Dynamics on the space of lattices and number theory

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Introduction

- in dynamics we study actions of groups with complicated orbits

- a particularly appealing case: flows on locally homogeneous spaces, an example of which is space of unimodular latices in $\mathbb{R}^n$(for example $n = 3$).

- study of these are very specific dynamical systems involves deep issues and has many applications, e.g. Oppenheim conjecture (proved), Littlewood conjecture (open), diophantine approximation, and more!

The space of lattices

- If $v_1, \ldots, v_n$ are n linearly independent vectors in $\mathbb{R}^n$ then
$$\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n$$
will be called a lattice (in $\mathbb{R}^n$).
- Many different $v_1, \ldots, v_n$ generates the same Λ.
$$\mathrm{covol}\,\Lambda := \det(v_1, \ldots, v_n) = \mathrm{vol}(\mathbb{R}^n/\Lambda)$$
does not depend on choice of generators.
- Space of (unimodal) lattices is
$$X_n = \{\text{ lattices } \Lambda < \mathbb{R}^n : \mathrm{covol}\,\Lambda = 1\} \cong \mathrm{SL}(n,\mathbb{Z})\backslash\,\mathrm{SL}(n,\mathbb{R})$$
isomorphism given by
$$\mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n \mapsto \mathrm{SL}(n,\mathbb{Z})(v_1, \ldots, v_n)^T$$
- Special case of $\Gamma\backslash G$ where G is Lie group and Γ discrete subgroup of G.

Properties of the space lattices

- SL$(n,\mathbb{R})$ acts on X_n by SL$(n,\mathbb{Z})h.g = $ SL$(n,\mathbb{Z})(hg)$; in terms of lattices $\Lambda.g = \left\{g^T v : v \in \Lambda\right\}$.

- Haar measure on SL$(n,\mathbb{R})$ projects to a SL$(n,\mathbb{R})$ invariant measure ν on X_n; unique up to a scalar. A nontrivial theorem says ν is finite.

- even though X_n as finite volume, it is not compact:

  Mahler's criterion: $\Lambda_i \to \infty$ if and only if there are $v_i \in \Lambda_i$ with $\|v_i\| \to 0$.

  for every $\epsilon > 0$, $\{\Lambda \in X_n : \forall v \in \Lambda,\ \|v\| \geq \epsilon\}$ is compact.

Example: the case $n = 2$

SL$(2, \mathbb{R})$ acts on $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az + b}{cz + d}.$$

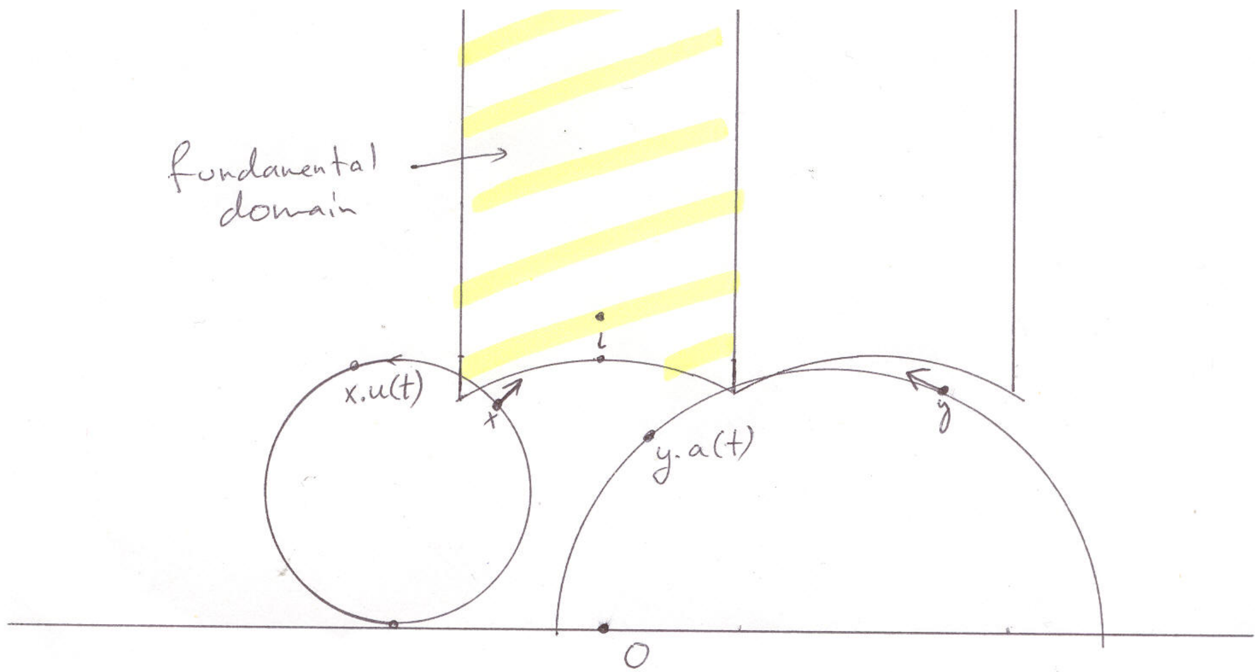
$i \in \mathbb{H}$ is fixed by SO$(2, \mathbb{R})$ so $\mathbb{H} \cong$ SL$(2, \mathbb{R}) /$ SO$(2, \mathbb{R})$ and unit tangent bundle $S\mathbb{H} \cong$ SL$(2, \mathbb{R}) / \{\pm I\}$.

In particular: X_2 can be identified with the unit tangent bundle of surface $M =$ SL$(2, \mathbb{Z}) \backslash \mathbb{H}$.

Two important 1-param subgroups acting on X_2:

$a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

$x \to x.a(t)$ gives geodesic flow on SM; $x \to a.u(t)$ gives horocyclic flow.



Geodesic flow and horocyclic flow are very different!

Theorem 1 (Hedlund). *Any orbit of the horocyclic flow is either periodic or dense. In particular any bounded orbit is periodic.*

Compare with folklore theorem:

Theorem 2. *For any $d \in [0, 2)$ there is nonperiodic bounded orbit $\{x.g(t)\}$ of the geodesic flow so that* $\dim_H \overline{\{x.g(t)\}} = d$.

Note: geodesic flow is ergodic, so for a.e. x we have that $\{x.g(t)\}$ is dense. However, unlike case of horocyclic flow, this gives nothing for specific x . . .

Unipotent flows

$g \in \mathsf{SL}(n, \mathbb{R})$ is unipotent if $(g - I)^n = 0$. Example: $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

A lot is known about actions of unipotent one parameter groups, for instance

Theorem 3 (Ratner). *G Lie group, Γ discrete subgroup with finite covolume, $X = \Gamma \backslash G$ (for example X_n), $u(t)$ unipotent one parameter group. For every $x \in X$ there is a connected subgroup $L < G$ so that xL is closed and $\{xu(t)\}$ is equidistributed (in particular, dense) in xL.*

This is only special case of Ratner's theorems, which also covers actions of groups generated by one parameter unipotent subgroups (such as $\mathsf{SL}(k, \mathbb{R})$ but not, e.g., group of $n \times n$ diagonal matrices).

Oppenheim's conjecture = Margulis' theorem

Let $Q(x) = \sum_{i,j} a_{i,j} x_i x_j$ be an indefinite quadratic form in $n \geq 3$ variables (interesting case: 3 vars). If $a_{i,j} \in \mathbb{Z}$ for all i, j then $Q(\mathbb{Z}^n) \subset \mathbb{Z}$.

Conjecture 4 (Oppenheim). *Suppose Q is not a scalar multiple of a quadratic form with integer coefficients. for every $\epsilon > 0$ there is a $z \in \mathbb{Z}^n$ so that $|Q(z)| < \epsilon$ (stronger form: $0 < |Q(z)| < \epsilon$).*

Margulis proved this using the following: Set $Q_0(x) = 2x_1x_3 - x_2^2$, $H = \{h \in \mathrm{SL}(3, \mathbb{R}) : Q_0 \circ h = Q_0\}$ ($H \cong \mathrm{SL}(2, \mathbb{R})$ & generated by unipotents).

Theorem 5 (Margulis). *Any H orbit in X_3 is either closed or dense.*

Key observation: write $Q(\mathbb{Z}^3) = cQ_0(\Lambda)$ with $\Lambda \in X_3$. If Λ contains small vectors, $Q_0(\Lambda)$ has small values.

Littlewood's conjecture

For $a \in \mathbb{R}$, let $\|a\| = \min_{n \in \mathbb{Z}} |a - n|$.
Conjecture 6 (Littlewood). *For any real α, β,*

$$\varliminf_{n \to \infty} n \|n\alpha\| \|n\beta\| = 0.$$

Will follow from dynamical statement:
Conjecture 7 (Margulis). *Let $A < \mathrm{SL}(3, \mathbb{R})$ be group of 3×3 diagonal matrices. Then any bounded A orbit in X_3 is actually closed.*

One reason this is difficult: false for any 1-param subgroup of A (which behave similarly to geodesic flow).

Connection: via the lattice $\Lambda_{\alpha,\beta}$ generated by

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Diophantine approximations

- $\mathbf{y} \in \mathbb{R}^n$ is said to be very well approximable (VWA) if $\exists \delta > 0$ and ∞ many $\mathbf{p} \in \mathbb{Z}^n, q \in \mathbb{Z}_+$ to

$$\|q\mathbf{y} - \mathbf{p}\| < q^{-\frac{1+\delta}{n}}.$$

- slightly less restrictive notion: $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is very well multiplicatively approximable (VWMA) if ∞ many solutions to

$$\prod_{i=1}^{n} |qy_i - p_i| < q^{-(1+\delta)}.$$

- set of very well multiplicatively approximable $\mathbf{y}$ has measure zero, $\dim_H = n$.

Diophantine approximations and the space of lattices

The Diophantine properties of $\mathbf{y} \in \mathbb{R}^n$ correspond to properties of orbit of lattice $\Lambda_{\mathbf{y}} \in X_{n+1}$ generated by

$$\begin{pmatrix} 1 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

For $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$ let $g_{\mathbf{t}} = diag(e^{-t}, e^{t_1}, \ldots, e^{t_n})$, $t = \sum_{i=1}^n t_i$.

Action of $g_{\mathbf{t}}$ on a lattice Λ contracts first component of every vector of Λ and expand the remaining components.

Proposition 8. *For $\mathbf{y} \in \mathbb{R}^n$, TFAE:*

(i) $\mathbf{y}$ is not very well multiplicatively approximable;

(ii) $\forall \gamma > 0$, $\min_{v \in g_{\mathbf{t}} \Lambda_{\mathbf{y}}} \|v\| > e^{-\gamma t}$ for $\mathbf{t} \in \mathbb{R}^n_+$ large.

Diophantine approximations on manifolds and fractal sets

- Mahler conjectured, and Sprindžuk proved (30 years later): for a.e. $x \in \mathbb{R}$, the point $(x, x^2, \ldots, x^n)$ is not VWA.

- A submanifold $M \subset \mathbb{R}^n$ is extremal (strongly extremal) if almost every point on M is not VWA (VWMA), nonplanar if $\not\subset$ any affine hyperplane. Sprindžuk proved any nonplanar algebraic variety is extremal.

- Compare with: for every $x \in \mathbb{R}$, the point $(x, x, \ldots, x)$ is VWA.

- Kleinbock-Margulis: nonplanar real analytic manifold $\subset \mathbb{R}^n$ are strongly extremal (conjectured by Sprindžuk). Proof via transition to space of lattices and using ideas developed to prove non-divergence of unipotent (!) flows.

- More generally: can consider measures. μ on $\mathbb{R}^n$ is strongly extremal if μ-a.e. pt. is not VWMA.
- Volume (or area etc.) on non planar real analytic manifolds is strongly extremal
- natural measure on Sierpinski Gasket, Cantor set, and many other fractals, and their images under real analytic maps (as long as it's not planar!) 

  All these measures μ satisfy three natural conditions (non planar, Federer, decaying) that can be shown to imply strong extremality (Kleinbock-L-Weiss).

A key fact used in Kleinbock-Margulis proof

Question: *How can you prove a lattice Λ has no short vectors?*

A flag is a chain of linear subspaces $0 < V_1 < \cdots < V_{n-1} < \mathbb{R}^n$ with $\dim V_i = i$. It is Λ rational if $\Lambda \cap V_i$ a lattice in V_i for all i.

Proposition 9. *If $\exists$ a Λ-rational flag $0 < V_1 < \cdots < V_{n-1} < \mathbb{R}^n$ with*

$$\theta < \text{covol}(\Lambda \cap V_i) < \Theta \qquad \textit{with } \theta < 1 < \Theta$$

then the size of every vector in Λ is at least $\|v\| < \frac{\Theta}{\theta}$.

Finding one such flag shows all vectors aren't too short.

Conclusion

- Many problems in number theory translates naturally to questions about orbits in the space of lattices.

- There are powerful methods from dynamical systems and ergodic theory to deal with these questions.

- There are also a lot of deep open questions.