigoplus_{i=1}^p(\mathfrak{a}_M/\mathfrak{a}_{M_i})$$

is an isomorphism. (See [**A8**, §7].) In case M is maximal, one combines (21.16) with estimates based on Selberg's positivity argument used to prove Theorem 14.1(a). (See [**A8**, §8–9].) □

It is a consequence of Langlands' construction of the discrete spectrum of M in terms of residues of cuspidal Eisenstein series that the sum over $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$ in Corollary 21.3 can be taken over a finite set. Lemma 21.5(b) asserts that for any π, the integral (21.10) converges absolutely. Combining the dominated convergence theorem with the formula of Corollary 21.3, we obtain the following theorem.

THEOREM 21.6. *For any $f \in \mathcal{H}(G)$, the linear form $J_\chi(f)$ equals the sum over $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$, and $s \in W^L(M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \tag{21.17}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\mathcal{M}_L(\lambda, P)M_P(s,0)\mathcal{I}_{P,\chi,\pi}(\lambda, f)\big)\mathrm{d}\lambda.$$

(See [**A8**, Theorem 8.2].) □

Remarks. 3. There is an error in [**A8**, §8]. It is the ill-considered inequality stated on p. 1329 of [**A8**], three lines above the expression (8.4), which was taken from [**A5**, (7.6)]. The inequality seems to be false if f lies in the complement of $\mathcal{H}(G)$ in $C_c^\infty\big(G(\mathbb{A})^1\big)$, and π is nontempered. Consequently, the last formula for $J_\chi(f)$ does not hold if f lies in the complement of $\mathcal{H}(G)$ in $C_c^\infty\big(G(\mathbb{A})^1\big)$.

The fine spectral expansion of $J(f)$ is the sum over $\chi \in \mathcal{X}$ of the formulas for $J_\chi(f)$ provided by the last theorem. It is convenient to express this expansion in terms of infinitesimal characters.

A representation $\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$ has an archimedean infinitesimal character, consisting of a W-orbit of points $\nu_\pi = X_\pi + iY_\pi$ in $\mathfrak{h}^*_{\mathbb{C}}/i\mathfrak{a}^*_G$. The imaginary part Y_π is really an $\mathfrak{a}^*_M$-coset in $\mathfrak{h}^*$, but as in §20, we can identify it with the unique point in the coset for which the norm $\|Y_\pi\|$ is minimal. We then define

$$\Pi_{t,\mathrm{unit}}\big(M(\mathbb{A})^1\big) = \big\{\pi \in \Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big) : \ \|\mathrm{Im}(\nu_\pi)\| = \|Y_\pi\| = t\big\},$$

for any nonnegative real number t.

Recall that a class $\chi \in \mathcal{X}$ is a W_0-orbit of pairs (M_1, π_1), with π_1 being a cuspidal automorphic representation of $M_1(\mathbb{A})^1$. Setting $\nu_\chi = \nu_{\pi_1}$, we define a linear form

$$J_t(f) = \sum_{\{\chi\in\mathcal{X}:\|\mathrm{Im}(\nu_\chi)\|=t\}} J_\chi(f), \qquad t \geq 0,\ f \in \mathcal{H}(G),$$

in which the sum may be taken over a finite set. Then

$$J(f) = \sum_{t\geq 0} J_t(f).$$

We also write $\mathcal{I}_{P,t}(\lambda, f)$ for the restriction of the operator $\mathcal{I}_P(\lambda, f)$ to the invariant subspace

$$\mathcal{H}_{P,t} = \bigoplus_{\{(\chi,\pi):\|\mathrm{Im}(\nu_\chi)\|=t\}} \mathcal{H}_{P,\chi,\pi},$$

of $\mathcal{H}_P$. It is again a consequence of Langlands' construction of the discrete spectrum that if $\|\mathrm{Im}(\nu_\chi)\| = t$, the space $\mathcal{H}_{P,\chi,\pi}$ vanishes unless π belongs to $\Pi_{t,\mathrm{unit}}\big(M(\mathbb{A})^1\big)$. In other words, the representation $\mathcal{I}_{P,t}(\lambda)$ is equivalent to a direct sum of induced representations of the form $\mathcal{I}_P(\pi_\lambda)$, for $\pi \in \Pi_{t,\mathrm{unit}}\big(M(\mathbb{A})^1\big)$. The fine spectral expansion is then given by the following corollary of Theorem 21.6.

Corollary 21.7. *For any $f \in \mathcal{H}(G)$, the linear form $J(f)$ equals the sum over $t \geq 0$, $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, and $s \in W^L(M)_{\mathrm{reg}}$ of the product of the coefficient (21.17) with the linear form*

$$\int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \mathrm{tr}\big(\mathcal{M}_L(\lambda, P)M_P(s, 0)\mathcal{I}_{P,t}(\lambda, f)\big)\mathrm{d}\lambda. \tag{21.18}$$ □

The fine spectral expansion is thus an explicit sum of integrals. Among these integrals, the ones that are discrete have special significance. They correspond to the terms with $L = G$. The discrete part of the fine spectral expansion attached to any t equals the linear form

(21.19)

$$I_{t,\mathrm{disc}}(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}^G_M}|^{-1}\mathrm{tr}\big(M_P(s, 0)\mathcal{I}_{P,t}(0, f)\big).$$

It contains the t-part of the discrete spectrum, as well as singular points in the t-parts of continuous spectra. Observe that we have not shown that the sum over t of these distributions converges. To do so, one would need to extend Müller's solution of the trace class conjecture [**Mul**], as has been done in the case $G = GL(n)$ by Müller and Speh [**MS**]. It is only after $I_{t,\mathrm{disc}}(f)$ has been enlarged to the linear form $J_t(f)$, by including the corresponding continuous terms, that the spectral arguments we have discussed yield the absolute convergence of the sum over t. However, it turns out that this circumstance does not effect our ability to use trace formulas to compare discrete spectra on different groups.

22. The problem of invariance

In the last four sections, we have refined both the geometric and spectral sides of the original formula (16.1). Let us now step back for a moment to assess the present state of affairs. The fine geometric expansion of Corollary 19.3 is transparent in its overall structure. It is a simple linear combination of weighted orbital integrals, taken over Levi subgroups $M \in \mathcal{L}$. The fine spectral expansion of Corollary 21.7 is also quite explicit, but it contains a more complicated double sum over Levi subgroups $M \subset L$. In order to focus our discussion on the next stage of development, we need to rewrite the spectral side so that it is parallel to the geometric side.

We shall first revisit the fine geometric expansion. This expansion is a sum of products of local distributions $J_M(\gamma, f)$ with global coefficients $a^M(S, \gamma)$, where $S \supset S_{\mathrm{ram}}$ is a large finite set of valuations depending on the support of f, and $\gamma \in \big(M(F)\big)_{M,S}$ is an (M, S)-equivalence class. Let us write

$$\Gamma(M)_S = \big(M(F)\big)_{M,S}, \qquad S \supset S_{\mathrm{ram}}, \tag{22.1}$$

in order to emphasize that this set is a quotient of the set $\Gamma(M)$ of conjugacy classes in $M(F)$. The semi-simple component γ_s of a class $\gamma \in \Gamma(M)_S$ can be identified with a semisimple conjugacy class in $M(F)$. By choosing S to be large, we guarantee that for any class γ with $J_M(\gamma, f) \neq 0$, the set

$$\mathrm{Int}\big(M(\mathbb{A}^S)\big)\gamma_s = \{m^{-1}\gamma_s m : \ m \in M(\mathbb{A}^S)\}$$

intersects the maximal compact subgroup K_M^S of $M(\mathbb{A}^S)$. If S is *any* finite set containing S_{ram}, and γ is a class in $\Gamma(M)_S$, we shall write

$$a^M(\gamma) = \begin{cases} a^M(S, \gamma), & \text{if } \mathrm{Int}\big(M(\mathbb{A}^S)\big)\gamma_s \cap K_M^S \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{22.2}$$

If f belongs to $\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})^1\big)$, we also write

$$J_M(\gamma, f) = J_M(\gamma, f_S),$$

where f_S is the restriction of f to the subgroup $G(F_S)^1$ of $G(\mathbb{A})^1$. We can then write the fine geometric expansion slightly more elegantly as the limit over increasing sets S of expressions

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) I_M(\gamma, f).$$

The limit stabilizes for large finite sets S.

To write the spectral expansion in parallel form, we have first to introduce suitable weighted characters $J_M(\pi, f)$. Suppose that π belongs to $\Pi_{\mathrm{unit}}\big(M(\mathbb{A})^1\big)$. Then π can be identified with an orbit π_λ of $i\mathfrak{a}_M^*$ in $\Pi_{\mathrm{unit}}\big(M(\mathbb{A})\big)$. In the last section, we defined normalized intertwining operators $R_{Q|P}(\pi_\lambda)$ in terms (21.11) and (21.13) of a suitable choice of local normalizing factors $\{r_{Q|P}(\pi_{v,\lambda})\}$. We now introduce the corresponding (G, M)-family

$$\mathcal{R}_Q(\Lambda, \pi_\lambda, P) = R_{Q|P}(\pi_\lambda)^{-1} R_{Q|P}(\pi_{\lambda+\Lambda}), \qquad Q \in \mathcal{P}(M),\ \Lambda \in i\mathfrak{a}_M^*,$$

of operators on $\mathcal{H}_P(\pi)$, which we use to define the linear form

$$J_M(\pi_\lambda, \widetilde{f}) = \operatorname{tr}\big(\mathcal{R}_M(\pi_\lambda, P)\mathcal{I}_P(\pi_\lambda, \widetilde{f})\big) \qquad \widetilde{f} \in \mathcal{H}\big(G(\mathbb{A})\big), \tag{22.3}$$

on $\mathcal{H}\big(G(\mathbb{A})\big)$. We then set

$$J_M(\pi, f) = \int_{i\mathfrak{a}_M^*} J_M(\pi_\lambda, \widetilde{f}) \mathrm{d}\lambda, \qquad f \in \mathcal{H}(G), \tag{22.4}$$

where $\widetilde{f}$ is any function in $\mathcal{H}\big(G(\mathbb{A})\big)$ whose restriction to $G(\mathbb{A})^1$ equals f. The last linear form does indeed depend only on π and f. It is the required weighted character.

The core of the fine spectral expansion is the t-discrete part $I_{t,\mathrm{disc}}(f)$, defined for any $t \geq 0$ and $f \in \mathcal{H}(G)$ by (21.19). The term "discrete" refers obviously to the fact that we can write the distribution as a linear combination

$$I_{t,\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{t,\mathrm{unit}}(G(\mathbb{A})^1)} a^G_{\mathrm{disc}}(\pi) f_G(\pi) \tag{22.5}$$

of irreducible characters, with complex coefficients $a^G_{\mathrm{disc}}(\pi)$. It is a consequence of Langlands' construction of the discrete spectrum that for any f, the sum may be taken over a finite set. (See [**A14**, Lemmas 4.1 and 4.2].) Let $\Pi_{t,\mathrm{disc}}(G)$ be the subset of irreducible constituents of induced representations

$$\sigma^G_\lambda, \qquad M \in \mathcal{L},\ \sigma \in \Pi_{t,\mathrm{unit}}\big(M(\mathbb{A})^1\big),\ \lambda \in i\mathfrak{a}^*_M / i\mathfrak{a}^*_G,$$

of $G(\mathbb{A})^1$, where the representation σ_λ of $M(\mathbb{A}) \cap G(\mathbb{A})^1$ satisfies the two conditions.

(i) $a^M_{\mathrm{disc}}(\sigma) \neq 0$.

(ii) there is an element $s \in W^G(\mathfrak{a}_M)_{\mathrm{reg}}$ such that $s\sigma_\lambda = \sigma_\lambda$.

As a discrete subset of $\Pi_{t,\mathrm{unit}}\big(G(\mathbb{A})^1\big)$, $\Pi_{t,\mathrm{disc}}(G)$ is a convenient domain for the coefficients $a^G_{\mathrm{disc}}(\pi)$.

It is also useful to introduce a manageable domain of induced representations in $\Pi_{t,\mathrm{unit}}\big(G(\mathbb{A})^1\big)$. We define a set

$$\Pi_t(G) = \{\pi^G_\lambda :\ M \in \mathcal{L},\ \pi \in \Pi_{t,\mathrm{disc}}(M),\ \lambda \in i\mathfrak{a}^*_M / i\mathfrak{a}^*_G\}, \tag{22.6}$$

equipped with the measure $d\pi^G_\lambda$ for which

$$\int_{\Pi_t(G)} \phi(\pi^G_\lambda) \mathrm{d}\pi^G_\lambda = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\pi \in \Pi_{t,\mathrm{disc}}(M)} \int_{i\mathfrak{a}^*_M / i\mathfrak{a}^*_G} \phi(\pi^G_\lambda) \mathrm{d}\lambda, \tag{22.7}$$

for any reasonable function ϕ on $\Pi_t(G)$. If π belongs to a set $\Pi_{t,\mathrm{disc}}(M)$, the global normalizing factors $r_{Q|P}(\pi_\lambda)$ can be defined by analytic continuation of a product (21.14). We can therefore form the (G, M)-family $\{r_Q(\Lambda, \pi_\lambda)\}$ as in (21.15). The associated function $r_M(\pi_\lambda) = r^G_M(\pi_\lambda)$ is analytic in λ, and satisfies the estimate of Lemma 21.5(a). We define a coefficient function on $\Pi_t(G)$ by setting

$$a^G(\pi^G_\lambda) = a^M_{\mathrm{disc}}(\pi) r^G_M(\pi_\lambda), \qquad M \in \mathcal{L},\ \pi \in \Pi_{t,\mathrm{disc}}(M),\ \lambda \in i\mathfrak{a}^*_M / i\mathfrak{a}^*_G. \tag{22.8}$$

It is not hard to show that the right hand side of this expression depends only on the induced representation π^G_λ, at least on the complement of a set of measure 0 in $\Pi_t(G)$.

For any $M \in \mathcal{L}$, we write $\Pi(M)$ for the union over $t \geq 0$ of the sets $\Pi_t(M)$. The analogues of (22.7) and (22.8) for M provide a measure $d\pi$ and a function $a^M(\pi)$ on $\Pi(M)$. Since we have now terminated our relationship with the earlier parameter of truncation, we allow ourselves henceforth to let T stand for a positive real number. With this notation, we write $\Pi(M)^T$ for the union over $t \leq T$ of the sets $\Pi_t(M)$. The refined spectral expansion then takes the form of a limit, as T approaches infinity, of a sum of integrals over the sets $\Pi(M)^T$.

We can now formulate the refined trace formula as an identity between two parallel expansions. We state it as a corollary of the results at the end of §19 and §21.

Corollary 22.1. *For any $f \in \mathcal{H}(G)$, $J(f)$ has a geometric expansion*

$$J(f) = \lim_S \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) J_M(\gamma, f) \tag{22.9}$$

and a spectral expansion

$$J(f) = \lim_T \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M)^T} a^M(\pi) J_M(\pi, f) \mathrm{d}\pi. \tag{22.10}$$

The geometric expansion (22.9) is essentially that of Corollary 19.3, as we noted above. The spectral expansion (22.10) is a straightforward reformulation of the expansion of Corollary 21.7, which is established in the first part of the proof of Theorem 4.4 of [**A14**]. One applies the appropriate analogue of the splitting formula (21.16) to the integral (21.18). This gives an expansion of $J_t(f)$ as a triple sum over Levi subgroups $M \subset L \subset S$ and a simple sum over $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$. One then observes that the sum over M gives rise to a form of the distribution $I^L_{t,\mathrm{disc}}$, for which one can substitute the analogue of (22.5). Having removed the original sum over M, we are free to write M in place of the index S. The expression (21.18) becomes a sum over $M \in \mathcal{L}(L)$ and an integral over $\lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$. The last step is to rewrite the integral as a double integral over the product of $i\mathfrak{a}_L^*/i\mathfrak{a}_M^*$ with $i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$. The spectral expansion (22.10) then follows from the definitions of the linear forms $J_M(\pi, f)$, the coefficients $a^M(\pi)$, and the measure $d\pi$. □

Although the refined trace formula of Corollary 22.1 is a considerable improvement over its predecessor (16.1), it still has defects. There are of course the questions inherent in the two limits. These difficulties were mentioned briefly in §19 (in the remark following Theorem 19.1) and in §21 (at the end of the section). The spectral problem has been solved for $GL(n)$, while the geometric problem is open for any group other than $GL(2)$. Both problems will be relevant to any attempt to exploit the trace formula of G in isolation. However, they seem to have no bearing on our ability to compare trace formulas on different groups. We shall not discuss them further.

Our concern here is with the failure of the linear forms $J_M(\gamma, f)$ and $J_M(\pi, f)$ to be invariant. There is also the disconcerting fact that they depend on a non-canonical choice of maximal compact subgroup K of $G(\mathbb{A})$. Of course, the domain $\mathcal{H}(G)$ of the linear forms already depends on K, through its archimedean component K_∞. However, even when we can extend the linear forms to the larger domain $C_c^\infty\big(G(\mathbb{A})^1\big)$, which we can invariably do in the geometric case, they are still fundamentally dependent on K.

To see why the lack of invariance is a concern, we recall the Jacquet-Langlands correspondence described in §3. Their mapping $\pi \to \pi^*$ of automorphic representations was governed by a correspondence $f \to f^*$ from functions f on the multiplicative group $G(\mathbb{A})$ of an adelic quaternion algebra, and functions f^* on the adelic group $G^*(\mathbb{A})$ attached to $G^* = GL(2)$. The correspondence of functions was defined by identifying invariant orbital integrals. It is expected that for any G, the set of strongly regular invariant orbital integrals spans a dense subspace of the entire space of invariant distributions. (The same is expected of the set of irreducible tempered characters.) We might therefore be able to transfer invariant distributions between suitably related groups. However, we cannot expect to be able to transfer distributions that are not invariant.

The problem is to transform the identity between the expansions (22.9) and (22.10) into a more canonical formula, whose terms are invariant distributions. How can we do this? The first thing to observe is that the weighted orbital integrals in (22.9) and the weighted characters in (22.10) fail to be invariant in a similar way. By the construction of §18, the weighted orbital integrals satisfy the relation

$$J_M(\gamma, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}),$$

for any $f \in C_c^\infty\big(G(\mathbb{A})^1\big)$, $\gamma \in \Gamma(M)_S$, and $y \in G(\mathbb{A})$. A minor technical lacuna arises here when we restrict f to the domain $\mathcal{H}(G)$ of the weighted characters, since the transformation $f \to f^y$ does not send $\mathcal{H}(G)$ to itself. However, the convolutions $L_h f = h * f$ and $R_h f = f * h$ of f by a fixed function $h \in \mathcal{H}(G)$ do preserve $\mathcal{H}(G)$. We define a linear form on $\mathcal{H}(G)$ to be invariant if for any such h it assumes the same values at $L_h f$ and $R_h f$. The relation above is equivalent to a formula

$$J_M(\gamma, L_h f) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, R_{Q,h} f), \tag{22.11}$$

where

$$R_{Q,h} f = \int_{G(\mathbb{A})^1} h(y)(R_{y^{-1}} f)_{Q,h} \mathrm{d}y$$

and $(R_{y^{-1}} f)(x) = f(xy)$, which applies equally well to functions f in either $C_c^\infty\big(G(\mathbb{A})^1\big)$ or $\mathcal{H}(G)$. It is no surprise to discover that the weighted characters satisfy a similar formula, since we know that the original distributions $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ satisfy the parallel variance formulas (16.2) and (16.3). It follows from Lemma 6.2 of [**A15**] that

$$J_M(\pi, L_h f) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\pi, R_{Q,h} f), \tag{22.12}$$

for any $f \in \mathcal{H}(G)$, $\pi \in \Pi(M)$ and $h \in \mathcal{H}(G)$.

We have just seen that the two families of linear forms in the trace formula satisfy parallel variance formulas. It seems entirely plausible that we could construct an invariant distribution by taking a typical noninvariant distribution from one of the two families, and subtracting from it some combination of noninvariant distributions from the other family. Two questions arise. What would be the precise mechanics of the process? At a more philosophical level, should we subtract some combination of weighted characters from a given weighted orbital integral, or should we start with a weighted character and subtract from it some combination of weighted orbital integrals? We shall discuss the second question in the rest of this section, leaving the first question for the beginning of the next section.

Consider the example that $G = GL(2)$, and M is the minimal Levi subgroup $GL(1) \times GL(1)$ of the diagonal matrices. Suppose that $f \in \mathcal{H}(G)$, and that S is a large finite set of valuations. We can then identify f with a function on $\mathcal{H}\big(G(F_S)^1\big)$. The weighted orbital integral $\gamma \to J_M(\gamma, f)$ is a compactly supported, locally integrable function on the group

$$M(F_S)^1 = \{(a,b) \in F_S \times F_S : \ |a| = |b| = 1\}.$$

The weighted character $\pi \to J_M(\pi, f)$ is a Schwartz function on the group $\Pi_{\text{unit}}\big(M(F_S)^1\big)$ of unitary characters on $M(F_S)^1$. We could form the distribution

$$J_M(\gamma, f) - \int_{\Pi_{\text{unit}}(M(F_S)^1)} \pi(\gamma^{-1}) J_M(\pi, f) \mathrm{d}\pi, \qquad \gamma \in M(F_S)^1, \tag{22.13}$$

by modifying the weighted orbital integral. We could also form the distribution

$$J_M(\pi, f) - \int_{M(F_S)^1} \pi(\gamma) J_M(\gamma, f) \mathrm{d}\gamma, \qquad \pi \in \Pi_{\text{unit}}\big(M(F_S)^1\big), \tag{22.14}$$

by modifying the weighted character. According to the variance formulas above, each of these distributions is invariant. Which one should we take?

The terms in the trace formula for $G = GL(2)$ that are not invariant are the ones attached to our minimal Levi subgroup M. They can be written as

$$\frac{1}{2}\mathrm{vol}\big(M(\mathfrak{o}_S)\backslash M(F_S)^1\big) \sum_{\gamma \in M(\mathfrak{o}_S)} J_M(\gamma, f)$$

and

$$\frac{1}{2} \sum_{\pi \in \Pi(M(\mathfrak{o}_S)\backslash M(F_S)^1)} J_M(\pi, f)$$

respectively, for the discrete, cocompact subring

$$\mathfrak{o}_S = \{\gamma \in F : \ |\gamma|_v \le 1, \ v \notin S\}$$

of F_S. Can we apply the Poisson summation formula to either of these expressions? Such an application to the first expression would yield an invariant trace formula for $GL(2)$ with terms of the form (22.14). An application of Poisson summation to the second expression would yield an invariant trace formula with terms of the form (22.13).

We need to be careful. Continuing with the example $G = GL(2)$, suppose that $\widetilde{f}$ lies in the Hecke algebra $\mathcal{H}\big(G(F_S)\big)$ on $G(F_S)$, and consider $J_M(\gamma, \widetilde{f})$ and $J_M(\pi, \widetilde{f})$ as functions on the larger groups $M(F_S)$ and $\Pi_{\text{unit}}\big(M(F_S)\big)$ respectively. The function $J_M(\gamma, \widetilde{f})$ is still compactly supported. However, it has singularities at points γ whose eigenvalues at some place $v \in S$ are equal. Indeed, in the example $v = \mathbb{R}$ examined in §18, we saw that the weighted orbital integral had a logarithmic singularity. If the logarithmic term is removed, the resulting function of γ is bounded, but it still fails to be smooth. Langlands showed that the function was nevertheless well enough behaved to be able to apply the Poisson summation formula. He made the trace formula for $GL(2)$ invariant in this way, using the distributions (22.14) in his proof of base change for $GL(2)$ [**Lan9**]. A particular advantage of this approach is a formulation of the contribution of weighted orbital integrals in terms of a continuous spectral variable, which can be separated from the discrete spectrum. For groups of higher rank, however, the singularities of weighted orbital integrals seem to be quite unmanageable.

The other function $J_M(\pi,\widetilde{f})$ belongs to the Schwartz space on $\Pi_{\mathrm{unit}}\big(M(F_S)\big)$, but it need not lie in the Paley-Wiener space. This is because the operator-valued weight factor

$$\mathcal{R}_M(\pi,P), \qquad \pi\in\Pi_{\mathrm{unit}}\big(M(F_S)\big),$$

is a rational function in the continuous parameters of π, which acquires poles in the complex domain $\Pi\big(M(F_S)\big)$. Therefore $J_M(\pi,\widetilde{f})$ is not the Fourier transform of a compactly supported function on $M(F_S)$. This again does not preclude applying Poisson summation in the case under consideration. However, it does not seem to bode well for higher rank.

What is one to do? I would argue that it is more natural in general to work with the geometric invariant distributions (22.13) than with their spectral counterparts (22.14). Weighted characters satisfy splitting formulas analogous to (18.7). In the example under consideration, the formula is

$$J_M(\pi,\widetilde{f})=\sum_{v\in S}\big(J_M(\pi_v,f_v)\cdot\prod_{w\neq v}f_{w,G}(\pi_w)\big),$$

where $\pi=\bigotimes_{v\in S}\pi_v$ and $\widetilde{f}=\prod_{v\in S}f_v$, and $J_M(\pi_v,f_v)$ is the local weighted character defined by the obvious analogue of (22.3). It follows from this that the Fourier transform

$$J_M^{\wedge}(\gamma,\widetilde{f})=\int_{\Pi_{\mathrm{unit}}(M(F_S))}\pi^{\vee}(\gamma)J_M(\pi,\widetilde{f})d\pi, \qquad \gamma\in M(F_S),$$

of $J_M(\pi,\widetilde{f})$ is equal to a sum of products

$$J_M^{\wedge}(\gamma,\widetilde{f})=\sum_{v\in S}\Big(J_M^{\wedge}(\gamma_v,f_v)\cdot\prod_{w\neq v}f_{w,G}(\gamma_w)\Big),$$

for $\gamma=\prod_{v\in S}\gamma_v$. The invariant orbital integrals $f_{w,G}(\gamma_w)$ are all compactly supported, even though the functions $J_M^{\wedge}(\gamma_v,f_v)$ are not. Remember that we are supposed to take the Poisson summation formula for the diagonal subgroup

$$M(F_S)^1=\Big\{\gamma\in M(F_S):\ H_M(\gamma)=\sum_{v\in S}H_M(\gamma_v)=0\Big\}$$

of $M(F_S)$. The intersection of this subgroup with any set that is a product of a noncompact subset of $M(F_v)$ with compact subsets of each of the complementary groups $M(F_w)$ is compact. It follows that if f belongs to $\mathcal{H}\big(G(F_S)^1\big)$, the weighted character

$$J_M(\pi,f)=\int_{i\mathfrak{a}_M^*}J_M(\pi_\lambda,\widetilde{f})d\lambda, \qquad \pi\in\Pi_{\mathrm{unit}}\big(M(F_S)^1\big),$$

that actually occurs in the trace formula belongs to the Paley-Wiener space on $\Pi_{\mathrm{unit}}\big(M(F_S)^1\big)$ after all.

Suppose now that G is arbitrary. It turns out that the phenomenon we have just described for $GL(2)$ holds in general. The underlying reason again is the fact that the weighted characters occur on the spectral side in the form of integrals (22.4), rather than as a discrete sum of linear forms (22.3). Otherwise said, the fine spectral expansion of Corollary 21.7 is composed of continuous integrals (21.18), while the fine geometric expansion of Corollary 19.3 is given by a discrete sum.

What if it had been the other way around? What if the weighted orbital integrals had occurred on the geometric side in the form of integrals

$$\int_{A^+_{M,\infty}} J_M(\gamma a, \widetilde{f})\mathrm{d}a, \qquad\qquad f \in \mathcal{H}(G),\ \gamma \in \Gamma(M)_S,$$

over the subgroup $A^+_{M,\infty}$ of $M(\mathbb{A})$, with $\widetilde{f}$ now being a function in $\mathcal{H}\big(G(\mathbb{A})\big)$ such that

$$f(x) = \int_{A^+_{M,\infty}} \widetilde{f}(xz)\mathrm{d}z,$$

while the weighted characters had occurred as a discrete sum of distributions (22.4)? It would then have been more natural to work with the general analogues of the spectral invariant distributions (22.14), rather than their geometric counterparts (22.13). Were this the case, we might want to identify $f \in \mathcal{H}(G)$ with a function on the quotient $A^+_{G,\infty}\backslash G(\mathbb{A})$. We would then identify $\Pi(M)$ with a family of representations of $A^+_{M,\infty}\backslash M(\mathbb{A})$. In the example $G = GL(2)$ above, this would lead to an application of the Poisson summation formula to the discrete image of $M(\mathfrak{o}_S)$ in $A^+_{M,\infty}\backslash M(\mathbb{A})$, rather than to the discrete subgroup $M(\mathfrak{o}_S)$ of $M(\mathbb{A})^1$.

These questions are not completely hypothetical. In the local trace formula [**A19**], which we do not have space to discuss here, weighted characters and weighted orbital integrals both occur continuously. One could therefore make the local trace formula invariant in one of two natural ways. One could equally well work with the general analogues of either of the two families (22.13) or (22.14) of invariant distributions.

23. The invariant trace formula

We have settled on trying to make the trace formula invariant by adding combinations of weighted characters to a given weighted orbital integral. We can now focus on the mechanics of the process.

For flexibility, we take S to be any finite set of valuations of F. The trace formula applies to the case that S is large, and contains S_{ram}. In the example of $G = GL(2)$ in §22, the correction term in the invariant distribution (22.13) is a Fourier transform of the function

$$J_M(\pi, f), \qquad\qquad \pi \in \Pi_{\mathrm{unit}}\big(M(F_S)\big).$$

In the general case, M of course need not be abelian. The appropriate analogue of the abelian dual group is not the set $\Pi_{\mathrm{unit}}\big(M(F_S)\big)$ of all unitary representations. It is rather the subset $\Pi_{\mathrm{temp}}\big(M(F_S)\big)$ of representations $\pi \in \Pi_{\mathrm{unit}}\big(M(F_S)\big)$ that are tempered, in the sense that the distributional character $f \to f_G(\pi)$ on $G(F_S)$ extends to a continuous linear form on Harish-Chandra's Schwartz space $\mathcal{C}\big(G(F_S)\big)$. Tempered representations are the spectral ingredients of Harish-Chandra's general theory of local harmonic analysis. They can be characterized as irreducible constituents of representations obtained by unitary induction from discrete series of Levi subgroups.

The tempered characters provide a mapping

$$f \longrightarrow f_G(\pi), \qquad\qquad f \in \mathcal{H}\big(G(F_S)\big),\ \pi \in \Pi_{\mathrm{temp}}\big(G(F_S)\big),$$

from $\mathcal{H}\big(G(F_S)\big)$ onto a space $\mathcal{I}\big(G(F_S)\big)$ of complex-valued functions on $\Pi_{\text{temp}}\big(G(F_S)\big)$. The image of this mapping has been characterized in terms of the internal parameters of $\Pi_{\text{temp}}\big(G(F_S)\big)$ ([**CD**], [**BDK**]). Roughly speaking, $\mathcal{I}\big(G(F_S)\big)$ is the space of all functions in $\Pi_{\text{temp}}\big(G(F_S)\big)$ that have finite support in all discrete parameters, and lie in the relevant Paley-Wiener space in each continuous parameter. Consider a linear form i on $\mathcal{I}\big(G(F_S)\big)$ that is continuous with respect to the natural topology. The corresponding linear form

$$f \longrightarrow i(f_G), \qquad\qquad f \in \mathcal{H}\big(G(F_S)\big),$$

on $\mathcal{H}\big(G(F_S)\big)$ is both continuous and invariant. Conversely, suppose that I is any continuous, invariant linear form on $\mathcal{H}\big(G(F_S)\big)$. We say that I is *supported on characters* if $I(f) = 0$ for any $f \in \mathcal{H}\big(G(F_S)\big)$ with $f_G = 0$. If this is so, there is a continuous linear form $\widehat{I}$ on $\mathcal{I}\big(G(F_S)\big)$ such that

$$\widehat{I}(f_G) = I(f), \qquad\qquad f \in \mathcal{H}\big(G(F_S)\big).$$

We refer to $\widehat{I}$ as the invariant Fourier transform of I. It is believed that every continuous, invariant linear form on $\mathcal{I}\big(G(F_S)\big)$ is supported on characters. This property is known to hold in many cases, but I do not have a comprehensive reference. The point is actually not so important here, since in making the trace formula invariant, one can show directly that the relevant invariant forms are supported on characters.

We want to apply these notions to Levi subgroups M of G. In particular, we use the associated embedding $\widehat{I} \to I$ of distributions as a substitute for the Fourier transform of functions in (22.13). However, we have first to take care of the problem mentioned in the last section. Stated in the language of this section, the problem is that the function

$$\pi \longrightarrow J_M(\pi, f), \qquad\qquad \pi \in \Pi_{\text{temp}}\big(M(F_S)\big),$$

attached to any $f \in \mathcal{H}\big(G(F_S)\big)$ does not generally lie in $\mathcal{I}\big(M(F_S)\big)$. To deal with it, we introduce a variant of the space $\mathcal{I}\big(M(F_S)\big)$.

We shall say that a set S has the closure property if it either contains an archimedean valuation v, or contains only nonarchimedean valuations with a common residual characteristic. We assume until further notice that S has this property. The image

$$\mathfrak{a}_{G,S} = H_G\big(G(F_S)\big)$$

of $G(F_S)$ in $\mathfrak{a}_G$ is then a closed subgroup of $\mathfrak{a}_G$. It equals $\mathfrak{a}_G$ if S contains an archimedean place, and is a lattice in $\mathfrak{a}_G$ otherwise. In spectral terms, the action $\pi \to \pi_\lambda$ of $i\mathfrak{a}_G^*$ on $\Pi_{\text{temp}}\big(G(F_S)\big)$ lifts to the quotient

$$i\mathfrak{a}_{G,S}^* = i(\mathfrak{a}_G^*/\mathfrak{a}_{G,S}^\vee), \qquad\qquad \mathfrak{a}_{G,S}^\vee = \text{Hom}(\mathfrak{a}_{G,S}, 2\pi\mathbb{Z}),$$

of $i\mathfrak{a}_G^*$. If ϕ belongs to $\mathcal{I}\big(G(F_S)\big)$, we set

$$\phi(\pi, Z) = \int_{i\mathfrak{a}_{G,S}^*} \phi(\pi_\lambda)e^{-\lambda(Z)}d\lambda \qquad\qquad \pi \in \Pi_{\text{temp}}\big(G(F_S)\big),\ Z \in \mathfrak{a}_{G,S}.$$

This allows us to identify $\mathcal{I}\big(G(F_S)\big)$ with a space of functions ϕ on $\Pi_{\text{temp}}\big(G(F_S)\big) \times \mathfrak{a}_{G,S}$ such that

$$\phi(\pi_\lambda, Z) = e^{\lambda(Z)}\phi(\pi, Z).$$

If f belongs to $\mathcal{H}\big(G(F_S)\big)$, we have

$$f_G(\pi, Z) = \mathrm{tr}\big(\pi(f^Z)\big) = \mathrm{tr}\Big(\int_{G(F_S)^Z} f(x)\pi(x)\mathrm{d}x\Big),$$

where f^Z is the restriction of f to the closed subset

$$G(F_S)^Z = \{x \in G(F_S) : \ H_G(x) = Z\}$$

of $G(F_S)$. In particular, $f_G(\pi, 0)$ is the character of the restriction of π to the subgroup $G(F_S)^1$ of $G(F_S)$.

We use the interpretation of $\mathcal{I}\big(G(F_S)\big)$ as a space of functions on $G(F_S) \times \mathfrak{a}_{G,S}$ to define a larger space $\mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big)$. It is clear that

$$\mathcal{I}\big(G(F_S)\big) = \varinjlim_{\Gamma} \mathcal{I}\big(G(F_S)\big)_\Gamma,$$

where Γ ranges over finite sets of irreducible representations of the compact group $K_S = \prod_{v\in S} K_v$, and $\mathcal{I}\big(G(F_S)\big)_\Gamma$ is the space of functions $\phi \in \mathcal{I}\big(G(F_S)\big)$ such that $\phi(\pi, Z)$ vanishes for any $\pi \in \Pi_{\mathrm{temp}}\big(G(F_S)\big)$ whose restriction to K_S does not contain some representation in Γ. For any Γ, we define $\mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big)_\Gamma$ to be the space of functions ϕ on $G(F_S) \times \mathfrak{a}_{G,S}$ with the property that for any function $b \in C_c^\infty(\mathfrak{a}_{G,S})$, the product

$$\phi(\pi, Z)b(Z), \qquad \pi \in \Pi_{\mathrm{temp}}\big(G(F_S)\big),\ Z \in \mathfrak{a}_{G,S},$$

lies in $\mathcal{I}\big(G(F_S)\big)_\Gamma$. We then set

$$\mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big) = \varinjlim_{\Gamma} \mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big)_\Gamma.$$

It is also clear that

$$\mathcal{H}\big(G(F_S)\big) = \varinjlim_{\Gamma} \mathcal{H}\big(G(F_S)\big)_\Gamma,$$

where $\mathcal{H}\big(G(F_S)\big)_\Gamma$ is the subspace of functions in $\mathcal{H}\big(G(F_S)\big)$ that transform on each side under K_S according to representations in Γ. We define $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)_\Gamma$ to be the space of functions f on $G(F_S)$ such that each product

$$f(x)b\big(H_G(x)\big), \qquad x \in G(F_S),\ b \in C_c^\infty(\mathfrak{a}_{G,S}),$$

belongs to $\mathcal{H}\big(G(F_S)\big)_\Gamma$. We then set

$$\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big) = \varinjlim_{\Gamma} \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)_\Gamma.$$

The functions $f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$ thus have "almost compact support", in the sense that f^Z has compact support for any $Z \in \mathfrak{a}_{G,S}$. If f belongs to $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$, we set

$$f_G(\pi, Z) = \mathrm{tr}\big(\pi(f^Z)\big), \qquad \pi \in \Pi_{\mathrm{temp}}\big(G(F_S)\big),\ Z \in \mathfrak{a}_{G,S}.$$

Then $f \to f_G$ is a continuous linear mapping from $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$ onto $\mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big)$. The mapping $I \to \widehat{I}$ can obviously be extended to an isomorphism from the space of continuous linear forms on $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$ that are supported on characters, and the space of continuous linear forms on $\mathcal{I}_{\mathrm{ac}}\big(G(F_S)\big)$.

Having completed these preliminary remarks, we are now in a position to interpret the set of weighted characters attached to M as a transform of functions. Suppose that $f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$. We first attach a general meromorphic function

$$J_M(\pi_\lambda, f^Z) = \mathrm{tr}\big(\mathcal{R}_M(\pi_\lambda, P)\mathcal{I}_P(\pi_\lambda, f^Z)\big), \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, \tag{23.1}$$

to any $M \in \mathcal{L}$, $\pi \in \Pi\big(M(F_S)\big)$ and $Z \in \mathfrak{a}_{G,S}$. We can then attach a natural linear form $J_M(\pi, X, f)$ to any $X \in \mathfrak{a}_{M,S}$. For example, if $J_M(\pi_\lambda, f^Z)$ is analytic for $\lambda \in i\mathfrak{a}^*_M$, we set

$$J_M(\pi, X, f) = \int_{i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}} J_M(\pi_\lambda, f^Z)\mathrm{e}^{-\lambda(X)}\mathrm{d}\lambda, \tag{23.2}$$

where Z is the image of X in $\mathfrak{a}_{G,S}$. (In general, one must take a linear combination of integrals over contours $\varepsilon_P + i\mathfrak{a}^*_{M,S}/i\mathfrak{a}^*_{G,S}$, for groups $P \in \mathcal{P}(M)$ and small points $\varepsilon_P \in (\mathfrak{a}^*_M)^+_P$. See [**A15**, §7].) The premise underlying (23.2) holds if π is unitary. If in addition, $S \supset S_{\mathrm{ram}}$ and $X = 0$, (23.2) reduces to the earlier definition (22.4). Our transform is given by the special case that π belongs to the subset $\Pi_{\mathrm{temp}}\big(M(F_S)\big)$ of $\Pi_{\mathrm{unit}}\big(M(F_S)\big)$. We define $\phi_M(f)$ to be the function

$$(\pi, X) \longrightarrow \phi_M(f, \pi, X) = J_M(\pi, X, f), \qquad \pi \in \Pi_{\mathrm{temp}}\big(M(F_S)\big),\ X \in \mathfrak{a}_{M,S},$$

on $\Pi_{\mathrm{temp}}\big(M(F_S)\big) \times \mathfrak{a}_{M,S}$.

PROPOSITION 23.1. *The mapping*

$$f \longrightarrow \phi_M(f), \qquad f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big),$$

is a continuous linear transformation from $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$ *to* $\mathcal{I}_{\mathrm{ac}}\big(M(F_S)\big)$.

This is Theorem 12.1 of [**A15**]. The proof in [**A15**] is based on a study of the residues of the meromorphic functions

$$\lambda \longrightarrow J_M(\pi_\lambda, f^Z), \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},\ \pi \in \Pi_{\mathrm{temp}}\big(M(F_S)\big).$$

A somewhat simpler proof is implicit in the results of [**A13**]. (See the remark on p. 370 of [**A13**].) It is based on the splitting and descent formulas for the functions (23.1), which are parallel to (18.7) and (18.8), and are consequences of Lemmas 17.5 and 17.6. These formulas in turn yield splitting and descent formulas for the linear forms (23.2), and consequently, for the functions $\phi_M(f, \pi, X)$. They reduce the problem to the special case that S contains one element v, M is replaced by a Levi subgroup M_v over F_v, and π is replaced by a tempered representation π_v of $M_v(F_v)$ that is not properly induced. The family of such representations can be parametrized by a set that is discrete modulo the action of the connected group $i\mathfrak{a}^*_{M_v,F_v} = i\mathfrak{a}^*_{M_v,\{v\}}$. The proposition can then be established from the definition of $\mathcal{I}_{\mathrm{ac}}\big(M(F_S)\big)$. □

It is the mappings ϕ_M that allow us to transform the various noninvariant linear forms to invariant forms. We state the construction as a pair of parallel theorems, to be followed by an extended series of remarks. The first theorem describes the general analogues of the invariant linear forms (22.13). The second theorem describes associated spectral objects. Both theorems apply to a fixed finite set of valuations S with the closure property, and a Levi subgroup $M \in \mathcal{L}$.

THEOREM 23.2. *There are invariant linear forms*

$$I_M(\gamma, f) = I_M^G(\gamma, f), \qquad \gamma \in M(F_S),\ f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big),$$

that are supported on characters, and satisfy

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big). \tag{23.3}$$

THEOREM 23.3. *There are invariant linear forms*

$$I_M(\pi, X, f) = I_M^G(\pi, X, f), \qquad \pi \in \Pi\big(M(F_S)\big),\ X \in \mathfrak{a}_{M,S},\ f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big),$$

that are supported on characters, and satisfy

$$I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\pi, X, \phi_L(f)\big). \tag{23.4}$$

Remarks. 1. In the special case that G equals $GL(2)$, M is minimal, and S contains the set $S_{\mathrm{ram}} = S_\infty$, the right hand side of (23.3) reduces to the original expression (22.13). For in this case, the value of $\phi_M(\pi, X, f)$ at $X = 0$ equals the function $J_M(\pi, f)$ in (22.13). Since the linear form $I_M^M(\gamma)$ in this case is just the evaluation map of a function on $M(F_S)^1$ at γ, $\widehat{I}_M^M\big(\gamma, \phi_M(f)\big)$ reduces to the integral in (23.13) by the Fourier inversion formula for the abelian group $M(F_S)^1$.

2. The formulas (23.3) and (23.4) amount to inductive definitions of $I_M(\gamma, f)$ and $I_M(\pi, X, f)$. We need to know that these linear forms are supported on characters in order that the summands on the right hand sides of the two formulas be defined.

3. The theorems give nothing new in the case that $M = G$ and $X = Z$. For it follows immediately from the definitions that

$$I_G(\gamma, f) = J_G(\gamma, f) = f_G(\gamma)$$

and

$$I_G(\pi, Z, f) = J_G(\pi, Z, f) = f_G(\pi, Z).$$

4. The linear forms $I_M(\gamma, f)$ of Theorem 23.2 are really the primary objects. We see inductively from (23.3) that $I_M(\gamma, f)$ depends only on f^Z, where $Z = H_G(\gamma)$. In particular, $I_M(\gamma, f)$ is determined by its restriction to the subspace $\mathcal{H}\big(G(F_S)\big)$ of $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$. One can in fact show that as a continuous linear form on $\mathcal{H}\big(G(F_S)\big)$, $I_M(\gamma, f)$ extends continuously to the Schwartz space $\mathcal{C}\big(G(F_S)\big)$ [**A21**]. In other words, $I_M(\gamma, f)$ is a tempered distribution. It has an independent role in local harmonic analysis.

5. The linear forms $I_M(\pi, X, f)$ of Theorem 23.3 are secondary objects, but they are still interesting. We see inductively from (23.4) that $I_M(\pi, X, f)$ depends only on f^Z, where Z is the image of X in $\mathfrak{a}_{G,S}$, so $I_M(\pi, X, f)$ is also determined by its restriction to $\mathcal{H}\big(G(F_S)\big)$. However, it is not a tempered distribution. If π is tempered,

$$J_M(\pi, X, f) = \phi_M(f, \pi, X) = \widehat{I}_M^M\big(\pi, X, \phi_M(f)\big),$$

by definition. It follows inductively from (23.4) that

$$I_M(\pi, X, f) = \begin{cases} f_G(\pi, Z), & \text{if } M = G, \\ 0, & \text{otherwise,} \end{cases} \tag{23.5}$$

in this case. But if π is nontempered, $I_M(\pi, X, f)$ is considerably more complicated. Suppose for example that G is semisimple, M is maximal, $F = \mathbb{Q}$, $S = \{v_\infty\}$, and $\pi = \sigma_\mu$, for $\sigma \in \Pi_{\text{temp}}\big(M(\mathbb{R})\big)$ and $\mu \in \mathfrak{a}^*_{M,\mathbb{C}}$. We assume that $\text{Re}(\mu)$ is in general position. Then

$$J_M(\pi, X, f) = \int_{i\mathfrak{a}^*_M} J_M(\sigma_{\mu+\lambda}, f)\mathrm{e}^{-\lambda(X)}\mathrm{d}\lambda,$$

while

$$\phi_M(f, \pi, X) = \mathrm{e}^{\mu(X)} \int_{i\mathfrak{a}^*_M} J_M(\sigma_\lambda, f)\mathrm{e}^{-\lambda(X)}\mathrm{d}\lambda.$$

It follows that $I_M(\pi, X, f)$ is the finite sum of residues

$$\sum_\eta \operatorname*{Res}_{\Lambda=\eta} \big(J_M(\sigma_\Lambda, f)\mathrm{e}^{(\mu-\Lambda)(X)}\big),$$

obtained in deforming one contour of integration to the other. In general, $I_M(\pi, X, f)$ is a more elaborate combination of residues of general functions $J^{L_2}_{L_1}(\sigma_\Lambda, f)$.

6. The linear forms $J_M(\gamma, f)$ and $J_M(\pi, X, f)$ are strongly dependent on the choice of maximal compact subgroup $K_S = \prod_{v\in S} K_v$ of $G(F_S)$. However, it turns out that the invariant forms $I_M(\gamma, f)$ and $I_M(\pi, X, f)$ are independent of K_S. The proof of this fact is closely related to that of invariance, which we will discuss presently. (See [**A24**, Lemma 3.4].) The invariant linear forms are thus canonical objects, even though their construction is quite indirect.

7. The trace formula concerns the case that $S \supset S_{\text{ram}}$, $\gamma \in M(F_S)^1$, and $X = 0$. In this case, the summands corresponding to L in (23.3) and (23.4) depend only on the image of $\phi_L(f)$ in the invariant Hecke algebra $\mathcal{I}\big(L(F_S)^1\big)$ on $L(F_S)^1$. We can therefore take f to be a function in $\mathcal{H}\big(G(F_S)^1\big)$, and treat ϕ_M as the mapping from $\mathcal{H}\big(G(F_S)^1\big)$ to $\mathcal{I}\big(L(F_S)^1\big)$ implicit in Proposition 23.1. In fact, since these spaces both embed in the corresponding adelic spaces, we can take f to be a function in the space $\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})^1\big)$, and ϕ_M to be a mapping from $\mathcal{H}(G)$ to the adelic space $\mathcal{I}(M) = \mathcal{I}\big(M(\mathbb{A})^1\big)$. This is of course the setting of the invariant trace formula. Recall that on the geometric side, γ represents a class in the subset $\Gamma(M)_S$ of conjugacy classes in $M(F_S)$. We write

$$I_M(\gamma, f) = I_M(\gamma, f_S) \tag{23.6}$$

as before, where f_S is the restriction of f to the subgroup $G(F_S)^1$ of $G(\mathbb{A})^1$. On the spectral side, π is a representation in the subset $\Pi(M)$ of $\Pi_{\text{unit}}\big(M(\mathbb{A})^1\big)$. In this case, we write

$$I_M(\pi, f) = I_M(\pi_S, 0, f_S), \tag{23.7}$$

where $S \supset S_{\text{ram}}$ is any finite set outside of which both f and π are unramified, and $\pi_S \in \Pi_{\text{unit}}\big(M(F_S)^1\big)$ is the $M(F_S)^1$-component of π, or rather a representative in $\Pi_{\text{unit}}\big(M(\mathbb{A})\big)$ of that component.

8. The distributions $I_M(\gamma, f)$ satisfy splitting and descent formulas. We have

$$I_M(\gamma, f) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2)\widehat{I}^{L_1}_M(\gamma_1, f_{1,L_1})\widehat{I}^{L_2}_M(\gamma_2, f_{2,L_2}), \tag{23.8}$$

and

$$I_M(\gamma^M_v, f_v) = \sum_{L_v \in \mathcal{L}(M_v)} d^G_{M_v}(M, L_v)\widehat{I}^{L_v}_{M_v}(\gamma_v, f_{v,L_v}), \tag{23.9}$$

under the respective conditions of (18.7) and (18.8). (In (23.8), we of course have also to ask each of the two subsets S_1 and S_2 of S satisfy the closure property.) The formulas are established from the inductive definition (23.3), the formulas (18.7) and (18.8), and corresponding formulas for the functions $J_M(\pi_\lambda, f)$. (See [**A13**, Proposition 9.1 and Corollary 8.2]. If f belongs to $\mathcal{H}\big(G(F_S)\big)$ and $L \in \mathcal{L}(M)$, f_L is the function

$$\pi \longrightarrow f_L(\pi) = f_G(\pi^G), \qquad \pi \in \Pi_{\mathrm{temp}}\big(L(F_S)\big),$$

in $\mathcal{I}\big(L(F_S)\big)$. It is the image in $\mathcal{I}\big(L(F_S)\big)$ of any of the functions $f_Q \in \mathcal{H}\big(G(F_S)\big)$, but is independent of the choice of $Q \in \mathcal{P}(L)$.) The linear forms $J_M(\pi, X, f)$ satisfy their own splitting and descent formulas. Since these are slightly more complicated to state, we simply refer the reader to [**A13**, Proposition 9.4 and Corollary 8.5]. One often needs to apply the splitting and descent formulas to the linear forms (23.6) and (23.7) that are relevant to the trace formula. This is why one has to formulate the definitions in terms of spaces $\mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$ and $\mathcal{I}_{\mathrm{ac}}\big(L(F_S)\big)$, for general sets S, even though the objects (23.6) and (23.7) can be constructed in terms of the simpler spaces $\mathcal{H}(G)$ and $\mathcal{I}(L)$.

The two theorems are really just definitions, apart from the assertions that the linear forms are supported on characters. These assertions can be established globally, by exploiting the invariant trace formula of which they are the terms. In so doing, one discovers relations between the linear forms (23.3) and (23.4) that are essential for comparing traces on different groups. We shall therefore state the invariant trace formula as a third theorem, which is proved at the same time as the other two.

The invariant trace formula is completely parallel to the refined noninvariant formula of Corollary 22.1. It consists of two different expansions of a linear form $I(f) = I^G(f)$ on $\mathcal{H}(G)$ that is the invariant analogue of the original form $J(f)$. We assume inductively that for any $L \in \mathcal{L}$ with $L \neq G$, I^L has been defined, and is supported on characters. We can then define $I(f)$ inductively in terms of $J(f)$ by setting

$$I(f) = J(f) - \sum_{\substack{L \in \mathcal{L} \\ L \neq G}} |W_0^L||W_0^G|^{-1} \widehat{I}^L\big(\phi_L(f)\big), \qquad f \in \mathcal{H}(G). \tag{23.10}$$

The (refined) invariant trace formula is then stated as follows.

THEOREM 23.4. *For any $f \in \mathcal{H}(G)$, $I(f)$ has a geometric expansion*

$$I(f) = \lim_S \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) I_M(\gamma, f), \tag{23.11}$$

and a spectral expansion

$$I(f) = \lim_T \sum_{M \in S} |W_0^M||W_0^G|^{-1} \int_{\Pi(M)^T} a^M(\pi) I_M(\pi, f) d\pi. \tag{23.12}$$

Remarks. 9. The limit in (23.11) stabilizes for large S. Moreover, for any such S, the corresponding sums over γ can be taken over finite sets. One can in fact be more precise. Suppose that f belongs to the subspace $\mathcal{H}\big(G(F_V)^1\big)$ of $\mathcal{H}(G)$, for some finite set $V \supset S_{\mathrm{ram}}$, and is supported on a compact subset Δ of $G(\mathbb{A})^1$. Then the double sum in (23.11) is independent of S, so long as S is large in a sense that depends only on V and Δ. Moreover, for any such S, each sum over

γ can be taken over a finite set that depends only on V and Δ. These facts can be established by induction from the corresponding properties of the noninvariant geometric expansion (22.9). Alternatively, they can be established directly from [**A14**, Lemma 3.2], as on p. 513 of [**A14**].

10. For any T, the integral in (23.12) converges absolutely. This follows by induction from the corresponding property of the noninvariant spectral expansion (22.10). There is a weak quantitative estimate for the convergence of the limit, which is to say the convergence of the sum

$$I(f) = \sum_{t \geq 0} I_t(f)$$

of the linear forms

$$I_t(f) = I_t^G(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t(M)} a^M(\pi) I_M(\pi, f) \mathrm{d}\pi,$$

in terms of the multipliers of §20. For any $r \geq 0$, set

$$\mathfrak{h}_u^*(r, T) = \{\nu \in \mathfrak{h}_u^* : \ \|\mathrm{Re}(\nu)\| \leq r, \ \|\mathrm{Im}\, \nu\| \geq T\},$$

where $\mathfrak{h}_u^*$ is a subset of $\mathfrak{h}_{\mathbb{C}}^*/i\mathfrak{a}_G^*$ that is defined as on p. 536 of [**A14**], and contains the infinitesimal characters of all unitary representations of $G(F_\infty)^1$. Then for any $f \in \mathcal{H}(G)$, there are positive constants C, k and r with the following property. For any positive numbers T and N, and any α in the subspace

$$C_N^\infty(\mathfrak{h}^1)^W = \{\alpha \in C_c^\infty(\mathfrak{h}^1)^W : \ \|\mathrm{supp}\, \alpha\| \leq N\}$$

of $\mathcal{E}(\mathfrak{h}^1)^W$, the estimate

$$\sum_{t > T} |I_t(f_\alpha)| \leq C \mathrm{e}^{kT} \sup_{\nu \in \mathfrak{h}_u^*(r,T)} \big(|\widehat{\alpha}(\nu)| \big) \tag{23.13}$$

holds. (See [**A14**, Lemma 6.3].) This "weak multiplier estimate" serves as a substitute for the absolute convergence of the spectral expansion. It is critical for applications.

As we noted above, the three theorems are proved together. We assume inductively that they all hold if G is replaced by a proper Levi subgroup L.

It is easy to establish that the various linear forms are invariant. Fix S and M as in the first two theorems, and let h be any function in $\mathcal{H}\big(G(F_S)\big)$. It follows easily from (22.12) that

$$\phi_L(L_h f) = \sum_{Q \in \mathcal{F}(L)} \phi_L^{M_Q}(R_{Q,h} f), \qquad f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big),$$

for any $L \in \mathcal{L}(M)$. It then follows from (22.11) and the definition (23.3) that

$$\begin{aligned} & I_M(\gamma, L_h f) \\ &= \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\gamma, R_{Q,h} f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \sum_{Q \in \mathcal{F}(L)} \widehat{I}_M^L\big(\gamma, \phi_L^{M_Q}(R_{Q,h} f)\big) \\ &= \sum_{Q \in \mathcal{F}(M)} \Big(J_M^{M_Q}(\gamma, R_{Q,h} f) - \sum_{\substack{L \in \mathcal{L}^{M_Q}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L^{M_Q}(R_{Q,h} f)\big) \Big), \end{aligned}$$

for any element $\gamma \in M(F_S)$. If $Q \neq G$, the associated summand can be written

$$J_M^{M_Q}(\gamma, R_{Q,h}f) - \sum_{L \in \mathcal{L}^{M_Q}(M)} \widehat{I}_M^L\big(\gamma, \phi_L^{M_Q}(R_{Q,h}f)\big)$$
$$= \Big(J_M^{M_Q}(\gamma, R_{Q,h}f) - \sum_{\substack{L \in \mathcal{L}^{M_Q}(M) \\ L \neq M_Q}} \widehat{I}_M^L\big(\gamma, \phi_L^{M_Q}(R_{Q,h}f)\big)\Big) - I_M^{M_Q}(\gamma, R_{Q,h}f).$$

It therefore vanishes by (23.3). If $Q = G$, the corresponding summand equals

$$I_M^G(\gamma, R_{G,h}f) = I_M(\gamma, R_h f),$$

again by (23.3). Therefore $I_M(\gamma, L_h f)$ equals $I_M(\gamma, R_h f)$. It follows that $I_M(\gamma, \cdot)$ is an invariant distribution. Similarly, $I_M(\pi, X, \cdot)$ is an invariant linear form for any $\pi \in \Pi\big(M(F_S)\big)$ and $X \in \mathfrak{a}_{M,S}$. A minor variant of the argument establishes that the linear form I in (23.10) is invariant as well.

It is also easy to establish the required expansions of Theorem 23.4. To derive the geometric expansion (23.11), we apply what we already know to the terms on the right hand side of the definition (23.10). That is, we substitute the geometric expansion (22.9) for $J(f)$, and we apply (23.11) inductively to the summand $\widehat{I}^L\big(\phi_L(f)\big)$ attached to any $L \neq G$. We see that $I(f)$ equals the difference between the expressions

$$\lim_S \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) J_M(\gamma, f)$$

and

$$\lim_S \sum_{L \neq G} |W_0^L||W_0^G|^{-1} \sum_{M \in \mathcal{L}^L} |W_0^M||W_0^L|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) \widehat{I}_M^L\big(\gamma, \phi_L(f)\big).$$

The second expression can be written as

$$\lim_S \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big).$$

Therefore $I(f)$ equals

$$\lim_S \sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) \Big(J_M(\gamma, f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big)\Big)$$
$$= \lim_S \sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) I_M(\gamma, f),$$

by (23.3). This is the required geometric expansion (23.11). An identical argument yields the spectral expansion (23.12).

We have established the required expansions of Theorem 23.4. We have also shown that the terms in the expansions are invariant linear forms. The identity between the two expansions can thus be regarded as an invariant trace formula. If we knew that any invariant linear form was supported on characters, the inductive definitions of Theorem 23.2 and Theorem 23.3 would be complete, and we would be finished. Lacking such knowledge, we use the invariant trace formula to establish the property directly for the specific invariant linear forms in question.

Proposition 23.5. *The linear forms of Theorem 23.3 can be expressed in terms of those of Theorem 23.2. In particular, if the linear forms* $\{I_M(\gamma)\}$ *are all supported on characters, so are the linear forms* $\{I_M(\pi, X)\}$.

The first assertion of the proposition might be more informative if it contained the phrase "in principle", since the algorithm is quite complicated. It is based on the fact that the various residues that determine the linear forms $\{I_M(\pi, X)\}$ are themselves determined by the asymptotic values in γ of the linear forms $\{I_M(\gamma)\}$. We shall be content to illustrate the idea in a very special case.

Suppose that $G = SL(2)$, and that M is minimal. Since M is also maximal, the observations of Remark 5 above are relevant. Assume then that $F = \mathbb{Q}$, $S = \{v_\infty\}$, and $\pi = \sigma_\mu$, as earlier. For simplicity, we assume also that $f \in \mathcal{H}\bigl(G(\mathbb{R})\bigr)$ is invariant under the central element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and that σ is the trivial representation of $M(\mathbb{R})$. It then follows from Remark 5 that for any $X \in \mathfrak{a}_M$, $I_M(\pi, X, f)$ equals the sum of residues of the function

$$\Lambda \longrightarrow \bigl(J_M(\sigma_\Lambda, f)\mathrm{e}^{(\mu-\Lambda)(X)}\bigr) \tag{23.14}$$

obtained in deforming a contour of integration from $(\mu + i\mathfrak{a}_M^*)$ to $i\mathfrak{a}_M^*$.

On the other hand,

$$\begin{aligned} I_M(\gamma, f) &= J_M(\gamma, f) - \widehat{I}_M^M\bigl(\gamma, \phi_M(f)\bigr) \\ &= J_M(\gamma, f) - \int_{i\mathfrak{a}_M^*} J_M(\sigma_\lambda, f)\mathrm{e}^{-\lambda(H_M(\gamma))}\mathrm{d}\lambda, \end{aligned}$$

for any $\gamma \in M(\mathbb{R})$. Given X, we choose γ so that $H_M(\gamma) = X$. Since f is compactly supported, $J_M(\gamma, f)$ is compactly supported in X. However, the integral over $i\mathfrak{a}_M^*$ is not generally compactly supported in X, since its inverse transform $\lambda \to J_M(\sigma_\lambda, f)$ can have poles in the complex domain. Therefore $I_M(\gamma, f)$ need not be compactly supported in X. In fact, it is the failure of $I_M(\gamma, f)$ to have compact support that determines the residues of the function (23.14). For if we apply the proof of the classical Paley-Wiener theorem to the integral over $i\mathfrak{a}_M^*$, we see that the family of functions

$$\gamma \longrightarrow I_M(\gamma_1\gamma, f), \qquad \gamma_1 \in C,$$

in which C is a suitable compact subset of $M(\mathbb{R})$ and γ is large relative to C and f, spans a finite dimensional vector space. Moreover, it is easy to see that this space is canonically isomorphic to the space of functions of X spanned by the space of residues of (23.14). It follows that the distributions $I_M(\gamma, f)$ determine the residues (23.14), and hence the linear forms $I_M(\pi, X, f)$. In particular, if $I_M(\gamma, f)$ vanishes for all such γ, then $I_M(\pi, X, f)$ vanishes for all X. Applied to the case that $f_G = 0$, this gives the second assertion of the proposition in the special case under consideration.

For general G and M, the ideas are similar, but the details are considerably more elaborate. When the dimension of $\mathfrak{a}_M/\mathfrak{a}_G$ is greater than 1, we have to be concerned with partial residues and with functions whose support is compact in various directions. These are best handled with the supplementary mappings and linear forms of [**A13**, §4]. The first assertion of the proposition is implicit in the results of [**A13**, §4–5]. The second assertion is part of Theorem 6.1 of [**A13**]. □

It remains to show that the distributions of Theorem 23.2 are supported on characters. From the splitting formula (23.8), one sees easily that it is enough to treat the case that S contains one valuation v. We therefore fix a function $f_v \in \mathcal{H}\big(G(F_v)\big)$ with $f_{v,G} = 0$. The problem is to show that $I_M(\gamma_v, f_v) = 0$, for any $M \in \mathcal{L}$ and $\gamma_v \in M(F_v)$. How can we use the invariant trace formula to do this? We begin by choosing an arbitrary function $f^v \in \mathcal{H}\big(G(\mathbb{A}^v)\big)$ and letting f be the restriction of $f_v f^v$ to $G(\mathbb{A})^1$. We have then to isolate the corresponding geometric expansion (23.11) in the invariant trace formula. But how is this possible, when our control of the spectral side provided by Proposition 23.5 requires an a priori knowledge of the terms on the geometric side?

The point is that the terms on the spectral side are not arbitrary members of the family defined by Theorem 23.3. They are of the form $I_M(\pi, 0, f)$, where $S \supset S_{\mathrm{ram}}$ is large enough that f belongs to $\mathcal{H}\big(G(F_S)^1\big)$, and $\pi \in \Pi_{\mathrm{unit}}\big(M(F_S)\big)$. We need to show only that these terms vanish. Combining an induction argument with the splitting formula [**A13**, Proposition 9.4], one reduces the problem to showing that $I_M(\pi_v, X_v, f_v)$ vanishes for any $\pi_v \in \Pi_{\mathrm{unit}}\big(M(F_v)\big)$ and $X_v \in \mathfrak{a}_{M_v,F_v}$. The fact that π_v is unitary is critical. The representation need not be tempered, but within the Grothendieck group it can be expressed as an integral linear combination of induced (standard) representations

$$\sigma^M_{v,\Lambda}, \qquad \sigma_v \in \Pi_{\mathrm{temp}}(M_v),\ \Lambda \in (\mathfrak{a}^M_{M_v})^*,$$

for Levi subgroups M_v of M over F_v. If $M_v = M$, Λ equals 0, and

$$I_M(\sigma^M_{v,\Lambda}, X_v, f_v) = I_M(\sigma_v, X_v, f_v) = 0,$$

by (23.5). If $M_v \neq M$, we use the descent formula [**A13**, Corollary 8.5] to write $I_M(\sigma^M_{v,\Lambda}, X_v, f_v)$ in terms of linear forms

$$\widehat{I}^{L_v}_{M_v}(\sigma_{v,\Lambda}, Y_v, f_{v,L_v}), \qquad Y_v \in \mathfrak{a}_{M_v,F_v},$$

for Levi subgroups $L_v \in \mathcal{L}(M_v)$ with $L_v \neq G$. It follows from Proposition 23.5 and our induction hypotheses that $I_M(\sigma^M_{v,\Lambda}, X_v, f_v)$ again equals 0. Therefore $I_M(\pi_v, X_v, f_v)$ vanishes, and so therefore do the integrands on the spectral side.

We conclude that for the given function f, the spectral expansion (23.12) of $I(f)$ vanishes. Therefore the geometric expansion (23.11) of $I(f)$ also vanishes. In dealing with the distributions $I_M(\gamma, f)$ in this expansion, we are free to apply the splitting formula (23.8) recursively to the valuations $v \in S$. If $L \in \mathcal{L}(M)$ is a proper Levi subgroup of G, the induction hypotheses imply that $\widehat{I}^L_M(\gamma_v, f_v)$ vanishes for any element $\gamma_v \in M(F_v)$. It follows that

$$I_M(\gamma, f) = I_M(\gamma_v, f_v) f^v_M(\gamma^v), \qquad \gamma \in \Gamma(M)_S,$$

where $\gamma = \gamma_v\gamma^v$ is the decomposition of γ relative to the product

$$M(F_S) = M(F_v) \times M(F^v_S).$$

Therefore

$$\lim_S \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) I_M(\gamma_v, f_v) f^v_M(\gamma^v) = 0. \tag{23.15}$$

We are attempting to show that $I_M(\gamma_v, f_v) = 0$, for any $M \in \mathcal{L}$ and $\gamma_v \in M(F_v)$. The definition (18.12) reduces the problem to the case that $M_{\gamma_v} = G_{\gamma_v}$. A further reduction based on invariant orbital integrals on $M(F_v)$ allows us to assume that γ_v is strongly G-regular, in the sense that its centralizer in G is a maximal torus T_v. Finally, in view of the descent formula (23.9), we can assume that T_v is elliptic in M over F_v, which is to say that T_v lies in no proper Levi subgroup of M over F_v. The problem is of course local. To solve it, one should really start with objects G_1, M_1, and T_1 over a local field F_1, together with a function $f_1 \in \mathcal{H}\big(G_1(F_1)\big)$ such that $f_{1,G_1} = 0$. One then chooses global objects F, G, M, and T such that $F_1 = F_v$, $G_1 = G_v$, $M_1 = M_v$, and $T_1 = T_v$ for some valuation v of F, as for example on p. 526 of [**A14**]. Among the general constraints on the choice of G, M, and T is a condition that $T(F)$ be dense in $T(F_v)$. This reduces the problem to showing that $I_M(\delta, f_v)$ vanishes for any G-regular element $\delta \in T(F)$.

We can now sketch the proof of the remaining global argument. To exploit the identity (23.15), we have to allow the complementary function $f^v \in \mathcal{H}\big(G(\mathbb{A}^v)\big)$ to vary. We first fix a large finite set V of valuations containing v, outside of which G and T are unramified. We then restrict f^v to functions of the form $f^v_V f^V$, with f^V being the product over $w \notin V$ of characteristic functions of K_w, whose support is contained in a fixed compact neighbourhood Δ^v of δ^v in $G(\mathbb{A}^v)$. According to Remark 9, the sums over $\gamma \in \Gamma(M)_S$ can be then taken over finite sets that are independent of f^v, for a fixed finite set of valuations $S \supset V$ that is also independent of f^v. Since the factors $f^v_M(\gamma^v)$ in (23.15) are actually distributions, we can allow f^v_V to be a function in $C^\infty_c\big(G(F^v_V)\big)$. We choose this function so that it is supported on a small neighborhood of the image δ^v_V of δ^v in $G(F^v_V)$, and so that $f^v_M(\delta^v) = 1$. It is then easy to see that (23.15) reduces to an identity

$$\sum_{\gamma} c(\gamma) I_M(\gamma_v, f_v) = 0,$$

where γ is summed over the conjugacy classes in $M(F)$ that are $G(F_w)$-conjugate to δ for any $w \in V - \{v\}$ and are $G(F_w)$-conjugate to a point in K_w for every $w \notin V$, and where each coefficient $c(\gamma)$ is positive. A final argument, based on the Galois cohomology of T, establishes that any such γ is actually $G(F)$-conjugate to δ. This means that $\gamma = w_s^{-1}\delta w_s$ for some element $w_s \in W(M)$, and hence that

$$I_M(\gamma, f_v) = I_M(\delta, f_v).$$

(See [**A14**, pp. 527–529].) It follows that

$$I_M(\delta, f_v) = 0,$$

as required.

We have completed our sketch of the proof that the linear forms of Theorems 23.2 and 23.3 are supported on characters. The proof is a generalization of an argument introduced by Kazhdan to study invariant orbital integrals. (See [**Ka1**], [**Ka2**].) With its completion, we have also finished the collective proof of the three theorems. □

We have just devoted what might seem to be a disproportionate amount of space to a fairly arcane point. We have done so deliberately. Our proof that the linear forms $I_M(\gamma, f)$ and $I_M(\pi, X, f)$ are supported on characters can serve as a model for a family of more sophisticated arguments that are part of the general comparison of trace formulas. Instead of showing that $I_M(\gamma, f)$ and $I_M(\pi, X, f)$ vanish for certain functions f, as we have done here, one has to establish identities among corresponding linear forms for suitably related functions on different groups.

The invariant trace formula of Theorem 23.4 simplifies if we impose local vanishing conditions on the function f. We say that a function $f \in \mathcal{H}(G)$ is *cuspidal* at a place w if it is the restriction to $G(\mathbb{A})^1$ of a finite sum of functions $\prod_v f_v$ whose w-component f_w is cuspidal. This means that for any proper Levi subgroup M_w of G over F_w, the function

$$f_{w,M_w}(\pi_w) = f_{w,G}(\pi_w^G), \qquad \pi_w \in \Pi_{\mathrm{temp}}\big(M(F_w)\big),$$

in $\mathcal{I}\big(M_w(F_w)\big)$ vanishes.

Corollary 23.6. (a) *If f is cuspidal at one place w, then*

$$I(f) = \lim_T \int_{\Pi_{\mathrm{disc}}(G)^T} a^G_{\mathrm{disc}}(\pi) f_G(\pi),$$

where $\Pi_{\mathrm{disc}}(G)^T$ is the intersection of $\Pi_{\mathrm{disc}}(G)$ with $\Pi(G)^T$.

(b) *If f is cuspidal at two places w_1 and w_2, then*

$$I(f) = \lim_S \sum_{\gamma \in \Gamma(G)_S} a^G(\gamma) f_G(\gamma).$$

To establish the simple form of the spectral expansion in (a), one applies the splitting formula [**A13**, Proposition 9.4] to the linear forms $I_M(\pi, f)$ in (23.12). Combined with an argument similar to that following Proposition 23.5 above, this establishes that $I_M(\pi, f) = 0$, for any $M \neq G$, and for f as in (a). Since the distribution

$$f_G(\pi) = I_G(\pi, f)$$

vanishes for any π in the complement of $\Pi_{\mathrm{disc}}(G)^T$ in $\Pi(G)^T$, the expansion (a) follows. To establish the simple form of the spectral expansion in (b), one applies the splitting formula (23.8) to the terms $I_M(\gamma, f)$ in (23.11). This establishes that $I_M(\gamma, f) = 0$, for any $M \neq G$, and for f as in (b). The expansion in (b) follows. (See the proof of Theorem 7.1 of [**A14**].) □

24. A closed formula for the traces of Hecke operators

In the next three sections, we shall give three applications of the invariant trace formula. The application in this section might be called the "finite case" of the trace formula. It is a finite closed formula for the traces of Hecke operators on general spaces of automorphic forms. The result can be regarded as an analogue for higher rank of Selberg's explicit formula for the traces of Hecke operators on classical spaces of modular forms.

In this section, we revert to the setting that $F = \mathbb{Q}$, in order to match standard notation for Shimura varieties. We also assume for simplicity that A_G is the split component of G over $\mathbb{R}$ as well as over $\mathbb{Q}$. The group

$$G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$$

then has compact center. The finite case of the trace formula is obtained by specializing the archimedean component of the function $f \in \mathcal{H}(G)$ in the general invariant trace formula. Before we do so, we shall formulate the problem in terms somewhat more elementary than those of recent sections.

Suppose that $\pi_{\mathbb{R}} \in \Pi_{\mathrm{unit}}\big(G(\mathbb{R})\big)$ is an irreducible unitary representation of $G(\mathbb{R})$, and that K_0 is an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. We can write

$$L^2_{\mathrm{disc}}\big(\pi_{\mathbb{R}}, G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big) \tag{24.1}$$

for the $\pi_{\mathbb{R}}$-isotypical component of $L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$, which is to say, the largest subspace of $L^2_{\mathrm{disc}}\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0\big)$ that decomposes under the action of $G(\mathbb{R})^1$ into a sum of copies of the restriction of $\pi_{\mathbb{R}}$ to $G(\mathbb{R})^1$. We can also write

$$L^2_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0) = L^2_{\mathrm{disc}}\big(\pi_{\mathbb{R}}, G(\mathbb{Q})\backslash G(\mathbb{A})/K_0, \zeta_{\mathbb{R}}\big)$$

for the space of functions ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_0$ such that

$$\phi(zx) = \xi_{\mathbb{R}}(z)\phi(x), \qquad z \in A_G(\mathbb{R})^0,$$

where $\zeta_{\mathbb{R}}$ is the central character of $\pi_{\mathbb{R}}$ on $A_G(\mathbb{R})^0$, and such that the restriction of ϕ to $G(\mathbb{A})^1$ lies in the space (24.1). The restriction mapping from $G(\mathbb{A})$ to $G(\mathbb{A})^1$ is then a $G(\mathbb{R})^1$-isomorphism from $L^2_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$ onto the space (24.1). The action of $G(\mathbb{R})$ by right translation on $L^2_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$ is isomorphic to a direct sum of copies of $\pi_{\mathbb{R}}$, with finite multiplicity $m_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$. One would like to compute the nonnegative integer $m_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$.

More generally, suppose that h belongs to the nonarchimedean Hecke algebra $\mathcal{H}\big(G(\mathbb{A}_{\mathrm{fin}}), K_0\big)$ attached to K_0. Let $R_{\mathrm{disc}}(\pi_{\mathbb{R}}, h)$ be the operator on $L^2_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$ obtained by right convolution of h. As an endomorphism of the $G(\mathbb{R})$-module $L^2_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$, $R_{\mathrm{disc}}(\pi_{\mathbb{R}}, h)$ can be regarded as a square matrix of rank equal to $m_{\mathrm{disc}}(\pi_{\mathbb{R}}, K_0)$. One would like a finite closed formula for its trace.

The problem just posed is too broad. However, it is reasonable to consider the question when $\pi_{\mathbb{R}}$ belongs to a restricted class of representations. We shall assume that $\pi_{\mathbb{R}}$ belongs to the subset $\Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ of representations in $\Pi_{\mathrm{unit}}\big(G(\mathbb{R})\big)$ that are square integrable modulo the center of $G(\mathbb{R})$. Selberg's formula [**Sel1**] describes the solution to this problem in the case that $G = SL(2)$, $K_0 = K_{\mathrm{fin}}$ is maximal, and $\pi_{\mathbb{R}}$ is any representation in the set $\Pi_2\big(G(\mathbb{R})\big) = \Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ that is also integrable.

The set $\Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ is known as the discrete series, since it consists of those unitary representations of $G(\mathbb{R})$ whose restrictions to $G(\mathbb{R})^1$ occur discretely in the local spectral decomposition of $L^2\big(G(\mathbb{R})^1\big)$. The set is nonempty if and only if G has a maximal torus T_G that is elliptic over $\mathbb{R}$, which is to say that $T_G(\mathbb{R})/A_G(\mathbb{R})$ is compact. Assume for the rest of this section that T_G exists, and that $T_G(\mathbb{R})$ is contained in the subgroup $K_{\mathbb{R}}A_G(\mathbb{R})$ of $G(\mathbb{R})$. Then $\Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ is a disjoint union of finite sets $\Pi_2(\mu)$, parametrized by the irreducible finite dimensional representations μ of $G(\mathbb{R})$ with unitary central character. For any such μ, the set $\Pi_2(\mu)$ consists of those representations in $\Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ with the same infinitesimal character and central character as μ. It is noncanonically bijective with the set of right cosets of the Weyl group $W(K_{\mathbb{R}}, T_G)$ of $K_{\mathbb{R}}$ in the Weyl group $W(G, T_G)$ of G. In particular, the number of elements in any packet $\Pi_2(\mu)$ equals the quotient

$$w(G) = |W(K_{\mathbb{R}}, T_G)|^{-1}|W(G, T_G)|.$$

The facts we have just stated are part of Harish-Chandra's classification of discrete series. The classification depends on a deep theory of characters that Harish-Chandra developed expressly for the purpose. We recall that the character of an arbitrary irreducible representation $\pi_{\mathbb{R}}$ of $G(\mathbb{R})$ is defined initially as the distribution

$$f_{\mathbb{R}} \longrightarrow f_{\mathbb{R},G}(\pi_{\mathbb{R}}) = \mathrm{tr}\big(\pi_{\mathbb{R}}(f_{\mathbb{R}})\big), \qquad f_{\mathbb{R}} \in C_c^\infty\big(G(\mathbb{R})\big),$$

on $G(\mathbb{R})$. Harish-Chandra proved the fundamental theorem that a character equals a locally integrable function $\Theta(\pi_{\mathbb{R}}, \cdot)$ on $G(\mathbb{R})$, whose restriction to the open dense set $G_{\mathrm{reg}}(\mathbb{R})$ of strongly regular elements in $G(\mathbb{R})$ is analytic [**Har1**], [**Har2**]. That is,

$$f_{\mathbb{R},G}(\pi_{\mathbb{R}}) = \int_{G_{\mathrm{reg}}(\mathbb{R})} f_{\mathbb{R}}(x)\Theta(\pi_{\mathbb{R}}, x)\mathrm{d}x, \qquad f_{\mathbb{R}} \in C_c^{\infty}\big(G(\mathbb{R})\big).$$

After he established his character theorem, Harish-Chandra was able to prove a simple formula for the character values of any representation $\pi_{\mathbb{R}} \in \Pi_{\mathrm{temp},2}\big(G(\mathbb{R})\big)$ in the discrete series on the regular elliptic set

$$T_{G,\mathrm{reg}}(\mathbb{R}) = T_G(\mathbb{R}) \cap G_{\mathrm{reg}}.$$

The formula is a signed sum of exponential functions that is remarkably similar to the formula of Weyl for the character of a finite dimensional representation μ. However, there are two essential differences. The first is that the sum over the full Weyl group $W(G, T_G)$ in Weyl's formula is replaced by a sum over the Weyl group $W(K_{\mathbb{R}}, T_G)$ of $K_{\mathbb{R}}$. This is the reason that there are $w(G)$ representations $\pi_{\mathbb{R}}$ associated to μ. The second difference is that the real group $G(\mathbb{R})$ generally has several conjugacy classes of maximal tori $T(\mathbb{R})$ over $\mathbb{R}$. This means that the character of $\pi_{\mathbb{R}}$ has also to be specified on tori other than T_G. Harish-Chandra gave an algorithm for computing the values of $\Theta(\pi_{\mathbb{R}}, \cdot)$ on any set $T_{\mathrm{reg}}(\mathbb{R})$ in terms of its values on $T_{G,\mathrm{reg}}(\mathbb{R})$. The resulting expression is again a linear combination of exponential functions, but now with more general integral coefficients, which can be computed explicitly from Harish-Chandra's algorithm. (For a different way of looking at the algorithm, see [**GKM**].)

We return to the problem we have been discussing. We are going to impose another restriction. Rather than evaluating the trace of a single matrix $R_{\mathrm{disc}}(\pi_{\mathbb{R}}, h)$, we have to be content at this point with a formula for the sum of such traces, taken over $\pi_{\mathbb{R}}$ in a packet $\Pi_2(\mu)$. (Given μ, we shall actually sum over the packet $\Pi_2(\mu^{\vee})$, where

$$\mu^{\vee}(x) = {}^t\mu(x)^{-1}, \qquad x \in G(\mathbb{R}),$$

is the contragredient of μ.) This restriction is dictated by the present state of the invariant trace formula. There is a further refinement of the trace formula, the stable trace formula, which we shall discuss in §29. It is expected that if the stable trace formula is combined with the results we are about to describe, explicit formulas for the individual traces can be established.

We fix the irreducible finite dimensional representation μ of $G(\mathbb{R})$. The formula for the corresponding traces of Hecke operators is obtained by specializing the general invariant trace formula. In particular, it will retain the general structure of a sum over groups $M \in \mathcal{L}$. Each summand contains a product of three new factors, which we now describe.

The most interesting factor is a local function

$$\Phi'_M(\mu, \gamma_{\mathbb{R}}), \qquad \gamma_{\mathbb{R}} \in M(\mathbb{R}),$$

on $M(\mathbb{R})$ attached to the archimedean valuation v_{∞}. Assume first that $\gamma_{\mathbb{R}}$ lies in $T_M(\mathbb{R}) \cap G_{\mathrm{reg}}$, where T_M is a maximal torus in M over $\mathbb{R}$ such that $T_M(\mathbb{R})/A_M(\mathbb{R})$ is compact. In this case, we set

$$\Phi'_M(\mu, \gamma_{\mathbb{R}}) = |D_M^G(\gamma_{\mathbb{R}})|^{\frac{1}{2}} \sum_{\pi_{\mathbb{R}} \in \Pi_2(\mu)} \Theta(\pi_{\mathbb{R}}, \gamma_{\mathbb{R}}), \tag{24.2}$$

where

$$D^G_M(\gamma_{\mathbb{R}}) = D^G(\gamma_{\mathbb{R}})D^M(\gamma_{\mathbb{R}})^{-1}$$

is the relative Weyl discriminant. It is a straightforward consequence of the character formulas for discrete series that $\Phi'_M(\mu,\gamma_{\mathbb{R}})$ extends to a continuous function on the torus $T_M(\mathbb{R})$. (See [**A16**, Lemma 4.2].) If $\gamma_{\mathbb{R}} \in M(\mathbb{R})$ does not belong to any such torus, we set $\Phi'_M(\mu,\gamma_{\mathbb{R}}) = 0$. The function $\Phi'_M(\mu,\gamma_{\mathbb{R}})$ on $M(\mathbb{R})$ is complicated enough to be interesting (because it involves characters of discrete series on nonelliptic tori in $G(\mathbb{R})$), but simple enough to be given explicitly (because there are concrete formulas for such characters). It is supported on the semisimple elements in $M(\mathbb{R})$, and is invariant under conjugation by $M(\mathbb{R})$.

The second factor is a local term attached to the nonarchimedean valuations. If γ is any semisimple element in $M(\mathbb{Q})$, we write

(24.3)

$$h'_M(\gamma) = \delta_P(\gamma_{\mathrm{fin}})^{\frac{1}{2}} \int_{K_{\mathrm{fin}}} \int_{N_P(\mathbb{A}_{\mathrm{fin}})} \int_{M_\gamma(\mathbb{A}_{\mathrm{fin}})\backslash M(\mathbb{A}_{\mathrm{fin}})} h(k^{-1}m^{-1}\gamma mnk)\mathrm{d}n\mathrm{d}n\mathrm{d}k,$$

where P is any group in $\mathcal{P}(M)$, $\delta_P(\gamma_{\mathrm{fin}})$ is the modular function on $P(\mathbb{A}_{\mathrm{fin}})$, and K_{fin} is our maximal compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$. $\big($ In [**A16**], this function was denoted $h_M(\gamma)$ rather than $h'_M(\gamma)$. However, the symbol $h_M(\gamma)$ has since been used to denote the normalized orbital integral

$$h_M(\gamma) = |D^M(\gamma_{\mathrm{fin}})|^{\frac{1}{2}} h'_M(\gamma).\big)$$

Since the integrals in (24.3) reduce to finite linear combinations of values assumed by the locally constant function h, $h'_M(\gamma)$ can in principle by computed explicitly.

The third factor is a global term. It is defined only for semisimple elements $\gamma \in M(\mathbb{Q})$ that lie in $T_M(\mathbb{R})$, for a maximal torus T_M in M over $\mathbb{R}$ such that $T_M(\mathbb{R})/A_M(\mathbb{R})$ is compact. For any such γ, we set

$$\chi(M_\gamma) = (-1)^{q(M_\gamma)}\mathrm{vol}\big(\overline{M}_\gamma(\mathbb{Q})\backslash \overline{M}_\gamma(\mathbb{A}_{\mathrm{fin}})\big)w(M_\gamma), \tag{24.4}$$

where

$$q(M_\gamma) = \frac{1}{2}\dim\big(M_\gamma(\mathbb{R})/K_{\gamma,\mathbb{R}}A_M(\mathbb{R})^0\big)$$

is one-half the dimension of the symmetric space attached to M_γ, while $\overline{M}_\gamma$ is an inner twist of M_γ over $\mathbb{Q}$ such that $\overline{M}_\gamma(\mathbb{R})/A_M(\mathbb{R})^0$ is compact, and $w(M_\gamma)$ is the analogue for M_γ of the positive integer $w(G)$ defined for G above. The volume in the product $\chi(M_\gamma)$ is taken with respect to the inner twist of a chosen Haar measure on $M_\gamma(\mathbb{A}_{\mathrm{fin}})$. We note that the product of the Haar measure on $M_\gamma(\mathbb{A}_{\mathrm{fin}})$ with the invariant measures in the definition of $h'_M(\gamma)$ determines a Haar measure on $G(\mathbb{A}_{\mathrm{fin}})$. This measure is supposed to coincide with the Haar measure used to define the original operator $R_{\mathrm{disc}}(\pi_{\mathbb{R}},h)$ by right convolution of h on $G(\mathbb{A})$.

Theorem 24.1. *Suppose that the highest weight of the finite dimensional representation μ of $G(\mathbb{R})$ is nonsingular. Then for any element $h \in \mathcal{H}\big(G(\mathbb{A}_{\mathrm{fin}}),K_0\big)$, the sum*

$$\sum_{\pi_{\mathbb{R}}\in\Pi_2(\mu^\vee)} \mathrm{tr}\big(R_{\mathrm{disc}}(\pi_{\mathbb{R}},h)\big) \tag{24.5}$$

equals the geometric expansion

$$\sum_{M\in\mathcal{L}} (-1)^{\dim(A_M/A_G)}|W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M)} |\iota^M(\gamma)|^{-1}\chi(M_\gamma)\Phi'_M(\mu,\gamma)h'_M(\gamma). \tag{24.6}$$

To establish the formula, one has to specialize the function f in the general invariant trace formula. The finite dimensional representation μ satisfies

$$\mu(zx) = \zeta_{\mathbb{R}}(z)^{-1}\mu(x), \qquad z \in A_G(\mathbb{R})^0,\ x \in G(\mathbb{R}),$$

for a unitary character $\zeta_{\mathbb{R}}$ on $A_G(\mathbb{R})^0$. Its contragredient $\mu^\vee$ has central character $\zeta_{\mathbb{R}}$ on $A_G(\mathbb{R})^0$. The associated packet $\Pi_2(\mu^\vee)$ is contained in the set $\Pi_{\mathrm{temp}}\big(G(\mathbb{R}), \zeta_{\mathbb{R}}\big)$ of tempered representations of $G(\mathbb{R})$ whose central character on $A_G(\mathbb{R})^0$ equals $\zeta_{\mathbb{R}}$. Now the characterization [**CD**] of the invariant image $\mathcal{I}\big(G(\mathbb{R})\big)$ of $\mathcal{H}\big(G(\mathbb{R})\big)$ applies equally well to the $\zeta_{\mathbb{R}}^{-1}$-equivariant analogue $\mathcal{H}\big(G(\mathbb{R}), \zeta_{\mathbb{R}}\big)$ of the Hecke algebra. It implies that there is a function $f_{\mathbb{R}}$ in $\mathcal{H}\big(G(\mathbb{R}), \zeta_{\mathbb{R}}\big)$ such that

$$f_{\mathbb{R},G}(\pi_{\mathbb{R}}) = \begin{cases} 1, & \text{if } \pi_{\mathbb{R}} \in \Pi_2(\mu^\vee), \\ 0, & \text{otherwise,} \end{cases} \tag{24.7}$$

for any representation $\pi_{\mathbb{R}} \in \Pi_{\mathrm{temp}}\big(G(\mathbb{R}), \zeta_{\mathbb{R}}\big)$. The restriction f of the product $f_{\mathbb{R}}h$ to $G(\mathbb{A})^1$ is then a function in $\mathcal{H}(G)$. We shall substitute it into the invariant trace formula.

Since $f_{\mathbb{R},G}$ vanishes on the complement of the discrete series in $\Pi_{\mathrm{temp}}\big(G(\mathbb{R}), \zeta_{\mathbb{R}}\big)$, $f_{\mathbb{R}}$ is cuspidal. By Corollary 23.6(a), the spectral expansion of $I(f)$ simplifies. We obtain

$$\begin{aligned} I(f) &= \lim_T \sum_{\pi \in \Pi_{\mathrm{disc}}(G)^T} a^G_{\mathrm{disc}}(\pi) f_G(\pi) \\ &= \sum_t I_{t,\mathrm{disc}}(f) \\ &= \sum_t \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s \in W(M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}\big(M_P(s,0)\mathcal{I}_{P,t}(0,f)\big). \end{aligned}$$

The irreducible constituents of the representation $\mathcal{I}_{P,t}(0,f)$ could well be nontempered. However, given that $s \in W(M)$ is regular, and that the tempered support of $f_{\mathbb{R},G}$ contains no representation with singular infinitesimal character, one deduces that

$$\mathrm{tr}\big(M_P(s,0)\mathcal{I}_{P,t}(0,f)\big) = 0,$$

as long as $M \neq G$. (See [**A16**, p. 268].) The terms with $M \neq G$ therefore vanish. The expansion reduces simply to

$$I(f) = \sum_t \sum_{\pi \in \Pi_{t,\mathrm{disc}}(G)} m_{\mathrm{disc}}(\pi) \mathrm{tr}\big(\pi(f_{\mathbb{R}}h)\big), \tag{24.8}$$

the contribution from the discrete spectrum. There can of course be nontempered representations π with $m_{\mathrm{disc}}(\pi) \neq 0$. But the condition that the highest weight of μ be nonsingular is stronger than the conditions on $f_{\mathbb{R},G}$ used to derive (24.8). It can be seen to imply that the summands in (24.8) corresponding to nontempered archimedean components $\pi_{\mathbb{R}}$ vanish. (The proof on p. 283 on [**A16**], which uses the classification of unitary representations $\pi_{\mathbb{R}}$ with cohomology, anticipates Corollary 24.2 below.) It follows that

$$I(f) = \sum_t \sum_{\{\pi : \pi_{\mathbb{R}} \in \Pi_2(\mu^\vee)\}} m_{\mathrm{disc}}(\pi) f_{\mathbb{R},G}(\pi_{\mathbb{R}}) h_M(\pi_{\mathrm{fin}}).$$

This in turn implies that $I(f)$ equals the sum (24.5).

The problem is then to compute the geometric expansion (23.11) of $I(f)$, for the chosen function f defined by $f_{\mathbb{R}}h$. Consider the terms

$$I_M(\gamma, f), \qquad\qquad M \in \mathcal{L},\ \gamma \in \Gamma(M)_S,$$

in (23.11). We apply the splitting formula (23.8) successively to the valuations in S. If $L \in \mathcal{L}(M)$ is proper in G, the contribution $\widehat{I}^L_M(\gamma_{\mathbb{R}}, f_{\mathbb{R},L})$ to the formula vanishes. It follows that

$$I_M(\gamma, f) = I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})h_M(\gamma_{\mathrm{fin}}).$$

The sum of traces (24.5) therefore equals

$$\lim_S \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M)_S} a^M(\gamma) I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})h_M(\gamma_{\mathrm{fin}}). \tag{24.9}$$

The problem reduces to that of computing the archimedean component $I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})$, for elements $\gamma_{\mathbb{R}} \in M(\mathbb{R})$.

Suppose that $\gamma_{\mathbb{R}} = t_{\mathbb{R}}$ is strongly G-regular. In this case, the main theorem of [**A1**] provides a formula for $I_M(t_{\mathbb{R}}, f_{\mathbb{R}})$ in terms of character values of discrete series at $t_{\mathbb{R}}$. The proof uses differential equations and boundary conditions satisfied by $I_M(t_{\mathbb{R}}, f_{\mathbb{R}})$ to reduce the problem to the case $M = G$, which had been solved earlier by Harish-Chandra [**Har3**]. A more conceptual proof of the same formula came later, as a consequence of the local trace formula [**A20**, Theorem 5.1]. (A p-adic analogue for Lie algebras of this result is contained in the lectures of Kottwitz [**Ko8**].) If $t_{\mathbb{R}}$ is elliptic in $M(\mathbb{R})$, the formula asserts that $I_M(t_{\mathbb{R}}, f_{\mathbb{R}})$ equals the product of

$$(-1)^{\dim(A_M/A_G)}\mathrm{vol}\big(T_M(\mathbb{R})/A_M(\mathbb{R})^0\big)^{-1}$$

with

$$\sum_{\pi_{\mathbb{R}}\in\Pi_2(G(\mathbb{R}),\zeta_{\mathbb{R}})} |D^G(t_{\mathbb{R}})|^{\frac{1}{2}}\Theta(\pi_{\mathbb{R}}, t_{\mathbb{R}}) f_{\mathbb{R},G}(\pi_{\mathbb{R}}),$$

where T_M is the centralizer of $t_{\mathbb{R}}$. It follows that

$$I_M(t_{\mathbb{R}}, f_{\mathbb{R}}) = (-1)^{\dim(A_M/A_G)}\mathrm{vol}\big(T_M(\mathbb{R})/A_M(\mathbb{R})^0\big)^{-1}|D^M(t_{\mathbb{R}})|^{\frac{1}{2}}\Phi'_M(\mu, t_{\mathbb{R}}). \tag{24.10}$$

If $t_{\mathbb{R}}$ is not elliptic in $M(\mathbb{R})$, the formula of [**A1**] (or just the descent formula (23.9)) tells us that $I_M(t_{\mathbb{R}}, f_{\mathbb{R}})$ vanishes. Since $\Phi'_M(\mu, t_{\mathbb{R}})$ vanishes by definition in this case, (24.10) holds for any strongly G-regular element $t_{\mathbb{R}}$.

It remains to sketch a generalization of (24.10) to arbitrary elements $\gamma_{\mathbb{R}} \in M(\mathbb{R})$. From the definitions (18.12) and (23.3), we deduce that

$$I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}}) = \lim_{a_{\mathbb{R}}\to 1} \sum_{L\in\mathcal{L}(M)} r^L_M(\gamma_{\mathbb{R}}, a_{\mathbb{R}}) I_L(a_{\mathbb{R}}\gamma_{\mathbb{R}}, f_{\mathbb{R}}),$$

for small points $a_{\mathbb{R}} \in A_M(\mathbb{R})$ in general position. Since $f_{\mathbb{R}}$ is cuspidal, the descent formula (23.9) implies that the summands on the right with $L \neq M$ vanish. Replacing $\gamma_{\mathbb{R}}$ by $a_{\mathbb{R}}\gamma_{\mathbb{R}}$, if necessary, we can therefore assume that the centralizer of $\gamma_{\mathbb{R}}$ in G is contained in M. In this case, $I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})$ can be approximated by functions $I_M(t_{\mathbb{R}}, f_{\mathbb{R}})$, for G-regular elements $t_{\mathbb{R}}$ in $M(\mathbb{R})$ that are close to the semisimple part $\sigma_{\mathbb{R}}$ of $\gamma_{\mathbb{R}}$. We can actually assume that $\sigma_{\mathbb{R}}$ lies in an elliptic torus T_M, again by the descent formula (23.9). The approximation of $I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})$ then takes the form of a limit formula

$$I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}}) = \lim_{t_{\mathbb{R}}\to\sigma_{\mathbb{R}}} \big(\partial(h_{u_{\mathbb{R}}}) I_M(t_{\mathbb{R}}, f_{\mathbb{R}})\big),$$

where $\partial(h_{u_{\mathbb{R}}})$ is a harmonic differential operator on $T_M(\mathbb{R})$ attached to the unipotent part $u_{\mathbb{R}}$ of $\gamma_{\mathbb{R}}$ [**A16**, Lemma 5.2]. One can compute the limit from the properties of the function $\Phi'_M(\mu, t_{\mathbb{R}})$ on the right hand side of (24.10). The fact that this function is constructed from a sum of characters of discrete series in the packet $\Pi_2(\mu)$ is critical. One uses it to show that the limit vanishes unless $u_{\mathbb{R}} = 1$. The conclusion [**A16**, Theorem 5.1] is that

$$I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}}) = (-1)^{\dim(A_M/A_G)} v(M_{\gamma_{\mathbb{R}}})^{-1} |D^M(\gamma_{\mathbb{R}})|^{\frac{1}{2}} \Phi'_M(\mu, \gamma_{\mathbb{R}}),$$

where

$$v(M_{\gamma_{\mathbb{R}}}) = (-1)^{q(M_{\gamma_{\mathbb{R}}})} \mathrm{vol}\big(\overline{M}_{\gamma_{\mathbb{R}}}(\mathbb{R})/A_M(\mathbb{R})^0\big) w(M_{\gamma_{\mathbb{R}}})^{-1}.$$

In particular, $I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})$ vanishes unless $\gamma_{\mathbb{R}}$ is semisimple and lies in an elliptic maximal torus T_M.

We substitute the general formula for $I_M(\gamma_{\mathbb{R}}, f_{\mathbb{R}})$ into the expression (24.9) for $I(f)$. We see that the summand in (24.9) corresponding to $\gamma \in \Gamma(M)_S$ vanishes unless γ is semisimple. Since (M, S)-equivalence of semisimple elements in $\Gamma(M)_S$ is the same as $M(\mathbb{Q})$-conjugacy, we can sum γ over the set $\Gamma(M)$ instead of $\Gamma(M)_S$, removing the limit over S at the same time. We can also write

$$\begin{aligned} &|D^M(\gamma_{\mathbb{R}})|^{\frac{1}{2}} \Phi'_M(\mu, \gamma_{\mathbb{R}}) h_M(\gamma_{\mathrm{fin}}) \\ &= |D^M(\gamma_{\mathbb{R}}) D^M(\gamma_{\mathrm{fin}})|^{\frac{1}{2}} \Phi'_M(\mu, \gamma_{\mathbb{R}}) h'_M(\gamma_{\mathrm{fin}}) \\ &= \Phi'_M(\mu, \gamma) h'_M(\gamma), \end{aligned}$$

for any semisimple element $\gamma \in M(\mathbb{Q})$, by the product formula for $\mathbb{Q}$. Finally, it follows from the definitions (19.5) and (22.2) of $a^M(\gamma)$, together with the main theorem of [**Ko6**], that

$$a^M(\gamma) v(M_{\gamma_{\mathbb{R}}})^{-1} = \chi(M_\gamma) |\iota^M(\gamma)|^{-1},$$

again for any semisimple element $\gamma \in M(\mathbb{Q})$. We conclude that $I(f)$ is equal to the required expression (24.6). Since it is also equal to the original sum (24.5), the theorem follows. □

Remarks. 1. The theorem from [**Ko6**] we have just appealed to is that the coefficient

$$a^G(1) = \mathrm{vol}\big(G(F)\backslash G(\mathbb{A})^1\big)$$

is invariant under inner twisting of G. Kottwitz was able to match the terms with $M = G$ and $\gamma = 1$ in the fine geometric expansion (22.9) for any two groups related by inner twisting. This completed the proof of the Weil conjecture on Tamagawa numbers, following a suggestion from [**JL**, §16]. It represents a different and quite striking application of the general trace formula, which clearly illustrates the need for a fine geometric expansion. Unfortunately, we do not have space to discuss it further.

2. The condition that the highest weight of μ be nonsingular was studied by F. Williams [**Wi**], in connection with multiplicity formulas for compact quotient. It is weaker than the condition that the relevant discrete series representations be integrable, which was used in the original multiplicity formulas of Langlands [**Lan2**].

If our condition on the highest weight of μ is removed, the expression (24.6) still makes sense. To what does it correspond?

Assume that

$$\mu: \ G \longrightarrow GL(V)$$

is an irreducible finite dimensional representation of G that is defined over $\mathbb{Q}$. This represents a slight change of perspective. On the one hand, we are asking that the restriction of μ to the center of G be algebraic, and that the representation itself be defined over $\mathbb{Q}$. On the other, we are relaxing the condition that the central character $\zeta_{\mathbb{R}}^{-1}$ of μ on $A_G(\mathbb{R})^0$ be unitary. The corresponding packet $\Pi_2(\mu)$ still exists, but it is now contained only in the set $\Pi_2\big(G(\mathbb{R})\big)$ of general (not necessarily tempered) representations of $G(\mathbb{R})$ that are square integrable modulo the center. We define the function $\Phi'_M(\gamma_{\mathbb{R}}, \mu)$ exactly as before.

If $K'_{\mathbb{R}} = K_{\mathbb{R}} A_G(\mathbb{R})^0$, the quotient

$$X = G(\mathbb{R})/K'_{\mathbb{R}}$$

is a globally symmetric space with respect to a fixed left $G(\mathbb{R})$-invariant metric. Let us assume that none of the simple factors of G is anisotropic over $\mathbb{R}$. We assume also that the open compact subgroup $K_0 \subset G(\mathbb{A}_{\mathrm{fin}})$ is small enough that the action of $G(\mathbb{Q})$ on the product of X with $G(\mathbb{A}_{\mathrm{fin}})/K_0$ has no fixed points. The quotient

$$\mathcal{M}(K_0) = G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_{\mathrm{fin}})/K_0\big)$$

is then a finite union of locally symmetric spaces. Moreover, the restriction of the representation μ to $G(\mathbb{Q})$ determines a locally constant sheaf

$$\mathcal{F}_\mu(K_0) = V(\mathbb{C}) \underset{G(\mathbb{Q})}{\times} \big(X \times G(\mathbb{A}_{\mathrm{fin}})/K_0\big)$$

on $\mathcal{M}(K_0)$.

One can form the L^2-cohomology

$$H^*_{(2)}\big(\mathcal{M}(K_0), \mathcal{F}_\mu(K_0)\big) = \bigoplus_{q \geq 0} H^q_{(2)}\big(\mathcal{M}(K_0), \mathcal{F}_\mu(K_0)\big)$$

of $\mathcal{M}(K_0)$ with values in $\mathcal{F}_\mu$. It is a finite dimensional graded vector space, which reduces to ordinary de Rham cohomology in the case that $\mathcal{M}(K_0)$ is compact. The element h in the Hecke algebra $\mathcal{H}\big(G(\mathbb{A}_{\mathrm{fin}}), K_0\big)$ acts by right convolution on any reasonable space of functions or differential forms on $\mathcal{M}(K_0)$. It yields an operator

$$H^*_{(2)}\big(h, \mathcal{F}_\mu(K_0)\big) = \bigoplus_{q} H^q_{(2)}\big(h, \mathcal{F}_\mu(K_0)\big)$$

on the L^2-cohomology space. Let

$$\mathcal{L}_\mu(h) = \sum_q (-1)^q \mathrm{tr}\big(H^q_{(2)}\big(h, \mathcal{F}_\mu(K_0)\big)\big) \tag{24.11}$$

be its Lefschetz number.

Corollary 24.2. *The Lefschetz number $\mathcal{L}_\mu(h)$ equals the product of $(-1)^{q(G)}$ with the geometric expression (24.6).*

The reduction of the corollary to the formula of the theorem depends on the spectral decomposition of L^2-cohomology [**BC**], and the Vogan-Zuckermann classification [**VZ**] of unitary representations of $G(\mathbb{R})$ with $\big(\mathfrak{g}(\mathbb{R}), K'_{\mathbb{R}}\big)$-cohomology. These matters are discussed in §2 of [**A16**]. We shall include only a few words here.

The space $H^q_{(2)}\big(\mathcal{M}(K_0), \mathcal{F}_\mu(K_0)\big)$ is defined by square-integrable differential q-forms on $\mathcal{M}(K_0)$. Consider the case that $\mathcal{M}(K_0)$ is compact. Elements in the space are then defined by smooth, differential q-forms on $\mathcal{M}(K_0)$ with values in $\mathcal{F}_\mu(K_0)$. By thinking carefully about the nature of such objects, one is led to a canonical isomorphism

$$H^q_{(2)}\big(\mathcal{M}(K_0), \mathcal{F}_\mu(K_0)\big) \cong \bigoplus_{\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}),\zeta_\mathbb{R})} m_{\mathrm{disc}}(\pi)\big(H^q\big(\mathfrak{g}(\mathbb{R}), K'_\mathbb{R}; \pi_\mathbb{R} \otimes \mu\big) \otimes \pi_{\mathrm{fin}}^{K_0}\big),$$

in which $\Pi_{\mathrm{unit}}\big(G(\mathbb{A}),\zeta_\mathbb{R}\big)$ denotes the set of representations in $\Pi\big(G(\mathbb{A}),\zeta_\mathbb{R}\big)$ that are unitary modulo $A_G(\mathbb{R})^0$, $H^q\big(\mathfrak{g}(\mathbb{R}), K'_\mathbb{R}; \cdot\big)$ represents the $\big(\mathfrak{g}(\mathbb{R}), K'_\mathbb{R}\big)$-cohomology groups defined in [**BW**, Chapter II], for example, and $\pi_{\mathrm{fin}}^{K_0}$ stands for the space of K_0-invariant vectors for the finite component π_{fin} of π. (See [**BW**, Chapter VII].) This isomorphism is compatible with the canonical action of the Hecke algebra $\mathcal{H}\big(G(\mathbb{A}_{\mathrm{fin}}), K_0\big)$ on each side. It follows that there is a canonical isomorphism of operators

$$H^q_{(2)}\big(h; \mathcal{F}_\mu(K_0)\big) \cong \bigoplus_{\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}),\zeta_\mathbb{R})} m_{\mathrm{disc}}(\pi) \cdot \dim\big(H^q\big(\mathfrak{g}(\mathbb{R}), K'_\mathbb{R}; \pi_\mathbb{R} \otimes \mu\big)\big) \cdot \pi_{\mathrm{fin}}(h).$$

In the paper [**BC**], Borel and Casselman show that this isomorphism carries over to the case of noncompact quotient (with our assumption that $G(\mathbb{R})$ has discrete series). Define

$$\chi_\mu(\pi_\mathbb{R}) = \sum_q (-1)^q \dim\big(H^q\big(\mathfrak{g}(\mathbb{R}), K'_\mathbb{R}; \pi_\mathbb{R} \otimes \mu\big)\big),$$

for any unitary representation $\pi_\mathbb{R}$ of $G(\mathbb{R})$. It then follows that

$$\mathcal{L}_\mu(h) = \sum_{\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}),\zeta_\mathbb{R})} m_{\mathrm{disc}}(\pi)\chi_\mu(\pi_\mathbb{R})\mathrm{tr}\big(\pi_{\mathrm{fin}}(h)\big). \tag{24.12}$$

The second step is to describe the integers $\chi_\mu(\pi_\mathbb{R})$. This is done in [**CD**]. The result can be expressed as an identity

$$\chi_\mu(\pi_\mathbb{R}) = (-1)^{q(G)} f_{\mathbb{R},G}(\pi_\mathbb{R}), \qquad \pi_\mathbb{R} \in \Pi_{\mathrm{unit}}\big(G(\mathbb{R}),\zeta_\mathbb{R}\big),$$

where $f_\mathbb{R} \in \mathcal{H}\big(G(\mathbb{R}),\zeta_\mathbb{R}\big)$ is a function that satisfies (24.7). It follows that if π is as in (24.8), then

$$\begin{aligned}\chi_\mu(\pi_\mathbb{R})\mathrm{tr}\big(\pi_{\mathrm{fin}}(h)\big) &= (-1)^{q(G)}\mathrm{tr}\big(\pi_\mathbb{R}(f_\mathbb{R})\big)\mathrm{tr}\big(\pi_{\mathrm{fin}}(h)\big)\\ &= (-1)^{q(G)}\mathrm{tr}\big(\pi(f_\mathbb{R} h)\big),\end{aligned}$$

where $\pi_\mathbb{R} \otimes \pi_{\mathrm{fin}}$ is the representation in $\Pi_{\mathrm{unit}}\big(G(\mathbb{A}),\zeta_\mathbb{R}\big)$ whose restriction to $G(\mathbb{A})^1$ equals π. It follows from (24.8) and (24.12) that

$$\mathcal{L}_\mu(h) = (-1)^{q(G)} I(f).$$

Since we have already seen that $I(f)$ equals the geometric expression (24.6), the corollary follows. □

The formula of Corollary 24.2 is relevant to Shimura varieties. The reader will recall from the lectures of Milne [**Mi**] that with further conditions on G, the space $\mathcal{M}(K_0)$ becomes the set of complex points of a Shimura variety. It is a fundamental problem for Shimura varieties to establish reciprocity laws between the analytic data contained in Hecke operators on L^2-cohomology, and the arithmetic data contained in ℓ-adic representations of Galois groups on étale cohomology. Following the strategy that was successful for $GL(2)$ [**Lan4**], one would try to compare geometric sides of two Lefschetz formulas. Much progress has been made in the case that $\mathcal{M}(K_0)$ is compact [**Ko7**]. In the general case, the formula of Corollary 24.2 could serve as the basic analytic Lefschetz formula. (One still has to "stabilize" this formula, a problem closely related to that of computing the individual summands in (24.5), as opposed to their sum.) The other ingredient would be a Lefschetz trace formula for Frobenius-Hecke correspondences on the ℓ-adic intersection cohomology of the Bailey-Borel compactification $\overline{\mathcal{M}(K_0)}$, and a comparison of its gemetric terms with those of the analytic formula.

The general problem is still far from being solved. However, Goresky, Kottwitz, and Macpherson have taken an important step. They have established a formula for the Lefschetz numbers of Hecke correspondences in the complex intersection cohomology of $\overline{\mathcal{M}(K_0)}$, whose geometric terms match those of the analytic formula [**GKM**]. Since one knows that the spectral sides of the two formulas match, by Zucker's conjecture [**Lo**], [**SS**], the results of Goresky, Kottwitz, and Macpherson can be regarded as a topological proof of the formula of Corollary 24.2. It is hoped that their methods can be applied to ℓ-adic intersection cohomology.

25. Inner forms of $GL(n)$

The other two applications each entail a comparison of trace formulas. They concern higher rank analogues of the Jacquet-Langlands correspondence, and the theorem of Saito-Shintani and Langlands on base change for $GL(2)$. These two applications are the essential content of the monograph [**AC**]. Since we are devoting only limited space to them here, our discussion will have to be somewhat selective.

The two comparisons were treated together in [**AC**]. However, it is more instructive to discuss them separately. In this section we will discuss a partial generalization of the Jacquet-Langlands correspondence from $GL(2)$ to $GL(n)$. We shall describe a term by term comparison of the invariant trace formula of the multiplicative group of a central simple algebra with that of $GL(n)$.

We return to the general setting of Part II, in which G is defined over a number field F. In this section, G^* will stand for the general linear group $GL(n)$ over F. We take G to be an inner twist of G^* over F. This means that G is equipped with an isomorphism $\psi\colon G \to G^*$ such that for every element τ in $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, the relation

$$\psi \circ \tau(\psi)^{-1} = \mathrm{Int}\big(a(\tau)\big)$$

holds for some element $a(\tau)$ in G^*.

The general classification of reductive groups over local and global fields assigns a family of invariants

$$\big\{\mathrm{inv}_v = \mathrm{inv}_v(G,\psi)\big\}$$

to (G,ψ), parametrized by the valuations v of F. The local invariant inv_v is attached to the localization of (G,ψ) at F_v, and takes values in the cyclic group $(\mathbb{Z}/n\mathbb{Z})$. It can assume any value if v is nonarchimedean, but satisfies the constraints $2\mathrm{inv}_v = 0$ if $F_v \cong \mathbb{R}$, and $\mathrm{inv}_v = 0$ if $F_v \cong \mathbb{C}$. The elements in the family $\{\mathrm{inv}_v\}$ vanish for almost all v, and satisfy the global constraint

$$\sum_v \mathrm{inv}_v = 0.$$

Conversely, given G^* and any set of invariants $\{\mathrm{inv}_v\}$ in $\mathbb{Z}/n\mathbb{Z}$ with these constraints, there is an essentially unique inner twist (G,ψ) of G^* with the given invariants. These assertions are special cases of Theorems 1.2 and 2.2 of [**Ko5**].

(To see this, one has to identify $(\mathbb{Z}/n\mathbb{Z})$ with the group of characters on the center $\widehat{Z}_{\mathrm{sc}}$ of the complex dual group $SL(n,\mathbb{C})$ of $G^*_{ad} = PGL(n)$.) We write $n = d_v m_v$, where d_v is the order of the element inv_v in $(\mathbb{Z}/n\mathbb{Z})$, and $n = dm$, where d is the least common multiple of the integers d_v.

The notation $\{\mathrm{inv}_v\}$ is taken from the older theory of central simple algebras. Since the inner automorphisms $\mathrm{Int}\big(a(\tau)\big)$ of G^* extend to the matrix algebra $M_n(\overline{F})$, one sees easily that ψ extends to an isomorphism from $A \bigotimes \overline{F}$ to $M_n(\overline{F})$, where A is a central simple algebra over F such that

$$G(k) = (A \bigotimes k)^*,$$

for any $k \supset F$, and the tensor products are taken over F. It is a consequence of the theory of such algebras [**We**] that A is isomorphic to $M_m(D)$, where D is a division algebra over F of degree d. Similarly, for any v, the local algebra $A_v = A \bigotimes F_v$ is isomorphic to $M_{m_v}(D_v)$. It follows that $G(F) \cong GL(m,D)$ and $G(F_v) \cong GL(m_v, D_v)$. (These facts can also be deduced from the two theorems quoted from [**Ko5**].) In particular, the minimal Levi subgroup M_0 of G we suppose to be fixed is isomorphic to a product of m copies of multiplicative groups of D.

It is easy to see that by replacing ψ with some conjugate

$$\mathrm{Int}(g)^{-1} \circ \psi, \qquad g \in G^*(\overline{F}),$$

if necessary, we can assume that the image $M_0^* = \psi(M_0)$ is defined over F. The mapping

$$M \longrightarrow M^* = \psi(M), \qquad M \in \mathcal{L},$$

is then a bijection from $\mathcal{L}$ onto the set of Levi subgroups $\mathcal{L}(M_0^*)$ in G^*. For any $M \in \mathcal{L}$, there is a bijection $P \to P^*$ from $\mathcal{P}(M)$ to $\mathcal{P}(M^*)$. Similar remarks apply to any completion F_v of F. We can choose a point $g_v \in M_0^*(F_v)$ such that the conjugate

$$\psi_v = \mathrm{Int}(g_v)^{-1} \circ \psi$$

maps a fixed minimal Levi subgroup $M_{v0} \subset M_0$ over F_v to a Levi subgroup $M^*_{v0} \subset M^*_0$ over F_v. The mapping $M_v \to M_v^*$ is then a bijection from $\mathcal{L}_v = \mathcal{L}(M_{v0})$ to $\mathcal{L}(M^*_{v0})$. In the special case that $\mathrm{inv}_v = 0$, the isomorphism ψ_v from G to G^* is defined over F_v.

In order to transfer functions from G to G^*, one has first to be able to transfer conjugacy classes. Working with a general field $k \supset F$, we start with a semisimple conjugacy class $\sigma \in \Gamma_{\mathrm{ss}}\big(G(k)\big)$ in $G(k)$. The image $\psi(\sigma)$ of σ in G^* generates a semisimple conjugacy class in $G(\overline{k})$. Since

$$\tau\big(\psi(\sigma)\big) = \tau(\psi)\big(\tau(\sigma)\big) = \mathrm{Int}\big(a(\tau)\big)^{-1}\psi(\sigma),$$

for any element $\tau \in \mathrm{Gal}(\overline{k}/k)$, the characteristic polynomial of this conjugacy class has coefficients in k. It follows from rational canonical form that the conjugacy class of $\psi(\sigma)$ intersects $G(k)$. It therefore determines a canonical semisimple conjugacy class $\sigma^* \in \Gamma_{\mathrm{ss}}\big(G^*(k)\big)$. We thus obtain a canonical injection $\sigma \to \sigma^*$ from $\Gamma_{\mathrm{ss}}\big(G(k)\big)$ into $\Gamma_{\mathrm{ss}}\big(G^*(k)\big)$. Now if σ is a semisimple element in $G(k)$, it is easy to see that $G_\sigma(k)$ is isomorphic to $GL(m_\sigma, D_\sigma)$, where D_σ is a division algebra of rank d_σ over an extension field k_σ of degree e_σ over k, with $n = d_\sigma \mathrm{e}_\sigma m_\sigma$, while $G^*_{\sigma^*}(k)$ is isomorphic to $GL(d_\sigma m_\sigma, k_\sigma)$. The unipotent classes u in $G_\sigma(k)$ correspond to partitions of m_σ. For any such u, let u^* be the unipotent class in $G^*_{\sigma^*}(k)$ that corresponds to the partition of $d_\sigma m_\sigma$ obtained by multiplying the components of the first partition by d_σ. Then

$$\gamma = \sigma u \;\longrightarrow\; \gamma^* = \sigma^* u^*$$

is a canonical injection from the set $\Gamma\big(G(k)\big)$ of all conjugacy classes in $G(k)$ into the corresponding set $\Gamma\big(G^*(k)\big)$ in $G^*(k)$.

Suppose that $k = F$. If $M \in \mathcal{L}$, ψ restricts to an inner twist from M to the Levi subgroup M^* of G^*. It therefore defines an injection $\gamma_M \to \gamma_M^*$ from $\Gamma(M)$ to $\Gamma(M^*)$, by the prescription above. If $\gamma \in \Gamma(G)$ is the induced class γ_M^G, it follows immediately from the definitions that γ^* is the induced class $(\gamma_M^*)^{G^*}$ in $\Gamma(G^*)$.

For the transfer of functions, we need to take k to be a completion F_v of F, and γ to be a strongly regular class $\gamma_v \in \Gamma_{\mathrm{reg}}\big(G(F_v)\big)$ in $G(F_v)$. Suppose that f_v is a function in $\mathcal{H}\big(G(F_v)\big)$. We define a function f_v^* on $\Gamma_{\mathrm{reg}}\big(G^*(F_v)\big)$ by setting

$$f_v^*(\gamma_v^*) = \begin{cases} f_{v,G}(\gamma_v), & \text{if } \gamma_v \text{ maps to } \gamma_v^*, \\ 0, & \text{if } \gamma_v^* \text{ is not in the image of } \Gamma\big(G(F_v)\big), \end{cases}$$

for any class $\gamma_v^* \in \Gamma_{\mathrm{reg}}\big(G^*(F_v)\big)$. In the case that $\mathrm{inv}_v = 0$, f_v^* is the image in $\mathcal{I}\big(G^*(F_v)\big)$ of the function $f_v \circ \psi_v^{-1}$ in $\mathcal{H}\big(G^*(F_v)\big)$. In particular, if v is also nonarchimedean (so that G is unramified at v), and f_v is the characteristic function of the maximal compact subgroup K_v, f_v^* is the image in $\mathcal{I}\big(G^*(F_v)\big)$ of the characteristic function of the maximal compact subgroup $K_v^* = \psi(K_v)$ of $G^*(F_v)$.

The next theorem applies to any valuation v of F.

Theorem 25.1. (Deligne, Kazhdan, Vigneras)

(a) *For any $f_v \in \mathcal{H}\big(G(F_v)\big)$, the function f_v^* belongs to $\mathcal{I}\big(G^*(F_v)\big)$. In other words, f_v^* represents the set of strongly regular orbital integrals of some function in $\mathcal{H}\big(G^*(F_v)\big)$.*

(b) *There is a canonical injection $\pi_v \to \pi_v^*$ from $\Pi_{\mathrm{temp}}\big(G(F_v)\big)$ into $\Pi_{\mathrm{temp}}\big(G^*(F_v)\big)$ such that*

$$f_v^*(\pi_v^*) = e(G_v) f_{v,G}(\pi_v), \qquad f_v \in \mathcal{H}\big(G(F_v)\big),$$

where $e(G_v)$ is the sign attached to the reductive group G over F_v by Kottwitz [**Ko2**].

These results were established in [**DKV**]. The largely global argument makes use of a simple version of the trace formula, such as the formula provided by Corollary 23.6 for functions $f \in \mathcal{H}(G)$ that are cuspidal at two places. Part (a) is Theorem B.2.c of [**DKV**]. Part (b) follows from Theorems B.2.a, B.2.c, and B.2.d of [**DKV**]. □

The assertions of the theorem remain valid if G is replaced by a Levi subgroup. This is because a Levi subgroup is itself a product of groups attached to central semisimple algebras.

Recall that the invariant trace formula depends on a choice of normalizing factors for local intertwining operators. In the case of the group $G^* = GL(n)$, Shahidi [**Sha2**] has shown that Langlands' conjectural definition of normalizing factors in terms of L-functions satisfies the required properties. Now if we are to be able to compare terms in the general trace formulas of G and G^*, we will need a set of local normalizing factors for G that are compatible with those of G^*. Suppose then that v is a valuation, and that $M_v \in \mathcal{L}_v$. It is enough to define normalizing factors $r_{P|Q}(\pi_{v,\lambda})$ for tempered representations $\pi_v \in \Pi_{\mathrm{temp}}\big(M(F_v)\big)$, by [**A15**, Theorem 2.1]. We set

$$r_{P|Q}(\pi_{v,\lambda}) = r_{P^*|Q^*}(\pi^*_{v,\lambda}), \qquad \pi_v \in \Pi_{\mathrm{temp}}\big(M(F_v)\big),\ P, Q \in \mathcal{P}(M), \tag{25.1}$$

where the right hand side is Langlands' canonical normalizing factor for $GL(n)$.

LEMMA 25.2. *The functions (25.1) give valid normalizing factors for G.*

This is [**AC**, Lemma 2.2.1]. One has to show that the functions (25.1) satisfy the conditions of Theorem 21.4. The main point is to establish the basic identity (21.12) that relates the normalizing factors to Harish-Chandra's μ-function $\mu_M(\pi_{v,\lambda})$. To establish this identity, one first deduces that

$$\mu_M(\pi_{v,\lambda}) = \mu_{M^*}(\pi^*_{v,\lambda})$$

from the formula

$$f_v(1) = e(G_v) f^*_v(1), \qquad f_v \in \mathcal{H}\big(G(F_v)\big),$$

the Plancherel formulas for $G(F_v)$ and $G^*(F_v)$, and the relationship between μ-functions and corresponding Plancherel densities. The required identity for G then follows from its analogue for G^* established by Shahidi. □

Suppose that f is the restriction to $G(\mathbb{A})^1$ of a function in $\mathcal{H}\big(G(\mathbb{A})\big)$ of the form $\prod f_v$. Let f^* be the corresponding restriction of the function $\prod f^*_v$. Then $f \to f^*$ extends to a linear mapping from $\mathcal{H}(G)$ to $\mathcal{I}(G^*)$. It takes any subspace $\mathcal{H}\big(G(F_S)^1\big)$ of $\mathcal{H}(G)$ to the corresponding subspace $\mathcal{I}\big(G^*(F_S)^1\big)$ of $\mathcal{I}(G)$.

We define

$$I^{\mathcal{E}}(f) = \widehat{I}^*(f^*), \qquad f \in \mathcal{H}(G^*),$$

where $I^* = I^{G^*}$ is the distribution given by either side of the invariant trace formula for G^*. We of course also have the corresponding distribution $I = I^G$ from the trace formula for G. One of the main problems is to show that $I^{\mathcal{E}}(f) = I(f)$. There seems to be no direct way to do this. One employs instead an indirect strategy of comparing terms, both geometric and spectral, in the two trace formulas.

If S is a finite set of valuations of F that contains S_{ram}, and γ belongs to $\Gamma(M)_S$, we define

$$a^{M,\mathcal{E}}(\gamma) = a^{M^*}(\gamma^*), \qquad M \in \mathcal{L}, \tag{25.2}$$

and

$$I^{\mathcal{E}}_M(\gamma, f) = \widehat{I}_{M^*}(\gamma^*, f^*), \qquad f \in \mathcal{H}(G). \tag{25.3}$$

More generally, the definition (25.3) applies to any finite set of valuations S with the closure property, any conjugacy class γ in $M(F_S)$, and any function $f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$.

LEMMA 25.3. *There is an expansion*

$$I^{\mathcal{E}}(f) = \lim_{S} \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^{M,\mathcal{E}}(\gamma) I^{\mathcal{E}}_M(\gamma, f). \tag{25.4}$$

This is Proposition 2.5.1 of [**AC**]. By definition,

$$\begin{aligned} I^{\mathcal{E}}(f) &= \widehat{I}^*(f^*) \\ &= \lim_S \sum_{L\in\mathcal{L}^*} |W_0^L||W_0^G|^{-1} \sum_{\beta\in\Gamma(L)_S} a^L(\beta)\widehat{I}_L(\beta, f^*), \end{aligned}$$

where $\mathcal{L}^*$ is the finite set of Levi subgroups of G^* that contain the standard minimal Levi subgroup. This can in turn be written

$$\lim_S \sum_{\{L\}} |W^{G^*}(L)|^{-1} \sum_{\beta\in\Gamma(L)_S} a^L(\beta)\widehat{I}_L(\beta, f^*),$$

where $\{L\}$ is a fixed set of representations of conjugacy classes in $\mathcal{L}^*$. A global vanishing property [**A5**, Proposition 8.1] asserts that $\widehat{I}_L(\beta, f^*)$ vanishes unless the pair (L, β) comes from G, in the sense that it is conjugate to the image (M^*, γ^*) of a pair (M, γ). We can assume in this case that our representative L actually equals M^*. Moreover, M^* is G^*-conjugate to another group M_1^* if and only if M is G-conjugate to M_1. Since $W^{G^*}(M^*) = W^G(M)$, we see that

$$\begin{aligned} I^{\mathcal{E}}(f) &= \lim_S \sum_{\{M\}} |W^G(M)|^{-1} \sum_{\gamma\in\Gamma(M)_S} a^{M^*}(\gamma^*)\widehat{I}_{M^*}(\gamma^*, f^*) \\ &= \lim_S \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M)_S} a^{M^*}(\gamma^*)\widehat{I}_{M^*}(\gamma^*, f^*), \end{aligned}$$

where $\{M\}$ is a fixed set of representatives of conjugacy classes in $\mathcal{L}$. This in turn equals the right hand side of (25.4). □

If we could somehow establish identities between the terms in (25.4) and their analogues in the geometric expansion of $I(f)$, we would know that $I^{\mathcal{E}}(f)$ equals $I(f)$. We could then try to compare the spectral expansions. In practice, one has to consider the two kinds of expansions simultaneously. Before we try to do this, however, we must first establish a spectral expansion of $I^{\mathcal{E}}(f)$ in terms of objects associated with G. The process is slightly more subtle than the geometric case just treated. This is because the local correspondence $\pi_v \to \pi_v^*$ works only for tempered representations, while nontempered representations occur on the two spectral sides.

We have been writing $\Pi\big(G(\mathbb{A})^1\big)$ for the set of irreducible representations of $G(\mathbb{A})^1$. If τ belongs to the corresponding set for G^*, we can write

$$f_G^*(\tau) = \sum_{\pi\in\Pi(G(\mathbb{A})^1)} \delta_G(\tau, \pi) f_G(\pi), \qquad f \in \mathcal{H}(G),$$

for uniquely determined complex numbers $\delta_G(\tau, \pi)$. This definition would be superfluous if we were concerned only with the tempered case. For if τ and π are tempered,

$$\delta_G(\tau, \pi) = \begin{cases} 1, & \text{if } \tau = \pi^*, \\ 0, & \text{otherwise,} \end{cases}$$

since the product $\prod_v e(G_v)$ of signs equals 1. If τ and π are nontempered, however, $\delta(\tau, \pi)$ could be more complicated. This is because the decompositions of irreducible representations into standard representations for G and G^* might not be compatible.

If $\pi \in \Pi\big(G(\mathbb{A})^1\big)$, we define

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) = \sum_{\tau \in \Pi_{\mathrm{disc}}(G^*)} a^{G^*}_{\mathrm{disc}}(\tau)\delta_G(\tau,\pi). \tag{25.5}$$

It is not hard to show that the sum may be taken over a finite set [**AC**, Lemma 2.9.1]. Using the coefficients $a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi)$ in place of $a^G_{\mathrm{disc}}(\pi)$, we modify the definition of the set $\Pi_{t,\mathrm{disc}}(G)$ in §22. This gives us a discrete subset $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G)$ of $\Pi\big(G(\mathbb{A})^1\big)$ for every $t \geq 0$. We then form the larger subset

$$\Pi^{\mathcal{E}}_t(G) = \big\{\pi^G_\lambda :\ M \in \mathcal{L},\ \pi \in \Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M),\ \lambda \in i\mathfrak{a}^*_M/i\mathfrak{a}^*_G\big\}$$

of $\Pi\big(G(\mathbb{A})^1\big)$, equipped with a measure $d\pi^G_\lambda$ defined as in (22.7). Finally, we define a function $a^{G,\mathcal{E}}$ on $\Pi^{\mathcal{E}}_t(G)$ by setting

$$a^{G,\mathcal{E}}(\pi^G_\lambda) = a^{M,\mathcal{E}}_{\mathrm{disc}}(\pi) r^G_M(\pi_\lambda), \tag{25.6}$$

as in (22.8). The ultimate aim, in some sense, is to show that the discrete coefficients $a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi)$ and $a^G_{\mathrm{disc}}(\pi)$ match. We now assume inductively that this is true if G is replaced by any proper Levi subgroup M. Then $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M)$ equals $\Pi_{t,\mathrm{disc}}(M)$, and in particular, consists of unitary representations of $M(\mathbb{A})^1$. It follows that the function $a^{G,\mathcal{E}}(\pi^G_\lambda)$ is analytic, and slowly increasing in the sense of Lemma 21.5.

The extra complication arises when we try to describe the function $a^{G,\mathcal{E}}$ as a pullback of the corresponding function for G^*. Suppose that $\pi \in \Pi\big(M(\mathbb{A})^1\big)$ and $\tau \in \Pi\big(M^*(\mathbb{A})^1\big)$ are representations with $\delta_M(\tau,\pi) \neq 0$. Given a point $\lambda \in i\mathfrak{a}^*_M/i\mathfrak{a}^*_G$ in general position, and groups $P, Q \in \mathcal{P}(M)$, we set

$$r_{Q|P}(\tau_\lambda,\pi_\lambda) = r_{Q^*|P^*}(\tau_{S,\lambda})^{-1} r_{Q|P}(\pi_{S,\lambda}),$$

where $S \supset S_{\mathrm{ram}}$ is a large finite set of valuations, and τ_S and π_S are the S-components of τ and π. The condition that $\delta_M(\tau,\pi) \neq 0$ implies that $\tau_v \cong \pi_v$ for almost all v [**AC**, Corollary 2.8.3], so that $r_{Q|P}(\tau_\lambda,\pi_\lambda)$ is independent of the choice of S. Moreover, $r_{Q|P}(\tau_\lambda,\pi_\lambda)$ is a rational function in the relevant variables $\lambda(\alpha^\vee)$ or $q_v^{-\lambda(\alpha^\vee)}$ attached to valuations v in S [**A15**, Proposition 5.2]. As Q varies, we obtain a (G,M)-family of functions

$$r_Q(\Lambda,\tau_\lambda,\pi_\lambda,P) = \delta_M(\tau,\pi) r_{Q|P}(\tau_{\lambda+\Lambda},\pi_{\lambda+\Lambda}) r_{Q|P}(\tau_\lambda,\pi_\lambda)^{-1}$$

of $\Lambda \in i\mathfrak{a}^*_M$, which we define for *any* τ and π.

Assume now that π belongs to $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M)$. For any representation $\tau \in \Pi\big(M^*(\mathbb{A})^1\big)$, the (G,M)-family of global normalizing factors

$$\delta_M(\tau,\pi) r_Q(\Lambda,\pi_\lambda,P), \qquad Q \in \mathcal{P}(M),$$

is defined, and equals the product of (G,M)-families

$$r_{Q^*}(\Lambda,\tau_\lambda,P^*) r_Q(\Lambda,\tau_\lambda,\pi_\lambda,P), \qquad Q \in \mathcal{P}(M).$$

It follows from the product formula (17.12) that

$$\delta_M(\tau,\pi) r^G_M(\pi_\lambda) = \sum_{L \in \mathcal{L}(M)} r^{L^*}_{M^*}(\tau_\lambda) r^G_L(\tau_\lambda,\pi_\lambda).$$

Multiplying each side of this last identity by $a^{M^*}_{\mathrm{disc}}(\tau)$, and then summing over τ, we obtain an identity

$$a^{G,\mathcal{E}}_M(\pi^G_\lambda) = \sum_{\tau \in \Pi_{t,\mathrm{disc}}(M^*)} \sum_{L \in \mathcal{L}(M)} a^{L^*}_{M^*}(\tau^{L^*}_\lambda) r^G_L(\tau_\lambda,\pi_\lambda). \tag{25.7}$$

The description of the coefficient $a_M^{G,\mathcal{E}}(\pi_\lambda^G)$ as a pullback of coefficients from G^* is thus more elaborate than its geometric counterpart. This has to be reflected in the construction of the corresponding linear forms that occur in the spectral expansion of $I^{\mathcal{E}}(f)$. Suppose that S is any finite set of valuations with the closure property. The function $r_M^L(\tau_\lambda, \pi_\lambda)$ can obviously be defined for representations $\tau \in \Pi\big(M^*(F_S)\big)$ and $\pi \in \Pi\big(M(F_S)\big)$. If either τ or π is in general position, $r_M^L(\tau_\lambda, \pi_\lambda)$ is an analytic function of λ in $i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{L,S}^*$. In this case, we define linear forms

$$I_M^{\mathcal{E}}(\pi, X, f), \qquad\qquad X \in \mathfrak{a}_{M,S},\ f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big),$$

inductively by setting

(25.8)

$$\widehat{I}_{M^*}(\tau, X, f^*) = \sum_{L\in\mathcal{L}(M)} \sum_{\pi\in\Pi(M(F_S))} \int_{i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{L,S}^*} r_M^L(\tau_\lambda, \pi_\lambda) I_L^{\mathcal{E}}(\pi_\lambda^L, X_L, f) \mathrm{e}^{-\lambda(X)} \mathrm{d}\lambda,$$

for any τ. (For arbitrary τ and π, the functions $r_M^L(\tau_\lambda, \pi_\lambda)$ can acquire poles in the domain of integration, and one has to take a linear combination of integrals over contours $\varepsilon_P + i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{L,S}^*$. See [**AC**, pp. 124–126]. The general definition in [**AC**] avoids induction, but is a three stage process that is based on standard representations.) It is of course the summands with $L \neq M$ in (25.8) that we assume inductively to be defined. The summand of $L = M$ equals

$$\sum_{\pi\in\Pi(M(F_S))} \delta_M(\tau, \pi) I_M^{\mathcal{E}}(\pi, X, f).$$

By applying the local vanishing property [**AC**, Proposition 2.10.3] to the left hand side of the relation (25.8), one shows without difficulty that $I_M^{\mathcal{E}}(\pi, X, f)$ is well defined by this relation. We extend the definition to adelic representations $\pi \in \Pi\big(M(\mathbb{A})^1\big)$ and functions $f \in \mathcal{H}(G)$ by taking $S \supset S_{\mathrm{ram}}$ to be large. If in addition, π is unitary, we write

$$I_M^{\mathcal{E}}(\pi, f) = I_M^{\mathcal{E}}(\pi, 0, f),$$

as before.

LEMMA 25.4. *There is an expansion*

$$I^{\mathcal{E}}(f) = \lim_T \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathcal{E}}(M)^T} a^{M,\mathcal{E}}(\pi) I_M^{\mathcal{E}}(\pi, f) \mathrm{d}\pi. \tag{25.9}$$

This is Proposition 2.12.2 of [**AC**]. The inductive definition (25.8) we have given here leads to a two step proof. The first step is a duplication of the proof of Lemma 25.3, while the second is an application of the formulas (25.7) and (25.8).

We begin by writing

$$\begin{aligned} I^{\mathcal{E}}(f) &= \widehat{I}^*(f^*) \\ &= \lim_T \sum_{L\in\mathcal{L}^*} |W_0^L||W_0^{G^*}|^{-1} \int_{\Pi(L)^T} a^L(\tau) \widehat{I}_L(\tau, f^*) \mathrm{d}\tau, \end{aligned}$$

by the spectral expansion (23.12) for G^*. The global vanishing property [**A14**, Proposition 8.2] asserts that $\widehat{I}_L(\tau, f^*)$ vanishes unless L is conjugate to the image of a group M in $\mathcal{L}$. Using the elementary counting argument from the proof of Lemma 25.3, we see that

$$I^{\mathcal{E}}(f) = \lim_{T} \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M^*)^T} a^{M^*}(\tau) \widehat{I}_{M^*}(\tau, f^*) \mathrm{d}\tau.$$

For the second step, we have to substitute the formula (25.8), with $S \supset S_{\mathrm{ram}}$ large and $X = 0$, for $\widehat{I}_{M^*}(\tau, f^*)$. More correctly, we substitute the version of (25.8) that is valid if any of the functions $r_M^L(\tau_\lambda, \pi_\lambda)$ have poles, since we do not know a priori that the representations π over which we sum are unitary. We then substitute the explicit form (22.7) of the measure $d\tau$ on $\Pi(M^*)^T$. In the resulting multiple (seven-fold, as a matter of fact) sum-integral, it is not difficult to recognize the expansion (25.7). There is some minor effort involved in keeping track of the various constants and domains of the integration. This accounts for the length of some of the arguments in [**AC**]. In the end, however, the expression collapses to the required expansion (25.9). □

THEOREM 25.5. *If γ belongs to $\Gamma(M)_S$ for some $S \supset S_{\mathrm{ram}}$, then*

$$I_M^{\mathcal{E}}(\gamma, f) = I_M(\gamma, f) \tag{25.10}$$

and

$$a^{M,\mathcal{E}}(\gamma) = a^M(\gamma). \tag{25.11}$$

THEOREM 25.6. *If π belongs to the union of $\Pi(M)^T$ and $\Pi^{\mathcal{E}}(M)^T$, for some $T > 0$, then*

$$I_M^{\mathcal{E}}(\pi, f) = I_M(\pi, f) \tag{25.12}$$

and

$$a^{M,\mathcal{E}}(\pi) = a^M(\pi). \tag{25.13}$$

Theorems 25.5 and 25.6 correspond to Theorems A and B in Sections 2.5 and 2.9 of [**AC**], which are the main results of Chapter 2 of [**AC**]. They are proved together, by an argument that despite its length sometimes seems to move forward of its own momentum. In following our sketch of the proof, the reader might keep in mind the earlier argument used in §21 to establish that the terms in the invariant trace formula are supported on characters.

The combined proof of the two theorems is by double induction on n and $\dim(A_M)$. The first induction hypothesis immediately implies that the global formulas (25.11) and (25.13) are valid for proper Levi subgroups $M \neq G$. If $M = G$, on the other hand, the local formulas (25.10) and (25.12) hold by definition, the two sides in each case being equal to $f_G(\gamma)$ and $f_G(\pi)$ respectively. We apply these observations to the identity obtained from the right hand sides of (25.4) and (25.9). Combining the resulting formula with the invariant trace formula for G, we see that the limit over S of the sum of

$$\sum_{M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) \big(I_M^{\mathcal{E}}(\gamma, f) - I_M(\gamma, f) \big) \tag{25.14}$$

and

$$\sum_{\gamma \in \Gamma(G)_S} \big(a^{G,\mathcal{E}}(\gamma) - a^G(\gamma) \big) f_G(\gamma) \tag{25.15}$$

equals the limit over T of the sum of

$$\sum_{M \neq G} |W_0^M||W_0^G|^{-1} \int_{\Pi(M)^T} a^M(\pi)\big(I_M^{\mathcal{E}}(\pi, f) - I_M(\pi, f)\big)\mathrm{d}\pi \tag{25.16}$$

and

$$\int_{\Pi^*(G)^T} \big(a^{G,\mathcal{E}}(\pi) - a^G(\pi)\big) f_G(\pi)\mathrm{d}\pi, \tag{25.17}$$

where $\Pi^*(G)^T$ is the union of $\Pi^{\mathcal{E}}(G)^T$ with $\Pi(G)^T$.

The linear forms $I_M(\gamma, f)$ and $I_M(\pi, X, f)$ were defined for any finite set S with the closure property, and any $f \in \mathcal{H}_{\mathrm{ac}}\big(G(F_S)\big)$. They each satisfy splitting and descent formulas. The linear forms $I_M^{\mathcal{E}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\pi, X, f)$ have been defined in the same context, and satisfy parallel splitting and descent formulas. The required local identities (25.10) and (25.12) can be broadened to formulas

$$I_M^{\mathcal{E}}(\gamma, f) = I_M(\gamma, f), \qquad \gamma \in \Gamma\big(M(F_S)\big), \tag{25.18}$$

and

$$I_M^{\mathcal{E}}(\pi, X, f) = I_M(\pi, X, f), \qquad \pi \in \Pi\big(M(F_S)\big),\ X \in \mathfrak{a}_{M,S}, \tag{25.19}$$

which we postulate for any $f \in \mathcal{H}\big(G(F_S)\big)$. These general identities were originally established only up to some undetermined constants [**AC**, Theorem 2.6.1], but they were later resolved by the local trace formula [**A18**, Theorem 3.C]. We assume inductively that (25.18) and (25.19) hold if n is replaced by a smaller integer. This allows us to simplify the local terms in (25.14) and (25.16). In so doing, we can assume that the function $f \in \mathcal{H}(G)$ is the restriction to $G(\mathbb{A})^1$ of a product of $\prod f_v$.

Consider first the expression (25.16). We recall that Proposition 23.5 applies to the linear forms $I_M(\gamma, f)$ and $I_M(\pi, X, f)$. This proposition can also be adapted to the linear forms $I_M^{\mathcal{E}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\pi, X, f)$ [**AC**, §2.8]. Its first assertion implies that either of the two spectral linear forms can be expressed in terms of its geometric counterpart. The analogue of the more specific second assertion of Proposition 23.5 can be formulated to say that if (25.18) holds for all M, S, γ and f, then so does (25.19) [**AC**, Theorem 2.10.2]. We combine this with the splitting and descent formulas satisfied by the terms in the brackets in (25.16). As in §23, the fact that the representations $\pi \in \Pi(M)$ are unitary is critical to the success of the argument. Following the corresponding discussion after Proposition 23.5, one deduces that the required local identity (25.12) is valid. The expression (25.14) therefore vanishes.

Now consider the expression (25.14). It follows from the splitting formulas (23.8) and [**AC**, $(2.3.4)^{\mathcal{E}}$], together with our induction hypotheses, that

$$I_M^{\mathcal{E}}(\gamma, f) - I_M(\gamma, f) = \sum_v \varepsilon_M(f_v, \gamma_v) f^v(\gamma^v),$$

where

$$\varepsilon_M(f_v, \gamma_v) = I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v),$$

and

$$f^v(\gamma^v) = \prod_{w \neq v} f_w(\gamma_w).$$

If v does not belong to the set S_{ram}, $\varepsilon_M(f_v, \gamma_v) = 0$, since G and G^* are isomorphic over F_v. The expression (25.14) therefore reduces to

$$\sum_{M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma)\Big(\sum_{v \in S_{\text{ram}}} \varepsilon_M(f_v, \gamma_v) f_G^v(\gamma^v)\Big). \tag{25.20}$$

The remaining global coefficients can also be simplified. Consider a class $\gamma \in \Gamma(G)_S$ in (25.15) whose semisimple part is represented by a noncentral element $\sigma \in G(F)$. Then G_σ is a proper subgroup of G. It follows from the definitions (19.6), (22.2), and (25.2), together with our induction hypothesis, that $a^{G,\mathcal{E}}(\gamma)$ equals $a^G(\gamma)$. The expression (25.15) therefore reduces to

$$\sum_{z \in A_G(F)} \sum_{u \in \Gamma_{\text{unip}}(G)_S} \big(a^{G,\mathcal{E}}(zu) - a^G(zu)\big) f_G(zu), \tag{25.21}$$

where $\Gamma_{\text{unip}}(G)_S = \big(\mathcal{U}_G(F)\big)_{G,S}$ is the set of unipotent classes in $\Gamma(G)_S$.

Consider a representation $\pi \in \Pi^*(G)^T$ in (25.17) that does not lie in the union $\Pi^*_{t,\text{disc}}(G)$ of $\Pi^{\mathcal{E}}_{t,\text{disc}}(G)$ and $\Pi_{t,\text{disc}}(G)$, for any t. The induction hypothesis we have taken on includes the earlier assumption that the coefficients $a^{M,\mathcal{E}}_{\text{disc}}$ and a^M_{disc} are equal, for any $M \neq G$. It follows from the definitions (22.8) and (25.6) that $a^{G,\mathcal{E}}(\pi)$ equals $a^G(\pi)$. The expression (25.17) therefore reduces to

$$\sum_{t \leq T} \sum_{\pi \in \Pi^*_{t,\text{disc}}(G)} \big(a^{G,\mathcal{E}}_{\text{disc}}(\pi) - a^G_{\text{disc}}(\pi)\big) f_G(\pi).$$

We conclude that the limit in T of the sum of (25.16) and (25.17) equals

$$\sum_{t} \sum_{\pi \in \Pi^*_{t,\text{disc}}(G)} \big(a^{G,\mathcal{E}}_{\text{disc}}(\pi) - a^G_{\text{disc}}(\pi)\big) f_G(\pi). \tag{25.22}$$

This expression is conditionally convergent, in the sense that the iterated sums converge absolutely.

Using the induction hypothesis, we have reduced the original four expressions to (25.20), (25.21), and (25.22). It follows that if $S \supset S_{\text{ram}}$ is large, in a sense that depends only on the support of f, the sum of (25.20) and (25.21) equals (25.22). The rest of the proof is harder. It consists of several quite substantial steps, each of which we shall attempt to sketch in a few words.

The first step concerns the summands in (25.20). The problem at this stage is to establish something weaker than the required vanishing of these summands. It is to show that for any $M \in \mathcal{L}$ and $v \in S_{\text{ram}}$, and for certain $f_v \in \mathcal{H}\big(G(F_v)\big)$, the function

$$\varepsilon_M(f_v) : \ \gamma_v \ \longrightarrow \ \varepsilon_M(f_v, \gamma_v) = I^{\mathcal{E}}_M(\gamma_v, f_v) - I_M(\gamma_v, f_v), \qquad \gamma_v \in \Gamma_{\text{reg}}\big(M(F_v)\big),$$

belongs to $\mathcal{I}_{\text{ac}}\big(M(F_v)\big)$. The functions $I^{\mathcal{E}}_M(\gamma_v, f_v)$ and $I_M(\gamma_v, f_v)$ are smooth on the strongly G-regular set in $M(F_v)$, but as γ_v approaches the boundary, they acquire singularities over and above those attached to invariant orbital integrals on $M(F_v)$. The problem is to show that these supplementary singularities cancel.

If v is nonarchimedean, let $\mathcal{H}\big(G(F_v)\big)^0$ be the subspace of functions $f_v \in \mathcal{H}\big(G(F_v)\big)$ such that for every central element $z_v \in A_G(F_v)$ and every nontrivial unipotent element $u_v \neq 1$ in $G(F_v)$, $f_G(z_v u_v)$ vanishes. If v is archimedean, we set $\mathcal{H}\big(G(F_v)\big)^0$ equal to $\mathcal{H}\big(G(F_v)\big)$. The result is that the correspondence

$$f_v \longrightarrow \varepsilon_M(f_v), \qquad\qquad f_v \in \mathcal{H}\big(G(F_v)\big)^0,$$

is a continuous linear mapping from $\mathcal{H}\big(G(F_v)\big)^0$ to $\mathcal{I}_{\mathrm{ac}}\big(M(F_v)\big)$. If v is archimedean, one establishes the result by combining the induction hypothesis with the differential equations and boundary conditions [**A12**, §11–13] satisfied by weighted orbital integrals . If v is nonarchimedean, one combines the induction hypotheses with the germ expansion [**A12**, §9] of weighted orbital integrals about a singular point. In this case, one has also to make use of the explicit formulas for weighted orbital integrals of supercuspidal matrix coefficients, in order to match the germs corresponding to $u_v = 1$. (See [**AC**, Proposition 2.13.2].) Let $\mathcal{H}\big(G(\mathbb{A})^0\big)$ be the subspace of $\mathcal{H}\big(G(\mathbb{A})\big)$ spanned by products $f = \prod f_v$ such that for every $v \in S_{\mathrm{ram}}$, f_v belongs to $\mathcal{H}\big(G(F_v)\big)^0$. The result above then implies that the correspondence

$$f \longrightarrow \varepsilon_M(f) = \sum_{v \in S_{\mathrm{ram}}} \varepsilon_M(f_v) f_G^v$$

is a continuous linear mapping from $\mathcal{H}\big(G(\mathbb{A})\big)^0$ to $\mathcal{I}_{\mathrm{ac}}\big(M(\mathbb{A})\big)$.

Suppose now that $M \in \mathcal{L}$ is fixed. We formally introduce the second induction hypothesis that the analogue of (25.18), for any $L \in \mathcal{L}$ with $\dim(A_L) < \dim(A_M)$, holds for any S. We define $\mathcal{H}\big(G(\mathbb{A}), M\big)$ to be the space of functions f in $\mathcal{H}\big(G(\mathbb{A})\big)$ that are M-*cuspidal* at two nonarchimedean places v, in the sense that the local functions $f_{v,L}$ vanish unless L contains a conjugate of M. We also define $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$ to be the space of functions f in the intersection

$$\mathcal{H}\big(G(\mathbb{A}), M\big) \cap \mathcal{H}\big(G(\mathbb{A})\big)^0$$

that satisfy one additional condition. We ask that f vanish at any element in $G(\mathbb{A})$ whose component at each finite place v belongs to $A_G(F_v)$. In combination with the definition of $\mathcal{H}\big(G(\mathbb{A})\big)^0$, this last condition is designed to insure that the terms $f_G(zu)$ in (25.21) all vanish. Notice that f may be modified at any archimedean place without affecting the condition that it lie in $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$.

Suppose that f belongs to $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$. The last induction hypothesis then implies that the summand in (25.20) corresponding to any Levi subgroup that is not conjugate to our fixed group M vanishes. The expression (25.20) reduces to

$$(25.23) \qquad |W(M)|^{-1} \sum_{\gamma \in \Gamma(M)_S} a^M(\gamma) \varepsilon_M(f, \gamma).$$

It is an easy consequence of the original induction hypothesis and the splitting formulas that the function $\varepsilon_M(f)$ in $\mathcal{I}_{\mathrm{ac}}\big(M(\mathbb{A})\big)$ is cuspidal at two places. It then follows from the simple form of the geometric expansion for M in Corollary 23.6 that the original expansion (25.20) equals the product of $|W(M)|^{-1}$ with $\widehat{I}^M\big(\varepsilon_M(f)\big)$. The conditions on f imply that the second expression (25.21) vanishes. Recall that the third expression (25.22) was the ultimate reduction of the spectral expansion of $I^{\mathcal{E}}(f) - I(f)$. Since the third expression equals the sum of the first two, we can write

$$(25.24) \qquad \sum_t \big(I_t^{\mathcal{E}}(f) - I_t(f)\big) - |W(M)|^{-1} \sum_t \widehat{I}_t^M\big(\varepsilon_M(f)\big) = 0,$$

in the notation of Remark 10 in §23. (See [**AC**, (2.15.1)].)

The second step is to apply the weak multiplier estimate (23.13) to the sums in (25.24). Suppose that $f \in \mathcal{H}\big(G(\mathbb{A}), M\big)^0$ is fixed. If $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$ is any multiplier, f_α also belongs to $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$, and the identity (25.24) remains valid with f_α in place of α. It is a consequence of the definitions that $I_t^{\mathcal{E}}(f_\alpha) = \widehat{I}_t(f_\alpha^*)$. One shows also that $\varepsilon_M(f_\alpha) = \varepsilon_M(f)_\alpha$ [**AC**, Corollary 2.14.4]. It then follows from (23.13) that there are positive constants C, k and r such that for any $T > 0$, any $N \geq 0$, and any α in the subspace $C_N^\infty(\mathfrak{h}^1)^W$ of $\mathcal{E}(\mathfrak{h}^1)^W$, the sum

$$\Big| \sum_{t \leq T} \big(I_t^{\mathcal{E}}(f_\alpha) - I_t(f_\alpha)\big) - |W(M)|^{-1} \sum_{t \leq T} \widehat{I}_t^M\big(\varepsilon_M(f_\alpha)\big) \Big| \tag{25.25}$$

is bounded by

$$C e^{kN} \sup_{\nu \in \mathfrak{h}_u^*(r,T)} \big(|\widehat{\alpha}(\nu)|\big). \tag{25.26}$$

To exploit the last inequality, one fixes a point ν_1 in $\mathfrak{h}_u^*$. Enlarging r if necessary, we can assume that ν_1 lies in the space $\mathfrak{h}_u^*(r) = \mathfrak{h}_u^*(r, 0)$. It is then possible to choose a function $\alpha_1 \in C_c^\infty(\mathfrak{h}^1)^W$ such that $\widehat{\alpha}_1$ maps $\mathfrak{h}_u^*(r)$ to the unit interval, and such that the inverse image of 1 under $\widehat{\alpha}_1$ is the W-orbit $W(\nu_1)$ of ν_1 [**AC**, Lemma 2.15.2]. If α_1 belongs to $C_{N_1}^\infty(\mathfrak{h}^1)^W$, and r and k are as in (25.26), we chose $T > 0$ so that

$$|\widehat{\alpha}_1(\nu)| \leq e^{-2kN_1}$$

for all $\nu \in \mathfrak{h}_u^*(r, T)$. We then apply the inequality, with α equal to the function α_m obtained by convolving α_1 with itself m times. Since $\widehat{\alpha}_m(\nu)$ equals $\widehat{\alpha}_1(\nu)^m$, the expression (25.26) approaches 0 as m approaches infinity. One shows independently that the second sum in (25.25) also approaches 0 as m approaches infinity [**AC**, p. 183–188]. Therefore, the first sum in (25.25) approaches 0 as m approaches infinity. But this first sum equals the double sum

$$\sum_{t \leq T} \sum_{\pi \in \Pi_{t,\mathrm{disc}}^*(G)} \big(a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^G(\pi)\big) f_G(\pi) \alpha_1(\nu_\pi)^m,$$

which can be taken over a finite set that is independent of m. We can assume that $T \geq \|\mathrm{Im}(\nu_1)\|$. It follows that the double sum approaches

$$\sum_{\pi \in \Pi_{\nu_1,\mathrm{disc}}^*(G)} \big(a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^G(\pi)\big) f_G(\pi)$$

as m approaches infinity, where $\Pi_{\nu_1,\mathrm{disc}}^*(G)$ is the set of representations π in the set $\Pi_{\mathrm{disc}}^*(G)$ with $\nu_\pi = \nu_1$. Summing over the infinitesimal characters ν_1 with $\|\mathrm{Im}(\nu_1)\| = t$, we conclude that

$$\sum_{\pi \in \Pi_{t,\mathrm{disc}}^*(G)} \big(a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^G(\pi)\big) f_G(\pi) = 0, \tag{25.27}$$

for any $t \geq 0$.

The identity (25.27) holds for any function f in $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$. The third step is to show that it extends to any f in the larger space $\mathcal{H}\big(G(\mathbb{A}), M\big)$. This is a fairly standard argument. On the one hand, the left hand side of (25.27) is a linear combination of point measures in the spectral variables of f_G. On the other hand, the linear forms whose kernels define the subspace $\mathcal{H}\big(G(\mathbb{A}), M\big)^0$ of $\mathcal{H}\big(G(\mathbb{A}), M\big)$ are easily seen to be continuous in the spectral variables. Playing one against the other, one sees that (25.27) does indeed remain valid for any f in $\mathcal{H}\big(G(\mathbb{A}), M\big)$. (See [**AC**, §2.16].) In particular, (25.22) vanishes for any such f. Since (25.20) equals (25.23), we deduce that the sum of (25.23) and (25.21) vanishes for any function f in $\mathcal{H}\big(G(\mathbb{A}), M\big)$.

The fourth step is to apply what we have just established to the expression (25.23). Suppose that for each $v \in S_{\mathrm{ram}}$, f_v is a given function in $\mathcal{H}\big(G(F_v)\big)$. Suppose also that γ_1 is a fixed G-regular element in $M(F)$ that is M-elliptic at two unramified places w_1 and w_2. At the places $w \notin S_{\mathrm{ram}}$, we choose functions $f_w \in \mathcal{H}\big(G(F_w)\big)$ so that $f_{w,G}(\gamma_1) = 1$, and so that the product $f = \prod f_v$ lies in $\mathcal{H}\big(G(\mathbb{A})\big)$. We fix f_w for w distinct from w_1 and w_2, but for w equal to w_1 or w_2, we allow the support of f_w to shrink around a small neighbourhood of γ_1 in $G(F_w)$. Then f belongs to $\mathcal{H}\big(G(\mathbb{A}), M\big)$. Since the support of f remains within a fixed compact set, we can take S to be some fixed finite set containing S_{ram}, w_1, and w_2. We can also restrict the sum in (25.23) to a finite set that is independent of f. (See Remark 9 in §23.)

Since we are shrinking f_{w_1} and f_{w_2} around γ_1, the terms $f_G(zu)$ in (25.21) all vanish. In addition, the function

$$\varepsilon_M(f, \gamma) = \sum_{v \in S_{\mathrm{ram}}} \varepsilon_M(f_v, \gamma) f_M^v(\gamma)$$

in (25.23) is supported on the subset $\Gamma_{G\text{-reg}}(M)$ of G-regular classes in $\Gamma(M)_S$. It is in fact supported on classes γ that are $G(F_{w_i})$-conjugate to γ_1. For the group G at hand, any such class is actually $G(F)$-conjugate to γ_1, and hence equal to $w_s^{-1}\gamma_1 w_s$, for some $s \in W(M)$. But

$$\varepsilon_M(f_v, w_s^{-1}\gamma_1 w_s) f_M^v(w_s^{-1}\gamma_1 w_s) = \varepsilon_M(f_v, \gamma_1) f_M^v(\gamma_1).$$

Moreover, since γ_1 is F-elliptic in M, the coefficients

$$a^M(w_s^{-1}\gamma_1 w_s) = a^M(\gamma_1) = \mathrm{vol}\big(M_{\gamma_1}(F)\backslash M_{\gamma_1}(\mathbb{A})^1\big)$$

are all positive. The vanishing of the sum of (25.23) and (25.21) thus reduces to the identity

$$\varepsilon_M(f, \gamma_1) = \sum_{v \in S_{\mathrm{ram}}} \varepsilon_M(f_v, \gamma_1) f_M^v(\gamma_1) = 0.$$

This holds for any choice of functions $f_v \in \mathcal{H}\big(G(F_v)\big)$ at the places $v \in S_{\mathrm{ram}}$.

Consider a fixed valuation $v \in S_{\mathrm{ram}}$. It follows from what we have just established that if $f_{v,G}(\gamma_1) = 0$, then $\varepsilon_M(f_v, \gamma_1) = 0$. This in turn implies that if f_v is arbitrary, then

$$\varepsilon_M(f_v, \gamma_1) = \varepsilon_v(\gamma_1) f_{v,M}(\gamma_1),$$

for a complex number $\varepsilon_v(\gamma_1)$ depending on the chosen element $\gamma_1 \in M(F)$. Now, it is known that $G(F)$ is dense in $G(F_S)$, for any finite set $S \supset S_{\mathrm{ram}}$. Letting the G-regular point $\gamma_1 \in M(F)$ vary, we see that

$$\varepsilon_M(f_v, \gamma_v) = \varepsilon_v(\gamma_v) f_{v,M}(\gamma_v), \qquad \gamma_v \in \Gamma_{G\text{-reg}}\big(M(F_v)\big),\ f_v \in \mathcal{H}\big(G(F_v)\big),$$

for a function ε_M on $\Gamma_{G\text{-reg}}\big(M(F_v)\big)$ that is smooth.

The last identity is a watershed. It represents a critical global contribution to a local problem. It is also the input for one of the elementary applications of the local trace formula in the article [**A18**]. The result in question is Theorem 3C of

[**A18**], which asserts that the function $\varepsilon_M(\gamma_v)$ actually vanishes. We can therefore conclude that the linear form

$$\varepsilon_M(f_v, \gamma_v) = I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v), \qquad f \in \mathcal{H}\big(G(F_v)\big),$$

vanishes for any G-regular class γ_v in $M(F_v)$. It is then not hard to see from the definitions (18.3), (18.12), (23.3) and (25.3) that the linear form vanishes for any element $\gamma_v \in M(F_v)$ at all.

The fourth step we have just sketched completes the induction argument on M. Indeed, the general identity (25.18) follows for any S from the splitting formula (23.8), and the case $S = \{v\}$ just established. In particular, the required identity (25.10) is valid for any M. We have already noted that (25.18) implies the companion identity (25.19). In particular, both required local identities (25.10) and (25.12) of the two theorems are valid for any M.

The last step is to extract what remains of the required global identities (25.11) and (25.13) from the properties of the expressions (25.21) and (25.23) we have found. Since we have completed the induction argument on M, and since $\mathcal{H}\big(G(\mathbb{A}), M_0\big)$ equals $\mathcal{H}\big(G(\mathbb{A})\big)$ by definition, the identity (25.27) holds for any function $f \in \mathcal{H}\big(G(\mathbb{A})\big)$. The sum in (25.27) can be taken over a finite set that depends only on a choice of open compact subgroup $K_0 \subset G(\mathbb{A}_{\mathrm{fin}})$ under which f is bi-invariant. It is then not hard to show that the coefficients

$$a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^{G}(\pi), \qquad \pi \in \Pi_{t,\mathrm{disc}}^{*}(G),$$

in (25.27) vanish. This completes the proof of (25.13). Since (25.27) vanishes for any f, so does the expression (25.22). We have already established that (25.20) vanishes. It follows that the remaining expression (25.21) vanishes for any $f \in \mathcal{H}(G)$. By varying f, one deduces that the coefficients

$$a^{G,\mathcal{E}}(zu) - a^{G}(zu), \qquad z \in A_G(F),\ u \in \Gamma_{\mathrm{unip}}(G)_S,$$

in (25.21) vanish. This completes the proof of (25.11). It also finishes the original induction argument on n. (See [**AC**, §2.16] and [**A18**, §2–3].) □

For global applications, the most important assertion of the two theorems is the identity (25.13) of global coefficients. It implies that

$$I_{t,\mathrm{disc}}(f) = I_{t,\mathrm{disc}}^{*}(f^{*}), \tag{25.28}$$

for any $t \geq 0$ and $f \in \mathcal{H}(G)$. Given the explicit definition (21.19) of $I_{t,\mathrm{disc}}(f)$, one could try to use (25.28) to establish an explicit global correspondence $\pi \to \pi^*$ from automorphic representations in the discrete spectrum of G to automorphic representations in the discrete spectrum of G^*. However, this has not been done. So far as I know, the best results are due to Vigneras [**Vi**], who establishes the correspondence in the special case that for any v, $G(F_v)$ is either the multiplicative group of a division algebra, or is equal to $GL(n, F_v)$. (See also [**HT**].) Since the local condition implies that $G(F)\backslash G(\mathbb{A})^1$ is compact, this special case relies only on the trace formula for compact quotient, and a simple version of the trace formula (such as that of Corollary 23.6) for $GL(n)$. The general problem seems to be accesssible, at least in part, and would certainly be interesting.

26. Functoriality and base change for $GL(n)$

The third application of the invariant trace formula is to cyclic base change for $GL(n)$. This again entails a comparison of trace formulas. The base change comparison is very similar to that for inner twistings of $GL(n)$. We recall that the two were actually treated together in [**AC**]. Having just discussed the inner twisting comparison in some detail, we shall devote most of this section to some broader questions related to base change.

Base change is a special case of Langlands' general principle of functoriality. It is also closely related to a separate case of functoriality, Langlands' conjectural formulation of nonabelian class field theory. We have alluded to functoriality earlier, without actually stating it. Let us make up for this omission now.

For the time being, G is to be a general group over the number field F. In fact, we regard G as a group over some given extension k of F. The theory of algebraic groups assigns to G a canonical based root datum

$$\Psi(G) = (X, \Delta, X^\vee, \Delta^\vee),$$

equipped with an action of the Galois group

$$\Gamma_k = \mathrm{Gal}(\overline{k}/k).$$

Recall that there are many based root data attached to G. They are in bijection with pairs (B, T), where T is a maximal torus in G, and B is a Borel subgroup of G containing T. However, there is a *canonical* isomorphism between any two of them, given by any inner automorphism of G between the corresponding two pairs. It is this property that gives rise to the canonical based root domain $\Psi(G)$. By construction, the group $\mathrm{Aut}\big(\Psi(G)\big)$ of automorphisms of $\Psi(G)$ is canonically isomorphic to the group

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$$

of outer automorphisms of G. The Γ_k-action on $\Psi(G)$ comes from a choice of isomorphism ψ_s from G to a split group G_s^*. It is given by the homomorphism from Γ_k to $\mathrm{Out}(G)$ defined by

$$\sigma \longrightarrow \psi_s \circ \sigma(\psi_s)^{-1}, \qquad \sigma \in \Gamma_k.$$

(See [**Spr2**, §1], [**Ko3**, (1.1)–(1.2)].)

Recall that a *splitting* of G is a pair (B, T), together with a set $\{X_\alpha : \alpha \in \Delta\}$ of nonzero vectors in the associated root spaces $\{\mathfrak{g}_\alpha : \alpha \in \Delta\}$. There is a canonical isomorphism from the group $\mathrm{Out}(G)$, and hence also the group $\mathrm{Aut}\big(\Psi(G)\big)$, onto the group of automorphisms of G that preserve a given splitting [**Spr2**, Proposition 2.13]. Recall also that an action of any finite group by automorphisms on G is called an *L-action* if it preserves some splitting of G. We define a *dual group* of G to be a complex reductive group $\widehat{G}$, equipped with an L-action of Γ_k, and a Γ_k-isomorphism from $\Psi(\widehat{G})$ to the dual

$$\Psi(G)^\vee = (X^\vee, \Delta^\vee, X, \Delta)$$

of $\Psi(G)$. Suppose for example that G is a torus T. Then

$$\Psi(T) = \big(X(T), \emptyset, X(T)^\vee, \emptyset\big),$$

where $X(T)^\vee = \mathrm{Hom}\big(X(T), \mathbb{Z}\big)$ is the dual of the additive character group $X(T)$. The dual group of T is the complex dual torus

$$\widehat{T} = X(T) \otimes \mathbb{C}^*,$$

defined as a tensor product over $\mathbb{Z}$ of two abelian groups. In general, $\widehat{G}$ comes with the structure that assigns to any pair (B,T) for G, and any pair $(\widehat{B},\widehat{T})$ for $\widehat{G}$, a Γ_k-isomorphism from $\widehat{T}$ to a dual torus for T.

An *L-group* for G can take one of several forms. The Galois form is a semidirect product

$$^L G = \widehat{G} \rtimes \Gamma_k,$$

with respect to the L-action of Γ_k on $\widehat{G}$. For many purposes, one can replace the profinite group Γ_k with a finite group $\Gamma_{k'/k} = \mathrm{Gal}(k'/k)$, for a Galois extension k'/k over which G splits. For example, if G is a group such as $GL(n)$ that splits over k, one can often work with $\widehat{G}$ instead of the full L-group. If k is a local or global field, one sometimes replaces Γ_k with the corresponding Weil group W_k, which we recall is a locally compact group equipped with a continuous homomorphism into Γ_k [**Tat2**]. The Weil form of the L-group is a semidirect product

$$^L G = \widehat{G} \rtimes W_k$$

obtained by pulling back the L-action from Γ_k to W_k. The symbol $^L G$ is generally used in this way to denote any of the forms of the L-group. Suppose that k is the completion F_v of F with respect to a valuation v. The local Galois group Γ_{F_v} or Weil group W_{F_v} comes with a conjugacy class of embeddings into its global counterpart Γ_F or W_F. There is consequently a conjugacy class of embeddings of the local L-group $^L G_v$ into $^L G$, which is trivial on $\widehat{G}$.

Suppose that as a group over F, G is unramified at a given place v. As we recall, this means that v is nonarchimedean, that G is quasisplit over F_v, and that G splits over a finite unramified extension F'_v of F_v. We recall also that $\Gamma_{F'_v/F_v}$ is a finite cyclic group, with a canonical generator the Frobenius automorphism Frob_v. We take the finite form

$$^L G_v = \widehat{G} \rtimes \Gamma_{F'_v/F_v}$$

of the L-group of G over F_v determined by the outer automorphism Frob_v of $\widehat{G}$. We can choose a pair (B_v, T_v) defined over F_v such that the torus T_v splits over F'_v, and a hyperspecial maximal compact subgroup K_v of $G(F_v)$ that lies in the apartment of T_v [**Ti**]. The *unramified* representations of $G(F_v)$ (relative to K_v) are the irreducible representations whose restrictions to K_v contain the trivial representation.

If λ belongs to the space $\mathfrak{a}^*_{T_v,\mathbb{C}}$, and $1_{v,\lambda}$ is the unramified quasicharacter

$$t_v \longrightarrow q_v^{-\lambda(H_{T_v}(t_v))}, \qquad t_v \in T_v(F_v),$$

the induced representation $\mathcal{I}_{B_v}(1_{v,\lambda})$ contains the trivial representation of K_v with multiplicity 1. This representation need not be irreducible. However, it does have a unique irreducible constituent $\pi_{v,\lambda}$ that contains the trivial representation of K_v, and is hence unramified. Obviously $\pi_{v,\lambda}$ depends only on the image of λ in the quotient of $\mathfrak{a}^*_{T_v,\mathbb{C}}$ by the discrete subgroup

$$\Lambda_v = \Big(\frac{2\pi i}{\log q_v}\Big)\mathrm{Hom}(\mathfrak{a}_{T_v,F_v},\mathbb{Z}) = \frac{i}{\log q_v}\mathfrak{a}^\vee_{T_v,F_v}.$$

It also depends only on the orbit of λ under the restricted Weyl group W_{v0} of (G, A_{T_v}). The correspondence $\lambda \to \pi_{v,\lambda}$ is thus a mapping from the quotient

$$W_{v0}\backslash\mathfrak{a}^*_{T_v,\mathbb{C}}/\Lambda_v \tag{26.1}$$

to the set of unramified representations of $G(F_v)$. One shows that the mapping is a bijection. (See [**Ca**], for example.) On the other hand, there is a canonical homomorphism $\lambda \to q_v^{-\lambda}$ from $\mathfrak{a}^*_{T_v,\mathbb{C}}/\Lambda_v$ to the complex torus $\widehat{T}_v$, which takes a point in (26.1) to a W_{v0}-orbit of points in $\widehat{T}_v$. One shows that the correspondence

$$\lambda \longrightarrow q_v^{-\lambda} \rtimes \mathrm{Frob}_v$$

is a bijection from (26.1) onto the set of semisimple conjugacy classes in ${}^LG_v$ whose image in $\Gamma_{F'_v/F_v}$ equals Frob_v. (See [**Bor3**, (6.4), (6.5)], for example.) It follows that there is a canonical bijection

$$\pi_v \longrightarrow c(\pi_v)$$

from the set of unramified representations of $G(F_v)$ onto the set of semisimple conjugacy classes in ${}^LG_v$ that project to Frob_v. This mapping is due to Langlands [**Lan3**], and in itself justifies the introduction of the L-group.

The reader may recall that the symbol $c(\pi_v)$ also appeared earlier. It was introduced in §2 (in the special case $F = \mathbb{Q}$) to denote the homomorphism from the unramified Hecke algebra $\mathcal{H}_v = \mathcal{H}(G_v, K_v)$ to $\mathbb{C}$ attached to π_v. The two uses of the symbol are consistent. They are related by the Satake isomorphism from $\mathcal{H}_v$ to the complex co-ordinate algebra on the space (26.1). (See [**Ca**, (4.2)], for example. By the co-ordinate algebra on (26.1), we mean the subalgebra of W_{v0}-invariant functions in the co-ordinate algebra of $\mathfrak{a}^*_{T_v,\mathbb{C}}/\Lambda_v$, regarded as a subtorus of $\widehat{T}_v$.) The complex valued homomorphisms of $\mathcal{H}_v$ are therefore bijective with the points in (26.1), and hence with the set of semisimple conjugacy classes in ${}^LG_v$ that project to Frob_v.

Suppose now that π is an automorphic representation of G. Then $\pi = \bigotimes \pi_v$, where π_v is unramified for almost all v. We choose a finite form ${}^LG = \widehat{G} \rtimes \Gamma_{F'/F}$ of the global L-group, for some finite Galois extension F' of F over which G splits, and a finite set of valuations S outside of which π and F' are unramified. For any $v \notin S$, we then write $c_v(\pi)$ for the image of $c(\pi_v)$ under the canonical conjugacy class of embeddings of ${}^LG_v = \widehat{G} \rtimes \Gamma_{F'_v/F_v}$ into ${}^LG$. This gives a correspondence

$$\pi \longrightarrow c(\pi) = \{c_v(\pi) : \ v \notin S\}$$

from automorphic representations of G to families of semisimple conjugacy classes in ${}^LG$. The construction becomes independent of the choice of F' and S if we agree to identify to families of conjugacy classes that are equal almost everywhere.

An automorphic representation thus carries some very concrete data, namely the complex parameters that determine the conjugacy classes in the associated family. The interest stems not so much from the values assumed by individual classes $c_v(\pi)$, but rather in the relationships among the different classes implicit in the requirement that π be automorphic. Following traditions from number theory and algebraic geometry, Langlands wrapped the data in analytic garb by introducing an unramified L-function

$$L^S(s, \pi, r) = \prod_{v \notin S} \det\big(1 - r\big(c_v(\pi)\big)q_v^{-s}\big)^{-1}, \tag{26.2}$$

for any automorphic representation π, any reasonable finite dimensional representation

$$r : \ {}^LG \longrightarrow GL(N, \mathbb{C}),$$

and any finite set S of valuations outside of which π and r are unramified. He observed that the product converged for $\mathrm{Re}(s)$ large, and conjectured that it had analytic continuation with functional equation.

Langlands' principle of functoriality [**Lan3**] postulates deep and quite unexpected reciprocity laws among the families $c(\pi)$ attached to different groups. Assume that G is quasisplit over F, and that G' is a second connected reductive group over F. Suppose that

$$\rho: \ {}^LG' \longrightarrow {}^LG$$

is an L-homomorphism of L-groups. (Besides satisfying the obvious conditions, an L-homomorphism between two groups that each project onto a common Galois or Weil group is required to be compatible with the two projections.) The principle of functoriality asserts that for any automorphic representation π' of G', there is an automorphic representation π of G such that

$$c(\pi) = \rho\big(c(\pi')\big). \tag{26.3}$$

In other words, $c_v(\pi) = \rho\big(c_v(\pi')\big)$ for every valuation v outside some finite set S. Functoriality thus postulates a correspondence $\pi' \to \pi$ of automorphic representations, which depends only on the $\widehat{G}$-orbit of ρ. We shall recall three basic examples.

Suppose that G is an inner form of a quasiplit group G^*, equipped with an inner twist

$$\psi: \ G \to G^*.$$

In other words, ψ is an isomorphism such that $\psi \circ \sigma(\psi)^{-1}$ is an inner automorphism of G^* for every $\sigma \in \Gamma_F$. It determines an L-isomorphism

$${}^L\psi: \ {}^LG \longrightarrow {}^LG^*,$$

which allows us to identify the two L-groups. Functoriality asserts that the set of automorphic families $\{c(\pi)\}$ of conjugacy classes for G is contained in the set of such families $\{c(\pi^*)\}$ for G^*. Our last section was devoted to the study of this question in the case $G^* = GL(n)$. It is pretty clear from the conclusion (25.28), together with the explicit formula for $I_{t,\mathrm{disc}}(f)$ and the fact that $f_v = f_v^*$ for almost all v, that something pretty close to the assertion of functoriality holds in this case. However, the precise nature of the correspondence remains open.

Langlands introduced the second example in his original article [**Lan3**], as a particularly vivid illustration of the depth of functoriality. It concerns the case that G is an arbitrary quasisplit group, and G' is the trivial group $\{1\}$. The L-group ${}^LG'$ need not be trivial, since it can take the form of the Galois group Γ_F. Functoriality applies to a continuous homomorphism

$$\rho: \ \Gamma_F \longrightarrow {}^LG$$

whose composition with the projection of ${}^LG$ on Γ_F equals the identity. Since Γ_F is totally disconnected, ρ can be identified with an L-homomorphism from $\Gamma_{F'/F}$ to the restricted form ${}^LG = \widehat{G} \rtimes \Gamma_{F'/F}$ of the L-group of G given by some finite Galois extension F' of F. Let S be any finite set of valuations v of F outside of which F' is unramified. Then for any $v \notin S$, $F'_{v'}/F_v$ is an unramified extension of local fields, for any (normalized) valuation v' of F' over v. Its Galois group is cyclic, with a canonical generator $\mathrm{Frob}_v = \mathrm{Frob}_{v,F'}$, whose conjugacy class in $\Gamma_{F'/F}$ is independent of the choice of v'. Thus, ρ gives rise to a family

$$\{\rho(\mathrm{Frob}_v) : \ v \notin S\}$$

of conjugacy classes of finite order in ${}^L G$. If π' is the trivial automorphic representation of $G' = \{1\}$ and $v \notin S$, the image of $c_v(\pi')$ in the group ${}^L G' = \Gamma_{F'/F}$ equals Frob_v, by construction. Functoriality asserts that there is an automorphic representation π of G such that for any $v \notin S$, the class $c_v(\pi)$ in ${}^L G$ equals $\rho(\mathrm{Frob}_v)$. A more general assertion applies to the Weil form of ${}^L G'$. In this form, functoriality attaches an automorphic representation π to any L-homomorphism

$$\phi : \ W_F \longrightarrow {}^L G$$

of the global Weil group into ${}^L G$.

The third example is general base change. It applies to an arbitrary group G' over F, and a finite extension E of F over which G' is quasisplit. Given these objects, we take G to be the group $R_{E/F}(G'_E)$ over F obtained from the quasisplit group G' over E by restriction of scalars. Following [**Bor3**, §4–5], we identify $\widehat{G}$ with the group of functions g from Γ_F to $\widehat{G}'$ such that

$$g(\sigma\tau) = \sigma g(\tau), \qquad \sigma \in \Gamma_E, \ \tau \in \Gamma_F,$$

with pointwise multiplication, and Γ_F-action

$$(\tau_1 g)(\tau) = g(\tau\tau_1), \qquad \tau, \tau_1 \in \Gamma_F.$$

We then obtain an L-homomorphism

$$\rho : \ {}^L G' \longrightarrow {}^L G$$

by mapping any $g' \in {}^L G'$ to the function

$$g(\tau) = \tau g', \qquad \tau \in \Gamma_F,$$

on Γ_F. This case of functoriality can be formulated in slightly more concrete terms. The restriction of scalars functor provides a canonical isomorphism from $G(\mathbb{A})$ onto $G'(\mathbb{A}_E)$, which takes $G(F)$ to $G'(E)$. The automorphic representations of G are therefore in bijection with those of G'_E. This means that we can work with the L-group ${}^L G'_E = \widehat{G}' \rtimes \Gamma_E$ of G'_E instead of ${}^L G$. Base change becomes a conjectural correspondence $\pi' \to \pi$ of automorphic representations of G' and G'_E such that for any valuation v of F for which π' and E are unramified, and any valuation w of E over v, the associated conjugacy classes are related by

$$c_v(\pi') = c_w(\pi)^{f_w}, \qquad f_w = \deg(E_w/F_v).$$

We should bear in mind that Langlands also postulated a local principle of functoriality. This takes the form of a conjectural correspondence $\pi'_v \to \pi_v$ of irreducible representations of $G'(F_v)$ and $G(F_v)$, for any v and any local L-homomorphism ρ_v of local L-groups, which is compatible with the global functoriality correspondence $\pi' \to \pi$. Representations π_v of the local groups $G(F_v)$ are important for the functional equations of L-functions (among many other things). Langlands conjectured the existence of local L-functions $L(s, \pi_v, r_v)$, which reduce to the relevant factors of (26.2) in the unramified case, and local ε-factors

$$\varepsilon(s, \pi_v, r_v, \psi_v) = a q_v^{-bs}, \qquad a \in \mathbb{C}, \ b \in \mathbb{Z},$$

which equal 1 in the unramified case, such that the finite product

$$\varepsilon(s,\pi,r) = \prod_v \varepsilon(s,\pi_v,r_v,\psi_v)$$

is independent of the nontrivial additive character ψ of $\mathbb{A}/F$ of which ψ_v is the restriction, and such that the product

$$L(s,\pi,r) = \Big(\prod_{v\notin S} L(s,\pi_v,r_v)\Big)L^S(s,\pi,r)$$

satisfies the functional equation

$$L(s,\pi,r) = \varepsilon(s,\pi,r)L(1-s,\pi,r^\vee). \tag{26.4}$$

We have written $r^\vee$ here for the contragredient of the representation r. The local L-functions and ε-factors should be compatible with the local version of functoriality, in the sense that

$$L(s,\pi'_v,r_v\circ\rho_v) = L(s,\pi_v,r_v)$$

and

$$\varepsilon(s,\pi_v,r_v\circ\rho_v,\psi_v) = \varepsilon(s,\pi_v,r_v,\psi_v).$$

These relations are obvious in the unramified case. In general, they imply corresponding relations for global L-functions and ε-factors.

Suppose now that $G = GL(n)$. The constructions above are, not surprisingly, more explicit in this case. There is no harm in reviewing them in concrete terms.

Let v be a nonarchimedean valuation, and take (B_v,T_v) to be the standard pair (B,M_0). If λ belongs to $\mathfrak{a}^*_{T_v,\mathbb{C}} \cong \mathbb{C}^n$, the induced representation $\mathcal{I}_{B_v}(1_{v,\lambda})$ acts by right translation on the space of functions ϕ on $G(F_v)$ such that

$$\phi(bx) = |b_{11}|^{\lambda_1+\frac{n-1}{2}}|b_{22}|^{\lambda_2+\frac{n-3}{2}}\dots|b_{nn}|^{\lambda_n-(\frac{n-1}{2})}\phi(x), \quad b\in B_v(F_v),\ x\in G(F_v).$$

It has a unique irreducible constituent $\pi_{v,\lambda}$ that contains the trivial representation of $K_v = GL(n,\mathfrak{o}_v)$. Two such representations $\pi_{v,\lambda'}$ and $\pi_{v,\lambda}$ are equivalent if and only if the corresponding vectors $\lambda',\lambda\in\mathbb{C}^n$ are related by

$$(\lambda'_1,\dots,\lambda'_n) \equiv (\lambda_{\sigma(1)},\dots,\lambda_{\sigma(n)})\Big(\mathrm{mod}\Big(\frac{2\pi i}{\log\, q_v}\Big)\mathbb{Z}^n\Big),$$

for some permutation $\sigma\in S_n$. The dual group $\widehat{G}$ equals $GL(n,\mathbb{C})$. We give it the canonical structure, which assigns to the standard pairs (B,T) and $(\widehat{B},\widehat{T})$ in G and $\widehat{G}$ the obvious isomorphism of $\widehat{T}$ with the complex dual torus of T. Since the action of Γ_F on $\widehat{G}$ is trivial, we can take the restricted form ${}^LG = \widehat{G}$ of the L-group. The semisimple conjugacy class of the representation $\pi_{v,\lambda}$ is then given by

$$c(\pi_{v,\lambda}) = \left\{\begin{pmatrix} q_v^{-\lambda_1} & & 0\\ & \ddots & \\ 0 & & q_v^{-\lambda_n}\end{pmatrix}\right\}.$$

Given an automorphic representation π of $GL(n)$, let S be any finite set of valuations outside of which π is unramified. Then π gives rise to a family

$$c(\pi) = \big\{c_v(\pi) = c(\pi_v):\ v\notin S\big\}$$

of semisimple conjugacy classes in $\widehat{G} = GL(n,\mathbb{C})$. It is known that if π occurs in the spectral decomposition of $L^2\big(G(F)\backslash G(\mathbb{A})\big)$, it is uniquely determined by the family $c(\pi)$ [**JaS**]. This remarkable property is particular to $G = GL(n)$.

Consider a continuous n-dimensional representation r of Γ_F. Then r lifts to a representation of a finite group $\Gamma_{F'/F}$, for a finite Galois extension F' of F. We may as well take F' to be the minimal such extension, for which $\Gamma_{F'}$ is the kernel of r. Let S be any finite set of valuations outside of which F' is unramified. The representation r then gives rise to a family

$$c(r) = \big\{c_v(r) = r(\mathrm{Frob}_v) : \ v \notin S\big\}$$

of semisimple conjugacy classes in $\widehat{G} = GL(n, \mathbb{C})$. As in the automorphic setting, the equivalence class of r is uniquely determined by $c(r)$. For the Tchebotarev density theorem characterizes F' as the Galois extension of F for which

$$\mathrm{Spl}_{F'/F} = \{v \notin S : \ c_v(r) = 1\}$$

is the set of valuations outside of S that split completely in F'. Since the Tchebotarev theorem deals in densities of subsets, the characterization is independent of the choice of S. The theorem also implies that every conjugacy class in the group $\Gamma_{F'/F}$ is of the form Frob_v, for some $v \notin S$. The character of r is therefore determined by the family $c(r)$.

According to the second example of functoriality above, specialized to the case that $G = GL(n)$, there should be an automorphic representation π attached to any r such that $c(\pi) = c(r)$. Consider the further specialization to the case that $n = 1$. The one dimensional characters of the group Γ_F are the characters of its abelianization Γ_F^{ab}. The case $n = 1$ of Langlands' Galois representation conjecture could thus be interpreted as the existence of a surjective dual homomorphism

$$GL(1, F)\backslash GL(1, \mathbb{A}) = F^*\backslash \mathbb{A}^* \ \longrightarrow \ \Gamma_F^{\mathrm{ab}}. \tag{26.5}$$

The condition $c(\pi) = c(r)$ specializes to the requirement that the composition of (26.5) with the projection of Γ_F^{ab} onto the Galois group of any finite abelian extension F' of F satisfy

$$x_v \ \longrightarrow \ (\mathrm{Frob}_v)^{\mathrm{ord}(x_v)}, \qquad\qquad x_v \in F_v^*,$$

where v is any valuation that is unramified in F', Frob_v is the corresponding Frobenius element in the abelian group $\Gamma_{F'/F}$, and

$$\mathrm{ord}(x_v) = -\log_{q_v}(|x_v|).$$

The mapping (26.5) has been known for many years. It is the Artin reciprocity law, which is at the heart of class field theory. (See [**Has**], [**Tat1**].) Langlands' Galois representation conjecture thus represents a nonabelian analogue of class field theory. If $n = 2$ and $\Gamma_{F'/F}$ is solvable, it was established as a consequence of cyclic base change for $GL(2)$ [**Lan9**], [**Tu**]. If n is arbitrary and $\Gamma_{F'/F}$ is nilpotent, it is a consequence [**AC**, Theorem 3.7.3] of cyclic base change for $GL(n)$, the ostensible topic of this section. Other cases for $n = 2$ have been established [**BDST**], as have a few other cases in higher rank.

Besides extending class field theory, Langlands' Galois representation conjecture has important implications for Artin L-functions

$$L^S(s, r) = \prod_{v \notin S} \det\big(1 - r(\mathrm{Frob}_v) q_v^{-s}\big)^{-1}.$$

If r corresponds to π, it is clear that

$$L^S(s, r) = L^S(s, \pi), \tag{26.6}$$

where $L^S(s,\pi)$ is the automorphic L-function for $GL(n)$ relative to the standard, n-dimensional representation of ${}^LG = GL(n,\mathbb{C})$. It has been known for some time how to construct the local L-functions and ε-factors in this case so that the functional equation (26.4) holds [**GoJ**]. These results are now part of the larger theory of Rankin-Selberg L-functions $L(s,\pi_1\times\pi_2)$, attached to representations $\pi_1\otimes\pi_2$ of $GL(n_1)\times GL(n_2)$, and the representation

$$(g_1,g_2):\ X \longrightarrow g_1Xg_2^{-1}, \qquad\qquad X\in M_{n_1\times n_2}(\mathbb{C}),$$

of $GL(n_1,\mathbb{C})\times GL(n_2,\mathbb{C})$ [**JPS**]. In fact, there is a broader theory still, known as the Langlands-Shahidi method, which exploits the functional equations from the theory of Eisenstein series. It pertains to automorphic L-functions of a maximal Levi subgroup M of a given group, and the representation of ${}^LM$ on the Lie algebra of a unipotent radical [**GS**]. Be that as it may, our refined knowledge of the automorphic L-function in the special case encompassed by the right hand side of (26.6) would establish critical analytic properties of the Artin L-function on the left hand side of (26.6).

The Langlands conjecture for Galois representations (which we re-iterate is but a special case of functoriality) is still far from being solved in general. However, it plays an important role purely as a conjecture in motivating independent operations on automorphic representations. Nowhere is this more evident than in the question of cyclic base change of prime order for the group $GL(n)$.

Suppose that E is a Galois extension of F, with cyclic Galois group $\{1,\sigma,\dots,\sigma^{\ell-1}\}$ of prime order ℓ. To be consistent with the description of base change above, we change notation slightly. We write r' instead of r for a continuous n-dimensional representation of Γ_F, leaving r to stand for a continuous n-dimensional representation of Γ_E. Regarded as equivalence classes of representations, these two families come with two bijections $r\to r^\sigma$ and $r'\to r'\otimes\eta$ of order ℓ, where $r^\sigma(\tau)=r(\sigma\tau\sigma^{-1})$, and η is the pullback to Γ_F of the character on $\Gamma_{E/F}$ that maps the generator σ to $e^{\frac{2\pi i}{\ell}}$. The main operation is the mapping $r'\to r$ obtained by restricting r' to the subgroup Γ_E of Γ_F. This mapping is characterized in terms of conjugacy classes by the relation

$$c_w(r)=\begin{cases} c_v(r'), & \text{if } v \text{ splits in } E,\\ c_v(r')^\ell, & \text{otherwise,}\end{cases}$$

for any valuation v of F at which r' and E are unramified and any valuation w over v, and satisfies the following further conditions.

(i) The image of the mapping is the set of r with $r^\sigma = r$.

(ii) If r' is irreducible, the fibre of its image equals

$$\{r', r'\otimes\eta,\dots,r'\otimes\eta^{\ell-1}\}.$$

(iii) If r' is irreducible, its image r is irreducible if and only if $r'\neq r'\otimes\eta$, which is to say that the fibre in (ii) contains ℓ elements.

(iv) If r' is irreducible and $r'=r'\otimes\eta$, its image equals a direct sum

$$r = r_1\oplus r_1^\sigma\oplus\cdots\oplus r_1^{\sigma^{\ell-1}},$$

for an irreducible representation r_1 of degree $n_1 = n\ell^{-1}$ such that $r_1^\sigma\neq r_1$. Conversely, the preimage of any such direct sum consists of a representation r' that is irreducible and satisfies $r'=r'\otimes\eta$.

These conditions are all elementary consequences of the fact that Γ_E is a normal subgroup of prime index in Γ_F. For example, the representation r' in (iv) is obtained by induction of the representation r_1 from Γ_E to Γ_F.

Base change is a mapping of automorphic representations with completely parallel properties. We write π' and π for (equivalence classes of) automorphic representations of $GL(n)_F$ and $GL(n)_E$ respectively. The two families come with bijections $\pi \to \pi^\sigma$ and $\pi' \to \pi' \otimes \eta$ of order ℓ, where $\pi^\sigma(x) = \pi\big(\sigma(x)\big)$, and η has been identified with the 1-dimensional automorphic representation of $GL(n)_F$ obtained by composing the determinant on $GL(n,\mathbb{A})$ with the pullback of η to $GL(1,\mathbb{A})$ by (26.5). The results of [**AC**] were established for cuspidal automorphic representations, and the larger class of "induced cuspidal" representations. For $GL(n)_E$, this larger class consists of induced representations

$$\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_p = \mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_p),$$

where P is the standard parabolic subgroup of $GL(n)$ corresponding to a partition $(n_1, \dots, n_p)$, and π_i is a *unitary* cuspidal automorphic representation of $GL(n_i)_E$. Any such representation is automorphic, by virtue of the theory of Eisenstein series.

THEOREM 26.1. (Base change for $GL(n)$). *There is a mapping $\pi' \to \pi$ from induced cuspidal automorphic representations of $GL(n)_F$ to induced cuspidal automorphic representations of $GL(n)_E$, which is characterized by the relation*

$$c_w(\pi) = \begin{cases} c_v(\pi'), & \text{if } v \text{ splits in } E, \\ c_v(\pi')^\ell, & \text{otherwise,} \end{cases} \tag{26.7}$$

for any valuation v of F at which π' and E are unramified and any valuation w of F over v, and which satisfies the following further conditions.

(i) *The image of the mapping is the set of π with $\pi^\sigma = \pi$.*

(ii) *If π' is cuspidal, the fibre of its image equals*

$$\{\pi', \pi' \otimes \eta, \dots, \pi' \otimes \eta^{\ell-1}\}.$$

(iii) *If π' is cuspidal, its image π is cuspidal if and only if $\pi' \neq \pi' \otimes \eta$, which is to say that the fibre in (ii) contains ℓ elements.*

(iv) *If π' is cuspidal and $\pi' = \pi' \otimes \eta$, its image equals a sum*

$$\pi = \pi_1 \boxtimes \pi_1^\sigma \boxtimes \cdots \boxtimes \pi_1^{\sigma^{\ell-1}},$$

for a cuspidal automorphic representation π_1 of $GL(n_1)_E$ such that $\pi_1^\sigma \neq \pi_1$. Conversely, the preimage of any such sum consists of a representation π' that is cuspidal and satisfies $\pi' = \pi' \otimes \eta$.

Remark. The theorem provides two mappings of cuspidal automorphic representations. Base change gives an ℓ to 1 mapping $\pi' \to \pi$, from the set of cuspidal representations of π' of $GL(n)_F$ with $\pi' \neq \pi' \otimes \eta$ onto the set of cuspidal representations π of $GL(n)_E$ with $\pi = \pi^\sigma$. The second mapping is given by (iv), and is known as *automorphic induction*. It is an ℓ to 1 mapping $\pi_1 \to \pi'$, from the set of cuspidal autmorphic representations π_1 of $GL(n_1)_E$ with $\pi_1 \neq \pi_1^\sigma$ onto the set of cuspidal automorphic representations π' of $GL(n)_E$ with $\pi' = \pi' \otimes \eta$.

Theorem 26.1 contains the main results of [**AC**]. It is proved by a comparison of two trace formulas. One is the invariant trace formula for the group $GL(n)_F$.

The other is the invariant twisted trace formula, applied to the automorphism of the group $R_{E/F}\big(GL(n)_E\big)$ determined by σ.

The twisted trace formula is a generalization of the ordinary trace formula. It applies to an F-rational automorphism θ of finite order of a connected reductive group G over F. The twisted trace formula was introduced by Saito for classical modular forms [**Sai**], by Shintani for the associated automorphic representations of $GL(2)$ (see [**Shin**]), and by Langlands for general automorphic representations of $GL(2)$ [**Lan9**]. The idea, in the special case of compact quotient, for example, is to express the trace of an operator

$$R(f) \circ \theta, \qquad f \in \mathcal{H}\big(G(\mathbb{A})\big),$$

in terms of twisted orbital integrals

$$\int_{G_{\theta\gamma}(A)\backslash G(\mathbb{A})} f\big(x^{-1}\gamma\theta(x)\big)\mathrm{d}x, \qquad \gamma \in G(F).$$

This gives a geometric expression for a sum of twisted characters

$$\sum_{\pi} m(\pi)\mathrm{tr}\big(\pi(f)\circ\theta\big),$$

taken over irreducible representations π of G such that $\pi^\theta = \pi$. It is exactly the sort of formula needed to quantify the proposed image of the base change map.

In general, our discussion that led to the invariant trace formula applies also to the twisted case. (See [**CLL**], [**A14**].) Most of the results in fact remain valid as stated, if we introduce a minor change in notation. We take G to be a connected component of a (not necessarily connected) reductive group over F such that $G(F)$ is not empty. We write G^+ for the reductive group generated by G, and G^0 for the connnected component of 1 in G^+. We then consider distributions on $G(\mathbb{A})$ that are invariant with respect to the action of $G^0(\mathbb{A})$ on $G(\mathbb{A})$ by conjugation. The analogue of the Hecke algebra becomes a space $\mathcal{H}(G)$ of functions on a certain closed subset $G(\mathbb{A})^1$ of $G(\mathbb{A})$. The objects of Theorems 23.2, 23.3, and 23.4 can all be formulated in this context, and the invariant twisted trace formula becomes the identity of Theorem 23.4. (See [**A14**].) It holds for any G, under one condition. We require that the twisted form of the archimedean trace Paley-Wiener theorem of Clozel-Delorme [**CD**] hold for G. This condition, which was established by Rogawski in the p-adic case [**Ro2**], is needed to characterize the invariant image $\mathcal{I}(G)$ of the twisted Hecke algebra $\mathcal{H}(G)$. (See also [**KR**].)

For base change, we take

$$G = G^0 \rtimes \theta, \qquad G^0 = R_{E/F}\big(GL(n)_E\big),$$

where θ is the automorphism of G^0 defined by the generator σ of $\Gamma_{E/F}$. We also set $G' = GL(n)_F$. Our task is to compare the invariant twisted trace formula of G with the invariant trace formula of G'. The problem is very similar to the comparison for inner twistings of $GL(n)$, treated at some length in §25. In fact, we recall that the two comparisons were actually treated together in [**AC**]. We shall add only a few words here, concentrating on aspects of the problem that are different from those of §25.

The first step is to define a mapping $\gamma \to \gamma'$, which for any $k \supset F$ takes the set $\Gamma\big(G(k)\big)$ of $G^0(k)$-orbits in $G(k)$ to the set $\Gamma\big(G'(k)\big)$ of conjugacy classes in $G'(k)$. The mapping is analogous to the injection $\gamma \to \gamma^*$ of §25. In place of the inner twist, one uses the norm mapping from number theory, which in the present context becomes the mapping $\gamma \to \gamma^\ell$ from G to G^0. Setting $k = F_v$, one uses the mapping $\gamma_v \to \gamma'_v$ to transfer twisted orbital integrals on $\Gamma_{\text{reg}}\big(G(F_v)\big)$. This gives a transformation $f_v \to f'_v$, from functions $f_v \in \mathcal{H}\big(G(F_v)\big)$ to functions f' on $\Gamma_{\text{reg}}\big(G'(F_v)\big)$. One then combines Theorem 25.1 with methods of descent to show that f'_v lies in the invariant Hecke algebra $\mathcal{I}\big(G'(F_v)\big)$.

One has then to combine the mappings $f_v \to f'_v$ into a global correspondence of adelic functions. This is more complicated than it was in §25. The problem is that $G'(F_v)$ is distinct from $G(F_v)$ at all places, not just the unramified ones. At almost all places v, we want f_v to be the characteristic function of the compact subset $K_v = K_v^0 \rtimes \theta = G(\mathfrak{o}_v)$ of $G(F_v)$, and f'_v to be the image in $\mathcal{I}\big(G'(F_v)\big)$ of the characteristic function of the maximal compact subgroup $K'_v = G'(\mathfrak{o}_v)$ of $G'(F_v)$. However, we do not know a priori that this is compatible with the transfer of orbital integrals. The assertion that the two mappings are in fact compatible is a special case of the twisted fundamental lemma. It was established in the case at hand by Kottwitz [**Ko4**]. The result of Kottwitz allows us to put the local mappings together. We obtain a mapping $f \to f'$ from $\mathcal{H}(G)$ to $\mathcal{I}(G')$, which takes any subspace $\mathcal{H}\big(G(F_S)^1\big)$ of $\mathcal{H}(G)$ to the corresponding subspace $\mathcal{I}\big(G'(F_S)^1\big)$ of $\mathcal{I}(G')$.

The next step is to extend the fundamental lemma to more general functions in an unramified Hecke algebra $\mathcal{H}\big(G(F_v), K_v^0\big)$. More precisely, one needs to show that at an unramified place v, the canonical mapping from $\mathcal{H}\big(G(F_v), K_v^0\big)$ to $\mathcal{H}\big(G'(F_v), K'_v\big)$ defined by Satake isomorphisms is compatible with the transfer of orbital integrals. This was established in [**AC**, §1.4], using the special case established by Kottwitz, and the simple forms of Corollary 23.6 of the two trace formulas. Further analysis of the two simple trace formulas allows one to establish local base change [**AC**, §1.6–1.7]. The result is a mapping $\pi'_v \to \pi_v$ of tempered representations, which satisfies local forms of the conditions of the theorem, and is the analogue of Theorem 25.1(b).

The expansions (23.11) and (23.12) represent the two sides of the invariant twisted trace formula for G. We define "endoscopic" forms $I_M^{\mathcal{E}}(\gamma, f)$, $a^{M,\mathcal{E}}(\gamma)$, $I_M^{\mathcal{E}}(\pi, f)$ and $a^{M,\mathcal{E}}(\pi)$ of the terms in the two expansions by using the mapping $f \to f'$ to pull back the corresponding terms from G'. The constructions are similar to those of §25, but with one essential difference. In the present situation, we have to average spectral objects $\widehat{I}_{M'}(\cdot, f')$ and $a^{M'}(\cdot)$ over representations $\tau \otimes \xi$, for characters ξ on $M'(\mathbb{A})$ obtained from the original character η on $\Gamma_{E/F}$. The reason for this is related to condition (ii) of the theorem, which in turn is a consequence of the fact that the norm mapping is not surjective. However, the averaging operation is not hard to handle. It is an essential part of the discussion in [**AC**, §2.10–2.12]. The identities of Theorems 25.5 and 25.6 can therefore be formulated in the present context. Their proof is more or less the same as in §25.

The analogue of the global spectral identity (25.13) (with $M = G$) is again what is most relevant for global applications. It leads directly to an identity

$$I_{t,\text{disc}}(f) = \widehat{I}'_{t,\text{disc}}(f'), \qquad f \in \mathcal{H}(G), \tag{26.8}$$

of t-discrete parts of the two trace formulas. One extracts global information from the last identity by allowing local components f_v of f to vary over unramified Hecke algebras $\mathcal{H}\big(G(F_v), K_v^0\big)$. By combining general properties of the distributions in (26.8) with operations on Rankin-Selberg L-functions $L(s, \pi_1 \times \pi_2)$, one establishes all the assertions of the theorem. (See [**AC**, Chapter 3].) □

We remark that the proof of base change in [**AC**] works only for cyclic extensions of prime degree (despite assertions in [**AC**] to the contrary). The mistake, which occurred in Lemma 6.1 of [**AC**], was pointed out by Lapid and Rogawski. In the case of $G = GL(2)$, they characterized the image of base change for a general cyclic extension by combining the special case of Theorem 26.1 established by Langlands [**Lan9**] with a second comparison of trace formulas [**LR**]. There is also a gap in the density argument at the top of p. 196 of [**AC**], which was filled in [**A29**, Lemma 8.2].

We know that the spectral decomposition for $GL(n)$ contains more than just induced cuspidal representations. In particular, the discete spectrum contains more than the cuspidal automorphic representations. The classification of the discrete spectrum for $GL(n)$ came after [**AC**]. It was established through a deep study by Moeglin and Waldspurger of residues of cuspidal Eisenstein series [**MW2**], following earlier work of Jacquet [**J**].

THEOREM 26.2. (Moeglin-Waldspurger). *The irreducible representations π of $GL(n,\mathbb{A})$ that occur in $L^2_{\mathrm{disc}}\big(GL(n,F)\backslash GL(n,\mathbb{A})^1\big)$ have multiplicity one, and are parametrized by pairs (k,σ), where $n = kp$ is divisible by k, and σ in an irreducible unitary cuspidal automorphic representation of $GL(k,\mathbb{A})$. If P is the standard parabolic subgroup of $GL(n)$ of type $(k,\dots,k)$, and ρ_σ is the nontempered representation*

$$(\sigma\otimes\cdots\otimes\sigma)\cdot\delta_P^{\frac{1}{2}}:\ m\to\big(\sigma(m_1)|\det m_1|^{\frac{p-1}{2}}\big)\otimes\cdots\otimes\big(\sigma(m_p)|\det m_p|^{-\frac{p-1}{2}}\big)$$

of $M_P(\mathbb{A}) \cong GL(k,\mathbb{A})^p$, then π is the unique irreducible quotient of the induced representation $\mathcal{I}_P(\rho_\sigma)$. □

If we combine Theorem 26.1 with Theorem 26.2 (and the theory of Eisenstein series), we obtain a base change mapping $\pi' \to \pi$ for any representation π' of $G'(\mathbb{A})$ that occurs in the spectral decomposition of $L^2\big(G'(F)\backslash G'(\mathbb{A})\big)$. It would be interesting, and presumably not difficult, to describe the general properties of this mapping. It would also be interesting to try to establish the last step of the proof of Theorem 26.1 without recourse to the argument based on L-functions in [**AC**, Chapter 3]. This might be possible with a careful study of the fine structure of the distributions on each side of (26.8).

As a postscript to this section on base change, we note that there is a suggestive way to look at the theorem of Moeglin and Waldspurger. It applies to those representations π in the discrete spectrum for which the underlying cuspidal automorphic representation σ is attached to an irreducible representation

$$\mu:\ W_F \longrightarrow GL(k,\mathbb{C})$$

of the global Weil group, according to the special case of functoriality we discussed earlier. One expects σ to be tempered. This means that μ is (conjugate to) a unitary representation, or equivalently, that its image in $GL(k,\mathbb{C})$ is bounded. We are assuming that $n = kp$, for some positive integer p. Let ν be the irreducible representation of the group $SL(2,\mathbb{C})$ of degree p. We then represent the automorphic representation π by the irreducible n-dimensional representation

$$\psi = \mu\otimes\nu:\ W_F\times SL(2,\mathbb{C}) \longrightarrow GL(n,\mathbb{C})$$

of the product of W_F with $SL(2,\mathbb{C})$. Set

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad w\in W_F,$$

where $|w|$ is the canonical absolute value on W_F. By comparing the unramified constituents of π with the unramified images of Frobenius classes in W_F, we see that π is the automorphic representation corresponding to the n-dimensional representation ϕ_ψ of W_F. Thus, according to functoriality, there is a mapping $\psi \to \pi$, from the set of irreducible n-dimensional representations of $W_F \times SL(2,\mathbb{C})$ whose restriction to W_F is bounded, into the set of automorphic representations π of $GL(n)$ that occur in the discrete spectrum. Removing the condition that ψ be irreducible gives rise to representations π that occur in the general spectrum.

27. The problem of stability

We return to the general trace formula. The invariant trace formula of Theorem 23.4 still has one serious deficiency. The invariant distributions on each side are not usually stable. We shall discuss the notion of stability, and why it is an essential consideration in any general attempt to compare trace formulas on different groups.

Stability was discovered by Langlands in attempting to understand how to generalize the Jacquet-Langlands correspondence. We discussed the extension of this correspondence from $GL(2)$ to $GL(n)$ in §25, but it is for groups other than $GL(n)$ that the problems arise. Suppose then that G is an arbitrary connected reductive group over our number field F. We fix an inner twist

$$\psi : \ G \ \longrightarrow \ G^*,$$

where G^* is a quasisplit reductive group over F. One would like to establish the reciprocity laws between automorphic representations of G and G^* predicted by functoriality.

To use the trace formula, we would start with a test function f for G. For the time being, we take f to be a function in $C_c^\infty\big(G(\mathbb{A})^1\big)$, which we assume is the restriction of a product of functions

$$\prod_v f_v, \qquad\qquad f_v \in C_c^\infty\big(G(F_v)\big).$$

If we were to follow the prescription of Jacquet-Langlands, we would map f to a function f^* on $G(\mathbb{A})^1$ obtained by restriction of a product $\prod f_v^*$ of functions on the local groups $G^*(F_v)$. Each function f_v^* would be attached to the associated function f_v on $G(F_v)$ by imposing a matching condition for the local invariant orbital integrals of f_v and f_v^*. This would in turn require a correspondence $\gamma_v \to \gamma_v^*$ between strongly regular conjugacy classes. How might such a correspondence be defined in general?

In the special case discussed in §25, the correspondence of strongly regular elements can be formulated explicitly in terms of characteristic polynomials. For any $k \supset F$, one matches a characteristic polynomial on the matrix algebra $M_n(k)$ with its variant for the central simple algebra that defines G. Now the coefficients of characteristic polynomials have analogues for the general group G. For example, one can take any set of generators of the algebra of G-invariant polynomials on G. These objects can certainly be used to transfer semisimple conjugacy classes from G to G^*. However, invariant polynomials measure only *geometric* conjugacy classes, that is, conjugacy classes in the group of points over an algebraically closed field. In general, if k is not algebraically closed, and G is just about any group other than $GL(n)$ (or one of its inner twists), there can be nonconjugate elements in $G(k)$ that are conjugate over an algebraic closure $G(\overline{k})$. For example, in the case that $G = SL(2)$ and $k = \mathbb{R}$, the relation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

represents conjugacy over $G(\mathbb{C})$ of nonconjugate elements in $G(\mathbb{R})$. This phenomenon obviously complicates the problem of transferring conjugacy classes.

Langlands defined two strongly regular elements in $G(k)$ to be *stably conjugate* if they were conjugate over an algebraic closure $G(\overline{k})$. Stable conjugacy is thus an equivalence relation that is weaker than conjugacy. Suppose that δ belongs to the set $\Delta_{\text{reg}}\big(G(k)\big)$ of (strongly regular) stable conjugacy classes in $G(k)$. The image $\psi(\delta)$ of δ in G^* yields a well defined conjugacy class in $G(\overline{k})$. If σ belongs to $\text{Gal}(\overline{k}/k)$,

$$\sigma\big(\psi(\delta)\big) = \sigma(\psi)\sigma(\delta) = \alpha(\sigma)^{-1}\psi(\delta),$$

for the inner automorphism $\alpha(\sigma) = \psi{\circ}\sigma(\psi)^{-1}$ of G^*. The geometric conjugacy class of $\psi(\delta)$ is therefore defined over k. Because G^* is quasisplit and $\psi(\delta)$ is semisimple, an important theorem of Steinberg [**Ste**] implies that the geometric conjugacy class has a representative in $G(k)$. This representative is of course not unique, but it does map to a well defined stable conjugacy class $\delta^* \in \Delta_{\text{reg}}\big(G^*(k)\big)$. We therefore have an injection $\delta \to \delta^*$ from $\Delta_{\text{reg}}\big(G(k)\big)$ to $\Delta_{\text{reg}}\big(G^*(k)\big)$, determined canonically by ψ. (The fact that Steinberg's theorem holds only for quasisplit groups is responsible for the mapping not being surjective.) We cannot however expect to be able to transfer ordinary conjugacy classes $\gamma \in \Gamma_{\text{reg}}\big(G(k)\big)$ from G to G^*.

Besides the subtle global questions it raises for the trace formula, stable conjugacy also has very interesting implications for local harmonic analysis. Suppose that k is one of the local fields F_v. In this case, there are only finitely many conjugacy classes in any stable class. One defines the stable orbital integral of a function $f_v \in C_c^\infty\big(G(F_v)\big)$ over a (strongly regular) stable conjugacy class δ_v as a finite sum

$$f_v^G(\delta_v) = \sum_{\gamma_v} f_{v,G}(\gamma_v)$$

of invariant orbital integrals, taken over the conjugacy classes γ_v in the stable class δ_v. (It is not hard to see how to choose compatible invariant measures on the various domains $G_{\gamma_v}(F_v)\backslash G(F_v)$.) An invariant distribution S_v on $G(F_v)$ is said to be *stable* if its value at f_v depends only on the set of stable orbital integrals $\{f_v^G(\delta_v)\}$ of f_v. Under this condition, there is a continuous linear form $\widehat{S}_v$ on the space of functions

$$S\mathcal{I}\big(G(F_v)\big) = \big\{f_v^G : \ f_v \in \mathcal{H}\big(G(F_v)\big)\big\}$$

on $\Delta_{\text{reg}}\big(G(F_v)\big)$ such that

$$S_v(f_v) = \widehat{S}_v(f_v^G), \qquad\qquad f_v \in \mathcal{H}\big(G(F_v)\big).$$

We thus have a whole new class of distributions on $G(F_v)$, which is more restrictive than the family of invariant distributions. Is there some other way to characterize it?

In general terms, one becomes accustomed to thinking of conjugacy classes as being dual to irreducible characters. From the perspective of local harmonic analysis, the semisimple conjugacy classes in $G(F_v)$ could well be regarded as dual analogues of irreducible tempered characters on $G(F_v)$. The relation of stable conjugacy ought then to determine a parallel relationship on the set of tempered characters. (In [**Ko1**], Kottwitz extended the notion of stable conjugacy to arbitrary semisimple elements.) Langlands called this hypothetical relationship *L-equivalence*, and referred to the corresponding equivalence classes as *L-packets*, since they seemed to preserve the local L-functions and ε-factors attached to irreducible representations of $G(F_v)$. He also realized that in the case $F_v = \mathbb{R}$, there was already a good candidate for this relationship in the work of Harish-Chandra.

Recall from §24 that $G(\mathbb{R})$ has a discrete series if and only if G has an elliptic maximal torus T_G over $\mathbb{R}$. In this case, the discrete series occur in finite packets $\Pi_2(\mu)$. On the other hand, any strongly regular elliptic conjugacy class for $G(\mathbb{R})$ intersects $T_{G,\mathrm{reg}}(\mathbb{R})$. Moreover, two elements in $T_{G,\mathrm{reg}}(\mathbb{R})$ are $G(\mathbb{R})$-conjugate if and only if they lie in the same $W(K_{\mathbb{R}}, T_G)$-orbit, and are stably conjugate if and only if they are in the same orbit under the full Weyl group $W(G, T_G)$. This is because $W(K_{\mathbb{R}}, T_G)$ is the subgroup of elements in $W(G, T_G)$ that are actually induced by conjugation from points in $G(\mathbb{R})$. It can then be shown from Harish-Chandra's algorithm for the characters of discrete series that the sum of characters

$$\Theta(\mu, \gamma) = \sum_{\pi_{\mathbb{R}} \in \Pi_2(\mu)} \Theta(\pi_{\mathbb{R}}, \gamma), \qquad \gamma \in G_{\mathrm{reg}}(\mathbb{R}),$$

attached to representations in a packet $\Pi_2(\mu)$, depends only on the stable conjugacy class of γ, rather than its actual conjugacy class. In other words, the distribution

$$f_{\mathbb{R}} \longrightarrow \sum_{\pi_{\mathbb{R}} \in \Pi_2(\mu)} f_{\mathbb{R},G}(\pi_{\mathbb{R}}), \qquad f_{\mathbb{R}} \in C_c^\infty\big(G(\mathbb{R})\big),$$

on $G(\mathbb{R})$ is stable. This fact justifies calling $\Theta(\mu, \gamma)$ a "stable character", and designating the sets $\Pi_2(\mu)$ the L-packets of discrete series. It also helps to explain why the sum over $\pi_{\mathbb{R}} \in \Pi_2(\mu)$, which occurs on each side of the "finite case" of the trace formula in Theorem 24.1, is a natural operation. Langlands used the L-packet structure of discrete series as a starting point for a classification of the irreducible representations of $G(\mathbb{R})$, and a partition of the representations into L-packets governed by their local L-functions [**Lan11**]. (Knapp and Zuckerman [**KZ2**] later determined the precise structure of the L-packets outside the discrete series.) The Langlands classification for real groups applies to all irreducible representations, but it is only for the tempered representations that the sum of the characters in an L-packet is stable.

Let us return to the invariant trace formula. The basic questions raised by the problem of stability can be posed for the simplest terms on the geometric side. Let $\Gamma_{\mathrm{reg,ell}}(G)$ be the set of conjugacy classes γ in $G(F)$ that are both strongly regular and elliptic. An element $\gamma \in G(F)$ represents a class in $\Gamma_{\mathrm{reg,ell}}(G)$ if and only if the centralizer G_γ is a maximal torus in G that is elliptic, in the usual sense that $A_{G_\gamma} = A_G$. It follows from the definitions that

$$\Gamma_{\mathrm{reg,ell}}(G) \subset \Gamma_{\mathrm{anis}}(G) \subset \Gamma(G)_S.$$

The elements in $\Gamma_{\mathrm{reg,ell}}(G)$ are in some sense the generic elements in the set $\Gamma(G)_S$, which we recall indexes the terms in the sum with $M = G$ on the geometric side. The *regular elliptic* part

$$I_{\mathrm{reg,ell}}(f) = \sum_{\gamma \in \Gamma_{\mathrm{reg,ell}}(G)} a^G(\gamma) f_G(\gamma) \tag{27.1}$$

of the trace formula therefore represents the generic part of this sum.

The first question that comes to mind is the following. Is the distribution

$$f \longrightarrow I_{\mathrm{reg,ell}}(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})^1\big),$$

stable? In other words, does $I_{\mathrm{reg,ell}}(f)$ depend only on the family of stable orbital integrals $\{f_v^G(\delta_v)\}$? An affirmative answer could solve many of the global problems created by stability. For in order to compare $I_{\mathrm{reg,ell}}$ with its analogue on G^*, it would then only be necessary to transfer f to a function f^* on $\Delta_{G\text{-reg}}\big(G^*(\mathbb{A})\big)$, something we could do by the local correspondence of stable conjugacy classes.

A cursory glance seems to suggest that the answer is indeed affirmative. The volume

$$a^G(\gamma) = \mathrm{vol}\big(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})^1\big), \qquad \gamma \in \Gamma_{\mathrm{reg,ell}}(G),$$

depends only on the stable conjugacy class δ of γ in $G(F)$, since it depends only on the F-isomorphism class of the maximal torus G_γ. We can therefore write

$$I_{\mathrm{reg,ell}}(f) = \sum_\delta a^G(\delta)\Big(\sum_{\gamma\to\delta} f_G(\gamma)\Big), \tag{27.2}$$

where δ is summed over the set $\Delta_{\mathrm{reg,ell}}(G)$ of elliptic stable classes in $\Delta_{\mathrm{reg}}\big(G(F)\big)$, γ is summed over the preimage of δ in $\Gamma_{\mathrm{reg,ell}}(G)$, and $a^G(\delta) = a^G(\gamma)$. The sum over γ looks as if it might be stable in f. However, a closer inspection reveals that it is not. For we are demanding that the distribution be stable in each component f_v of f. If

$$\delta_{\mathbb{A}} = \prod_{v\in S} \delta_v, \qquad \delta_v \in \Delta_{\mathrm{reg}}\big(G(F_v)\big),$$

is a product of local stable classes with a rational representative δ, each ordinary conjugacy class $\gamma_{\mathbb{A}} = \prod \gamma_v$ in $\delta_{\mathbb{A}}$ would also have to have a rational representative γ. It turns out that there are not enough rational conjugacy classes γ for this to happen. Contrary to our initial impression then, the distribution $I_{\mathrm{reg,ell}}(f)$ is not generally stable in f.

Since $I_{\mathrm{reg,ell}}(f)$ need not be stable, the question has to be reformulated in terms of stabilizing this distribution. The problem may be stated in general terms as follows.

Express $I_{\mathrm{reg,ell}}(f)$ as the sum of a canonical stable distribution $S^G_{\mathrm{reg,ell}}(f)$ with an explicit error term.

The first group to be investigated was $SL(2)$. Labesse and Langlands stabilized the full trace formula for this group, as well as for its inner forms, and showed that the solution had remarkable implications for the corresponding spectral decompositions [**Lab1**], [**She1**], [**LL**]. Langlands also stabilized $I_{\mathrm{reg,ell}}$ in the general case, under the assumption of two conjectures in local harmonic analysis [**Lan10**].

In his general stabilization of $I_{\mathrm{reg,ell}}(f)$, Langlands constructed the stable component $S^G_{\mathrm{reg,ell}}$ explicitly. He expressed the error term in terms of corresponding stable components attached to groups G' of dimension smaller than G. The groups $\{G'\}$ are all quasisplit. Together with the group $G' = G^*$ of dimension equal to G, they are known as *elliptic endoscopic groups* for G. For each G', Langlands formulated a conjectural correspondence $f \to f'$ between test functions for G and G'. His stabilization then took the form

$$I_{\text{reg,ell}}(f) = \sum_{G'} \iota(G, G') \widehat{S}'_{G\text{-reg,ell}}(f'), \tag{27.3}$$

for explicitly determined coefficients $\iota(G, G')$, and stable linear forms $S'_{G\text{-reg,ell}}$ attached to G'. In case $G' = G^*$, the corresponding terms satisfy $\iota(G, G') = 1$ and $S^*_{G\text{-reg,ell}} = S^*_{\text{reg,ell}}$. The stable component of $I_{\text{reg,ell}}(f)$ is the associated summand

$$S^G_{\text{reg,ell}}(f) = \widehat{S}^*_{\text{reg,ell}}(f^*).$$

For arbitrary G', $S'_{G\text{-reg,ell}}$ is the strongly G-regular part of $S'_{\text{reg,ell}}$, obtained from classes in $\Delta_{\text{reg,ell}}(G')$ whose image in G remains strongly regular.

Langlands' stabilization is founded on class field theory. Specifically, it depends on the application of Tate-Nakayama duality to the Galois cohomology of algebraic groups. The basic relationship is easy to describe. Suppose that δ is a strongly regular element in $G(k)$, for some $k \supset F$. The centralizer of δ in G is a maximal torus T over k. Suppose that $\gamma \in G(k)$ is stably conjugate to δ. Then γ equals $g^{-1}\delta g$, for some element $g \in G(\overline{k})$. If σ belongs to $\text{Gal}(\overline{k}/k)$, we have

$$\delta = \sigma(\delta) = \sigma(g\gamma g^{-1}) = \sigma(g)\gamma\sigma(g)^{-1} = t(\sigma)^{-1}\delta t(\sigma),$$

where $t(\sigma)$ is the 1-cocycle $g\sigma(g)^{-1}$ from $\text{Gal}(\overline{k}/k)$ to $T(\overline{k})$. One checks that a second element $\gamma_1 \in G(k)$ in the stable class of δ is $G(k)$-conjugate at γ if and only if the corresponding 1-cocycle $t_1(\sigma)$ has the same image as $t(\sigma)$ in the Galois cohomology group

$$H^1(k, T) = H^1\big(\Gamma_k, T(\overline{k})\big), \qquad \Gamma_k = \text{Gal}(\overline{k}/k).$$

Conversely, an arbitrary class in $H^1(k, T)$ comes from an element γ if and only if it is represented by a 1-cocycle of the form $g\sigma(g)^{-1}$. The mapping $\gamma \to t$ therefore defines a bijection from the set of $G(k)$-conjugacy classes in the stable conjugacy class of δ to the kernel

$$\mathcal{D}(T) = \mathcal{D}(T/k) = \ker\big(H^1(k, T) \to H^1(k, G)\big). \tag{27.4}$$

Keep in mind that $H^1(k, G)$ is only a set with distinguished element 1, since G is generally nonabelian. The preimage $\mathcal{D}(T)$ of this element in $H^1(k, T)$ therefore need not be a subgroup. However, $\mathcal{D}(T)$ is contained in the subgroup

$$\mathcal{E}(T) = \mathcal{E}(T/k) = \text{im}\big(H^1(k, T_{\text{sc}}) \to H^1(k, T)\big)$$

of $H^1(k, T)$, where T_{sc} is the preimage of T in the simply connected cover G_{sc} of the derived group of G. This is because the canonical map $\mathcal{D}(T_{\text{sc}}) \to \mathcal{D}(T)$ is surjective. If $H^1(k, G_{\text{sc}}) = \{1\}$, which is the case whenever k is a nonarchimedean local field [**Spr1**, §3.2], $\mathcal{D}(T)$ actually equals the subgroup $\mathcal{E}(T)$. This is one of the reasons why one works with the groups $\mathcal{E}(T)$ in place of $H^1(T, G)$, and why the simply connected group G_{sc} plays a significant role in the theory.

In the case that k is a local or global field, Tate-Nakayama duality applies class field theory to the groups $H^1(k, T)$. If k is a completion F_v of F, it provides a canonical isomorphism

$$H^1(F_v, T) \xrightarrow{\sim} \pi_0(\widehat{T}^{\Gamma_v})^*$$

of $H^1(F_v,T)$ with the group of characters on the finite abelian group $\pi_0(\widehat{T}^{\Gamma_v})$. We have written Γ_v here for the Galois group $\Gamma_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$, which acts on the complex dual torus

$$\widehat{T} = X(T) \otimes \mathbb{C}^*$$

through its action on the group of rational characters $X(T)$. As usual, $\pi_0(\cdot)$ denotes the set of connected components of a topological space. If $k = F$, Tate-Nakayama duality characterizes the group

$$H^1\big(F, T(\overline{\mathbb{A}})/T(\overline{F})\big) = H^1\big(\Gamma_F, T(\mathbb{A}_{\bar{F}})/T(\overline{F})\big) = H^1\big(\Gamma_{F'/F}, T(\mathbb{A}_{F'})/T(F')\big),$$

where F' is some finite Galois extension of F over which T splits. It provides a canonical isomorphism

$$H^1\big(F, T(\overline{\mathbb{A}})/T(\overline{F})\big) \ \xrightarrow{\sim}\ \pi_0(\widehat{T}^{\Gamma})^*,$$

where the Galois group $\Gamma = \Gamma_F = \mathrm{Gal}(\overline{F}/F)$ again acts on the complex torus $\widehat{T}$ through its action on $X(T)$. If we combine this with the long exact sequence of cohomology attached to the exact sequence of Γ-modules

$$1 \longrightarrow T(\overline{F}) \longrightarrow T(\overline{\mathbb{A}}) \longrightarrow T(\overline{\mathbb{A}})/T(\overline{F}) \longrightarrow 1,$$

and the isomorphism

$$H^1\big(F, T(\overline{\mathbb{A}})\big) \cong \bigoplus_v H^1(F_v, T)$$

provided by Shapiro's lemma, we obtain a characterization of the diagonal image of $H^1(F,T)$ in the direct sum over v of the groups $H^1(F_v,T)$. It is given by a canonical isomorphism from the cokernel

$$\mathrm{coker}^1(F,T) = \mathrm{coker}\Big(H^1(F,T) \longrightarrow \bigoplus_v H^1(F_v,T)\Big) \tag{27.5}$$

onto the image

$$\mathrm{im}\Big(\bigoplus_v \pi_0(\widehat{T}^{\Gamma_v})^* \longrightarrow \pi_0(\widehat{T}^{\Gamma})^*\Big).$$

If these results are combined with their analogues for T_{sc}, they provide similar assertions for the subgroups $\mathcal{E}(T/k)$ of $H^1(k,T)$. In the local case, one has only to replace $\pi_0(\widehat{T}^{\Gamma_v})$ by the group $\mathcal{K}(T/F_v)$ of elements in $\pi_0\big((\widehat{T}/Z(\widehat{G}))^{\Gamma_v}\big)$ whose image in $H^1\big(F_v, Z(\widehat{G})\big)$ is trivial. In the global case, one replaces $\pi_0(\widehat{T}^{\Gamma})$ by the group $\mathcal{K}(T/F)$ of elements in $\pi_0\big((\widehat{T}/Z(\widehat{G}))^{\Gamma}\big)$ whose image in $H^1\big(F, Z(\widehat{G})\big)$ is locally trivial, in the sense that their image in $H^1\big(F_v, Z(\widehat{G})\big)$ is trivial for each v. (See [**Lan10**], [**Ko5**].)

To simplify the discussion, assume for the present that $G = G_{\mathrm{sc}}$. Then $\mathcal{E}(T/k) = H^1(k,T)$, for any k. Moreover, $\mathcal{K}(T/F_v) = \pi_0(\widehat{T}^{\Gamma_v})$ and $\mathcal{K}(T/F) = \pi_0(\widehat{T}^{\Gamma})$, since $Z(\widehat{G}) = 1$. In fact, $\pi_0(\widehat{T}^{\Gamma})$ equals $\widehat{T}^{\Gamma}$ if T is elliptic in G over F.

We recall that Langlands' stabilization (27.3) of $I_{\mathrm{reg,ell}}(f)$ was necessitated by the failure of each $G(\mathbb{A})$-conjugacy class in the $G(\mathbb{A})$-stable class of $\delta \in \Delta_{\mathrm{reg,ell}}(G)$ to have a representative in $G(F)$. The cokernel (27.5) gives a measure of this failure. Langlands' construction treats the quantity in brackets on the right hand side of (27.2) as the value at 1 of a function on the finite abelian group $\mathrm{coker}^1(F,T)$. The critical step is to expand this function according to Fourier inversion on

$\mathrm{coker}^1(F,T)$. One has to keep track of the $G(F)$-conjugacy classes in the $G(\mathbb{A})$-conjugacy class of δ, which by the Hasse principle for $G = G_{\mathrm{sc}}$ are in bijection with the finite abelian group

$$\ker^1(F,T) = \ker\Big(H^1(F,T) \;\longrightarrow\; \bigoplus_v H^1(F_v,T)\Big).$$

The formula (27.2) becomes an expansion

$$I_{\mathrm{reg,ell}}(f) = \sum_{\delta\in\Delta_{\mathrm{reg,ell}}(G)} a^G(\delta)\iota(T)\sum_{\kappa\in\widehat{T}^\Gamma} f_G^\kappa(\delta), \tag{27.6}$$

where $T = G_\delta$ denotes the centralizer of (some fixed representative of) δ, $\iota(\tau)$ equals the product of $|\widehat{T}^\Gamma|^{-1}$ with $|\ker^1(F,T)|$, and

$$f_G^\kappa(\delta) = \sum_{\{\gamma_{\mathbb{A}}\in\Gamma(G(\mathbb{A})):\gamma_{\mathbb{A}}\sim\delta\}} f_G(\gamma_{\mathbb{A}})\kappa(\gamma_{\mathbb{A}}).$$

The last sum is of course over the $G(\mathbb{A})$-conjugacy classes $\gamma_{\mathbb{A}} = \prod\gamma_v$ in the stable class of δ in $G(\mathbb{A})$. For any such $\gamma_{\mathbb{A}}$, it can be shown that γ_v is $G(F_v)$-conjugate to δ for almost all v. It follows that $\gamma_{\mathbb{A}}$ maps to an element $t_{\mathbb{A}} = \bigoplus t_v$ in the direct sum of the groups $H^1(F_v,T)$. This in turn maps to a point in the cokernel (27.5), and hence to a character in $(\widehat{T}^\Gamma)^*$. The coefficient $\kappa(\gamma_{\mathbb{A}})$ is the value of this character at κ.

Suppose for example that $G = SL(2)$. The eigenvalues of δ then lie in a quadratic extension E of F, and $T = G_\delta$ is the one-dimensional torus over F such that

$$T(F) \cong \{t\in E^*:\ t\sigma(t) = 1\}, \qquad\qquad \Gamma_{E/F} = \{1,\sigma\}.$$

The nontrivial element $\sigma\in\Gamma_{E/F}$ acts on $X(T)\cong\mathbb{Z}$ by $m\to(-m)$, and therefore acts on $\widehat{T} = \mathbb{Z}\otimes\mathbb{C}^*\cong\mathbb{C}^*$ by $z\to z^{-1}$. It follows that $\pi_0(\widehat{T}^\Gamma) = \widehat{T}^\Gamma$ is isomorphic to the subgroup $\{\pm1\}$ of $\mathbb{C}^*$. Similarly, $\pi_0(\widehat{T}^{\Gamma_v}) = \widehat{T}^{\Gamma_v}\cong\{\pm1\}$ if v does not split in E, while $\pi_0(\widehat{T}^{F_v}) = \pi_0(\widehat{T}) = \{1\}$ if v does split. In particular, if κ is the nontrivial element in $\pi_0(\widehat{T}^\Gamma)$, the local κ-orbital integral $f^\kappa_{v,G}(\delta) = f^\kappa_{v,G}(\delta_v)$ equals a difference of two orbital integrals if v does not split, and is a simple orbital integral otherwise. The characterization we have described here for the various groups $H^1(F_v,T)$, and for the diagonal image of $H^1(F,T)$ in their direct sum, is typical of what happens in general. In the present situation $\ker^1(F,T) = \{1\}$, so that $H^1(F,T)$ can in fact be identified with its diagonal image.

The expression (27.6) is part of the stabilization (27.3) of $I_{\mathrm{reg,ell}}(f)$. We need to see how it gives rise to the quasisplit groups G' of (27.3).

Suppose that T and κ are as in (27.6). We choose an embedding $\widehat{T}\subset\widehat{G}$ of the dual torus of T into $\widehat{G}$ that is admissible, in the sense that it is the mapping assigned to a choice of some pair $(\widehat{B},\widehat{T})$ in $\widehat{G}$, and some Borel subgroup B of G containing T. Let s' be the image of κ in $\widehat{G}$, and let $\widehat{G}' = \widehat{G}_{s'}$ be its connected centralizer in $\widehat{G}$. Then $\widehat{G}'$ is a reductive subgroup of $\widehat{G}$. It is known that there is an L-embedding

$${}^LT = \widehat{T}\rtimes W_F \;\hookrightarrow\; {}^LG = \widehat{G}\rtimes W_F,$$

for the Weil forms of the L-groups of T and G, which restricts to the given embedding of $\widehat{T}$ into $\widehat{G}$ [**LS1**, (2.6)]. Fix such an embedding, and set

$$\mathcal{G}' \;=\; {}^L T\widehat{G}'.$$

Then $\mathcal{G}'$ is an L-subgroup of ${}^L G$, which commutes with s'. It provides a split extension

$$1 \longrightarrow \widehat{G}' \longrightarrow \mathcal{G}' \longrightarrow W_F \longrightarrow 1 \tag{27.7}$$

of W_F by $\widehat{G}'$. In particular, it determines an action of W_F on $\widehat{G}'$ by outer automorphisms, which factors through a finite quotient of Γ_F. Let G' be any quasisplit group over F for which $\widehat{G}'$, with the given action of Γ_F, is a dual group. We have obtained a correspondence

$$(T,\kappa) \longrightarrow (G',\mathcal{G}',s').$$

We can choose a maximal torus $T' \subset G'$ in G' over F, together with an isomorphism from T' to T over F that is admissible, in the sense that the associated isomorphism $\widehat{T}' \to \widehat{T}$ of dual groups is the composition of an admissible embedding $\widehat{T}' \subset \widehat{G}'$ with an inner automorphism of $\widehat{G}$ that takes $\widehat{T}'$ to $\widehat{T}$. Let $\delta' \in T'(F)$ be the associated preimage of the original point $\delta \in T(F)$. The tori T and T' are the centralizers in G and G' of δ and δ'. The two points δ and δ' are therefore the primary objects. They become part of a larger correspondence

$$(\delta,\kappa) \longrightarrow \big((G',\mathcal{G}',s'),\delta'\big). \tag{27.8}$$

Elements $\delta' \in G'(F)$ obtained in this way are said to be *images* from G [**LS1**, (1.3)].

Suppose now that G is arbitrary. Motivated by the last construction, one defines an *endoscopic datum* for G to be a triplet $(G',\mathcal{G}',s',\xi')$, where G' is a quasisplit group over F, $\mathcal{G}'$ is a split extension of W_F by a dual group $\widehat{G}'$ of G', s' is a semisimple element in $\widehat{G}$, and ξ' is an L-embedding of $\mathcal{G}'$ into ${}^L G$. It is required that $\xi'(\widehat{G}')$ be equal to the connected centralizer of s' in $\widehat{G}$, and that

$$\xi'(u')s' = s'\xi'(u')a(u'), \qquad u' \in \mathcal{G}', \tag{27.9}$$

where a is a 1-cocycle from W_F to $Z(\widehat{G})$ that is locally trivial, in the sense that its image in $H^1\big(W_{F_v}, Z(\widehat{G})\big)$ is trivial for every v. The quasisplit group G' is called an *endoscopic group* for G. An *isomorphism* of endoscopic data $(G',\mathcal{G}',s',\xi')$ and $(G'_1,\mathcal{G}'_1,s'_1,\xi'_1)$ is an isomorphism α: $G' \to G'_1$ over F for which, roughly speaking, there is dual isomorphism induced by some element in $\widehat{G}$. More precisely, it is required that there be an L-isomorphism β: $\mathcal{G}'_1 \to \mathcal{G}'$ such that the corresponding mappings $\Psi(G') \xrightarrow{\alpha} \Psi(G'_1)$ and $\Psi(\widehat{G}'_1) \xrightarrow{\beta} \Psi(\widehat{G}')$ of based root data are dual, and an element $g \in \widehat{G}$ such that

$$\xi'\big(\beta(u'_1)\big) = g^{-1}\xi'_1(u'_1)g, \qquad u'_1 \in \mathcal{G}'_1,$$

and

$$s' = g^{-1}s'_1 g z, \qquad z \in Z(\widehat{G})Z(\xi'_1)^0,$$

where $Z(\xi'_1)^0$ is the connected component of 1 in the centralizer in $\widehat{G}$ of $\xi'_1(\mathcal{G}'_1)$. (See [**LS1**, (1.2)].) We write $\mathrm{Aut}_G(G')$ for the group of isomorphisms α: $G' \to G'$ of G' as a endoscopic datum for G.

We say that $(G', \mathcal{G}', s', \xi')$ is *elliptic* if $Z(\xi')^0 = 1$. This means that the image of ξ' is ${}^LG$ is not contained in ${}^LM$, for any proper Levi subgroup of G over F. We write $\mathcal{E}_{\text{ell}}(G)$ for the set of isomorphism classes of elliptic endoscopic data for G. It is customary to denote an element in $\mathcal{E}_{\text{ell}}(G)$ by G', even though G' is really only the first component of a representative $(G', \mathcal{G}', s', \xi')$ of an isomorphism class. Any $G' \in \mathcal{E}_{\text{ell}}(G)$ then comes with a finite group

$$\text{Out}_G(G') = \text{Aut}_G(G')/\text{Int}(G'),$$

of outer automorphisms of G' as an endoscopic datum.

Suppose for example that $G = GL(n)$. The centralizer of any semisimple element s' in $\widehat{G} = GL(n, \mathbb{C})$ is a product of general linear groups. It follows that any endoscopic datum for G is represented by a Levi subgroup M. In particular, there is only one element in $\mathcal{E}_{\text{ell}}(G)$, namely the endoscopic datum represented by G itself. This is why the problem of stability is trivial for $GL(n)$.

The general definitions tend to obscure the essential nature of the construction. Suppose again that $G = G_{\text{sc}}$. The dual group $\widehat{G}$ is then adjoint, and $Z(\widehat{G}) = 1$. In general, any $G' \in \mathcal{E}_{\text{ell}}(G)$ can be represented by an endoscopic datum for which $\mathcal{G}'$ is a subgroup of ${}^LG$, and ξ' is the identity embedding ι'. The condition (27.9) reduces in the case at hand to the requirement that $\mathcal{G}'$ commute with s'. To construct a general element in $\mathcal{E}_{\text{ell}}(G)$, we start with the semisimple element $s' \in \widehat{G}$. The centralizer ${}^LG_{s',+}$ of s' in ${}^LG$ is easily seen to project onto W_F. Its quotient by the connected centralizer $\widehat{G}' = \widehat{G}_{s'}$ is an extension of W_F by a finite group. To obtain an endoscopic datum, we need only choose a section

$$\omega' :\ W_F \ \longrightarrow \ {}^LG_{s',+}/\widehat{G}'$$

that can be inflated to a homomorphism $W_F \to \ {}^LG_{s',+}$. For the product

$$\mathcal{G}' = \widehat{G}'\omega'(W_F)$$

is then a split extension of W_F by $\widehat{G}'$. It determines an L-action of W_F on $\widehat{G}'$, and hence a quasisplit group G' over F of which $\widehat{G}'$ is a dual group. The endoscopic datum $(G', \mathcal{G}', s', \iota')$ thus obtained is elliptic if and only if the centralizer of $\mathcal{G}'$ in $\widehat{G}$ is finite, a condition that reduces considerably the possibilities for the pairs (s', ω'). The mapping

$$(s', \omega') \ \longrightarrow \ (G', \mathcal{G}', s', \iota')$$

becomes a bijection from the set of $\widehat{G}$-orbits of such pairs and $\mathcal{E}_{\text{ell}}(G)$. We note that a point $g \in \widehat{G}$ represents an element in $\text{Out}_G(G')$ if and only if it stabilizes $\mathcal{G}'$ and commutes with s'.

For purposes of illustration, suppose that G is split as well as being simply connected. We have then to consider semisimple elements $s' \in \widehat{G}$ whose centralizer $\widehat{G}_{s',+}$ has finite center. It is an interesting exercise (which I confess not to have completed) to classify the $\widehat{G}$-orbits of such elements in terms of the extended Coxeter-Dynkin diagram of $\widehat{G}$. For example, elements s' that satisfy the stronger condition that $\widehat{G}' = \widehat{G}_{s'}$ has finite center are represented by vertices in the affine diagram (although in the adjoint group $\widehat{G}$, some of these elements are conjugate). Once we have chosen s', we then select a homomorphism ω' from Γ_F to the finite abelian group $\pi_0(\widehat{G}_{s',+}) = \widehat{G}_{s',+}/\widehat{G}'$ whose image pulls back to a subgroup of $\widehat{G}_{s',+}$ that still has finite center. Suppose for example that $G = SL(2)$, and that s' is the image of $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ in $\widehat{G} = PGL(2,\mathbb{C})$. Then $\widehat{G}_{s',+}$ consists of the group of diagonal matrices, together with a second component generated by the element $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Since the center of $\widehat{G}_{s',+}$ equals $\{1, s'\}$, we obtain elliptic endoscopic data for G by choosing nontrivial homomorphisms from Γ_F to the group $\pi_0(\widehat{G}_{s',+}) \cong \mathbb{Z}/2\mathbb{Z}$. The classes in $\mathcal{E}_{\mathrm{ell}}(G)$ other than G itself, are thus parametrized by quadratic extensions E of F.

We return to the case of a general group G. We have spent most of this section trying to motivate some of the new ideas that arose with the problem of stability. This leaves only limited space for a brief description of the details of Langlands' stabilization (27.3) of $I_{\mathrm{reg,ell}}(f)$.

The general form of the expansion (27.6) is

$$I_{\mathrm{reg,ell}}(f) = \sum_{\delta\in\Delta_{\mathrm{reg,ell}}(G)} a^G(\delta)\iota(T,G)\sum_{\kappa\in\mathcal{K}(T/F)} f_G^\kappa(\delta),$$

where

$$\iota(T,G) = \Big|\ker\Big(\mathcal{E}(T/F) \longrightarrow \bigoplus_v \mathcal{E}(T/F_v)\Big)\Big||\mathcal{K}(T/F)|^{-1},$$

and $f_G^\kappa(\delta)$ is defined as in (27.6). The correspondence (27.8) is easily seen to have an inverse, which in general extends to a bijection

$$\{(G',\delta')\} \xrightarrow{\sim} \{(\delta,\kappa)\}.$$

The domain of this bijection is the set of equivalence classes of pairs (G',δ'), where G' is an elliptic endoscopic datum for G, δ' is a strongly G-regular, elliptic element in $G'(F)$ that is an image from G, and equivalence is defined by isomorphisms of endoscopic data. The range is the set of equivalence classes of pairs (δ,κ), where δ belongs to $\Delta_{\mathrm{reg,ell}}(G)$, κ lies in $\mathcal{K}(G_\delta/F)$, and equivalence is defined by conjugating by $G(\overline{F})$. (See [**Lan10**], [**Ko5**, Lemma 9.7].) Given (G',δ'), we set

$$f'(\delta') = f_G^\kappa(\delta) = \sum_{\{\gamma_{\mathbb{A}}\in\Gamma(G(\mathbb{A})):\gamma_{\mathbb{A}}\sim\delta\}} f_G(\gamma_{\mathbb{A}})\kappa(\gamma_{\mathbb{A}}). \tag{27.10}$$

We can then write

$$I_{\mathrm{reg,ell}}(f) = \sum_{G'\in\mathcal{E}_{\mathrm{ell}}(G)} |\mathrm{Out}_G(G')|^{-1} \sum_{\delta'\in\Delta_{G\text{-reg,ell}}(G')} a^G(\delta)\iota(G_\delta,G)f'(\delta'),$$

with the understanding that $f'(\delta') = 0$ if δ' is not an image from G. Langlands showed that for any pair (G',δ'), the number

$$\iota(G,G') = \iota(G_\delta,G)\iota(G'_{\delta'},G')^{-1}|\mathrm{Out}_G(G')|^{-1}$$

was independent of δ' and δ. (Kottwitz later expressed the product of the first two factors on the right as a quotient $\tau(G)\tau(G')^{-1}$ of Tamagawa numbers [**Ko3**, Theorem 8.3.1].) Set

$$\widehat{S}'_{G\text{-reg,ell}}(f') = \sum_{\delta'\in\Delta_{G\text{-reg,ell}}(G')} b'(\delta')f'(\delta'), \tag{27.11}$$

where

$$b'(\delta') = a^G(\delta)\iota(G'_{\delta'},G') = \mathrm{vol}\big(G'_{\delta'}(F)\backslash G'_{\delta'}(\mathbb{A})^1\big)\iota(G'_{\delta'},G').$$

Then

$$I_{\mathrm{reg,ell}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G') \widehat{S}'_{G\text{-reg,ell}}(f').$$

We have now sketched how to derive the formula (27.3). However, the term $f'(\delta')$ in (27.11) is defined in (27.10) only as a function on $\Delta_{G\text{-reg,ell}}(G')$. One would hope that it is the stable orbital integral at δ' of a function in $C_c^\infty\big(G'(\mathbb{A})\big)$. The sum in (27.10) can be taken over adelic products $\gamma_{\mathbb{A}} = \prod \gamma_v$, where γ_v is a conjugacy class in $G(F_v)$ that lies in the stable class of the image δ_v of δ in $G(F_v)$. It follows that

$$f'(\delta') = f_G^\kappa(\delta) = \prod_v f_{v,G}^\kappa(\delta_v),$$

where

$$f_{v,G}^\kappa(\delta_v) = \sum_{\gamma_v \sim \delta_v} f_{v,G}(\gamma_v)\kappa(\gamma_v).$$

Are the local components $\delta'_v \to f_{v,G}^\kappa(\delta_v)$ stable orbital integrals of functions in $C_c^\infty\big(G'(F_v)\big)$? The question concerns the singularities that arise, as the strongly regular points approach 1, for example. Do enough of the singularities of the orbital integrals $f_{v,G}(\gamma_v)$ disappear from the sum so that only singularities of stable orbital integrals on the smaller group $G'(F_v)$ remain?

The question is very subtle. We have been treating δ as both a stable class in $\Delta_{\mathrm{reg,ell}}(G)$ and a representative in $G(F)$ of that class. The distinction has not mattered so far, since $f'(\delta') = f_G^\kappa(\delta)$ depends only on the class of δ. However, the coefficients $\kappa(\gamma_v)$ in the local functions $f_{v,G}^\kappa(\delta_v)$ are defined in terms of the relative position of γ_v and δ_v. The local functions do therefore depend on the choice of δ_v within its stable class in $G(F_v)$. The solution of Langlands and Shelstad was to replace $\kappa(\gamma_v)$ with a function $\Delta_G(\delta'_v, \gamma_v)$ that they called a *transfer factor.* This function is defined as a product of $\kappa(\gamma_v)$ with an explicit but complicated factor that depends on δ'_v and δ_v, but not γ_v. The product $\Delta_G(\delta'_v, \gamma_v)$ then turns out to be independent of the choice of δ_v, and depends only on the local stable class of δ'_v and local conjugacy class of γ_v. Moreover, if δ'_v is the local image of $\delta' \in \Delta_{G\text{-reg,ell}}(G')$, for every v, the product over v of the corresponding local transfer factors is equal to the coefficient $\kappa(\gamma_{\mathbb{A}})$ in (27.10). (See [**LS1**, §3,§6].)

There is one further technical complication we should mention. The Langlands-Shelstad transfer factor depends on a choice of L-embedding of ${}^LG'$ into ${}^LG$. If G' represents an endoscopic datum $(G', \mathcal{G}', s', \xi')$ with $\mathcal{G}' \subset {}^LG$ and $\xi' = \iota'$, this amounts to a choice of L-isomorphism $\widetilde{\xi}'\colon \mathcal{G}' \to {}^LG'$. In the case that G_{der} is simply connected, such an L-isomorphism exists [**Lan8**]. However, it is not canonical, and one does have to choose $\widetilde{\xi}'$ in order to specify the transfer factors. The general situation is more complicated. The problem is that there might not be any such L-isomorphism. In this case, one has to modify the construction slightly. One replaces the group G' by a central extension

$$1 \longrightarrow \widetilde{C}' \longrightarrow \widetilde{G}' \longrightarrow G' \longrightarrow 1$$

of G', where $\widetilde{C}'$ is a suitable torus over F, and $\widetilde{G}'_{\mathrm{der}}$ is simply connected. One can then take $\widetilde{\xi}'$ to be an L-embedding

$$\widetilde{\xi}' : \mathcal{G}' \hookrightarrow {}^L\widetilde{G}',$$

whose existence is again implied by [**Lan8**]. This determines a character $\widetilde{\eta}'$ on $\widetilde{C}'(\mathbb{A})/\widetilde{C}'(F)$, which is dual to the global Langlands parameter defined by the composition

$$W_F \longrightarrow \mathcal{G}' \xrightarrow{\widetilde{\xi}'} {}^L\widetilde{G}' \longrightarrow {}^L\widetilde{C}',$$

for any section $W_F \to \mathcal{G}'$. The transfer factor at v becomes a function $\Delta_G(\delta'_v, \gamma_v)$ of $\delta'_v \in \Delta_{G\text{-reg}}\big(\widetilde{G}'(F_v)\big)$ and $\gamma_v \in \Gamma_{\text{reg}}\big(G(F_v)\big)$, such that

$$\Delta_G(c'_v\delta'_v, \gamma_v) = \widetilde{\eta}'_v(c'_v)^{-1}\Delta_G(\delta'_v, \gamma_v), \qquad c'_v \in \widetilde{C}'(F_v).$$

It vanishes unless δ'_v is an image of a stable conjugacy class $\delta_v \in \Delta_{\text{reg}}\big(G(F_v)\big)$ in $G(F_v)$, in which case it is supported on those conjugacy classes γ_v in $G(F_v)$ that lie in δ_v. In particular, $\Delta_G(\delta'_v, \gamma_v)$ has finite support in γ_v, for any δ'_v.

Transfer factors play the role of a kernel in a transform of functions. Consider a function f_v in $G(F_v)$, which we now take to be in the Hecke algebra $\mathcal{H}\big(G(F_v)\big)$. For any such f_v, we define an $(\widetilde{\eta}'_v)^{-1}$-equivariant function

$$f'_v(\delta'_v) = f_v^{\widetilde{G}'}(\delta'_v) = \sum_{\gamma_v \in \Gamma_{\text{reg}}(G(F_v))} \Delta_G(\delta'_v, \gamma_v) f_{v,G}(\gamma_v) \qquad (27.12)$$

of $\delta'_v \in \Delta_{G\text{-reg}}\big(\widetilde{G}'(F_v)\big)$. Langlands and Shelstad conjecture that f'_v lies in the space $S\mathcal{I}\big(\widetilde{G}'(F_v), \widetilde{\eta}'_v\big)$ [**LS1**]. In other words, $f'_v(\delta'_v)$ can be identified with the stable orbital integral at δ'_v of some fixed function h'_v in the $(\widetilde{\eta}'_v)^{-1}$-equivariant Hecke algebra $\mathcal{H}\big(\widetilde{G}'(F_v), \widetilde{\eta}'_v\big)$ on $\widetilde{G}'(F_v)$. (Langlands' earlier formulation of the conjecture [**Lan10**] was less precise, in that it postulated the existence of suitable transfer factors.) For archimedean v, the conjecture was established by Shelstad [**She3**]. In fact, it was Shelstad's results for real groups that motivated the construction of the general transfer factors $\Delta_G(\delta'_v, \gamma_v)$. (Shelstad actually worked with the Schwartz space $\mathcal{C}\big(G(F_v)\big)$. However, she also characterized the functions f'_v in spectral terms, and in combination with the main theorem of [**CD**], this establishes the conjecture for the space $\mathcal{H}\big(G(F_v)\big)$.)

If v is nonarchimedean, the Langlands-Shelstad conjecture remains open. Consider the special case that G, G' and $\widetilde{\eta}'$ are unramified at v, and that f_v is the characteristic function of a (hyperspecial) maximal compact subgroup K_v of $G(F_v)$. Then one would like to know not only that h'_v exists, but also that it can be taken to be the characteristic function of a (hyperspecial) maximal compact subgroup $\widetilde{K}'_v$ of $\widetilde{G}'(F_v)$ (or rather, the image of such a function in $\mathcal{H}(\widetilde{G}'_v, \widetilde{\eta}'_v)$.) This variant of the Langlands-Shelstad conjecture is what is known as the *fundamental lemma*. It is discussed in the lectures [**Hal1**] of Hales. Waldspurger has shown that the fundamental lemma actually implies the general transfer conjecture [**Wa2**]. To be precise, if the fundamental lemma holds for sufficiently many unramified pairs (G_v, G'_v), the Langlands-Shelstad transfer conjecture holds for an arbitrary given pair (G_v, G'_v).

The two conjectures together imply the existence of a global mapping

$$f = \prod_v f_v \longrightarrow f' = \prod_v f'_v$$

from $\mathcal{H}(G)$ to the space

$$S\mathcal{I}\big(\widetilde{G}'(\mathbb{A}), \widetilde{\eta}'\big) = \varinjlim_S S\mathcal{I}(\widetilde{G}'_S, \widetilde{\eta}'_S).$$

Such a mapping would complete Langlands' stabilization (27.3) of the regular elliptic term. It would express $I_{\mathrm{reg,ell}}(f)$ as the sum of a stable component, and pullbacks of corresponding stable components for proper endoscopic groups. We shall henceforth assume the existence of the mapping $f \to f'$. The remaining problem of stabilization is then to establish similar relations for the other terms in the invariant trace formula. We would like to show that any such term I_* has a stable component

$$S_* = S_*^G = S_*^{G^*},$$

now regarded as a stable linear form on the Hecke algebra, such that

$$I_*(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}'_*(f'), \tag{27.13}$$

for any $f \in \mathcal{H}\big(G(\mathbb{A})\big)$. The identity obtained by replacing the terms in the invariant trace formula by their corresponding stable components would then be a stable trace formula. We shall describe the solution to this problem in §29.

In recognition of the inductive nature of the putative identity (27.13), we ought to modify some of the definitions slightly. In the case of $I_* = I_{\mathrm{reg,ell}}$, for example, the term $\widehat{S}'_{G\text{-reg,ell}}(f')$ in (27.3) is not the full stable component of $I'_{\mathrm{reg,ell}}$. We could rectify this minor inconsistency by replacing $I_{\mathrm{reg,ell}}(f)$ with its H-regular part $I_{H\text{-reg,ell}}(f)$, for some reductive group H that shares a maximal torus with G, and whose roots contain those of G. The resulting version

$$I_{H\text{-reg,ell}}(f) = \sum_{G'} \iota(G, G') \widehat{S}'_{H\text{-reg,ell}}(f')$$

of (27.3) is then a true inductive formula.

Another point concerns the function f'. We are assuming that f' is the stable image of a function in the $(\widetilde{\eta}')^{-1}$-equivariant algebra $\mathcal{H}\big(\widetilde{G}'(\mathbb{A}), \widetilde{\eta}'\big)$. However, the original function f belongs to the ordinary Hecke algebra $\mathcal{H}(G)$. To put the two functions on an even footing, we fix a central torus Z in G over F, and a character ζ on $Z(\mathbb{A})/Z(F)$. We then write $\widetilde{Z}'$ for the preimage in $\widetilde{G}'$ of the canonical image of Z in G'. Global analogues of the local constructions in [**LS1**, (4.4)] provide a canonical extension of $\widetilde{\eta}'$ to a character on $\widetilde{Z}'(\mathbb{A})/\widetilde{Z}'(F)$. We write $\widetilde{\zeta}'$ for the character on $\widetilde{Z}'(\mathbb{A})/\widetilde{Z}'(F)$ obtained from the product of $\widetilde{\eta}'$ with the pullback of ζ. The presumed correspondence $f \to f'$ then takes the form of a mapping from $\mathcal{H}\big(G(\mathbb{A}), \zeta\big)$ to $S\mathcal{I}\big(\widetilde{G}'(\mathbb{A}), \widetilde{\zeta}'\big)$. At the beginning of §29, we shall describe a version of the invariant trace formula that applies to equivariant test functions f.

28. Local spectral transfer and normalization

We have now set the stage for the final refinement of the trace formula. We shall describe it over the course of the next two sections. This discussion, as well as that of the applications in §30, contains much that is only implicit. However, it also contains remarks that are intended to provide general orientation. A reader who is not an expert should ignore the more puzzling points at first pass, and aim instead at acquiring a sense of the underlying structure.

The problem is to stabilize the invariant trace formula for a general connected group G over F. In the case that G is an inner form of $GL(n)$, Theorems 25.5 and 25.6 represent a solution of the problem. They provide a term by term identification of the trace formula for G with the relevant part of the trace formula for the group

$G^* = GL(n)$. In this case, all invariant distributions are stable, and G^* is the only elliptic endoscopic group. The stabilization problem therefore reduces to the comparison of G with its quasisplit inner form.

The case of an inner form of $GL(n)$ is also simpler for the existence of a local correspondence $\pi_v \to \pi_v^*$ of tempered representations. Among other things, this allows us to define normalizing factors for intertwining operators for G in terms of those for G^*. We recall that the invariant distributions in the trace formula depend on a choice of normalizing factors. So therefore do any identities among these distributions. For general G, the theory of endoscopy predicts a refined local correspondence, which would yield compatible normalizations as a biproduct. However, the full form of this correspondence is presently out of reach. We nevertheless do require some analogue of it in any attempt to stabilize the trace formula.

We shall first describe a makeshift substitute for the local correspondence, which notwithstanding its provisional nature, still depends on the fundamental lemma. We will then review how the actual correspondence is supposed to work. After seeing the two side by side, the reader will probably agree that it is not reasonable at this point to try to construct compatible normalizing factors. Fortunately, there is a second way to normalize weighted orbital integrals, which does not depend on a normalization of intertwining operators. We shall discuss the construction at the end of the section. At the beginning of the next section, we shall describe how the construction leads to another form of the invariant trace formula. It will be this second form of the trace formula that we actually stabilize.

The global stabilization of the next section will be based on two spaces of invariant distributions, which reflect the general duality between conjugacy classes and characters. We may as well introduce them here. We are assuming that G is arbitrary. If V is a finite set of valuations of F, we shall write $G_V = G(F_V)$ for simplicity. Suppose that Z is a torus in G over F that is contained in the center, and that ζ_V is a character on Z_V. Let $\mathcal{D}(G_V, \zeta_V)$ be the space of invariant distributions that are ζ_V-equivariant under translation by Z_V, and are supported on the preimage in G_V of a finite union of conjugacy classes in $\overline{G}_V = G_V/Z_V$. Let $\mathcal{F}(G_V, \zeta_V)$ be the space of invariant distributions that are ζ_V-equivariant under translation by Z_V, and are spanned by irreducible characters on G_V. This second space is obviously spanned by the characters attached to the set $\Pi(G_V, \zeta_V)$ of irreducible representations of G_V whose central character on Z_V equals ζ_V. Now, the Hecke algebra on G_V has ζ_V^{-1}-equivariant analogue $\mathcal{H}(G_V, \zeta_V)$, composed of functions f such that

$$f(zx) = \zeta_V(z)^{-1} f(x), \qquad z \in Z_V.$$

Likewise, the invariant Hecke algebra has a ζ_V^{-1}-analogue $\mathcal{I}(G_V, \zeta_V)$. A distribution D in either of the spaces $\mathcal{D}(G_V, \zeta_V)$ or $\mathcal{F}(G_V, \delta_V)$ can be regarded as a linear form

$$D(f) = f_G(D), \qquad f \in \mathcal{H}(G_V, \zeta_V),$$

on either $\mathcal{H}(G_V, \zeta_V)$ or $\mathcal{I}(G_V, \zeta_V)$.

The notation $f_G(D)$ requires further comment. On the one hand, it generalizes the way we have been denoting both invariant orbital integrals $f_G(\gamma)$ and irreducible characters $f_G(\pi)$. But it also has the more subtle interpretation as the value of a linear form on the function f_G in $\mathcal{I}(G_V, \zeta_V)$. Since we have defined $\mathcal{I}(G_V, \zeta_V)$ as a space of functions on $\Pi_{\text{temp}}(G_V, \zeta_V)$, we need to know that D is supported on characters. This is clear if D belongs to $\mathcal{F}(G_V, \zeta_V)$. If D belongs to the other space $\mathcal{D}(G_V, \zeta_V)$, results of Harish-Chandra and Bouaziz [**Bou**] imply that it can be expressed in terms of strongly regular invariant orbital integrals. (See [**A30**, Lemma 1.1].) Since invariant orbital integrals are supported on characters, by the special case of Theorem 23.2 with $M = G$, D is indeed supported on characters. Incidentally, by the special case of Theorem 23.2 and the fact that characters are locally integrable functions, we can identify $\mathcal{I}(G_V, \zeta_V)$ with the space

$$\big\{f_G(\gamma) : \ \gamma \in \Gamma_{G\text{-reg}}(G_V), \ f \in \mathcal{H}(G_V, \zeta_V)\big\}.$$

We have in fact already implicitly done so in our discussion of inner twists and base change for GL_n in §25 and §26. However, we do not have a geometric analogue of [**CD**] that would allow us to characterize $\mathcal{I}(G_V, \zeta_V)$ explicitly as a space of functions on $\Gamma_{G\text{-reg}}(G_V)$.

We write $S\mathcal{D}(G_V, \zeta_V)$ and $S\mathcal{F}(G_V, \zeta_V)$ for the subspaces of stable distributions in $\mathcal{D}(G_V, \zeta_V)$ and $\mathcal{F}(G_V, \zeta_V)$ respectively. We also write $S\mathcal{I}(G_V, \zeta_V)$ for the ζ_V^{-1}-analogue of the stably invariant Hecke algebra. Any distribution S in either $S\mathcal{D}(G_V, \zeta_V)$ or $S\mathcal{F}(G_V, \zeta_V)$ can then be identified with a linear form

$$f^G \longrightarrow f^G(S), \qquad f \in \mathcal{H}(G_V, \zeta_V),$$

on $S\mathcal{I}(G_V, \zeta_V)$. We recall $S\mathcal{I}(G_V, \zeta_V)$ is presently just a space of functions on $\Delta_{G\text{-reg}}(G_V)$. One consequence of the results we are about to describe is a spectral characterization of $S\mathcal{I}(G_V, \zeta_V)$.

Our focus for the rest of this section will be entirely local. We shall consider the second space $\mathcal{F}(G_V, \zeta_V)$, under the condition that V consist of one valuation v. We shall regard G and Z as groups over the local field $k = F_v$, and we shall write $\zeta = \zeta_v$, $G = G_v = G(F_v)$, $\mathcal{F}(G, \zeta) = \mathcal{F}(G_v, \zeta_v)$, $\Pi(G, \zeta) = \Pi(G_v, \zeta_v)$, $\mathcal{H}(G, \zeta) = \mathcal{H}(G_v, \zeta_v)$, and so on, for simplicity. With this notation, we write $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ for the subspace of functions in $\mathcal{I}(G, \zeta)$ that are supported on the k-elliptic subset $\Gamma_{\mathrm{reg,ell}}(G)$ of $\Gamma_{\mathrm{reg}}(G) = \Gamma_{\mathrm{reg}}(G_v)$. We also write $S\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ for the image of $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ in $S\mathcal{I}(G, \zeta)$, and $\mathcal{H}_{\mathrm{cusp}}(G, \zeta)$ for the preimage of $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ in $\mathcal{H}(G, \zeta)$. Keep in mind that any element $D \in \mathcal{F}(G, \zeta)$ has a (virtual) character. It is a locally integrable, invariant function $\Theta(D, \cdot)$ on G_v such that

$$f_G(D) = \int_{G_v} f(x)\Theta(D, x)dx, \qquad f \in \mathcal{H}(G, \zeta).$$

Assume for a moment that Z contains the split component A_G (over k) of the center of G. The space $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$ then has the noteworthy property that it is a canonical linear image of $\mathcal{F}(G, \zeta)$. To be precise, there is a surjective linear mapping

$$\mathcal{F}(G, \zeta) \longrightarrow \mathcal{I}_{\mathrm{cusp}}(G, \zeta)$$

that assigns to any element $D \in \mathcal{F}(G, \zeta)$ the elliptic part

$$I_{\mathrm{ell}}(D, \gamma) = \begin{cases} I(D, \gamma), & \text{if } \gamma \in \Gamma_{\mathrm{reg,ell}}(G) \\ 0, & \text{otherwise,} \end{cases}$$

of its normalized character

$$I(D, \gamma) = |D^G(\gamma)|^{\frac{1}{2}}\Theta(D, \gamma).$$

This assertion follows from the case $M = G$ of the general result [**A20**, Theorem 5.1]. What is more, the mapping has a canonical linear section

$$\mathcal{I}_{\mathrm{cusp}}(G,\zeta) \longrightarrow \mathcal{F}(G,\zeta).$$

This is defined by a natural subset $T_{\mathrm{ell}}(G,\zeta)$ [**A22**, §4] of $\mathcal{F}(G,\zeta)$ whose image in $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ forms a basis.

The set $T_{\mathrm{ell}}(G,\zeta)$ contains the family $\Pi_2(G,\zeta)$ of square integrable representations of G_v with central character ζ. However, it also contains certain linear combinations of irreducible constituents of induced representations. We can define $T_{\mathrm{ell}}(G,\zeta)$ as the set of G_v-orbits of triplets (L,σ,r), where L is a Levi subgroup of G over $k = F_v$, σ belongs to $\Pi_2(L,\zeta)$, and r is an element in the R-group R_σ of σ whose null space in $\mathfrak{a}_M$ equals $\mathfrak{a}_G$. The R-group is an important object in local harmonic analysis that was discovered by Knapp. In general terms, it can be represented as a subgroup of the stabilizer of σ in $W(L)$, for which corresponding normalized intertwining operators

$$R_Q(r,\sigma) = A(\sigma_r)R_{r^{-1}Qr|Q}(\sigma), \qquad Q \in \mathcal{P}(L),\ r \in R_\sigma,$$

form a basis of the space of all operators that intertwine the induced representation $\mathcal{I}_Q(\sigma)$. We write σ_r for an extension of the representation σ to the group generated by L_v and a representative $\widetilde{w}_r$ of r in K_v. Then

$$A(\sigma_r):\ \mathcal{H}_{r^{-1}Qr}(\sigma) \longrightarrow \mathcal{H}_Q(\sigma)$$

is the operator defined by

$$\big(A(\sigma_r)\phi'\big)(x) = \sigma_r(\widetilde{w}_r)\phi'(\widetilde{w}_r^{-1}x), \qquad \phi' \in \mathcal{H}_{r^{-1}Qr}(\sigma).$$

(See [**A20**, §2].)

We identify elements $\tau \in T_{\mathrm{ell}}(G,\zeta)$ with the distributions

$$f_G(\tau) = \mathrm{tr}\big(R_Q(r,\sigma)\mathcal{I}_Q(\sigma,f)\big), \qquad f \in C_c^\infty(G),$$

in $\mathcal{F}(G,\zeta)$. It is the associated set of functions

$$I_{\mathrm{ell}}(\tau,\cdot), \qquad \tau \in T_{\mathrm{ell}}(G,\zeta),$$

that provides a basis of $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$. In fact, by Theorem 6.1 of [**A20**], these functions form an orthogonal basis of $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ with respect to a canonical measure $d\gamma$ on $\Gamma_{\mathrm{reg,ell}}(G/Z)$, whose square norms

$$\|I_{\mathrm{ell}}(\tau)\|^2 = \int_{\Gamma_{\mathrm{reg,ell}}(G/Z)} I_{\mathrm{ell}}(\tau,\gamma)\overline{I_{\mathrm{ell}}(\tau,\gamma)}\mathrm{d}\gamma = n(\tau), \qquad \tau \in T_{\mathrm{ell}}(G,\zeta),$$

satisfy

$$n(\tau) = |R_{\sigma,r}||\det(1-r)_{\mathfrak{a}_L/\mathfrak{a}_G}|.$$

(As usual $R_{\sigma,r}$ denotes the centralizer of r in R_σ. See [**A21**, §4].)

The set $T_{\mathrm{ell}}(G,\zeta)$ is part of a natural basis $T(G,\zeta)$ of $\mathcal{F}(G,\zeta)$. This can either be defined directly [**A20**, §3], or built up from elliptic sets attached to Levi subgroups. To remove the dependence on Z, we should really let ζ vary. The union

$$T_{\mathrm{temp,ell}}(G) = \coprod_\zeta T_{\mathrm{ell}}(G,\zeta)$$

is a set of tempered distributions, which embeds in the set

$$T_{\mathrm{ell}}(G) = \big\{\tau_\lambda :\ \tau \in T_{\mathrm{temp,ell}}(G),\ \lambda \in \mathfrak{a}^*_{G,\mathbb{C}}\big\}$$

that parametrizes nontempered elliptic characters

$$\Theta(\tau_\lambda, \gamma) = \Theta(\tau, \gamma) e^{\lambda(H_G(\gamma))}.$$

These two elliptic sets are in turn contained in respective larger sets

$$T_{\text{temp}}(G) = \coprod_{\{M\}} T_{\text{temp,ell}}(M)/W(M)$$

and

$$T(G) = \coprod_{\{M\}} T_{\text{ell}}(M)/W(M),$$

where $\{M\}$ represents the set of conjugacy classes of Levi subgroups of G over $k = F_v$. If $T_*(G)$ is any of the four sets above, we obviously have an associated subset $T_*(G, \zeta)$ attached to any Z and ζ. The distributions $f \to f_G(\tau)$ parametrized by the largest set $T(G, \zeta)$ form a basis of $\mathcal{F}(G, \zeta)$, while the distributions parametrized by $T_{\text{temp}}(G, \zeta)$ give a basis of the subset of tempered distributions in $\mathcal{F}(G, \zeta)$. We thus have bases that are parallel to the more familiar bases $\Pi(G, \zeta)$ and $\Pi_{\text{temp}}(G, \zeta)$ of these spaces given by irreducible characters.

Assume now that $k = F_v$ is nonarchimedean. In this case, one does not have a stable analogue for the set $T_{\text{ell}}(G, \zeta)$. As a substitute, in case G is quasisplit and Z contains A_G, we write $\Phi_2(G, \zeta)$ for an indexing set $\{\phi\}$ that parametrizes a fixed family of functions $\{S_{\text{ell}}(\phi, \cdot)\} \subset S\mathcal{I}_{\text{cusp}}(G, \zeta)$ for which the products

$$n(\delta) S_{\text{ell}}(\phi, \delta), \qquad \delta \in \Delta_{G\text{-reg,ell}}(G),\ \phi \in \Phi_2(G, \zeta),$$

form an orthogonal basis of $S\mathcal{I}_{\text{cusp}}(G, \zeta)$. (The number $n(\delta)$ stands for the number of conjugacy classes in the stable class δ, and is used to form the measure $d\delta$ on $\Delta_{G\text{-reg,ell}}(G/Z)$. The subscript *2* is used in place of *ell* because the complement of $\Pi_2(G, \zeta)$ in $T_{\text{ell}}(G, \zeta)$ is believed to be purely unstable.) We fix the family $\{S_{\text{ell}}(\phi, \cdot)\}$, subject to certain natural conditions [**A22**, Proposition 5.1]. We then form larger sets

$$\Phi_{\text{temp},2}(G) = \coprod_{\zeta} \Phi_2(G, \zeta),$$

$$\Phi_2(G) = \{\phi_\lambda :\ \phi \in \Phi_{\text{temp},2}(G),\ \lambda \in \mathfrak{a}^*_{G,\mathbb{C}}\},$$

$$\Phi_{\text{temp}}(G) = \coprod_{\{M\}} \Phi_{\text{temp},2}(M)/W(M),$$

and

$$\Phi(G) = \coprod_{\{M\}} \Phi_2(M)/W(M),$$

where $S_{\text{ell}}(\phi_\lambda, \delta) = S_{\text{ell}}(\phi, \delta) e^{\lambda(H_G(S))}$, as well as corresponding subsets $\Phi_*(G, \zeta)$ of $\Phi_*(G)$ attached to any Z and ζ. The analogy with the sets $T_*(G, \zeta)$ is clear. What is not obvious, however, is that the elements in $\Phi_*(G, \zeta)$ give stable distributions. The first step in this direction is to define

$$f^G(\phi) = \int_{\Delta_{\text{reg,ell}}(G/Z)} f^G(\delta) S_{\text{ell}}(\phi, \delta) d\delta, \tag{28.1}$$

for any $f \in \mathcal{H}_{\text{cusp}}(G, \zeta)$ and $\phi \in \Phi_2(G, \zeta)$.

We shall now apply the Langlands-Shelstad transfer of functions. One introduces endoscopic data G' for G over the local field $k = F_v$ by copying the definitions of §27 for the global field F. (The global requirement that a certain class in

$H^1\big(F, Z(\widehat{G})\big)$ be locally trivial is replaced by the simpler condition that the corresponding class in $H^1\big(F_v, Z(\widehat{G})\big)$ be trivial, but this is the only difference.) We follow the same notation as in the global constructions of §27. In particular, we write $\mathcal{E}_{\text{ell}}(G)$ for the set of isomorphism classes of elliptic endoscopic data for G over k.

We are assuming that the fundamental lemma holds, for units of Hecke algebras at unramified places of any group over F that is isomorphic to G over $k = F_v$. The theorem of Waldspurger mentioned at the end of the last section asserts that this global hypothesis (augmented to allow for induction arguments) implies the Langlands-Shelstad transfer conjecture for any endoscopic datum G' for G over k. We suppose that for each elliptic endoscopic datum $G' \in \mathcal{E}_{\text{ell}}(G)$ of G over k, we have chosen sets $\Phi(\widetilde{G}', \widetilde{\zeta}')$, as above. If f belongs to $\mathcal{H}_{\text{cusp}}(G,\zeta)$, f' belongs to $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}')$, by our assumption. Since the orbital integrals of f are supported on the elliptic set, f' in fact belongs to the subspace $S\mathcal{I}_{\text{cusp}}(\widetilde{G}', \widetilde{\zeta}')$ of $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}')$. We can therefore define $f'(\phi')$ by (28.1), for any element $\phi' \in \Phi_2(\widetilde{G}', \widetilde{\zeta}')$. As a linear form on $\mathcal{H}_{\text{cusp}}(G,\zeta)$, $f'(\phi')$ is easily seen to be the restriction of some distribution in $\mathcal{F}(G,\zeta)$. It therefore has an expression

$$f'(\phi') = \sum_{\tau \in T_{\text{ell}}(G,\zeta)} \Delta_G(\phi', \tau) f_G(\tau), \qquad f \in \mathcal{H}_{\text{cusp}}(G,\zeta), \tag{28.2}$$

in terms of the basis $T_{\text{ell}}(G,\zeta)$.

The coefficients $\Delta_G(\phi',\tau)$ in (28.2) are to be regarded as spectral transfer factors. They are defined a priori for elements $\phi' \in \Phi_2(\widetilde{G}', \widetilde{\zeta}')$ and $\tau \in T_{\text{ell}}(G,\zeta)$. However, it is easy to extend the construction to general elements $\phi' \in \Phi(\widetilde{G}', \widetilde{\zeta}')$ and $\tau \in T(G,\zeta)$. To do so, we represent ϕ' and τ respectively as Weyl orbits $\{\phi'_{M'}\}$ and $\{\tau_M\}$ of elliptic elements $\phi'_M \in \Phi_2(\widetilde{M}', \widetilde{\zeta}')$ and $\tau_M \in T_{\text{ell}}(M,\zeta)$ attached to Levi subgroups $\widetilde{M}' \subset \widetilde{G}'$ and $M \subset G$. We then define $\Delta_G(\phi',\sigma) = 0$ unless M' can be identified with an elliptic endoscopic group for M, in which case we set

$$\Delta_G(\phi',\tau) = \sum_{w \in W(M)} \Delta_M(\phi'_{M'}, w\tau_M).$$

It is not hard to deduce that for a fixed value of one of the arguments, $\Delta_G(\phi',\tau)$ has finite support in the other.

Suppose now that f is any function in $\mathcal{H}(G,\zeta)$. For any $G' \in \mathcal{E}_{\text{ell}}(G)$, we define the *spectral transfer* of f to be the function

$$f'_{\text{gr}}(\phi') = \sum_{\tau \in T(G,\zeta)} \Delta_G(\phi',\tau) f_G(\tau), \qquad \phi' \in \Phi'(\widetilde{G}', \widetilde{\zeta}').$$

(The subscript *gr* stands for the grading on the space $\mathcal{I}(G,\zeta)$ provided by the basis $T(G,\zeta)$ of $\mathcal{F}(G,\zeta)$.) It is by no means clear, a priori, that f'_{gr} coincides with the Langlands-Shelstad transfer f'. The problem is this. We defined the coefficients $\Delta_G(\phi',\tau)$ by stabilizing elliptic (virtual) characters $T_{\text{ell}}(G,\zeta)$ on the elliptic set. However, these characters also take values at nonelliptic elements. Why should their stabilization on the elliptic set, where they are uniquely determined, induce a corresponding stabilization on the nonelliptic set? The answer is provided by the following theorem.

THEOREM 28.1. (a) *Suppose that G is quasisplit and that $\phi \in \Phi(G,\zeta)$. Then the distribution*

$$f \longrightarrow f^G_{\mathrm{gr}}(\phi), \qquad f \in \mathcal{H}(G,\zeta),$$

is stable, and therefore lifts to a linear form

$$f^G \longrightarrow f^G(\phi), \qquad f \in \mathcal{H}(G,\zeta),$$

on $S\mathcal{I}(G,\zeta)$.

(b) *Suppose that G is arbitrary, that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$, and that $\phi' \in \Phi(\widetilde{G}',\widetilde{\zeta}')$. Then*

$$f'(\phi') = f'_{\mathrm{gr}}(\phi'), \qquad f \in \mathcal{H}(G,\zeta).$$

Remark. The theorem asserts that for any $\phi' \in \Phi(\widetilde{G}',\widetilde{\zeta}')$, the mapping $f \to f'(\phi')$ is a well defined element in $\mathcal{F}(G,\zeta)$, with an expansion

$$f'(\phi') = \sum_{\tau \in T(G,\zeta)} \Delta_G(\phi',\tau) f_G(\tau), \qquad f \in \mathcal{H}(G,\zeta). \tag{28.3}$$

Since $\Pi(G,\zeta)$ is another basis of $\mathcal{F}(G,\zeta)$, we could also write

$$f'(\phi') = \sum_{\pi \in \Pi(G,\zeta)} \Delta_G(\phi',\pi) f_G(\pi), \tag{28.4}$$

for complex numbers $\Delta_G(\phi',\pi)$.

The two assertions (a) and (b) of the theorem coincide with Theorems 6.1 and 6.2 of [**A22**], the main results of that paper. The proof is global. One chooses a suitable group over F that is isomorphic to G over $k = F_v$. By taking a global test function that is cuspidal at two places distinct from v, one can apply the simple trace formula of Corollary 23.6. The fundamental lemma and the Langlands-Shelstad transfer mapping provide a transfer of global test functions to endoscopic groups. One deduces the assertions of the theorem by a variant of the arguments used to establish Theorem 25.1(b) [**DKV**] and local base change [**AC**, §1]. □

We have taken some time to describe a weak form of spectral transfer. This is of course needed to stabilize the general trace formula. However, we would also like to contrast it with the stronger version expected from the theory of endoscopy, which among many other things, ought to give rise to compatible normalizing factors. For we are trying to see why we need another form of the invariant trace formula.

One expects to be able to identify $\Phi(G)$ with the set of Langlands parameters. A Langlands parameter for G is a $\widehat{G}$-conjugacy class of relevant L-homomorphisms

$$\phi : \; L_k \longrightarrow {}^L G,$$

from the local Langlands group

$$L_k = W_k \times SU(2)$$

to the Weil form ${}^L G = \widehat{G} \rtimes W_k$ of G over $k = F_v$. (In this context, an L-homomorphism is a continuous homomorphism for which the image in $\widehat{G}$ of any element is semisimple, and which commutes with the projections of L_k and ${}^L G$ onto W_k. *Relevant* means that if the image of ϕ is contained in a Levi subgroup ${}^L M$ of ${}^L G$, then ${}^L M$ must be the L-group of a Levi subgroup M of G over k.) Let us temporarily let $\Phi(G)$ denote the set of such parameters, rather than the abstract indexing set above. Then $\Phi_{\mathrm{temp}}(G)$ corresponds to those homomorphisms whose image projects to a relatively compact subset of $\widehat{G}$. The subset $\Phi_2(G)$ corresponds to mappings whose images are contained in no proper Levi subgroup ${}^LM$ of ${}^LG$, while $\Phi_{\mathrm{temp},2}(G)$ is of course the intersection of $\Phi_{\mathrm{temp}}(G)$ with $\Phi_2(G)$. For any ϕ, one writes S_ϕ for the centralizer in $\widehat{G}$ of the image of ϕ, and $\mathcal{S}_\phi$ for the group of connected components of the quotient $\overline{S}_\phi = S_\phi/Z(\widehat{G})^{\Gamma_k}$. The *R-group* R_ϕ of ϕ is defined as the quotient of $\mathcal{S}_\phi$ by the subgroup of components that act by inner automorphism on $\overline{S}^0_\phi$. A choice of Borel subgroup in the connected reductive group $\overline{S}_\phi$ induces an embedding of R_ϕ into $\mathcal{S}_\phi$.

In the case that G is abelian, Langlands constructed a natural bijection $\phi \to \pi$ from the set of parameters $\Phi(G)$ onto the set $\Pi(G)$ of quasicharacters on G [**Lan12**]. We can therefore set

$$S_{\mathrm{ell}}(\phi,\gamma) = \Theta(\pi,\gamma) = \pi(\gamma), \qquad \gamma \in G(k),$$

in this case. For example, if $G = GL(1)$, a parameter in $\Phi(G)$ is tantamount to a continuous homomorphism

$$L_k = W_k \times SU(2) \longrightarrow \widehat{G} = \mathbb{C}^*.$$

Since $SU(2)$ is its own derived group, and the abelianization of W_k is isomorphic to $k^* \cong G(k)$, a parameter does indeed correspond to a quasicharacter. If G is a general group, with central torus Z, there is a canonical homomorphism from ${}^LG$ to ${}^LZ$. A parameter in $\Phi(G)$ then yields a quasicharacter ζ on Z, whose corresponding parameter is the composition

$$L_k \xrightarrow{\phi} {}^LG \longrightarrow {}^LZ.$$

The entire set of parameters $\Phi(G)$ thus decomposes into a disjoint union over ζ of the subsets $\Phi(G,\zeta)$ with central quasicharacter ζ on Z.

Suppose that ζ is a character on Z. For each parameter $\phi \in \Phi_{\mathrm{temp}}(G,\zeta)$, it is expected that there is a canonical nonnegative integer valued function $d_\phi(\pi)$ on $\Pi_{\mathrm{temp}}(G,\zeta)$ with finite support, such that the distribution

$$f \longrightarrow f^G(\phi) = \sum_\pi d_\phi(\pi) f_G(\pi), \qquad f \in \mathcal{H}(G,\zeta),$$

is stable. The sum

$$S(\phi,\delta) = \sum_\pi d_\phi(\pi) I(\pi,\gamma), \qquad \gamma \in \Gamma_{\mathrm{reg}}(G),$$

would then depend only on the stable conjugacy class δ of γ. Moreover, the finite packets

$$\Pi_\phi = \big\{\pi \in \Pi_{\mathrm{temp}}(G,\zeta) : \ d_\phi(\pi) > 0\big\}, \qquad \phi \in \Phi_{\mathrm{temp}}(G,\zeta),$$

are supposed to be disjoint, and have union equal to $\Pi_{\mathrm{temp}}(G,\zeta)$. The subset $\Pi_{\mathrm{temp},2}(G,\zeta)$ of $\Pi_{\mathrm{temp}}(G,\zeta)$ should be the disjoint union of packets Π_ϕ, in which ϕ ranges over the subset $\Phi_{\mathrm{temp},2}(G,\zeta)$ of $\Phi_{\mathrm{temp}}(G,\zeta)$.

Suppose that these properties hold in general, and that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ and $\phi' \in \Phi_{\mathrm{temp}}(\widetilde{G}',\widetilde{\zeta}')$. Then $f'(\phi')$ is a well defined linear form in $f \in \mathcal{H}(G,\zeta)$. The pair $(\widetilde{G}',\widetilde{\zeta}')$ is constructed in such a way that ϕ' maps to a parameter $\phi \in \Phi_{\mathrm{temp}}(G,\zeta)$. For example, if $\widetilde{G}'$ happens to equal G', ϕ is just the composition of

ϕ' with the underlying embedding of ${}^LG'$ into ${}^LG$. It is believed that the expansion of $f'(\phi')$ into irreducible characters on G takes the form

$$f'(\phi') = \sum_{\pi\in\Pi_\phi} \Delta_G(\phi', \pi) f_G(\pi), \tag{28.5}$$

for complex coefficients $\Delta_G(\phi', \pi)$ that are supported on the packet Π_ϕ.

The other basis $T_{\text{temp}}(G, \zeta)$ would also have a packet structure. For the elements of Π_ϕ ought to be irreducible constituents of induced representations

$$\mathcal{I}_P(\sigma), \qquad P \in \mathcal{P}(M),\ \sigma \in \Pi_{\phi_M}, \tag{28.6}$$

where $M \subset G$ is a minimal Levi subgroup whose L-group ${}^LM \subset {}^LG$ contains the image of ϕ, and ϕ_M is the parameter in $\Phi_{\text{temp},2}(M, \zeta)$ determined by ϕ. Recall that as a representation in $\Pi_2(M, \zeta)$, σ has its own R-group R_σ. In terms of the R-group of ϕ, R_σ ought to be the stabilizer of σ under the dual action of R_ϕ on M. Let T_ϕ be the subset of $T_{\text{temp}}(G, \zeta)$ represented by triplets

$$(M, \sigma, r), \qquad \sigma \in \Pi_{\phi_M},\ r \in R_\sigma.$$

If the packet Π_ϕ is defined as above, the packet T_ϕ gives rise to a second basis of the subspace of $\mathcal{F}(G, \zeta)$ spanned by Π_ϕ. It provides a second expansion

$$f'(\phi') = \sum_{\tau\in T_\phi} \Delta_G(\phi', \tau) f_G(\tau), \tag{28.7}$$

for complex coefficients $\Delta_G(\phi', \tau)$. As ϕ varies over $\Phi_{\text{temp}}(G, \zeta)$, $T_{\text{temp}}(G, \zeta)$ is a disjoint union of the corresponding packets T_ϕ.

Given their expected properties, Langlands' parameters become canonical indexing sets. If G is quasisplit and Z contains A_G, we can set

$$S_{\text{ell}}(\phi, \delta) = \begin{cases} S(\phi, \delta), & \text{if } \delta \in \Delta_{\text{reg,ell}}(G), \\ 0, & \text{otherwise}, \end{cases}$$

for any $\phi \in \Phi_2(G, \zeta)$. The family $\{S_{\text{ell}}(\phi, \cdot)\}$ then serves as the basis of $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ chosen earlier. The improvement of the conjectural transfer (28.5) or (28.7) over the weaker version (28.4) or (28.3) that one can actually prove (modulo the fundamental lemma) is obvious. For example, the hypothetical coefficients in (28.5) are supported on disjoint sets parametrized by $\Phi_{\text{temp}}(G, \zeta)$. However, the actual coefficents in (28.4) could have overlapping supports, for which we have no control.

The hypothetical coefficients in (28.5) are expected to have further striking properties. Suppose for example that G is quasisplit. In this case, it seems to be generally believed that coefficients will give a bijection from Π_ϕ onto the set $\widehat{\mathcal{S}}_\phi$ of irreducible characters on $\mathcal{S}_\phi$. This bijection would depend on a noncanonical choice of any base point π_1 in Π_ϕ at which the integer $d_\phi(\pi_1) = \Delta_G(\phi, \pi_1)$ equals 1. The irreducible character attached to any $\pi \in \Pi_\phi$ ought then to be the function

$$s \to \langle s, \pi | \pi_1 \rangle = \Delta(\phi', \pi)\Delta(\phi', \pi_1)^{-1}, \qquad s \in \mathcal{S}_\phi, \tag{28.8}$$

where s is the projection onto $\mathcal{S}_\phi$ of the semisimple element $s' \in S_\phi$ attached to the elliptic endoscopic datum G'. There is also a parallel interpretation that relates the hypothetical coefficients (28.7) and the packets T_ϕ with the representation theory of the finite groups $\mathcal{S}_\phi$. In the case that G is not quasisplit, similar properties are expected, but they are weaker and not completely understood. (See [**LL**], [**Lan10**, §IV.2].)

We have been assuming that the parameter ϕ is tempered. Suppose now that ϕ is a general parameter. Then ϕ is the image in $\Phi(G)$ of a twist $\phi_{M,\lambda}$, for a Levi subgroup $M \subset G$, a tempered parameter $\phi_M \in \Phi_{\mathrm{temp}}(M)$, and a point λ in the chamber $(\mathfrak{a}_M^*)_P^+$ in $\mathfrak{a}_M^*$ attached to a parabolic subgroup $P \in \mathcal{P}(M)$. The packet Π_ϕ can be defined to be the set of irreducible representations obtained by taking the unique irreducible quotient (the Langlands quotient) of each representation

$$\mathcal{I}_P(\pi_{M,\lambda}), \qquad \pi_M \in \Pi_{\phi_M}. \tag{28.9}$$

Similar constructions allow one to define the packet T_ϕ in terms of the tempered packet T_{ϕ_M}. One can thus attach conjectural packets to nontempered parameters. The Langlands classification for real groups [**Lan11**] extends to p-adic groups, to the extent that it reduces the general classification to the tempered case [**BW**, §XI.2]. Combined with the expected packet structure of tempered representations, it then gives a conjectural classification of $\Pi(G)$ into a disjoint union of finite packets Π_ϕ, indexed by parameters $\phi \in \Phi(G)$. Moreover, for ϕ, ϕ_M, and λ as above, the finite group $\mathcal{S}_\phi$ equals the corresponding group $\mathcal{S}_{\phi_M}$ attached to the tempered parameter ϕ_M for M. We can therefore relate the representations in Π_ϕ to characters on $\mathcal{S}_\phi$, if we are able to relate the representations in the tempered packet Π_{ϕ_M} with characters in $\mathcal{S}_{\phi_M}$. However, the nontempered analogues of the character relations (28.5) and (28.7) will generally be false.

Suppose that $G = GL(n)$. In this case, the centralizer S_ϕ of the image of any parameter $\phi \in \Phi(G)$ is connected. The group $\mathcal{S}_\phi$ is therefore trivial, and the corresponding packet Π_ϕ should consequently contain exactly one element. The Langlands classification for $G = GL(n)$ thus takes the form of a bijection between parameters $\phi \in \Phi(G)$ and irreducible representations $\pi \in \Pi(G)$. It has recently been established by Harris and Taylor [**HT**] and Henniart [**He**].

We have assumed that the local field k was nonarchimedean. The analogues for archimedean fields $k = F_v$ of the conjectural properties described above have all been established. They are valid as stated, except that L_k is just the Weil group W_k, and the correspondence $\Pi_\phi \to \widehat{\mathcal{S}}_\phi$ is an injection rather than a bijection. As we mentioned earlier, the classification of irreducible representations $\Pi(G)$ in terms of parameters $\phi \in \Phi(G)$ was established by Langlands and Knapp-Zuckermann. (See [**KZ1**].) The transfer identities (28.5) and (28.7) for tempered parameters ϕ, together with the description of packets in terms of characters $\widehat{\mathcal{S}}_\phi$, were established by Shelstad [**She2**], [**She3**]. In particular, there is a classification of irreducible representations of $G(k)$ in terms of simple invariants attached to the dual group ${}^L G$. One would obviously like to have a similar classification for nonarchimedean fields.

One reason for wanting such a classification is to give a systematic construction of L-functions for irreducible representations. Suppose that $k = F_v$ is any completion of F. One can attach a local L-function $L(s,r)$ and ε-factor $\varepsilon(s,r,\psi)$ of the complex variable s to any (continuous, semisimple) representation r of the local Weil group W_k, and any nontrivial additive character $\psi\colon k \to \mathbb{C}$. The ε-factors are needed for the functional equations of L-functions attached to representations of the global Weil group W_F. Deligne's proof [**D1**] that they exist and have the appropriate properties in fact uses global arguments. Suppose that the local Langlands conjecture holds for $G = G_v$. That is, any irreducible representation $\pi \in \Pi(G)$ lies in the packet Π_ϕ attached to a unique parameter ϕ. We write ϕ_W for the restriction of ϕ to the subgroup W_k of L_k. Suppose that ρ is a finite dimensional representation of the L-group ${}^LG$. We can then define a local L-function

$$L(s, \pi, \rho) = L(s, \rho \circ \phi_W) \tag{28.10}$$

and ε-factor

$$\varepsilon(s, \pi, \rho, \psi) = L(s, \rho \circ \phi_W, \psi) \tag{28.11}$$

in terms of corresponding objects for W_k. For example, suppose that $k = F_v$ is nonarchimedean, and that π, ρ, and ψ are unramified. Then π is parametrized by a semisimple conjugacy class $c = c(\pi)$ in ${}^LG$. The associated parameter ϕ: $L_k \to {}^LG$ is trivial on both $SU(2)$ and the inertia subgroup I_k of W_k. It maps the element Frob_k that generates the cyclic quotient W_k/I_k to c. In this case, $\varepsilon(s, \pi, \rho, \psi) = 1$, and

$$L(s, \pi, \rho) = \det\big(1 - \rho(c)q^{-s}\big),$$

where $q = q_v$.

Langlands has conjectured that local L-functions give canonical normalizing factors for induced representations. Suppose that $\pi \in \Pi(M)$ is an irreducible representation of a Levi subgroup M of G over $k = F_v$. Recall that the unnormalized intertwining operators

$$J_{Q|P}(\pi_\lambda) : \mathcal{I}_P(\pi_\lambda) \longrightarrow \mathcal{I}_Q(\pi_\lambda), \qquad P, Q \in \mathcal{P}(M),$$

between induced representations are meromorphic functions of a complex variable $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Let $\rho_{Q|P}$ be the adjoint representation of ${}^LM$ on the Lie algebra of the intersection of the unipotent radicals of the parabolic subgroups $\widehat{P}$ and $\widehat{\overline{Q}}$ of $\widehat{G}$. We can then set

$$r_{Q|P}(\pi_\lambda) = L(0, \pi_\lambda, \rho_{Q|P})\big(\varepsilon(0, \pi_\lambda, \rho^\vee_{Q|P}, \psi)L(1, \pi_\lambda, \rho_{Q|P})\big)^{-1}, \tag{28.12}$$

assuming of course that the functions on the right have been defined. Langlands conjectured [**Lan5**, Appendix II] that for a suitable normalization of Haar measures on the groups $N_Q \cap N_{\bar{P}}$, these meromorphic functions of λ are an admissible set of normalizing factors, in the sense that they satisfy the conditions of Theorem 21.4. It is this conjecture that Shahidi established in case $G = GL(n)$, and that was used in the applications described in §25 and §26. (We recall that for $GL(n)$, the relevant local L and ε-factors were defined independently of Weil groups. Part of the recent proof of the local Langlands classification for $GL(n)$ by Harris-Taylor and Henniart was to show that these L and ε-factors were the same as the ones attached to representations of W_k.)

However, we do not have a general classification of representations in the packets Π_ϕ. We therefore cannot use (28.10) and (28.11) to define the factors on the right hand side of (28.12). The canonical normalization factors are thus not available. This is our pretext for normalizing the weighted characters in a different way.

Instead of normalizing factors $r = \{r_{Q|P}(\pi_\lambda)\}$, we use Harish-Chandra's canonical family $\mu = \{\mu_{Q|P}(\pi_\lambda)\}$ of μ-functions. We recall that

$$\mu_{Q|P}(\pi_\lambda) = \big(J_{Q|P}(\pi_\lambda)J_{P|Q}(\pi_\lambda)\big)^{-1} = \big(r_{Q|P}(\pi_\lambda)r_{P|Q}(\pi_\lambda)\big)^{-1},$$

for any $Q, P \in \mathcal{P}(M)$, $\pi \in \Pi(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Suppose that π is in general position, in the sense that the unnormalized intertwining operators $J_{Q|P}(\pi_\lambda)$ are analytic for $\lambda \in i\mathfrak{a}_M^*$. For fixed P, the operator valued family

$$\mathcal{J}_Q(\Lambda, \pi, P) = J_{Q|P}(\pi)^{-1} J_{Q|P}(\pi_\Lambda), \qquad Q \in \mathcal{P}(M),$$

is a (G, M)-family of functions of $\Lambda \in i\mathfrak{a}_M^*$. The normalized weighted characters used in the original invariant trace formula were constructed from the product (G, M)-family

$$\mathcal{R}_Q(\Lambda, \pi, P) = r_Q(\Lambda, \pi, P)^{-1} \mathcal{J}_Q(\Lambda, \pi, P),$$

where

$$r_Q(\Lambda, \pi, P) = r_{Q|P}(\pi)^{-1} r_{Q|P}(\pi_\Lambda).$$

The normalized weighted characters for the second version are to be constructed from the product (G, M)-family

$$\mathcal{M}_Q(\Lambda, \pi, P) = \mu_Q(\Lambda, \pi, P) \mathcal{J}_Q(\Lambda, \pi, P), \tag{28.13}$$

where

$$\mu_Q(\Lambda, \pi, P) = \mu_{Q|P}(\pi)^{-1} \mu_{Q|P}(\pi_{\frac{1}{2}\Lambda}).$$

They are defined by setting

$$J_M(\pi, f) = \operatorname{tr}\big(\mathcal{M}_M(\pi, P) \mathcal{I}_P(\pi, f)\big), \qquad f \in \mathcal{H}(G), \tag{28.14}$$

where

$$\mathcal{M}_M(\pi, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{M}_Q(\Lambda, \pi, P) \theta_Q(\Lambda)^{-1}, \tag{28.15}$$

as usual. Notice that we are using the same notation for the two sets of normalized weighted characters. It there is any danger of confusion, we can always denote the original objects by $J_M^r(\pi, f)$, and the ones we have just constructed by $J_M^\mu(\pi, f)$.

PROPOSITION 28.2. *The linear form $J_M(\pi, f) = J_M^\mu(\pi, f)$, defined for $\pi \in \Pi(M)$ in general position, is independent of the fixed group $P \in \mathcal{P}(M)$. Moreover, if $\pi \in \Pi_{\mathrm{unit}}(M)$ is any unitary representation, $J_M(\pi_\lambda, f)$ is an analytic function of $\lambda \in i\mathfrak{a}_M^*$.*

The two assertions are among the main results of [**A24**]. We know that for the original weighted characters $J_M^r(\pi, \lambda)$, the assertions are simple consequences of the properties of the normalizing factors r. We form a second (G, M)-family

$$r_Q(\Lambda, \pi) = r_{Q|\overline{Q}}(\pi)^{-1} r_{Q|\overline{Q}}(\pi_{\frac{1}{2}\Lambda}), \qquad Q \in \mathcal{P}(M),$$

from the normalizing factors. The new weighted characters are then related to the original ones by an expansion

$$J_M^\mu(\pi, f) = \sum_{L \in \mathcal{L}(M)} r_M^L(\pi) J_L^r(\pi^L, f),$$

which one derives easily from the relations between the functions $\{r_{Q|P}(\pi_\lambda)\}$ and $\{\mu_{Q|P}(\pi_\lambda)\}$ [**A24**, Lemma 2.1]. The first assertion follows immediately [**A24**, Corollary 2.2]. To establish the second assertion, one shows that for $\pi \in \Pi_{\mathrm{unit}}(M)$, the functions $r_M^L(\pi_\lambda)$ are analytic on $i\mathfrak{a}_M^*$ [**A24**, Proposition 2.3]. □

29. The stable trace formula

In this section, we shall discuss the solution to the problem posed at the end of §27. We shall describe how to stabilize all of the terms in the invariant trace formula. The stabilization is conditional upon the fundamental lemma. It is also contingent upon a generalization of the fundamental lemma, which applies to unramified weighted orbital integrals.

The results are contained in the three papers [**A27**], [**A26**], and [**A29**]. They depend on other papers as well, including some still in preparation. Our discussion will therefore have to be quite limited. However, we can at least try to give a coherent statement of the results. The techniques follow the model of inner twistings of $GL(n)$, outlined in some detail in §25. However, the details here are considerably more elaborate. The results discussed in this section are in fact the most technical of the paper.

We have of course to return to the global setting, with which we were preoccupied before the local interlude of the last section. Then G is a fixed reductive group over the number field F. There are two preliminary matters to deal with before we can consider the main problem.

The first is to reformulate the invariant trace formula for G. Since it is based on the construction at the end of the last section, this second version does not depend on the normalization of intertwining operators. In some ways, it is slightly less elegant than the original version, but the two are essentially equivalent. In particular, our stabilization of the second version would no doubt give a stabilization of the first, if we had the compatible normalizing factors provided by a refined local correspondence of representations.

Our reformulation of the invariant trace formula entails a couple of other minor changes. It applies to test functions f on the group $G_V = G(F_V)$, where V is a finite set of valuations of F that contains the set $S_{\mathrm{ram}} = S_{\mathrm{ram}}(G)$ of places at which G is ramified. We can take V to be large. However, we want to distinguish it from the large finite set S that occurs on the geometric side of the original formula. In relating the two versions of the formula, S would be a finite set of places that is large relative to both V and the support of some chosen test function on G_V. The terms in our second version will be indexed by conjugacy classes in M_V (rather than $M(\mathbb{Q})$-conjugacy classes or (M, S)-classes) and irreducible representations of M_V (rather than automorphic representations of $M(\mathbb{A})$). In order to allow for induction arguments, we also need to work with equivariant test functions on G_V. We fix a suitable central torus $Z \subset G$ over F, and a character ζ on $Z(\mathbb{A})/Z(F)$. We then assume that V contains the larger finite set $S_{\mathrm{ram}}(G, \zeta)$ of valuations at which any of G, Z or ζ ramifies. We write G_V^Z for the subgroup of elements $x \in G_V$ such that $H_G(x)$ lies in the image of $\mathfrak{a}_Z$ in $\mathfrak{a}_G$, and ζ_V for the restriction of ζ to Z_V. Our test functions are to be taken from the Hecke algebra

$$\mathcal{H}(G, V, \zeta) = \mathcal{H}(G_V^Z, \zeta_V),$$

and its invariant analogue

$$\mathcal{I}(G, V, \zeta) = \mathcal{I}(G_V^Z, \zeta_V).$$

Observe that if Z equals 1, G_V^Z equals the group $G(F_V)^1$. In this case, $\mathcal{H}(G, V, \zeta)$ embeds in the original space $\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})^1\big)$ of test functions.

There is a natural projection from the subspace $\mathcal{H}(G, V) = \mathcal{H}\big(G(F_V)^1\big)$ of $\mathcal{H}(G)$ onto $\mathcal{H}(G, V, \zeta)$. Let J be the basic linear form on $\mathcal{H}(G)$ whose two expansions give the noninvariant trace formula. If f lies in $\mathcal{H}(G)$, and f_z denotes the translate of f by a point $z \in Z(\mathbb{A})^1$, the integral

$$\int_{Z(F)\backslash Z(\mathbb{A})^1} J(f_z)\zeta(z)\mathrm{d}z$$

is well defined. If f belongs to the subspace $\mathcal{H}(G, V)$, the integral depends only on the image of f in $\mathcal{H}(G, V, \zeta)$. It therefore determines a linear form on $\mathcal{H}(G, V, \zeta)$, which we continue to denote by J. To make this linear form invariant, we define mappings

$$\phi_M : \mathcal{H}(G, V, \zeta) \longrightarrow \mathcal{I}(M, V, \zeta), \qquad M \in \mathcal{L}, \tag{29.1}$$

in terms of the weighted characters at the end of last section. In other words, the operator valued weight factor is to be attached to a product over $v \in V$ of (G, M)-families (28.13), rather than the (G, M)-family defined in §23 in terms of normalized intertwining operators. The mapping itself is defined by an integral analogous to (23.2) (with $X = 0$), but over a domain $i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}$ (where $i\mathfrak{a}^*_{M,Z}$ is the subspace of elements in $i\mathfrak{a}^*_M$ that vanish on the image of $i\mathfrak{a}^*_Z$ on $i\mathfrak{a}^*_M$). It follows from the proof of Propositions 23.1 and 28.2 that ϕ_M does indeed map $\mathcal{H}(G, V, \zeta)$ to $\mathcal{I}(M, V, \zeta)$. We can therefore define an invariant linear form $I = I^G$ on $\mathcal{H}(G, V, \zeta)$ by the analogue of (23.10). The problem is to transform the two expansions of Theorem 23.4 into two expansions of this new linear form.

We define weighted orbital integrals $J_M(\gamma, f)$ for functions $f \in \mathcal{H}(G, V, \zeta)$ exactly as in §18. The element γ is initially a conjugacy class in M^Z_V. However, $J_M(\gamma, f)$ depends only on the image of γ in the space $\mathcal{D}(M^Z_V, \zeta_V)$ of invariant distributions on M^Z_V, defined as at the beginning of §28. We can therefore regard $J_M(\cdot, f)$ as a linear form on the subspace $\mathcal{D}_{\mathrm{orb}}(M^Z_V, \zeta_V)$ of $\mathcal{D}(M^Z_V, \zeta_V)$ generated by conjugacy classes. There is actually a more subtle point, which we may as well raise here. As it turns out, stabilization requires that $J_M(\gamma, f)$ be defined for *all* elements in the space $\mathcal{D}(M^Z_V, \zeta)$. If v is nonarchimedean, $\mathcal{D}_{\mathrm{orb}}(M_v, \zeta_v)$ equals $\mathcal{D}(M_v, \zeta_v)$. In this case, there is nothing further to do. However, if v is archimedean, $\mathcal{D}(M_v, \zeta_v)$ is typically much larger than $\mathcal{D}_{\mathrm{orb}}(M_v, \zeta_v)$, thanks to the presence of normal derivatives along conjugacy classes. The construction of weighted orbital integrals at distributions in this larger space demands a careful study of the underlying differential equations. Nevertheless, one can in the end extend $J_M(\gamma, f)$ to a canonical linear form on the space $\mathcal{D}(M^Z_V, \zeta_V)$. (See [**A31**].) One then uses the mappings (29.1) as in (23.3), to define invariant distributions

$$I_M(\gamma, f), \qquad \gamma \in \mathcal{D}(M^Z_V, \zeta_V),\ f \in \mathcal{H}(G, V, \zeta).$$

These distributions, with γ restricted to the subspace $\mathcal{D}_{\mathrm{orb}}(M^Z_V, \zeta_V)$ of $\mathcal{D}(M^Z_V, \zeta_V)$, will be the terms in the geometric expansion.

The coefficients in the geometric expansion should really be regarded as elements in $\mathcal{D}(M^Z_V, \zeta_V)$, or rather, the appropriate completion $\widehat{\mathcal{D}}(M^Z_V, \zeta_V)$ of $\mathcal{D}(M^Z_V, \zeta_V)$. As such, they have a natural pairing with the linear forms $I_M(\cdot, f)$ on $\mathcal{D}(M^Z_V, \zeta_V)$. However, we would like to work with an expansion like that of (23.11). We therefore identify $\widehat{\mathcal{D}}(M^Z_V, \zeta_V)$ with the dual space of $\mathcal{D}(M^Z_V, \zeta_V)$ by fixing a suitable basis of $\Gamma(M^Z_V, \zeta_V)$ of $\mathcal{D}(M^Z_V, \zeta_V)$. Since we can arrange that the elements in

$$\Gamma_{\mathrm{orb}}(M_V^Z,\zeta_V)=\Gamma(M_V^Z,\zeta_V)\cap\mathcal{D}_{\mathrm{orb}}(M_V^Z,\zeta_V)$$

be parametrized by conjugacy classes in $\overline{M}_V=M_V/Z_V$, we will still be dealing essentially with conjugacy classes. We define coefficient functions on $\Gamma(M_V^Z,\zeta_V)$ by compressing the corresponding coefficients in (23.11). It is done in two stages. For a given $\gamma_M\in\Gamma(M_{\underline{V}},\zeta_V)$, we choose a large finite set $S\supset V$, and take k to be a conjugacy class in $\overline{M}(F_V^S)$ that meets K_V^S. We then define a function $a_{\mathrm{ell}}^M(\gamma_M\times k)$ as a certain finite linear combination of coefficients $a^M(\gamma)$ in (23.11), taken over those (M,S)-equivalence classes $\gamma\in\Gamma(M)_S$ that map to $\gamma_M\times k$ [**A27**, (2.6)]. For any given k, we can form the unramified weighted orbital integral

$$r_M^G(k)=J_M(k,u_S^V), \tag{29.2}$$

where $u_S^V=u_S^{V,\zeta}$ is the projection onto $\mathcal{H}(G_S^V,\zeta_S^V)$ of the characteristic function of K_S^V. If γ is now an element in $\Gamma(G_V^Z,\zeta_V)$, we set

$$a^G(\gamma)=\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_k a_{\mathrm{ell}}^M(\gamma_M\times k)r_M^G(k), \tag{29.3}$$

where $\gamma\to\gamma_M$ is the restriction operator that is adjoint to induction of conjugacy classes (and invariant distributions). (See [**A27**, (2.8), (1.9)].)

PROPOSITION 29.1. *Suppose that $f\in\mathcal{H}(G,V,\zeta)$. Then*

$$I(f)=\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)}a^M(\gamma)I_M(\gamma,f),$$

where $\Gamma(M,V,\zeta)$ is a discrete subset of $\Gamma(M_V^Z,\zeta_V)$ that contains the support of $a^M(\gamma)$, and on which $I_M(\gamma,f)$ has finite support.

See [**A27**, Proposition 2.2]. □

The spectral expansion of $I(f)$ begins with the decomposition

$$I(f)=\sum_{t\geq 0}I_t(f),\qquad f\in\mathcal{H}(G,V,\zeta), \tag{29.4}$$

relative to the norms t of archimedean infinitesimal characters. The summand $I_t(f)$ is as in Remark 10 of §23, the invariant version of a linear form $J_t(f)$ on $\mathcal{H}(G,V,\zeta)$ defined as at the end of §21. The sum itself satisfies the weak multiplier estimate (23.13), and hence converges absolutely. We shall describe the spectral expansion of $I_t(f)$.

We define weighted characters $J_M(\pi,f)$, for functions $f\in\mathcal{H}(G,V,\zeta)$, by a minor modification of the construction of §22. The element π lies in $\Pi_{\mathrm{unit}}(M_V,\zeta_V)$, and can therefore be regarded as a distribution in the space $\mathcal{F}(G_V^Z,\zeta_V)$. As with the mappings (29.1), $J_M(\pi,f)$ is defined in terms of the product over $v\in V$ of (G,M)-families in (28.13), and an integral analogous to (22.4), but over a domain $i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*$. We then form corresponding invariant distributions $I_M(\pi,f)$ from the mappings (29.1) as in (23.4) (or rather the special case of (23.4) with $X=0$).

The coefficients in the spectral expansion are parallel to those in the geometric expansion. The analogues of the classes k in (29.3) are families

$$c=\{c_v:\ v\notin V\}$$

of semisimple conjugacy classes in ${}^LM$. We allow only those classes of the form $c=c(\pi^V)$, where $\pi^V=\pi^V(c)$ is an unramified representation of $M^V=M(\mathbb{A}^V)$

whose Z^V-central character is equal to the corresponding component ζ^V of ζ. There is an obvious action

$$c \longrightarrow c_\lambda = \{c_{v,\lambda} : \ v \notin S\}, \qquad \lambda \in i\mathfrak{a}^*_{M,Z},$$

such that $\pi^V(c_\lambda) = \pi^V(c)_\lambda$. If $\pi^V(c)$ is unitary, we write

$$\pi \times c = \pi \otimes \pi^V(c)$$

for the representation in $\Pi_{\mathrm{unit}}\big(M(\mathbb{A}), \zeta\big)$ attached to any representation π in $\Pi_{\mathrm{unit}}(M_V, \zeta_V)$. Similar notation holds if π belongs to the quotient $\Pi_{\mathrm{unit}}(M_V^Z, \zeta_V)$ of $\Pi_{\mathrm{unit}}(M_V, \zeta_V)$, with the understanding that π is identified with a representative in $\Pi_{\mathrm{unit}}(M_V, \zeta_V)$. We define $\Pi_{t,\mathrm{disc}}(M, V, \zeta)$ to be the set of representations $\pi \in \Pi_{\mathrm{unit}}(M_V^Z, \zeta_V)$ such that for some c, $\pi \times c$ belongs to the subset $\Pi_{t,\mathrm{disc}}(M, \zeta)$ of $\Pi_{t,\mathrm{disc}}(M)$ attached to ζ. We also define $\mathcal{C}^V_{\mathrm{disc}}(M, \zeta)$ to be the set of c such that $\pi \times c$ belongs to $\Pi_{t,\mathrm{disc}}(M, \zeta)$, for some t and some $\pi \in \Pi_{t,\mathrm{disc}}(M, \zeta)$.

If c belongs to $\mathcal{C}^V_{\mathrm{disc}}(M, \zeta)$ and $\lambda \in \mathfrak{a}^*_{M,Z,\mathbb{C}}$, the unramified L-function

$$L(s, c_\lambda, \rho) = \prod_{v \notin V} \det\big(1 - \rho(c_{v,\lambda}) q_v^{-s}\big)^{-1}$$

converges absolutely for $\mathrm{Re}(s)$ large. In case ρ is the representation $\rho_{Q|P}$ of ${}^L M$, it is known that $L(s, c_\lambda, \rho)$ has analytic continuation as a meromorphic function of s, and that for any fixed s, $L(s, c_\lambda, \rho)$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,Z,\mathbb{C}}$. Following (28.12), we define the unramified normalizing factor

$$r_{Q|P}(c_\lambda) = L(0, c_\lambda, \rho_{Q|P}) L(1, c_\lambda, \rho^\vee_{Q|P})^{-1}, \qquad P, Q \in \mathcal{P}(M).$$

We then define a (G, M)-family

$$r_Q(\Lambda, c_\lambda) = r_{Q|\bar{Q}}(c_\lambda)^{-1} r_{Q|\bar{Q}}(c_{\lambda + \frac{1}{2}\Lambda}), \qquad Q \in \mathcal{P}(M),$$

and a corresponding meromorphic function

$$r^G_M(c_\lambda) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\Lambda, c_\lambda) \theta_Q(\Lambda)^{-1} \tag{29.5}$$

of λ. One shows that $r^G_M(c_\lambda)$ is an analytic function of $\lambda \in i\mathfrak{a}^*_{M,Z}$, whose integral against any rapidly decreasing function of λ converges [**A27**, Lemma 3.2]. If π is now a representation in $\Pi_{t,\mathrm{unit}}(G_V, \zeta_V)$, we define

$$a^G(\pi) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_c a^M_{\mathrm{disc}}(\pi_M \times c) r^G_M(c), \tag{29.6}$$

where $\pi \to \pi_M$ is the restriction operation that is adjoint to induction of characters. We define a subset $\Pi_t(G, V, \zeta)$ of $\Pi_{t,\mathrm{unit}}(G_V, \zeta_V)$, which contains the support of $a^G(\pi)$, and a measure $d\pi$ on $\Pi_t(G, V, \zeta)$ by following the appropriate analogues of (22.6) and (22.7). (See [**A27**, p. 205].)

PROPOSITION 29.2. *Suppose that $f \in \mathcal{H}(G, V, \zeta)$. Then*

$$I_t(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi) I_M(\pi, f) d\pi.$$

(See [**A27**, Proposition 3.3].) □

The identity obtained from (29.4) and Propositions 29.1 and 29.2 is the required reformulation of the invariant trace formula. Observe that the spectral factors $r_M^G(c)$ in the coefficients (29.6) are constructed from canonical unramified normalizing factors, while their counterparts $r_M^G(\pi_\lambda)$ in the earlier coefficients (22.8) were constructed from noncanonical global normalizing factors. This is a consequence of the modified definition of the mappings (29.1). The geometric factors $r_M^G(k)$ in the coefficients (29.3) have no counterparts in the earlier coefficients (19.6). They occur in the original geometric expansion (23.11) instead as implicit factors of the distributions $I_M(\gamma, f)$. This is because the set V is fixed, whereas S is large, in a sense that depends on the support of $f \in \mathcal{H}(G, V, \zeta)$.

The second preliminary matter pertains directly to the notion of stability. If T is a maximal torus in G over F, and v is archimedean, the subset $\mathcal{D}(T/F_v)$ of $\mathcal{E}(T/F_v)$ in (27.4) can be proper. On the other hand, the v-components of the summands $f_G^\kappa(\delta)$ in Langlands' stabilization (27.6) are parametrized by points κ_v in the dual group $\mathcal{K}(T/F_v)$ of $\mathcal{E}(T/F_v)$. If $\mathcal{D}(T/F_v)$ is proper in $\mathcal{E}(T/F_v)$, the mapping

$$f_v \longrightarrow f_{v,G}^{\kappa_v}(\delta_v), \qquad \kappa_v \in \mathcal{K}(T/F_v),\ f_v \in \mathcal{H}(G_v),$$

from functions $f_{v,G} \in \mathcal{I}\big(G(F_v)\big)$ to functions on $\mathcal{K}(T/F_v)$, is not surjective. This makes it difficult to characterize the image of the collective transfer mappings

$$\mathcal{I}(G, V, \zeta) \longrightarrow \bigoplus_{G'} S\mathcal{I}(\widetilde{G}', V, \widetilde{\zeta}').$$

It was pointed out by Vogan that the missing elements in $\mathcal{D}(T/F_v)$ could be attached to other groups. He observed that $\mathcal{E}(T/F_v)$ could be expressed as a disjoint union

$$\mathcal{E}(T/F_v) = \coprod \mathcal{D}_{\alpha_v}(T/F_v),$$

over sets $\mathcal{D}_{\alpha_v}(T/F_v)$ attached to a finite collection of groups G_{α_v} over F_v related by inner twisting. (See [**AV**] and [**ABV**] for extensions and applications of this idea.) Kottwitz then formulated the observations of Vogan in terms of the transfer factors. His formulation gives rise to a notion that was called a K-group in [**A25**]. Over the global field F, a K-group is an algebraic variety

$$G = \coprod_\alpha G_\alpha, \qquad \alpha \in \pi_0(G),$$

whose connected components are reductive algebraic groups G_α over F, and which is equipped with two kinds of supplementary structure. One consists of cohomological data, which include inner twists $\psi_{\alpha_\beta}\colon G_\beta \to G_\alpha$ between any two components. The other is a local product structure, which for any finite set $V \supset V_{\mathrm{ram}}(G)$ allows us to identify the set

$$G_V = \coprod_\alpha G_{\alpha,V} = \coprod_\alpha G_\alpha(F_V)$$

with a product

$$\prod_{v\in V} G_v = \prod_{v \in V} G_v(F_v)$$

of F_v-points in local K-groups G_v over F_v. The local K-group G_v is a finite disjoint union

$$G_v = \coprod_{\alpha_v} G_{\alpha_v}$$

of connected groups if v is archimedean, but it is just a connected group if v is nonarchimedean. In particular, the set $V_{\mathrm{ram}}(G) = V_{\mathrm{ram}}(G_\alpha)$ is independent of α. (See [**A27**, p. 209–211].)

We assume for the rest of this section that G is a K-group over F. Many concepts for connected groups carry over to this new setting without change in notation. For example, we define $\Gamma_{\mathrm{reg}}(G_V)$ to be the disjoint union over α of the corresponding sets $\Gamma_{\mathrm{reg}}(G_{\alpha,V})$ for the connected groups G_α. Similar conventions apply to the sets $\Pi(G_V)$, $\Pi_{\mathrm{unit}}(G_V)$, and $\Pi_{\mathrm{temp}}(G_V)$ of irreducible representations. We define compatible central character data $Z = \{Z_\alpha\}$ and $\zeta = \{\zeta_\alpha\}$ for G by choosing data Z_α and ζ_α for any one component G_α. This allows us to form the sets $\Pi(G_V, \zeta_V)$, $\Pi_{\mathrm{unit}}(G_V, \zeta_V)$, and $\Pi_{\mathrm{temp}}(G_V, \zeta_V)$ as disjoint unions of corresponding sets attached to components G_α. We can also define the vector spaces $\mathcal{H}(G_V, \zeta_V)$, $\mathcal{H}(G, V, \zeta)$, $\mathcal{I}(G_V, \zeta_V)$, $\mathcal{I}(G, V, \zeta)$, $\mathcal{D}(G_V^Z, \zeta_V)$, $\mathcal{F}(G_V^Z, \zeta_V)$, etc., by taking direct sums of the corresponding spaces attached to components G_α. Finally, we define sets $\Gamma(G_V^Z, \zeta_V)$, $\Gamma(G, V, \zeta)$, $\Pi_t(G_V^Z, \zeta_V)$, $\Pi_t(G, V, \zeta)$, and $\Pi_{t,\mathrm{disc}}(G, V, \zeta)$, again as disjoint unions of corresponding sets attached to components G_α.

There is also a notion of Levi subgroup (or more correctly, Levi K-subgroup) M of G. For any such M, the objects $\mathfrak{a}_M$, A_M, $W(M)$, $\mathcal{P}(M)$, $\mathcal{L}(M)$, and $\mathcal{F}(M)$ all have meaning, and play a role similar to that of the connected case. (See [**A25**, §1].) We again write $\mathcal{L}$ for the set $\mathcal{L}(M_0)$ attached to a fixed minimal Levi subgroup M_0 of G. With these conventions, the objects in the expansions of Proposition 29.1 and 29.2 now all have meaning for the K-group G over F. The invariant trace formula for G is an identity

$$
\begin{aligned}
&\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f) \\
&= \sum_t \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi) I_M(\pi, f) d\pi,
\end{aligned}
\tag{29.7}
$$

which holds for any $f \in \mathcal{H}(G, V, \zeta)$. It is obtained by applying (29.4) and Propositions 29.1 and 29.2 to the components $f_\alpha \in \mathcal{H}(G_{\alpha,V}, \zeta_{\alpha,V})$ of f, and then summing the resulting expansions over α.

Stable conjugacy in G_V has to be formulated slightly differently. We define two strongly regular elements $\gamma \in G_{\alpha,V}$ and $\delta \in G_{\beta,V}$ to be *stably conjugate* if $\psi_{\alpha\beta}(\delta)$ is stably conjugate in $G_{\alpha,V}$ to γ. We then define $S\mathcal{I}(G_V, \zeta_V)$ as a space of functions on the set $\Delta_{\mathrm{reg}}(G_V)$ of strongly regular stable conjugacy classes in G_V. This leads to the notion of a stable distribution on G_V, and allows us to define the subspaces $S\mathcal{D}(G_V, \zeta_V)$ and $S\mathcal{F}(G_V, \zeta_V)$ of stable distributions in $\mathcal{D}(G_V, \zeta_V)$ and $\mathcal{F}(G_V, \zeta_V)$ respectively. The conventions here are just minor variations of what we used for connected groups. We define a quasisplit inner twist of G to be a *connected*, quasisplit group G^* over F, together with a family of inner twists ψ_α: $G_\alpha \to G^*$ of connected groups such that $\psi_\beta = \psi_\alpha \circ \psi_{\alpha\beta}$. For any such G^*, there is a canonical injection $\delta \to \delta^*$ from $\Delta_{\mathrm{reg}}(G_V)$ to $\Delta_{\mathrm{reg}}(G_V^*)$. There is also a surjective mapping $S^* \to S$ from the space of stable distributions on G_V^* to the space of stable distributions on G_V. We say that G is *quasisplit* if one of the components G_α is quasisplit. In this case, the mapping $\delta \to \delta^*$ is a bijection, and the mapping $S^* \to S$ is an isomorphism.

Because the components G_α of G are related by inner twists, they can all be assigned a common dual group $\widehat{G}$, and a common L-group ${}^LG$. We recall that endoscopic data were defined entirely in terms of $\widehat{G}$. We can therefore regard them as objects G' attached to the K-group G. The same holds for auxiliary data $\widetilde{G}'$ and $\widetilde{\xi}'$ attached to G'. Similarly, local endoscopic data G'_v, with auxiliary data $\widetilde{G}'_v$ and $\widetilde{\xi}'_v$, are objects attached to the local K-group G_v.

The main new property is a natural extension of the Langlands-Shelstad construction of local transfer factors to K-groups. For any G_v, $\widetilde{G}'_v$ and $\widetilde{\xi}'_v$, it provides a function $\Delta_{G_v}(\delta'_v, \gamma_v)$ of $\delta'_v \in \Delta_{G\text{-reg}}(\widetilde{G}'_v)$ and $\gamma_v \in \Gamma_{\text{reg}}(G_v)$. (See [**A25**, §2].) This is the essence of the observations of Kottwitz and Vogan. It has two implications. One is that the transfer factors are now built around sets $\mathcal{D}(T/F_v)$, which are attached to the local K-group G_v, and are equal to the subgroups $\mathcal{E}(T/F_v)$ of $H^1(F_v, T)$. This places the theory of real and p-adic groups on an even footing. The other concerns a related point, which we did not raise earlier. The original Langlands-Shelstad transfer factor attached to G'_v (and $(\widetilde{G}'_v, \widetilde{\xi}'_v)$) depends on an arbitrary multiplicative constant. If G'_v is the localization of a global endoscopic datum, the product over v of these constants equals 1. However, if G'_v is taken in isolation, the constant reflects an intrinsic lack of uniqueness in the correspondence $f_v \to f'_v$. The extension of the transfer factors to G_v still depends on an arbitrary multiplicative constant. However, the constants for the components G_{α_v} of G_v can all be specified in terms of the one constant for G_v.

Thus, despite their ungainly appearance, K-groups streamline some aspects of the study of connected groups. This is the reason for introducing them. If we are given a connected reductive group G_1 over F, we can find a K-group G over F such that $G_{\alpha_1} = G_1$ for some $\alpha_1 \in \pi_1(G)$. Moreover, G is uniquely determined by G_1, up to a natural notion of isomorphism. In particular, for any connected quasisplit group G^*, there is a quasisplit K-group G such that $G_{\alpha^*} = G^*$, for some $\alpha^* \in \pi_0(G)$.

Let V be a fixed finite set of valuations that contains $S_{\text{ram}}(G, Z, \zeta)$. Suppose that for each $v \in V$, G'_v represents an endoscopic datum $(G'_v, \mathcal{G}'_v, s'_v, \xi'_v)$ for G over F_v, equipped with auxiliary data $\widetilde{G}'_v \to G'_v$ and $\widetilde{\xi}'_v \colon \mathcal{G}'_v \to {}^L\widetilde{G}'_v$, and a corresponding choice of local transfer factor $\Delta_v = \Delta_{G_v}$. We are assuming the Langlands-Shelstad transfer conjecture. Applied to each of the components G_{α_v} of G_v, it gives a mapping $f_v \to f'_v = f_v^{\widetilde{G}'}$ from $\mathcal{H}(G_v, \zeta_v)$ to $S\mathcal{I}(\widetilde{G}'_v, \widetilde{\zeta}'_v)$, which can be identified with a mapping $a_v \to a'_v$ from $\mathcal{I}(G_v, \zeta_v)$ to $S\mathcal{I}(\widetilde{G}'_v, \widetilde{\zeta}'_v)$. We write $\widetilde{G}'_V$, $\widetilde{\zeta}'_V$, and $\widetilde{\xi}'_V$ for the product over $v \in V$ of $\widetilde{G}'_v$, $\widetilde{\zeta}'_v$, and $\widetilde{\xi}'_v$ respectively. The product

$$\prod_v a_v \longrightarrow \prod_v a'_v, \qquad\qquad a_v \in \mathcal{I}(G_v, \zeta_v),$$

then gives a linear transformation $a \to a'$ from $\mathcal{I}(G_V, \zeta_V)$ to $S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$. This mapping is attached to the product G'_V of data G'_v, which we can think of as an endoscopic datum for G over F_V, equipped with auxiliary data $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$, and a corresponding product Δ_V of local transfer factors. We can think of the transfer factor Δ_V over F_V as the primary object, since it presupposes a choice of the other objects G'_V, $\widetilde{G}'_V$, $\widetilde{\zeta}'_V$ and $\widetilde{\xi}'_V$.

Letting G'_V vary, we obtain a mapping

$$\mathcal{I}(G_V, \zeta_V) \longrightarrow \prod_{G'_V} S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V) \tag{29.8}$$

by putting together all of the individual images a'. Notice that we have taken a direct product rather than a direct sum. This is because G'_V ranges over the infinite set of endoscopic data, equipped with auxiliary data $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$, rather than the finite set of isomorphism classes. However, the fact that G is a K-group makes it possible to characterize the image of $\mathcal{I}(G_V, \zeta_V)$ in this product. The image fits into a sequence of inclusions

$$\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V) \subset \bigoplus_{\{G'_V\}} \mathcal{I}^{\mathcal{E}}(G'_V, G_V, \zeta_V) \subset \prod_{\Delta_V} S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$$

in which the summand $\mathcal{I}^{\mathcal{E}}(G'_V, G_V, \zeta_V)$ depends only on the F_V-isomorphism class of G'_V. Roughly speaking, $\mathcal{I}^{\mathcal{E}}(G'_V, G_V, \zeta_V)$ is the subspace of products $\prod a'_v$ of functions attached to choices of transfer factors Δ_V for $\{G'_V\}$ that have the appropriate equivariance properties relative to variations in these choices. The space $\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V)$ is defined as the subspace of functions in the direct sum whose various components are compatible under restriction to common Levi subgroups. One shows that the transfer mapping gives an isomorphism

$$a \longrightarrow a^{\mathcal{E}} = \prod_{\Delta_V} a', \qquad a \in \mathcal{I}(G_V, \zeta_V),$$

from $\mathcal{I}(G_V, \zeta_V)$ onto $\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V)$. This in turn determines an isomorphism from the quotient

$$\mathcal{I}(G, V, \zeta) = \mathcal{I}(G_V^Z, \zeta_V)$$

of $\mathcal{I}(G_V, \zeta_V)$ onto the corresponding quotient

$$\mathcal{I}^{\mathcal{E}}(G, V, \zeta) = \mathcal{I}^{\mathcal{E}}(G_V^Z, \zeta_V)$$

of $\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V)$. (See [**A31**].) The image fits into a sequence of inclusions

$$\mathcal{I}^{\mathcal{E}}(G_V^Z, \zeta_V) \subset \bigoplus_{\{G'_V\}} \mathcal{I}^{\mathcal{E}}(G'_V, G_V^Z, \zeta_V) \subset \prod_{G'_V} S\mathcal{I}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big). \tag{29.9}$$

The mappings of functions we have described have dual analogues for distributions. Given G'_V (with auxiliary data $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$), assume that δ' belongs to the space of stable distributions $S\mathcal{D}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$. If f belongs to $\mathcal{H}(G, V, \zeta)$, the transfer f' of f can be evaluated at δ'. Since $f \to f'(\delta')$ belongs to $\mathcal{D}(G_V^Z, \zeta_V)$, we can write

$$f'(\delta') = \sum_{\gamma \in \Gamma(G_V^Z, \zeta_V)} \Delta_G(\delta', \gamma) f_G(\gamma), \tag{29.10}$$

for complex numbers $\Delta_G(\delta', \gamma)$ that depend linearly on δ'. Now (29.9) is dual to a sequence of surjective linear mappings

$$\prod_{G'_V} S\mathcal{D}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big) \ \mapsto\ \bigoplus_{\{G'_V\}} \mathcal{D}^{\mathcal{E}}(G'_V, G_V^Z, \zeta_V) \ \mapsto\ \mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$$

between spaces of distributions. Since f' is the image of the function $f_G \in \mathcal{I}(G, V, \zeta)$, $f'(\delta')$ depends only on the image δ of δ' in $\mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$. In other words, $f'(\delta')$ equals $f_G^{\mathcal{E}}(\delta)$, where $f_G^{\mathcal{E}}$ is the image of f_G in $\mathcal{I}^{\mathcal{E}}(G, V, \zeta)$. The same is therefore true of the coefficients $\Delta_G(\delta', \gamma)$. We can write

$$\Delta_G(\delta, \gamma) = \Delta_G(\delta', \gamma), \qquad \gamma \in \Gamma(G_V^Z, \zeta_V),$$

for complex numbers $\Delta_G(\delta,\gamma)$ that depend linearly on $\delta \in \mathcal{D}^{\mathcal{E}}(G_V^Z,\zeta_V)$. We note that the image in $\mathcal{D}^{\mathcal{E}}(G_V^Z,\zeta_V)$ of the subspace

$$S\mathcal{D}\big((G_V^*)^{Z^*},\zeta_V^*\big) \overset{\sim}{\mapsto} S\mathcal{D}(G_V^*,G_V^Z,\zeta_V)$$

can be identified with the space $S\mathcal{D}(G_V^Z,\zeta_V)$ of stable distributions in $\mathcal{D}(G_V^Z,\zeta_V)$.

The constructions above were given in terms of products G'_V of local endoscopic data for G. The stabilization of the trace formula is based primarily on global endoscopic data, particularly the subset $\mathcal{E}_{\mathrm{ell}}(G,V)$ of global isomorphism classes in $\mathcal{E}_{\mathrm{ell}}(G)$ that are unramified outside of V. If G' is any endoscopic datum for G over F, we can form the product G'_V of its completions. We can also attach auxiliary data $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$ for G'_V to global auxiliary data $\widetilde{G}'$ and $\widetilde{\xi}'$ for G'. The datum G'_V, together with $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$, indexes a component on the right hand side of (29.9). There are of course other components in (29.9) that do not come from global endoscopic data.

We are trying to formulate stable and endoscopic analogues of the terms in the invariant trace formula (29.7). We start with the local terms $I_M(\gamma,f)$ on the geometric side. Specializing the distributional transfer coefficients above to Levi subgroups $M\in\mathcal{L}$, we can define a linear form

$$I_M(\delta,f) = \sum_{\gamma\in\Gamma(M_V^Z,\zeta_V)} \Delta_M(\delta,\gamma) I_M(\gamma,f), \tag{29.11}$$

for any $\delta\in\mathcal{D}^{\mathcal{E}}(M_V^Z,\zeta_V)$. However, the true endoscopic analogue of $I_M(\gamma,f)$ is a more interesting object. It is defined inductively in terms of an important family $\mathcal{E}_{M'}(G)$ of global endoscopic data for G.

Suppose that M' represents a global endoscopic datum $(M',\mathcal{M}',s'_M,\xi'_M)$ for M, which is elliptic and unramified outside of V. We assume that $\mathcal{M}'$ is an L-subgroup of ${}^LM$ and that ξ'_M is the identity embedding. We define $\mathcal{E}_{M'}(G)$ to be the set of endoscopic data $(G',\mathcal{G}',s',\xi')$ for G, taken up to translation of s' by $Z(\widehat{G})^\Gamma$, in which s' lies in $s'_M Z(\widehat{M})^\Gamma$, $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\widehat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ and ${}^LG$. For each $G'\in\mathcal{E}_{M'}(G)$, we fix an embedding $M'\subset G'$ for which $\widehat{M}'\subset\widehat{G}'$ is a dual Levi subgroup. We also fix auxiliary data $\widetilde{G}'\to G'$ and $\widetilde{\xi}'\colon \mathcal{G}'\to {}^L\widetilde{G}'$ for G'. These objects restrict to auxiliary data $\widetilde{M}'\to M'$ and $\widetilde{\xi}'_M\colon \mathcal{M}'\to {}^L\widetilde{M}'$ for M', whose central character data $\widetilde{Z}'$ and $\widetilde{\zeta}'$ are the same as those for G'. Observe that G^* belongs to $\mathcal{E}_{M'}(G)$ if and only if M' equals M^*. We write

$$\mathcal{E}^0_{M'}(G) = \begin{cases} \mathcal{E}_{M'}(G) - \{G^*\}, & \text{if } G \text{ is quasisplit},\\ \mathcal{E}_{M'}(G), & \text{otherwise.}\end{cases}$$

For any $G'\in\mathcal{E}_{M'}(G)$, we also define a coefficient

$$\iota_{M'}(G,G') = |Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma||Z(\widehat{G}')^\Gamma/Z(\widehat{G})^\Gamma|^{-1}.$$

Suppose that δ' belongs to $S\mathcal{D}\big((\widetilde{M}'_V)^{\tilde{Z}'},\widetilde{\zeta}'_V\big)$. We assume inductively that for every $G'\in\mathcal{E}^0_{M'}(G)$, we have defined a stable linear form $S^{\tilde{G}'}_{\tilde{M}'}(\delta',\cdot)$ on $\mathcal{H}\big((\widetilde{G}'_V)^{\tilde{Z}'},\widetilde{\zeta}'_V\big)$. We impose natural conditions of equivariance on $S^{\tilde{G}'}_{\tilde{M}'}(\delta',\cdot)$, which imply that the linear form

$$f \longrightarrow \widehat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta',f'), \qquad f\in\mathcal{H}(G,V,\zeta),$$

on $\mathcal{H}(G,V,\zeta)$ depends only on the image of δ' in the space $\mathcal{D}^{\mathcal{E}}(M'_V, M^Z_V, \zeta_V)$. In particular, the last linear form is independent of the choice of auxiliary data $\widetilde{G}'$ and $\widetilde{\xi}'$. If G is not quasisplit, we define an "endoscopic" linear form

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \widehat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f'). \tag{29.12}$$

In the case that G is quasisplit, we define a linear form

$$S^G_M(M', \delta', f) = I_M(\delta, f) - \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \widehat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f'), \tag{29.13}$$

where δ is the image of δ' in $\mathcal{D}^{\mathcal{E}}(M^Z_V, \zeta_V)$. In this case, we also define the endoscopic linear form by the trivial relation

$$I^{\mathcal{E}}_M(\delta', f) = I_M(\delta, f). \tag{29.14}$$

These definitions represent the first stage of an extensive generalization of the constructions of §25. To see this more clearly, we need to replace the argument δ' in $I^{\mathcal{E}}_M(\delta', f)$ by an element $\gamma \in \mathcal{D}(M^Z_V, \zeta_V)$. It turns out that there is a canonical bilinear form $I^{\mathcal{E}}_M(\gamma, f)$ in γ and f such that

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{\gamma \in \Gamma(M^Z_V, \zeta)} \Delta_M(\delta', \gamma) I^{\mathcal{E}}_M(\gamma, f), \tag{29.15}$$

for any (M', δ'). Since M' was chosen to be an endoscopic datum over F, $I^{\mathcal{E}}_M(\gamma, f)$ is not uniquely determined by (29.15). However, the definitions (29.13) and (29.14) apply more generally if M' is replaced by an endoscopic datum M'_V over F_V. (See [**A25**, §5].) One shows directly that the resulting linear form

$$I^{\mathcal{E}}_M(\delta, f) = I^{\mathcal{E}}_M(\delta', f)$$

depends only on the image δ of δ' in $\mathcal{D}^{\mathcal{E}}(M^Z_V, \zeta_V)$. The distribution $I^{\mathcal{E}}_M(\gamma, f)$ is then defined by inversion from the corresponding extension of (29.15). (See [**A31**].)

To complete the inductive definition, one still has to prove something in the special case that G is quasisplit and $M' = M^*$. Then $\delta' = \delta^*$ belongs to $S\mathcal{D}\big((M^*_V)^{Z^*}, \zeta^*_V\big)$, and the image δ of δ' in $\mathcal{D}^{\mathcal{E}}(M^Z_V, \zeta_V)$ lies in the subspace $S\mathcal{D}(M^Z_V, \zeta_V)$ of stable distributions. The problem in this case is to show that the linear form

$$S^G_M(\delta, f) = S^G_M(M^*, \delta^*, f) \tag{29.16}$$

is stable. Only then would we have a linear form

$$\widehat{S}^{G^*}_{M^*}(\delta^*, f^*) = S^G_M(\delta, f)$$

on $S\mathcal{I}\big((G^*_V)^{Z^*}, \zeta^*_V\big)$ that is the analogue for (G^*, M^*) of the terms $\widehat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f')$ in (29.12) and (29.13). This property is deep, and is a critical part of the stabilization of the general trace formula. In the case that G is quasi-split but $M' \neq M^*$, there is a second question which is as deep as the first. The problem in this case is to show that $S^G_M(M', \delta', f)$ vanishes for any δ' and f.

The analogue for unramified valuations $v \notin V_{\mathrm{ram}}(G)$ of this second problem is of special interest. It represents the generalization of the fundamental lemma to weighted oribital integrals. To state it, we write

$$r^{G_v}_{M_v}(k_v) = J_{M_v}(k_v, u_v), \qquad k_v \in \Gamma_{G\text{-reg}}(M_v),$$

where u_k is the characteristic function of K_v in $G_v(F_v)$, and M_v is a Levi subgroup of G_v. Since v is nonarchimedean, the associated component G_v of G is a connected reductive group over F_v. In this context, we may as well take $Z_v = 1$, since for any endoscopic datum G'_v over F_v, there is a canonical class of L-embeddings of ${}^L G'_v$ in ${}^L G_v$ [**Hal1**, §6]. If M'_v is an unramified elliptic endoscopic datum for M_v, and $\ell'_v \in \Delta_{G\text{-reg}}(M'_v)$, we write

$$r^{G_v}_{M_v}(\ell'_v) = \sum_{k_v} \Delta_{M_v}(\ell'_v, k_v) r^{G_v}_{M_v}(k_v).$$

We can also obviously write $\mathcal{E}_{M'_v}(G_v)$ and $\iota_{M_v}(G_v, G'_v)$ for the local analogues of the global objects defined earlier.

Conjecture. (Generalized fundamental lemma). *For any M'_v and ℓ'_v, there is an identity*

$$r^{G_v}_{M_v}(\ell'_v) = \sum_{G'_v \in \mathcal{E}_{M'_v}(G_v)} \iota_{M'_v}(G_v, G'_v) s^{G'_v}_{M'_v}(\ell'_v), \tag{29.17}$$

for functions $s^{G'_v}_{M'_v}(\ell'_v)$ that depend only on G'_v, M'_v and ℓ'_v.

If $M'_v = M^*_v$ and $\ell'_v = \ell^*_v$, G^*_v belongs to $\mathcal{E}_{M'_v}(G_v)$, and (29.17) represents an inductive definition of $s^{G_v}_{M_v}(\ell^*_v)$. If $M'_v \neq M^*_v$, G^*_v does not belong to $\mathcal{E}_{M'_v}(G_v)$, and (29.17) becomes an identity to be proved. The reader can check that when $M_v = G_v$, the identity reduces to the standard fundamental lemma, which we described near the end of §27. We assume from now on that this conjecture holds for G, at least at almost all valuations $v \notin S_{\text{ram}}(G)$, as well as for any other groups that might be required for induction arguments. Since this includes the usual fundamental lemma, it also encompasses our assumption that the Langlands-Shelstad transfer conjecture is valid [**Wa2**].

We can now state the first of four theorems, which together comprise the stabilization of the invariant trace formula. They are all dependent on our assumption that the generalized fundamental lemma holds.

THEOREM 29.3. (a) *If G is arbitrary,*

$$I^{\mathcal{E}}_M(\gamma, f) = I_M(\gamma, f), \qquad \gamma \in \mathcal{D}(M^Z_V, \zeta_V),\ f \in \mathcal{H}(G, V, \zeta).$$

(b) *Suppose that G is quasisplit, and that δ' belongs to $S\mathcal{D}\big((\widetilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V\big)$, for some $M' \in \mathcal{E}_{\text{ell}}(M, V)$. Then the linear form*

$$f \longrightarrow S^G_M(M', \delta', f), \qquad f \in \mathcal{H}(G, V, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

The linear forms $I^{\mathcal{E}}_M(\gamma, f)$ and $S^G_M(\delta, f)$ ultimately become terms in endoscopic and stable analogues of the geometric side of (29.7). These objects are to be regarded as the local components of the expansions. The global components are endoscopic and stable analogues of the coefficients $a^G(\gamma)$ in (29.7). As before, the new coefficients really belong to a completion of the appropriate space of distributions. However, we again identify them with elements in a dual space by choosing bases of the relevant spaces of distributions. We fix a basis $\Delta\big((\widetilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V\big)$ of $S\mathcal{D}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V\big)$ for any F_V-endoscopic datum G'_V, with auxiliary data $\widetilde{G}'_V$ and

$\widetilde{\xi}'_V$. We also fix a basis $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$ of the space $\mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$. Among various conditions, we require that the subset

$$\Delta(G_V^Z, \zeta_V) = \Delta^{\mathcal{E}}(G_V^Z, \zeta_V) \cap S\mathcal{D}(G_V^Z, \zeta_V)$$

of $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$ be a basis of $S\mathcal{D}(G_V^Z, \zeta_V)$, and in the case that G is quasisplit, that $\Delta(G_V^Z, \zeta_V)$ be the isomorphic image of the basis $\Delta\big((G_V^*)^{Z^*}, \zeta_V^*\big)$.

We assume inductively that for every G' in the set

$$\mathcal{E}^0_{\mathrm{ell}}(G,V) = \begin{cases} \mathcal{E}_{\mathrm{ell}}(G,V) - \{G^*\}, & \text{if } G \text{ is quasisplit,} \\ \mathcal{E}_{\mathrm{ell}}(G,V), & \text{otherwise,} \end{cases}$$

we have defined a function $b^{\tilde{G}'}(\delta')$ on $\Delta\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$. If G is not quasisplit, we can then define the "endoscopic" coefficient

$$a^{G,\mathcal{E}}(\gamma) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G,V)} \sum_{\delta'} \iota(G,G') b^{\tilde{G}'}(\delta') \Delta_G(\delta', \gamma), \tag{29.18}$$

as a function of $\gamma \in \Gamma(G_V^Z, \zeta_V)$. In the case that G is quasisplit, we define a "stable" coefficient function $b^G(\delta)$ of $\delta \in \Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$ by requiring that

$$\sum_{\delta} b^G(\delta)\Delta_G(\delta,\gamma) = a^G(\gamma) - \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \sum_{\delta'} \iota(G,G') b^{\tilde{G}'}(\delta') \Delta_G(\delta', \gamma), \tag{29.19}$$

for any $\gamma \in \Gamma(G_V^Z, \zeta_V)$. In this case, we also define the endoscopic coefficient by the trivial relation

$$a^{G,\mathcal{E}}(\gamma) = a^G(\gamma).$$

In both (29.18) and (29.19), the numbers $\iota(G,G')$ are Langlands' original global coefficients from (27.3), while δ' and δ are summed over $\Delta\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$ and $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$ respectively. To complete the inductive definition, we set

$$b^{G^*}(\delta^*) = b^G(\delta), \qquad \delta^* \in \Delta\big((G_V^*)^{Z^*}, \zeta_V^*\big),$$

when G is quasisplit and δ is the preimage of δ^* in the subset $\Delta(G_V^Z, \zeta_V)$ of $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$.

THEOREM 29.4. (a) *If G is arbitrary,*

$$a^{G,\mathcal{E}}(\gamma) = a^G(\gamma), \qquad \gamma \in \Gamma(G_V^Z, \zeta_V).$$

(b) *If G is quasisplit, $b^G(\delta)$ vanishes for any δ in the complement of $\Delta(G_V^Z, \zeta_V)$ in $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$.*

We have completed our description of the geometric ingredients that go into the stabilization of the trace formula. The spectral ingredients are entirely parallel. In place of the spaces of distributions $\mathcal{D}(G_V^Z, \zeta_V)$, $S\mathcal{D}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, $\mathcal{D}^{\mathcal{E}}(G'_V, G_V^Z, \zeta_V)$, and $\mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$, we have spectral analogues $\mathcal{F}(G_V^Z, \zeta_V)$, $S\mathcal{F}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, $\mathcal{F}^{\mathcal{E}}(G'_V, G_V^Z, \zeta_V)$, and $\mathcal{F}^{\mathcal{E}}(G_V^Z, \zeta_V)$. The subspace $S\mathcal{D}(G_V^Z, \zeta_V)$ of $\mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$ is replaced by a corresponding subspace $S\mathcal{F}(G_V^Z, \zeta_V)$ of $\mathcal{F}^{\mathcal{E}}(G_V^Z, \zeta_V)$. In place of the prescribed basis $\Gamma(G_V^Z, \zeta_V)$ of $\mathcal{D}(G_V^Z, \zeta_V)$, we have the basis

$$\Pi(G_V^Z, \zeta_V) = \coprod_{t \geq 0} \Pi_t(G_V^Z, \zeta_V)$$

of $\mathcal{F}(G_V^Z, \zeta_V)$ consisting of irreducible characters. If ϕ' belongs to $S\mathcal{F}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, the distribution $f \to f'(\phi')$ belongs to $\mathcal{F}(G_V^Z, \zeta_V)$. It therefore has an expansion

$$f'(\phi') = \sum_{\pi \in \Pi(G_V^Z, \zeta_V)} \Delta(\phi', \pi) f_G(\pi)$$

that is parallel to (29.15). The coefficients

$$\Delta(\phi, \pi) = \Delta(\phi', \pi), \qquad \pi \in \Pi(G_V^Z, \zeta_V),$$

are products over v of local coefficients in (28.4) (or rather, linear extensions in ϕ'_v of such coefficients), and depend only on the image ϕ of ϕ' in $\mathcal{F}^{\mathcal{E}}(G_V^Z, \zeta_V)$.

The definitions (29.13)–(29.16) have obvious spectral variants. They provide linear forms $I_M(\phi, f)$, $I_M^{\mathcal{E}}(\phi', f)$, $S_M^G(M', \phi', f)$, $I_M^{\mathcal{E}}(\pi, f)$, and $S_M(\phi, f)$ in $f \in \mathcal{H}(G, V, \zeta)$, which also depend linearly on the distributions ϕ, ϕ' and π.

THEOREM 29.5. (a) *If G is arbitrary,*

$$I_M^{\mathcal{E}}(\pi, f) = I_M(\pi, f), \qquad \pi \in \mathcal{F}(M_V^Z, \zeta_V),\ f \in \mathcal{H}(G, V, \zeta).$$

(b) *Suppose that G is quasisplit, and that ϕ' belongs to $S\mathcal{F}\big((\widetilde{M}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, for some $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$. Then the linear form*

$$f \;\longrightarrow\; S_M^G(M', \phi', f), \qquad f \in \mathcal{H}(G, V, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

The linear forms $I_M^{\mathcal{E}}(\pi, f)$ and $S_M^G(\phi, f)$ ultimately become local terms in endoscopic and stable analogues of the spectral side of (29.7). The global terms are endoscopic and stable analogues of the coefficients $a^G(\pi)$ in (29.7). We fix a basis $\Phi\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$ of the space $S\mathcal{F}\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, for each G'_V, $\widetilde{G}'_V$ and $\widetilde{\xi}'_V$, which we can form from local bases $\Phi(\widetilde{G}'_v, \widetilde{\zeta}'_v)$. If v is nonarchimedean, we take $\Phi(\widetilde{G}'_v, \widetilde{\zeta}'_v)$ to be the abstract basis discussed in §28. If v is archimedean, we can identify $\Phi(\widetilde{G}'_v, \widetilde{\zeta}'_v)$ with the relevant set of archimedean Langlands parameters ϕ_v, thanks to the work of Shelstad. Since any such ϕ_v has an archimedean infinitesimal character, there is a decomposition

$$\Phi\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big) = \coprod_{t \geq 0} \Phi_t\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big).$$

We also fix a basis $\Phi^{\mathcal{E}}(G_V^Z, \zeta_V)$ of the space $\mathcal{D}^{\mathcal{E}}(G_V^Z, \zeta_V)$, which can in fact be taken to be a set of equivalence classes in the union of the various bases $\Phi\big((\widetilde{G}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$. Among other things, this implies that the subset

$$\Phi(G_V^Z, \zeta_V) = \Phi^{\mathcal{E}}(G_V^Z, \zeta_V) \cap S\mathcal{F}(G_V^Z, \zeta_V)$$

of $\Phi^{\mathcal{E}}(G_V^Z, \zeta_V)$ is a basis of $S\mathcal{F}(G_V^Z, \zeta_V)$, and in the case that G is quasisplit, is the isomorphic image of the basis $\Phi\big((G_V^*)^{Z^*}, \zeta_V^*\big)$.

Having fixed bases, we can apply the obvious spectral variants of the definitions (29.18) and (29.19). We thereby obtain functions $a^{G,\mathcal{E}}(\pi)$ and $b^G(\phi)$ of $\pi \in \Pi(G_V^Z, \zeta_V)$ and $\phi \in \Phi^{\mathcal{E}}(G_V^Z, \zeta_V)$ respectively.

THEOREM 29.6. (a) *If G is arbitrary,*

$$a^{G,\mathcal{E}}(\pi) = a^G(\pi), \qquad \pi \in \Pi(G_V^Z, \zeta_V).$$

(b) *If G is quasisplit, $b^G(\phi)$ vanishes for any ϕ in the complement of $\Phi(G^Z_V, \zeta_V)$ in $\Phi^{\mathcal{E}}(G^Z_V, \zeta_V)$.*

Theorems 29.3 and 29.4 are general analogues of Theorem 25.5 for inner twistings of $GL(n)$. The extra assertions (b) of theorems were not required earlier, since the question of stability is trivial for $GL(n)$. Similarly, Theorems 29.5 and 29.6 are general analogues of Theorem 25.6. Taken together, Theorems 29.3–29.6 amount to a stabilization of the general trace formula. This will become clearer after we have stated the general analogues of Lemmas 25.3 and 25.4.

The four theorems are proved together. As in the special case in §25, the argument is by double induction on $\dim(G/Z)$ and $\dim(A_M)$. The first stage of the proof is to obtain endoscopic and stable analogues of the expansions on each side of (29.7). For this, one needs only the induction assumption that the global assertions (b) of Theorems 29.4 and 29.6 be valid if (G, ζ) is replaced by $(\widetilde{G}', \widetilde{\zeta}')$, for any $G' \in \mathcal{E}^0_{\text{ell}}(G, V)$.

Let I be the invariant linear form on $\mathcal{H}(G, V, \zeta)$ defined by either of the two sides of (29.7). If G is not quasisplit, we define an "endoscopic" linear form inductively by setting

$$I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G,V)} \iota(G, G')\widehat{S}'(f'), \tag{29.20}$$

for stable linear forms $\widehat{S}' = \widehat{S}^{\tilde{G}'}$ on $S\mathcal{I}(\widetilde{G}', V, \widetilde{\zeta}')$. In the case that G is quasisplit, we define a linear form

$$S^G(f) = I(f) - \sum_{G' \in \mathcal{E}^0_{\text{ell}}(G,V)} \iota(G, G')\widehat{S}'(f'). \tag{29.21}$$

We also define the endoscopic linear form by the trivial relation

$$I^{\mathcal{E}}(f) = I(f). \tag{29.22}$$

In the case G is quasisplit, we need to show that the linear form S^G on $\mathcal{I}(G, V, \zeta)$ is stable. Only then will we have a linear form

$$\widehat{S}^{G^*}(f^*) = S^G(f)$$

on $S\mathcal{I}(G^*, V, \zeta^*)$ that is the analogue for G^* of the summands in (29.20) and (29.21) needed to complete the inductive definition. We would also like to show that $I^{\mathcal{E}}(f) = I(f)$. These properties are obviously related to the assertions of the four theorems.

The reader will recognize in the definitions (29.20)–(29.22), taken with the assertions that $S^G(f)$ is stable and $I^{\mathcal{E}}(f) = I(f)$, an analogue of Langlands' stabilization (27.3) of the regular elliptic terms. This construction is in fact a model for the stabilization of any part of the trace formula. For example, let

$$I_{\text{orb}}(f) = \sum_{\gamma \in \Gamma(G,V,\zeta)} a^G(\gamma) f_G(\gamma) \tag{29.23}$$

be the component with $M = G$ in the geometric expansion in (29.7). This sum includes the regular elliptic terms, as well as orbital integrals over more general conjugacy classes. Its complement $I(f) - I_{\text{orb}}(f)$ in $I(f)$, being a sum over M in the complement $\mathcal{L}^0$ of $\{G\}$ in $\mathcal{L}$, can be regarded as the "parabolic" part of the geometric expansion. We define linear forms $I^{\mathcal{E}}_{\text{orb}}(f)$ and $S^G_{\text{orb}}(f)$ on $\mathcal{H}(G, V, \zeta)$ by the obvious analogues of (29.20)–(29.22).

PROPOSITION 29.7. (a) *If G is arbitrary,*

$$I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{orb}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma) I^{\mathcal{E}}_M(\gamma, f),$$

where $\Gamma^{\mathcal{E}}(M,V,\zeta)$ is a natural discrete subset of $\Gamma(M^Z_V, \zeta_V)$ that contains the support of $a^{M,\mathcal{E}}(\gamma)$.

(b) *If G is quasisplit,*

$$S^G(f) - S^G_{\mathrm{orb}}(f)$$
$$= \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \sum_{\delta' \in \Delta(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\delta') S^G_M(M', \delta', f),$$

where $\Delta(\widetilde{M}', V, \widetilde{\zeta}')$ is a natural discrete subset of $\Delta\big((\widetilde{M}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$ that contains the support of $b^{\tilde{M}'}(\delta')$.

See [**A27**, Theorem 10.1]. □

Let $I_t(f)$ be the summand of t on the spectral side of (29.7). We attach linear forms $I^{\mathcal{E}}_t(f)$ and $S^G_t(f)$ to $I_t(f)$ by the analogues of (29.20)–(29.22). The decomposition in (29.7) of $I(f)$ as a sum over $t \geq 0$ of $I_t(f)$ leads to corresponding decompositions

(29.24(a)) $$I^{\mathcal{E}}(f) = \sum_{t \geq 0} I^{\mathcal{E}}_t(f)$$

and

(29.24(b)) $$S^G(f) = \sum_{t \geq 0} S^G_t(f)$$

of $I^{\mathcal{E}}(f)$ and $S^G(f)$. Each of these sums satisfies the analogue of the weak multiplier estimate (23.13), and hence converges absolutely. (See [**A27**, Proposition 10.5].) For any t, we write

(29.25) $$I_{t,\mathrm{unit}}(f) = \int_{\Pi_t(G,V,\zeta)} a^G(\pi) f_G(\pi) \mathrm{d}\pi$$

for the component with $M = G$ for the spectral expansion of $I_t(f)$ in (29.7). We then define corresponding linear forms $I^{\mathcal{E}}_{t,\mathrm{unit}}(f)$ and $S^G_{t,\mathrm{unit}}(f)$ on $\mathcal{H}(G,V,\zeta)$, again by the obvious analogues of (29.20)–(29.22).

PROPOSITION 29.8. (a) *If G is arbitrary,*

$$I^{\mathcal{E}}_t(f) - I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathcal{E}}_t(M,V,\zeta)} a^{M,\mathcal{E}}(\pi) I^{\mathcal{E}}_M(\pi, f) \mathrm{d}\pi,$$

where $\Pi^{\mathcal{E}}_t(M,V,\zeta)$ is a subset of $\Pi_t(M^Z_V, \zeta_V)$, equipped with a natural measure $\mathrm{d}\pi$, that contains the support of $a^{M,\mathcal{E}}(\pi)$.

(b) *If G is quasisplit,*

$$S^G_t(f) - S^G_{t,\mathrm{unit}}(f)$$
$$= \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \int_{\Phi_{t'}(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\phi') S^G_M(M', \phi', f) \mathrm{d}\phi',$$

where t' is a translate of t, and $\Phi_{t'}(\widetilde{M}', V, \widetilde{\zeta}')$ is a subset of $\Phi_{t'}\big((\widetilde{M}'_V)^{\tilde{Z}'}, \widetilde{\zeta}'_V\big)$, equipped with a natural measure $\mathrm{d}\phi'$, which contains the support of $b^{\tilde{M}'}(\phi')$.

See [**A27**, Theorem 10.6]. □

In contrast to the special cases of Lemmas 25.3 and 25.4, we have excluded the terms with $M = G$ from the expansions of Propositions 29.7 and 29.8. This was only to keep the notation slightly simpler in the assertions (b). It is a consequence of the definitions that

$$I^{\mathcal{E}}_{\mathrm{orb}}(f) = \sum_{\gamma \in \Gamma^{\mathcal{E}}(G,V,\zeta)} a^{G,\mathcal{E}}(\gamma) f_G(\gamma) \tag{29.26(a)}$$

and

$$S^G_{\mathrm{orb}}(f) = \sum_{\delta \in \Delta^{\mathcal{E}}(G,V,\zeta)} b^G(\delta) f^{\mathcal{E}}_G(\delta), \tag{29.26(b)}$$

where $\Delta^{\mathcal{E}}(G, V, \zeta)$ is a certain discrete subset of $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$ that contains the support of b^G. Similarly, we have

$$I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = \int_{\Pi^{\mathcal{E}}_t(G,V,\zeta)} a^{G,\mathcal{E}}(\pi) f_G(\pi) \mathrm{d}\pi \tag{29.27(a)}$$

and

$$S^G_{t,\mathrm{unit}}(f) = \int_{\Phi^{\mathcal{E}}_t(G,V,\zeta)} b^G(\phi) f^{\mathcal{E}}_G(\phi) \mathrm{d}\phi, \tag{29.27(b)}$$

where $\Phi^{\mathcal{E}}_t(G, V, \zeta)$ is a subset of $\Phi^{\mathcal{E}}_t(G^Z_V, \zeta_V)$, equipped with a natural measure $\mathrm{d}\phi$, that contains the support of $b^G(\phi)$. (See [**A27**, Lemmas 7.2 and 7.3].) We can obviously combine (29.26(a)) and (29.27(a)) with the expansions (a) of Propositions 29.7 and 29.8. This provides expressions for $I^{\mathcal{E}}(f)$ and $I^{\mathcal{E}}_t(f)$ that are more clearly generalizations of those of Lemmas 25.3 and 25.4. On the other hand, the sums in (29.26(b)) and (29.27(b)) are not of the same form as those in the expansions (b) of Propositions 29.7 and 29.8. Their substitution into these expansions leads to expressions for $S^G(f)$ and $S^G_t(f)$ that, without the general assertions (b) of the four theorems, are more ungainly.

We shall say only a few words about the proof of the four theorems. If G is not quasisplit, one works with the identity obtained from (29.24(a)), (29.26(a)), (29.27(a)), and Propositions 29.7(a) and 29.8(a). The problem is to compare the terms in this identity with those of the invariant trace formula (29.7). If G is quasisplit, one works with the identity obtained from (29.24(b)), (29.26(b)), (29.27(b)), and Propositions 29.7(b) and 29.8(b). The problem here is to show that if $f^G = 0$, the appropriate terms in the identity vanish. The arguments are long and complicated, but they do follow the basic model established in §25. In particular, they frequently move forward under their own momentum.

There is one point we should mention explicitly. The geometric coefficients $a^G(\gamma)$ are compound objects, defined (29.3) in terms of the original coefficients $a^M_{\mathrm{ell}}(\gamma_M \times k)$. The identities stated in Theorem 29.4 have analogues that apply to endoscopic and stable forms of the coefficients $a^G_{\mathrm{ell}}(\gamma \times k)$. The role of the generalized fundamental lemma is to reduce Theorem 29.4 to these basic identities [**A27**, Proposition 10.3]. (The case $M = G$ of the generalized fundamental lemma, namely the ordinary fundamental lemma, carries the more obvious burden of establishing the existence of the mappings $f \to f'$.) One has then to reduce these basic identities further to the special case of classes in $G(F_S)$ that are purely unipotent. This turns out to be a major undertaking [**A26**], which depends heavily on Langlands-Shelstad descent for transfer factors [**LS2**]. The reduction to unipotent classes can be regarded as an extension of the stabilization of the semisimple elliptic terms by Langlands [**Lan10**] and Kottwitz [**Ko5**].

The spectral coefficients $a^G(\pi)$ are also compound objects. They are defined (29.6) in terms of the original spectral coefficients $a^M_{\mathrm{disc}}(\pi_M \times c)$. The identities stated in Theorem 29.6 have analogues for endoscopic and stable forms of the coefficients $a^G_{\mathrm{disc}}(\pi \times c)$. It is interesting to note that the generalized fundamental lemma has a spectral variant [**A27**, Proposition 8.3], albeit one which is much less deep, and which has a straightforward proof. (For example, the case $M = G$ of this spectral result is entirely vacuous. The cases with $M \neq G$ reflect relatively superficial aspects of the deeper geometric conjecture.) The role of the spectral result is to reduce Theorem 29.6 to the identities for endoscopic and stable forms of the coefficients $a^G_{\mathrm{disc}}(\gamma \times k)$ [**A27**, Proposition 10.7].

We have touched on a couple of aspects of the first half of the argument. The second half of the proof is contained in [**A29**]. It is based on a comparison of the expansions in Propositions 29.7 and 29.8 with those in (29.7). Among the many reductions on the geometric sides, one establishes the required cancellation of almost all of the terms in $I_{\mathrm{orb}}(f)$, $I^{\mathcal{E}}_{\mathrm{orb}}(f)$, and $S^G_{\mathrm{orb}}(f)$ by appealing to the reductions of Theorem 29.4 described above. Those that remain correspond to unipotent elements. They can be separated from the complementary terms in the expansions by an approximation argument. Among the spectral reductions, one sees that many of the terms in $I_{t,\mathrm{unit}}(f)$, $I^{\mathcal{E}}_{t,\mathrm{unit}}(f)$, and $S^G_{t,\mathrm{unit}}(f)$ also cancel, thanks to the reduction of Theorem 29.6 we have mentioned. Those that remain occur discretely. They can be separated from the complementary terms in the expansions by the appropriate forms of the weak multiplier estimate (23.13).

These sparse comments convey very little sense of the scope of the argument. It will suffice for us to reiterate that much of the collective proof of the four theorems is in attempting to generalize arguments described in the special case of §25. □

Corollary 29.9. (a) (Endoscopic trace formula). *The identity*

(29.28(a))
$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma) I^{\mathcal{E}}_M(\gamma,f) = \sum_{t\geq 0}\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathcal{E}}_t(M,V,\zeta)} a^{M,\mathcal{E}}(\pi) I^{\mathcal{E}}_M(\pi,f)$$

holds for any $f \in \mathcal{H}(G,V,\zeta)$. Each term in the identity is equal to its corresponding analogue in the invariant trace formula (29.7).

(b) (Stable trace formula). *If G is quasisplit, the identity*

(29.28(b))
$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta) S_M(\delta,f) = \sum_{t\geq 0}\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi_t(M,V,\zeta)} b^M(\phi) S_M(\phi,f)\,\mathrm{d}\phi$$

holds for any $f \in \mathcal{H}(G,V,\zeta)$. The terms in the identity are all stable in f.

The identity (29.28(a)) follows immediately from Propositions 29.7(a) and 29.8(a) and expansions (29.26(a)) and (29.27(a)), as we have already noted. Assertions (a) of the four theorems give the term by term identification of this identity with the invariant trace formula.

To establish (29.28(b)), we combine the expansions of Propositions 29.7(b) and 29.8(b) with (29.26(b)) and (29.27(b)). This yields a rather complicated formula. However, assertions (b) of the four theorems imply immediately that the formula collapses to the required identity (29.28(b)). Supplementary assertions in Theorems 29.3(b) and 29.5(b) tell us that the linear forms $S_M(\delta, f)$ and $S_M(\phi, f)$ in (29.28(b)) are stable in f. □

The endoscopic trace formula (29.28(a)) is a priori quite different from the original formula (29.7). In case G is not quasisplit, it is defined as a linear combination of stable trace formulas for endoscopic groups G'. Our conclusion that it is in fact equal to the original formula amounts to a stabilization of the trace formula.

We recall that $G = \coprod G_\alpha$ is a K-group over F. However, if f is supported on a component $G_\alpha(F_V)$, the sums in (29.28(a)) can be taken over geometric and stable objects attached to G_α. Moreover, if G is quasisplit, the stable distributions on G_V are in bijective correspondence with those on G^*_V. It follows that the assertions of Corollary 29.9 hold as stated if G is an ordinary connected group over F.

There is one final corollary. To state it, we return to the setting of earlier sections. We take G to be a connected reductive group over F, and f to be a function in the adelic Hecke algebra $\mathcal{H}(G, \zeta) = \mathcal{H}\big(G(\mathbb{A})^Z, \zeta\big)$. The t-discrete part $I_{t,\mathrm{disc}}(f)$ of the trace formula (21.19) represents its spectral core. It is the part that is actually used for applications.

COROLLARY 29.10. *There are stable linear forms*

$$S^G_{t,\mathrm{disc}}(f), \qquad\qquad f \in \mathcal{H}(G, V),\ t \geq 0,$$

defined whenever G is quasisplit, such that

$$I_{t,\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}^{\tilde{G}'}_{t,\mathrm{disc}}(f'), \tag{29.29}$$

for any G, t and f.

We define linear forms $I^{\mathcal{E}}_{t,\mathrm{disc}}$ and $S^G_{t,\mathrm{disc}}$ inductively by analogues of (29.20)–(29.22). Recall that there is an expansion

$$I_{t,\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{t,\mathrm{disc}}(G)} a^G_{\mathrm{disc}}(\pi) f_G(\pi),$$

which serves as the definition of the coefficients $a^G_{\mathrm{disc}}(\pi)$, and is parallel to the definition (29.25) of $I_{t,\mathrm{unit}}(f)$. This leads to corresponding expansions of $I^{\mathcal{E}}_{t,\mathrm{disc}}(f)$ and $S^G_{t,\mathrm{disc}}(f)$, which are parallel to (29.27(a)) and (29.27(b)). We have already noted that the assertions of Theorem 29.6 reduce to corresponding assertions for the coefficients of these latter expansions. Theorem 29.6 therefore implies that $I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = I_{t,\mathrm{disc}}(f)$, and that $S^G_{t,\mathrm{disc}}(f)$ is stable in case G is quasisplit. The identity (29.29) then follows from the definition of $I^{\mathcal{E}}_{t,\mathrm{disc}}(f)$. □

30. Representations of classical groups

To give some sense of the power of the stable trace formula, we shall describe a broad application. It concerns the representations of classical groups. We shall describe a classification of automorphic representations of classical groups G in terms of those of general linear groups $GL(N)$. Since it depends on the stable trace formula for G, the classification is conditional on the fundamental lemma (both the standard version and its generalization (29.17)) for each of the classical groups in question. It also depends on the stabilization of a twisted trace formula for $GL(N)$. The classification is therefore conditional also on the corresponding twisted fundamental lemma (both standard and generalized) for $GL(N)$, as well as twisted analogues (yet to be established) of the results of §29.

It is possible to work in a more general context. One could take a product of general linear groups, equipped with a pair $\alpha = (\theta, \omega)$, where θ is an outer automorphism, and ω is an automorphic character of $GL(1)$. This is the setting adopted by Kottwitz and Shelstad in their construction of twisted transfer factors [**KoS**]. There is much to be learned by working in such generality. However, we shall adopt the more restricted setting in which $\alpha = \theta$ is the standard outer automorphism of $GL(N)$. For reasons on induction, it is important to allow N to vary. The groups G will then range the quasisplit classical groups in the three infinite families $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$. The results have yet to be published. My notes apply only to the special case under discussion, but I will try to write them up in greater generality.

The groups G arise as twisted endoscopic groups. For computational purposes, we represent θ as the automorphism

$$\theta(x) \longrightarrow {}_t x^{-1} = J\, {}^t x^{-1} J^{-1}, \qquad x \in GL(N),$$

of $GL(N)$, where

$${}_t x = J\, {}^t x J = J\, {}^t x J^{-1}, \qquad J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix},$$

is the "second transpose" of x, about the second diagonal. Then θ stabilizes the standard Borel subgroup of $GL(N)$. (For theoretical purposes [**KoS**], it is sometimes better to work with the automorphism

$$\theta'(x) = J'\, {}^t x^{-1} (J')^{-1}, \qquad J' = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{N+1} & & 0 \end{pmatrix},$$

that stabilizes the standard splitting in $GL(N)$ as well.) We form the connected component

$$\widetilde{G} = \widetilde{G}_N = GL(N) \rtimes \theta$$

in the nonconnected semidirect product

$$\widetilde{G}^+ = \widetilde{G}^+_N = GL(N) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

whose identity component we denote by $\widetilde{G}^0$. Twisted endoscopic data are like ordinary endoscopic data, except that their dual groups are connected centralizers of semisimple automorphisms within the inner class defined by $\widetilde{G}$, rather than the earlier identity class of inner automorphisms. We have then to consider semisimple elements s in the component $\widehat{\widetilde{G}} = \widehat{\widetilde{G}}{}^0 \rtimes \theta$, acting by conjugation on $\widetilde{G}^0$. It suffices to work here with the Galois form of L-groups. In the present context, a twisted endoscopic datum for $\widetilde{G}$ can be taken to be a quasisplit group G, together with an admissible L-embedding ξ of $\mathcal{G} = {}^LG$ into the centralizer in ${}^L\widetilde{G}^0 = GL(N,\mathbb{C}) \times \Gamma_F$ of some element s. We define G to be *elliptic* if $A_G = \{1\}$, which is to say that the group $Z(\widehat{G})^{\Gamma_F}$ is finite. We then write $\mathcal{E}_{\mathrm{ell}}(\widehat{G})$ for the set of isomorphism classes of elliptic (twisted) endoscopic data for $\widetilde{G}$.

Suppose for example that N is odd, and that $s = \theta$. Then the centralizer of s in $\widehat{\widetilde{G}}^0$ is a group we will denote by $O(N,\mathbb{C})$, even though it is really the orthogonal group with respect to the symmetric bilinear attached to J. The element s therefore yields a twisted endoscopic group G for which $\widehat{G}$ is the special orthogonal group $SO(N,\mathbb{C})$. Since N is odd, G is isomorphic to the split group $Sp(N-1)$. The group $O(N,\mathbb{C})$ has a second connected component, represented by the central element $(-I)$ in $GL(N,\mathbb{C})$. This means that there are many admissible ways to embed ${}^LG$ into ${}^L\widetilde{G}^0$. They are parametrized by isomorphisms from Γ_F to $\mathbb{Z}/2\mathbb{Z}$, which by class field theory correspond to characters η on $F^*\backslash\mathbb{A}^*$ with $\eta^2 = 1$. The set of such η parametrizes the subset of $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ attached to s. This phenomenon illustrates a second point of departure in the twisted case. The different embeddings represent distinct isomorphism classes of twisted endoscopic data, even though the underlying twisted endoscopic groups and associated elements s are all the same.

To describe the full set $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$, we consider decompositions of N into a sum $N_s + N_o$ of nonnegative integers, with N_s even. We then take the diagonal matrix

$$s = \begin{pmatrix} -I_s & & 0 \\ & I_o & \\ 0 & & I_s \end{pmatrix},$$

where I_s is the identity matrix of rank $(N_{s/2})$, and I_o is the identity matrix of rank N_o. The centralizer of s in $\widehat{\widetilde{G}}^0$ is a product

$$Sp(N_s,\mathbb{C}) \times O(N_o,\mathbb{C})$$

of complex classical groups, defined again by bilinear forms supported on the second diagonal. It corresponds to a twisted endoscopic group G with dual group

$$\widehat{G} = Sp(N_s,\mathbb{C}) \times SO(N_o,\mathbb{C}).$$

The group $O(N_o,\mathbb{C})$ has two connected components if $N_o > 0$. We have then also to specifiy an idèle class character η with $\eta^2 = 1$. If N_o is odd, the twisted endoscopic group is the split group

$$G = SO(N_s+1) \times Sp(N_o-1)$$

over F. In this case, η serves to specify the embedding of ${}^LG$ into ${}^L\widetilde{G}^0$, as in the special case above. We emphasize again that η is an essential part of the associated endoscopic datum. If N_o is even, the nonidentity component of $O(N_o,\mathbb{C})$ acts on the identity component $SO(N_o,\mathbb{C})$ as an outer automorphism. In this case, the twisted endoscopic group is the quasisplit group

$$G = SO(N_s+1) \times SO(N_o,\eta),$$

where $SO(N_o,\eta)$ is the outer twist of the split group $SO(N_o)$ determined by η. The character η also determines an L-embedding of ${}^LG$ into ${}^L\widetilde{G}^0$ in this case. If $N_o = 2$, the group $SO(N_o)$ is abelian. In this case, η must be nontrivial in order that corresponding twisted endoscopic datum be elliptic. In all other cases, η can be arbitrary. It is a straightforward exercise to check that the twisted endoscopic data obtained from triplets (N_s, N_o, η) in this way give a complete set of representatives of $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$.

It is possible to motivate the discussion above in more elementary terms. One does so by analyzing continuous representations

$$r: \ \Gamma_F \longrightarrow GL(N, \mathbb{C})$$

that are self-contragredient, in the sense that the representation

$${}_t r^{-1}: \ \sigma \longrightarrow {}_t r(\sigma)^{-1}, \qquad \sigma \in \Gamma_F,$$

is equivalent to r. Since r is continuous, it factors through a finite quotient of Γ_F. The analysis is therefore essentially that of the self-contragredient representations of an abstract finite group. One sees that twisted endoscopic data arise naturally in terms of decompositions of r into symplectic and orthogonal components. (See [**A23**, §3].)

The general results are proved by induction on N. We therefore have a particular interest in elements $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ that are *primitive*, in the sense either N_s or N_o equals zero. There are three cases. They correspond to $N = N_s$ even, $N = N_o$ odd, and $N = N_o$ even. The associated twisted endoscopic groups are the split group $G = SO(N+1)$ with dual group $\widehat{G} = Sp(N, \mathbb{C})$, the split group $G = Sp(N-1)$ with dual group $\widehat{G} = SO(N, \mathbb{C})$, and the quasisplit group $G = SO(N, \eta)$ with dual group $\widehat{G} = SO(N, \mathbb{C})$. We write $\mathcal{E}_{\mathrm{prim}}(\widetilde{G})$ for the subset of primitive elements in $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$.

Suppose that $G \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G})$. Regarding G simply as a reductive group over F, we can calculate its (standard) elliptic endoscopic data $G' \in \mathcal{E}_{\mathrm{ell}}(G)$. It suffices to consider diagonal matrices $s' \in \widehat{G}$ with entries ± 1. For example, in the first case that $G = SO(N+1)$ and $\widehat{G} = Sp(N, \mathbb{C})$ (for N even), it is enough to take diagonal elements

$$s' = \begin{pmatrix} -I' & & 0 \\ & I'' & \\ 0 & & -I' \end{pmatrix},$$

where I' is the identity matrix of rank $(N'/2)$, and I'' is the identity matrix of rank N''. The set $\mathcal{E}_{\mathrm{ell}}(G)$ is parametrized by pairs (N', N'') of nonnegative even integers, with $0 \leq N' \leq N''$ and $N = N' + N''$. The corresponding endoscopic groups are the split groups

$$G' = SO(N'+1) \times SO(N''+1),$$

with dual groups

$$\widehat{G}' = Sp(N', \mathbb{C}) \times Sp(N'', \mathbb{C}) \subset Sp(N, \mathbb{C}) = \widehat{G}.$$

In the second case that $G = Sp(N-1)$ and $\widehat{G} = SO(N, \mathbb{C})$, $\mathcal{E}_{\mathrm{ell}}(G)$ is parametrized by pairs of (N', N'') of nonnegative even integers with $N = N' + (N''+1)$, and idèle class characters η' with $(\eta')^2 = 1$. The corresponding endoscopic groups are the quasisplit groups

$$G' = SO(N', \eta') \times Sp(N''),$$

with dual groups

$$\widehat{G}' = SO(N', \mathbb{C}) \times SO(N''+1, \mathbb{C}) \subset \widehat{G} = SO(N, \mathbb{C}).$$

In the third case that $G = SO(N,\eta)$ and $\widehat{G} = SO(N,\mathbb{C})$, $\mathcal{E}_{\text{ell}}(G)$ is parametrized pairs of nonnegative even integers (N', N'') with $0 \leq N' \leq N''$ and $N = N' + N''$, and pairs (η', η'') of idèle class characters with $(\eta')^2 = (\eta'')^2 = 1$ and $\eta = \eta'\eta''$. The corresponding endoscopic groups are the quasisplit groups

$$G' = SO(N', \eta') \times SO(N'', \eta''),$$

with dual groups

$$\widehat{G}' = SO(N', \mathbb{C}) \times SO(N'', \mathbb{C}) \subset SO(N, \mathbb{C}) = \widehat{G}.$$

In the second and third cases, the character η^* has to be nontrivial if the corresponding integer N^* equals 2, and in the case $N' = 0$, η' must of course be trivial.

Our goal is to try to classify automorphic representations of a group $G \in \mathcal{E}_{\text{prim}}(\widetilde{G})$ by means of the trace formula. The core of the trace formula for G is the t-discrete part

(30.1)

$$I_{t,\text{disc}}(f) = \sum_{\{M\}} |W(M)|^{-1} \sum_{s \in W(M)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}_M}|^{-1} \operatorname{tr}\big(M_P(s,0)\mathcal{I}_{P,t}(0,f)\big),$$

of its spectral side. We recall that f is a test function in $\mathcal{H}(G) = \mathcal{H}\big(G(\mathbb{A})\big)$, while t is a nonnegative number that restricts the automorphic constituents of $\mathcal{I}_{P,t}(0,f)$ by specifying the norm of their archimedean infinitesimal characters. The stabilization described in §29 yields the decomposition

$$I_{t,\text{disc}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^{G'}_{t,\text{disc}}(f') \tag{30.2}$$

stated in Corollary 29.10. We recall that $S^{G'}_{t,\text{disc}}$ is a stable distribution on $G'(\mathbb{A})$, while $f' = f^{G'}$ is the Langlands-Shelstad transfer of f. This is the payoff. It is our remuneration for the work done in stabilizing the other terms in the trace formula. But really, how valuable is it? Since G is quasisplit, $G = G^*$ is an element in $\mathcal{E}(G)$. The stabilization does not provide an independent characterization of the distribution $S^G_{t,\text{disc}}$. In fact, (30.2) can be regarded as an inductive definition of $S^G_{t,\text{disc}}$ in terms of $I_{\text{disc},t}$ and corresponding distributions for groups G' of dimension smaller than G. Thus, (30.2) amounts to the assertion that one can modify $I_{t,\text{disc}}(f)$ by adding some correction terms, defined inductively in terms of Langlands-Shelstad transfer, so that it becomes stable. A useful property, no doubt, but not something that in itself could classify the automorphic representations of G.

What saves the day is the twisted trace formula for $\widetilde{G}$. Let $\widetilde{f}$ be a test function in the Hecke space $\mathcal{H}(\widetilde{G}) = \mathcal{H}\big(\widetilde{G}(\mathbb{A})\big)$ attached to the component $\widetilde{G} = GL(N) \rtimes \theta$. The twisted trace formula is an identity of linear forms whose spectral side also has a discrete part

(30.3)

$$I_{t,\text{disc}}(\widetilde{f}) = \sum_{\{\widetilde{M}^0\}} |W(\widetilde{M}^0)|^{-1} \sum_{s \in W(\widetilde{M}^0)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}^{\widetilde{G}}_{\widetilde{M}^0}}|^{-1} \operatorname{tr}\big(M_{\widetilde{P}^0}(s,0)\mathcal{I}_{\widetilde{P}^0,t}(0)\big)(\widetilde{f})$$

with the same general structure as (30.1). (The first sum is over the set of $\widetilde{G}^0$-orbits of Levi subgroups $\widetilde{M}^0$, while the second sum is over the regular elements in the relevant twisted Weyl set. The other terms are also twisted forms of their analogues in (30.1), for which the reader can consult [**CLL**] and [**A14**, §4].) We assume that the twisted fundamental lemma (both ordinary and weighted) holds for

$\widetilde{G}$, as well as twisted analogues of the other results described in §29. These include the twisted analogue of Waldspurger's theorem that the fundamental lemma implies transfer. We can therefore suppose that the transfer mapping $\widetilde{f} \to \widetilde{f}^G$, defined for any $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ by the twisted transfer factors of Kottwitz-Shelstad [**KoS**], sends $\mathcal{H}(\widetilde{G})$ to the space $S\mathcal{I}(G)$. The stabilization of the twisted trace formula for $\widetilde{G}$ then yields a decomposition

$$I_{t,\mathrm{disc}}(\widetilde{f}) = \sum_{G \in \mathcal{E}_{\mathrm{ell}}(\tilde{G})} \iota(\widetilde{G}, G) \widehat{S}^G_{t,\mathrm{disc}}(\widetilde{f}^G), \tag{30.4}$$

where $S^G_{t,\mathrm{disc}}$ is the (untwisted) stable distribution on $G(\mathbb{A})$ that appears in (30.2), and $\iota(\widetilde{G}, G)$ is an explicit constant. This gives an a priori relationship among the terms $S^G_{t,\mathrm{disc}}$ defined in the formulas (30.2).

By combining the global identities (30.2) and (30.4), one obtains both local and global results. In the end, the interplay between the two formulas yields a classification of representations of odd orthogonal and symplectic groups, and something close to a classification in the even orthogonal case. We shall say little more about the proofs. We shall instead use the rest of the section to try to give a precise statement of the results.

Since everything ultimately depends on the automorphic spectrum of $GL(N)$, we begin with this group. We need to formulate the results of Moeglin and Waldspurger in a way that can be extended to the classical groups in question.

We shall represent the discrete spectrum of $GL(N)$ by a set of formal objects that are parallel to the global parameters at the end of §26. Let $\Psi_2\big(GL(N)\big)$ be the set of formal tensor products

$$\psi = \mu \boxtimes \nu,$$

where μ is an irreducible, unitary, cuspidal automorphic representation of $GL(m)$, and ν is the unique irreducible n-dimensional representation of the group $SL(2, \mathbb{C})$, for positive integers m and n such that $N = mn$. For any such ψ, we form the induced representation

$$\mathcal{I}^G_P\big((\underbrace{\mu \otimes \cdots \otimes \mu}_{n})\delta_P^{\frac{1}{2}}\big), \tag{30.5}$$

of $GL(N, \mathbb{A})$, where P is the standard parabolic subgroup of type $(m, \dots, m)$. We then write π_ψ for the unique irreducible quotient of this representation. The theorem of Moeglin and Waldspurger asserts that the mapping $\psi \to \pi_\psi$ is a bijection from $\Psi_2\big(GL(N)\big)$ onto the set of automorphic representations of $GL(N)$ that occur in the discrete spectrum. Set

$$c(\psi) = \big\{c_v(\psi) : \ v \notin S\big\},$$

for any finite set $S \supset S_\infty$ of valuations outside of which μ is unramified, and semisimple conjugacy classes

$$c_v(\psi) = c_v(\mu) \otimes c_v(\phi_\nu) = c_v(\mu) q_v^{(\frac{n-1}{2})} \oplus \cdots \oplus c_v(\mu) q_v^{-(\frac{n-1}{2})}$$

in $GL(N, \mathbb{C})$. The family $c(\psi)$ then equals the family $c(\pi_\psi)$ attached to π_ψ in §26.

We also represent the entire automorphic spectrum of $GL(N)$ by a larger set of formal objects. Let $\Psi\big(GL(N)\big)$ be the set of formal (unordered) direct sums

$$\psi = \ell_1 \psi_1 \boxplus \cdots \boxplus \ell_r \psi_r \tag{30.6}$$

for positive integers ℓ_i, and distinct elements $\psi_i = \mu_i \boxtimes \nu_i$ in $\Psi_2\big(GL(N_i)\big)$. The ranks N_i are positive integers of the form $N_i = m_i n_i$ such that

$$N = \ell_1 N_1 + \cdots + \ell_r N_r = \ell_1 m_1 n_1 + \cdots + \ell_r m_r n_r.$$

For any ψ as in (30.6), take P to be the standard parabolic subgroup with Levi component

$$M = \big(\underbrace{GL(N_1) \times \cdots \times GL(N_1)}_{\ell_1}\big) \times \cdots \times \big(\underbrace{GL(N_r) \times \cdots \times GL(N_r)}_{\ell_r}\big),$$

and form the corresponding induced representation

$$\pi_\psi = \mathcal{I}_P^G\big((\underbrace{\pi_{\psi_1} \otimes \cdots \otimes \pi_{\psi_1}}_{\ell_1}) \otimes \cdots \otimes (\underbrace{\pi_{\psi_r} \otimes \cdots \otimes \pi_{\psi_r}}_{\ell_r})\big). \tag{30.7}$$

As a representation of $GL(N, \mathbb{A})$ induced from a unitary representation, π_ψ is known to be irreducible [**Be**]. It follows from the theory of Eisenstein series, and Theorem 7.2 in particular, that $\psi \to \pi_\psi$ is a bijection from $\Psi\big(GL(N)\big)$ onto the set of irreducible representations of $GL(N, \mathbb{A})$ that occur in the spectral decomposition of $L^2\big(GL(N, F)\backslash GL(N, \mathbb{A})\big)$. We set

$$c(\psi) = \{c_v(\psi) :\ v \notin S\},$$

for any finite set $S \supset S_\infty$ outside of which each μ_i is unramified, and semisimple conjugacy classes

$$c_v(\psi) = \big(\underbrace{c_v(\psi_1) \oplus \cdots \oplus c_v(\psi_1)}_{\ell_1}\big) \oplus \cdots \oplus \big(\underbrace{c_v(\psi_r) \oplus \cdots \oplus c_v(\psi_r)}_{\ell_r}\big),$$

in $GL(N, \mathbb{C})$. Then $c(\psi)$ is again equal to $c(\pi_\psi)$. The theorem of Jacquet and Shalika mentioned in §26 [**JaS**] tells us that the mapping

$$\psi \longrightarrow c(\psi), \qquad\qquad \psi \in \Psi\big(GL(N)\big),$$

from $\Psi\big(GL(N)\big)$ to the set of (equivalence classes of) semisimple conjugacy classes in $GL(N, \mathbb{C})$, is injective.

There is an action $\pi_\psi \to \pi_\psi^\theta$ of the outer automorphism θ on the set of representations π_ψ. If ψ is an element (30.6) in $\Psi\big(GL(N)\big)$, set

$$\begin{aligned}\psi^\theta &= \ell_1(\mu_1^\theta \boxtimes \nu_1^\theta) \boxplus \cdots \boxplus \ell_r(\mu_r^\theta \boxtimes \nu_r^\theta)\\ &= \ell(\mu_1^\theta \boxtimes \nu_1) \boxplus \cdots \boxplus \ell_r(\mu_r^\theta \boxtimes \nu_r),\end{aligned}$$

where μ_i^θ is the contragredient of the cuspidal automorphic representation μ_i of $GL(m_i)$. (We can write $\nu_i^\theta = \nu_i$, since any irreducible representation of $SL(2, \mathbb{C})$ is self dual.) Then $\pi_\psi^\theta = \pi_{\psi^\theta}$. We introduce a subset

$$\widetilde{\Psi} = \Psi(\widetilde{G}) = \big\{\psi \in \Psi\big(GL(N)\big) :\ \psi^\theta = \psi\big\}$$

of elements in $\Psi\big(GL(N)\big)$ associated to the component

$$\widetilde{G} = \widetilde{G}_N = GL(N) \rtimes \theta.$$

It corresponds to those representations π_ψ of $GL(N, \mathbb{A})$ that extend to group $\widetilde{G}(\mathbb{A})^+$ generated by $\widetilde{G}(\mathbb{A})$. We shall say that ψ is *primitive* if $r = \ell_1 = n_1 = 1$. In other words, $\psi = \mu_1$ is a self-dual cuspidal automorphic representation of $GL(N)$. In this case ψ has a central character η_ψ of order 1 or 2.

We would like to think of the elements in $\widetilde{\Psi}$ as parameters. They ought to correspond to self dual, N-dimensional representations of a group $L_F \times SL(2,\mathbb{C})$, where L_F is a global analogue of the local Langlands group L_{F_v}. The global Langlands group L_F is purely hypothetical. It should be an extension of the global Weil group W_F, equipped with a conjugacy class of embeddings

$$\begin{array}{ccc} L_{F_v} & \longrightarrow & W_{F_v} \\ \downarrow & & \downarrow \\ L_F & \longrightarrow & W_F \end{array}$$

of each local group. The hypothetical group L_F should ultimately play a fundamental role in the automorphic representation theory of any G. In the meantime, we attach an ad hoc substitute for L_F to any ψ.

The proofs of the results we are going to describe include an extended induction argument. There are in fact both local and global induction hypotheses. We introduce the global hypothesis first, in order to define our substitutes for L_F.

Global induction hypothesis. *Suppose that $\psi \in \widetilde{\Psi}$ is primitive. Then there is a unique class $G_\psi = (G_\psi, {}^LG_\psi, s_\psi, \xi_\psi)$ of (twisted) elliptic endoscopic data in $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ such that*

$$c(\psi) = \xi_\psi\big(c(\pi)\big),$$

for some irreducible representation π of $G(\mathbb{A})$ that occurs in $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$. Moreover G_ψ is primitive.

The assertion is quite transparent. Among all the (twisted) elliptic endoscopic data G for $\widetilde{G}$, there should be exactly one source for the conjugacy class data of ψ. If ψ happens to be attached to an irreducible, self-dual representation of a group L_F, it is an elementary exercise in linear algebra to show that the assertion is valid. That is, ψ factors through the L-group of a unique $G_\psi \in \mathcal{E}_{\mathrm{ell}}(G)$, with G_ψ being primitive. Of course, we do not know that ψ is of this form. We do know that if G_ψ is primitive, the dual group $\widehat{G}_\psi \subset GL(N,\mathbb{C})$ is purely orthogonal or symplectic. If $\eta_\psi \neq 1$ or N is odd, $\widehat{G}_\psi$ is orthogonal, and η_ψ determines G_ψ uniquely. However, if $\eta_\psi = 1$ and N is even, $\widehat{G}_\psi$ could be either symplectic or orthogonal. In this case, we will require a deeper property of ψ to characterize G_ψ.

In proving the results, one fixes N, and assumes inductively that the hypothesis holds if N is replaced by a positive integer $m < N$. The completion of the induction argument is of course part of what needs to be proved. Our purpose here is simply to state the results. Therefore, in order to save space, we shall treat the hypothesis as a separate theorem. In other words, we shall assume that it holds for $m = N$ as well.

Suppose that ψ is an arbitrary element in $\widetilde{\Psi}$. Then θ acts by permutation on the indices $1 \leq i \leq \ell$ in (30.6). Let I be the set of i with $\psi_i^\theta = \psi_i$. The complement of I is a disjoint union of two sets J and J', with a bijection $j \to j'$ from J to J', such that $\psi_j^\theta = \psi_{j'}$ for every $j \in J$. We can then write

$$\psi = \Big(\boxplus_{i\in I} \ell_i\psi_i\Big) \boxplus \Big(\boxplus_{j\in J} \ell_j(\psi_j \boxplus \psi_{j'})\Big).$$

If i belongs to I, we apply the global induction hypothesis to the self-dual, cuspidal automorphic representation μ_i of $GL(m_i)$. This gives us a canonical datum $G_i = G_{\mu_i}$ in $\mathcal{E}_{\mathrm{prim}}(\widetilde{G}_{m_i})$. If j belongs to J, we simply set $G_j = GL(m_j)$. We thus obtain a group G_α over F for any index α in I or J. Let ${}^L G_\alpha$ be the Galois form of its L-group. We can then form the fibre product

$$\mathcal{L}_\psi = \prod_{\alpha \in I \cup J} \big({}^L G_\alpha \longrightarrow \Gamma_F\big) \tag{30.8}$$

of these groups over Γ_F. If i belongs to I, the endoscopic datum G_i comes with the standard embedding

$$\widetilde{\mu}_i : \; {}^L G_i \longrightarrow {}^L\big(GL(m_i)\big) = GL(m_i, \mathbb{C}) \times \Gamma_F.$$

If j belongs to J, we define a standard embedding

$$\widetilde{\mu}_j : \; {}^L G_j \longrightarrow {}^L\big(GL(2m_j)\big) = GL(2m_j, \mathbb{C}) \times \Gamma_F$$

by setting

$$\widetilde{\mu}_j(g_j \times \sigma) = (g_j \oplus {}_t g_j^{-1}) \times \sigma \qquad\qquad g_j \in \widehat{G}_j = GL(m_j, \mathbb{C}), \; \sigma \in \Gamma_F.$$

We then define the L-embedding

$$\widetilde{\psi} : \; \mathcal{L}_\psi \times SL(2, \mathbb{C}) \longrightarrow {}^L\big(GL(N)\big) = GL(N, \mathbb{C}) \times \Gamma_F \tag{30.9}$$

by taking the appropriate direct sum

$$\widetilde{\psi} = \Big(\bigoplus_{i \in I} \ell_i(\widetilde{\mu}_i \otimes \nu_i)\Big) \oplus \Big(\bigoplus_{j \in J} \ell_j(\widetilde{\mu}_j \otimes \nu_j)\Big).$$

We can of course interpret the embedding $\widetilde{\psi} = \psi_{\tilde{G}}$ also as an N-dimensional representation of $\mathcal{L}_\psi \times SL(2, \mathbb{C})$. With either interpretation, we are primarily interested in the equivalence class of $\widetilde{\psi}$, which is a $GL(N, \mathbb{C})$-conjugacy class of homomorphisms from $\mathcal{L}_\psi \times SL(2, \mathbb{C})$ to either $GL(N, \mathbb{C})$ or ${}^L\big(GL(N)\big)$.

Suppose that G belongs to $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$. We write $\widetilde{\Psi}(G)$ for the set of $\psi \in \widetilde{\Psi}$ such that $\widetilde{\psi}$ factors through ${}^L G$. By this, we mean that there exists an L-homomorphism

$$\psi_G : \; \mathcal{L}_\psi \times SL(2, \mathbb{C}) \longrightarrow {}^L G \tag{30.10}$$

such that

$$\xi \circ \psi_G = \widetilde{\psi},$$

where ξ is the embedding of ${}^L G$ into ${}^L\big(GL(N)\big)$ that is part of the twisted endoscopic datum represented by G. Since $\widetilde{\psi}$ and ξ are to be regarded as $GL(N, \mathbb{C})$-conjugacy classes of homomorphisms, ψ_G is determined only up to conjugacy by a subgroup of $GL(N, \mathbb{C})$. We define $\mathrm{Aut}_{\tilde{G}}(G)$ to be the group of automorphisms of ${}^L G$ induced by conjugation of elements in $GL(N, \mathbb{C})$ that normalize the image of ${}^L G$. Then ψ_G is to be regarded as an $\mathrm{Aut}_{\tilde{G}}(G)$-orbit of L-homomorphisms (30.10). One sees easily that the quotient

$$\mathrm{Out}_{\tilde{G}}(G) = \mathrm{Aut}_{\tilde{G}}(G) / \mathrm{Int}(\widehat{G})$$

is trivial unless the integer N_o attached to G is even and positive, in which case it equals $\mathbb{Z}/2\mathbb{Z}$. In particular, if G is primitive and equals an even orthogonal group, there can be two $\widehat{G}$-orbits of homomorphisms in the class of ψ_G. It is for this reason that we write $\widetilde{\Psi}(G)$ in place of the more natural symbol $\Psi(G)$.

If ψ belongs to $\widetilde{\Psi}(G)$, we form the subgroup

$$S_\psi = S_\psi(G) = \mathrm{Cent}\big(\widehat{G}, \psi_G\big(\mathcal{L}_\psi \times SL(2, \mathbb{C})\big)\big) \tag{30.11}$$

of elements in $\widehat{G}$ that centralize the image of ψ_G. The quotient

$$\mathcal{S}_\psi = \mathcal{S}_\psi(G) = S_\psi / S_\psi^0 Z(\widehat{G})^\Gamma \tag{30.12}$$

is a finite abelian group, which plays a central role in the theory. Notice that there is a canonical element

$$s_\psi = \psi_G \left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{30.13}$$

in S_ψ. Its image in $\mathcal{S}_\psi$ (which we denote also by s_ψ) will be part of the description of nontempered automorphic representations.

Let $\widetilde{\Psi}_2$ be the subset of elements $\psi \in \widetilde{\Psi}$ such that the indexing set J is empty, and such that $\ell_i = 1$ for each $i \in I$. A general element $\psi \in \widetilde{\Psi}$ always belongs to a set $\widetilde{\Psi}(G)$, for some datum $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$. It belongs to a unique such set if and only if it lies in $\widetilde{\Psi}_2$. If G belongs to $\mathcal{E}_{\mathrm{ell}}(\widetilde{G})$, the intersection

$$\widetilde{\Psi}_2(G) = \widetilde{\Psi}(G) \cap \widetilde{\Psi}_2$$

is clearly the set of elements $\psi \in \widetilde{\Psi}(G)$ such that the group S_ψ is finite. We shall write $\widetilde{\Psi}_{\mathrm{prim}}$ for the set of primitive elements in $\widetilde{\Psi}$. Then

$$\widetilde{\Psi}_{\mathrm{prim}} \subset \widetilde{\Psi}_2 \subset \widetilde{\Psi},$$

and

$$\widetilde{\Psi}_{\mathrm{prim}}(G) \subset \widetilde{\Psi}_2(G) \subset \widetilde{\Psi}(G),$$

where

$$\widetilde{\Psi}_{\mathrm{prim}}(G) = \widetilde{\Psi}(G) \cap \widetilde{\Psi}_{\mathrm{prim}}.$$

Suppose that

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$$

belongs to $\widetilde{\Psi}_2$. How do we determine the group $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ such that ψ lies in $\widetilde{\Psi}_2(G)$? To answer the question, we have to be able to write $N = N_s + N_o$ and $\psi = \psi_s \boxplus \psi_o$, where $\psi_s \in \Psi_2(\widetilde{G}_{N_s})$ is the sum of those components ψ_i of symplectic type, and $\psi_o \in \Psi_2(\widetilde{G}_{N_o})$ is the sum of the components ψ_i of orthogonal type. Consider a general component

$$\psi_i = \mu_i \boxtimes \nu_i.$$

The representation $\mu_i \in \Psi_{\mathrm{prim}}(\widetilde{G}_{m_i})$ has a central character $\eta_i = \eta_{\mu_i}$ of order 1 or 2. It gives rise to a datum $G_i \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G}_{m_i})$, according to the global inductive hypothesis, and hence a complex, connected classical group $\widehat{G}_i \subset GL(m, \mathbb{C})$. The n_i-dimensional representation ν_i of $SL(2, \mathbb{C})$ gives rise to a complex, connected classical group $\widehat{H}_i \subset GL(n_i, \mathbb{C})$, which contains its image. By considering principal unipotent elements, for example, the reader can check that $\widehat{H}_i$ is symplectic when n_i is even, and is orthogonal when n_i is odd. The tensor product of the bilinear forms that define $\widehat{G}_i$ and $\widehat{H}_i$ is a bilinear form on $\mathbb{C}^{N_i} = \mathbb{C}^{m_i n_i}$. This yields a complex, connected classical group $\widehat{G}_{\psi_i} \subset GL(N_i, \mathbb{C})$, which contains the image of $\widehat{G}_i \times \widehat{H}_i$ under the tensor product of the two standard representations. In concrete terms, $\widehat{G}_{\psi_i}$ is symplectic if one of $\widehat{G}_i$ and $\widehat{H}_i$ is symplectic and the other is orthogonal, and is orthogonal if both $\widehat{G}_i$ and $\widehat{H}_i$ are of the same type. This allows us to designate ψ_i as either symplectic or orthogonal. It therefore gives us our decomposition $\psi = \psi_s \oplus \psi_o$. The component ψ_s lies in the subset $\widetilde{\Psi}_2(G_s)$ of $\Psi_2(\widetilde{G}_{N_s})$, for the datum $G_s \in \mathcal{E}_{\text{prim}}(\widetilde{G}_{N_s})$ with dual group $\widehat{G}_s = Sp(N_s, \mathbb{C})$. The component ψ_o lies in the subset $\widetilde{\Psi}_2(G_o)$ of $\Psi_2(\widetilde{G}_{N_o})$, for the datum $G_o \in \mathcal{E}_{\text{prim}}(\widetilde{G}_{N_o})$ with dual group $\widehat{G}_o = SO(N_o, \mathbb{C})$, and character

$$\eta_o = \prod_{i=1}^{r} (\eta_i)^{n_i}.$$

The original element ψ therefore lies in $\widetilde{\Psi}_2(G)$, where G is the product datum $G_s \times G_o$ in $\mathcal{E}_{\text{ell}}(\widetilde{G})$. We note that

$$S_\psi(G) = \begin{cases} (Z/2\mathbb{Z})^r, & \text{if each } N_i \text{ is even,} \\ (\mathbb{Z}/2\mathbb{Z})^{r-1}, & \text{otherwise.} \end{cases}$$

Suppose now that F is replaced by a completion $k = F_v$ of F. With this condition, we treat $\widetilde{G}^0 = GL(N)$ and $\widetilde{G} = GL(N) \rtimes \theta$ as objects over k, to which we add a subscript v if there is any chance of confusion. As we noted in §28, one can introduce endoscopic data over k by copying the definitions for the global field F. Similarly, one can introduce twisted endoscopic data for $\widetilde{G}$ over k. This gives local forms of the sets $\mathcal{E}_{\text{prim}}(\widetilde{G}) \subset \mathcal{E}_{\text{ell}}(\widetilde{G})$.

We can also construct the sets $\Psi_2\big(GL(N)\big)$, $\Psi\big(GL(N)\big)$, and $\widetilde{\Psi} = \Psi(\widetilde{G})$ as objects over k. We define $\Psi_2\big(GL(N)\big)$ to be the set of formal tensor products $\psi = \mu \boxtimes \nu$, where μ is now an element in the set $\Pi_{\text{temp},2}\big(GL(m,k)\big)$ of tempered irreducible representations of $GL(m,k)$ that are square integrable modulo the center. The other component ν remains an irreducible, n-dimensional representation of $SL(2,\mathbb{C})$, for a positive integer n with $N = mn$. For any such ψ, we form the induced representation

$$\mathcal{I}_P^G\big((\underbrace{\mu \otimes \cdots \otimes \mu}_{n})\delta_P^{\frac{1}{2}}\big)$$

of $GL(N,k)$, as in (30.5). It has a unique irreducible quotient π_ψ, which is known to be unitary. The larger set $\Psi\big(GL(N)\big)$ is again the set of formal direct sums

$$\psi = \ell_1\psi_1 \boxplus \cdots \boxplus \ell_r\psi_r,$$

for positive integers ℓ_i, and distinct elements $\psi_i = \mu_i \boxtimes \nu_i$ in $\Psi_2\big(GL(N_i)\big)$. For any such ψ, we form the induced representation

$$\pi_\psi = \mathcal{I}_P^G\big(\underbrace{(\pi_{\psi_1} \otimes \cdots \otimes \pi_{\psi_1})}_{\ell_1} \otimes \cdots \otimes \underbrace{(\pi_{\psi_r} \otimes \cdots \otimes \pi_{\psi_r})}_{\ell_r}\big)$$

of $GL(N,k)$, as in (30.7). It is irreducible and unitary. Finally, the local set $\widetilde{\Psi}$ is again the subset of elements ψ in the local set $\Psi\big(GL(N)\big)$ such that $\psi^\theta = \psi$. It has subsets

$$\widetilde{\Psi}_{\text{prim}} \subset \widetilde{\Psi}_2 \subset \widetilde{\Psi},$$

defined as in the global case.

We require a local form of our ad hoc substitute for the global Langlands group. Given the results of Harris-Taylor and Henniart, it is likely that one could work with the actual local Langlands group

$$L_{F_v} = \begin{cases} W_{F_v} \times SU(2), & \text{if } v \text{ is nonarchimedean,} \\ W_{F_v}, & \text{if } v \text{ is archimedean.} \end{cases}$$

However, the proof of the results described in this section still requires a local companion to the global induction hypothesis above. We may as well therefore use the local induction hypothesis to define analogues of the groups $\mathcal{L}_\psi$.

The local induction hypothesis depends on being able to attach a twisted character $\widetilde{f} \to \widetilde{f}_{\tilde{G}}(\psi)$ to any $\widetilde{f}$ and ψ in the local sets $\mathcal{H}(\widetilde{G})$ and $\widetilde{\Psi}$. Suppose first that $\psi = \mu \otimes \nu$. By applying the theory of local Whittaker models to the local form of the induced representation (30.5), one can define a canonical extension of the quotient π_ψ to $\widetilde{G}^+(k)$. This in turn provides a canonical extension of π_ψ to $\widetilde{G}^+(k)$ for a general parameter $\psi \in \widetilde{\Psi}$. We define

$$\widetilde{f}_{\tilde{G}}(\psi) = \mathrm{tr}\big(\pi_\psi(\widetilde{f})\big), \qquad \widetilde{f} \in \mathcal{H}(\widetilde{G}).$$

On the other hand, we are assuming that the twisted form of the Langlands-Shelstad transfer conjecture holds for $k = F_v$. This gives a mapping

$$\widetilde{f} \longrightarrow \widetilde{f}^G$$

from $\mathcal{H}(\widetilde{G})$ to $S\mathcal{I}(G)$, for any twisted endoscopic datum G for $\widetilde{G}$ over k.

Local Induction Hypothesis. *Suppose that $\psi \in \widetilde{\Psi}$ is primitive. Then there is a unique class $G_\psi \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ such that $\widetilde{f}_{\tilde{G}}(\psi)$ is the pullback of some stable distribution*

$$h \longrightarrow h^{G_\psi}(\psi), \qquad h \in \mathcal{H}(G_\psi),$$

on $G_\psi(k)$. In other words,

$$\widetilde{f}_{\tilde{G}}(\psi) = \widetilde{f}^{G_\psi}(\psi), \qquad \widetilde{f} \in \mathcal{H}(\widetilde{G}).$$

Moreover, G_ψ is primitive.

The assertion is less transparent than its global counterpart, for it is tailored to the fine structure of the terms in the spectral identities (30.1) and (30.3). It nonetheless serves the same purpose. Among all the local endoscopic data $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$ for $\widetilde{G}$, it singles out one that we can attach to ψ. As with the global hypothesis, we shall treat the local induction hypothesis as a separate theorem. In particular, we assume that it holds for $\Psi_{\mathrm{prim}}(\widetilde{G}_m)$, for any $m \leq N$.

We can now duplicate the constructions from the global case. If $\psi \in \widetilde{\Psi}$ is a general local parameter for the component $\widetilde{G} = \widetilde{G}_N$ over k, we obtain groups $G_i = G_{\mu_i}$ in $\mathcal{E}(\widetilde{G}_{m_i})$ for each i. We can then define the local form of the group $\mathcal{L}_\psi$. It is an extension of the local Galois group Γ_k, and comes with an L-embedding

$$\widetilde{\psi}: \ \mathcal{L}_\psi \times SL(2,\mathbb{C}) \longrightarrow {}^L GL(N) = GL(N,\mathbb{C}) \times \Gamma_k.$$

We again attach a subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}$ to any $G \in \mathcal{E}_{\mathrm{ell}}(\widetilde{G})$. Any $\psi \in \widetilde{\Psi}(G)$ comes with an $\mathrm{Aut}_{\tilde{G}}(G)$-orbit of local L-embeddings (30.10), with $\xi \circ \psi_G = \widetilde{\psi}$. It also comes with the reductive group $S_\psi = S_\psi(G)$, the finite abelian group $\mathcal{S}_\psi = \mathcal{S}_\psi(G)$, and the element s_ψ in either S_ψ or $\mathcal{S}_\psi$, defined by (30.11), (30.12), and (30.13) respectively.

There are a few more observations to be made in the case $k = F_v$, before we can state the theorems. We first note that the definitions above make sense if G is a general endoscopic datum for $\widetilde{G}$, rather than one that is just elliptic. The more general setting is required in the local context under discussion, since the localization of an elliptic global endoscopic datum need not remain elliptic.

Suppose that $G \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G})$. The putative Langlands-Shelstad mapping $\widetilde{f} \to \widetilde{f}^G$ takes $\mathcal{H}(\widetilde{G}) = \mathcal{H}\big(\widetilde{G}(k)\big)$ to the subspace $S\widetilde{\mathcal{I}}(G) = S\widetilde{\mathcal{I}}\big(G(k)\big)$ of functions in $S\mathcal{I}(G)$ that are invariant under the group $\mathrm{Out}_{\tilde{G}}(G)$. We recall that this group is trivial unless G is an even special orthogonal group $SO(N)$, in which case it is of order 2. In the latter case, the nontrivial element in $\mathrm{Out}_{\tilde{G}}(G)$ is induced by conjugation of the nontrivial connected component in $O(N)$. By choosing a k-rational element in this component, we obtain an outer automorphism of $G(k)$ (regarded as an abstract group). We can therefore identify $\mathrm{Out}_{\tilde{G}}(G)$ as a group of outer automorphisms of $G(k)$ of order 1 or 2. We write $\widetilde{\mathcal{I}}(G) = \widetilde{\mathcal{I}}\big(G(k)\big)$ for the space of functions in $\mathcal{I}(G)$ that are symmetric under $\mathrm{Out}_{\tilde{G}}(G)$, and $\widetilde{\mathcal{H}}(G) = \widetilde{\mathcal{H}}\big(G(k)\big)$ for the space of functions in $\mathcal{H}(G)$ that are symmetric under the image of $\mathrm{Out}_{\tilde{G}}(G)$ in $\mathrm{Aut}\big(G(k)\big)$ (relative to a suitable section). The mapping $f \to f_G$ then takes $\widetilde{\mathcal{H}}(G)$ onto $\widetilde{\mathcal{I}}(G)$, while the stable orbital integral mapping $f \to f^G$ takes $\widetilde{\mathcal{H}}(G)$ onto $S\widetilde{\mathcal{I}}(G)$. Let $\widetilde{\Pi}(G)$ denote the set of $\mathrm{Out}_{\tilde{G}}(G)$-orbits in the set $\Pi(G) = \Pi\big(G(k)\big)$ of irreducible representations. We also write $\widetilde{\Pi}_{\mathrm{fin}}(G)$ for the set of formal, finite, nonnegative integral combinations of elements in $\widetilde{\Pi}(G)$. Any element $\pi \in \widetilde{\Pi}_{\mathrm{fin}}$ then determines a linear form

$$f \longrightarrow f_G(\pi), \qquad f \in \widetilde{\mathcal{H}}(G),$$

on $\widetilde{\mathcal{H}}(G)$. We write $\widetilde{\Pi}_{\mathrm{unit}}(G)$ and $\widetilde{\Pi}_{\mathrm{fin,unit}}(G)$ for the subsets of $\widetilde{\Pi}(G)$ and $\widetilde{\Pi}_{\mathrm{fin}}(G)$ built out of unitary representations. By taking the appropriate product, we can extend these definitions to any endoscopic datum G for $\widetilde{G}$.

Suppose again that $G \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G})$, and that ψ belongs to $\widetilde{\Psi}(G)$. Suppose also that s' is a semisimple element in $S_\psi(G)$. Let $\widehat{G}'$ be the connected centralizer $\widehat{G}_{s'}$ of s' in $\widehat{G}$, and set

$$\mathcal{G}' = \widehat{G}'\psi_G(\mathcal{L}_\psi).$$

Then $\mathcal{G}'$ is an L-subgroup of ${}^LG$, for which the identity embedding ξ' is an L-homomorphism. We take $G' = G_{s'}$ to be a quasisplit group for which $\widehat{G}'$, with the L-action of Γ_F induced by $\mathcal{G}'$, is a dual group. We thus obtain an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G. Now the set $\widetilde{\Psi}(G')$ can be defined as an obvious Cartesian product of sets we have already constructed. Since s' lies in the centralizer of the image of $\mathcal{L}_\psi$ in $\widehat{G}$, ψ_G factors through ${}^LG'$. We obtain an L-embedding

$$\psi_{G'}:\ \mathcal{L}_\psi \times SL(2,\mathbb{C}) \longrightarrow {}^LG'$$

such that

$$\xi' \circ \psi_{G'} = \psi_G,$$

and a corresponding element $\psi' = \psi_{s'}$ in $\widetilde{\Psi}(G')$. Once again, this construction extends to the case that G is a general twisted endoscopic datum for $\widetilde{G}$.

There is one final technical complication. We want the local objects ψ over $k = F_v$ to represent local components at v of global parameters associated to automorphic representations of $GL(N)$. Because we do not know that the extension to $GL(N)$ of Ramanujan's conjecture is valid, we do not know that the local components are tempered. This requires a minor generalization of the local set $\widetilde{\Psi}$ attached to $k = F_v$. We define a larger set $\widetilde{\Psi}^+ = \Psi^+(\widetilde{G})$ of formal direct sums

$$\psi = \ell_1\psi_1 \boxplus \cdots \boxplus \ell_r\psi_r,$$

by relaxing the condition on the representations μ_i in components $\psi_i = \mu_i \boxtimes \nu_i$. We require only that μ_i belong to the set $\Pi_2\big(GL(m_i,k)\big)$. In other words, μ_i is an irreducible representation of $GL(m_i,k)$ that is square integrable modulo the center, but whose central character need not be unitary. This condition applies only to the components μ_i such that $\mu_i^\theta \neq \mu_i$, since the central character of μ_i would otherwise have order 2. If ψ belongs to $\widetilde{\Psi}^+$, the twisted character $\widetilde{f} \to \widetilde{f}_{\tilde{G}}(\psi)$ is defined from the tempered case by analytic continuation in the central characters of the components μ_i. The various other objects we have associated to the set $\widetilde{\Psi}$ are also easily formulated for the larger set $\widetilde{\Psi}^+$.

We shall now state the results as three theorems. They are conditional on the fundamental lemma, and the further requirements discussed at the beginning of the section.

THEOREM 30.1. *Assume that $k = F_v$ is local, and that $G \in \mathcal{E}_{\rm prim}(\widetilde{G})$.*

(a) *For each $\psi \in \widetilde{\Psi}(G)$, there is a stable linear form $h \to h^G(\psi)$ on $\widetilde{\mathcal{H}}(G)$ such that*

$$\widetilde{f}_{\tilde{G}}(\psi) = \widetilde{f}^G(\psi), \qquad \widetilde{f} \in \mathcal{H}(\widetilde{G}).$$

(b) *For each $\psi \in \widetilde{\Psi}(G)$, there is a finite subset $\widetilde{\Pi}_\psi$ of $\widetilde{\Pi}_{\rm fin,unit}(G)$, together with an injective mapping*

$$\pi \longrightarrow \langle \cdot, \pi \rangle, \qquad \pi \in \widetilde{\Pi}_\psi,$$

from $\widetilde{\Pi}_\psi$ to the group of characters $\widehat{\mathcal{S}}_\psi(G)$ on $\mathcal{S}_\psi(G)$ that satisfies the following condition. For any $s' \in S_\psi(G)$,

$$f^{G'}(\psi') = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi s, \pi \rangle f_G(\pi), \qquad f \in \widetilde{\mathcal{H}}(G), \tag{30.14}$$

where $G' = G'_{s'}$, $\psi' = \psi'_{s'}$, and s is the image of s' in $\mathcal{S}_\psi(G)$.

(c) *Let $\widetilde{\Phi}_{\rm temp}(G)$ denote the subset of elements in $\widetilde{\Psi}(G)$ for which each of the $SL(2,\mathbb{C})$ components ν_i is trivial. Then if $\phi \in \widetilde{\Phi}_{\rm temp}(G)$, the elements in $\widetilde{\Pi}_\phi$ are tempered and irreducible, in the sense that they belong to the set $\widetilde{\Pi}_{\rm temp}(G)$ of ${\rm Out}_{\tilde{G}}(G)$-orbits in $\Pi_{\rm temp}(G)$. Moreover, every element in $\widetilde{\Pi}_{\rm temp}(G)$ belongs to exactly one packet $\widetilde{\Pi}_\phi$. Finally, if k is nonarchimedean, the mapping $\widetilde{\Pi}_\phi \to \widehat{\mathcal{S}}_\phi$ is bijective.* □

Remarks: 1. The assertions (b) and (c) of the theorem are new only in the nonarchimedean case. (For archimedean v, they are special cases of results of Shelstad [**She3**] and Adams, Barbasch, and Vogan [**ABV**].) If v is nonarchimedean, assertion (c) can be combined with the local Langlands conjecture for $GL(N)$ [**HT**], [**He**]. This ought to yield the local Langlands conjecture for G, at least in the case that ${\rm Out}_{\tilde{G}}(G) = 1$.

2. The transfer mapping $f \to f^{G'}$ in (b) depends on a normalization for the transfer factors $\Delta_G(\delta', \gamma)$ for the quasisplit group G'. We assume implicitly that $\Delta_G(\delta', \gamma)$ equals the function denoted $\Delta_0(\delta', \gamma)$ on p. 248 of [**LS1**]. This is the reason that the characters $\langle \cdot, \pi \rangle$ on $\mathcal{S}_\varphi$ attached to an element $\phi \in \widetilde{\Phi}_{\rm temp}(G)$ are slightly simpler than in the general formulation (28.8).

3. Suppose that ψ lies in the larger set $\widetilde{\Psi}^+(G)$. We can then combine the theorem with a discussion similar to that of (28.9). In particular, we can identify ψ with the image in $\widetilde{\Psi}^+(G)$ induced from a nontempered twist $\psi_{M,\lambda}$, where M is a Levi subgroup of G, ψ_M is an element in $\widetilde{\Psi}(M)$, and λ is a point in $(\mathfrak{a}_M^*)_P^+$. We can then form the corresponding induced packet

$$\widetilde{\Pi}_\psi = \{\mathcal{I}_P^G(\pi_{M,\lambda}) : \ \pi_M \in \widetilde{\Pi}_{\psi_M}\}$$

for $G(k)$. Since we are dealing with full induced representations, rather than Langlands quotients, the assertions of the theorem extend to $\widetilde{\Pi}_\psi$.

THEOREM 30.2. *Assume that $k = F$ is global, and that $G \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G})$.*

(a) *Suppose that $\psi \in \widetilde{\Psi}(G)$. If v is any valuation of F, the localization ψ_v of ψ, defined in the obvious way as an element in the set $\widetilde{\Psi}_v^+ = \Psi^+(\widetilde{G}_v)$, has the property that $\mathcal{L}_{\psi_v}$ is contained in $\mathcal{L}_\psi$. In particular, ψ_v belongs to $\widetilde{\Psi}^+(G_v)$, $S_\psi(G)$ is contained in $S_{\psi_v}(G_v)$, and there is a canonical homomorphism $s \to s_v$ from $\mathcal{S}_\psi(G)$ to $\mathcal{S}_{\psi_v}(G_v)$. We can therefore define a global packet*

$$\widetilde{\Pi}_\psi = \Big\{ \bigotimes_v \pi_v : \ \pi_v \in \widetilde{\Pi}_{\psi_v}, \ \langle \cdot, \pi_v \rangle = 1 \text{ for almost all } v \Big\},$$

and for each element $\pi = \bigotimes_v \pi_v$ in $\widetilde{\Pi}_\psi$, a character

$$\langle s, \pi \rangle = \prod_v \langle s_v, \pi_v \rangle, \qquad s \in \mathcal{S}_v,$$

on $\mathcal{S}_\psi = \mathcal{S}_\psi(G)$.

(b) *Define a subalgebra of $\mathcal{H}(G)$ by taking the restricted tensor product*

$$\widetilde{\mathcal{H}}(G) = \bigotimes_v^{\mathrm{rest}} \widetilde{\mathcal{H}}(G_v).$$

Then there is an $\widetilde{\mathcal{H}}(G)$-module isomorphism

$$L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big) \cong \bigoplus_{\psi \in \tilde{\Psi}_2(G)} m_\psi \Big(\bigoplus_{\{\pi \in \tilde{\Pi}_\psi : \langle \cdot, \pi \rangle = \varepsilon_\psi\}} \pi \Big), \tag{30.15}$$

where m_ψ equals 1 or 2, and

$$\varepsilon_\psi : \ \mathcal{S}_\psi \ \longrightarrow \ \{\pm 1\}$$

is a linear character defined explicitly in terms of symplectic root numbers. □

Remarks. 4. The multiplicity m_ψ is defined to be the number of $\widehat{G}$-orbits of embeddings

$$\mathcal{L}_\psi \times SL(2, \mathbb{C}) \ \longrightarrow \ {}^L G$$

in the $\mathrm{Aut}_{\widetilde{G}}(G)$-orbit of ψ_G. We leave the reader to check that m_ψ equals 1 unless N is even, $\widehat{G} = SO(N, \mathbb{C})$, and the rank N_i of each of the components $\psi_i = \mu_i \otimes \nu_i$ of ψ is also even, in which case $m_\psi = 2$.

5. The sign character ε_ψ is defined as follows. We first define an orthogonal representation

$$\tau_\psi : \ S_\psi \times \mathcal{L}_\psi \times SL(2, \mathbb{C}) \ \longrightarrow \ GL(\widehat{\mathfrak{g}})$$

on the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ by setting

$$\tau_\psi(s,g,h) = \mathrm{Ad}\big(s\psi_G(g\times h)\big), \qquad s\in S_\psi,\ g\in\mathcal{L}_\psi,\ h\in SL(2,\mathbb{C}).$$

We then write

$$\tau_\psi = \bigoplus_\alpha \tau_\alpha = \bigoplus_\alpha \big(\lambda_\alpha\otimes\mu_\alpha\otimes\nu_\alpha\big),$$

for irreducible representations λ_α, μ_α and ν_α of S_ψ, $\mathcal{L}_\psi$ and $SL(2,\mathbb{C})$ respectively. Given the definition of the global group $\mathcal{L}_\psi$, we can regard $L(s,\mu_\alpha)$ as an automorphic L-function for a product of general linear groups. One checks that it is among those L-functions for which one has analytic continuation, and a functional equation

$$L(s,\mu_\alpha) = \varepsilon(s,\mu_\alpha)L(1-s,\mu_\alpha^\vee).$$

In particular, if $\mu_\alpha^\vee = \mu_\alpha$, $\varepsilon(\frac{1}{2},\mu_\alpha) = \pm 1$. Let $\mathcal{A}$ be the set of indices α such that

(i) $\tau_\alpha^\vee = \tau_\alpha$ (and hence $\mu_\alpha^\vee = \mu_\alpha$),
(ii) $\dim(\nu_\alpha)$ is even (and hence ν_α is symplectic),
(iii) $\varepsilon(\frac{1}{2},\mu_\alpha) = -1$.

Then

$$\varepsilon_\psi(s) = \prod_{\alpha\in\mathcal{A}} \det\big(\lambda_\alpha(s)\big), \qquad s\in S_\psi. \tag{30.16}$$

Theorem 30.3. *Assume that F is global.*

(a) *Suppose that $G\in\mathcal{E}_{\mathrm{prim}}(\widetilde{G})$, and that $\psi=\mu$ belongs to $\widetilde{\Psi}_{\mathrm{prim}}(G)$. Then $\widehat{G}$ is orthogonal if and only if the symmetric square L-function $L(s,\mu,S^2)$ has a pole at $s=1$, while $\widehat{G}$ is symplectic if and only if the skew-symmetric L-function $L(s,\mu,\Lambda^2)$ has a pole at $s=1$.*

(b) *Suppose that for $i=1,2$, $G_i\in\mathcal{E}_{\mathrm{prim}}(\widetilde{G}_{N_i})$ and that $\psi_i=\mu_i$ belongs to $\widetilde{\Psi}_{\mathrm{prim}}(G_i)$. Then the corresponding Rankin-Selberg ε-factor satisfies*

$$\varepsilon(\tfrac{1}{2},\mu_1\times\mu_2) = 1,$$

provided that $\widehat{G}_1$ and $\widehat{G}_2$ are either both orthogonal or both symplectic. □

Remarks: 6. Suppose that μ is as in (i). It follows from the fact $\mu^\theta=\mu$ that

$$L(s,\mu\times\mu) = L(s,\mu,S^2)L(2,\mu,\Lambda^2).$$

The Rankin-Selberg L-function on the left is known to have a pole of order 1 at $s=1$. One also knows that neither of the two L-functions on the right can have a zero at $s=1$. The assertion of (a) is therefore compatible with our a priori knowledge of the relevent L-functions. It is also compatible with properties of the corresponding Artin L-functions, in case μ is attached to an irreducible N-dimensional representation of Γ_F or W_F. The assertion is an essential part of both the resolution of the global induction hypothesis and the proof of the multiplicity formula (30.15).

7. Consider the assertion of (b). If μ_1 and μ_2 are both attached to irreducible representations of W_F, the conditions of (b) reduce to the requirement that the tensor product of the two representations be orthogonal. The assertion of (b) is known in this case [**D2**]. The general assertion (b) is again intimately related to the global induction hypothesis and the multiplicity formula (30.15).

We shall add a few observations on the "tempered" case of the multiplicity formula (30.15). Assume that $G \in \mathcal{E}_{\mathrm{prim}}(\widetilde{G})$, as in Theorem 30.2. Let us write $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ for the subspace of $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ whose irreducible constituents transfer to *cuspidal* Eisenstein series for $\widetilde{G}^0 = GL(N)$. (The notation anticipates a successful resolution of the Ramanujan conjecture for $GL(N)$, which given our theorems, would imply that $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ is indeed the subspace of $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ whose irreducible constituents are tempered.) Let $\widetilde{\Phi}_2(G) = \widetilde{\Phi}_{\mathrm{temp},2}(G)$ be the subset of elements in the global set $\widetilde{\Psi}_2(G)$ for which the $SL(2,\mathbb{C})$-components ν_i are all trivial. Then $\varepsilon_\phi = 1$ for every $\phi \in \widetilde{\Phi}_2(G)$. The formula (30.15) therefore provides an $\widetilde{\mathcal{H}}(G)$-module isomorphism

$$L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big) \cong \bigoplus_{\phi\in\widetilde{\Phi}_2(G)} m_\phi\Big(\bigoplus_{\{\pi\in\widetilde{\Pi}_\phi:\langle\cdot,\pi\rangle=1\}} \pi\Big). \tag{30.17}$$

Suppose that N is odd, or that $\widehat{G} = Sp(N,\mathbb{C})$. Then $m_\phi = 1$. It is also easy to see that $\widetilde{\mathcal{H}}(G_v) = \mathcal{H}(G_v)$ for any v, so that $\widetilde{\mathcal{H}}(G) = \mathcal{H}(G)$ in this case. Moreover, the local packets $\widetilde{\Pi}_{\phi_v} = \Pi_{\phi_v}$ attached to elements ϕ_v in the set $\widetilde{\Phi}_{\mathrm{temp}}(G_v) = \Phi_{\mathrm{temp}}(G_v)$ contain only irreducible representations of $G(F_v)$. Now the local component ϕ_v of an element ϕ in the global set $\widetilde{\Phi}_2(G) = \Phi_2(G)$ could lie in a set $\Phi^+_{\mathrm{temp}}(G_v) \subset \Phi^+(G_v)$ that properly contains $\Phi_{\mathrm{temp}}(G_v)$. However, it is likely that the induced representations that comprise the corresponding packet Π_{ϕ_v} are still irreducible. (I have not checked this point in general, but it should be a straightforward consequence of the well known structure of generic, irreducible, unitary representations of $GL(N,F_v)$.) Taking the last point for granted, we see that the global packet

$$\Pi_\phi = \Big\{\bigotimes_v \pi_v :\ \pi_v \in \Pi_{\phi_v},\ \langle\cdot,\pi_v\rangle = 1 \text{ for almost all } v\Big\}$$

attached to any $\phi \in \Phi_2(G)$ contains only irreducible representations of $G(\mathbb{A})$. The injectivity of the mapping $\pi \to c(\pi)$ implies that the global packets are disjoint. It then follows from (30.17) that $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ decomposes with multiplicity 1 in this case.

In the remaining case, N is even and $\widehat{G} = SO(N,\mathbb{C})$. If one of the integers N_i attached to a given global element $\phi \in \widetilde{\Phi}_2(G)$ is odd, m_ϕ equals 1. An argument like that above then implies that the irreducible constituents of $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ attached to ϕ have multiplicity 1. However, if the integers N_i attached to ϕ are all even, m_ϕ equals 2. The multiplicity formula (30.17) then becomes more interesting. It depends in fact on the integers

$$N_{v,i}, \qquad\qquad 1 \le i \le \ell_v,$$

attached to the local components ϕ_v of ϕ. If for some v, all of these integers are even, (30.17) can be used to show that the irreducible constituents of $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ attached to ϕ again have multiplicity 1. However, it could also happen that for every v, one of the integers $N_{v,i}$ is odd. A slightly more elaborate analysis of (30.17) then leads to the conclusion that the irreducible constituents of $L^2_{\mathrm{temp,disc}}\big(G(F)\backslash G(\mathbb{A})\big)$ attached to ϕ all have multiplicity 2. This represents a quantitative description of a phenomenon investigated by M. Larsen in terms of representations of Galois groups [**Lar**, p. 253].

The discussion of this section has been restricted to quasisplit orthogonal and symplectic groups. It is of course important to treat other classical groups as well. For example, there ought to be a parallel theory for quasisplit unitary groups over F. The case of unitary groups is in fact somewhat simpler. Moreover, a proof of the fundamental lemma for unitary groups has been announced recently by Laumon and Ngo [**LN**]. It is quite possible that their methods could be extended to weighted orbital integrals and their twisted analogues. The goal would be to extend the results of Rogawski for $U(3)$ [**Ro2**], [**Ro3**] to general rank.

Finally, we note that there has been considerable progress recently in applying other methods to classical groups. These methods center around the theory of L-functions, and a generalization [**CP**] of Hecke's converse theorem for $GL(2)$. They apply primarily to generic representations (both local and global) of classical groups, but they do not depend on the fundamental lemma. We refer the reader to [**Co**] for a general introduction, and to selected papers [**CKPS1**], [**CKPS2**], [**JiS**] and [**GRS**].

Afterword: beyond endoscopy

The principle of functoriality is one of the pillars of the Langlands program. It is among the deepest problems in mathematics, and has untold relations to other questions. For example, the work of Wiles suggests that functoriality is inextricably intertwined with that second pillar of the Langlands program, the general analogue of the Shimura-Taniyama-Weil conjecture [**Lan7**].

The theory of endoscopy, which is still largely conjectural, analyzes representations of G in terms of representations of its endoscopic groups G'. In its global form, endoscopy amounts to a comparison of trace formulas, namely the invariant (or twisted) trace formula for G with stable trace formulas for G'. It includes the applications we discussed in §25, §26, and §30 as special cases. The primary aim of endoscopy is to organize the representations of G into packets. It can be regarded as a first attempt to describe the fibres of the mapping

$$\pi \longrightarrow c(\pi)$$

from automorphic representations to families of conjugacy classes. However, it also includes functorial correspondences for the L-homomorphisms

$$\xi' : \ {}^LG' \longrightarrow {}^LG$$

attached to endoscopic groups G' for G (in cases where $\mathcal{G}'$ can be identified with an L-group ${}^LG'$).

The general principle of functoriality applies to an L-homomorphism

$$\rho : \ {}^LG' \longrightarrow {}^LG \tag{A.1}$$

attached to any pair G' and G of quasisplit groups. As a strategy for attacking this problem, the theory of endoscopy has obvious theoretical limitations. It pertains, roughly speaking, to the case that ${}^LG'$ is the group of fixed points of a semisimple L-automorphism of ${}^LG$. Most mappings ρ do not fall into this category.

Suppose for example that $G' = GL(2)$ and $G = GL(m+1)$, and that ρ is the $(m+1)$-dimensional representation of $\widehat{G}' = GL(2,\mathbb{C})$ defined by the m^{th} symmetric power of the standard two-dimensional representation. If $m = 2$, the image of $GL(2,\mathbb{C})$ in ${}^LG = GL(3,\mathbb{C})$ is essentially an orthogonal group. In this case, the problem is endoscopic, and is included in the theory of classical groups discussed in §30. (In fact, functoriality was established in this case by other means some years ago [**GeJ**].) In the case $m = 3$ and $m = 4$, functoriality was established recently by Kim and Shahidi [**KiS**] and Kim [**Ki**]. These results came as a considerable surprise. They were proved by an ingenious combination of the converse theorems of Cogdell and Piatetskii-Shapiro with the Langlands-Shahidi method. If $m \geq 5$, however, these methods do not seem to work. Since the problem is clearly not endoscopic in this case, none of the known techniques appear to hold any hope of success.

We are going to conclude with a word about some recent ideas of Langlands[1] [**Lan13**], [**Lan15**]. The ideas are quite speculative. They have yet to be shown to apply even heuristically to new cases of functoriality. However, they have the distinct advantage that everything else appears to fail in principle. The ideas are in any case intriguing. They are based on applications of the trace formula that have never before been considered.

[1] I thank Langlands for enlightening conversations on the topic.

The difficulty in attacking the general case (A.1) of functoriality is that it is hard to characterize the image of ${}^LG'$ in ${}^LG$. If $\rho({}^LG')$ is the group of fixed points of some outer automorphism, G' will be related to a twisted endoscopic group for G. The corresponding twisted trace formula isolates automorphic representations of G that are fixed by the outer automorphism. A comparison of this formula with stable trace formulas for the associated collection of twisted endoscopic groups is aimed, roughly speaking, at those L-subgroups of ${}^LG$ fixed by automorphisms in the given inner class. If the image of ${}^LG'$ in ${}^LG$ is more general, however, the problem becomes much more subtle. Is it possible to use the trace formula in a way that counts only the automorphic representations of $G(\mathbb{A})$ that are functorial images of automorphic representations of G'?

Suppose that r is some finite dimensional representation of ${}^LG$. We write $V_{\mathrm{ram}}(G,r)$ for the finite set of valuations v of F at which either G or r is ramified. For ρ as in (A.1), the composition $r\circ\rho$ is a finite dimensional representation of ${}^LG'$. If this representation contains the trivial representation of ${}^LG'$, and the L-function $L(s,\pi',r\circ\rho)$ attached to a given automorphic representation π' of G' has the expected analytic continuation, the L-function will have a pole at $s=1$. The same would therefore be true of the L-function $L(s,\pi,r)$ attached to an automorphic representation π of G that is a functorial image of π' under ρ. On the other hand, so long as r does not contain the trivial representation of ${}^LG$, there will be many automorphic representations π of G for which $L(s,\pi,r)$ does not have a pole at $s=1$. One would like to have a trace formula that includes only the automorphic representations π of G for which $L(s,\pi,\rho)$ has a pole at $s=1$.

The objects of interest are of course automorphic representations π of G that occur in the discrete spectrum. The case that π is nontempered is believed to be more elementary, in the sense that it should reduce to the study of tempered automorphic representations of groups G_ψ of dimension smaller than G [**A17**]. The primary objects are therefore the representations π that are tempered, and hence cuspidal. If π is a tempered, cuspidal automorphic representation of G, $L(s,\pi,r)$ should have a pole at $s=1$ of order equal to that of the unramified L-function

$$L^V(s,\pi,r)=\prod_{v\notin V}\det\big(1-r(c(\pi_v))q_v^{-s}\big)^{-1},$$

attached to any finite set $V\supset S_{\mathrm{ram}}(G,r)$ outside of which π is unramified. The partial L-function $L^V(s,\pi,r)$ is not expected to have a zero at $s=1$. The order of its pole will thus equal

$$n(\pi,r)=\operatorname*{Res}_{s=1}\Big(-\frac{d}{ds}\log\,L^V(s,\pi,r)\Big),$$

a nonnegative integer that is independent of V.

We can write

$$\begin{aligned}&-\frac{d}{ds}\log\,L^V(s,\pi,r)\\&=\sum_{v\notin V}\frac{d}{ds}\log\big(\det(1-r(c(\pi_v))q_v^{-s})\big)\\&=\sum_{v\notin V}\sum_{k=1}^{\infty}\log(q_v)\mathrm{tr}\big(r(c(\pi_v))^k\big)q_v^{-ks},\end{aligned}$$

for $\mathrm{Re}(s)$ large. Since π is assumed to be tempered, the projection of any conjugacy class $c(\pi_v)$ onto $\widehat{G}$ is bounded, in the sense that it intersects any maximal compact subgroup of $\widehat{G}$. It follows that the set of coefficients

$$\big\{\mathrm{tr}\big(r(c(\pi_v))^k\big) :\ v \notin V,\ k \geq 1\big\}$$

is bounded, and hence that the last Dirichlet series actually converges for $\mathrm{Re}(s) > 1$. Since π is also assumed to be cuspidal automorphic, $L^V(s, \pi, r)$ is expected to have analytic continuation to a meromorphic function on the complex plane. The last Dirichlet series will then have at most a simple pole at $s = 1$, whose residue can be described in terms of the coefficients. Namely, by a familiar application of the Wiener-Ikehara tauberian theorem, there would be an identity

$$n(\pi, r) = \lim_{N\to\infty} \Big(V_N^{-1} \sum_{\{v\notin V:\, q_v \leq N\}} \mathrm{tr}\big(r(c(\pi_v))\big)\Big), \tag{A.2}$$

where

$$V_N = |\{v \notin V :\ q_v \leq N\}|.$$

(See [**Ser1**, p. I-29]. Observe that the contribution of the coefficients with $k > 1$ to the Dirchlet series is analytic at $s = 1$, and can therefore be ignored.)

Langlands proposes to apply the trace formula to a family of functions f_N that depend on the representation r. We begin with an arbitrary function $f \in \mathcal{H}\big(G(\mathbb{A})\big)$. If $V \supset S_{\mathrm{ram}}(G)$ is a finite set of valuations such that f belongs to the subspace $\mathcal{H}\big(G(F_V)\big)$ of $\mathcal{H}\big(G(\mathbb{A})\big)$, and ϕ belongs to the unramified Hecke algebra $\mathcal{H}\big(G(\mathbb{A}^V), K^V\big)$, the product

$$f_\phi :\ x \ \rightarrow\ f(x)\phi(x^V), \qquad x \in G(\mathbb{A}),$$

also belongs to $\mathcal{H}\big(G(\mathbb{A})\big)$. We choose the function $\phi = \phi_N$ so that it depends on r, as well as a positive integer N. Motivated by (A.2), and assuming that V contains the larger finite set $S_{\mathrm{ram}}(G, r)$, we define ϕ_N by the requirement that

$$(\phi_N)_G(\pi^V) = \sum_{\{v\notin V:\, q_v\leq N\}} r\big(c(\pi_v)\big),$$

for any unramified representation π^V of $G(\mathbb{A}^V)$. Then

$$n(\pi, r) = \lim_{N\to\infty} \big((\phi_N)_G(\pi^V)V_N^{-1}\big),$$

for any π as in (A.2). The products

$$f_N = f_N^r = f_{\phi_N}, \qquad N \geq 1,$$

or rather their images in $\mathcal{H}(G)$, are the relevant test functions.

Set

$$I_{\mathrm{temp,cusp}}(f) = \mathrm{tr}\big(R_{\mathrm{temp,cusp}}(f)\big),$$

where $R_{\mathrm{temp,cusp}}$ is the representation of $G(\mathbb{A})^1$ on the subspace of $L^2_{\mathrm{cusp}}\big(G(F)\backslash G(\mathbb{A})^1\big)$ that decomposes into tempered representations π of $G(\mathbb{A})^1$. Suppose that we happen to know that $L^V(s, \pi, r)$ has analytic continuation for each such π. Then the sum

$$\sum_\pi n(\pi, r) m_{\mathrm{cusp}}(\pi) f_G(\pi), \tag{A.3}$$

taken over irreducible tempered representations π of $G(\mathbb{A})^1$, equals the limit

$$I^r_{\mathrm{temp,cusp}}(f) = \lim_{N\to\infty}\big(I_{\mathrm{temp,cusp}}(f_N)V_N^{-1}\big).$$

However, it is conceivable that one could investigate the limit $I^r_{\mathrm{temp,cusp}}(f)$ without knowing the analytic continuation of the L-functions. The term $I_{\mathrm{temp,cusp}}(f_N)$ in this limit is part of the invariant trace formula for G. It is the sum over $t \geq 0$ of the tempered, cuspidal part of the term with $M = G$ in the t-discrete part $I_{t,\mathrm{disc}}(f_N)$. (We recall that the linear form $I_{t,\mathrm{disc}}$ is defined by a sum (21.19) over Levi subgroups M of G.) For each N, one can replace $I_{\mathrm{temp,cusp}}(f_N)$ by the complementary terms of the trace formula. Langlands' hope (referred to as a pipe dream in [**Lan15**]) is that the resulting limit might ultimately be shown to exist, through an analysis of these complementary terms. The expression for the limit so obtained would then provide a formula for the putative sum (A.3).

In general, it will probably be necessary to work with the stable trace formula, rather than the invariant trace formula. This is quite appropriate, since we are assuming that G is quasisplit. The t-discrete part

$$S_{t,\mathrm{disc}}(f) = S^G_{t,\mathrm{disc}}(f)$$

of the stable trace formula, defined in Corollary 29.10, has a decomposition

$$S_{t,\mathrm{disc}}(f) = \sum_{c\in\mathcal{C}_{t,\mathrm{disc}}(G)} S_c(f), \qquad f\in\mathcal{H}(G),$$

into Hecke eigenspaces. The indices c here range over "t-discrete" equivalence classes of families

$$c^V = \{c_v :\ v\notin V\}, \qquad V\supset S_{\mathrm{ram}}(G),$$

of semisimple conjugacy classes in ${}^LG$ attached to unramified representations $\pi^V = \pi(c^V)$ of $G(\mathbb{A}^V)$. We recall that two such families are equivalent if they are equal for almost all v. The eigendistribution $S_c(f)$ is characterized by the property that

$$S_c(f_\phi) = S_c(f)\phi_G(c^V), \qquad \phi\in\mathcal{H}\big(G(\mathbb{A}^V),K^V\big),$$

where V is a large finite set of valuations depending on f, c^V is some representative of the equivalence class c, and

$$\phi_G(c^V) = \phi_G\big(\pi(c^V)\big).$$

For any f and t, the sum in c can be taken over a finite set. Let $\mathcal{C}_{\mathrm{temp,cusp}}(G)$ be the subset of classes in the union

$$\mathcal{C}_{\mathrm{disc}}(G) = \bigcup_{t\geq 0}\mathcal{C}_{t,\mathrm{disc}}(G)$$

that do not lie in the image of $\mathcal{C}_{\mathrm{disc}}(M)$ in $\mathcal{C}_{\mathrm{disc}}(G)$ for any $M\neq G$, and whose components c_v are bounded in $\widehat{G}$. The sum

$$S_{\mathrm{temp,cusp}}(f) = S^G_{\mathrm{temp,cusp}}(f) = \sum_{c\in\mathcal{C}_{\mathrm{temp,cusp}}(G)} S_c(f), \qquad f\in\mathcal{H}(G),$$

is then easily seen to be absolutely convergent.

The sum (A.3) and the limit $I^r_{\mathrm{cusp,temp}}(f)$ have obvious stable analogues. If the partial L-function

$$L(s,c^V,r) = L^V(s,\pi,r), \qquad \pi^V = \pi(c^V),$$

attached to a class $c \in \mathcal{C}_{\mathrm{temp,cusp}}(G)$ has analytic continuation, set $n(c,r)$ equal to $n(\pi,r)$. Then

$$n(c,r) = \lim_{N\to\infty}\Big(V_N^{-1}\sum_{\{v\notin V:\, q_v\le N\}}\mathrm{tr}\big(r(c_v)\big)\Big) = \lim_{N\to\infty}\big((\phi_N)_G(c^V)V_N^{-1}\big).$$

The notation here reflects the fact that the limit is independent of both V and the representative c^V of c. If $L(s,c^V,r)$ has analytic continuation for every c, the sum

$$\sum_{c\in\mathcal{C}_{\mathrm{temp,cusp}}(G)} n(c,r)S_c(f) \tag{A.4}$$

equals the limit

$$S^r_{\mathrm{temp,cusp}}(f) = \lim_{N\to\infty}\big(S_{\mathrm{temp,cusp}}(f_N)V_N^{-1}\big).$$

The remarks for $I^r_{\mathrm{temp,cusp}}(f)$ above apply again to the limit $S^r_{\mathrm{temp,cusp}}(f)$ here. Namely, it might be possible to investigate this limit without knowing the analytic continuation of the L-functions. Since $S_{\mathrm{temp,cusp}}(f_N)$ is part of the stable trace formula for G, we could replace it by the complementary terms in the formula. The ultimate goal would be to show that the limit exists, and that it has an explicit expression given by these complementary terms.

An important step along the way would be to deal with the complementary terms attached to nontempered classes c. These terms represent contributions to $S_{t,\mathrm{disc}}(f_N)$ from nontempered representations of $G(\mathbb{A})$ that occur in the discrete spectrum. The conjectural classification in [**A17**] suggests that they can be expressed in terms of groups G_ψ of dimension smaller than G. One can imagine that the total contribution of a group $H = G_\psi$ might take the form of a sum

$$\Big(\sum_{\{\psi:G_\psi=H\}}\widehat{S}^H_\psi(f_N^\psi)\Big)V_N^{-1}, \tag{A.5}$$

where S^H_ψ is a component of the linear form $S^H_{\mathrm{temp,cusp}}$ on $\mathcal{H}(H)$, and $f_N \to f_N^\psi$ is a transform from $\mathcal{H}(G)$ to $S\mathcal{I}(H)$. For example, the one-dimensional automorphic representations χ of $G(\mathbb{A})$ are represented by parameters

$$\psi:\ \Gamma_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

in which $\psi\left(1,\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$ is a principal unipotent element in $\widehat{G}$. In this case, $H = G_\psi$ is the co-center of G, and

$$\widehat{S}^H_\psi(f_N^\psi) = \int_{G(\mathbb{A})} f_N(x)\chi(x)\mathrm{d}x.$$

In general, the transform f_N^ψ would be defined by nontempered stable characters, and the contribution (A.5) of G_ψ will not have a limit in N. One would have to combine the sum over H of these contributions with the sum obtained from the remaining terms in the stable trace formula. More precisely, one would need to show that the difference of the two sums does have a limit in N, for which there is an explicit expression. In the process, one could try to establish the global conjectures in [**A17**], in the more exotic cases where endoscopy gives only partial information.

This is a tall order indeed. The most optimistic prediction might be that the program can be carried out with a great deal of work by many mathematicians over a long period of time! However, the potential rewards seem to justify any amount of effort. A successful resolution to the questions raised so far would be spectacular. It would give a complicated, but presumably quite explicit, formula for the linear form $S^r_{\mathrm{temp,cusp}}(f)$ in terms that are primarily geometric. The result would be a stable trace formula for the tempered, cuspidal automorphic representations π of G such that $L(s,\pi,r)$ has a pole at $s=1$.

The lesson we have learned from earlier applications is that a complicated trace formula is more useful when it can be compared with something else. The case at hand should be no different. One could imagine that for any L-embedding ρ as in (A.1), there might be a mapping $f \to f^{r,\rho}$ from $\mathcal{H}(G)$ to $S\mathcal{I}(G')$ by which one could detect functorial contributions of ρ to $S^r_{\mathrm{temp,cusp}}(f)$. The mapping might perhaps be defined locally. It should certainly vanish unless ρ is unramified outside of V, for any finite set V such that f lies in the subspace $\mathcal{H}(G_V)$ of $\mathcal{H}(G)$. We would include only those ρ that are *elliptic*, in the sense that their image is contained in no proper parabolic subgroup of ${}^LG$.

From the theory of endoscopy, we know that we have to treat a somewhat larger class of embeddings ρ. We consider the set of $\widehat{G}$-orbits of elliptic L-embeddings

$$\rho:\ \mathcal{G}' \longrightarrow {}^LG, \tag{A.1*}$$

where $\mathcal{G}'$ is an extension

$$1 \longrightarrow \widehat{G}' \longrightarrow \mathcal{G}' \longrightarrow W_F \longrightarrow 1$$

for which there is an L-embedding $\mathcal{G}' \hookrightarrow {}^L\widetilde{G}'$. It is assumed that $\widehat{G}'$ is the L-group of a quasisplit group G', and that $\widetilde{G}' \to G'$ is a z-extension of quasisplit groups. For each such ρ, we suppose that there is a mapping $f \to f^{r,\rho}$ from $\mathcal{H}(G)$ to $S\mathcal{I}(\widetilde{G}',\widetilde{\eta}')$, for the appropriate character $\widetilde{\eta}'$ on the kernel of the projection $\widetilde{G}' \to G'$, which vanishes unless ρ is unramified outside of V. One might hope ultimately to establish an identity

$$S^r_{\mathrm{temp,cusp}}(f) = \sum_{\rho} \sigma(r,\rho)\widehat{S}'_{\mathrm{prim}}(f^{r,\rho}), \tag{A.6}$$

where ρ ranges over classes of elliptic L-embeddings (A.1)*, $\sigma(r,\rho)$ are global coefficients determined by r and ρ, and S'_{prim} is a stable linear form on $\mathcal{H}(\widetilde{G}',\widetilde{\eta}')$ that depends only on $\widetilde{G}'$ and $\widetilde{\eta}'$. In fact, S'_{prim} should be defined by a stable sum of the tempered, cuspidal, automorphic representations $\pi' \in \Pi_{\mathrm{temp}}(\widetilde{G}',\widetilde{\eta}')$ such that for any finite dimensional representation r' of ${}^L\widetilde{G}'$, the order of the pole of $L(s,\pi',r')$ at $s=1$ equals the multiplicity of the trivial representation of ${}^L\widetilde{G}'$ in r'. For each G', one would try to construct a trace formula for S'_{prim} inductively from the formulas for the analogues for $\widetilde{G}'$ of the linear forms $S^r_{\mathrm{temp,cusp}}$. The goal would be to compare the contribution of these formulas to the right hand side of (A.6) with the formula one hopes to obtain for the left hand side. If one could show that the two primarily geometric expressions cancel, one would obtain an identity (A.6).

A formula (A.6) for any G would presumably lead to the general principle of functoriality. Functoriality in turn implies the analytic continuation of the L-functions $L(s,\pi,r)$ (for cuspidal automorphic representations π) and Ramanujan's conjecture (for those cuspidal automorphic representations π not attached to the

$SL(2,\mathbb{C})$-parameters of [**A17**]). Both of these implications were drawn in Langlands' original paper [**Lan3**]. It is interesting to note that Langlands' ideas are based on the intuition gained from the analytic continuation and the Ramanujan conjecture. However, his strategy is to bypass these two conjectures, leaving them to be deduced from the principle of functoriality one hopes eventually to establish.

The existence of a formula (A.6) would actually imply something beyond functoriality. Let $\Pi_{\mathrm{prim}}(G)$ be the set of tempered, cuspidal, automorphic representations of G that are *primitive*, in the sense that they are not functorial images of representations $\pi' \in \Pi(\widetilde{G}', \widetilde{\eta}')$, for any L-embedding $(\mathrm{A}.1)^*$ with proper image in ${}^LG$. An identity of the form (A.6) implies that if $\pi \in \Pi_{\mathrm{prim}}(G)$, and r is any finite dimensional representation of ${}^LG$, the order of the pole of $L(s,\pi,r)$ at $s = 1$ equals the multiplicity of the trivial representation of ${}^LG$ in r. This condition represents a kind of converse to functoriality. It implies that any tempered, cuspidal, automorphic representation π of G is a functional image under some ρ of a representation π' in the associated set $\Pi_{\mathrm{prim}}(\widetilde{G}', \widetilde{\eta}')$. The condition is closely related to the existence of the automorphic Langlands group L_F. If it fails, the strategy for attacking the functoriality we have described would seem also to fail.

All of this is implicit in Langlands' paper [**Lan13**], if I have understood it correctly. Langlands is particularly concerned with the case that $G = PGL(2)$, a group for which the stable trace formula is the same as the invariant trace formula, and r is the irreducible representation of $\widehat{G} = SL(2,\mathbb{C})$ of dimension $(m+1)$. In this case, an elliptic homomorphism ρ will be of dihedral, tetrahedral, octahedral, or icosahedral type. For each of the last three types, the image of ρ is actually finite. The poles that any of these three types would contribute to L-functions $L(s,\pi,r)$ are quite sparse. (See [**Lan13**, p. 24].) For example, to detect the contribution of an icosahedral homomorphism ρ, one would have to take a 12-dimensional representation r. For a representation of $\widehat{G}$ of this size, there will be many terms in the putative limit $I^r_{\mathrm{temp,cusp}}(f) = S^r_{\mathrm{temp,cusp}}(f)$ that overwhelm the expected contribution of ρ. The analytic techniques required to rule out such terms are well beyond anything that is presently understood. Techniques that can be applied to smaller representations r are discussed in [**Lan13**] and [**Lan15**], and also in the letter [**Sar**] of Sarnak.

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