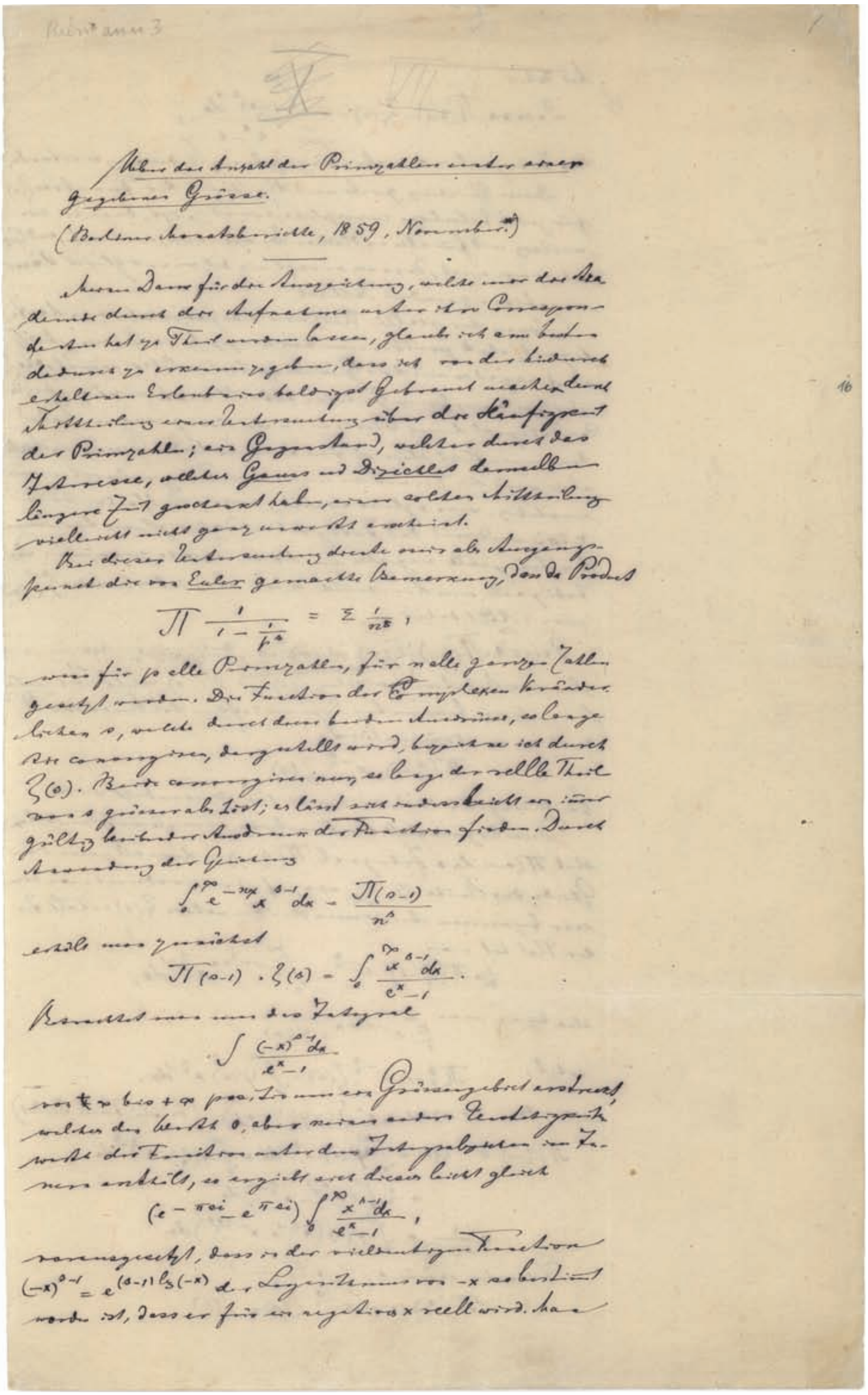
Below is the manuscript, just seven pages in length, of Riemann's paper. The photos of the manuscript are courtesy of the Niedersächsische Staats- und Universtätsbibliothek Göttingen. A complete (but lower resolution) digital facsimile courtesy of the same source is available online at www.claymath.org/millennium/Riemann/



Courtesy of the Niedersächsische Staats- und Universtätsbibliothek Göttingen.

The Riemann hypothesis, the oldest of the Millennium Prize Problems, was formulated in 1859 by Bernhard Riemann as part of his attempt to understand how prime numbers are distributed along the number line. In some sense, the problem goes back to Euclid, in whose Elements one finds that the number of primes is infinite (theorem IX.20). Now, there are just four primes less than ten, namely 2, 3, 5, 7; twenty-five primes less than one hundred; etc. But what is the general pattern? Relying on "experimental data" obtained at the cost of much computation, Gauss, some time around 1800, formulated a conjecture: that the number of primes less than x is approximately $x/\log(x)$, where the logarithm is the natural one. This quantity is proportional to x divided by the number of digits in x. Moreover, Gauss stated that the relative error between his $x/\log(x)$ guess and the number of primes tends to zero as x tends to infinity.

It was this conjecture of Gauss that Riemann attempted to prove using bold new ideas, of which his hypothesis was a byproduct. Although Riemann never formulated a proof, two great trees of mathematics grew from his work. First was Gauss' conjecture, proved independently by Hadamard and de la Vallée Poussin in 1896, and now baptized the "Prime Number Theorem." Both men relied heavily on Riemann's methods. Second was the set of profound results inspired by the effort to prove the Riemann hypothesis itself. Deepest of all are the Weil conjectures, proved by Pierre Deligne in 1974.

Peter Sarnak's article, beginning on page 5, and Enrico Bombieri's official problem description for the Clay Institute give precise statements of the Riemann hypothesis. Together they provide an account of its history and of recent work on it. But what is this hypothesis in a few, if less precise, words? We state it in an equivalent form: the difference between the actual number of primes and Gauss's formula is not only small compared to x, it is smaller by a considerable amount. More precisely, it is smaller by a factor proportional to the reciprocal of the square root of x, divided by $\log(x)$. This factor is rather like one encountered in the theory of random walks, gambling, and Brownian motion. In such statistical phenomena the square root of x enters as a measure of the deviation about some average value. Thus, the Riemann hypothesis is a statement about the "statistical regularities" one can expect of the prime numbers.