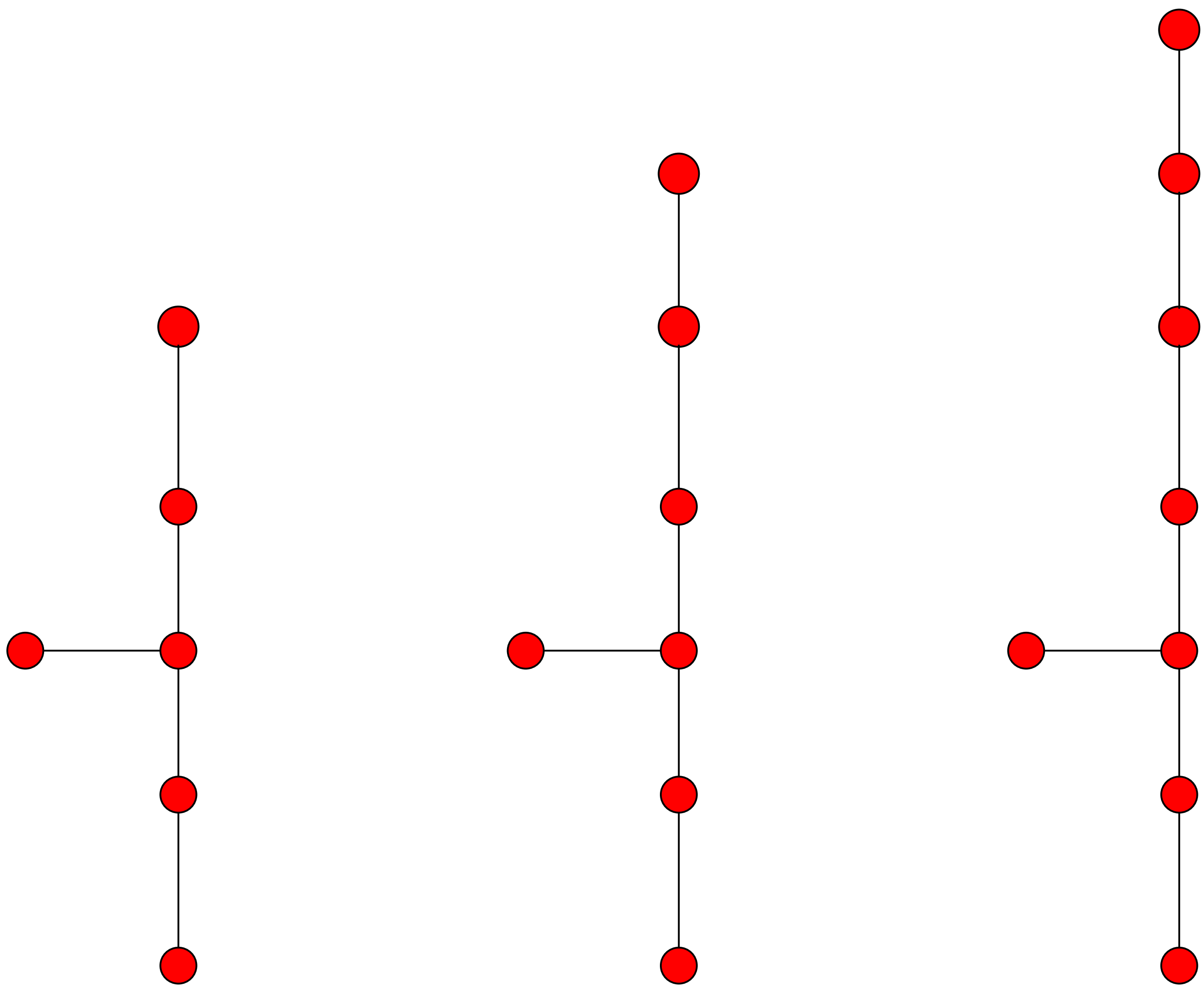
E6, E7, E8

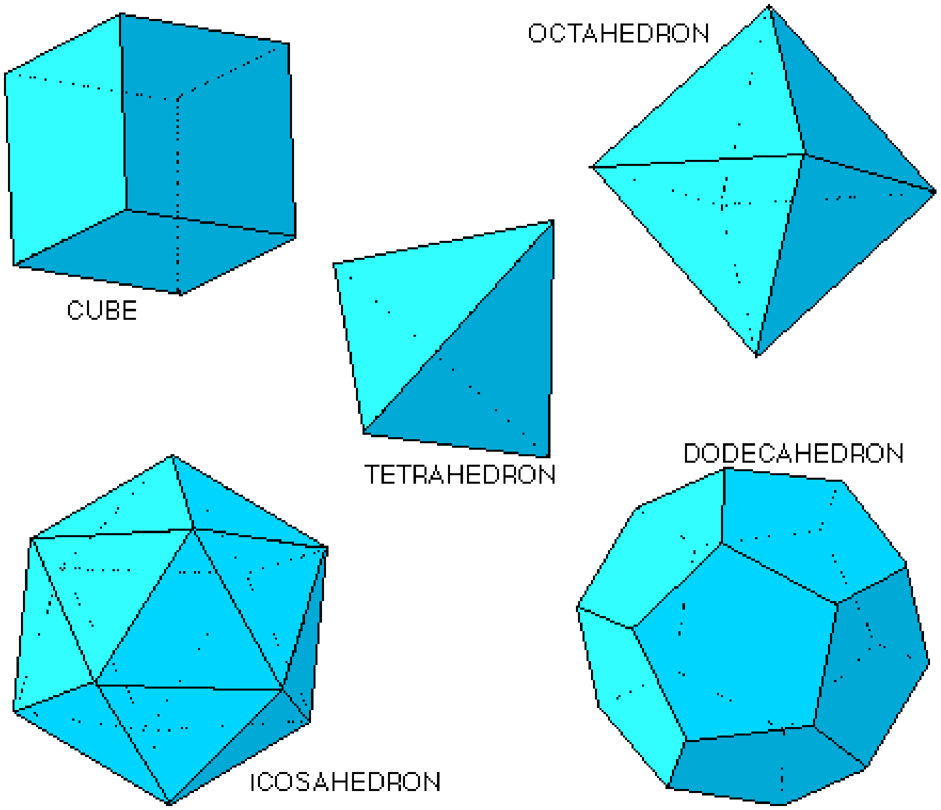
Nigel Hitchin (Oxford)

CLAY ACADEMY LECTURE

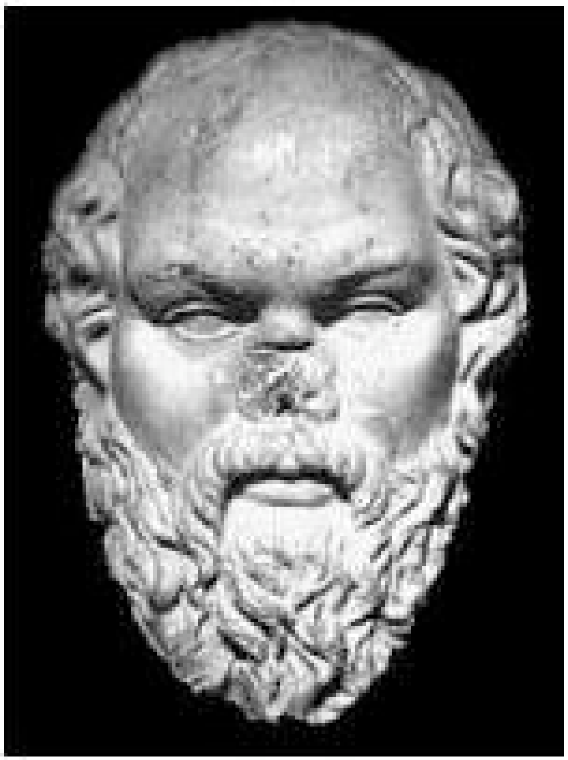
April 24th 2005

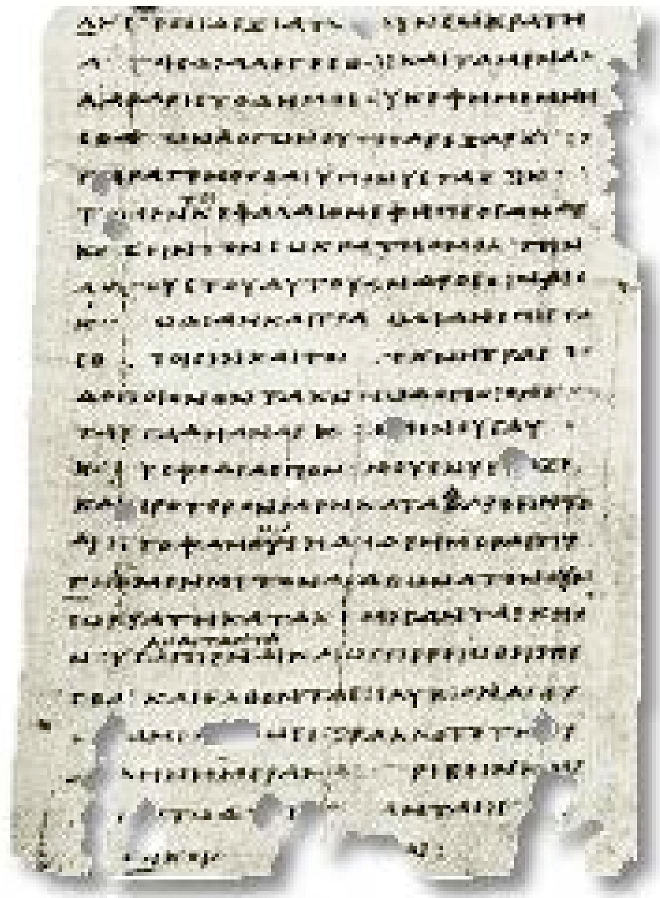


THE PLATONIC SOLIDS

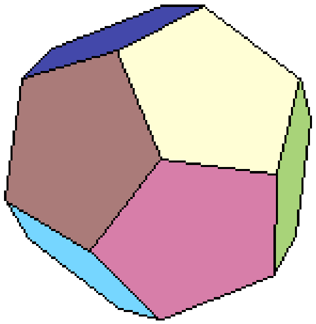


PLATO 427-347 BC





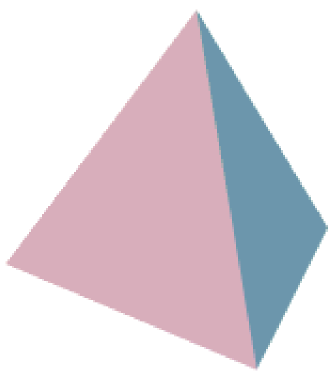
HIPPASUS 500 BC

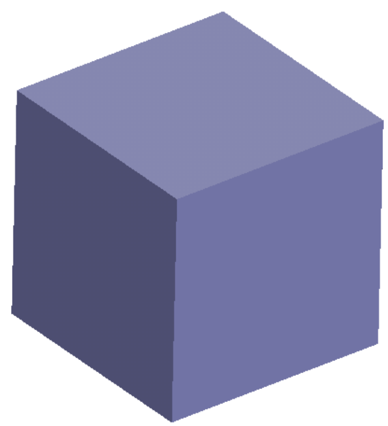


"... because he was the first to make public the secret of the sphere of twelve pentagons, he perished at sea"

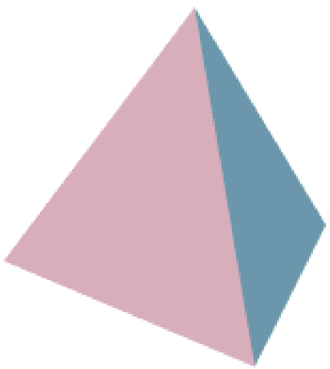
Iamblicus 300 AD

SKARA BRAE, ORKNEY 3000 BC

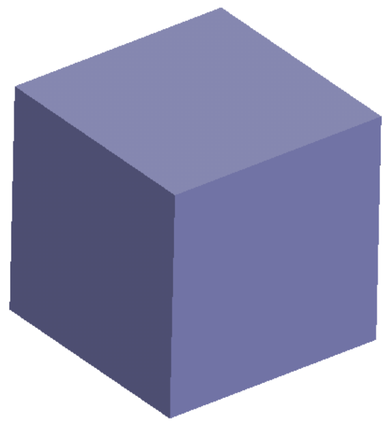




E6

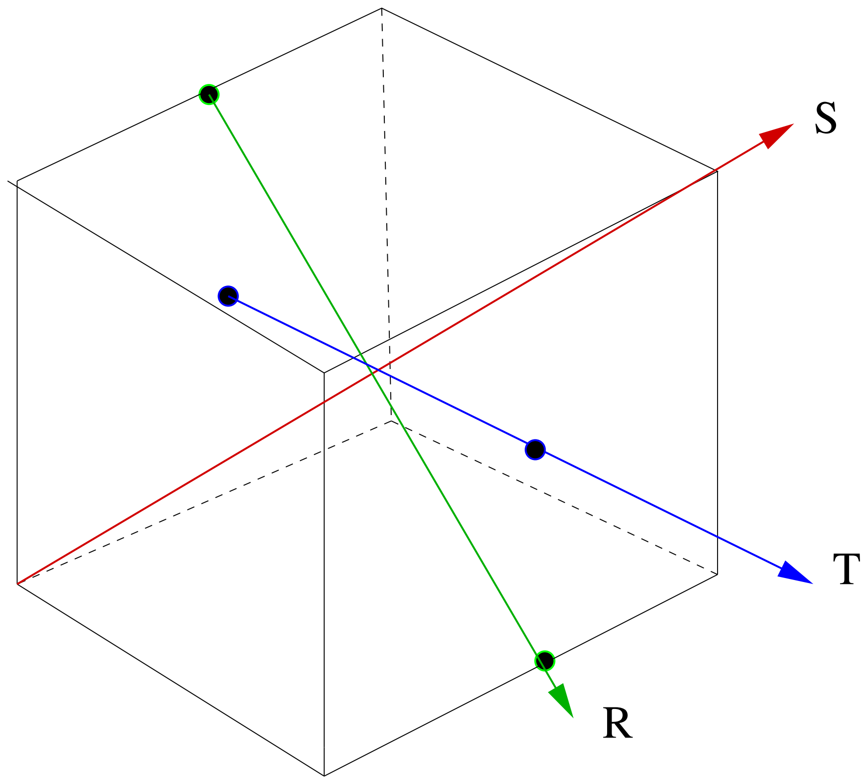


E7



E8

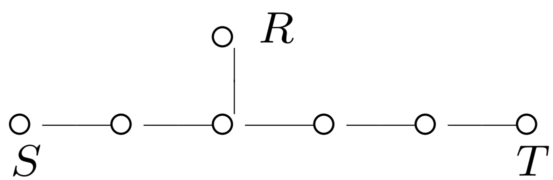
SYMMETRIES OF THE CUBE



$$R^2 = S^3 = T^4 = RST = 1$$

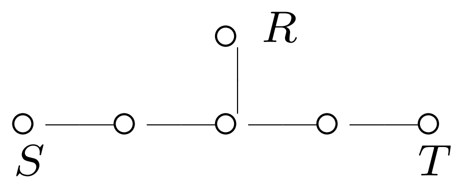
DYNKIN DIAGRAM

- $\mathbf{E_7}$



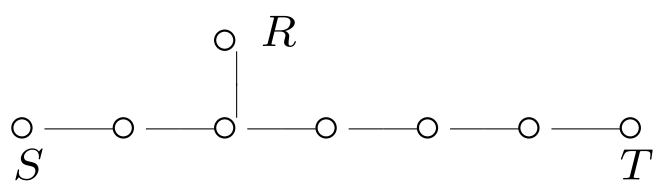
$$R^2 = S^3 = T^4 = RST = 1$$

- $\mathbf{E_6}$



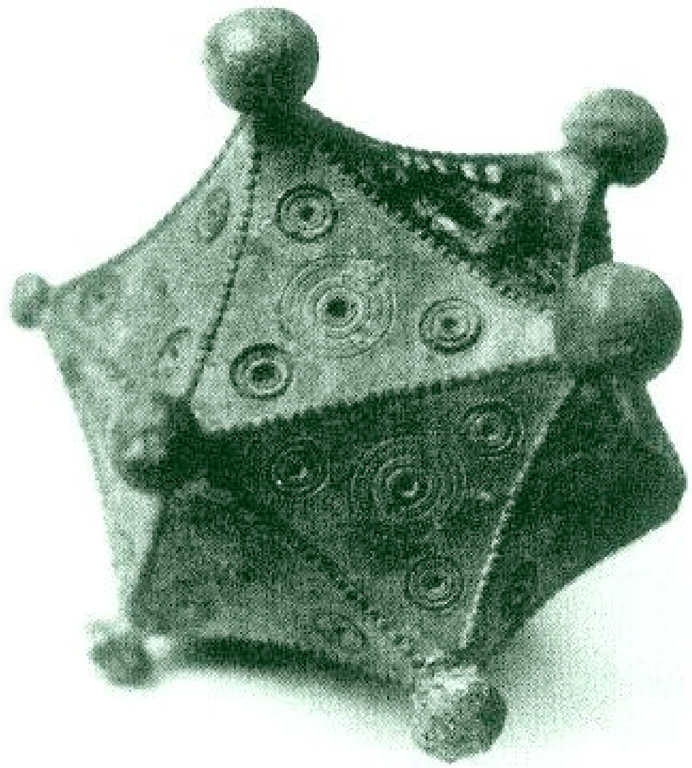
$$R^2 = S^3 = T^3 = RST = 1$$

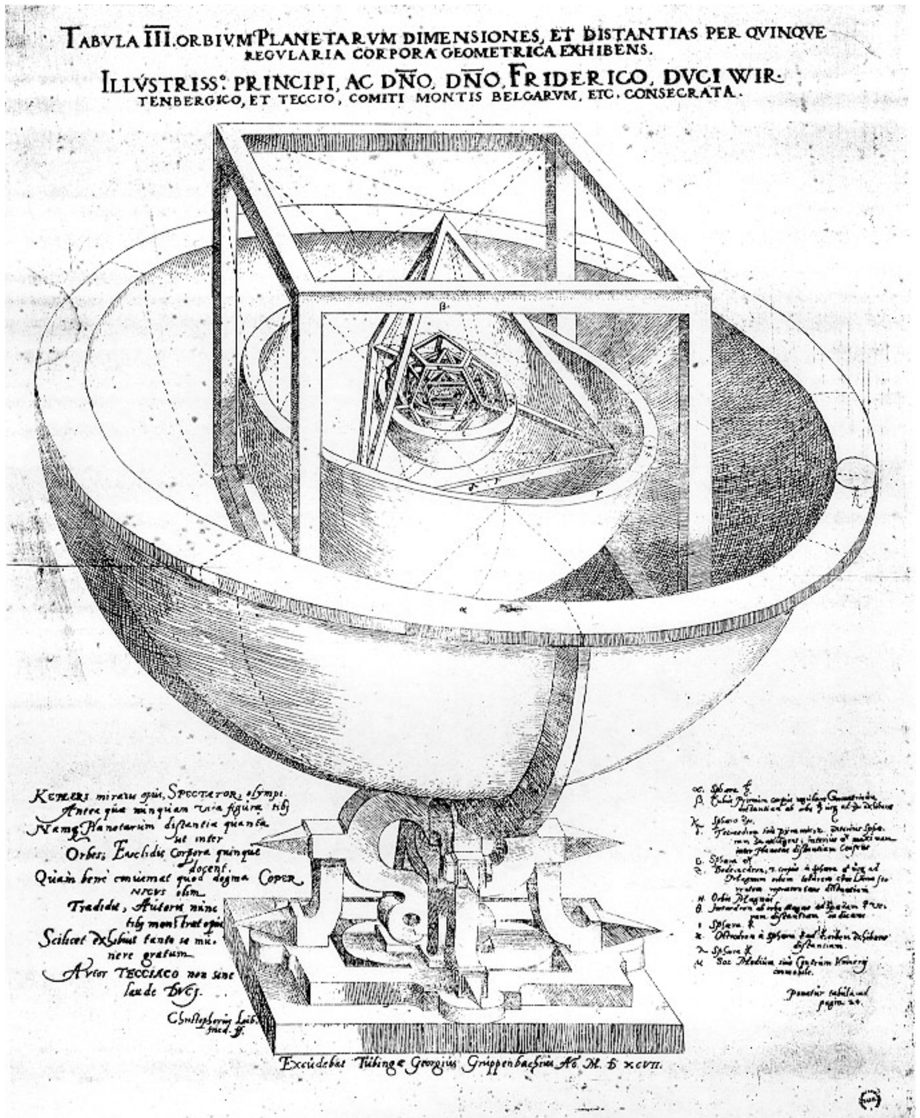
- $\mathbf{E_8}$



$$R^2 = S^3 = T^5 = RST = 1$$

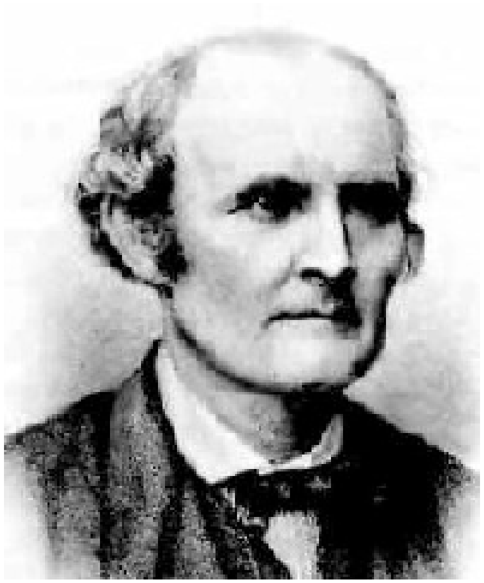
... THE ROMANS




TABVLA III. ORBIVM PLANETARVM DIMENSIONES, ET DISTANTIAS PER QVINQUE REGVLARIA CORPORA GEOMETRICA EXHIBENS.
ILLVSTRISS: PRINCIPI, AC DÑO. DÑO. FRIDERICO, DVCI WIRTENBERGICO, ET TECCIO, COMITI MONTIS BELGARVM, ETC. CONSECRATA.

KEPLER ...

... THE 19th CENTURY



27 lines on a cubic surface (A.Cayley and G.Salmon, 1849)

 28 bitangents to a plane quartic curve (C.Jacobi, 1850)

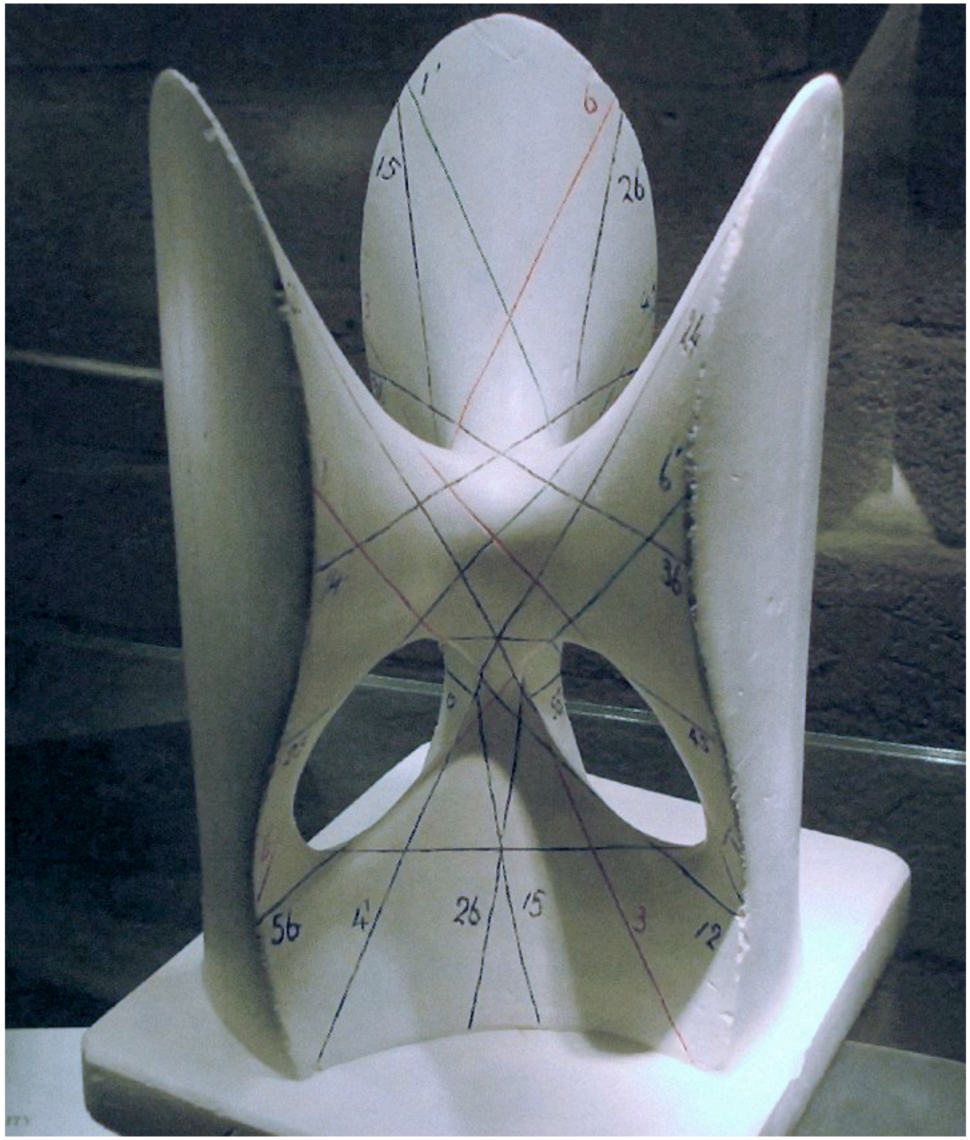
 120 tritangent planes to the intersection of a cubic surface and a quadric surface (A.Clebsch, 1863)

THE CUBIC SURFACE

In 1849 Cayley pointed out that there was a definite number of lines on a cubic surface. Salmon proved that this number was 27.

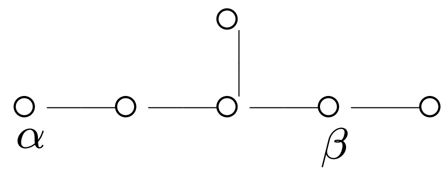
"Surely with as good reason as had Archimedes to have the cylinder, cone and sphere engraved on his tombstone might our distinguished countrymen leave testamentary direction for the cubic eikosiheptagram to be engraved on theirs".

J.Sylvester


6
15
26
36
56
4'
26
15
3
12

- 6 disjoint red lines, 6 disjoint green ones
- 15 black ones intersecting two of each
- 72 ways of choosing 6 disjoint lines
- a group of order $51,840$ changes the labellings of the 27 lines preserving their intersection properties

$\mathbf{E_6}$



- $s_\alpha s_\alpha = 1$
- $(s_\alpha s_\beta)^2 = 1$ if α, β are not adjacent
- $(s_\alpha s_\beta)^3 = 1$ if α, β *are* adjacent

group = symmetries of the 27 lines.

THE QUARTIC CURVE

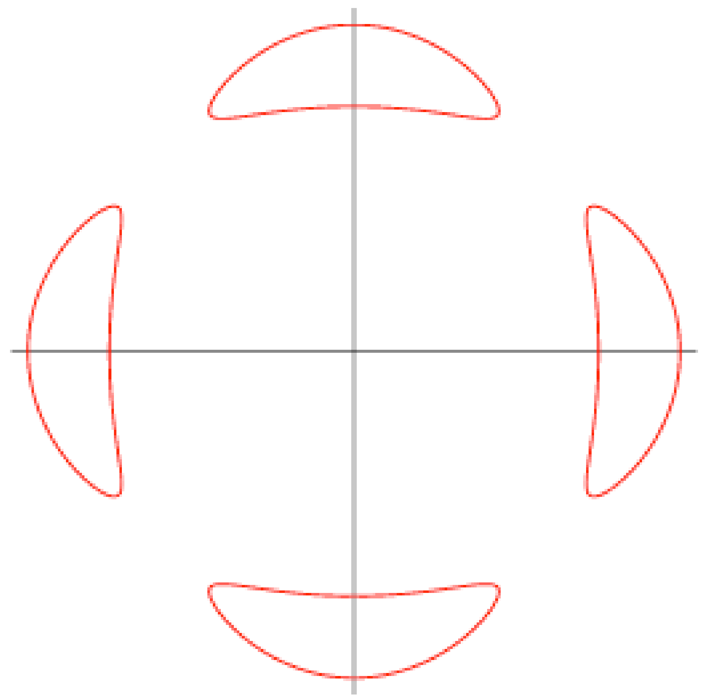
- $ax^4 + by^4 + c + fy^2 + gx^2 + hx^2y^2 + lx^2y +$

  $+my^2x + nxy + px^3y + qx^3 + ry^3x + sy^3 + tx + uy = 0$

- Every quartic has precisely 28 bitangents.

- There are 1260 combinations of three bitangents whose points of intersection with the quartic lie on a conic.

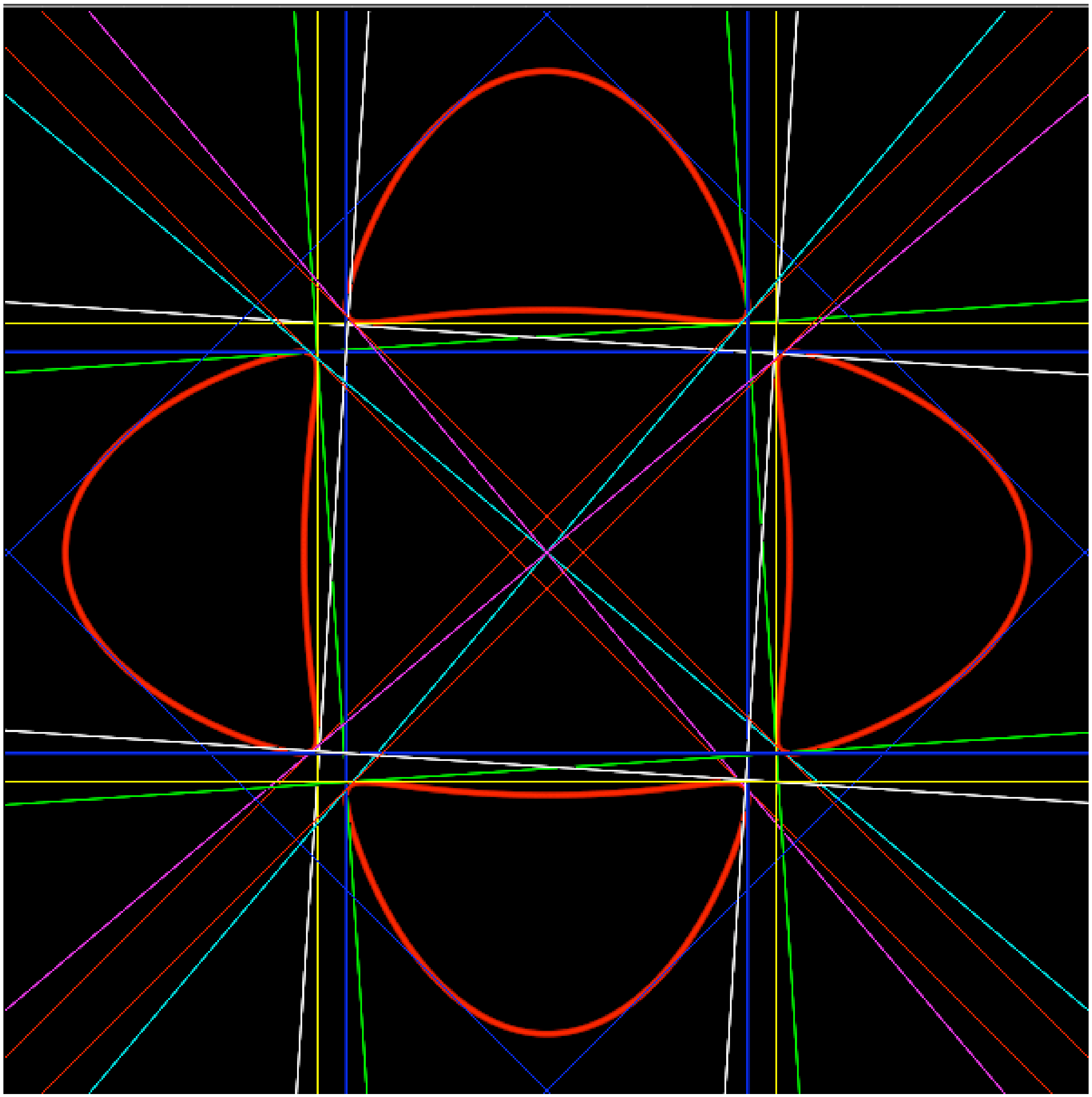
- There are 2016 combinations which do not.

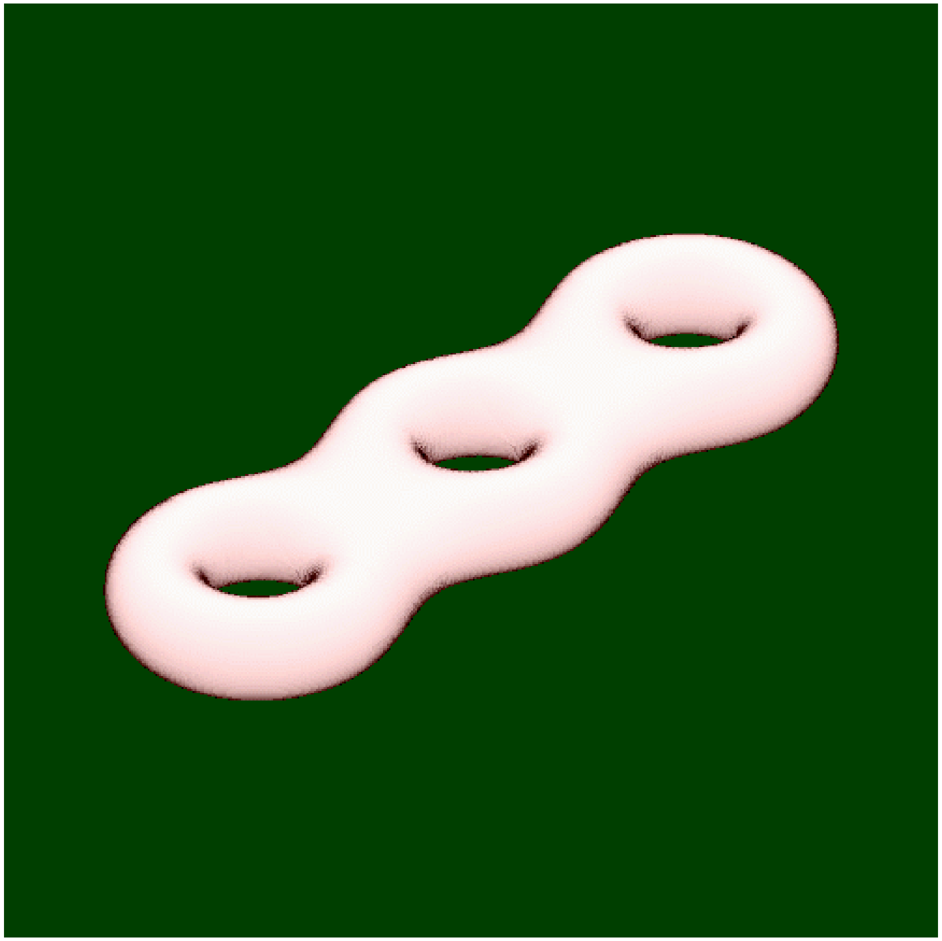


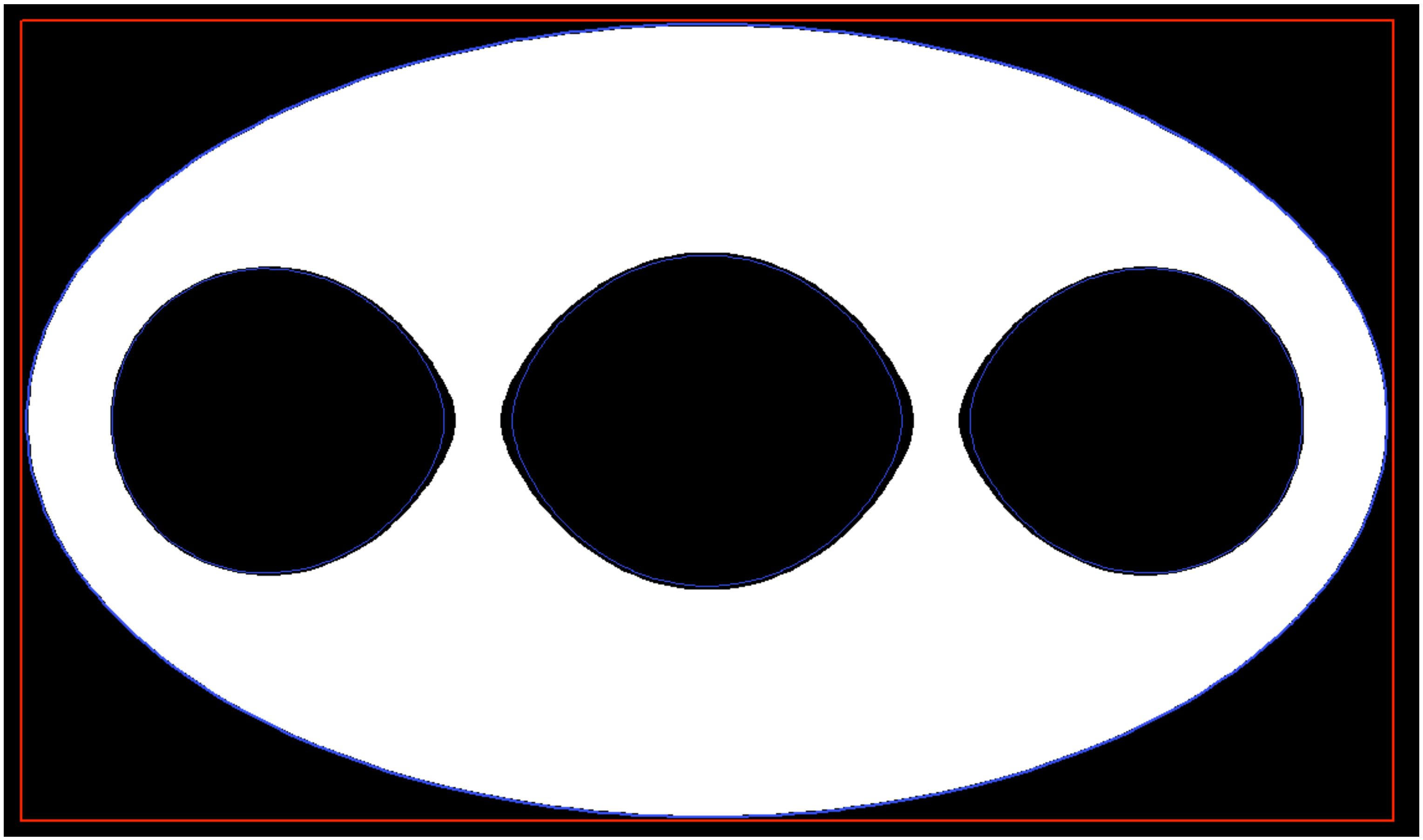
$$x^4 + y^4 - 6(x^2 + y^2) + 10 = 0$$

28 bitangents

- $x = \pm(2 \pm \sqrt{5})y$
- $x - y = 2, -2, \sqrt{2}, -\sqrt{2}, \qquad x + y = 2, -2, \sqrt{2}, -\sqrt{2}$
- $y = \pm 1 \pm \sqrt{2}, \qquad x = \pm 1 \pm \sqrt{2}$
- $y \pm \sqrt{5} = \pm x\sqrt{2}, \qquad x \pm \sqrt{5} = \pm y\sqrt{2}$







- $x^4 + y^4 + 1 = 0$
-
-
-

- $x^4 + y^4 + 1 = 0$ quartic with no real points
-
-
-

- $x^4 + y^4 + 1 = 0$ quartic with no real points

- $(x+y+1)(-x+y+1)(x-y+1)(x+y-1)+2(x^4+y^4+1) \equiv (x^2+y^2+1)^2$

-

-

- $x^4 + y^4 + 1 = 0$ quartic with no real points

- $(x+y+1)(-x+y+1)(x-y+1)(x+y-1)+2(x^4+y^4+1) \equiv (x^2+y^2+1)^2$

- $x+y+1=0 \Leftrightarrow (x^2+(x+1)^2+1)^2=0$

-

- $x^4 + y^4 + 1 = 0$ quartic with no real points

- $(x+y+1)(-x+y+1)(x-y+1)(x+y-1)+2(x^4+y^4+1) \equiv (x^2+y^2+1)^2$

- $x+y+1=0 \Leftrightarrow (x^2+(x+1)^2+1)^2=0$

- If there are no real points there are precisely four real bitangents:

- $x^4 + y^4 + 1 = 0$ quartic with no real points

- $(x+y+1)(-x+y+1)(x-y+1)(x+y-1)+2(x^4+y^4+1) \equiv (x^2+y^2+1)^2$

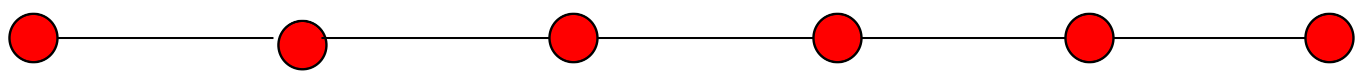
- $x+y+1=0 \Leftrightarrow (x^2+(x+1)^2+1)^2=0$

- If there are no real points there are precisely four real bitangents:

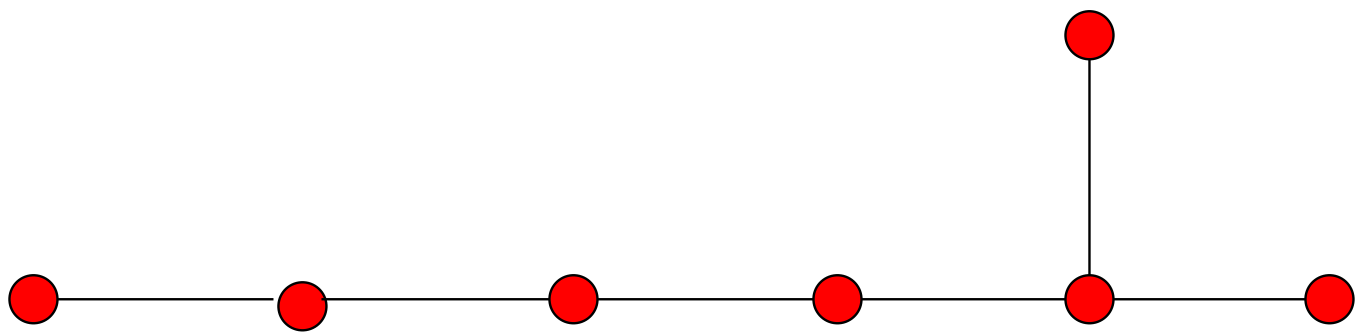
H.G.Zeuthen, *Sur les différentes formes de courbes planes du quatrième ordre*, Math.Ann. **7** (1874).

M.F.Atiyah, *Riemann surfaces and spin structures*, Ann.Scient.Éc.Norm. **4** (1971).

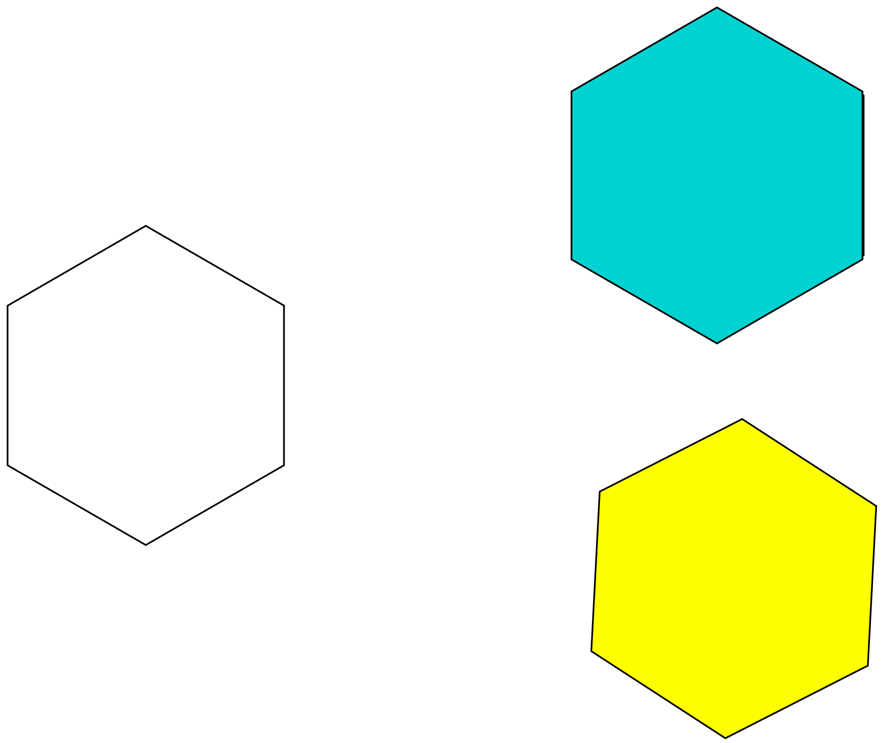
A_k

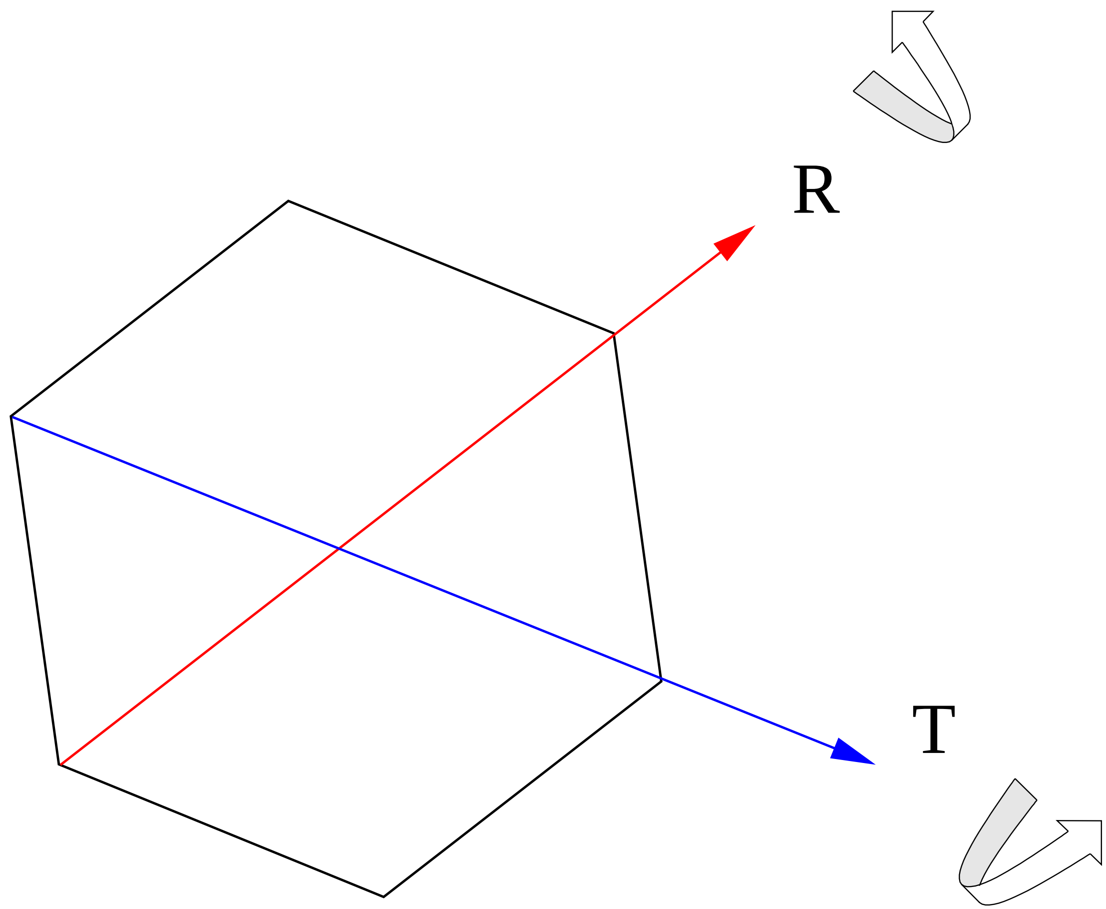


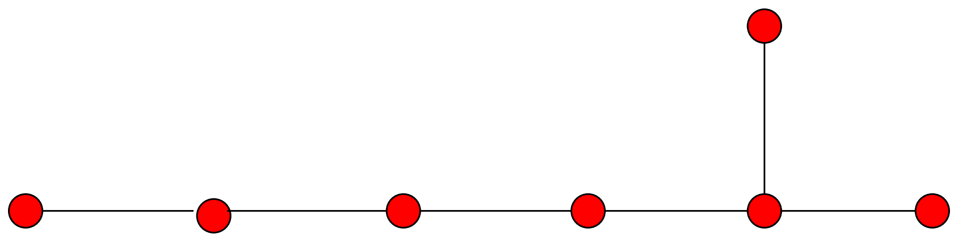
D_k



POLYGON AND DIHEDRON



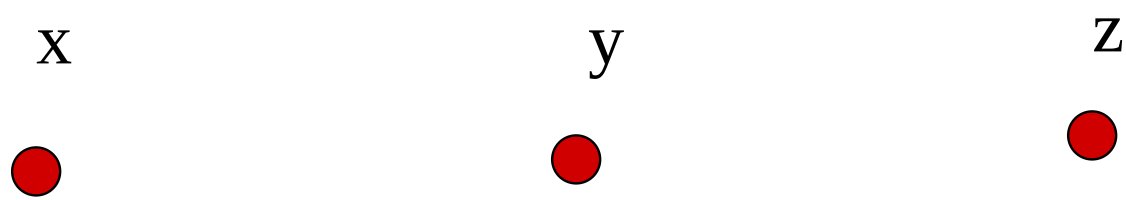

R
T



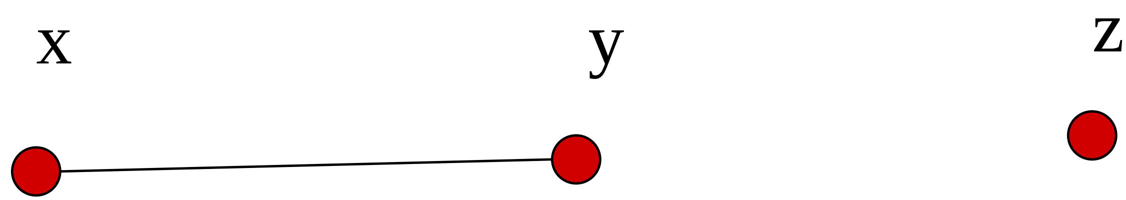
$$R^2 = S^k = T^2 = RST = 1$$

$$\Rightarrow (RT)^k = 1$$

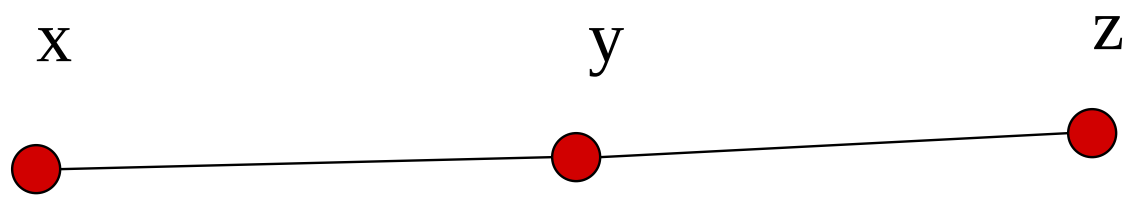
... AN EXERCISE



$$x^2 + y^2 + z^2$$

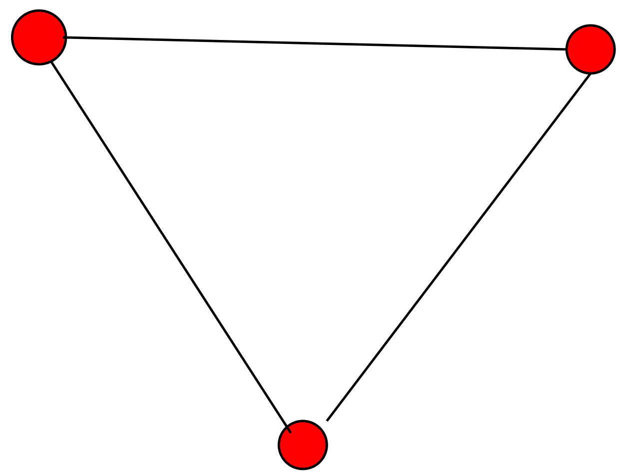


$$x^2 - xy + y^2 + z^2$$



$$x^2 - xy + y^2 - yz + z^2 =$$

$$= (x - y/2)^2 + (z - y/2)^2 + y^2/2 > 0$$



$$x^2 + y^2 + z^2 - xy - yz - zx$$

$$1 + 1 + 1 - 1 - 1 - 1 = 0$$

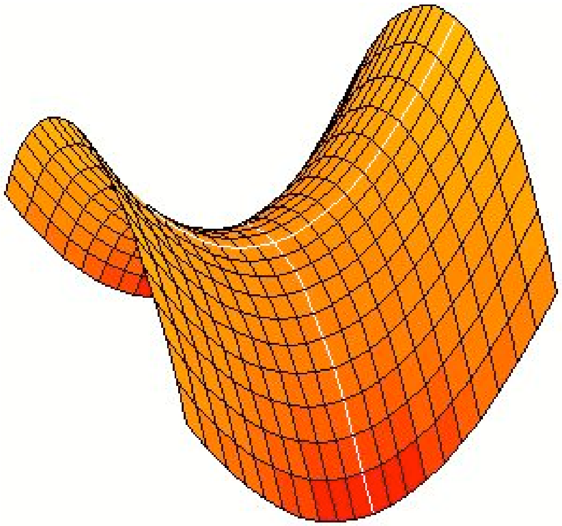
PROBLEM

FOR WHICH CONNECTED GRAPHS IS THIS FUNCTION POSITIVE?

CRITICAL POINTS OF FUNCTIONS

V.I.Arnold, *Normal forms for functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities*, Funktsional'nyi Analiz i ego Prilozheniya **6** (1972)

$$f(x,y) : \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$



$$f(x,y) = x^2 - y^2 + (4x^3 + x^2 \sin y + \ldots)$$

Change variables $(x,y) \mapsto (u(x,y), v(x,y))$ to get $f(x,y) = u^2 - v^2$ (Morse lemma)

- $f(x,y) = x^3 - y^2$: degenerate critical point

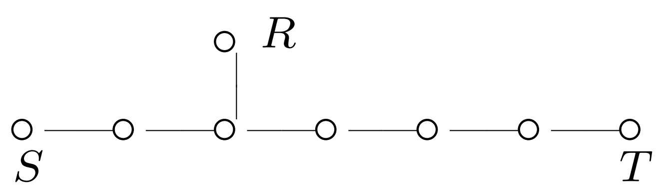
- $f_t(x,y) = x^3 - y^2 - tx^2 \sim u^2 - v^2$

- *Simple* critical points $\Leftrightarrow$ nearby functions have only a *finite* number of types

SIMPLE CRITICAL POINTS (*V.I.Arnold*)

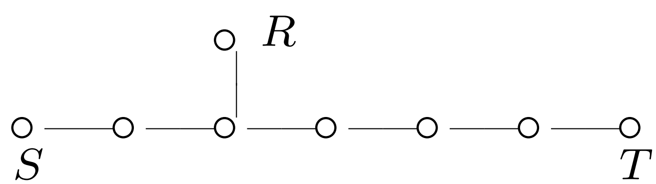
- $\mathbf{A_k}$: $x^{k+1} + quadratic$
- $\mathbf{D_k}$: $x^2y + y^{k-1} + quadratic$
- $\mathbf{E_6}$: $x^3 + y^4 + quadratic$
- $\mathbf{E_7}$: $x^3 + xy^3 + quadratic$
- $\mathbf{E_8}$: $x^3 + y^5 + quadratic$

CRITICAL POINTS AND PLATONIC SOLIDS

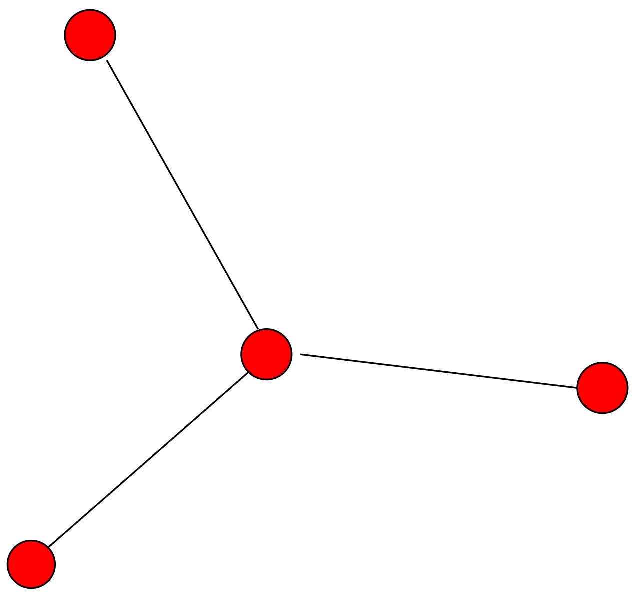


$$R^2 = S^3 = T^5 = RST = -1$$

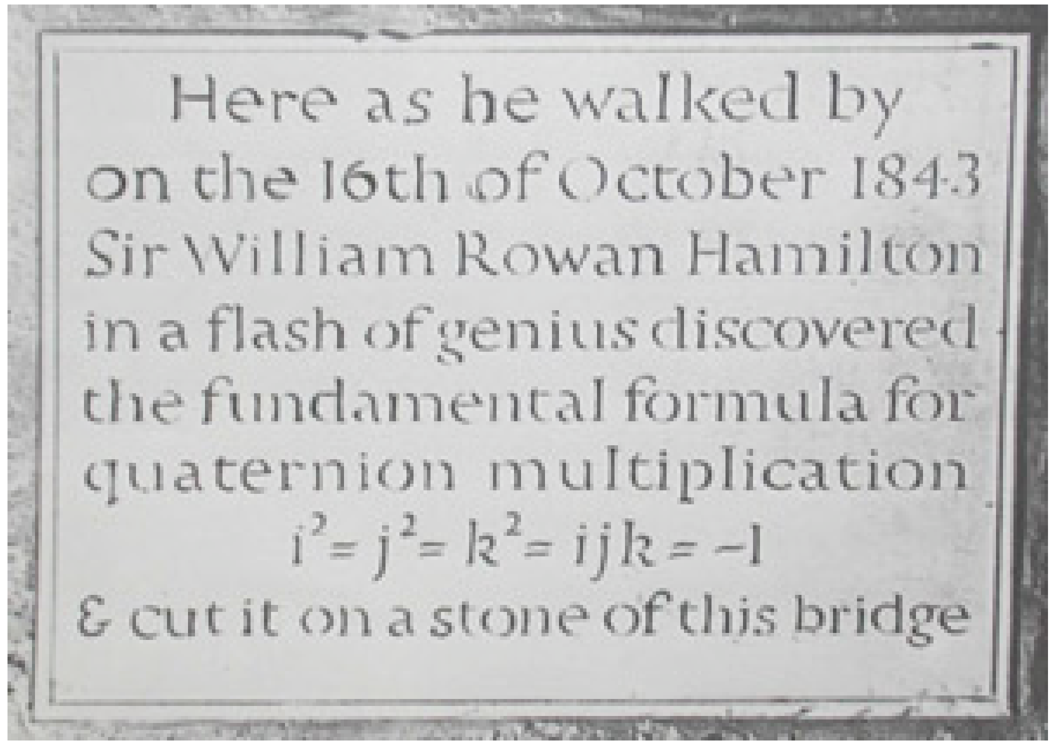
CRITICAL POINTS AND PLATONIC SOLIDS

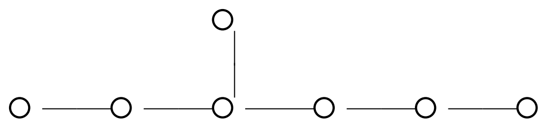


$$R^2 = S^3 = T^5 = RST = -1$$



$$i^2 = j^2 = k^2 = ijk = -1$$


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge



$$R^2 = S^3 = T^4 = RST = -1$$

$$R = \frac{i+j}{\sqrt{2}}, \qquad S = \frac{1}{2}(1 - i - j + k), \qquad T = \frac{1-i}{\sqrt{2}}$$

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$p(z)$ polynomial of degree k

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad p(z) \mapsto p\left(\frac{az+b}{cz+d}\right)(cz+d)^k$$

polynomials p, q, r invariant under R, S, T

$$p(z) = z^2(z^4 - 1)^2/4, \qquad r(z) = (z^8 + 14z^4 + 1)/4$$

$$q(z) = z(z^8 - 1)(z^8 - 34z^4 + 1)/16$$

$$q^2 = p(r^3 - 27p^2) \sim x^3 + xy^3 + \text{quadratic}$$

- 27 lines, 28 bitangents, 120 tritangent planes

- Lie algebras

- Coxeter groups

- quivers

- simple singularities

- regular polyhedra
- theoretical physics
- connected graphs with eigenvalues less than 2.
- planar algebras with $\delta \leq 2$

- regular polyhedra
- theoretical physics
- connected graphs with eigenvalues less than 2.
- planar algebras with $\delta \leq 2$

- regular polyhedra

- theoretical physics

- connected graphs with eigenvalues less than 2.

- planar algebras with $\delta \leq 2$

"Mathematicians thought of them as somehow connected with $A - D - E$ groups.... Dynkin diagrams are all very well and good but they are a method of studying groups, not the other way around."


Edward Witten

"Small instantons in String Theory" (1999)

THE UNIVERSE?

2000 AD "The Seiberg-Witten curve for the E-string describes the low-energy dynamics of a six-dimensional $(1,0)$ SUSY theory when compactified on $R^4 \times T^2$. It has a manifest affine E_8 global symmetry with modulus τ and E_8 Wilson line parameters $\{m_i\}, i = 1, 2, \ldots, 8$,..."

THE UNIVERSE?

1600 AD "Put an icosahedron inside the Earth's sphere, then Venus will move on a sphere contained in it."

Kepler spent 20 years trying to make this model work...and failed: the data just would not agree with the model. Hard as this was, he dropped this line of investigation.

This work, however, was of some use: he was recognized as "someone" and, in 1600, was hired by Tycho Brahe..."

THE UNIVERSE?

400 BC “Four of the five regular solids, according to Plato, represented the four elements, while the dodecahedron represented the universe as a whole....”