# Harmonic Analysis and Group Representations

*James Arthur*

Harmonic analysis can be interpreted broadly as a general principle that relates geometric objects and spectral objects. The two kinds of objects are sometimes related by explicit formulas, and sometimes simply by parallel theories. This principle runs throughout much of mathematics. The rather impressionistic table at the top of the opposite page provides illustrations from different areas.

The table gives me a pretext to say a word about the Langlands program. In very general terms, the Langlands program can be viewed as a series of far-reaching but quite precise conjectures, which describe relationships among two kinds of spectral objects—motives and automorphic representations—at the end of the table. Wiles's spectacular work on the Shimura-Taniyama-Weil conjecture, which established the proof of Fermat's Last Theorem, can be regarded as confirmation of such a relationship in the case of elliptic curves. In general, the arithmetic information wrapped up in motives comes from solutions of polynomial equations with rational coefficients. It would not seem to be amenable to any sort of classification. The analytic information from automorphic representations, on the other hand, is backed up by the rigid structure of Lie theory. The Langlands program represents a profound organizing scheme for fundamental arithmetic data in terms of highly structured analytic data.

I am going to devote most of this article to a short introduction to the work of Harish-Chandra. I have been motivated by the following three considerations.

(i) Harish-Chandra's monumental contributions to representation theory are the analytic foundation of the Langlands program. For many people, they are the most serious obstacle to being able to work on the many problems that arise from Langlands's conjectures.

(ii) The view of harmonic analysis introduced above, at least insofar as it pertains to group representations, was a cornerstone of Harish-Chandra's philosophy.

(iii) It is more than fifteen years since the death of Harish-Chandra. As the creation of one of the great mathematicians of our time, his work deserves to be much better known.

I shall spend most of the article discussing Harish-Chandra's ultimate solution of what he long regarded as the central problem of representation theory, the Plancherel formula for real groups. I shall then return briefly to the Langlands program, where I shall try to give a sense of the role played by Harish-Chandra's work.

## Representations

A *representation* of a group $G$ is a homomorphism

$$R : G \longrightarrow GL(V),$$

where $V = V_R$ is a complex vector space that one often takes to be a Hilbert space. We take for granted the notions of *irreducible*, *unitary*, *direct sum*, and *equivalence*, all applied to representations of a fixed group $G$. Representations of a finite group $G$ were studied by Frobenius, as a tool for investigating $G$. More recently, it was the representations themselves that became the primary objects of study. From this point of view, there are

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| | **Geometric objects** | **Spectral objects** |
|---|---|---|
| linear algebra | sum of diagonal entries of a square matrix | sum of eigenvalues of the matrix |
| finite groups | conjugacy classes | irreducible characters |
| topology | singular homology | deRham cohomology |
| differential geometry | lengths of closed geodesics | eigenvalues of the Laplacian |
| physics | particles (classical mechanics) | waves (quantum mechanics) |
| number theory | logarithms of powers of prime numbers | zeros of $\zeta(s)$ |
| algebraic geometry | algebraic cycles | motives |
| automorphic forms | rational conjugacy classes | automorphic representations |

**Table of Illustrations**

always two general problems to consider, for any given $G$.

1. Classify the set $\Pi(G)$ of equivalence classes of irreducible unitary representations of $G$.
2. If $R$ is some natural unitary representation of $G$, decompose $R$ explicitly into irreducible representations; that is, find a $G$-equivariant isomorphism $V_R \xrightarrow{\sim} V_{\hat{R}}$, where $V_{\hat{R}} = \hat{V}$ is a space built explicitly out of irreducible representations, as a direct sum

$$\bigoplus_{\pi \in \Pi(G)} n_\pi V_\pi \,, \qquad n_\pi \in \{0, 1, 2, \ldots, \infty\},$$

or possibly in some more general fashion.

***Example 1.*** $G = \mathbb{R}/\mathbb{Z}$, $V_R = L^2(\mathbb{R}/\mathbb{Z})$, and

$$\Big(R(y)f\Big)(x) \;=\; f(x+y), \qquad y \in G,\; f \in V_R.$$

This is the *regular representation* that underlies classical Fourier analysis. The set $\Pi(G)$ is parametrized by $\mathbb{Z}$ as follows:

$$\pi \in \Pi(G) \iff V_\pi = \mathbb{C},$$
$$\pi(y)v \;=\; e^{-2\pi i n y} v, \qquad v \in V_\pi,\; n \in \mathbb{Z}.$$

The space

$$\hat{V} \;=\; L^2(\mathbb{Z}) \;=\; \Big\{c = (c_n) :\; \Sigma |c_n|^2 < \infty\Big\}$$

supports a representation

$$\Big(\hat{R}(y)c\Big)_n \;=\; e^{2\pi i n y} c_n \,,$$

of $G$ that is a direct sum of all irreducible representations, each occurring with multiplicity one. The *Fourier coefficients*

$$f \;\longrightarrow\; \hat{f}_n \;=\; \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi i n x}\, dx \,,$$

then provide an isomorphism from $V$ to $\hat{V}$ that makes $R$ equivalent to $\hat{R}$. Moreover, this isomorphism satisfies the Plancherel formula

$$\int_{\mathbb{R}/\mathbb{Z}} |f(x)|^2\, dx \;=\; \sum_n |\hat{f}_n|^2 \,.$$

***Example 2.*** $G = \mathbb{R}$, $V_R = L^2(\mathbb{R})$, and

$$\Big(R(y)f\Big)(x) \;=\; f(x+y)\,, \qquad y \in G,\; f \in V_R.$$

In this case $\Pi(G)$ is parametrized by $\mathbb{R}$:

$$\pi \in \Pi(G) \iff V_\pi = \mathbb{C}\,,$$
$$\pi(y)v \;=\; e^{-i\lambda y} v\,, \qquad v \in V_\pi,\; \lambda \in \mathbb{R}.$$

Here we define $\hat{V} = L^2(\mathbb{R})$ and

$$\Big(\hat{R}(y)\phi\Big)(\lambda) = e^{i\lambda y}\phi(\lambda), \,, \qquad \phi \in \hat{V}, \lambda \in \mathbb{R}.$$

Then $\hat{V}$ is a "continuous direct sum", or *direct integral* of irreducible representations. The *Fourier transform*

$$f \;\longrightarrow\; \hat{f}(\lambda) \;=\; \int_{\mathbb{R}} f(x) e^{-i\lambda x}\, dx\,, \qquad f \in C_c^\infty(\mathbb{R}),$$

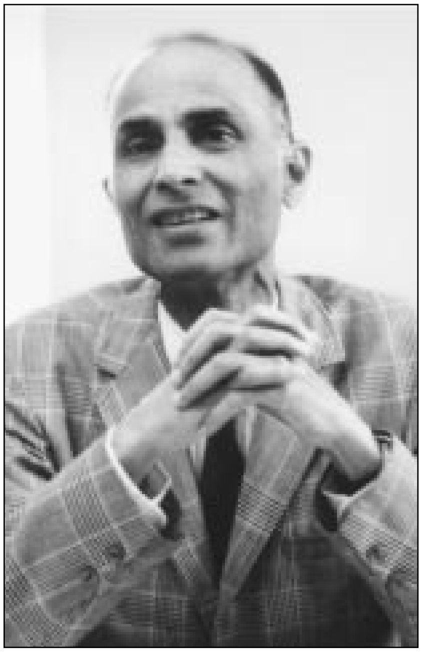
extends to an isomorphism from $V$ to $\hat{V}$ that satisfies the relevant Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2\, dx \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\lambda)|^2\, d\lambda \,.$$

These two examples were the starting point for a general theory of representations of locally compact abelian groups, which was established in the earlier part of the twentieth century. Attention then turned to the study of general nonabelian locally compact groups. Representations of nonabelian groups have the following new features.

(i) Representations $\pi \in \Pi(G)$ are typically infinite dimensional.

(ii) Decompositions of general representations $R$ typically have both a discrete part (like Fourier series) and a continuous part (like Fourier transforms).


Photograph by Herman Landshoff, courtesy of the Institute for Advanced Study.

**Harish-Chandra.**

## Problem of the Plancherel Formula

Against the prevailing opinion of the time, Harish-Chandra realized early in his career that a rich theory would require the study of a more restricted class of nonabelian groups. From the very beginning, he confined his attention to the class of *semisimple Lie groups*, or slightly more generally, *reductive Lie groups*. These include the general linear groups $GL(n, \mathbb{R})$, the special orthogonal groups $SO(p, q; \mathbb{R})$, the symplectic groups $Sp(2n, \mathbb{R})$, and the unitary groups $U(p, q; \mathbb{C})$. For the purposes of the present article, the reader can in fact take $G$ to be one of these familiar matrix groups.

Harish-Chandra's long-term goal became that of finding an explicit Plancherel formula for any such $G$. As in the two examples above, one takes $V_R = L^2(G)$, with respect to a fixed Haar measure on $G$. One can then take $R$ to be the 2-sided regular representation

$$\Big(R(y_1, y_2)f\Big)(x) = f(y_1^{-1} x y_2), \quad y_1, y_2 \in G,\ f \in V_R,$$

of $G \times G$ on $V$. The regular representation is special among arbitrary representations in that it already comes with a candidate for an isomorphism with a direct integral. This is provided by the general Fourier transform

$$f \to \hat{f}(\pi) = \int_G f(x)\pi(x)\,dx, \quad f \in C_c^\infty(G),\ \pi \in \Pi(G),$$

which is defined on a dense subspace $C_c^\infty(G)$ of $L^2(G)$, and takes values in the vector space of families of operators on the spaces $\{V_\pi\}$.

The problem of the Plancherel formula is to compute the measure $d\pi$ on $\Pi(G)$ such that the norm

$$\|f\|_2^2 = \int_G |f(x)|^2\,dx$$

equals the dual norm

$$\|\hat{f}\|_2^2 = \int_{\Pi(G)} \|\hat{f}(\pi)\|_2^2\,d\pi,$$

for any function $f \in C_c^\infty(G)$. In the second integrand, $\|\hat{f}(\pi)\|_2$ denotes the Hilbert-Schmidt norm of the operator $\hat{f}(\pi)$. The norm $\|\hat{f}\|_2$ then defines a Hilbert space $\hat{V}$, on which $G \times G$ acts pointwise by left and right translation on the spaces of Hilbert-Schmidt operators on $\{V_\pi\}$. The problem was known to be well posed. A general theorem of I. Segal from 1950, together with Harish-Chandra's proof in 1953 that $G$ is of "type I", ensures that the *Plancherel measure* $d\pi$ exists and is unique. The point is to calculate $d\pi$ explicitly. This includes the problem of giving a parametrization of $\Pi(G)$, at least up to a set of Plancherel measure zero.

The first person to consider the problem and to make significant progress was I. M. Gelfand. He established Plancherel formulas for a number of matrix groups, and laid foundations for much of the later work in representation theory and automorphic forms. However, some of the most severe difficulties arose in groups that he did not consider. Harish-Chandra worked in the category of general semisimple (or reductive) groups. His eventual proof of the Plancherel formula for these groups was the culmination of twenty-five years of work. It includes many beautiful papers, and many ideas and constructions that are of great importance in their own right. I shall describe, in briefest terms, a few of the main points of Harish-Chandra's overall strategy, as it applies to the example $G = GL(n, \mathbb{R})$.

## Geometric Objects

In Harish-Chandra's theory of the Plancherel formula, the *geometric objects* are parametrized by the regular, semisimple conjugacy classes in $G$. In the example $G = GL(n, \mathbb{R})$ we are considering, these conjugacy classes are the ones that lie in the open dense subset

$$G_{\text{reg}} = \left\{ \gamma \in G : \begin{array}{l} \text{the eigenvalues } \gamma_i \in \mathbb{C} \\ \text{of } \gamma \text{ are distinct} \end{array} \right\}$$

of $G$. They are classified by the characteristic polynomial as a disjoint union of orbits

$$\coprod_P \Big(T_{P,\text{reg}}/W_P\Big),$$

where $P$ ranges over certain partitions

$$\Big\{P = (\underbrace{1, \dots, 1}_{r_1}, \underbrace{2, \dots, 2}_{r_2}) : r_1 + 2r_2 = n\Big\}$$

of $n$. For a given $P$,

$$T_P = \left\{ \gamma = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{r_1+r_2} \end{pmatrix} \right\},$$

where $t_k$ belongs to $\mathbb{R}^*$ if $1 \leq k \leq r_1$, and is of the form

$$\begin{pmatrix} \rho_k \cos\theta_k & \rho_k \sin\theta_k \\ -\rho_k \sin\theta_k & \rho_k \cos\theta_k \end{pmatrix}$$

if $r_1 + 1 \leq k \leq r_1 + r_2$, while $T_{P,\mathrm{reg}}$ stands for the intersection of $T_P$ with $G_{\mathrm{reg}}$. The group

$$W_P = S_{r_1} \times S_{r_2} \times (\mathbb{Z}/2\mathbb{Z})^{r_2},$$

in which $S_k$ denotes the symmetric group on $k$ letters, acts in the obvious way by permutation of the elements $\{t_k : 1 \leq k \leq r_1 + r_2\}$ and by sign changes in the coordinates $\{\theta_k : r_1 + 1 \leq k \leq r_1 + r_2\}$.

The complement of $G_{\mathrm{reg}}$ in $G$ has Haar measure 0. By calculating a Jacobian determinant, Harish-Chandra decomposed the restriction of the Haar measure to $G_{\mathrm{reg}}$ into measures on the coordinates defined by conjugacy classes. The resulting formula is

$$\int_G f(x)dx = \sum_P |W_P|^{-1} \int_{T_{P,\mathrm{reg}}} \left( |D(\gamma)| \int_{T_P \backslash G} f(x^{-1}\gamma x)\,dx \right) d\gamma,$$

for any $f \in C_c^\infty(G)$. Here $d\gamma$ is a Haar measure on $T_P$, and $T_P \backslash G$ represents the right cosets of $T_P$ in $G$, a set that can be identified with the conjugacy class of any $\gamma \in T_{P,\mathrm{reg}}$. The function

$$D(\gamma) = \prod_{1 \leq i < j \leq n} (\gamma_i - \gamma_j)^2$$

is the discriminant of the characteristic polynomial of $\gamma$. This generalizes the integration formula proved by Weyl in his elegant classification of the irreducible representations of compact Lie groups. We can regard it as a starting point for Harish-Chandra's study of the much more difficult case of noncompact groups.

Motivated by the integration formula, Harish-Chandra introduced a distribution

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{T_P \backslash G} f(x^{-1}\gamma x)\,dx, \qquad f \in C_c^\infty(G),$$

for any element $\gamma$ in $T_{P,\mathrm{reg}}$. This distribution is now known as Harish-Chandra's *orbital integral*, and is at the heart of much of his work. Harish-Chandra needed to prove many deep theorems about orbital integrals. The questions concern the extension of these linear forms to the Schwartz space $\mathcal{C}(G)$ on $G$ (which he later defined), and their behavior as $\gamma$ approaches the singular set in $T_P$. What is perhaps surprising at first glance is that the problems are not always amenable to direct attack. Harish-Chandra often established concrete inequalities by deep and remarkably indirect methods, that fully exploited the duality between the geometric objects $f_G(\gamma)$ and their corresponding spectral analogues.

## Spectral Objects

The *spectral objects* for Harish-Chandra were the characters of representations $\pi$ in $\Pi(G)$. Here, he was immediately faced with the problem that the space $V_\pi$ is generally infinite dimensional, in which case the sum determining the trace of the unitary operators $\pi(x)$ on $V_\pi$ can diverge. His answer was to prove that the average $\hat{f}(\pi)$ of these operators against a function $f \in C_c^\infty(G)$ is in fact of trace class. He then defined the *character* of $\pi$ to be the distribution

$$f_G(\pi) = \mathrm{tr}\left(\hat{f}(\pi)\right), \qquad f \in C_c^\infty(G).$$

However, this was by no means sufficient for the purposes he had in mind.

*Differential equations* play a central role in Harish-Chandra's analysis of both characters and orbital integrals. Let $\mathcal{Z}$ be the algebra of differential operators on $G$ that commute with both left and right translation. One of Harish-Chandra's earliest theorems, for which he won the AMS Cole Prize in 1954, was to describe the structure of $\mathcal{Z}$ as an algebra over $\mathbb{C}$. Let $\mathfrak{t}_P \cong \mathbb{C}^n$ be the complexification of the Lie algebra of the Cartan subgroup $T_P$ of $G = GL(n, \mathbb{R})$. By definition there is then a canonical isomorphism $\partial$ from the symmetric algebra $S(\mathfrak{t}_P)$ to the algebra of invariant differential operators on $T_P$.

The following theorem combines several results of Harish-Chandra on differential equations, including the basic structure theorem.

***Theorem***. *There is an isomorphism $z \to h_P(z)$, from $\mathcal{Z}$ onto the subalgebra of elements in $S(\mathfrak{t}_P)$ that are invariant under the symmetric group $S_n$, such that*

(i) $$(zf)_G(\gamma) = \partial\left(h_P(z)\right) f_G(\gamma), \qquad \gamma \in T_{P,\mathrm{reg}}.$$

*Moreover,*

(ii) $$(zf)_G(\pi) = \left\langle h_P(z), \lambda_\pi \right\rangle f_G(\pi), \qquad \pi \in \Pi(G),$$

*for some linear functional $\lambda_\pi$ on $\mathfrak{t}_P \cong \mathbb{C}^n$ that is unique up to the action of $S_n$.*

The equation (i) can be interpreted in terms of the traditional technique of separation of variables. The relevant differential operators are of course the elements in $\mathcal{Z}$, while the variables of separation are defined by the coordinates of conjugacy classes in $G_{\mathrm{reg}}$. The equation (ii) is a variant of Schur's lemma, which says that any operator commuting with the action of a finite group under an irreducible representation is a scalar. The functional $\lambda_\pi$ comes from the characterization of homomorphisms $\mathcal{Z} \to \mathbb{C}$ that is given by the isomorphism $z \to h_P(z)$.

Harish-Chandra also used the separation of variables technique to study the distribution on $G_{\mathrm{reg}}$ obtained by restricting the character of any $\pi$. It is easy to see that many of the differential operators $\partial\big(h_T(z)\big)$ on a given $T_P$ are actually elliptic. This allowed him to apply the well known theorem that eigendistributions of elliptic operators are actually real analytic functions. In this way, he was able to prove that

$$f_G(\pi) = \int_{G_{\mathrm{reg}}} f(x)\Theta_\pi(x)\,dx, \qquad f \in C_c^\infty(G_{\mathrm{reg}}),$$

for a real analytic function $\Theta_\pi$ on $G_{\mathrm{reg}}$. The separation of variables that is part of his argument then implies that for any $T_P$, the function

$$\Phi_\pi(\gamma) = |D(\gamma)|^{\frac{1}{2}}\Theta_\pi(\gamma), \qquad \gamma \in T_{P,\mathrm{reg}},$$

satisfies the differential equations

$$\partial\Big(h_P(z)\Big)\Phi_\pi(\gamma) = \Big\langle h_P(z), \lambda_\pi\Big\rangle\Phi_\pi(\gamma), \qquad z \in \mathcal{Z}.$$

Thus, $\Phi_\pi(\gamma)$ is a simultaneous eigenfunction of a large family of invariant differential operators on the abelian group $T_P$. From this, it is not hard to deduce, at least in the case that the coordinates of $\lambda_\pi$ in $\mathbb{C}^n$ are distinct, that the restriction of $\Phi_\pi(\gamma)$ to any connected component of $T_{P,\mathrm{reg}}$ has a simple formula of the form

$$\Phi_\pi(\gamma) = \sum_{s\in S_n} c_s e^{(s\lambda_\pi)(H)}, \qquad \gamma = \exp H,$$

for complex coefficients $\{c_s\}$.

We can see that the differential equations give detailed information about characters. To be able to apply this information to the study of the Plancherel formula, however, Harish-Chandra required the following fundamental theorem.

***Theorem***. *The character of any representation $\pi \in \Pi(G)$ is actually a function on $G$. In other words, $\Theta_\pi$ extends to a locally integrable function on $G$ such that*

$$f_G(\pi) = \int_G f(x)\Theta_\pi(x)\,dx\,, \qquad f \in C_c^\infty(G).$$

The proof of this theorem required many new ideas, which Harish-Chandra developed over the course of nine years. Atiyah and Schmid later gave a different proof of the theorem, by combining some of Harish-Chandra's techniques with methods from geometry.

The theorem provides a more concrete formula for the character of $\pi$. It follows from the original integration formula, and the invariance of $\Theta_\pi$ under conjugation by $G$, that

$$f_G(\pi) = \sum_P |W_P|^{-1} \int_{T_P,\mathrm{reg}} f_G(\gamma)\Phi_\pi(\gamma)\,d\gamma\,, \qquad f \in C_c^\infty(G).$$

This is a particularly vivid illustration of the duality between the geometric objects $f_G(\gamma)$ and the spectral objects $f_G(\pi)$. The formula becomes more explicit if we substitute the expansion above for $\Phi_\pi(\gamma)$. The resulting expression reduces the study of characters to the determination of the linear functionals $\lambda_\pi$ and the families of coefficients $\{c_s\}$.

## Plancherel Formula and Discrete Series

We can now state Harish-Chandra's Plancherel formula (for the group $G = GL_n(\mathbb{R})$) as follows.

***Theorem*** *(Plancherel formula). For each character $c$ in the dual group*

$$\hat{T}_P \cong (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})^{r_1} \times (\mathbb{Z} \times \mathbb{R})^{r_2}\,,$$

*there is an irreducible representation $\pi_c$ of $G$ such that*

$$\int_G |f(x)|^2 dx = \sum_P |W_P|^{-1} \int_{\hat{T}_P} \|\hat{f}(\pi_c)\|_2^2\, m(c)\,dc\,, \qquad f \in C_c^\infty(G),$$

*for an explicit real analytic function $m(c)$ on $\hat{T}_P$.*

***Remarks***.

1. The Plancherel density $m(c)$ actually vanishes if the image of $c$ in $(\mathbb{Z} \times \mathbb{R})^{r_2}$ has any $\mathbb{Z}$-component equal to zero. For any such $c$, the representation $\pi_c$ is not well defined by the formula, and can be taken to be 0.

2. The linear function $\lambda_\pi$ attached to $\pi = \pi_c$ is equal to the differential of $c$.

3. Harish-Chandra actually stated the theorem in the form of a Fourier inversion formula

$$f(1) = \sum_P |W_P|^{-1} \int_{\hat{T}_P} f_G(\pi_c) m(c)\,dc\,, \qquad f \in C_c^\infty(G).$$

To recover the Plancherel formula one needs only replace $f$ by the function

$$(f * f^*)(x) = \int_G f(y)\overline{f(x^{-1}y)}\,dy$$

in the inversion formula. Note the duality with the earlier integration formula, which can be written in the form

$$\int_G f(x)\,dx = \sum_P |W_P|^{-1} \int_{T_P,\mathrm{reg}} f_G(\gamma)|D(\gamma)|^{\frac{1}{2}}\,d\gamma\,, \quad f \in C_c^\infty(G).$$

This short introduction does not begin to convey a sense of the difficulties Harish-Chandra encountered, and was able to overcome. The most famous is the construction of the *discrete series*, the family of representations $\pi \in \Pi(G)$ to which the Plancherel measure $d\pi$ attaches positive mass. We ought to say something about these objects, since they are really at the heart of the Plancherel formula.

It might be helpful first to recall Weyl's classification of representations of compact groups, as it applies to the special case of the unitary group $G = U(n, \mathbb{C})$. By elementary linear algebra, any unitary matrix can be diagonalized, so there is only the one Cartan subgroup

$$T = \left\{ \gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} : |\gamma_i| = 1 \right\}$$

in $G$ to consider. We can otherwise use notation similar to that of $GL(n, \mathbb{R})$. Weyl's classification is provided by a canonical bijection $\pi \leftrightarrow \lambda_\pi$ between the irreducible representations $\pi \in \Pi(G)$ and the subset of points $\lambda = (\lambda_1, \dots, \lambda_n)$ in $\mathbb{Z}^n$ such that $\lambda_i > \lambda_{i+1}$ for each $i$. This bijection is determined uniquely by a simple formula Weyl established for the value of the character

$$\Theta_\pi(\gamma) = \mathrm{tr}\Big(\pi(\gamma)\Big)\,,$$

at any element $\gamma \in T_{\mathrm{reg}}$. (Since $U(n, \mathbb{C})$ is compact, $\pi$ is in fact finite dimensional.) The *Weyl character formula* is the identity

$$\Theta_\pi(\gamma) = \Big(D(\gamma)^{\frac{1}{2}}\Big)^{-1} \Big( \sum_{s \in S_n} \mathrm{sign}(s) \gamma^{s(\lambda_\pi)} \Big)\,,$$

where for any $\lambda \in \mathbb{Z}^n$, $\gamma^\lambda$ denotes the product $\gamma_1^{\lambda_1} \cdots \gamma_n^{\lambda_n}$. (The denominator $D(\gamma)^{\frac{1}{2}}$ is the canonical square root $\prod_{i<j}(\gamma_i - \gamma_j)$ of the discriminant. One could easily write the Weyl character formula less elegantly in the framework of the previous section, as a formula for the function $\Phi_\pi(\gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta_\pi(\gamma)$ on any connected component of $T_{\mathrm{reg}}$.)

Harish-Chandra's construction of the discrete series is a grand generalization of Weyl's theorem, in both its final statement and its methods of proof. In particular, Harish-Chandra constructed the characters of discrete series representations explicitly, starting from the considerations of the previous section. In the classification he eventually achieved, Harish-Chandra proved that a group $G$ has a discrete series if and only if it has a Cartan subgroup $T$ that is compact. Moreover, he specified the representations in the discrete series uniquely by a simple expression for their characters on $T_{\mathrm{reg}}$ that is a striking generalization of the Weyl character formula.

The group $GL(n, \mathbb{R})$ does not have a discrete series. The representations that appear in its Plancherel formula are all constructed from discrete series of $SL(2, \mathbb{R})$ and characters of $\mathbb{R}^*$. (For a given partition $P$, there are $r_2$ copies of $SL(2, \mathbb{R})$ to consider; the representations $\pi_c$ are defined by "parabolic induction" from representations of the subgroup of block diagonal matrices in $GL(n, \mathbb{R})$ of type $P$.) The example of $GL(n, \mathbb{R})$ is therefore relatively simple. Groups that have discrete series, such as $Sp(2n, \mathbb{R})$ and $U(p, q; \mathbb{C})$, are much more difficult. What is remarkable is that the final statement of the general Plancherel formula, suitably interpreted, is completely parallel to that of $GL(n, \mathbb{R})$.

After he established the Plancherel formula for real groups, Harish-Chandra worked almost exclusively on the representation theory of $p$-adic groups. This subject is extremely important for the analytic side of the Langlands program, but it has a more arithmetic flavor. Harish-Chandra was able to establish a version of the Plancherel formula for $p$-adic groups. However, it is less explicit than his formula for real groups, for the reason that he did not classify the discrete series. The problem of discrete series for general $G$ is still wide open, in fact, as is much of the theory for $p$-adic groups.[1]

## Nature of the Langlands Program

The analytic side of the Langlands program is concerned with automorphic forms. The language of the general theory of automorphic forms, as opposed to classical modular forms, is that of the representation theory of reductive groups. It is a language created largely by Harish-Chandra. Harish-Chandra's influence on the theory of automorphic forms is pervasive. It is not so much in the actual statement of his Plancherel formula, but rather in the enormously powerful methods and constructions (including the discrete series) that he created in order to establish the Plancherel formula.

The object of interest for automorphic forms is the regular representation $R_\Gamma$ of $G$ on the Hilbert space $V_{R_\Gamma} = L^2(\Gamma \backslash G)$, where $\Gamma$ is a congruence subgroup of $\mathbf{G}(\mathbb{Z})$. (We assume that the real reductive

[1] *The Langlands conjectures include a classification of discrete series for $p$-adic groups. A report on the recent proof of this classification for the group $G = GL(n)$, which in the $p$-adic case does have a discrete series, is given by Rogawski in this issue of the* Notices. *A separate classification for $G = GL(n)$, based on quite different algebraic criteria, has been known for some time from results of Bushnell and Kutzko. It is an open problem to compare the two classifications directly.*

group $G = \mathbf{G}(\mathbb{R})$ has been equipped with structure necessary to define $\mathbf{G}(\mathbb{Z})$.) As above, one seeks information about the decomposition of $R_\Gamma$ into irreducible representations. In this case, however, there is some interesting extra structure. The space $L^2(\Gamma\backslash G)$ comes with a family of semisimple operators $\{T_{p,i}\}$, the *Hecke operators*, which are parametrized by a cofinite set $\{p: p \notin \mathcal{P}_\Gamma\}$ of prime numbers, and a supplementary set of indices $\{i: 1 \le i \le n_p\}$ that depends on $p$ and has order bounded by the rank of $G$. These operators commute with $R_\Gamma$, and also with each other. If $\pi \in \Pi(G)$ is a representation that occurs discretely in $R_\Gamma$ with multiplicity $m(\pi)$, the Hecke operators then provide a family

$$\{T_{p,i}(\pi): p \notin \mathcal{P}_\Gamma, 1 \le i \le n_p\}$$

of mutually commuting $\big(m(\pi) \times m(\pi)\big)$-matrices. It is the eigenvalues of these matrices that are thought to carry the fundamental arithmetic information.

The most powerful tool available at present for the study of $R_\Gamma$ (and the Hecke operators) is the trace formula. The trace formula plays the role here of the Plancherel formula, and is the analogue of the Poisson summation formula for the discrete subgroup $\mathbb{Z}$ of $\mathbb{R}$. It is an explicit but quite complicated formula for the trace of the restriction of the operator

$$R_\Gamma(f) = \int_G f(x) R_\Gamma(x)\, dx, \qquad f \in C_c^\infty(G),$$

and more generally, the composition of $R_\Gamma(f)$ with several Hecke operators, to the subspace of $L^2(\Gamma\backslash G)$ that decomposes discretely. The formula is really an identity of two expansions. One is a sum of terms parametrized by rational conjugacy classes, while the other is a sum of terms parametrized by automorphic representations. The trace formula is thus a clear justification of the last line of our original table. It is also a typical (if elaborate) example of the kind of explicit formula that relates geometric and spectral objects on other lines of the table.

I mention the trace formula mainly to point out its dependence on the work of Harish-Chandra. The geometric side is composed of orbital integrals, together with some more general objects. The spectral side includes the required trace, as well as some supplementary distributions. All of these terms rely in one way or another on the work of Harish-Chandra, for both their construction and their analysis in future applications of the trace formula.

I shall say no more about the trace formula. It is also not possible in the dwindling allotment of space to give any kind of introduction to the Langlands program. I shall instead comment briefly on one specific example of the influence of Harish-Chandra's work—that of the discrete series.

Assume that $G$ does have a compact Cartan subgroup. The Hecke operators $\{T_{p,i}(\pi)\}$ associated to discrete series representations $\pi$ of $G$ are expected to be related to arithmetic objects attached to algebraic varieties. In many cases, it is known how to construct algebraic varieties for which this is so. Let $K$ be a maximal compact subgroup of $G$, and assume that $G$ is such that the space of double cosets

$$S_\Gamma = \Gamma\backslash G/K$$

has a complex structure.[2] This is the case, for example, if $G$ equals $Sp(2n, \mathbb{R})$ or $U(p, q; \mathbb{C})$. Then $S_\Gamma$ is the set of complex points of an algebraic variety. Moreover, it is known that this variety can be defined in a canonical way over some number field $F$ (equipped with an embedding $F \subset \mathbb{C}$). If $G = SL(2, \mathbb{R})$, $S_\Gamma$ is just a quotient of the upper half plane, and as $\Gamma$ varies, the varieties in this case determine the modular elliptic curves of the Shimura-Taniyama-Weil conjecture. The varieties in general were introduced and investigated extensively by Shimura. Their serious study was later taken up by Deligne, Langlands, Kottwitz, and others.

It is a key problem to describe the cohomology $H^*(S_\Gamma)$ of the space $S_\Gamma$, and more generally, various arithmetic objects associated with this cohomology. The discrete series representations $\pi$ are at the heart of the problem. There is a well defined procedure, based on differential forms, for passing from the subspace of $L^2(\Gamma\backslash G)$ defined by $\pi$ (of multiplicity $m(\pi)$) to a subspace, possibly 0, of $H^*(S_\Gamma)$. Different $\pi$ correspond to orthogonal subspaces of $H^*(S_\Gamma)$, and as $\pi$ ranges over all representations in the discrete series, these subspaces span the part of the cohomology of $H^*(S_\Gamma)$ that is primitive and is concentrated in the middle dimension. Moreover, the Hodge structure on this part of the cohomology can be read off from the parametrization of discrete series. Finally, there has been much progress on the deeper problem of establishing reciprocity laws between the eigenvalues of the Hecke operators $T_{p,i}(\pi)$ and arithmetic data attached to the corresponding subspaces of $H^*(S_\Gamma)$. These are serious results, due to Langlands and others, that I have not stated precisely, or even quite correctly.[3] The point is that the results provide answers to fundamental questions,

[2] *By replacing $\Gamma$ with a subgroup of finite index, if necessary, one also assumes that $\Gamma$ has no nontrivial elements of finite order.*

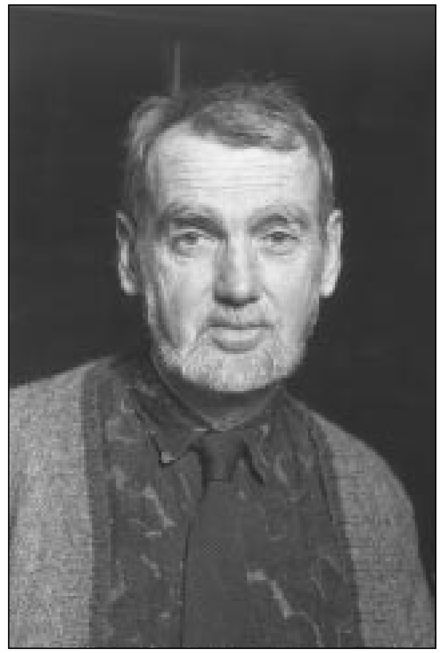
[3] *The axioms for a Shimura variety are somewhat more complicated than I have indicated. They require a slightly modified discussion, which applies to groups with noncompact center. Moreover, if $S_\Gamma$ is itself noncompact, $H^*(S_\Gamma)$ should really be replaced by the corresponding $L^2$-cohomology.*

which could not have been broached without Harish-Chandra's classification of discrete series.

The discussion raises further questions. What about the rest of the cohomology of $S_\Gamma$? What about the representations $\pi \in \Pi(G)$ in the complement of the discrete series? Harish-Chandra's Plancherel formula included a classification of the representations that lie in the natural support of the Plancherel measure (up to some questions of reducibility of induced representations, which were later resolved by Knapp and Zuckerman). Such representations are said to be *tempered*, because their characters are actually tempered distributions on $G$—they extend to continuous linear forms on the Schwartz space of $G$. Tempered representations that lie in the complement of the discrete series are certainly interesting for automorphic forms, but they do not contribute to the cohomology of $S_\Gamma$. Nontempered representations, on the other hand, have long been known to play an important role in cohomology. Can one classify the nontempered representations $\pi \in \Pi(G)$ that occur discretely in $L^2(\Gamma\backslash G)$?

To motivate the answers, let me go back to the last line of the original table. A conjugacy class in $\mathbf{G}(\mathbb{Q})$ has a *Jordan decomposition* into a semisimple part and a unipotent part. (Recall that an element $x \in GL(n,\mathbb{Q})$ is unipotent if some power of the matrix $x - I$ equals $0$. The Jordan decomposition for $GL(n,\mathbb{Q})$ is given by the elementary divisor decomposition of linear algebra.) Since automorphic representations are dual in some sense to rational conjugacy classes, it is not unreasonable to ask whether they too have some kind of Jordan decomposition.

I can no longer avoid giving at least a provisional definition of an automorphic representation. Assume for simplicity that $\mathbf{G}(\mathbb{C})$ is simply connected. In general, one would like an object that combines a representation $\pi \in \Pi(G)$ with any one of the $m(\pi)$ families $\{\lambda_{p,i} :\ p \notin \mathcal{P}_\Gamma, 1 \le i \le n_p\}$ of simultaneous eigenvalues of the Hecke operators. (It is these complex numbers, after all, that are supposed to carry arithmetic information.) It turns out that any such $\pi$ and any such family, as well as some (noncommutative) algebras of operators $\{U_q :\ q \in \mathcal{P}_\Gamma\}$ obtained from the ramified primes, can be packaged neatly together in the form of a representation of the adelic group $\mathbf{G}(\mathbb{A})$. Here $\mathbb{A}$ is a certain locally compact ring that contains $\mathbb{R}$, and also the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to $p$-adic absolute values. The rational field $\mathbb{Q}$ embeds diagonally as a discrete subring of $\mathbb{A}$. Let us define an *automorphic representation* restrictively as an irreducible representation $\pi$ of $\mathbf{G}(\mathbb{A})$ that occurs discretely in the decomposition of $L^2\big(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})\big)$. Any such $\pi$ determines a representation $\pi = \pi_\mathbb{R}$ in $\Pi(G)$ and a discrete subgroup $\Gamma \subset G$ such that $\pi$ occurs discretely in $L^2(\Gamma\backslash G)$. It also determines irreducible representations $\{\pi_p\}$ of the $p$-adic groups $\mathbf{G}(\mathbb{Q}_p)$, from which one can recover the complex numbers $\{\lambda_{p,i}\}$ and the algebras $\{U_q\}$.[4]



Photograph © 1996 Randall Hagadorn, courtesy of the Institute for Advanced Study.

**Robert Langlands**

An automorphic representation $\pi$ is said to be *tempered* if its components $\pi_\mathbb{R}$ and $\pi_p$ are all tempered. I did not give the definition of tempered representations for $p$-adic groups, but it is the same as for real groups. For the unramified primes $p \notin \mathcal{P}_\Gamma$, it is equivalent to a certain set of bounds on the absolute values of the complex numbers $\{\lambda_{p,i}\}$. The validity of these bounds for one particular automorphic representation of the group $\mathbf{G} = SL(2)$ is equivalent to a famous conjecture of Ramanujan, which was proved by Deligne in 1973.

The conjectures of Langlands include a general parametrization of tempered automorphic representations. In the early 1980s, I gave a conjectural characterization of automorphic representations that are nontempered. Among other things, this characterization describes the failure of a representation $\pi$ to be tempered in terms of a certain unipotent conjugacy class. It is not a conjugacy class in $\mathbf{G}(\mathbb{Q})$—such objects are only dual to automorphic representations—but rather in the complex dual group $\hat{\mathbf{G}}$ of $\mathbf{G}$. Here $\hat{\mathbf{G}}$ is the identity component of the $L$-group ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that is at the center of Langlands's conjectures. In this way, one can construct a conjectural Jordan decomposition for automorphic representations that is dual to the Jordan decomposition for conjugacy classes in $\mathbf{G}(\mathbb{Q})$. The conjectures for nontempered representations contain some character identities for the local components $\pi_\mathbb{R}$ and $\pi_p$ of representations $\pi$ of $\mathbf{G}(\mathbb{A})$. They also include a global formula for the multiplicity of $\pi$ in $L^2\big(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})\big)$ that implies qualitative properties for the eigenvalues of Hecke operators. The local conjectures for $\pi_\mathbb{R}$ have been established by Adams, Barbasch, and Vogan, by very interesting methods from intersection

[4] *The proper definition of automorphic representation also includes representations that occur continuously in $L^2\big(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})\big)$, as well as analytic continuations of such representations. If $\mathbf{G}(\mathbb{C})$ is not simply connected, $\pi$ actually determines several discrete subgroups of $\mathbf{G}$.*

homology. The remaining assertions are open. However, the contribution of nontempered representations to cohomology is quite well understood. The unipotent class that measures the failure of a representation to be tempered turns out to be the same as the unipotent class obtained from the action of a Lefschetz hyperplane section on cohomology. One can in fact read off the Lefschetz structure on $H^*(S_\Gamma)$, as well as the Hodge structure, from the parametrization of representations.

## References

For a reader who is able to invest the time, the best overall reference for Harish-Chandra's work is still his collected papers.


Harish-Chandra, *Collected Papers*, Volumes I–IV, Springer-Verlag, 1984.


Harish-Chandra's papers are very carefully written, and are not difficult to follow step by step. On the other hand, they are highly interdependent (even in their notation), and it is sometimes hard to see where they are leading. The excellent technical introduction of Varadarajan goes some way towards easing this difficulty.

Weyl's classification of representations of compact groups is proved concisely (in the special case of $U(n, \mathbb{C})$), in


H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, 1950, pp. 377–385.


The following two articles are general introductions to the Langlands program


S. Gelbart, An elementary introduction to the Langlands program, *Bull. Amer. Math. Soc. N.S.* **10** (1984), 177–219.

J. Arthur and S. Gelbart, Lectures on automorphic $L$-functions, Part I, *L-functions and Arithmetic*, London Math. Soc. Lecture Note Series, vol. 153, Cambridge Univ. Press, 1991, pp. 2–21.


Other introductory articles on the Langlands program and on the work of Harish-Chandra are contained in the proceedings of the Edinburgh instructional conference


*Representation Theory and Automorphic Forms*, Proc. Sympos. Pure Math., vol. 61, Amer. Math. Soc., 1996, edited by T. N. Bailey and A. W. Knapp.


The general axioms for a Shimura variety are summarized in the Deligne's Bourbaki lecture


P. Deligne, Travaux de Shimura, *Séminaire Bourbaki, 23ème Année (1970/71)*, Lecture Notes in Math., vol. 244, Springer-Verlag, 1971, pp. 123–165.


Finally, the most general conjecture relating motives with automorphic representations is stated at the end of §2 of Langlands's article


R. P. Langlands, Automorphic representations, Shimura varieties and motives. Ein Märchen, *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math., vol. 33, Part 2, Amer. Math. Soc., 1979, pp. 205–246.