CHAPTER 4

AUTOMORPHIC REPRESENTATIONS OF GSp(4)

By JAMES ARTHUR

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1. In this note, we shall describe a classification for automorphic representations of GSp(4), the group of similitudes of four-dimensional symplectic space. The results are part of a project [A3] on the automorphic representations of general classical groups. The monograph [A3] is still in preparation. When complete, it will contain a larger classification of representations, subject to a general condition on the fundamental lemma.

In the case of GSp(4), the standard fundamental lemma for invariant orbital integrals has been established [Ha], [W]. However, a natural variant of the standard fundamental lemma is also needed. To be specific, the theorem we will announce here is contingent upon a fundamental lemma for twisted, weighted, orbital integrals on the group $\mathrm{GL}(4) \times \mathrm{GL}(1)$ (relative to a certain outer automorphism). This has not been established. However, it seems likely that by methods of descent, perhaps in combination with other means, one could reduce the problem to known cases of the standard fundamental lemma. (The papers [BWW], [F], and [Sc] apply such methods to the twisted analogue of the fundamental lemma, but not its generalization to weighted orbital integrals.)

The general results of [A3] are proved by a comparison of spectral terms in the stabilized trace formula. It is for the existence of the stabilized trace formula [A2] (and its twisted analogues) that the fundamental lemma is required. However, any discussion of such methods would be outside the scope of this paper. We shall be content simply to state the classification for GSp(4) in reasonably elementary terms. The paper will in fact be somewhat expository. We shall try to motivate the classification by examining the relevant mappings from a Galois group (or some extension thereof) to the appropriate L-groups.

Representations of the group GSp(4) have been widely studied. The papers [HP], [Ku], [Y], [So], and [Ro] contain results that were established directly for GSp(4). Results for groups of higher rank in [CKPS] and [GRS] could also be applied (either now or in the near future) to the special case of GSp(4).

2. Let F be a local or global field of characteristic zero. If N is any positive integer, the general linear group $\mathrm{GL}(N)$ has an outer automorphism

$$g \to g^\vee = {}^t g^{-1}, \qquad g \in \mathrm{GL}(N),$$

over F. Standard classical groups arise as fixed point groups of automorphisms in the associated inner class. In this paper, we shall be concerned with classical groups of similitudes. We therefore take the slightly larger group

$$\widetilde{G} = \mathrm{GL}(N) \times \mathrm{GL}(1)$$

over F, equipped with the outer automorphism

$$\alpha : (x, y) \to (x^\vee, \det(x)y), \qquad x \in \mathrm{GL}(N), \qquad y \in \mathrm{GL}(1).$$

The corresponding complex dual group

$$\widehat{\widetilde{G}} = \mathrm{GL}(N, \mathbb{C}) \times \mathbb{C}^*$$

comes with the dual outer automorphism

$$\widehat{\alpha} : (g, z) \to (g^\vee z, z), \qquad g \in \mathrm{GL}(N, \mathbb{C}), \qquad z \in \mathbb{C}^*.$$

Motivated by Langlands's conjectural parametrization of representations, we consider homomorphisms

$$\tilde{\psi} : \Gamma_F \to \widehat{\widetilde{G}},$$

from the Galois group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ into $\widehat{\widetilde{G}}$. Each $\tilde{\psi}$ is required to be continuous, which is to say that it factors through a finite quotient $\Gamma_{E/F} = \mathrm{Gal}(E/F)$ of Γ_F, and is to be taken up to conjugacy in $\widehat{\widetilde{G}}$. Any $\tilde{\psi}$ may therefore be decomposed according to the representation theory of finite groups. We first write

$$\tilde{\psi} = \psi \oplus \chi : \sigma \to \psi(\sigma) \oplus \chi(\sigma), \qquad \sigma \in \Gamma_F,$$

where ψ is a (continuous) N-dimensional representation of Γ_F, and χ is a (continuous) 1-dimensional character on Γ_F. We then break ψ into a direct sum

$$\psi = \ell_1 \psi_1 \oplus \cdots \oplus \ell_r \psi_r,$$

for inequivalent irreducible representations

$$\psi_i : \Gamma_F \to \mathrm{GL}(N_i, \mathbb{C}), \qquad 1 \leq i \leq r,$$

and multiplicities ℓ_i such that

$$N = \ell_1 N_1 + \cdots + \ell_r N_r.$$

We shall be interested in maps $\tilde{\psi}$ that are *$\widehat{\alpha}$-stable*, in the sense that the homomorphism $\widehat{\alpha} \circ \tilde{\psi}$ is conjugate to $\tilde{\psi}$. It is clear that $\tilde{\psi}$ is $\widehat{\alpha}$-stable if and only if the N-dimensional representation

$$\psi^\vee \otimes \chi : \sigma \to \psi(\sigma)^\vee \chi(\sigma), \qquad \sigma \in \Gamma_F,$$

is equivalent to ψ. This in turn is true if and only if there is an involution $i \leftrightarrow i^\vee$ on the indices such that for any i, the representation $\psi_i^\vee \otimes \chi$ is equivalent to $\psi_{i^\vee}$, and ℓ_i equals $\ell_{i^\vee}$. We shall say that $\tilde{\psi}$ is *$\widehat{\alpha}$-discrete* if it satisfies the further constraint that for each i, $i^\vee = i$ and $\ell_i = 1$.

Suppose that $\tilde{\psi}$ is $\widehat{\alpha}$-discrete. Then

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$

for distinct irreducible representations ψ_i of degree N_i that are χ-self dual, in the sense that ψ_i is equivalent to $\psi_i^\vee \otimes \chi$. We write

$$\psi_i(\sigma)^\vee \chi(\sigma) = A_i^{-1}\psi_i(\sigma)A_i, \qquad \sigma \in \Gamma_F, \qquad 1 \leq i \leq r,$$

for fixed intertwining operators $A_i \in \mathrm{GL}(N_i, \mathbb{C})$. Applying the automorphism $g \to g^\vee$ to each side of the last equation, we deduce from Schur's lemma that

$${}^t A_i = c_i A_i,$$

for some complex number c_i with $c_i^2 = 1$. The operator A_i can thus be identified with a bilinear form on $\mathbb{C}^n$ that is symmetric if $c_i = 1$ and skew-symmetric if $c_i = -1$. We are of course free to replace any ψ_i by a conjugate

$$B_i^{-1}\psi_i(w)B_i, \qquad B_i \in \mathrm{GL}(N_i, \mathbb{C}).$$

This has the effect of replacing A_i by the matrix

$$B_i A_i {}^t B_i.$$

We can therefore assume that the intertwining operator takes a standard form

$$A_i = \begin{pmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & 0 \end{pmatrix}, \qquad \text{if } c_i = 1,$$

and

$$A_i = \begin{pmatrix} 0 & & & & & 1 \\ & & & & -1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ & 1 & & & & \\ -1 & & & & & 0 \end{pmatrix}, \qquad \text{if } c_i = -1,$$

We shall say that ψ_i is *orthogonal* or *symplectic* according to whether c_i equals 1 or -1.

We have shown that the image of the homomorphism $\tilde{\psi}_i = \psi_i \oplus \chi$ is contained in the subgroup

$$\{(g, z) \in \mathrm{GL}(N_i, \mathbb{C}) \times \mathbb{C}^* : \ g^\vee z = A_i^{-1} g A_i\}$$

of $\mathrm{GL}(N_i, \mathbb{C}) \times \mathbb{C}^*$. If (g, z) belongs to this subgroup, z is the image

$$g \to z = \Lambda(g)$$

of g under a rational character Λ. The subgroup of $\mathrm{GL}(N_i, \mathbb{C}) \times \mathbb{C}^*$ therefore projects isomorphically onto the subgroup

$$\{g \in \mathrm{GL}(N_i, \mathbb{C}) : A_i{}^t g A_i^{-1} g = \Lambda(g) I\}$$

of $\mathrm{GL}(N_i, \mathbb{C})$. This subgroup is, by definition, the group $\mathrm{GO}(N_i, \mathbb{C})$ of orthogonal similitudes if $c_i = 1$, and the group $\mathrm{GSp}(N_i, \mathbb{C})$ of symplectic similitudes if $c_i = -1$. In each case, the rational character Λ is called the *similitude character* of the group. Set

$$N_\pm = \sum\nolimits_{i \in I_\pm} N_i, \qquad I_\pm = \{i : \ c_i = \pm 1\}.$$

We then obtain a decomposition

$$\psi = \psi_+ \oplus \psi_-,$$

where ψ_+ takes values in a subgroup of $\mathrm{GL}(N_+, \mathbb{C})$ that is isomorphic to $\mathrm{GO}(N_+, \mathbb{C})$, while ψ_- takes values in a subgroup of $\mathrm{GL}(N_-, \mathbb{C})$ that is isomorphic to $\mathrm{GSp}(N_-, \mathbb{C})$. The original representation takes a form

$$\tilde{\psi} = \psi_+ \oplus \psi_- \oplus \chi,$$

in which the two similitude characters satisfy

$$\Lambda(\psi_+(\sigma)) = \Lambda(\psi_-(\sigma)) = \chi(\sigma), \qquad \sigma \in \Gamma_F.$$

The complex group $\mathrm{GSp}(N_-, \mathbb{C})$ is connected. It is therefore isomorphic to the complex dual group $\widehat{G}_-$ of a split group G_- over F. If N_+ is odd, $\mathrm{GO}(N_+, \mathbb{C})$ is also connected. It is again isomorphic to a complex dual group $\widehat{G}_+$, for a split group G_+ over F. If $N_+ = 2n_+$ is even, however, the mapping

$$\nu : g \to \Lambda(g)^{-n_+} \det(g), \qquad g \in \mathrm{GO}(N_+, \mathbb{C}),$$

is a nontrivial homomorphism from $\mathrm{GO}(N_+, \mathbb{C})$ to the group $\{\pm 1\}$, whose kernel $\mathrm{SGO}(N_+, \mathbb{C})$ is connected. (See [Ra, §2].) The composition of ψ_+ with ν then provides a homomorphism from Γ_F to a group of outer automorphisms of $\mathrm{SGO}(N_+, \mathbb{C})$. In this case, we take G_+ to be the quasisplit group over F whose dual group is isomorphic to the group $\mathrm{SGO}(N_+, \mathbb{C})$, equipped with the given action of Γ_F. Having defined G_+ and G_- in all cases, we write G for the quotient of $G_+ \times G_-$ whose dual group is isomorphic to the subgroup

$$\widehat{G} = \{g_+, g_-, z) \in \widehat{G}_+ \times \widehat{G}_- \times \mathbb{C}^* : \ \Lambda(g_+) = \Lambda(g_-) = z\}$$

of $\widehat{G}_+ \times \widehat{G}_- \times \mathbb{C}^*$.

The quasisplit groups G over F, obtained from $\widehat{\alpha}$-discrete homomorphisms $\tilde{\psi}$ as above, are called the (elliptic, α-twisted) endoscopic groups for $\widetilde{G}$. Any such G is determined up to isomorphism by a partition $N = N_+ + N_-$, and an extension E of F of degree at most two (with $E = F$ unless N_+ is even). One sees easily that there is a natural L-embedding

$$^L G = \widehat{G} \rtimes \Gamma_F \hookrightarrow {}^L\widetilde{G} = (\mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^*) \times \Gamma_F$$

of L-groups. Given the $\widehat{\alpha}$-discrete parameter $\tilde{\psi}$, we conclude that the mapping $\sigma \to \tilde{\psi}(\sigma) \times \sigma$ from Γ_F to ${}^L\widetilde{G}$ factors through a subgroup ${}^L G$, for a unique endoscopic group G. The discussion above can also be carried out for $\widehat{\alpha}$-stable maps $\tilde{\psi}$ that are not $\widehat{\alpha}$-discrete. In this case, however, the mapping $\sigma \to \tilde{\psi}(\sigma) \times \sigma$ could factor through several subgroups ${}^L G$ of ${}^L\widetilde{G}$.

The classification of $\widehat{\alpha}$-discrete maps $\tilde{\psi}$ has been a simple exercise in elementary representation theory. The group Γ_F plays no special role, apart from the property that its quotients of order two parametrize quadratic extensions of F. The discussion would still make sense if Γ_F were replaced by a product of the group $\mathrm{SL}(2, \mathbb{C})$ with the Weil group W_F, or more generally, the Langlands group L_F of F. We recall [Ko, §12] that L_F equals W_F in the case that F is local archimedean, and equals the product of W_F with the group SU(2) if F is local nonarchimedean. If F is global, L_F is a hypothetical group, which is believed to be an extension of W_F by a product of compact, semisimple, simply connected groups. We assume its existence in what follows. Then in all cases, L_F comes with a projection $w \to \sigma(w)$ onto a dense subgroup of Γ_F.

Having granted the existence of L_F, we consider continuous homomorphisms

$$\tilde{\psi} = \psi \oplus \chi : L_F \times \mathrm{SL}(2, \mathbb{C}) \to \widehat{\widetilde{G}} = \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^*.$$

In this context, we also impose the condition that the restriction of $\tilde{\psi}$ to L_F be unitary, or equivalently, that the image of L_F in $\widehat{\widetilde{G}}$ be relatively compact. Assume that $\tilde{\psi}$ is $\widehat{\alpha}$-stable and $\widehat{\alpha}$-discrete. The discussion above then carries over verbatim. We obtain a decomposition

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$

for distinct irreducible representations

$$\psi_i : L_F \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(N_i, \mathbb{C}),$$

such that ψ_i is equivalent to $\psi_i^\vee \otimes \chi$. We shall again say that ψ_i is *symplectic* or *orthogonal*, according to whether its image is contained in the subgroup $\mathrm{GSp}(N_i, \mathbb{C})$ or $\mathrm{GO}(N_i, \mathbb{C})$ of $\mathrm{GL}(N_i, \mathbb{C})$. Combining the symplectic and orthogonal components as before, we see that $\tilde{\psi}$ factors through a subgroup ${}^L G$ of ${}^L\widetilde{G}$, for a unique (elliptic, α-twisted) endoscopic group G for $\widetilde{G}$.

We are working with a product $L_F \times \mathrm{SL}(2, \mathbb{C})$, in place of the original group Γ_F. This means that the irreducible components of ψ decompose into tensor products

$$\psi_i = \mu_i \otimes \nu_i, \qquad 1 \leq i \leq r,$$

for irreducible representations $\mu_i \colon L_F \to \mathrm{GL}(m_i, \mathbb{C})$ and $\nu_i \colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(n_i, \mathbb{C})$ such that $N_i = m_i n_i$. Any irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ is automatically self dual. This means that for any i, μ_i is equivalent to $\mu_i{}^\vee \otimes \chi$. Moreover, the representation ν_i of $\mathrm{SL}(2, \mathbb{C})$ is symplectic or orthogonal according to whether it is even or odd dimensional. It follows that ψ_i is symplectic if and only if either μ_i is symplectic and ν_i is odd dimensional, or μ_i is orthogonal and ν_i is even dimensional.

3. We have classified the $\widehat{\alpha}$-discrete representations of the group $L_F \times \mathrm{SL}(2, \mathbb{C})$ in order to motivate a classification of symplectic automorphic representations. At this point, we may as well specialize to the case that ψ is purely symplectic, which is to say that the corresponding group $\widehat{G}$ is purely symplectic. We assume henceforth that N is even, and that G is the split group over F such that $\widehat{G}$ is isomorphic to $\mathrm{GSp}(N, \mathbb{C})$. Then G is isomorphic to the general spin group

$$\mathrm{GSpin}(N+1) = (\mathrm{Spin}(N+1) \times \mathbb{C}^*)/\{\pm 1\}$$

over F. Our ultimate concern will in fact be the special case that N equals 4. In this case, there is an exceptional isomorphism between $\mathrm{GSpin}(N+1)$ and $\mathrm{GSp}(N)$, so that G is isomorphic to the group $\mathrm{GSp}(4)$ of the title.

For the given group $G \cong \mathrm{GSpin}(N+1)$, we write $\Psi(G) = \Psi(G/F)$ for the set of continuous homomorphisms ψ from $L_F \times \mathrm{SL}(2, \mathbb{C})$ to $\widehat{G}$, taken up to conjugacy in $\widehat{G}$, such that the image of L_F is relatively compact. For any such ψ, we set

$$\chi(w) = \Lambda(\psi(w, u)), \qquad w \in L_F, \qquad u \in \mathrm{SL}(2, \mathbb{C}),$$

where $\Lambda\colon\ \widehat{G} \to \mathbb{C}^*$ is the similitude character on $\widehat{G}$. Then $\chi = \chi_\psi$ is a one-dimensional unitary character on L_F. The correspondence $\psi \to \tilde{\psi} = \psi \oplus \chi_\psi$ gives a bijection from $\Psi(G)$ to the subset of (equivalence classes of) $\widehat{\alpha}$-stable representations $\tilde{\psi}$ such that the mapping

$$(w, u) \to \tilde{\psi}(w, u) \times \sigma(w), \qquad w \in L_F, \qquad u \in \mathrm{SL}(2, \mathbb{C}),$$

factors through the subgroup ${}^L G$ of ${}^L \widetilde{G}$. We shall write $\Psi_2(G)$ for the subset of elements $\psi \in \Psi(G)$ such that $\tilde{\psi}$ is $\widehat{\alpha}$-discrete. For any unitary 1-dimensional character χ on L_F, we also write $\Psi(G, \chi)$ and $\Psi_2(G, \chi)$ for the subsets of elements ψ in $\Psi(G)$ and $\Psi_2(G)$, respectively, such that $\chi_\psi = \chi$.

If ψ belongs to $\Psi(G)$, we set

$$S_\psi = \mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\psi)) = \{s \in \widehat{G}\colon\ s\psi(w, u) = \psi(w, u)s, (w, u) \in L_F \times \mathrm{SL}(2, \mathbb{C})\},$$

and also

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G}),$$

where $Z(\widehat{G}) \cong \mathbb{C}^*$ is the center of $\widehat{G}$. Then ψ belongs to $\Psi_2(G)$ if and only if the connected group S_ψ^0 equals $Z(\widehat{G})$, which is to say that the group S_ψ is finite modulo $Z(\widehat{G})$. It is not hard to compute S_ψ directly in terms of the irreducible components ψ_i of ψ. For example, if ψ belongs to the subset $\Psi_2(G)$, and has r components, then S_ψ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1} \times \mathbb{C}^*$, while $\mathcal{S}_\psi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1}$.

We assume from now on that F is global. We write V_F for the set of valuations of F, and $V_{F,\infty}$ for the finite subset of archimedean valuations in V_F. The Langlands group L_F is supposed to come with an embedding $L_{F_v} \hookrightarrow L_F$ for each $v \in V_F$. This embedding is determined up to conjugacy in L_F, and extends the usual conjugacy classes of embeddings $W_{F_v} \hookrightarrow W_F$ and $\Gamma_{F_v} \hookrightarrow \Gamma_F$. It gives rise to a restriction mapping

$$\psi \to \psi_v = \psi|_{L_{F_v} \times \mathrm{SL}(2,\mathbb{C})}, \qquad \psi \in \Psi(G),$$

from $\Psi(G) = \Psi(G/F)$ to $\Psi(G/F_v)$, which in turn provides an injection $S_{\psi_v} \to S_\psi$, and a homomorphism $\mathcal{S}_{\psi_v} \to \mathcal{S}_\psi$. Consider the special case that ψ is unramified at v. This means that v lies in the complement of $V_{F,\infty}$, and that for each $u \in \mathrm{SL}(2,\mathbb{C})$, the function

$$w_v \to \psi_v(w_v, u), \qquad w_v \in L_{F_v},$$

depends only on the image of w_v in the quotient

$$W_{F_v}/I_{F_v} \cong F_v^*/\mathcal{O}_v^*$$

of $W_{F_v} = L_{F_v}$. Following standard notation, we write ϖ_v for a fixed uniformizing element in F_v^*. Then ϖ_v maps to a generator of the cyclic group W_{F_v}/I_{F_v}, and can also be mapped to the element

$$\begin{pmatrix} |\varpi_v|^{\frac{1}{2}} & 0 \\ 0 & |\varpi_v|^{-\frac{1}{2}} \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{C})$. Composed with ψ_v, these mappings yield a semisimple conjugacy class

$$c_v(\psi) = c(\psi_v) = \psi_v\left(\varpi_v, \begin{pmatrix} |\varpi_v|^{\frac{1}{2}} & 0 \\ 0 & |\varpi_v|^{-\frac{1}{2}} \end{pmatrix}\right)$$

in $\widehat{G}$.

The Langlands group is assumed to have the property that its finite dimensional representations are unramified almost everywhere. It follows that any $\psi \in \Psi(G)$ determines a family

$$c(\psi) = \{c_v(\psi) = c(\psi_v) \colon \ v \notin V_\psi\}$$

of semisimple conjugacy classes in $\widehat{G}$, indexed by the complement of a finite subset $V_\psi \supset V_{F,\infty}$ of V_F. We note that if ψ belongs to a subset $\Psi_2(G,\chi)$ of $\Psi(G)$, the family of complex numbers $c(\chi) = \{c_v(\chi)\}$ is equal to the image $\Lambda(c(\psi)) = \{\Lambda(c_v(\psi))\}$ of $c(\psi)$ under the similitude character. In general the relationships among the different elements in any family $c(\psi)$ convey much of the arithmetic information that is wrapped up in the homomorphism ψ.

There is of course another source of semisimple conjugacy classes in $\widehat{G}$, namely automorphic representations. If π is an automorphic representation of G, the Frobenius–Hecke conjugacy classes provide a family

$$c(\pi) = \{c_v(\pi) = c(\pi_v) \colon \ v \notin V_\pi\}$$

of semisimple conjugacy classes in $\widehat{G}$, indexed by the complement of a finite subset $V_\pi \supset V_{F,\infty}$ of V_F. The elements in $c(\pi)$ are constructed in a simple way from the inducing data attached to unramified constituents π_v of π. (See [B], for example.) Now a one dimensional character χ of L_F amounts to an idèle class character of F, and this in turn can be identified with a character on the center of $G(\mathbb{A})$. Let $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}), \chi)$ be the space of χ-equivariant, square integrable functions on $G(F)\backslash G(\mathbb{A})$ that decompose discretely under the action of $G(\mathbb{A})$. We write $\Pi_2(G, \chi)$ for the set of equivalence classes of automorphic representations of G that are constituents of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}), \chi)$. If π belongs to $\Pi_2(G, \chi)$, the family $c(\chi)$ is equal to the image $\Lambda(c(\pi)) = \{\Lambda(c_v(\pi))\}$ of $c(\pi)$ under Λ. In general, the relationships among the different elements in any family $c(\pi)$ convey much of the arithmetic information that is wrapped up in the automorphic representation π.

The following conjecture was an outgrowth of Langlands's conjectural theory of endoscopy. We have stated it here somewhat informally. A more precise assertion, which applies to any group, is given in [A1] and [AG].

Conjecture. (i) For any ψ, there is a canonical mapping $\pi \to \psi$ from $\Pi_2(G, \chi)$ to $\Psi_2(G, \chi)$ such that

$$c(\pi) = c(\psi), \qquad \pi \in \Pi_2(G, \chi).$$

(ii) Any fiber of the mapping is of the form

$$\{\pi \in \Pi_2(G, \chi) \colon \ c(\pi) = c(\psi)\}, \qquad \psi \in \Psi_2(G, \chi),$$

and can be characterized explicitly in terms of the groups $\mathcal{S}_{\psi_v}$, the diagonal image of the map

$$\mathcal{S}_\psi \to \prod_v \mathcal{S}_{\psi_v},$$

and a character

$$\varepsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$$

attached to certain symplectic root numbers [A1, §8].

4. The conjecture describes a classification of the automorphic χ-discrete spectrum of G in terms of mappings

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r = (\mu_1 \otimes \nu_1) \oplus \cdots \oplus (\mu_r \otimes \nu_r)$$

in $\Psi_2(G, \chi)$. It is the simplest way to motivate what one might try to prove. The conjecture would actually be very difficult to establish in the form stated above (and in [A1] and [AG]). Indeed, one would first have to establish the existence and fundamental properties of the global Langlands group L_F. However, there is a natural way to reformulate the conjecture as a classification of automorphic representations of G in terms of those of general linear groups. The idea is to reinterpret the "semisimple" constituents μ_i of ψ.

We may as well keep the idèle class character χ fixed from this point on. In the formulation above, μ_i stands for an irreducible unitary representation of L_F of dimension m_i that is χ-self dual. The main hypothetical property of L_F is that its irreducible unitary representations of dimension m should be in canonical bijection with the unitary cuspidal automorphic representations of GL(m). We could therefore bypass L_F altogether by interpreting μ_i as an automorphic representation. From now on, μ_i will stand for a unitary, cuspidal automorphic representation of GL(m_i) that is χ-self dual, in the sense that the representation

$$x \to \mu_i(x^\vee)\chi(\det x), \qquad x \in \mathrm{GL}(m_i, \mathbb{A}),$$

is equivalent to μ_i. As an automorphic representation of GL(m_i), μ_i comes with a family

$$c(\mu_i) = \{c_v(\mu_i)\colon\ v \notin V_{\mu_i}\}$$

of Frobenius–Hecke conjugacy classes in $\mathrm{GL}(m_i, \mathbb{C})$. The family satisfies

$$c_v(\mu_i)^{-1} c_v(\chi) = c_v(\mu_i), \qquad v \notin V_{\mu_i},$$

since μ_i is χ-self dual, and it determines μ_i uniquely, by the theorem of strong multiplicity one.

As an example, we shall describe the classification of χ-self dual, unitary, cuspidal automorphic representations of GL(2).

Example. Suppose that E is a quadratic extension of F, and that θ is an idèle class character of E. We assume that θ is not fixed by $\mathrm{Gal}(E/F)$. Then there is a (unique) χ-self dual, unitary, cuspidal automorphic representation $\mu = \mu(\theta)$ of GL(2) such that

$$c(\mu) = \{c_v(\mu) = \rho(c_v(\theta))\colon\ v \notin V_\theta\},$$

where θ is regarded as an automorphic representation of the group $K_E = \mathrm{Res}_{E/F}(\mathrm{GL}(1))$, and ρ is the standard two dimensional representation of ${}^L K_E$. Conversely, suppose that μ is a χ-self dual, unitary, cuspidal automorphic representation of GL(2). We write χ_μ for the central character of μ. It follows from the definitions that $\chi_\mu^2 = \chi^2$, or in other words, that the idèle class character $\eta_\mu = \chi_\mu \chi^{-1}$ of F has order one or two. If $\eta_\mu \neq 1$, it is known that μ equals $\mu(\theta)$, for an idèle class character θ of the class field E of η_μ. In this case, we shall say that μ is of *orthogonal type*. If $\eta_\mu = 1$, μ is to be regarded as symplectic, for the obvious reason that $\mathrm{GL}(2) \cong \mathrm{GSp}(2)$. This is really the generic case, since if μ is *any* automorphic representation with central character χ_μ equal to χ, μ is automatically χ-self dual.

For the group GL(4), some χ-self dual, unitary, cuspidal automorphic representations can also be described in relatively simple terms. They are given by the following theorem of Ramakrishnan.

THEOREM: [Ra]. *Let E be an extension of F of degree at most two, and set*

$$H_E = \begin{cases} \mathrm{GL}(2) \times \mathrm{GL}(2), & \text{if } E = F, \\ \mathrm{Res}_{E/F}(\mathrm{GL}(2)), & \text{if } E \neq F. \end{cases}$$

Let ρ be the homomorphism from ${}^L H_E$ to the group $\mathrm{GL}(4, \mathbb{C}) \cong \mathrm{GL}(M_2(\mathbb{C}))$ *defined by setting*

$$\rho(g_1, g_2)X = g_1 X^t g_2, \qquad g_1, g_2 \in \mathrm{GL}(2, \mathbb{C}),$$

and

$$\rho(\sigma)X = \begin{cases} X, & \text{if } \sigma_E = 1, \\ {}^t X, & \text{if } \sigma_E \neq 1, \end{cases}$$

for any $X \in M_2(\mathbb{C})$, and for any $\sigma \in \Gamma_F$ with image σ_E in $\Gamma_{E/F}$. Suppose that τ is a unitary, cuspidal automorphic representation of H_E that is not a transfer from GL(2) (*relative to the natural embedding of* $\mathrm{GL}(2, \mathbb{C})$ *into ${}^L H_E$), and whose central character is the pullback of χ* (*relative to the natural mapping from the center of H_E to* GL(1)). *Then there is a unique χ-self dual, unitary, cuspidal automorphic representation μ of* GL(4) *such that*

$$c(\mu) = \{c_v(\mu) = \rho(c_v(\tau)) : \ v \notin V_\tau\}.$$

We shall say that a χ-self dual, unitary, cuspidal automorphic representation μ of GL(4) is of *orthogonal type* if it is given by the construction of Ramakrishnan's theorem. This is of course because μ is a transfer to GL(4) of a representation of the group GO(4).

Returning to our group $G \cong \mathrm{GSpin}(N + 1)$, with N even, we can try to construct objects ψ for G purely in terms of automorphic representations. We now define $\Psi_2(G, \chi)$ to be the set of formal (unordered) sums

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$$

of distinct, formal, χ-self dual tensor products

$$\psi_i = \mu_i \boxtimes \nu_i, \qquad 1 \le i \le r,$$

of symplectic type. More precisely, ν_i is an irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ of dimension m_i, and μ_i is a χ-self dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(m_i)$ that is of symplectic type if n_i is odd and orthogonal type if n_i is even, for integers m_i and n_i such that

$$N = N_1 + \cdots + N_r = m_1 n_1 + \cdots + m_r n_r.$$

To complete the definition, one would of course have to be able to say what it means for μ_i to be of symplectic or orthogonal type. In general, this must be done in terms of whether a certain automorphic L-function for μ_i (essentially the symmetric square or skew-symmetric square) has a pole at $s = 1$. The necessary consistency arguments for such a characterization are inductive, and require higher cases of the fundamental lemma, even when $N = 4$. However, if m_i equals either 2 or 4, we can give an ad hoc characterization. In these cases, we have already defined what it means for μ_i to be of orthogonal type. We declare μ_i to be of symplectic type simply if it is not of orthogonal type. This expedient allows us to construct the family $\Psi_2(G, \chi)$ in the case that $N = 4$.

Suppose that N is such that the set $\Psi_2(G, \chi)$ has been defined, and that

$$\psi = (\mu_1 \boxtimes \nu_1) \boxplus \cdots \boxplus (\mu_r \boxtimes \nu_r)$$

is an element in this set. For any i, μ_i comes with a family $c(\mu_i)$ of Frobenius–Hecke conjugacy classes in $\mathrm{GL}(m_i, \mathbb{C})$. The representation ν_i of $\mathrm{SL}(2, \mathbb{C})$ gives rise to its own family

$$c(\nu_i) = \left\{ c_v(\nu_i) = \nu_i \begin{pmatrix} |\varpi_v|^{\frac{1}{2}} & 0 \\ 0 & |\varpi_v|^{-\frac{1}{2}} \end{pmatrix} : \ v \notin V_{F,\infty} \right\}$$

of conjugacy classes in $\mathrm{GL}(n_i, \mathbb{C})$. The tensor product family

$$c(\psi_i) = \{ c_v(\psi_i) = c_v(\mu_i) \otimes c_v(\nu_i) : \ v \notin V_{\mu_i} \}$$

is then a family of semisimple conjugacy classes in the group $\mathrm{GL}(N_i, \mathbb{C}) = \mathrm{GL}(m_i n_i, \mathbb{C})$. Taking the direct sum over i, we obtain a family

$$c(\psi) = \bigoplus_{i=1}^{r} c(\psi_i) = \left\{ c_v(\psi) = \bigoplus_{i=1}^{r} c_v(\psi_i) : \ v \notin V_\psi \right\}$$

of semisimple conjugacy classes in $\mathrm{GL}(N, \mathbb{C})$. This is of course parallel to the family of conjugacy classes constructed with the earlier interpretation of ψ as a representation of $L_F \times \mathrm{SL}(2, \mathbb{C})$. We also set

$$\mathcal{S}_\psi = (\mathbb{Z}/2\mathbb{Z})^{r-1},$$

as before. We can then define a character

$$\varepsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$$

in terms of symplectic root numbers by copying the prescription in [A1, §8].

The local Langlands conjecture has now been proved for the general linear groups $\mathrm{GL}(m_i)$ [HT], [He]. We can therefore identify any local component

$$\psi_v = (\mu_{1,v} \boxtimes \nu_1) \boxplus \cdots \boxplus (\mu_{r,v} \boxtimes \nu_r)$$

of an element $\psi \in \Psi_2(G, \chi)$ with an N-dimensional representation of the group $L_{F_v} \times \mathrm{SL}(2, \mathbb{C})$. In the process of proving the classification theorem stated below, one shows that if μ_i is either symplectic or orthogonal (in the sense alluded to above), the same holds for the local components $\mu_{i,v}$ (as representations of L_{F_v}). It follows that ψ_v can be identified with a homomorphism from $L_{F_v} \times \mathrm{SL}(2, \mathbb{C})$ into $\widehat{G}$. In particular, we can define the groups S_{ψ_v} and $\mathcal{S}_{\psi_v}$ as before, in terms of the centralizer of the image of ψ_v. Moreover, there is a canonical homomorphism $s \to s_v$ from $\mathcal{S}_\psi$ to $\mathcal{S}_{\psi_v}$. The groups $\mathcal{S}_\psi$ and $\mathcal{S}_{\psi_v}$ are always abelian, and in fact are products of groups $\mathbb{Z}/2\mathbb{Z}$.

5. We shall now specialize to the case that $N = 4$. Thus, $\widehat{G}$ is isomorphic to $\mathrm{GSp}(4, \mathbb{C})$, and

$$G \cong \mathrm{GSpin}(5) \cong \mathrm{GSp}(4).$$

As we have noted, the set $\Psi_2(G, \chi)$ can be defined explicitly in this case in terms of certain cuspidal automorphic representations of general linear groups.

The object of this article has been to announce the following classification theorem for automorphic representations of G. The theorem is contingent upon cases of the fundamental lemma that are in principle within reach, and I should also admit, the general results in [A3] that have still to be written up in detail.

Classification Theorem. (i) *There exist canonical local packets*

$$\Pi_{\psi_v}, \qquad \psi \in \Psi_2(G, \chi), \qquad v \in V_F,$$

of (possibly reducible) representations of the groups $G(F_v)$, together with injections

$$\pi_v \to \xi_{\pi_v}, \qquad \pi_v \in \Pi_{\psi_v},$$

from these packets to the associated finite groups $\widehat{\mathcal{S}}_{\psi_v}$ of characters on $\mathcal{S}_{\psi_v}$.

(ii) *The automorphic discrete spectrum attached to χ has an explicit decomposition*

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}), \chi) = \bigoplus_{\psi \in \Psi_2(G,\chi)} \bigoplus_{\{\pi \in \prod_\psi :\ \xi_\pi = \varepsilon_\psi\}} \pi$$

in terms of the global packets

$$\Pi_\psi = \{\pi = \bigotimes_v \pi_v :\ \pi_v \in \Pi_{\psi_v}, \xi_{\pi_v} = 1 \textit{ for almost all } v\}$$

of (possibly reducible) representations of $G(\mathbb{A})$, and corresponding characters

$$\xi_\pi :\ s \to \prod_v \xi_{\pi_v}(s_v), \qquad s \in \mathcal{S}_\psi, \qquad \pi \in \Pi_\psi,$$

on the groups $\mathcal{S}_\psi$.

(iii) *The global packets*

$$\Pi_\psi, \qquad \psi \in \Pi_2(G, \chi),$$

are disjoint, in the sense that no irreducible representation of $G(\mathbb{A})$ is a constituent of representations in two distinct packets. Moreover, if ψ belongs to the subset $\Psi_{ss,2}(G,\chi)$ of elements in $\Psi_2(G,\chi)$ that are trivial on SL(2, $\mathbb{C}$)*, the packet Π_ψ contains only irreducible representations. Thus, for any $\psi \in \Psi_{ss,2}(G,\chi)$, any representation $\pi \in \Pi_\psi$ occurs in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}),\chi)$ with multiplicity* 1 *or* 0*.*

Remarks. 1. The local packets Π_{ψ_v} in part (i) are defined by the endoscopic transfer of characters. More precisely, the characters of representations in Π_{ψ_v} are defined in terms of Langlands–Shelstad (and Kottwitz–Shelstad) transfer mappings of functions, and the groups $\mathcal{S}_{\psi_v}$. I do not know whether the representations in Π_{ψ_v} are generally irreducible. However, in the case that ψ is unramified at v, one can at least show that the preimage of the trivial character in $\widehat{\mathcal{S}}_{\psi_v}$ under the mapping $\pi_v \to \xi_{\pi_v}$ is irreducible.

2. If ψ belongs to the complement of $\Psi_{ss,2}(G,\chi)$ in $\Psi_2(G,\chi)$, the representations in the packet Π_ψ are all nontempered. On the other hand, if ψ belongs to $\Psi_{ss,2}(G,\chi)$, the generalized Ramanujan conjecture (applied to the groups GL(2) and GL(4)) implies that the representations in the packet Π_ψ are tempered. Thus, the multiplicity assertion at the end of the theorem pertains to what ought to be the tempered constituents of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}),\chi)$. If ψ is a more general element in $\Psi_2(G,\chi)$, and if the direct sum of the representations in each local packet Π_{ψ_v} is multiplicity free, the irreducible constituents of the representations in Π_ψ also occur with multiplicity 1 or 0.

The multiplicity formula of the theorem is a quantitative description of the decomposition of the discrete spectrum. The general structure of the parameters ψ also provides useful qualitative information about the spectrum. We shall conclude with a list of the six general families of automorphic representations that occur in the discrete spectrum. In each case, we shall describe the relevant parameters ψ, the corresponding families of Frobenius–Hecke conjugacy classes, the groups $\mathcal{S}_\psi$, and the sign characters ε_ψ on $\mathcal{S}_\psi$. (The characters ε_ψ are in fact trivial for all but one of the six families.) We shall write $\nu(n)$ for the irreducible representation of SL(2, $\mathbb{C}$) of dimension n. Observe that the Frobenius–Hecke conjugacy classes

$$c(\nu(n)) = \left\{ c_v(\nu(n)) = \begin{pmatrix} |\varpi_v|^{\frac{n-1}{2}} & & & 0 \\ & |\varpi_v|^{\frac{n-3}{2}} & & \\ & & \ddots & \\ 0 & & & |\varpi_v|^{-\frac{n-1}{2}} \end{pmatrix} \right\}$$

of $\nu(n)$ have positive real eigenvalues. This is in contrast to the case of a unitary, cuspidal automorphic representation μ of GL(m), which according to the generalized

Ramanujan conjecture, has Frobenius–Hecke conjugacy classes

$$c(\mu) = \left\{ c_v(\mu) = \begin{pmatrix} c_v^1(\mu) & & 0 \\ & \ddots & \\ 0 & & c_v^m(\mu) \end{pmatrix} \right\}$$

whose eigenvalues lie on the unit circle.

We list the six families according to how they behave with respect to stability (for the multiplicities of representations $\pi \in \Pi_\psi$) and the implicit Jordan decomposition (for elements $\psi \in \Psi_2(G, \chi)$). I have also taken the liberty of assigning proper names to some of the families, which I hope give fair reflection of their history.

(a) Stable, semisimple (general type)

$$\psi = \psi_1 = \mu \boxtimes 1,$$

where μ is a χ-self dual, unitary cuspidal automorphic representation of GL(4) that is not of orthogonal type,

$$c(\psi) = c(\mu) = \left\{ \begin{pmatrix} c_v^1(\mu) & & 0 \\ & \ddots & \\ 0 & & c_v^4(\mu) \end{pmatrix} \right\},$$

$$\mathcal{S}_\psi = 1,$$

$$\varepsilon_\psi = 1.$$

(b) Unstable, semisimple (Yoshida type [Y])

$$\psi = \psi_1 \boxplus \psi_2 = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1),$$

where μ_1 and μ_2 are *distinct*, unitary, cuspidal automorphic representations of GL(2) whose central characters satisfy $\chi_{\mu_1} = \chi_{\mu_2} = \chi$,

$$c(\psi) = c(\mu_1) \oplus c(\mu_2) = \left\{ \begin{pmatrix} c_v^1(\mu_1) & & & 0 \\ & c_v^1(\mu_2) & & \\ & & c_v^2(\mu_2) & \\ & 0 & & c_v^2(\mu_1) \end{pmatrix} \right\},$$

$$\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z},$$

$$\varepsilon_\psi = 1.$$

(c) Stable, mixed (Soudry type [So])

$$\psi = \psi_1 = \mu \boxtimes \nu(2),$$

where $\mu = \mu(\theta)$ is a unitary, cuspidal automorphic representation of GL(2) of orthogonal type with $\chi_\mu^2 = \chi$,

$$\begin{aligned} c(\psi) &= c(\mu) \otimes c(\nu(2)) \\ &= \left\{ \begin{pmatrix} c_v^1(\mu)|\varpi_v|^{\frac{1}{2}} & & & 0 \\ & c_v^2(\mu)|\varpi_v|^{\frac{1}{2}} & & \\ & & c_v^1(\mu)|\varpi_v|^{-\frac{1}{2}} & \\ 0 & & & c_v^2(\mu)|\varpi_v|^{-\frac{1}{2}} \end{pmatrix} \right\}, \\ \mathcal{S}_\psi &= 1, \\ \varepsilon_\psi &= 1. \end{aligned}$$

(d) Unstable, mixed (Saito, Kurokawa type [Ku])

$$\psi = \psi_1 \boxplus \psi_2 = (\lambda \boxtimes \nu(2)) \boxplus (\mu \boxtimes 1),$$

where λ is an idèle class character of F and μ is a unitary, cuspidal automorphic representation of GL(2), with $\lambda^2 = \chi_\mu = \chi$,

$$\begin{aligned} c(\psi) &= (c(\lambda) \otimes c(\nu(2)) \oplus (c(\mu)) \\ &= \left\{ \begin{pmatrix} c_v(\lambda)|\varpi_v|^{\frac{1}{2}} & & & 0 \\ & c_v^1(\mu) & & \\ & & c_v^2(\mu) & \\ 0 & & & c_v(\lambda)|\varpi_v|^{-\frac{1}{2}} \end{pmatrix} \right\}, \\ \mathcal{S}_\psi &= \mathbb{Z}/2\mathbb{Z}, \\ \varepsilon_\psi &= \begin{cases} 1, & \text{if } \varepsilon\left(\frac{1}{2}, \mu \otimes \lambda^{-1}\right) = 1, \\ \mathrm{sgn}, & \text{if } \varepsilon\left(\frac{1}{2}, \mu \otimes \lambda^{-1}\right) = -1, \end{cases} \end{aligned}$$

where sgn is the nontrivial character on $\mathbb{Z}/2\mathbb{Z}$.

(e) Unstable, almost unipotent (Howe, Piatetski-Shapiro type [HP])

$$\psi = \psi_1 \boxplus \psi_2 = (\lambda_1 \boxtimes \nu(2)) \boxplus (\lambda_2 \boxtimes \nu(2)),$$

where λ_1 and λ_2 are distinct idèle class characters of F with $\lambda_1^2 = \lambda_2^2 = \chi$,

$$\begin{aligned} c(\psi) &= (c(\lambda_1) \otimes c(\nu(2))) \oplus (c(\lambda_2) \otimes c(\nu(2))) \\ &= \left\{ \begin{pmatrix} c_v(\lambda_1)|\varpi_v|^{\frac{1}{2}} & & & 0 \\ & c_v(\lambda_2)|\varpi_v|^{\frac{1}{2}} & & \\ & & c_v(\lambda_2)|\varpi_v|^{-\frac{1}{2}} & \\ 0 & & & c_v(\lambda_1)|\varpi_v|^{-\frac{1}{2}} \end{pmatrix} \right\}, \\ \mathcal{S}_\psi &= \mathbb{Z}/2\mathbb{Z}, \\ \varepsilon_\psi &= 1. \end{aligned}$$

(f) Stable, almost unipotent (one dimensional type)

$$\psi = \psi_1 = \lambda \boxtimes \nu(4),$$

where λ is an idèle class character of F with $\lambda^4 = \chi$,

$$\begin{aligned} c(\psi) &= c(\lambda) \otimes c(\nu(4)) \\ &= \left\{ \begin{pmatrix} c_v(\lambda)|\varpi_v|^{\frac{3}{2}} & & & 0 \\ & c_v(\lambda)|\varpi_v|^{\frac{1}{2}} & & \\ & & c_v(\lambda)|\varpi_v|^{-\frac{1}{2}} & \\ 0 & & & c_v(\lambda)|\varpi_v|^{-\frac{3}{2}} \end{pmatrix} \right\}, \\ \mathcal{S}_\psi &= 1, \\ \varepsilon_\psi &= 1. \end{aligned}$$

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