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The Problem of Classifying Automorphic Representations of Classical Groups

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In this article we shall give an elementary introduction to an important problem in representation theory. The problem is to relate the automorphic representations of classical groups to those of the general linear group. Thanks to the work of a number of people over the past twenty-five years, the automorphic representation theory of $\mathrm{GL}(n)$ is in pretty good shape. The theory for $\mathrm{GL}(n)$ now includes a good understanding of the analytic properties of Rankin-Selberg L-functions, the classification of the discrete spectrum, and cyclic base change. One would like to establish similar things for classical groups. The goal would be an explicit comparison between the automorphic spectra of classical groups and $\mathrm{GL}(n)$ through the appropriate trace formulas. There are still obstacles to be overcome. However with the progress of recent years, there is also reason to be optimistic.

We shall not discuss the techniques here. Nor will we consider the possible applications. Our modest aim is to introduce the problem itself, in a form that might be accessible to a nonspecialist. In the process we shall review some of the basic constructions and conjectures of Langlands that underlie the theory of automorphic representations.

1. We shall begin with a few of the basic concepts from the theory for the general linear group. For the present, then, we take $G = \mathrm{GL}(n)$. The adèles of $\mathbb{Q}$ form a locally compact ring

$$\mathbb{A} = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots$$

in which $\mathbb{Q}$ embeds diagonally as a discrete subring. Consequently $G(\mathbb{A})$ is a locally compact group which contains $G(\mathbb{Q})$ as a discrete subgroup. One can form the Hilbert space $L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$ of functions which are square integrable with respect to the right $G(\mathbb{A})$-invariant measure. The primary object of study is the regular representation

$$\big(R(y)f\big)(x) = f(xy), \quad f \in L^2\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big), \quad x, y \in G(\mathbb{A}),$$

on the Hilbert space.

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The unitary representation R is highly reducible. For this discussion we shall define an *automorphic representation* informally as an irreducible unitary representation π of $G(\mathbb{A})$ which occurs in the decomposition of R. This notion would be precise certainly if π occurred as a discrete summand of R. However, the irreducible constituents of R depend on several continuous parameters and one wants to include all of these. The proper definition [**14**] in fact includes irreducible representations of $G(\mathbb{A})$ which come from the analytic continuation of these parameters, but there is no need to consider such objects here. It is known [**5**] that any such π has a decomposition

$$\pi = \pi_{\mathbb{R}} \otimes \pi_2 \otimes \pi_3 \otimes \pi_5 \otimes \cdots$$

as a restricted tensor product, with each π_p being an irreducible unitary representation of the group $G(\mathbb{Q}_p)$.

Anyone seeing these definitions for the first time could well ask why automorphic representations are interesting. To get a feeling for the situation, we fix a prime p and recall the construction of the unramified representations of $G(\mathbb{Q}_p)$—the simplest family of irreducible representations $\{\pi_p\}$ of this group.

The representations in the family are determined by elements $u = (u_1, \ldots, u_n)$ in $\mathbb{C}^n$. Such an element defines a character of the Borel subgroup

$$B(\mathbb{Q}_p) = \left\{ b = \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} \right\} \subseteq G(\mathbb{Q}_p)$$

of $\mathrm{GL}(n, \mathbb{Q}_p)$ by

$$\chi_u(b) = |b_{11}|_p^{u_1+(n-1)/2} \, |b_{22}|_p^{u_2+(n-3)/2} \cdots |b_{nn}|_p^{u_n-(n-1)/2}.$$

Let $\pi_{p,u}^+$ be the corresponding induced representation of $G(\mathbb{Q}_p)$. It acts on a space of functions $f_p : G(\mathbb{Q}_p) \to \mathbb{C}$ which satisfy

$$f_p(bx) = \chi_u(b) f_p(x), \quad b \in B(\mathbb{Q}_p), \quad x \in G(\mathbb{Q}_p),$$

be right translation—

$$\big(\pi_{p,u}^+(y) f_p\big)(x) = f_p(xy), \quad x, y \in G(\mathbb{Q}_p).$$

The vector $\frac{1}{2}\big(n-1, n-3, \ldots, -(n-1)\big)$ comes from the usual Jacobian factor, and is included so that $\pi_{p,u}^+$ will be unitary if u is purely imaginary. If u is purely imaginary, $\pi_{p,u}^+$ is known to be irreducible as well as unitary. In general, $\pi_{p,u}^+$ can have several irreducible constituents, but there is a canonical one—the irreducible constituent $\pi_{p,u}$ of $\pi_{p,u}^+$ which contains a $G(\mathbb{Z}_p)$-fixed vector. Thus, any u determines an irreducible representation $\pi_{p,u}$ of $G(\mathbb{Q}_p)$. Since the p-adic absolute values in the definition of χ_u are powers of p, it is clear that $\pi_{p,u}$ remains the same if u is translated by a vector in $(2\pi i/\log p)\mathbb{Z}^n$. In fact if u' is any other vector in $\mathbb{C}^n$, it is known that $\pi_{p,u'}$ is equivalent to $\pi_{p,u}$ if and only if

$$(u_1', \ldots, u_n') \equiv (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \left(\mathrm{mod} \left(\frac{2\pi i}{\log p} \right) \mathbb{Z}^n \right),$$

for some permutation σ in S_n.

By definition, the *unramified* representations of $G(\mathbb{Q}_p)$ are the ones in the family $\{\pi_{p,u} : u \in \mathbb{C}^n\}$. Set

$$t(\pi_{p,u}) = \begin{pmatrix} p^{-u_1} & & 0 \\ & \ddots & \\ 0 & & p^{-u_n} \end{pmatrix},$$

regarded as a semisimple conjugacy class in $\mathrm{GL}(n, \mathbb{C})$. This is a special case of a general construction [**13**] of Langlands. In the present situation it gives a bijection between the unramified representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ and the semisimple conjugacy classes in $\mathrm{GL}(n, \mathbb{C})$.

Now suppose that π is an automorphic representation of $\mathrm{GL}(n, \mathbb{A})$. It is known that the local components π_p of π are unramified for almost all p. In other words, π determines a family

$$t(\pi) = \{t(\pi_p) : p \notin S\}$$

of semisimple conjugacy classes in $\mathrm{GL}(n, \mathbb{C})$. Here $S = S_\pi$ is a finite set of completions of $\mathbb{Q}$ which includes the Archimedean place $\mathbb{R}$. Returning to the original question, automorphic representations are interesting because the corresponding families $t(\pi)$ are believed to carry fundamental arithmetic information. What is important is not the fact that almost all π_p are unramified—this would be true of any irreducible representation of $G(\mathbb{A})$ with some weak continuity hypothesis—but that π is automorphic. It is only then that the semisimple conjugacy classes $\{t(\pi_p)\}$ will be related one to another in a way that is governed by fundamental arithmetic phenomena.

In order to package the data $t(\pi)$ conveniently, one defines the local L-function

$$L(s, \pi_p) = \det\bigl(I - t(\pi_p)p^{-s}\bigr)^{-1}, \quad s \in \mathbb{C}, \quad p \in S,$$

as the reciprocal (evaluated at p^{-s}) of the characteristic polynomial of the conjugacy class $t(\pi_p)$. One can then define a global L-function

$$L_S(s, \pi) = \prod_{p \notin S} L(s, \pi_p)$$

as an Euler product which converges in some right half plane. It is known that $L_S(s, \pi)$ has analytic continuation as a meromorphic function of $s \in \mathbb{C}$, and satisfies a functional equation [**9**]. The basic proof is a generalization of the one used by Tate for $\mathrm{GL}(1)$. It exploits the embedding of $\mathrm{GL}(n)$ into the space of $(n \times n)$-matrices. The proof entails defining local L-functions $L(s, \pi_p)$ for every p (including $p = \mathbb{R}$). If one forms the product

$$L(s, \pi) = \prod_p L(s, \pi_p)$$

over all p, the functional equation takes the form

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \tilde{\pi}),$$

where $\tilde{\pi}$ is the contragredient representation of π, and the ϵ-factor is a simple function of the form

$$\epsilon(s, \pi) = a_\pi (p^{r_\pi})^s, \quad a_\pi \in \mathbb{C}, \quad r_\pi \in \mathbb{Z}.$$

For an elementary example, take $G = \mathrm{GL}(1)$. Then $G(\mathbb{A}) = \mathbb{A}^*$ is the group of idèles, while $G(\mathbb{Q})\backslash G(\mathbb{A}) = \mathbb{Q}^*\backslash \mathbb{A}^*$ is the quotient group of idèle classes. We shall consider an automorphic representation $\pi = \bigotimes_p \pi_p$ with $S_\pi = \{\mathbb{R}, 2\}$. Then if $p \notin$

S, π_p is determined by an unramified character on the group $B(\mathbb{Q}_p) = \mathbb{Q}_p^* = G(\mathbb{Q}_p)$. Any such prime is of course odd. Set

$$t(\pi_p) = p^{-u} = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ -1, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

In other words,

$$\pi_p(x_p) = |x_p|_p^u = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{v(x_p)}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $v(x_p) \in \mathbb{Z}$ is the valuation of a point $x_p \in \mathbb{Q}_p^*$. It is then easy to define characters $\pi_{\mathbb{R}}$ and π_2 on $\mathbb{R}^*$ and $\mathbb{Q}_2^*$ respectively so that $\pi = \bigotimes_p \pi_p$ is trivial on the subgroup $\mathbb{Q}^*$ of $\mathbb{A}^*$, and is hence an automorphic representation of GL(1). Observe that the definition of π_p for $p \notin S$ matches the splitting law of the prime p in the Gaussian integers $\mathbb{Z}\left[\sqrt{-1}\right]$; p is of the form

$$p = \left(a + \sqrt{-1}b\right)\left(a - \sqrt{-1}b\right) = a^2 + b^2, \quad a, b \in \mathbb{Z},$$

if and only if p is congruent to 1 modulo 4. This is no co-incidence. The Kronecker-Weber theorem can be read as the construction of an automorphic representation for any cyclic extension of $\mathbb{Q}$ in terms of how rational primes behave in the extension. The Artin reciprocity law gives a similar construction in the more general case that $\mathbb{Q}$ is replaced by an arbitrary number field F. It can be regarded as a classification of abelian extensions of F in terms of automorphic representations of GL(1) (relative to F).

This is a good point to recall Langlands' nonabelian generalization of the Artin reciprocity law. Suppose that

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

is an n-dimensional representation of the Galois group of an algebraic closure of $\mathbb{Q}$ which is continuous, that is, which factors through a finite quotient $\mathrm{Gal}(E/\mathbb{Q})$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then ϕ is unramified outside a finite set $S = S_\phi$ of primes. For any prime $p \notin S$, there is a Frobenius conjugacy class Fr_p in $\mathrm{Gal}(E/\mathbb{Q})$, and hence a conjugacy class $\phi(\mathrm{Fr}_p)$ in $\mathrm{GL}(n, \mathbb{C})$. Langlands conjectured that for any ϕ there is an automorphic representation π of $\mathrm{GL}(n)$ such that

$$t(\pi_p) = \phi(\mathrm{Fr}_p), \quad p \notin S_\pi \cap S_\phi.$$

This conjecture is very difficult, and has been established in only a limited number of cases [**15, 16, 4**]. It is known, however, that there is at most one π with this property [**10**].

We recall also that there is an Artin L-function attached to ϕ which is completely parallel to the construction of an automorphic L-function. It is defined by an Euler product

$$L(s, \phi) = \prod_p L(s, \phi_p)$$

which converges in a right half plane, with local factors given by

$$L(s, \phi_p) = \det\left(I - \phi(\mathrm{Fr}_p)p^{-s}\right)^{-1}$$

if p does not belong to S_ϕ. The function $L(s,\phi)$ has analytic continuation and satisfies a functional equation

$$L(s,\phi) = \epsilon(s,\phi)L(1-s,\tilde{\phi}),$$

with an ϵ-factor of the form

$$\epsilon(s,\phi) = a_\phi (p^{r_\phi})^s, \quad a_\phi \in \mathbb{C}, \quad r_\phi \in \mathbb{Z}.$$

Langlands' conjectural reciprocity law, which is actually a special case of his functoriality principle, was formulated for all places p. It asserts that $L(s,\pi_p) = L(s,\phi_p)$ for all p, or in global form, that

$$L(s,\pi) = L(s,\phi).$$

In other words, every Artin L-function is an automorphic L-function.

2. Suppose now that G belongs to one of the other three families $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n)$ of classical groups. We shall assume that G is quasi-split. Then G will actually be split if it is of the form $\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(2n)$. In the remaining case, G could be a nonsplit form of $\mathrm{SO}(2n)$ which splits over a quadratic extension E of $\mathbb{Q}$. (We exclude the exceptional quasi-split forms of $\mathrm{SO}(8)$.)

With suitable modifications, the constructions of §1 all carry over to G. (They were introduced by Langlands for any reductive group over any global field F [**13**].) In particular, an automorphic representation of $G(\mathbb{A})$ has a decomposition $\pi = \bigotimes_p \pi_p$, in which π_p is an unramified representation of $G(\mathbb{Q}_p)$ for all p outside a finite set $S = S_\pi$. Each such π_p is a constituent of a representation induced from an unramified quasi-character of a Borel subgroup $B(\mathbb{Q}_p)$ of $G(\mathbb{Q}_p)$. The reader unfamiliar with these things could try at this point to construct a semisimple conjugacy class $t(\pi_p)$, in analogy with $\mathrm{GL}(n)$. He/she will discover that such a conjugacy class exists, but that it occurs naturally in a complex group which is dual to G. If G is split, one can take the dual group $\widehat{G}$ given by the table

G	$\widehat{G}$
$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(2n,\mathbb{C})$
$\mathrm{Sp}(2n)$	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n,\mathbb{C})$

If G is not split, one must take a semi-direct product

$$\widehat{G} \rtimes \mathrm{Gal}(E/\mathbb{Q}),$$

in which $\mathrm{Gal}(E/\mathbb{Q})$ acts on $\widehat{G} = \mathrm{SO}(2n,\mathbb{C})$ by conjugation through the isomorphism

$$\mathrm{Gal}(E/\mathbb{Q}) \simeq \mathrm{O}(2n,\mathbb{C})/\,\mathrm{SO}(2n,\mathbb{C}).$$

The two cases are combined in Langlands' original construction of the L-group

$${}^LG = \widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on $\widehat{G}$ in case G is split, and acts on $\widehat{G}$ through its quotient $\mathrm{Gal}(E/\mathbb{Q})$ if G is not split. In the case of the general linear group, one obviously takes ${}^L\big(\mathrm{GL}(n)\big)$ to be the direct product of $\mathrm{GL}(n,\mathbb{C})$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Thus, to any automorphic representation π of $G(\mathbb{A})$ there is associated a family

$$t(\pi) = \big\{t(\pi_p) : p \notin S_\pi\big\}$$

of semisimple conjugacy classes in the complex reductive group ${}^LG$. We have to remind ourselves that the situation is more concrete than the final notation suggests; if G is split, for example, one can always replace ${}^LG$ by the complex connected group $\widehat{G}$. As with $\mathrm{GL}(n)$, the numerical data which determine these conjugacy classes are believed to carry fundamental arithmetic information. In fact, the data obtained in this way ought to be a subset of the data obtained from general linear groups. This is the essence of the problem we shall presently discuss, and is also a special case of Langlands' functoriality principle. (For an introduction to the functoriality principle, see [**1**].)

If the automorphic representations of classical groups are to be understood in terms of $\mathrm{GL}(n)$, why study them at all? There are compelling reasons to do so. Suppose for example that $G = \mathrm{Sp}(2n)$. One can form the Siegel moduli space

$$S(N) = \Gamma(N)\backslash\mathcal{H},$$

where $\mathcal{H}$ is the Siegel upper half space of genus n, and $\Gamma(N)$ is the congruence subgroup

$$\{\gamma \in \mathrm{Sp}(2n,\mathbb{Z}) : \gamma \equiv I \pmod N\}$$

of $\mathrm{Sp}(2n,\mathbb{R})$. Then $S(N)$ is a complex algebraic variety. The L^2-cohomology of $S(N)$, $H^*_{(2)}(S(N))$, is a very interesting object which is directly related to certain automorphic representations π of $\mathrm{Sp}(2n,\mathbb{A})$. For such π, the conjugacy classes $t(\pi_p)$ are governed by the eigenvalues of Hecke operators acting on the cohomology. (See [**3**] for an introduction to these and related questions.) In this way one studies quite different properties of π than one could get from the corresponding object on a general linear group.

To attach an L-function to an automorphic representation π of $G(\mathbb{A})$, one has first to embed ${}^LG$ in a general linear group. Suppose that

$${}^Lr : {}^LG \longrightarrow \mathrm{GL}(V)$$

is a complex analytic, finite dimensional representation of ${}^LG$. This determines local L-factors

$$L(s,\pi_p,{}^Lr) = \det\big(1 - {}^Lr\big(t(\pi_p)\big)p^{-s}\big)^{-1}, \quad p \notin S_\pi,$$

for almost all p. One would like to be able to define L-factors for all p, and to show that the Euler product

$$L(s,\pi,{}^Lr) = \prod_p L(s,\pi_p,{}^Lr)$$

has analytic continuation and functional equation. The case of $G = \mathrm{GL}(n)$ and ${}^Lr$ the standard n-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ was discussed in §1. Despite considerable progress [**8**], however, the general case is still far from solved.

Finally, we recall that the Langlands reciprocity conjecture applies equally well to L-homomorphisms

$$\phi_G : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow {}^LG$$

attached to G. (An *L-homomorphism* is one which is compatible with projections of the domain and co-domain onto $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) For each ϕ_G there should exist an automorphic representation π_G of $G(\mathbb{A})$ with the property that for any ${}^Lr$: ${}^LG \to \mathrm{GL}(V)$, the Artin L-function $L(s,{}^Lr\circ\phi_G)$ equals the automorphic L-function $L(s,\pi_G,{}^Lr)$.

This completes our discussion of some of the general properties of automorphic representations. We can now formulate the problem we set out to describe.

Observe that for our classical group G there is a canonical embedding

$$r_G : \widehat{G} \hookrightarrow \mathrm{GL}(N, \mathbb{C}),$$

with N equal to either $2n$ or $2n+1$. This can be extended to an L-embedding

$${}^L r_G : {}^L G = \widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{GL}(N, \mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = {}^L\big(\mathrm{GL}(N)\big).$$

By composing with ${}^L r_G$, we obtain a map $\phi_G \to \phi$,

$$\begin{array}{ccc} & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \\ {\scriptstyle \phi_g} \swarrow & & \searrow {\scriptstyle \phi} \\ {}^L G & \xrightarrow{\quad {}^L r_G \quad} & {}^L\big(\mathrm{GL}(N)\big) \end{array}$$

between L-homomorphisms into the two L-groups. We shall identify ϕ with its projection onto $\mathrm{GL}(N, \mathbb{C})$, that is, with an N-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As such it is self-contragredient. Conversely suppose that ϕ is an arbitrary self-contragredient N-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We assume also that ϕ is irreducible. Then ϕ factors through an orthogonal or a symplectic group. More precisely, there is a unique G, and an L-homomorphism ϕ_G for G, such that ϕ is equivalent to ${}^L r_G \circ \phi_G$. (This is an easy consequence of the self-contragredience of ϕ—see for example §3 below.)

The problem is to show that there is a similar mapping $\pi_G \to \pi$ between automorphic representations. The mapping should reduce to $\phi_G \to \phi$ for the automorphic representations attached (by Langlands' conjectural reciprocity law) to L-homomorphisms. As in this special case, the general mapping will be defined in terms of the families $t(\pi)$ of conjugacy classes.

Problem.

(i) *If π_G is an automorphic representation of the classical group G, show that there is an automorphic representation π of* $\mathrm{GL}(N, \mathbb{A})$ *such that*

$${}^L r_G\big(t(\pi_{G,p})\big) = t(\pi_p)$$

for almost all p.

(ii) *Conversely, suppose that π is a self-contragredient automorphic representation of* $\mathrm{GL}(N, \mathbb{A})$. *If π is cuspidal, show that π is the image of an automorphic representation π_G of $G(\mathbb{A})$, for a unique G as above.*

The problem is analogous to the base change problem, solved originally for GL(2) by Langlands [**15**]. That a similar question could be posed for the outer automorphism

$$x \to \tilde{x} = {}^t x^{-1}, \quad x \in \mathrm{GL}(N),$$

of $\mathrm{GL}(N)$ was I believe first noticed by Jacquet. However, there are some new phenomena here. The most obvious is the possibility of lifting representations from more than one G to a given $\mathrm{GL}(N)$. If $N = 2n$ is even, $\widehat{G}$ could be either $\mathrm{SO}(2n, \mathbb{C})$ or $\mathrm{Sp}(2n, \mathbb{C})$; that is, G could be either $\mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$. It was pointed out by Shalika that one ought to be able to separate these two cases by looking at the symmetric square and alternating square L-functions.

Let S^2 (respectively Λ^2) be the finite dimensional representation

$$g : X \to {}^t gXg, \quad g \in \mathrm{GL}(2n, \mathbb{C}),$$

of $\mathrm{GL}(2n, \mathbb{C})$ on the space of symmetric (resp. skew-symmetric) $(2n \times 2n)$-matrices. Consider a self-contragredient irreducible Galois representation

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(2n, \mathbb{C}).$$

Then ϕ factors through $\mathrm{O}(2n, \mathbb{C})$ (resp. $\mathrm{Sp}(2n, \mathbb{C})$) if and only if the representation $S^2 \circ \phi$ (resp. $\Lambda^2 \circ \phi$) of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains the trivial representation. This is the case if and only if the Artin L-function $L(s, S^2 \circ \phi)$ (resp. $L(s, \Lambda^2 \circ \phi)$) has a pole at $s = 1$. This suggests the following supplement to the problem.

(iii) *Suppose that π is a self-contragredient cuspidal automorphic representation of $\mathrm{GL}(2n)$. Show that π is the image of an automorphic representation π_G of $\mathrm{SO}(2n)$ (respectively $\mathrm{SO}(2n+1)$) if and only if the automorphic L-function $L(s, \pi, S^2)$ (resp. $L(s, \pi, \Lambda^2)$) has a pole at $s = 1$.*

We shall state a second supplement to the problem that concerns automorphic ϵ-factors. Suppose that

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(N, \mathbb{C})$$

is an irreducible Galois representation. If we apply the functional equation of the Artin L-function $L(s, \phi)$ twice, we obtain

$$\epsilon(s, \phi)\epsilon(1 - s, \tilde{\phi}) = 1.$$

Assume that ϕ is self-contragredient. Setting $s = \frac{1}{2}$, we see that

$$\epsilon\left(\frac{1}{2}, \phi\right) = \pm 1.$$

The self-contragredience of ϕ means that it factors through an orthogonal or a symplectic group. If ϕ factors through $\mathrm{Sp}(N, \mathbb{C})$, $\epsilon\left(\frac{1}{2}, \phi\right)$ can be either 1 or -1; the actual value of this sign has interesting number theoretic implications [**6**]. If ϕ factors through $\mathrm{O}(N, \mathbb{C})$, however, $\epsilon\left(\frac{1}{2}, \phi\right)$ is known to equal 1 [**7**]. One would like to establish the automorphic version of this property.

(iv) *Suppose that π is a self-contragredient cuspidal automorphic representation of $\mathrm{GL}(N)$. If π is the image of an automorphic representation π_G of a group G with $\widehat{G} = \mathrm{SO}(N, \mathbb{C})$, show that $\epsilon\left(\frac{1}{2}, \pi\right) = 1$.*

3. It is known that an automorphic representation π of $\mathrm{GL}(N)$ is uniquely determined by the family $t(\pi)$ of conjugacy classes. In other words, the map

$$\pi \longrightarrow t(\pi),$$

from the automorphic representations of $\mathrm{GL}(N)$ to families of semisimple conjugacy classes in $\mathrm{GL}(N, \mathbb{C})$, is injective. (The objects in the range are to be regarded as equivalence classes, two families being equivalent if they are equal at almost all p.) This is a theorem of Jacquet-Shalika [**10**], which is an extension of the earlier result for cuspidal automorphic representations. (Keep in mind that we have adopted a restrictive definition of automorphic representation. What we are calling an automorphic representation really includes an extra condition, that of being *globally tempered*; it is only with this condition that the injectivity is valid.)

The corresponding assertion for a classical group G is generally false. If

$$t_G = \{t_{G,p} : p \notin S\}$$

is a family of semisimple conjugacy classes in ${}^LG$, the set of automorphic representations π_G of $G(\mathbb{A})$ such that $t(\pi_G) = t_G$ could be an infinite packet. In particular, the mapping $\pi_G \to \pi$ of our problem could have large fibres. An important part of the problem is to determine these fibres. There is a precise conjectural description of the preimage of any π, based on the theory of endoscopy [**12**] and its extension to nontempered representations [**2**]. We shall not repeat it here. It suffices to say that the description is motivated by the case that π is attached to a self-contragredient Galois representation. We shall conclude this article with a few remarks on the structure of such Galois representations.

Consider an L-homomorphism

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow {}^L\big(\mathrm{GL}(N)\big).$$

We have agreed not to distinguish between such an object and the corresponding N-dimensional Galois representation. Thus, ϕ has a decomposition

$$\phi = \ell_1\phi_1 \oplus \cdots \oplus \ell_r\phi_r$$

into irreducible Galois representations

$$\phi_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow {}^L\big(\mathrm{GL}(N_i)\big),$$

which occur with multiplicities ℓ_i. Suppose that ϕ is self-contragredient. Then there is a permutation $i \to \tilde{\imath}$ of period 2 on the set of indices such that $\tilde{\phi}_i = \phi_{\tilde{\imath}}$ and $\ell_i = \ell_{\tilde{\imath}}$.

We are going to confine our attention to a special case. We assume that for every i, $\ell_i = 1$ and $\tilde{\phi}_i = \phi_i$. In particular, the irreducible representation

$$\sigma \longrightarrow \tilde{\phi}_i(\sigma) = {}^t\phi_i(\sigma)^{-1}, \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

is equivalent to ϕ_i. It follows that for each i, there is a matrix $A_i \in \mathrm{GL}(N_i, \mathbb{C})$ such that

$${}^t\phi_i(\sigma)^{-1} = A_i\phi_i(\sigma)A_i^{-1}, \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Applying this equation twice, we see that ${}^tA_i^{-1}A_i$ is an intertwining operator for the representation ϕ_i. It follows from Schur's lemma that ${}^tA_i = cA_i$ for some $c \in \mathbb{C}^*$. Applying this last identity twice, we find that $c^2 = 1$, so that A_i is either skew-symmetric or symmetric. Therefore ϕ_i is either of symplectic or orthogonal type. More precisely, if we replace ϕ_i by a suitable $\mathrm{GL}(N_i, \mathbb{C})$-conjugate, we can assume that either

$$\mathrm{Image}(\phi_i) \subseteq \mathrm{Sp}(N_i, \mathbb{C}) \subseteq \mathrm{GL}(N_i, \mathbb{C})$$

or

$$\mathrm{Image}(\phi_i) \subseteq \mathrm{O}(N_i, \mathbb{C}) \subseteq \mathrm{GL}(N_i, \mathbb{C}).$$

Separating the indices i into two disjoint sets I^1 and I^2 according to whether ϕ_i is symplectic or orthogonal, we obtain a decomposition

$$\phi = \phi^1 \oplus \phi^2,$$

where

$$\phi^1 = \bigoplus_{j \in I^1} \phi_j : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \prod_j \mathrm{Sp}(N_j, \mathbb{C}) \subseteq \mathrm{Sp}(N^1, \mathbb{C})$$

and

$$\phi^2 = \bigoplus_{k \in I^2} \phi_k : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \prod_k \mathrm{O}(N_k, \mathbb{C}) \subseteq \mathrm{O}(N^2, \mathbb{C}),$$

in which $N^1 = \sum_j N_j$ and $N^2 = \sum_k N_k$.

The maps ϕ^1 and ϕ^2 can be analyzed separately. For the first one, we note that $\mathrm{Sp}(N^1, \mathbb{C})$ is connected and equals $(\widehat{G^1})$, where $G^1 = \mathrm{SO}(N^1+1)$. There is nothing more to say in this case. For the second case, observe that the map

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{O}(N^2, \mathbb{C})/\,\mathrm{SO}(N^2, \mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$$

obtained from ϕ^2 by projection, determines a quadratic character η of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Suppose first that N^2 is odd. Then $\mathrm{O}(N^2, \mathbb{C})$ is the direct product of $\mathrm{SO}(N^2, \mathbb{C})$ with $\mathbb{Z}/2\mathbb{Z}$. Setting $G^2 = \mathrm{Sp}(N^2-1)$, we use η to define an embedding of

$$^L(G^2) = \mathrm{SO}(N^2, \mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

into

$$^L\big(\mathrm{GL}(N^2)\big) = \mathrm{GL}(N^2, \mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

so that ϕ^2 factors through $^L(G^2)$. Next suppose that N^2 is even. Then $\mathrm{O}(N^2, \mathbb{C})$ is a semi-direct product of $\mathrm{SO}(N^2, \mathbb{C})$ with $\mathbb{Z}/2\mathbb{Z}$. Let G^2 be the quasi-split form of $\mathrm{SO}(N^2)$ obtained from η and the action of the nonidentity component of $\mathrm{O}(N^2)$ on $\mathrm{SO}(N^2)$. Again there is an embedding of

$$^L(G^2) = \mathrm{SO}(N^2, \mathbb{C}) \underset{\eta}{\rtimes} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

into

$$^L\big(\mathrm{GL}(N^2)\big) = \mathrm{GL}(N^2, \mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

such that ϕ^2 factors through $^L(G^2)$.

We have shown that the original Galois representation factors through ^LG, for a unique classical group $G = G^1 \times G^2$. The groups obtained in this way (taken together with the embeddings $^LG \hookrightarrow {}^L(\mathrm{GL}(N)))$ are called the twisted endoscopic groups for $\mathrm{GL}(N)$. (See [**11**]). They arise naturally from the twisted trace formula for $\mathrm{GL}(N)$, which of course is where one would begin the study of our problem. If one is interested in the image and fibres of the maps $\pi_G \to \pi$, one should really state the problem in terms of these general endoscopic groups. However, for the study of classical groups, the primitive case that G equals G^1 or G^2 is obviously what is important.

The conjectural description of the contribution of ϕ to the spectrum of G we have alluded to (that is, the preimage in G of the automorphic representation π of $\mathrm{GL}(N)$ attached to ϕ) is given in terms of a group

$$S_\phi = S_\phi(G) = \mathrm{Cent}\big(\mathrm{Image}(\phi), \widehat{G}\big),$$

the centralizer in $\widehat{G}$ of the image of ϕ [**2**, Conjecture 8.1]. For example, ϕ should contribute to the discrete spectrum of G if and only if $S_\phi(G)$ is finite. It is clear that for ϕ as above,

$$S_\phi\big(\mathrm{GL}(N)\big) = (\mathbb{C}^*)^r.$$

One also sees easily that

$$S_\phi(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^r, & \text{if each } N_i \text{ is even,} \\ (\mathbb{Z}/2\mathbb{Z})^{r-1}, & \text{if some } N_i \text{ is odd.} \end{cases}$$

Thus, ϕ contributes to the continuous spectrum of $\mathrm{GL}(N)$, but ought to contribute to the discrete spectrum of G. This property actually characterizes the special case we have been considering. If $\ell_i > 1$ or $\tilde{\phi}_i \neq \phi_i$ for some i, and if ϕ factors through ^LG, the group $S_\phi(G)$ will be infinite. Then ϕ should contribute only to the continuous spectrum of G. In this more general situation, there could also be several different G such that ϕ factors through ${}^L G$.

What is apparent is that one will need some analogue of the group $S_\phi(G)$ to determine the fibres of the map $\pi_G \to \pi$. It is no solution to use $S_\phi(G)$ itself—the Langlands reciprocity law is far from being established, and even if it were, it would not be surjective. What we need instead is the construction of a group $S_\pi(G)$, for any self-contragredient automorphic representation π of $\mathrm{GL}(N)$, which reduces to $S_\phi(G)$ in case π comes from ϕ. Now we can write π formally as

$$\pi = \ell_1\pi_1 \oplus \cdots \oplus \ell_r\pi_r,$$

where each π_i is a (unitary) *cuspidal* automorphic representation of $\mathrm{GL}(N_i)$. The notation means that π is a representation induced from a parabolic subgroup with Levi component

$$\mathrm{GL}(N_1)^{\ell_1} \times \cdots \times \mathrm{GL}(N_r)^{\ell_r},$$

and embedded into $L^2\big(\mathrm{GL}(N,\mathbb{Q})\backslash \mathrm{GL}(N,\mathbb{A})\big)$ by an Eisenstein series. If we would handle the cuspidal components π_i, we could copy the construction above; we would be able to attach twisted endoscopic groups $G = G^1 \times G^2$ to π, and to define the groups $S_\pi(G)$. It is enough to treat the case that π_i is self-contragredient. One would need to show that each such π_i is attached to a unique endoscopic group G_i for $\mathrm{GL}(N_i)$, and that G_i is *primitive* in the sense that $\widehat{G}_i$ equals either $\mathrm{Sp}(N_i,\mathbb{C})$ or $\mathrm{SO}(N_i,\mathbb{C})$. This is essentially part (ii) of the problem stated above.

The remarks of this section have been concerned with setting up the definitions. One needs to define the group $S_\pi(G)$ in order even to state what the image and fibres of the maps $\pi_G \to \pi$ should be. These groups are therefore at the heart of things. The required properties of the cuspidal components π_i will have to be established as part of the full solution of the problem. One can foresee an elaborate inductive argument on the rank N of $\mathrm{GL}(N)$, which is based on the interplay of the stabilized twisted trace formula of $\mathrm{GL}(N)$, and the stabilized trace formulas of the endoscopic groups G.

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