Impossibility theorems for elementary integration

Brian Conrad

Abstract. Liouville proved that certain integrals, most famously $\int e^{-x^2}\mathrm{d}x$, cannot be expressed in elementary terms. We explain how to give precise meaning to the notion of integration "in elementary terms", and we formulate Liouville's theorem that characterizes the possible form of elementary antiderivatives. Using this theorem, we deduce a practical criterion for proving such impossibility results in special cases.

This criterion is illustrated for the Gaussian integral $\int e^{-x^2}\mathrm{d}x$ from probability theory, the logarithmic integral $\int \mathrm{d}t/\log(t)$ from the study of primes, and elliptic integrals. Our exposition is aimed at students who are familiar with calculus and elementary abstract algebra (at the level of polynomial rings $F[t]$ over a field F).

1. Introduction

The Central Limit Theorem in probability theory assigns a special significance to the cumulative area function $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\mathrm{d}u$ under the Gaussian bell curve $y = (1/\sqrt{2\pi})\cdot e^{-u^2/2}$. It is known that $\Phi(\infty) = 1$ (i.e., the total area under the bell curve is 1), as must be the case for applications in probability theory, but this value is not determined by computing $\Phi(x)$ as an "explicit" function of x and finding its limit as $x \to \infty$. It is a theorem (to be made precise later) that there is *no* elementary formula for $\Phi(x)$, so the evaluation of $\Phi(\infty)$ must proceed by a method different from the calculation of anti-derivatives as in calculus. One uses a trick from multivariable calculus to express $\Phi(\infty)^2$ in terms of an integral over the plane, and this planar integral is computed by a switch to polar coordinates. Note also that by a change of variable, the study of $\Phi(x)$ can be recast as the study of the anti-derivative of e^{-u^2} (in terms of which the identity $\Phi(\infty) = 1$ takes the form $\int_{-\infty}^{\infty} e^{-u^2}\mathrm{d}u = \sqrt{\pi}$).

Let us give another example of an important indefinite integral that lacks an elementary formula. In number theory, there is much interest in the study of the step function $\pi(x) = \#\{1 \leq n \leq x \,|\, n \text{ is prime}\}$ of a real variable x that counts the number of primes up to x. The *prime number theorem* provides a

---

1991 *Mathematics Subject Classification.* Primary 12H05; Secondary 12H20.

Key words and phrases. Elementary integration, differential field.

The author was supported by the NSF and the Sloan Foundation during the writing of this paper, and many thanks are due to the Clay Mathematics Institute for providing the opportunity to speak on this charming topic that deserves to be more widely understood.

remarkable asymptotic formula: $\pi(x) \sim x/\log(x)$ as $x \to \infty$. That is, as $x \to \infty$, $\pi(x)/(x/\log(x)) \to 1$. Asymptotic quantities have relative error that tends to 0, but the actual difference can explode; for example, $x^2 \sim x^2 + 3x$ as $x \to \infty$, even though the difference $3x$ explodes in absolute value. In this sense, one can seek better asymptotic approximations to $\pi(x)$, and the first in a sequence of better such approximations is given by the *logarithmic integral* $\mathrm{Li}(x) = \int_2^x \mathrm{d}t/\log(t)$ for $x > 2$. (It is an exercise to show $\mathrm{Li}(x) \sim x/\log(x)$ as $x \to \infty$.) The prime number theorem was first conjectured by the 14-year-old Gauss in the form $\pi(x) \sim \mathrm{Li}(x)$ as $x \to \infty$. As with the Gaussian integral from probability theory, the logarithmic integral likewise admits no elementary formula. Observe also that with the change of variable $u = \log t$, we have $\int \mathrm{d}t/\log(t) = \int (e^u/u)\mathrm{d}u$.

The above examples show the interest in computing $\int e^{-u^2}\mathrm{d}u$ and $\int (e^u/u)\mathrm{d}u$, and in both cases we have said that there is no elementary formula for such anti-derivatives. How can one prove such assertions? Of course, to prove impossibility results of this sort it is necessary to first give a precise definition of "elementary formula". Roughly speaking, an elementary formula should be built from the familiar operations and functions in calculus: addition, multiplication, division, root-extraction, trigonometric functions and their inverses, exponential and logarithmic functions, and arbitrary composition among such functions. For example, the function

$$\frac{\pi x^2 - 3x\log x}{\sqrt{e^x - \sin(x/(x^3-7))}} \tag{1.1}$$

should qualify as an elementary function *on an open interval where it makes sense.* It is a minor but important technical issue to keep track of the interval on which we are working; for example, the expression $1/(x^2-1)^{1/6}$ has two possible meanings, depending on whether we work on the interval $(-\infty, -1)$ or $(1, \infty)$. We will generally suppress explicit mention of open intervals of definition, leaving it to the reader to make such adjustments as are necessary in order that various algebraic manipulations with functions make sense.

Liouville proved that if a function can be integrated in elementary terms, then such an elementary integral has to have a very special form. For functions of the form fe^g with rational functions f and g (e.g., e^{-u^2} with $f = 1$ and $g = -u^2$, or e^u/u with $f = 1/u$ and $g = u$), Liouville's theorem gives rise to an "elementary integrability" criterion in terms of solving a first-order differential equation with a *rational function.* This criterion is especially well-suited to the two motivating integration problems considered above, and in each case we work out Liouville's criterion to deduce the asserted impossibility result (i.e., that neither the Gaussian bell curve integral nor the logarithmic integral are elementary functions). We also briefly discuss another example, for more advanced readers: the non-elementarity of elliptic integrals and more generally $\int \mathrm{d}x/\sqrt{P(x)}$ for polynomials $P(X)$ with degree ≥ 3 and no double roots.

The reader who wishes to follow up on further details of the theory described in these notes is encouraged to read [**1**]; our proof in §5 of Liouville's "elementary integrability" criterion for functions of the form fe^g with rational functions f and g is a variant on an argument in [**1**] but it is written in a manner that avoids the abstract formalism of differential fields and so is easier to understand for readers who do not have extensive experience with abstract algebra.

NOTATION. We write $\mathbf{R}$ and $\mathbf{C}$ to denote the real and complex numbers respectively. The notation $F[X]$ denotes the polynomial ring in one variable over a field F, and we write $F(X)$ to denote its fraction field (the "rational functions" in one variable over F). Similar notation $F[X_1, \dots, X_n]$ and $F(X_1, \dots, X_n)$ is used with several variables. For example, we will work with the polynomial ring $\mathbf{C}(X)[Y]$ consisting of polynomials $\sum_j a_j(X)Y^j$ in Y with coefficients that are rational functions of X over $\mathbf{C}$. We may and do view $\mathbf{C}(X)[Y]$ as a subring of the field $\mathbf{C}(X,Y)$ of two-variable rational functions over $\mathbf{C}$. An element of $\mathbf{C}(X)[Y]$ is *monic* if it is nonzero and as a polynomial in Y has leading coefficient (in $\mathbf{C}(X)$) equal to 1, such as $Y^3 + ((3X^2 - 2i)/(4X^2 + 7X))Y - (-2 + 7i)/X$; any nonzero element of $\mathbf{C}(X)[Y]$ has a unique $\mathbf{C}(X)$-multiple that is monic (divide by the $\mathbf{C}(X)$-coefficient of the highest-degree monomial in Y). We generally write $\mathbf{C}(X)$ when we are thinking algebraically and $\mathbf{C}(x)$ when we are thinking function-theoretically.

2. Calculus with C-valued functions

Though it may seem to complicate matters, our work will be much simplified by systematically using $\mathbf{C}$-valued functions of a *real* variable x (such as $e^{ix} = \cos(x) + i\sin(x)$ or $(2+3i)x^3 - 7ix + 12$) rather than $\mathbf{R}$-valued functions. As one indication of the simplification attained in this way, observe that the formulas

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{2.1}$$

express trigonometric functions in terms of $\mathbf{C}$-valued exponentials. In general, the advantage of using $\mathbf{C}$-valued functions is that they encode all trigonometric functions and inverse-trigonometric functions in terms of exponentials and logarithms. (See Example 2.1.) One also gets algebraic simplifications; for example, the addition formulas for $\sin(x)$ and $\cos(x)$ may be combined into the single appealing formula $e^{i(x+y)} = e^{ix}e^{iy}$.

The reader may be concerned that allowing $\mathbf{C}$-valued functions will permit a more expansive notion of elementary function than one may have wanted to consider within the framework of $\mathbf{R}$-valued functions, but it is also the case that working with a more general notion of "elementary function" will *strenghten* the meaning of impossibility theorems for elementary integration problems. Since we wish to allow $\mathbf{C}$-valued functions, we must carry over some notions of calculus to this more general setting. Briefly put, we carry over definitions using real and imaginary parts. For example, a $\mathbf{C}$-valued function can be written in the form $f(x) = u(x) + iv(x)$ via real and imaginary parts, and we say f is *continuous* when u and v are so, and likewise for differentiability. In the differentiable case, we make definitions such as $f' = u' + iv'$, and $\mathbf{C}$-valued integration is likewise defined in terms of ordinary integration of real and imaginary parts. (The crutch of real and imaginary parts can be avoided, but we do not dwell on the matter here.)

A $\mathbf{C}$-valued function $f(x)$ is *analytic* if its real and imaginary parts $u(x)$ and $v(x)$ are locally expressible as convergent Taylor series. The property that an $\mathbf{R}$-valued function is locally expressible as a convergent Taylor series is preserved under the usual operations on functions (sums, products, quotients, composition, differentiation, integration, inverse function with non-vanishing derivative), so all $\mathbf{C}$-valued or $\mathbf{R}$-valued functions that we easily "write down" are analytic. For example, (1.1) is analytic on the interval $(\sqrt[3]{7}, \infty)$, and if $f(x)$ is an $\mathbf{R}$-valued analytic function then $\sin(f(x))$ and $\tan^{-1}(f(x))$ are analytic functions. For any positive integer n

the function $\sqrt[n]{x}$ that is inverse to the function x^n is likewise analytic on $(0, \infty)$, so root extraction of positive functions preserves analyticity. By using the usual definition $e^{u(x)+iv(x)} = e^{u(x)}(\cos v(x) + i \sin v(x))$, it likewise follows that if a **C**-valued function $f(x)$ is analytic then so is $e^{f(x)}$. Some elementary algebraic manipulations with real and imaginary parts show moreover that $(e^{f(x)})' = f'(x)e^{f(x)}$, as one would expect.

If a **C**-valued function $f(x)$ is analytic and non-vanishing, then f'/f is analytic as well, so upon choosing a point x_0 the integral $(\log f)(x) = \int_{x_0}^{x} (f'(t)/f(t))\mathrm{d}t$ is an analytic function called a *logarithm* of f. Such a function depends on the choice of x_0 up to an additive constant (since replacing x_0 with another point x_1 changes the function by the constant $\int_{x_0}^{x_1} (f'(t)/f(t))\mathrm{d}t$), but such ambiguity is irrelevant for our purposes so we will ignore it. Thus, we can equivalently consider a logarithm of f to be a solution to the differential equation $y' = f'/f$. In the special case $x_0 = 1$ and $f(t) = t$ (on the interval $(0, \infty)$) this recovers the traditional logarithm function. If we add a suitable constant to a logarithm of f then we can arrange that $e^{\log f} = f$, so the terminology is reasonable.

EXAMPLE 2.1. The formulas (2.1) show that exponentiation recovers trigonometric functions. One checks by differentiation that $2i \tan^{-1}(x) + i\pi$ is a logarithm of the non-vanishing function $(x - i)/(x + i)$, so by forming logarithms we may recover the inverse trigonometric function $\tan^{-1}$. Using identities such as

$$\cos^{-1}(x) = \tan^{-1}\left(\sqrt{\frac{1}{x^2} - 1}\right), \quad \sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$

for $|x| < 1$, we recover all of the usual inverse trigonometric functions via logarithms of non-vanishing analytic functions. Thus, we can describe trigonometric functions and their inverses in terms of exponentials and logarithms of analytic functions.

We manipulate ratios of polynomials (with nonzero denominator) "as if" they are genuine functions by simply omitting the few points where the denominators vanish, and we likewise will treat ratios of analytic functions "as if" they are genuine functions by omitting the isolated points where the denominators vanish. Such ratios of analytic functions on a fixed non-empty open interval I in **R** are called *meromorphic* functions on I. For example, e^x/x is a meromorphic function on the real line. We will usually not explicitly mention the fixed open interval of definition being used; the context will always make it clear. The set of meromorphic functions (on a fixed non-empty open interval) is a *field*, and on this field we can define the derivative operator $f \mapsto f'$ by using the quotient rule in the usual manner. This derivative operator satisfies the usual properties (e.g., product rule, quotient rule). The field of meromorphic functions equipped with the derivative operator is the setting in which Liouville's theorem takes place.

3. Elementary fields and elementary functions

It is convenient to introduce the following general notation.

DEFINITION 3.1. If $f_1, \ldots, f_n$ are meromorphic functions then $\mathbf{C}(f_1, \ldots, f_n)$ denotes the set of meromorphic functions h of the form

$$h = \frac{p(f_1, \ldots, f_n)}{q(f_1, \ldots, f_n)} = \frac{\sum a_{e_1,\ldots,e_n} f_1^{e_1} \cdots f_n^{e_n}}{\sum b_{j_1,\ldots,j_n} f_1^{j_1} \cdots f_n^{j_n}}$$

for n-variable polynomials

$$p(X_1,\dots,X_n)=\sum a_{e_1,\dots,e_n}X_1^{e_1}\cdots X_n^{e_n},\ \ q(X_1,\dots,X_n)=\sum b_{j_1,\dots,j_n}X_1^{j_1}\cdots X_n^{j_n}$$

in $\mathbf{C}[X_1,\dots,X_n]$ with $q(f_1,\dots,f_n)\neq 0$.

The set $\mathbf{C}(f_1,\dots,f_n)$ is clearly a field, via the usual algebraic operations on functions.

Example 3.2. The field $K=\mathbf{C}(x,\sin(x),\cos(x))$ is the set of ratios

$$\frac{p(x,\sin(x),\cos(x))}{q(x,\sin(x),\cos(x))}$$

for polynomials $p,q\in\mathbf{C}[X,Y,Z]$ such that $q(x,\sin(x),\cos(x))\neq 0$. For example, we cannot use $q=Y^2+Z^2-1$ since $\sin(x)^2+\cos(x)^2-1=0$. By (2.1) we have $K=\mathbf{C}(x,e^{ix})$, so elements of K can also be written in the form $g(x,e^{ix})/h(x,e^{ix})$ with $g,h\in\mathbf{C}[X,Y]$ and $h\neq 0$. There is no restriction imposed on the nonzero polynomial h by the requirement that the function $h(x,e^{ix})$ not vanish identically, due to Lemma 5.1 below.

Definition 3.3. A field K of meromorphic functions is an *elementary field* if $K=\mathbf{C}(x,f_1,\dots,f_n)$ with each f_j either an exponential or logarithm of an element of $K_{j-1}=\mathbf{C}(x,f_1,\dots,f_{j-1})$ or else algebraic over K_{j-1} in the sense that $P(f_j)=0$ for some $P(T)=T^m+a_{m-1}T^{m-1}+\dots+a_0\in K_{j-1}[T]$ with all $a_k\in K_{j-1}$. A meromorphic function f is an *elementary function* if it lies in an elementary field of meromorphic functions.

Example 3.4. Consider the function f given by (1.1). An elementary field containing f is

$$K=\mathbf{C}(x,\log x,e^x,e^{ix/(x^3-7)},\sqrt{e^x-\sin(x/(x^3-7))}).$$

Example 3.5. Root-extractions such as $\sqrt[3]{\sin(x)-7x}$ are examples of the algebraic case in Definition 3.3 (this solves the cubic equation $T^3-(\sin(x)-7x)=0$ with coefficients in the elementary field $\mathbf{C}(x,e^{ix})$). A more elaborate example of such an "algebraic function" is $K=\mathbf{C}(x,f(x))$ with $f=\sqrt{x}+\sqrt[3]{x}$: we may take

$$P(T)=T^6-3xT^4-(x^3-x^2)T^3-3x^2T^2-6x^2T-(x^3-x^2)$$

in $\mathbf{C}(x)[T]$ as a polynomial satisfied by f over $\mathbf{C}(x)$. A simpler approach to this latter example is to view $\sqrt{x}+\sqrt[3]{x}$ as lying in the field $\mathbf{C}(x,\sqrt{x},\sqrt[3]{x})$, with $\sqrt{x}$ algebraic over $\mathbf{C}(x)$ and $\sqrt[3]{x}$ algebraic over $\mathbf{C}(x,\sqrt{x})$ (and even over $\mathbf{C}(x)$).

The notion of elementary function as just defined includes all functions that one might ever want to consider to be elementary. It is *not* obvious is how to prove that there exist non-elementary (meromorphic) functions, and Liouville's results will give a method to prove that specific meromorphic functions are not elementary.

Theorem 3.6. *If K is an elementary field, then it is closed under the operation of differentiation.*

Proof. We write $K=\mathbf{C}(x,f_1,\dots,f_n)$ as in Definition 3.3 and we induct on n. The case $n=0$ is the case $K=\mathbf{C}(x)$. It follows from the usual formulas for derivatives of sums, products, and ratios that $\mathbf{C}(x)$ is closed under differentiation. For the general case, by induction $K_0=\mathbf{C}(x,f_1,\dots,f_{n-1})$ is closed under differentiation, and we have $K = K_0(f_n)$ with f_n either algebraic over K_0 or a logarithm or exponential of an element of K_0. Let us now check that it suffices to prove $f_n' \in K_0(f_n)$. Under this assumption, for any polynomial $P(T) = \sum_{j\geq 0} a_j T^j \in K_0[T]$ we have

$$P(f_n)' = a_0' + \sum_{j\geq 1}(a_j' f_n^j + j a_{j-1} f_n^{j-1} f_n') \in K_0(f_n)$$

since $a_j' \in K_0$ for all j (as K_0 is closed under differentiation). Thus, if $P, Q \in K_0[T]$ are polynomials over K_0 and $Q(f_n) \neq 0$ then

$$\left(\frac{P(f_n)}{Q(f_n)}\right)' = \frac{Q(f_n)P(f_n)' - P(f_n)Q(f_n)'}{Q(f_n)^2} \in K_0(f_n) = K$$

since the numerator and denominator lie in $K_0(f_n)$.

It remains to check that the function f_n that is either algebraic over K_0 or is an exponential or logarithm of an element of K_0 has derivative f_n' that lies in $K_0(f_n)$. If $f_n = e^g$ for some $g \in K_0$ then $f_n' = g' f_n$. Thus, $f_n' \in K_0(f_n)$ since $g' \in K_0$ (as K_0 is closed under differentiation). If f_n is a logarithm of some $g \in K_0$ then $f_n' = g'/g \in K_0 \subseteq K_0(f_n)$. Finally, we treat the algebraic case. Suppose $P(f_n) = 0$ for a polynomial $P = T^m + a_{m-1}(x)T^{m-1} + \cdots + a_0(x) \in K_0[T]$. Take P with minimal degree, so $P'(T) := mT^{m-1} + (m-1)a_{m-1}(x)T^{m-2} + \cdots + 2a_2(x)T + a_1(x)$ with degree $m-1$ satisfies $P'(f_n) \neq 0$. But

$$0 = P(f_n)' = \sum_{j>0} j a_j(x) f_n^{j-1} f_n' + \sum_{j<m} a_j'(x) f_n^j = P'(f_n) f_n' + \sum_{j<m} a_j'(x) f_n^j,$$

so $P'(f_n)f_n' = -\sum_{j<m} a_j'(x) f_n^j \in K_0(f_n)$. Since $P'(f_n) \neq 0$ and $P'(f_n) \in K_0(f_n)$, we get $f_n' \in K_0(f_n)$ by division. □

Remark 3.7. A field K of meromorphic functions that is closed under differentiation is called a *differential field*. The preceding theorem says that elementary fields are examples of differential fields, but the method of proof shows more: if $K = K_0(f)$ with K_0 any differential field and f either algebraic over K_0 or an exponential or logarithm of an element of K_0 then K is a differential field. The field $\mathbf{C}(x, \sin(x), \cos(x))$ is a differential field, since $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. (Alternatively, this field coincides with $\mathbf{C}(x, e^{ix})$, and $(e^{ix})' = ie^{ix}$.) In contrast, $\mathbf{C}(x, \sin(x))$ is *not* a differential field. More specifically, $\sin'(x) = \cos(x)$ but $\cos(x)$ is not an element of $\mathbf{C}(x, \sin(x))$; that is, $\cos(x)$ does not admit a rational expression in terms of x and $\sin(x)$ (with coefficients in $\mathbf{C}$). The verification of this "obvious" fact requires Lemma 5.1 below and the irreducibility of $U^2 + V^2 - 1$ in $\mathbf{C}(U)[V]$; it is left as an exercise. Liouville's work on integration in elementary terms is best viewed as a contribution to the theory of differential fields, and this theory is the point of departure in [**1**].

Definition 3.8. A meromorphic function f can be *integrated in elementary terms* if $f = g'$ for an elementary function g (and so f is necessarily elementary, by Theorem 3.6).

Definition 3.8 captures any reasonable intuitive notion of an "elementary formula" for an anti-derivative of a function of the sort considered in calculus, though the use of $\mathbf{C}$-valued functions permits a much wider class of anti-derivatives to be considered as elementary than one may have wanted to allow in the classical setting of $\mathbf{R}$-valued functions. What is more important is that if we can prove an elementary function such as e^{-x^2} cannot be integrated in elementary terms in the sense of the preceding definition (that allows **C**-valued functions) then it is certainly not susceptible to "elementary" integration in any reasonable **R**-valued sense! That is, by proving an impossibility theorem with the preceding definitions we are proving something even stronger than we might have hoped to be true in the **R**-valued setting.

In case the reader is still disturbed by our use of **C**-valued functions, we should point out that using only **R**-valued functions in Definition 3.8 would give the *wrong* concept of elementary integrability. For example, under any reasonable definition we would want to say that $1/(1+x^2)$ admits an elementary integral (such as $\tan^{-1}(x)$). This is the case in the **C**-valued setting, since we know that $\tan^{-1}(x)$ is obtained from a logarithm of $(x+i)/(x-i)$ (see Example 2.1). However, if we work in the **R**-valued setting and permit *only* the operations of exponentiation, logarithm, and solving of algebraic equations (as in Definition 3.8, via Definition 3.3) then it can be proved that $1/(1+x^2)$ is *not* integrable (over **R**) in such elementary terms. (See [**1**, p. 968] for a rigorous proof.) A way around this technical glitch in the **R**-valued case is to incorporate *all* of the usual trigonometric functions and their inverses (and not merely exponentials and logarithms) in an **R**-valued definition of "integration in elementary terms". Unfortunately, this change in definitions is disastrous for the attempt to push through an **R**-valued analogue of Liouville's results because such trigonometric functions and their inverses are not solutions to simple first-order differential equations. Since our main interest is in impossibility results, Liouville's work in the **C**-valued setting will give what we require.

4. Integrability criterion and applications

Liouville's main theorem asserts that if an elementary function f is integrable in elementary terms then there are severe constraints on the possible form of an elementary anti-derivative of f:

Theorem 4.1 (Liouville). *Let f be an elementary function and let K be an elementary field containing f. The function f can be integrated in elementary terms if and only if there exist nonzero $c_1, \dots, c_n \in \mathbf{C}$, nonzero $g_1, \dots, g_n \in K$, and an element $h \in K$ such that*

$$f = \sum c_j \frac{g_j'}{g_j} + h'.$$

The key point is that the g_j's and h can be found in *any* elementary field K containing f; $\sum c_j \log(g_j) + h$ is then an elementary integral of f.

Example 4.2. Consider $f = e^{-x^2}$. This lies in the elementary field $K = \mathbf{C}(x, e^{-x^2})$. Hence, Liouville's theorem says that an elementary anti-derivative of f *must* have the special form $\sum c_j \log g_j + h$ for some $h \in \mathbf{C}(x, e^{-x^2})$ and nonzero $c_j \in \mathbf{C}$ and $g_j \in \mathbf{C}(x, e^{-x^2})$. It is not obvious how to prove the non-existence of such h and g_j's, but this still represents a significant advance over the problem of contemplating *all* elementary functions as candidates for elementary anti-derivatives of e^{-x^2}. We will soon see that the possible form of such an elementary anti-derivative of e^{-x^2} can be made even more special, and so it becomes a problem that we can solve without too much difficulty.

Example 4.3. There is a very interesting class of integrals for which Liouville's result in the above form is immediately applicable without an extra simplification:

elliptic integrals. Just as trignometric functions may be introduced through inversion of integral functions of the form $\int \mathrm{d}x/\sqrt{x^2-1}$ that arise from calculation of arc length along a unit circle, the theory of elliptic functions grew out of a study of inversion of integral functions of the form $\int \mathrm{d}x/\sqrt{P(x)}$ for certain cubic and quartic polynomials $P(X) \in \mathbf{R}[X]$ without repeated roots; such integrals arise in the calculation of arc length along an ellipse. In general, if $P(X) \in \mathbf{R}[X]$ is any monic polynomial with degree ≥ 3 and no repeated roots then we claim that $\int \mathrm{d}x/\sqrt{P(x)}$ is *not* an elementary function. Since $K = \mathbf{C}(x, \sqrt{P})$ is an elementary field, by the criterion in Theorem 4.1 it suffices to prove that there does not exist an identity of the form

$$\frac{1}{\sqrt{P(x)}} = \sum c_j \frac{g_j'}{g_j} + h'$$

with nonzero $c_1, \dots, c_n \in \mathbf{C}$, nonzero $g_1, \dots, g_n \in K$, and $h \in K$. Such impossibility is a consequence of general facts from the theory of compact Riemann surfaces. More specifically, for the advanced reader who knows a bit about this theory, the above identity is equivalent to the equality of meromorphic 1-forms

$$\frac{\mathrm{d}x}{y} = \sum c_j \frac{\mathrm{d}g_j}{g_j} + \mathrm{d}h$$

on the compact Riemann surface C associated to the equation $y^2 = P(x)$, and for $\deg(P) > 2$ the left side is a nonzero holomorphic 1-form on C. But a nonzero holomorphic 1-form on a compact Riemann surface never admits an expression as a linear combination of logarithmic meromorphic differentials $\mathrm{d}g/g$ and exact meromorphic differentials $\mathrm{d}h$.

The proof of Liouville's theorem rests on an inductive algebraic study of differential fields. We refer the reader to [**1**, §4–§5] for the proof of Theorem 4.1, and we now focus our attention on using and proving the following criterion that emerges as a consequence:

Theorem 4.4. *Choose $f, g \in \mathbf{C}(X)$ with $f \neq 0$ and g nonconstant. The function $f(x)e^{g(x)}$ can be integrated in elementary terms if and only if there exists a rational function $R \in \mathbf{C}(X)$ such that $R'(X) + g'(X)R(X) = f(X)$ in $\mathbf{C}(X)$.*

The content of the criterion in this theorem is not that the differential equation $R'(x) + g'(x)R(x) = f(x)$ has a solution as a $\mathbf{C}$-valued differentiable function of x (since we can *always* write down a simple integral formula for a solution via integrating factors), but rather than there is a solution with the special property that it is a *rational function* in x. (For any such $R \in \mathbf{C}(X)$, $R(x)e^{g(x)}$ is an elementary anti-derivative of $f(x)$.) In specific examples, as we shall see, it can be proved that there does *not* exist a rational function in x that solves the differential equation $R'(x) + g'(x)R(x) = f(x)$; this is how we will verify that some functions of the form $f(x)e^{g(x)}$ *cannot* be integrated in elementary terms. The deduction of Theorem 4.4 from Theorem 4.1 is given in §5, and the remainder of this section is devoted to applying Theorem 4.4 to the problem of computing the Gaussian integral and the logarithmic integral in elementary terms.

Remark 4.5. In what follows, we will systematically work with the field $\mathbf{C}(X)$ as an "abstract" field (also identified in this obvious manner with the more concrete field $\mathbf{C}(x)$ of rational $\mathbf{C}$-valued functions on an open interval in $\mathbf{R}$). On this field the operation of differentiation is *defined* by the standard formula on $\mathbf{C}[X]$ and is extended to $\mathbf{C}(X)$ via the quotient rule; one checks without difficulty that this is a well-defined operation on $\mathbf{C}(X)$ and that it satisfies all of the usual formulas with respect to sums, products, and quotients. The purpose of this algebraic point of view is to permit the use of differentiation in algebraic identities in $\mathbf{C}(X)$ that we may then specialize at points $z \in \mathbf{C}$ that are possibly *not* in $\mathbf{R}$. (The theory of differentiation for $\mathbf{C}$-valued functions of a *complex* variable has some surprising features; we do not require it for our purposes.)

Example 4.6. We now prove that e^{-x^2} cannot be integrated in elementary terms. Taking $f = 1$ and $g = -x^2$ in Theorem 4.4, we have to prove that the differential equation $R'(X) - 2XR(X) = 1$ in $\mathbf{C}(X)$ has no solution (in $\mathbf{C}(X)$). The method of integrating factors gives a formula for the general function solution, namely $R_c(x) = -e^{x^2}(\int e^{-x^2}\mathrm{d}x + c)$ with $c \in \mathbf{C}$, but we cannot expect to show by inspection that this is never a rational function since we have no way to describe $\int e^{-x^2}\mathrm{d}x$ in the first place! However, this formula for the general solution still provides a genuine simplification on the initial problem because the necessary and sufficient condition that the function $R_c(x)$ is a *rational* function in x for some choice of constant c says exactly that if e^{-x^2} is to have an elementary anti-derivative then there *must* be such an anti-derivative having the form $e^{-x^2}r(x)$ for a *rational function* $r \in \mathbf{C}(X)$. This is a severe constraint on the possible form of an elementary anti-derivative. (Explicitly, the condition that $e^{-x^2}r(x)$ be an anti-derivative to e^{-x^2} is precisely the condition that $r'(x) - 2xr(x) = 1$.)

To prove the non-existence of solutions to $R'(X) - 2XR(X) = 1$ in $\mathbf{C}(X)$, we shall argue by contradiction. If $R \in \mathbf{C}(X)$ is such a solution, then certainly R is non-constant and we claim that R cannot be a polynomial (in X). Indeed, if $R(X)$ is a polynomial with some degree $n > 0$ then $R'(X) - 2XR(X)$ is a polynomial with degree $n + 1$ and hence it cannot equal 1. Thus, in reduced form we must have $R(X) = p(X)/q(X)$ for nonzero relatively prime polymomials $p(X), q(X) \in \mathbf{C}[X]$ with $q(X)$ nonconstant. The identity $R'(X) - 2XR(X) = 1$ in $\mathbf{C}(X)$ says $(p(X)/q(X))' - 2X(p(X)/q(X)) = 1$.

Since $q(X)$ is a nonconstant polynomial, by the Fundamental Theorem of Algebra it has a root $z_0 \in \mathbf{C}$; of course, usually z_0 is not in $\mathbf{R}$. Relative primality of p and q in $\mathbf{C}[X]$ implies $p(z_0) \neq 0$. Hence, if z_0 is a root of q with multiplicity $\mu \geq 1$ then $p(X)/q(X) = h(X)/(X - z_0)^\mu$ with $h(X) \in \mathbf{C}(X)$ having numerator and denominator that are non-vanishing at z_0. Differentiation gives

$$\left(\frac{p(X)}{q(X)}\right)' = \frac{-h(X)}{\mu(X - z_0)^{\mu+1}} + \frac{h'(X)}{(x - z_0)^\mu},$$

so as $z \to z_0$ in $\mathbf{C}$ we see that $(p(X)/q(X))'|_{X=z}$ has absolute value that blows up like $A/|z - z_0|^{\mu+1}$ with $A = |h(z_0)/\mu| \neq 0$. But $|-2z \cdot (p(z)/q(z))|$ has growth bounded by a constant multiple of $1/|z - z_0|^\mu$ as $z \to z_0$ in $\mathbf{C}$, so

$$\left| \left(\left(\frac{p(X)}{q(X)}\right)' - 2X \cdot \left(\frac{p(X)}{q(X)}\right) \right) |_{X=z} \right| \sim \frac{A}{|z - z_0|^{\mu+1}}$$

as $z \to z_0$ in $\mathbf{C}$. This contradicts the identity $(p(X)/q(X))' - 2X(p(X)/q(X)) = 1$.

Example 4.7. Consider the logarithmic integral $\int \mathrm{d}t/\log(t)$, or equivalently $\int (e^x/x)\mathrm{d}x$ (with $x = \log t$). Taking $f = 1/x$ and $g = x$ in Theorem 4.4, to prove that the logarithmic integral cannot be expressed in elementary terms it suffices to show that the differential equation $R'(X) + R(X) = 1/X$ does not have a solution in $\mathbf{C}(X)$. If there exists $R \in \mathbf{C}(X)$ such that $R'(X) + R(X) = 1/X$ then obviously $R(X)$ cannot be a polynomial in X. Writing $R = p/q$ in reduced form with $q \in \mathbf{C}[X]$ a nonconstant polynomial, q has a root of order $\mu \geq 1$ at some $z_0 \in \mathbf{C}$. Thus, $R'(X)$ has a zero of order $\mu + 1$ at z_0 and so as with the previous example we see that $1/X = R'(X) + R(X)$ considered as a function of $z \in \mathbf{C}$ has absolute value that explodes like a nonzero constant multiple of $1/|z - z_0|^{\mu+1}$ as $z \to z_0$. But the only $w \in \mathbf{C}$ for which $|1/z|$ has explosive growth in absolute value as z approaches w in $\mathbf{C}$ is $w = 0$, so $z_0 = 0$. Hence, as $z \to 0$ it follows that $|1/z|$ grows like a nonzero constant multiple of $1/|z - z_0|^{\mu+1} = 1/|z|^{\mu+1}$, a contradiction since $\mu + 1 \geq 2$.

5. Proof of integrability criterion

We now show how to deduce Theorem 4.4 from Theorem 4.1. Assuming fe^g admits an elementary antiderivative, we have to find $R \in \mathbf{C}(X)$ such that $R'(X) + g'(X)R(X) = f(X)$. Theorem 4.1 with $K = \mathbf{C}(x, e^g)$ implies

$$f(x)e^{g(x)} = \sum c_j \frac{g_j'(x)}{g_j(x)} + h'(x) \tag{5.1}$$

(on a suitable non-empty open interval in $\mathbf{R}$) with nonzero $c_j \in \mathbf{C}$ and nonzero $g_j \in \mathbf{C}(x, e^g)$, and an element $h \in \mathbf{C}(x, e^g)$. We need to massage this expression to get it into a more useful form. Note that each g_j may be taken to be non-constant, as otherwise $g_j' = 0$ and so g_j'/g_j does not contribute anything for such j.

Choose nonzero $p_j, q_j \in \mathbf{C}[X, Y]$ such that $g_j(x) = p_j(x, e^{g(x)})/q_j(x, e^{g(x)})$. Since

$$\frac{g_j'(x)}{g_j(x)} = \frac{p_j(x, e^{g(x)})'}{p_j(x, e^{g(x)})} - \frac{q_j(x, e^{g(x)})'}{q_j(x, e^{g(x)})},$$

we have

$$\sum c_j \frac{g_j'(x)}{g_j(x)} = \sum_j c_j \frac{p_j(x, e^{g(x)})'}{p_j(x, e^{g(x)})} + \sum_j -c_j \frac{q_j(x, e^{g(x)})'}{q_j(x, e^{g(x)})}.$$

We write this as a sum over twice as many indices to reduce to the case when each g_j has the form $p_j(x, e^{g(x)})$ with a nonzero polynomial $p_j(X, Y) \in \mathbf{C}[X, Y]$. We can assume each p_j is nonconstant, as otherwise $p_j(x, e^{g(x)})' = 0$ and so this term contributes nothing. Considering $p_j(X, Y)$ as a nonconstant element in $\mathbf{C}(X)[Y]$, we may factor p_j as a product $\prod_k r_{kj}(X, Y)$ of irreducible monics in $\mathbf{C}(X)[Y]$ and (possibly) a nonconstant element of $\mathbf{C}(X)$. Using the general identity $(\prod h_k)'/\prod h_k = \sum h_k'/h_k$ for meromorphic functions h_k (on a common non-empty open interval in $\mathbf{R}$), we get

$$\frac{p_j(x, e^{g(x)})'}{p_j(x, e^{g(x)})} = \sum_k \frac{r_{kj}(x, e^{g(x)})'}{r_{kj}(x, e^{g(x)})}.$$

Upon renaming terms and indices once again we may suppose that each $p_j \in \mathbf{C}(X)[Y]$ is either a monic irreducible over $\mathbf{C}(X)$ or lies in $\mathbf{C}(X)$. It can also be assumed that the p_j's are pairwise distinct in $\mathbf{C}(X)[Y]$ by collecting repeated appearances of the same term and multiplying c_j by a positive integer that counts the number of repetitions.

Writing $h(x) = p(x, e^{g(x)})/q(x, e^{g(x)})$ for relatively prime $p, q \in \mathbf{C}(X)[Y]$ with q having leading coefficient equal to 1 as a polynomial in Y with coefficients in

$\mathbf{C}(X)$, (5.1) now says

$$\begin{aligned} f(x)e^{g(x)} &= \sum c_j \frac{p_j(x, e^{g(x)})'}{p_j(x, e^{g(x)})} + \left(\frac{p(x, e^{g(x)})}{q(x, e^{g(x)})}\right)' \\ &= \sum c_j \frac{p_j(x, e^{g(x)})'}{p_j(x, e^{g(x)})} + \frac{q(x, e^{g(x)})p(x, e^{g(x)})' - p(x, e^{g(x)})q(x, e^{g(x)})'}{q(x, e^{g(x)})^2}. \end{aligned} \tag{5.2}$$

Recall that our goal is to find $R \in \mathbf{C}(X)$ such that $R'(X) + g'(X)R(X) = f(X)$ in $\mathbf{C}(X)$, and a key step in this direction is to reformulate (5.2) in more algebraic terms. This rests on a lemma that passes between functional identities and algebraic identities:

LEMMA 5.1. *For nonconstant $g(X) \in \mathbf{C}(X)$, if $H_1, H_2 \in \mathbf{C}[X, Y]$ satisfy $H_1(x, e^{g(x)}) = H_2(x, e^{g(x)})$ as functions of x in a non-empty open interval I in $\mathbf{R}$ then $H_1 = H_2$ in $\mathbf{C}[X, Y]$.*

PROOF. Passing to $H_1 - H_2$ reduces us to showing that if $H \in \mathbf{C}[X, Y]$ satisfies $H(x, e^{g(x)}) = 0$ for $x \in I$ then $H = 0$ in $\mathbf{C}[X, Y]$. Suppose $H \neq 0$. Certainly H has to involve Y, as otherwise $H(X, 0) = H(X, Y)$ is a nonzero polynomial in X and the condition $H(x, 0) = H(x, e^{g(x)}) = 0$ says that the nonzero polynomial $H(X, 0)$ vanishes at the infinitely many points of I, an absurdity. Viewing H as an element in $\mathbf{C}[X, Y] \subseteq \mathbf{C}(X)[Y]$ and (without loss of generality) dividing through by the coefficient in $\mathbf{C}(X)$ for the top-degree monomial in Y appearing in H, we get a relation of the form

$$e^{ng(x)} + a_{n-1}(x)e^{(n-1)g(x)} + \cdots + a_1(x)e^{g(x)} + a_0(x) = 0 \tag{5.3}$$

for all x in a non-empty open interval, with $a_0, \ldots, a_{n-1} \in \mathbf{C}(X)$ and some $n > 0$. We consider such a relation with *minimal* $n \geq 0$. Obviously we must have $n \geq 1$, and some $a_j \in \mathbf{C}(X)$ must be nonzero.

Differentiating (5.3) gives

$$\begin{aligned} ng'(x)e^{ng(x)} + (a'_{n-1}(x) + (n-1)g'(x)a_{n-1}(x))e^{(n-1)g(x)} + \ldots \\ + (a'_1(x) + g'(x)a_1(x))e^{g(x)} + a'_0(x) = 0, \end{aligned}$$

yet $ng'(X) \in \mathbf{C}(X)$ is nonzero (as $g(X) \notin \mathbf{C}$), so dividing by $ng'(x)$ gives

$$e^{ng(x)} + \sum_{j=1^{n-1}} \frac{a'_j(x) + jg'(x)a_j(x)}{ng'(x)} e^{jg(x)} + \frac{a'_0(x)}{ng'(x)} = 0 \tag{5.4}$$

for x in a non-empty open interval in $\mathbf{R}$ on which (5.3) also holds.

We have now produced *two* "degree n" relations (5.3) and (5.4) for e^g with coefficients in $\mathbf{C}(X)$ and the same "degree n" term e^{ng}, so taking the difference gives a relation of degree $\leq n-1$ in e^g with coefficients in $\mathbf{C}(X)$. By the minimality of n, it follows that all $\mathbf{C}(X)$-coefficients in this difference relation must be identically zero (as otherwise we could divide by the nonzero $\mathbf{C}(X)$-coefficient corresponding to the highest degree in e^g to get a polynomial relation for e^g over $\mathbf{C}(X)$ with degree $< n$ in e^g and some nonzero coefficient in $\mathbf{C}(X)$, contrary to the minimality with respect to which n was chosen). Hence, for $0 \leq j \leq n-1$ we must have $a'_j + jg'a_j = ng'a_j$ in $\mathbf{C}(X)$. That is, $a'_j = (n-j)g'a_j$ in $\mathbf{C}(X)$ for all $0 \leq j \leq n-1$.

There is some j_0 such that $a_{j_0} \neq 0$ in $\mathbf{C}(X)$, and for such a j_0 we have

$$(n - j_0)g'(X) = \frac{a'_{j_0}(X)}{a_{j_0}(X)}$$

in $\mathbf{C}(X)$. By the Fundamental Theorem of Algebra, $a_{j_0}(X) = c\prod(X-\rho_k)^{e_k}$ for some nonzero $c \in \mathbf{C}$, nonzero integers e_k, and pairwise distinct $\rho_k \in \mathbf{C}$, so

$$(n-j_0)g'(X) = \frac{a'_{j_0}(X)}{a_{j_0}(X)} = \sum \frac{e_k}{X-\rho_k}.$$

Since g is *nonconstant* and $n-j_0 \neq 0$, so $(n-j_0)g'(X) \neq 0$, there must be some ρ_k's and hence $g'(x)$ behaves like a nonzero constant multiple of $1/|x-\rho_1|$ as $x \to \rho_1$. But this inverse-linear growth for the derivative of a nonconstant rational function is impossible: we may write $g(X) = (X-\rho_1)^{\mu}G(X)$ in $\mathbf{C}(X)$, where $\mu \in \mathbf{Z}$ and the nonzero $G(X) \in \mathbf{C}(X)$ has numerator and denominator that are non-vanishing at ρ_1, and by separately treating the cases $\mu < 0$, $\mu = 0$, and $\mu > 0$ we never get inverse-linear growth for $|g'(x)|$ as $x \to \rho_1$. This is a contradiction. □

By Lemma 5.1, if $H \in \mathbf{C}[X,Y]$ is nonzero then $1/H(x, e^{g(x)})$ makes sense as a meromorphic function. Hence, for any rational function $F = G(X,Y)/H(X,Y)$ with $G, H \in \mathbf{C}[X,Y]$ and $H \neq 0$, we get a well-defined meromorphic function $F(x, e^{g(x)}) = G(x, e^{g(x)})/H(x, e^{g(x)})$ for x in a suitable non-empty open interval in $\mathbf{R}$. We now use this to recast (5.2) is more algebraic terms by eliminating the appearance of exponentials as follows. Observe that $(e^{g(x)})' = g'(x)e^{g(x)}$, so more generally for any $h(X,Y) \in \mathbf{C}(X)[Y]$ we have $h(x, e^{g(x)})' = (\partial h)(x, e^{g(x)})$ where $\partial : \mathbf{C}(X)[Y] \to \mathbf{C}(X)[Y]$ is the operator

$$\partial(\sum r_j(X)Y^j) = \sum(r'_j(X) + jg'(X)r_j(X))Y^j.$$

This leads to the following reformulation of (5.2):

Lemma 5.2. *The identity*

$$f(X)Y = \sum_j c_j \frac{\partial(p_j(X,Y))}{p_j(X,Y)} + \frac{q(X,Y)\partial(p(X,Y)) - p(X,Y)\partial(q(X,Y))}{q(X,Y)^2} \tag{5.5}$$

holds in the two-variable rational function field $\mathbf{C}(X,Y)$.

Proof. By converting both sides into functions of x upon replacing Y with $e^{g(x)}$ we get the equality (5.2) since $(\partial h)(x, e^{g(x)}) = h(x, e^{g(x)})'$ for any $h(X,Y) \in \mathbf{C}(X)[Y]$. Thus, after cross-multiplying by a common denominator on both sides we may use Lemma 5.1 to complete the proof. □

Our aim is to use the identity (5.5) to find some $R \in \mathbf{C}(X)$ such that $f(X) = R'(X) + g'(X)R(X)$ in $\mathbf{C}(X)$. Since (5.5) is an algebraic identity that does not involve the intervention of exponentials, this is a *purely algebraic* problem and so is good progress! The idea for solving the algebraic problem is to analyze how the terms on the right side of (5.5) can possibly have enough cancellation in the denominators so that the right side can be equal to the left side whose denominator lies in $\mathbf{C}[X]$ (i.e., it involves no Y's). Once we take into account how such cancellation can arise, a comparison of Y-coefficients on both sides of (5.5) will give the desired identity $f(X) = R'(X) + g'(X)R(X)$ for a suitable $R \in \mathbf{C}(X)$.

We now look at denominators on the right side of (5.5). The jth term in the sum on the right side contributes a *possible* denominator of $p_j(X,Y)$ when $p_j(X,Y)$ is a monic irreducible in $\mathbf{C}(X)[Y]$. (Recall that each $p_j(X,Y)$ is an irreducible in $\mathbf{C}(X)[Y]$ that is monic in Y or is a nonconstant element of $\mathbf{C}(X)$). The reason we speak of a "possible" denominator $p_j(X,Y)$ for such j is that we have to take into account that such an irreducible p_j might divide $\partial(p_j(X,Y))$ in $\mathbf{C}(X)[Y]$. Suppose such divisibility occurs for some j_0. Writing $p_{j_0} = Y^m + r_{m-1}(X)Y^{m-1} + \cdots + r_0(X)$ with $r_k(X) \in \mathbf{C}(X)$ and $m > 0$, we compute

$$\partial(p_{j_0}) = mg'Y^m + (r'_{m-1} + (m-1)g'r_{m-1})Y^{m-1} + \cdots + (r'_1 + g'r_1)Y + r'_0$$

and so consideration of Y-degrees (and the nonvanishing of both $g' \in \mathbf{C}(X)$ and m) implies that the irreducible p_{j_0} divides $\partial(p_{j_0})$ in $\mathbf{C}(X)[Y]$ if and only if $\partial(p_{j_0}) = mg'p_{j_0}$. This says $mg'r_k = (r'_k - kg'r_k)$ for all $0 \leq k < m$, or in other words $r'_k = (m-k)g'r_k$ for all such k. As we saw in the proof of Lemma 5.1, since $g \in \mathbf{C}(X)$ is nonconstant and $m > 0$ such a condition with all $r_k \in \mathbf{C}(X)$ can only hold if $r_k = 0$ for all $k < m$, which is to say that the monic irreducible $p_{j_0} \in \mathbf{C}(X)[Y]$ is equal to Y.

To summarize, for each j_0 such that $p_{j_0} \in \mathbf{C}(X)[Y]$ is a monic irreducible *distinct* from Y, the sum over j on the right in (5.5) *does* contribute a denominator of p_{j_0} to the first power. There is no such term in the denominator on the left side of (5.5), so if such p_{j_0}'s arise then they must cancel against a contribution from a factor of p_{j_0} in the denominator q in the final term on the right in (5.5). But if any such factor appears in q with some multiplicity $\mu \geq 1$ then in the *reduced form* of the ratio $-p\partial(q)/q^2$ this term appears in the denominator with multiplicity exactly $\mu + 1$ (as we have just seen above that p_{j_0} does *not* divide $\partial(p_{j_0})$ for such j_0, and the operator ∂ satisfies the "Leibnitz rule" $\partial(G_1G_2) = G_1\partial(G_2) + G_2\partial(G_1)$). This multiplicity of order $\mu + 1$ is not cancelled on the right side of (5.5), and so this contradicts the absence of such a term in the denominator on the left side of (5.5). Hence, we have proved that (i) there is at most one p_j in the sum over j on the right side of (5.5) that is not in $\mathbf{C}(X)$, with the unique exceptional term (if any occurs) equal to Y, and (ii) the monic $q \in \mathbf{C}(X)[Y]$ cannot have any irreducible factor other than Y. Hence, $q = a(X)Y^n$ with $n \geq 0$ and $a(X) \in \mathbf{C}(X)$ a nonzero element.

We conclude that $h = p/q = \sum_k s_kY^k$ with $s_k \in \mathbf{C}(X)$ and k ranging through a finite set of integral values (possibly negative). Among the terms p_j that arise, the ones that lie in $\mathbf{C}(X)$ do not contribute to any Y-terms, and since the only other possible term $p_j = Y$ satisfies $c_j\partial(p_j)/p_j = c_jg'(X)$ it also does not contribute to any Y-terms. Hence, the left side $f(X)Y$ in (5.5) must coincide with the linear Y-term in $\partial(h) = \sum_k(s'_k + kg's_k)Y^k$, which is to say $f(X)Y = (s'_1 + g's_1)Y$ in $\mathbf{C}(X)[Y]$. This gives $s'_1 + g's_1 = f$ with $s_1 \in \mathbf{C}(X)$, so $R(X) = s_1(X) \in \mathbf{C}(X)$ satisfies $R'(X) + g'(X)R(X) = f(X)$.

References

[1] M. Rosenlicht, *Integration in finite terms*, American Math. Monthly **79** (1972), 963–972.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043
E-mail address: bconrad@umich.edu