Bott periodicity II

A ring with unit

$\Phi(A)$ semi-group of $\approx$ classes of f.g.p modules

$K(A) = S(\Phi(A))$ group associated to $\Phi(A)$

$= \Phi(A) \times \Phi(A) / \sim$

$(a, b) \sim (c, d)$

$\iff \exists e \mid a + d + e = b + c + e$

A	$K(A)$
field	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$
principal ring	$\mathbb{Z}$
$\mathbb{Z}(\sqrt{-5})$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$\mathbb{C}[G]$ G finite	$\mathbb{Z}^N$, $N = \#$ irreducible representations of G
$\mathbb{C}[\mathfrak{S}_n]$	same, $N = \#$ partitions of n
$C_{\mathbb{C}}(S^1)$	$\mathbb{Z}$
$C_{\mathbb{C}}(S^2)$	$\mathbb{Z} \oplus \mathbb{Z}$
$C_{\mathbb{R}}(S^1)$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$C_{\mathbb{R}}(S^2)$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$M_n(A)$	$K(A)$ (Morita equivalence)
$\mathcal{K}$	$\mathbb{Z}$ (ring of compact operators in H)
$\mathcal{B}(H)$	0

Q] Is there an "algebraic" definition of $K_n(A)$, A discrete?

Note: if A is a Banach algebra

$$K_n(A) \approx \pi_{n-1}(GL(A))$$

Q] What is an "algebraic" definition of hty groups for such objects like GL(A)?

Why? Many rings are NOT Banach algebras.

From now on, we assume that A is a "nice" ring (regular noetherian)

__ex__ $\mathbb{C}[x,y]/(P(x,y))$ curve in a plane without singularities

__1st case__ $\pi_0(GL(A))$

$$[0,1] \xrightarrow{?} GL(A)$$

The answer is to take Polynomial paths

<u>Def</u> A polynomial path is just an element of $GL(A[x])$

<u>ex</u> $\begin{pmatrix} 1 & x \\ 0 & a \end{pmatrix}$ a invertible

<u>Def</u> $\alpha_0, \alpha_1 \in GL(A)$ are homotopic $\iff \exists \beta(x) \in GL(A[x])$

s.t. $\beta(0) = \alpha_0 \quad \beta(1) = \alpha_1$

$$\pi_0^{alg}(GL(A)) = GL(A)/\sim$$

<u>Ex</u> $A =$ field F

$$\pi_0^{alg}(GL(F)) \xrightarrow[\text{determinant}]{\simeq} F^*$$

Any matrix in $GL_2(F)$ is a product of elementary matrices by a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(Gauss reduction)

$$\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

<u>Def</u> $K_1(A) = \pi_0^{alg}(GL(A))$

(2) Beware, if A is a Banach algebra, we have 2 definitions of $K_1(A)$. We call the first one $K_1^{top}(A)$

$$K_1(A) \longrightarrow K_1^{top}(A)$$

$$\begin{array}{ccc} K_1(\mathbb{C}) & \longrightarrow & K_1^{top}(\mathbb{C}) \\ \| & & \| \\ \mathbb{C}^* & & 0 \end{array}$$

Ex A ring of integers in a # field.

Then
$$K_1(A) \simeq A^*$$

(Bass-Milnor-Serre)

Note A^* has been computed by Dirichlet.

Ex $K_1(\mathbb{Z}) = \mathbb{Z}/2$ (easy)

$$K_1\left(\mathbb{Z}[x]/(1+x+\dots+x^{p-1})\right)$$
$$\simeq \mathbb{Z}^{\frac{p-1}{2}} \times \mathbb{Z}/p$$

Note $K_1(A)$ can be extended to non unital rings

$$K_1(A) = \mathrm{Ker}\left(K_1(A^+) \longrightarrow K_1(k)\right)$$

loop space $\Longleftrightarrow$ loop ring

$[0,1] \xrightarrow{f} X$

$f(0) = f(1) = x_0$

$\alpha \in A[x]$

$\alpha(0) = \alpha(1) = 1$

$$\Omega A = \{\alpha \in A[x] \mid \alpha(0) = \alpha(1) = 0\}$$

Def $K_n(A) = K_0(\Omega^n A)$

$= K_1(\Omega^{n-1} A)$

(Q) Bott periodicity?

$$K_n(A) \simeq K_{n+\alpha}(A)$$

A field $= F$

$K_1(F) = F^*$, $K_2(F) = ?$

$\exists$ a cup-product (A commutative)

$$K_n(A) \times K_m(A) \longrightarrow K_{n+m}(A)$$

$$\begin{array}{ccccc} K_1(F) & \times & K_1(F) & \longrightarrow & K_2(F) \\ \| & & & & \\ F^* & \times & F^* & \longrightarrow & K_2(F) \\ (u,v) & & & \longmapsto & \{u,v\} \end{array}$$

Thm (Matsumoto)

$K_2(F)$ is generated by

$\{u, v\}$

$\{uu', v\} = \{u, v\} + \{u', v\}$

$\{u, vv'\} = \{u, v\} + \{u, v'\}$

$\{u, 1-u\} = 0 \qquad \text{if } u \in F - \{0, 1\}$

Ex $K_2(\mathbb{Q}) = \mathbb{Z}/2 \oplus \bigoplus_p (\mathbb{Z}/p)^*$

Ex $K_2(\mathbb{Z}) = \mathbb{Z}/2$ (Magnus)

$K_3(\mathbb{Z}) = \mathbb{Z}/48$ (Lee-Szczarba)

i	$K_i(\mathbb{Z})$	
0	$\mathbb{Z}$	easy
1	$\mathbb{Z}/2$	easy
2	$\mathbb{Z}/2$	diff.
3	$\mathbb{Z}/48$	v. diff
4	0	
5	$\mathbb{Z}$	
6	0	
7	$\mathbb{Z}/24$ (?)	
8		

Bernoulli numbers $B_k = c_k/d_k$

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k!)} x^{2k} + \dots$$

Conjecture

$$K_{8n}(\mathbb{Z}) = 0 \qquad (n>0)$$

$$K_{8n+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/_2 \qquad (n>0)$$

$$K_{8n+2}(\mathbb{Z}) = \mathbb{Z}/_2 \oplus \mathbb{Z}/_{c_k} \qquad (k=2n+1)$$

$$K_{8n+3}(\mathbb{Z}) = \mathbb{Z}/8k.d_k \qquad (k=2n+1)$$

$$K_{8n+4}(\mathbb{Z}) = 0$$

$$K_{8n+5}(\mathbb{Z}) = \mathbb{Z}$$

$$K_{8n+6}(\mathbb{Z}) = \mathbb{Z}/_{c_k} \qquad (k=2n+2)$$

$$K_{8n+7}(\mathbb{Z}) = \mathbb{Z}/4k.d_k$$

$$(k=2n+2)$$

Ex

$$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$$

$$K_{23}(\mathbb{Z}) = \mathbb{Z}/65520$$

???

Bott periodicity?

FALSE in general (naive sense)

Finite field $\mathbb{F}_q$ $(q = p^n)$

$$K_{2i}(\mathbb{F}_q) = 0$$

$$K_{2i-1}(\mathbb{F}_q) \simeq \mathbb{Z}/(q^i - 1)\mathbb{Z}$$

(Quillen)

"K-theory with finite coefficients"

$$\xrightarrow{\cdot n} K_i(A) \longrightarrow K_i(A; \mathbb{Z}/n) \longrightarrow K_{i-1}(A) \xrightarrow{\cdot n}$$

e.g. F field

$$\Rightarrow K_1(F; \mathbb{Z}/n) = F^* / F^{*n}$$

Thm (Suslin)

F alg. closed field, then

$$K_{2i-1}(F; \mathbb{Z}/n) = 0$$

$$K_{2i}(F; \mathbb{Z}/n) \approx \mathbb{Z}/n \simeq \mu_n^{\otimes i}$$

(n prime to char F)

$$K_i(\mathbb{C};\mathbb{Z}/n) \xrightarrow{\simeq} K_i^{top}(\mathbb{C};\mathbb{Z}/n)$$

Analogous result for $\mathbb{R}$

$$K_i(\mathbb{R};\mathbb{Z}/n) \simeq K_i^{top}(\mathbb{R};\mathbb{Z}/n)$$

periodic of period 8!

<u>Another formulation</u>

$$\cdot K_i(F) \simeq \mathbb{Z}/n \oplus G \qquad i \text{ odd}$$

$$\simeq G' \qquad i \text{ even}$$

G, G' uniquely divisible groups

(if F is algebraically closed)

what about $K_i(F)$
for an arbitrary field F?

Conjecture (almost proved)

$\bar{F}$ alg. closure

Then $K_i(F; \mathbb{Z}/n)$ can be
computed (spectral sequence)
from $K_i(\bar{F}; \mathbb{Z}/n)$

(hidden Bott periodicity)

Hermitian K-theory

$GL(A) \longrightarrow$ orthogonal group $O(A)$

A ring with involution $a \mapsto \bar{a}$

$2n \times 2n$

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad M^* = \begin{pmatrix} {}^t\bar{\delta} & +{}^t\bar{\beta} \\ +{}^t\bar{\gamma} & {}^t\bar{\alpha} \end{pmatrix}$$

$$O_{2n}(A) = \{ M \mid MM^* = M^*M = Id \}$$

$$O(A) = \cup \, O_{2n}(A)$$

$$\boxed{L_i(A) = \pi_{i-1}(O(A))}$$

$$O(A) \subset GL(A)$$

$$0 \to W_i(A) \to L_i(A) \to K_i(A)$$

$$W_i(A) = \mathrm{Ker}\,(L_i(A) \to K_i(A))$$

<u>Thm</u> $\exists \quad W_i \underset{v}{\overset{u}{\rightleftarrows}} W_{i+4}$

$u.v$ and $v.u$ are 16.

$\Rightarrow$ There is a true periodicity for W_i (up to 2 torsion)

i	$W_i(\mathbb{Z}')$
0	$\mathbb{Z} \oplus \mathbb{Z}/2$
1	$(\mathbb{Z}/2)^2$
2	$\mathbb{Z}/2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

$(\mathbb{Z}' = \mathbb{Z}[\tfrac{1}{2}])$

$$W_i(\mathbb{Z}') \simeq W_{i+8}(\mathbb{Z}')$$

$$W_i(C_{\mathbb{R}}(X)) \xrightarrow{\simeq} W_i^{top}(C_{\mathbb{R}}(X)) = K_i^{top}(C_{\mathbb{R}}(X))$$

Applications

M manifold $= S^n$

$M \times [0,1] \circlearrowleft$ diffeo $=$ Id on $M \times \{0\}$

(pseudo isotopies)

$$\pi_i(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_{i+2}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$K_2(F)/nK_2(F) \simeq H^2(G; \mu_n^{\otimes 2})$$

In particular, if $\mu_n \subset F$

$$Br(F)_n \simeq K_2(F)/nK_2(F)$$

(Merkurjev. Suslin)

why "K" theory?

It comes from

"Klassen"

in German

(Grothendieck mother's tongue)