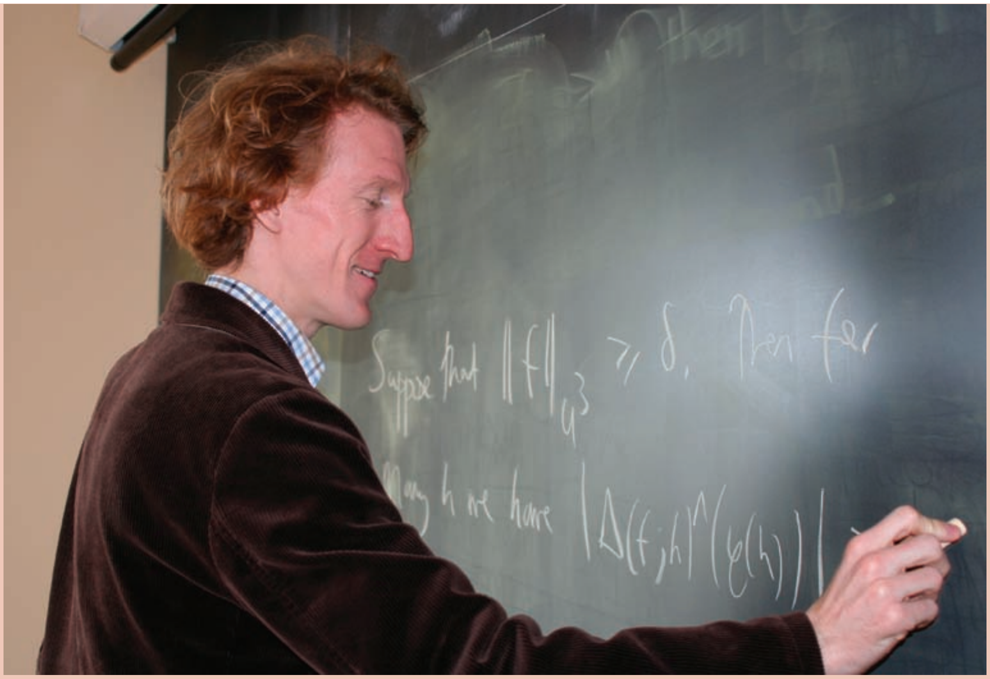
Interview with Research Fellow Ben Green

Profile



Ben Green was born in 1977 in Bristol, England, and educated at Trinity College, Cambridge, first as an undergraduate and later as a research student of Fields Medalist Tim Gowers. Since 2001 he has been a Fellow of Trinity College, and in that time he has made extended research visits to Princeton, the Rényi Institute in Budapest, the University of British Columbia, and the Pacific Institute of Mathematics (PIMS), where he was a postdoctoral fellow. In February 2005 Green was named a Clay Research Fellow. In January 2005, he took up a Chair in Pure Mathematics at the University of Bristol. He began his appointment as a Clay Research Fellow in July 2005, the first year of which he spent at MIT. Ben also spent from February to March of 2006 at CMI working with his student Tom Sanders. In the Spring of 2007, Ben and his student Julia Wolf visited CMI for two weeks.

What first drew you to mathematics? What are some of your earliest memories of mathematics?

I was always very interested in numbers as a small child — my mother tells me that I used to demand "sums" from the age of about 3 and I took an interest in such things as car registration plates and distances on signs which would not, perhaps, be regarded as normal for a young boy. Apparently the head teacher of my primary school (ages 5–11 in the UK) used me as an example of why it is not a good idea to try to teach your children at home, since I had learnt to subtract "the wrong way" (I don't recall the method I was using but, in my parents' defense, it was one I had discovered myself). I first started discovering "real" mathematics around the age of thirteen. The Olympiad movement — taking part in national competitions — was very important to me in this respect. However, I also started paying regular visits to the city library in Bristol, which contained a surprisingly large, if somewhat eccentric, collection of mathematics books. Thankfully, my father could always be persuaded to take me there so that he could indulge his interest in obscure folk and blues music at the same time. Two books which particularly influenced me were Richard Guy's *Unsolved Problems in Number Theory* and Albert Beiler's *Recreations in the Theory of Numbers*.

Could you talk about your mathematical education in the UK? What experiences and people were especially influential? Can you comment on your experiences at Cambridge as an undergraduate? Is there something special in the college system that had a particular impact on your development?

As I said above, the Olympiad movement was very important to me. I was very lucky that there were two teachers at my secondary school, Julie Kirby and Frank Burke, who took an interest in my mathematical development and ensured that I was entered for the national competitions. They (and I) were rather surprised when I obtained the highest mark in one of these competitions (for students under the age of thirteen). My school is currently ranked somewhere around 2000th in the UK academically so we were quite pleased to have scored this very minor victory over the famous schools like Eton and St Paul's. This was when I realized that I had a particular aptitude for mathematics and started taking it more seriously. Subsequently I took part in more senior mathematics competitions and twice represented the UK at the International Mathematical Olympiad. In doing this I made many lasting friends and was influenced by several wonderful teachers. Among these I would single out Tony Gardiner, Christopher Bradley and David Monk who would regularly send me sets of interesting problems by post. At the time the training system in the UK was delightfully low-key and personal, and refreshingly non-intensive. There was a long weekend at Trinity College, Cambridge, but nothing like the "hothouse" training camps some other countries employ.

However, I also started paying regular visits to the city library in Bristol, which contained a surprisingly large, if somewhat eccentric, collection of mathematics books. Thankfully, my father could always be persuaded to take me there so that he could indulge his interest in obscure folk and blues music.

Cambridge is an excellent place to be an undergraduate in mathematics. The course is hard and interesting, and moreover one is surrounded by other good and serious students. Essentially all of my close friends at university have gone on to tenured positions in mathematics of one kind or another. One aspect of the Cambridge education that I like personally is the fact that it is quite hands-off. The example sheets contain tough problems, and one is expected to bash one's head against them repeatedly as one would a research problem. You won't generally find Cambridge supervisors (people who conduct tutorials) giving away the key to the more interesting problems on a sheet unless the student has made a real effort.

At the time the training system in the UK was delightfully low-key and personal, and refreshingly non-intensive. There was a long weekend at Trinity College, Cambridge, but nothing like the "hothouse" training camps some other countries employ.

The collegiate system gives students the opportunity to come in close contact with world-class mathematicians. When I was a first-year undergraduate I was taught as one of a pair by both Tim Gowers and Bela Bollobas, eight times each: that keeps you on your toes, and exposes you to some pretty interesting mathematics.

Did you have a mentor? Who helped you develop your interest in mathematics, and how?

I've mentioned a few great teachers that I had whilst at school. When at university I was heavily influenced by Tim Gowers, who later became my thesis advisor. Towards the end of my thesis I gained a lot by talking to Imre Ruzsa in Budapest – I found we were interested in exactly the same types of questions.

What attracted you to the particular problems you have studied?

I very nearly opted to do a Ph.D. in algebraic number theory, but some somewhat negative experiences of this area in my last year as an undergraduate, coupled with the recent award of a Fields Medal to Tim Gowers, persuaded me to work under Gowers in the area now known as additive combinatorics. The area is appealing in that the problems may be stated quite easily to a general mathematical audience. A particular attraction for me was that I could embark on research straight away – I did not need to go and read Hartshorne, let alone SGA.

It is hard to say exactly what it is that attracts me to a problem nowadays. I am particularly fond of instances in which it is possible to extract "rigid" structure from rather soft information – in fact most of the questions I am working on right now have this kind of flavor. A theorem of this type that I very much admire (though I don't quite know how to prove, I'm ashamed to say) is Marina Ratner's theorem on the closures of orbits of unipotent flows. She related these to exact subgroups — that is, she took soft information (in this case a dynamical system) and found algebraic structure in it. Terry Tao and I are working on Freiman's theorem and on inverse theorems for the so-called Gowers norms — in both of these one starts with something very combinatorial and produces an algebraic object from it.

Another thing we try and do is make "robust" versions of algebraic results. What is meant by an approximate group? An approximate homomorphism? How do these relate to the corresponding "exact" structures? Often much can be gained by enlarging one's universe to include these approximate algebraic objects, provided one is able to handle the requisite approximate algebra.

Of course I am also motivated by the desire to prove results on the basic questions in number theory, say about prime numbers. But my results with Tao in this area have really come out of an attempt to understand the underlying structures in a more general context.

Can you describe your research in accessible terms? Does it have applications to other areas?

Right now I am working with Tao on generalizing the Hardy-Littlewood method for primes as far as we can. Using this method, Vinogradov proved in 1937 that every large odd number N can be written as the sum of three primes. We have a program which should eventually allow us to count solutions to a more-or-less arbitrary system of linear equations in primes (an example that we have already dealt with is the system $p_1 + p_3 = 2p_2$, $p_2 + p_4 = 2p_3$, which defines an arithmetic progression of four primes). There is one important exception — we do not have a feasible plan for handling certain "degenerate" systems, which include the system $p_1 - p_2 = 2$ (twin primes) and $p_1 + p_2 = N$ (Goldbach conjecture).

Although people seem to like results about the primes, from a mathematician's point of view the underlying methods are much more interesting. Our work, together with the work of many other people, has hinted at deep connections between several areas of mathematics: analytic number theory, graph theory, ergodic theory and Lie groups.

What research problems and areas are you likely to explore in the future?

There is plenty of work left to be done on the program I have just described, and a really serious amount of work to be done on the general area of "rigidity" results in additive combinatorics and their applications. A proper quantitative understanding of three main types of result in this vein (Freiman-type theorems, inverse theorems for Gowers-type norms and Ratner's theorem) is probably decades away. In the longer term I want to become more competent with "non-abelian" tools and questions, that is to say the theory of "multiplicative combinatorics". Who knows what may be brought to bear here — given the prevalence of Fourier-analytic methods in additive combinatorics, it seems likely that representation theory will have a major role to play. I also have quite a long list of miscellaneous problems that I would like to think about at some point.

Could you comment on collaboration versus solo work as a research style? Are certain kinds of problems better suited to collaboration? What do you find most rewarding or productive?

I just noticed, looking at my webpage, that almost all of my first ten papers had just me as an author, whereas my ten latest are all coauthored. I have never written a three-author paper, but have found collaboration in pairs very productive. It took me a while to realize that collaboration works best when both parties are completely open to sharing their best ideas — when I was a Ph.D. student I was terrified that people might steal my ideas, or jump in on a paper that I had 95 percent finished. That attitude was probably fairly sensible at that stage, but with the luxury of a tenured job I take a much more open position. My joint paper with Tao on arithmetic progressions of primes was a memorable example of collaboration (it was mostly done in a rapid-fire exchange of emails). I am sure Terry would agree that this result could never have been proved by either of us individually.

Although people seem to like results about the primes, from a mathematician's point of view the underlying methods are much more interesting. Our work, together with the work of many other people, has hinted at deep connections between several areas of mathematics: analytic number theory, graph theory, ergodic theory and Lie groups.

You have taken on thesis students at a very early stage in your career. Was that a conscious decision? How did you first start working with research students?

Does working with students have rewards as well as responsibilities?

I currently have three Ph.D. students and also talk quite a bit to other students in additive combinatorics at Cambridge. I started working with Tim Gowers' student Tom Sanders about four years ago, largely because he bugged me quite persistently with questions about the projects he was thinking about. After a while I came to realize that I rather enjoyed these discussions and resolved to take on a few good students should any come my way. I have a theory that having two children is less work than having one, as they can play with one another (I currently have none, so this hasn't been tested very thoroughly). I believe that this carries over in a reasonably obvious way to graduate students — we hold regular reading seminars as a group and they can talk amongst themselves when I am not available.

How has the Clay Fellowship made a difference for you?

It allowed me to spend the whole academic year 2005–06 at MIT, which was handy since my girlfriend is doing a Ph.D. at Harvard. I was also able to bring Tom Sanders over for a few months during this time, and we had a very productive period leading to an Annals paper that I'm very happy with. There is no doubt that the Clay Research Fellowship has some of the best conditions of any postdoc out there — no teaching duties, excellent funds for travel, and so on — and this allows the Fellow to work very intensively on research.

What advice would you give to young people starting out in math (i.e., high school students and young researchers)?

A few tips that I have found handy, in no particular order: 1. At high school, it's good to have the experience of tackling really hard problems (and failing, more often than not). Real mathematics is not as "safe" as Olympiad mathematics in that you don't have an *a priori* upper bound for the difficulty of the problem. I've listed a few books that I enjoyed reading at school in one of my answers below. 2. Follow your nose, not necessarily what other people tell you, when you choose what questions you work on. I have worked on some questions which even people in my own subject would probably think uninteresting. I've certainly written papers on questions that nobody (before me) asked. Naturally, over the course of a career (and to get a job) you want to have some results that a lot of people *are* interested in. Let me just say, however, that I can trace my line of thought that eventually led to my joint paper on arithmetic progressions of primes back to a paper Ruzsa and I wrote in answer to a question of Jacques Verstraete: how many of the subsets of $\mathbf{Z}/p\mathbf{Z}$ have the form $A + A$, for some set A in $\mathbf{Z}/p\mathbf{Z}$? I think most people would think of that question as more of a "puzzle" than a serious problem. 3. Check the ArXiv every day and use MathSciNet obsessively. The latter is a wonderful resource — all the papers in mathematics (certainly all those in the last 60 years) are indexed, cross-linked and reviewed.

What advice would you give laypersons who would like to know more about mathematics — what it is, what its role in our society has been and is, etc.? What should they read? How should they proceed?

Well, I find it hard to do better than recommend my advisor Tim Gowers' little book entitled *Mathematics, A Very Short Introduction*, the aim of which is pretty much to answer those questions. A couple of books that I really enjoyed as a teenager, long before I had any real understanding of what mathematics was about, are *The Mathematical Experience* by Davis and Hersh and *Game, Set and Math: Enigmas and Conundrums* by Ian Stewart. Both of these books do have *some* mathematics in them but they are certainly accessible to bright high-school students. Concerning the history of mathematics, I recall getting a lot from *Makers of Mathematics* by Stuart Hollingdale. Maybe some of these choices are eccentric — perhaps they were just the books that Bristol library had in stock — but I certainly enjoyed them myself.

There was a TV program in Britain about Wiles' proof of Fermat's last theorem which gave a wonderful insight into the personalities and mode of working of mathematicians.[1] I don't know how widely available it is.

1. Ben refers to the BBC documentary *Fermat's Last Theorem* that was written and produced by Simon Singh and John Lynch. Later, the same documentary (reversioned for American audiences and renamed *The Proof*) aired on PBS as part of the NOVA series. For more information, see http://www.pbs.org/wgbh/nova/proof/.

To get some sense of the way mathematicians talk to one another, it could be fun to check out one of the increasing number of mathematicians' blogs. Terry Tao has recently created one which attracts a lot of attention, and I have followed Luca Trevisan's "In Theory" for a while.

And of course the Clay Institute has some pretty interesting and accessible lectures linked from its website.

How do you think mathematics benefits culture and society?

Though this question seems like an invitation to say something wildly pretentious, I'll try and avoid doing so. I think one only needs to look at the attractiveness of mathematics graduates on the job market to realize that the mathematician's way of thinking is something that can be extremely useful in many areas of society. I doubt that most jobs require a specific knowledge of homological algebra (say) but the ability to think creatively within the confines of logic and to think "out of the box" are clearly very important everywhere. Let me stop before I start sounding like a management consultant.

I personally find that mathematics is a wonderful way of breaking down cultural barriers. For example I spent several months working in Hungary even though I speak (almost) no Hungarian. I doubt that would have been possible in many other walks of life.

Please tell us about things you enjoy when not doing mathematics.

Unlike quite a lot, possibly even most, other mathematicians, I almost completely avoid activities like chess, bridge or computer programming. When I'm not doing mathematics I like to do something that doesn't use my brain so intensively. I'm a keen cyclist and outdoor enthusiast, I enjoy playing cricket (in the summer) and I play jazz saxophone to a rather mediocre standard.

You were recently appointed a full professor at Cambridge. Congratulations! What are you planning next?

Well I was very pleased to get the job at Cambridge and I don't anticipate moving on for at least ten years or so. I want to develop a group of students and postdocs here, a seminar series, and graduate courses. I'm very happy with the way my career has gone so far but it is important to avoid burnout. I believe that diversity in research is the key to that — I always like to feel that one of my projects could be completely taken away (solved by someone else or studied from a totally new perspective that I don't understand, say) and I'd still have a decent portfolio of research projects.

Tom Sanders and Ben Green at the Clay Mathematics Institute.

Recent Research Articles

"Linear Equations in primes," with Terence Tao, to appear in *Annals of Math.*

"A quantitative version of the idempotent theorem in harmonic analysis," with T. Sanders, to appear in *Annals of Math.*

"Freiman's theorem in finite fields via extremal set theory," with Terence Tao, arXiv:math/0703668

"A note on the Freiman and Balog-Szemeredi-Gowers theorems in finite fields," with Terence Tao, arXiv: math/0701585

"New bounds for Szemeredi's theorem, II: A new bound for $r_4(N)$," with Terence Tao, arXiv:math/0610604